Predicate Logic

&

Math Background

Predicate Logic

Predicate logic over integer expressions:

a language of logical assertions, for example

$$\forall \mathsf{x}.\ \mathsf{x} + 0 = \mathsf{x}$$

Why discuss predicate logic?

- It is an example of a simple language
- It has simple denotational semantics
- We will use it later in program specifications

Abstract Syntax

Describes the structure of a phrase
ignoring the details of its representation.

An abstract grammar for predicate logic over integer expressions:

$$
\begin{aligned}
\mathit{intexp} ::= {} & \mathsf{0} \mid \mathsf{1} \mid \ldots \\
& \mid \mathit{var} \\
& \mid -\mathit{intexp} \mid \mathit{intexp} + \mathit{intexp} \mid \mathit{intexp} - \mathit{intexp} \mid \ldots
\end{aligned}
$$

$$
\begin{aligned}
\mathit{assert} ::= {} & \mathbf{true} \mid \mathbf{false} \\
& \mid \mathit{intexp} = \mathit{intexp} \mid \mathit{intexp} < \mathit{intexp} \mid \mathit{intexp} \leq \mathit{intexp} \mid \ldots \\
& \mid \neg \mathit{assert} \mid \mathit{assert} \wedge \mathit{assert} \mid \mathit{assert} \vee \mathit{assert} \\
& \mid \mathit{assert} \Rightarrow \mathit{assert} \mid \mathit{assert} \Leftrightarrow \mathit{assert} \\
& \mid \forall \mathit{var}.\ \mathit{assert} \mid \exists \mathit{var}.\ \mathit{assert}
\end{aligned}
$$

Resolving Notational Ambiguity

- Using parentheses: $(\forall \mathsf{x}.\ ((((\mathsf{x}) + (\mathsf{0})) + \mathsf{0}) = (\mathsf{x})))$

- Using precedence and parentheses: $\forall \mathsf{x}.\ (\mathsf{x} + \mathsf{0}) + \mathsf{0} = \mathsf{x}$

  arithmetic operators ($*$ / $\mathbf{rem}\ldots$) with the usual precedence
  relational operators ($= \neq < \leq\ \ldots$)
  $\neg$
  $\wedge$
  $\vee$
  $\Rightarrow$
  $\Leftrightarrow$

- The body of a quantified term extends to a delimiter.

Carriers and Constructors

- Carriers: sets of abstract phrases (e.g. $intexp$, $assert$)
- Constructors: specify abstract grammar productions

$$intexp ::= 0 \longrightarrow c_0 \in \{\langle\rangle\} \to intexp$$
$$intexp ::= intexp + intexp \longrightarrow c_+ \in intexp \times intexp \to intexp$$

Note: Independent of the concrete pattern of the production:

$$intexp ::= \textbf{plus}\ intexp\ intexp \longrightarrow c_+ \in intexp \times intexp \to intexp$$

- Constructors must be injective and have disjoint ranges

- Carriers must be either predefined or their elements must be constructible in finitely many constructor applications

Inductive Structure of Carrier Sets

With these properties of constructors and carriers,

carriers can be defined inductively:

$$
\begin{aligned}
intexp^{(0)} &= \{\} \\
intexp^{(j+1)} &= \{c_0\langle\rangle, \ldots\} \cup \{c_+(x_0, x_1) \mid x_0, x_1 \in intexp^{(j)}\} \cup \ldots \\
assert^{(0)} &= \{\} \\
assert^{(j+1)} &= \{c_{\mathbf{true}}\langle\rangle,\ c_{\mathbf{false}}\langle\rangle\} \\
&\quad \cup \{c_=(x_0, x_1) \mid x_0, x_1 \in intexp^{(j)}\} \cup \ldots \\
&\quad \cup \{c_\neg(x_0) \mid x_0 \in assert^{(j)}\} \cup \ldots
\end{aligned}
$$

$$
\begin{aligned}
intexp &= \bigcup_{j=0}^{\infty} intexp^{(j)} \\
assert &= \bigcup_{j=0}^{\infty} assert^{(j)}
\end{aligned}
$$

Denotational Semantics of Predicate Logic

The meaning of a term $e \in \mathit{intexp}$ is $[\![e]\!]_{\mathit{intexp}}$

i.e. the function $[\![-]\!]_{\mathit{intexp}}$ maps intexp objects to their meanings.

What is the set of meanings?

The meaning $[\![5 + 37]\!]_{\mathit{intexp}}$ of the term $\underbrace{5 + 37}_{\text{(that is, } c_{+}(c_{5}\langle\rangle,\, c_{37}\langle\rangle))}$ could be the integer 42.

However the term $x + 5$ contains the free variable x,

so the meaning of an intexp in general cannot be an integer...

Mathematical Background

- Sets
- Relations
- Functions
- Sequences
- Products and Sums

Sets

$x \in S$	membership	$\{\}$	the empty set
$x \in! S$	$S = \{x\}$	$\mathbf{N}$	natural numbers
$S \subseteq T$	inclusion	$\mathbf{Z}$	integers
$S \subseteq^{\mathsf{fin}} T$	finite subset	$\mathbf{B}$	$= \{\mathbf{true}, \mathbf{false}\}$

$\{E \mid P\}$	set comprehension	
$S \cap T$	intersection	$= \{x \mid x \in S \text{ and } x \in T\}$ x is a bound variable
$S \cup T$	union	$= \{x \mid x \in S \text{ or } x \in T\}$
$S - T$	difference	$= \{x \mid x \in S \text{ and not } x \in T\}$
$\mathcal{P}\, S$	powerset	$= \{T \mid T \subseteq S\}$
m **to** n	integer range	$= \{x \mid m \leq x \text{ and } x \leq n\}$

Generalized Set Operations

$$\cup S \stackrel{\text{def}}{=} \{x \mid \exists T \in S.\, x \in T\} \qquad \cap S \stackrel{\text{def}}{=} \{x \mid \forall T \in S.\, x \in T\}$$

$$\bigcup_{i \in I} S \stackrel{\text{def}}{=} \cup\{S \mid i \in I\} \qquad \bigcap_{i \in I} S \stackrel{\text{def}}{=} \cap\{S \mid i \in I\} \ldots$$

$$\bigcup_{i=m}^{n} S \stackrel{\text{def}}{=} \bigcup_{i \in m \textbf{ to } n} S \qquad \bigcap_{i=m}^{n} S \stackrel{\text{def}}{=} \bigcap_{i \in m \textbf{ to } n} S$$

$$\cup\{\} = \{\} \qquad \cap\{\} \quad \text{meaningless}$$

Examples:

$$A \cup B = \cup\{A, B\}$$

$$\cup\left\{i \textbf{ to } (i+1) \;\middle|\; i \in \{j^2 \mid j \in 1 \textbf{ to } 3\}\right\} = \{1, 2, 4, 5, 9, 10\}$$

Relations

A relation ρ is a set of primitive pairs $[x, y]$.

$$\rho \text{ relates } x \text{ and } y \iff x\,\rho\,y \iff [x, y] \in \rho$$

$$\rho \text{ is an identity relation} \iff (\forall x, y.\, x\,\rho\,y \Rightarrow x = y)$$

$$\begin{array}{rcl}
\text{the identity on } S & I_S & \stackrel{\text{def}}{=} \{[x, x] \mid x \in S\} \\
\text{the domain of } \rho & \operatorname{dom} \rho & \stackrel{\text{def}}{=} \{x \mid \exists y.\, x\,\rho\,y\} \\
\text{the range of } \rho & \operatorname{ran} \rho & \stackrel{\text{def}}{=} \{x \mid \exists y.\, y\,\rho\,x\} \\
\text{composition of } \rho \text{ with } \rho' & \rho' \cdot \rho & \stackrel{\text{def}}{=} \{[x, z] \mid \exists y.\, x\,\rho\,y \text{ and } y\,\rho'\,z\} \\
\text{reflection of } \rho & \rho^\dagger & \stackrel{\text{def}}{=} \{[y, x] \mid [x, y] \in \rho\}
\end{array}$$

Relations: Properties and Examples

$$(\rho_3 \cdot \rho_2) \cdot \rho_1 = \rho_3 \cdot (\rho_2 \cdot \rho_1)$$
$$\rho \cdot I_S \subseteq \rho \supseteq I_T \cdot \rho$$
$$\mathrm{dom}\, I_S = S = \mathrm{ran}\, I_S$$
$$I_T \cdot I_S = I_{T \cap S}$$
$${I_S}^\dagger = I_S$$
$$(\rho^\dagger)^\dagger = \rho$$
$$(\rho_2 \cdot \rho_1)^\dagger = {\rho_1}^\dagger \cdot {\rho_2}^\dagger$$
$$\rho \cdot \{\} = \{\} = \{\} \cdot \rho$$
$$I_{\{\}} = \{\} = \{\}^\dagger$$
$$\mathrm{dom}\, \rho = \{\} \;\Rightarrow\; \rho = \{\}$$

$$I_{\mathbf{N}} = \{[0,0],[1,1],[2,2],\ldots\}$$
$$< \; = \{[0,1],[0,2],[1,2],\ldots\}$$
$$\leq \; = \{[0,0],[0,1],[1,1],[0,2],\ldots\}$$
$$\geq \; = \{[0,0],[1,0],[1,1],[2,0],\ldots\}$$
$$< \; \subseteq \; \leq$$
$$< \cup I_{\mathbf{N}} = \; \leq$$
$$\leq \cap \geq \; = I_{\mathbf{N}}$$
$$< \cap \geq \; = \{\}$$
$$< \cdot \leq \; = \; <$$
$$\leq \cdot \leq \; = \; \leq$$
$$\geq \; = \; \leq^\dagger$$

Functions

A relation f is a function if

$$\forall x, x', x''.\, ([x, x'] \in f \text{ and } [x, x''] \in f) \Rightarrow x' = x''$$

If f is a function,

$$f\ x = y \iff f_x = y \iff f \text{ maps } x \text{ to } y \iff [x, y] \in f$$

I_S and $\{\}$ are functions.

If f and g are functions, then $g \cdot f$ is a function: $(g \cdot f)\ x = g(f\ x)$

$f^\dagger$ is not necessarily a function:

consider $f = \{[\mathbf{true}, \{\}], [\mathbf{false}, \{\}]\}$

f is an injection if both f and $f^\dagger$ are functions.

Notation for Functions

Typed abstraction: $\lambda x \in S.\, E \stackrel{\text{def}}{=} \{[x, E] \mid x \in S\}$

Defined only when E is defined for all $x \in S$

(consider $\lambda g \in \mathbf{N}.\, g\ 3$)

$I_S = \lambda x \in S.\, x$

$g \cdot f = \lambda x \in \text{dom } f.\, g(f\ x)$, if ran $f \subseteq$ dom g.

Placeholder: E with a dash $(-)$ standing for the bound variable

$g\ (-)\ h \;=\; \lambda x \in S.\, (g\ (x))\ h \qquad - + 42 \;=\; \lambda x \in \mathbf{N}.\, x + 42$

Variation of a function f: $[f \mid x : y]\ z = \begin{cases} y, & \text{if } z = x \\ f\ z, & \text{otherwise} \end{cases}$

$$\text{dom } [f \mid x : y] \;=\; (\text{dom } f) \cup \{x\}$$

$$\text{ran } [f \mid x : y] \;=\; ((\text{ran } f) - \{z \mid [x, z] \in f\}) \cup \{y\}$$

Sequences

$$[f \mid x_1 : y_1 \mid \ldots \mid x_n : y_n] \stackrel{\text{def}}{=} [\ldots[f \mid x_1 : y_1] \ldots \mid x_n : y_n]$$

$$[x_1 : y_1 \mid \ldots \mid x_n : y_n] \stackrel{\text{def}}{=} [\{\} \mid x_1 : y_1 \mid \ldots \mid x_n : y_n]$$

$$\langle x_0, \ldots x_{n-1} \rangle \stackrel{\text{def}}{=} [0 : x_0 \mid \ldots n-1 : x_{n-1}]$$

$[\,] = \{\}$ — the empty function

$\langle\rangle = [\,] = \{\}$ — the empty sequence

$\langle x_0, \ldots x_{n-1} \rangle$ — an n-tuple

$\langle x, y \rangle$ — a (non-primitive) pair

$\mathrm{dom}\ \langle x_0, \ldots x_{n-1} \rangle = 0 \textbf{ to } (n-1)$

$\langle x_0, \ldots x_{n-1} \rangle_i = x_i$ when $i \in 0 \textbf{ to } (n-1)$

Products

Let θ be an indexed family of sets (a function with sets in its range).

The Cartesian product of θ is

$$\prod \theta \stackrel{\text{def}}{=} \{f \mid \text{dom } f = \text{dom } \theta \text{ and } \forall i \in \text{dom } \theta.\, f\, i \in \theta\, i\}$$

$$\begin{aligned}
&\prod \langle \mathbf{B}, \mathbf{B} \rangle \\
&\quad = \prod(\lambda x \in 0 \text{ to } 1.\, \mathbf{B}) \\
&\quad = \{[0 : \mathbf{true}, 1 : \mathbf{true}],\ [0 : \mathbf{true}, 1 : \mathbf{false}], \\
&\qquad [0 : \mathbf{false}, 1 : \mathbf{true}],\ [0 : \mathbf{false}, 1 : \mathbf{false}]\} \\
&\quad = \{\langle \mathbf{true}, \mathbf{true} \rangle,\ \langle \mathbf{true}, \mathbf{false} \rangle,\ \langle \mathbf{false}, \mathbf{true} \rangle,\ \langle \mathbf{false}, \mathbf{false} \rangle\}
\end{aligned}$$

More Products

$$\prod_{x \in T} S \stackrel{\text{def}}{=} \prod \lambda x \in T.\, S \qquad S_1 \times \ldots \times S_n \stackrel{\text{def}}{=} \prod_{i=1}^{n} \text{“}S_i\text{”}$$

$$\prod_{i=m}^{n} S \stackrel{\text{def}}{=} \prod_{i \in (m \textbf{ to } n)} S \qquad S^T \stackrel{\text{def}}{=} \prod_{x \in T} S$$

$$S^n \stackrel{\text{def}}{=} S^{0 \textbf{ to } (n-1)} = \underbrace{S \times \ldots \times S}_{n \text{ times}}$$

$$\prod \langle \mathbf{B}, \mathbf{B} \rangle = \mathbf{B} \times \mathbf{B} = \mathbf{B}^2$$

$$S^0 = S^{\{\}} = \{\langle\rangle\} = \{\{\}\}$$

Sets of Sequences

$$S^{+} \stackrel{\text{def}}{=} \bigcup_{i=1}^{\infty} S^{i}$$

$$S^{*} \stackrel{\text{def}}{=} S^{0} \cup S^{+}$$

$$S^{\infty} \stackrel{\text{def}}{=} S^{*} \cup S^{\mathbf{N}}$$

Let $\mathbf{U} = \{\langle\rangle\}$

$$\mathbf{U}^{+} = \{\langle\langle\rangle\rangle,\ \langle\langle\rangle, \langle\rangle\rangle, \langle\langle\rangle, \langle\rangle, \langle\rangle\rangle, \ldots \text{(finite)}\}$$

$$\mathbf{U}^{*} = \{\langle\rangle, \langle\langle\rangle\rangle,\ \langle\langle\rangle, \langle\rangle\rangle, \langle\langle\rangle, \langle\rangle, \langle\rangle\rangle, \ldots \text{(finite)}\}$$

$$\mathbf{U}^{\infty} = \{\langle\rangle, \langle\langle\rangle\rangle,\ \langle\langle\rangle, \langle\rangle\rangle, \langle\langle\rangle, \langle\rangle, \langle\rangle\rangle, \ldots \text{(infinite)}\}$$

Sums

The disjoint union (sum) of θ is

$$\textstyle\sum \theta \stackrel{\text{def}}{=} \{\langle i, x\rangle \mid i \in \mathrm{dom}\,\theta \ \text{ and } \ x \in \theta\, i\}$$

$$\sum_{x \in T} S \stackrel{\text{def}}{=} \textstyle\sum \lambda x \in T.\, S \qquad\qquad S_1 + \ldots + S_n \stackrel{\text{def}}{=} \sum_{i=1}^{n} \text{“}S_i\text{”}$$

$$\sum_{i=m}^{n} S \stackrel{\text{def}}{=} \sum_{i \in (m \ \mathbf{to}\ n)} S \qquad\qquad T \times S = \sum_{x \in T} S$$

$$n \times S = (0\ \mathbf{to}\ (n-1)) \times S = \underbrace{S + \ldots + S}_{n \text{ times}}$$

$$\begin{aligned} \mathbf{B} + \mathbf{B} &= \textstyle\sum \langle \mathbf{B}, \mathbf{B}\rangle = \{\langle 0, \mathbf{true}\rangle, \langle 0, \mathbf{false}\rangle, \langle 1, \mathbf{true}\rangle, \langle 1, \mathbf{false}\rangle\} \\ &= 2 \times \mathbf{B} \end{aligned}$$

Sums

Let θ be an indexed family of sets (a function with sets in its range).

The disjoint union (sum) of θ is

$$\textstyle\sum \theta \stackrel{\text{def}}{=} \{\langle i, x\rangle \mid i \in \text{dom}\,\theta \text{ and } x \in \theta\, i\}$$

$$\sum_{x \in T} S \stackrel{\text{def}}{=} \sum \lambda x \in T.\, S \qquad\qquad S_1 + \ldots + S_n \stackrel{\text{def}}{=} \sum_{i=1}^{n} \text{“}S_i\text{”}$$

$$\sum_{i=m}^{n} S \stackrel{\text{def}}{=} \sum_{i \in (m \textbf{ to } n)} S \qquad\qquad T \times S = \sum_{x \in T} S$$

$$n \times S = (0 \textbf{ to } (n-1)) \times S = \underbrace{S + \ldots + S}_{n \text{ times}}$$

$$\begin{aligned} \mathbf{B} + \mathbf{B} = \textstyle\sum \langle \mathbf{B}, \mathbf{B}\rangle &= \{\langle 0, \mathbf{true}\rangle, \langle 0, \mathbf{false}\rangle, \langle 1, \mathbf{true}\rangle, \langle 1, \mathbf{false}\rangle\} \\ &= 2 \times \mathbf{B} \end{aligned}$$

Functions of Multiple Arguments

- Use tuples instead of multiple arguments:

$$f\ (a_0,\ \ldots a_{n-1}) \longrightarrow f\ \langle a_0,\ \ldots a_{n-1}\rangle$$

  Syntactic sugar:

$$\begin{aligned}&\lambda\langle x_0 \in S_0,\ \ldots,\ x_{n-1} \in S_{n-1}\rangle.\ E\\ &\quad \stackrel{\text{def}}{=}\ \lambda x \in S_0 \times \ldots \times S_{n-1}.\ (\lambda x_0 \in S_0.\ \ldots \lambda x_{n-1} \in S_{n-1}.\ E)\\ &\qquad\qquad\qquad\qquad\qquad\qquad (x\ 0) \ldots (x (n-1))\end{aligned}$$

- Use Currying:

$$\begin{aligned}f\ (a_0,\ \ldots a_{n-1}) \longrightarrow\ & f\ a_0\ \ldots\ a_{n-1}\\ =\ & (\ldots (f\ a_0)\ \ldots)\ a_{n-1}\end{aligned}$$

  where f is a Curried function $\lambda x_0 \in S_0.\ \ldots \lambda x_{n-1} \in S_{n-1}.\ E$.

Relations Between Sets

ρ is a relation from S to T

$$\iff \rho \in S \underset{\text{REL}}{\longrightarrow} T$$

$$\iff \text{dom}\,\rho \subseteq S \text{ and } \text{ran}\,\rho \subseteq T.$$

Relation on S $\stackrel{\text{def}}{=}$ relation from S to S.

$$I_S \in S \underset{\text{REL}}{\longrightarrow} S$$

$$\rho \in S \underset{\text{REL}}{\longrightarrow} T \;\Rightarrow\; \rho^\dagger \in T \underset{\text{REL}}{\longrightarrow} S$$

For all S and T, $\{\} \in S \underset{\text{REL}}{\longrightarrow} T$

$$\{\} \in! \; S \underset{\text{REL}}{\longrightarrow} \{\}$$

$$\{\} \in! \; \{\} \underset{\text{REL}}{\longrightarrow} T$$

Total Relations

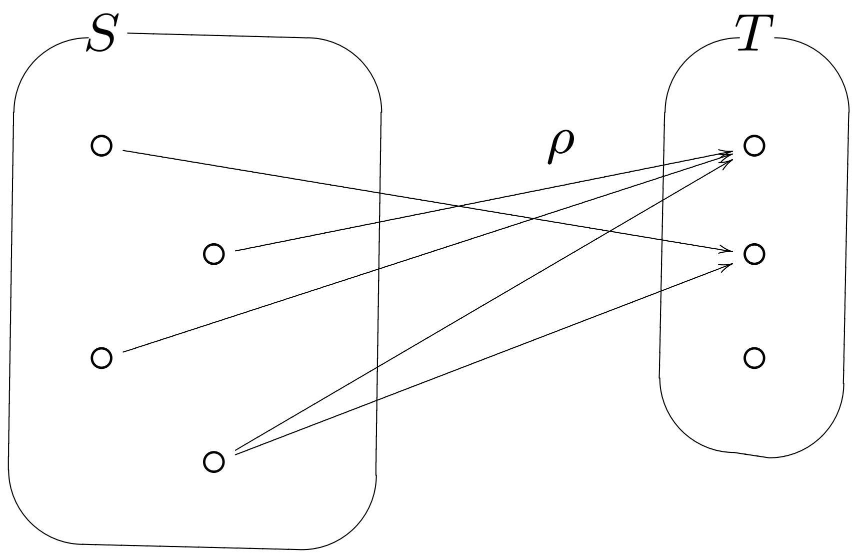
$\rho \in S \underset{\text{REL}}{\longrightarrow} T$ is a total relation from S to T

$$\iff \rho \in S \underset{\text{TREL}}{\longrightarrow} T$$

$$\iff \forall x \in S.\, \exists y \in T.\, x \; \rho \; y$$

$$\iff \text{dom } \rho = S$$

$$\iff I_S \subseteq \rho^\dagger \cdot \rho$$



$$\rho \in (\text{dom } \rho) \underset{\text{TREL}}{\longrightarrow} T \iff T \supseteq \text{ran } \rho$$

Functions Between Sets

f is a partial function from S to T

$$\iff f \in S \xrightarrow[\text{PFUN}]{} T$$

$$\iff f \in S \xrightarrow[\text{REL}]{} T \text{ and } f \text{ is a function.}$$

"Partial": $f \in S \xrightarrow[\text{REL}]{} T \;\Rightarrow\; \text{dom } f \subseteq S$

$f \in S \xrightarrow[\text{PFUN}]{} T$ is a (total) function from S to T

$$\iff f \in S \to T$$

$$\iff \text{dom } f = S.$$

- $S \to T = T^S = \prod_{x \in S} T$
- $S \to T \to U = S \to (T \to U)$

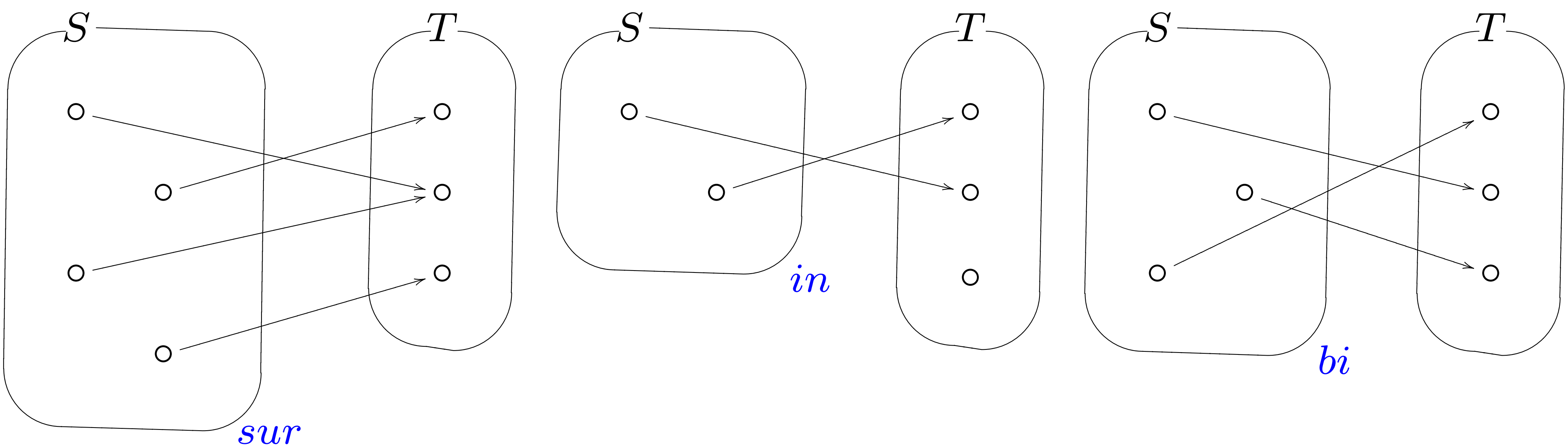
Surjections, Injections, Bijections

$$f \text{ is a surjection from } S \text{ to } T \iff \operatorname{ran} f = T$$

$$f \text{ is a injection from } S \text{ to } T \iff f^\dagger \in T \xrightarrow[\text{PFUN}]{} S$$

$$f \text{ is a bijection from } S \text{ to } T \iff f^\dagger \in T \to S$$

$$\iff f \text{ is an isomorphism from } S \text{ to } T$$



Back to Predicate Logic

$$
\begin{aligned}
intexp ::= {} & \mathsf{0} \mid \mathsf{1} \mid \ldots \\
& \mid var \\
& \mid -intexp \mid intexp + intexp \mid intexp - intexp \mid \ldots \\
assert ::= {} & \mathbf{true} \mid \mathbf{false} \\
& \mid intexp = intexp \mid intexp < intexp \mid intexp \leq intexp \mid \ldots \\
& \mid \neg assert \mid assert \wedge assert \mid assert \vee assert \\
& \mid assert \Rightarrow assert \mid assert \Leftrightarrow assert \\
& \mid \forall var.\, assert \mid \exists var.\, assert
\end{aligned}
$$

Denotational Semantics of Predicate Logic

The meaning of term $e \in \mathit{intexp}$ is $[\![e]\!]_{\mathit{intexp}}$

i.e. the function $[\![-]\!]_{\mathit{intexp}}$ maps objects from intexp to their meanings.

What is the set of meanings?

The meaning $[\![\mathsf{5}+\mathsf{37}]\!]_{\mathit{intexp}}$ of the term $\mathsf{5}+\mathsf{37}$ could be the integer 42.

But the term $\mathsf{x}+\mathsf{5}$ contains the free variable x…

Environments

…hence we need an environment (variable assignment, state)

$$\sigma \in \Sigma \stackrel{\text{def}}{=} var \to \mathbf{Z}$$

to give meaning to free variables.

The meaning of a term is a function from the states to $\mathbf{Z}$ or $\mathbf{B}$.

$$\begin{array}{lcl} [\![-]\!]_{intexp} & \in & intexp \to \Sigma \to \mathbf{Z} \\ [\![-]\!]_{assert} & \in & assert \to \Sigma \to \mathbf{B} \end{array}$$

$$\text{if } \sigma = [\mathsf{x} : 3, \mathsf{y} : 4], \quad \text{then} \quad \begin{array}{l} [\![\mathsf{x{+}5}]\!]_{intexp}\, \sigma = 8 \\ [\![\exists \mathsf{z}.\, \mathsf{x} < \mathsf{z} \wedge \mathsf{z} < \mathsf{y}]\!]\, \sigma = \mathbf{false} \end{array}$$

Direct Semantics Equations for Predicate Logic

$v \in var \qquad e \in intexp \qquad p \in assert$

$$
\begin{aligned}
[\![0]\!]_{intexp}\sigma &= 0 \\
\text{(really } [\![c_0\langle\rangle]\!]_{intexp}\sigma &= 0) \\
[\![v]\!]_{intexp}\sigma &= \sigma v \\
[\![e_0 + e_1]\!]_{intexp}\sigma &= [\![e_0]\!]_{intexp}\sigma + [\![e_1]\!]_{intexp}\sigma \\
[\![\mathbf{true}]\!]_{assert}\sigma &= \mathbf{true} \\
[\![e_0 = e_1]\!]_{assert}\sigma &= [\![e_0]\!]_{intexp}\sigma = [\![e_1]\!]_{intexp}\sigma \\
[\![\neg p]\!]_{assert}\sigma &= \neg([\![p]\!]_{assert}\sigma) \\
[\![p_0 \wedge p_1]\!]_{assert}\sigma &= [\![p_0]\!]_{assert}\sigma \wedge [\![p_1]\!]_{assert}\sigma \\
[\![\forall v.\, p]\!]_{assert}\sigma &= \forall n \in \mathbf{Z}.\, [\![p]\!]_{assert}[\sigma | v : n]
\end{aligned}
$$

Example: The Meaning of a Term

$$
\begin{aligned}
&[\![\forall \mathsf{x}.\ \mathsf{x{+}0{=}x}]\!]_{assert}\sigma \\
&\qquad = \forall n \in \mathbf{Z}.\ [\![\mathsf{x{+}0{=}x}]\!]_{assert}[\sigma|\mathsf{x} : n] \\
&\qquad = \forall n \in \mathbf{Z}.\ [\![\mathsf{x{+}0}]\!]_{intexp}[\sigma|\mathsf{x} : n] = [\![\mathsf{x}]\!]_{intexp}[\sigma|\mathsf{x} : n] \\
&\qquad = \forall n \in \mathbf{Z}.\ [\![\mathsf{x}]\!]_{intexp}[\sigma|\mathsf{x} : n] + [\![\mathsf{0}]\!]_{intexp}[\sigma|\mathsf{x} : n] = [\![\mathsf{x}]\!]_{intexp}[\sigma|\mathsf{x} : n] \\
&\qquad = \forall n \in \mathbf{Z}.\ [\sigma|\mathsf{x} : n](\mathsf{x}) + 0 = [\sigma|\mathsf{x} : n](\mathsf{x}) \\
&\qquad = \forall n \in \mathbf{Z}.\ n + 0 = n \\
&\qquad = \mathbf{true}
\end{aligned}
$$

Properties of the Semantic Equations

- They are syntax-directed (homomorphic):
  - exactly one equation for each abstract grammar production (constructor)
  - result expressed using functions (meanings) of subterms only (arguments of constructor)

  $\Rightarrow$ they have exactly one solution $\langle [\![-]\!]_{intexp}, [\![-]\!]_{assert} \rangle$ (proof by induction on the structure of terms).

- They define compositional semantic functions

  (depending only on the meaning of the subterms)

  $\Rightarrow$ "equivalent" subterms can be substituted

Validity of Assertions

p holds/is true in σ $\iff$ σ satisfies p $\iff$ $[\![p]\!]_{assert}\sigma = \mathbf{true}$

p is valid $\iff$ $\forall\sigma \in \Sigma.\ p$ holds in σ

p is unsatisfiable $\iff$ $\forall\sigma \in \Sigma.\ [\![p]\!]_{assert}\sigma = \mathbf{false}$

$\iff$ $\neg p$ is valid

p is stronger than p' $\iff$ $\forall\sigma \in \Sigma.\ (p'$ holds if p holds $)$

$\iff$ $(p \Rightarrow p')$ is valid

p and p' are equivalent $\iff$ p is stronger than p'
and p' is stronger than p

Inference Rules

Class	*Examples*
$\vdash p$ (Axiom)	$\vdash \mathsf{x} + 0 = \mathsf{x}$ (xPlusZero)
$\dfrac{}{\vdash p}$ (Axiom Schema)	$\dfrac{}{\vdash e_1 = e_0 \Rightarrow e_0 = e_1}$ (SymmObjEq)
$\dfrac{\vdash p_0 \quad \ldots \quad \vdash p_{n-1}}{\vdash p}$ (Rule)	$\dfrac{\vdash p \quad \vdash p \Rightarrow p'}{\vdash p'}$ (ModusPonens)
	$\dfrac{\vdash p}{\vdash \forall v.\, p}$ (Generalization)

Formal Proofs

A set of inference rules defines a logical theory $\vdash$.

A formal proof (in a logical theory):

a sequence of instances of the inference rules, where
the premisses of each rule occur as conclusions earlier in the sequence.

1.	$\vdash \mathsf{x} + 0 = \mathsf{x}$	(xPlusZero)
2.	$\vdash \mathsf{x} + 0 = \mathsf{x} \Rightarrow \mathsf{x} = \mathsf{x} + 0$	(SymmObjEq) $[e_0 : \mathsf{x} \mid e_1 : \mathsf{x} + 0]$
3.	$\vdash \mathsf{x} = \mathsf{x} + 0$	(ModusPonens, 1, 2) $[p : \mathsf{x} + 0 = \mathsf{x} \mid p' : \mathsf{x} = \mathsf{x} + 0]$
4.	$\vdash \forall \mathsf{x}.\, \mathsf{x} = \mathsf{x} + 0$	(Generalization, 3) $[v : \mathsf{x} \mid p : \mathsf{x} = \mathsf{x} + 0]$

Tree Representation of Formal Proofs

$$
\dfrac{\vdash \mathsf{x} + 0 = \mathsf{x} \qquad \dfrac{}{\vdash \mathsf{x} + 0 = \mathsf{x} \Rightarrow \mathsf{x} = \mathsf{x} + 0}\,(\text{SymmObjEq})}{\dfrac{\vdash \mathsf{x} = \mathsf{x} + 0}{\vdash \forall \mathsf{x}.\, \mathsf{x} = \mathsf{x} + 0}\,(\text{Gen})}\,(\text{MP})
$$

Soundness of a Logical Theory

An inference rule is sound if in every instance of the rule

the conclusion is valid if all the premisses are.

A logical theory $\vdash$ is sound if all inference rules in it are sound.

If $\vdash$ is sound and there is a formal proof of $\vdash p$, then p is valid.

Object vs Meta implication:

$\vdash p \Rightarrow \forall v.\, p$ is not a sound rule, although $\dfrac{\vdash p}{\vdash \forall v.\, p}$ is.

Completeness of a Logical Theory

A logical theory $\vdash$ is complete if

for every valid p there is a formal proof of $\vdash p$.

A logical theory $\vdash$ is axiomatizable if

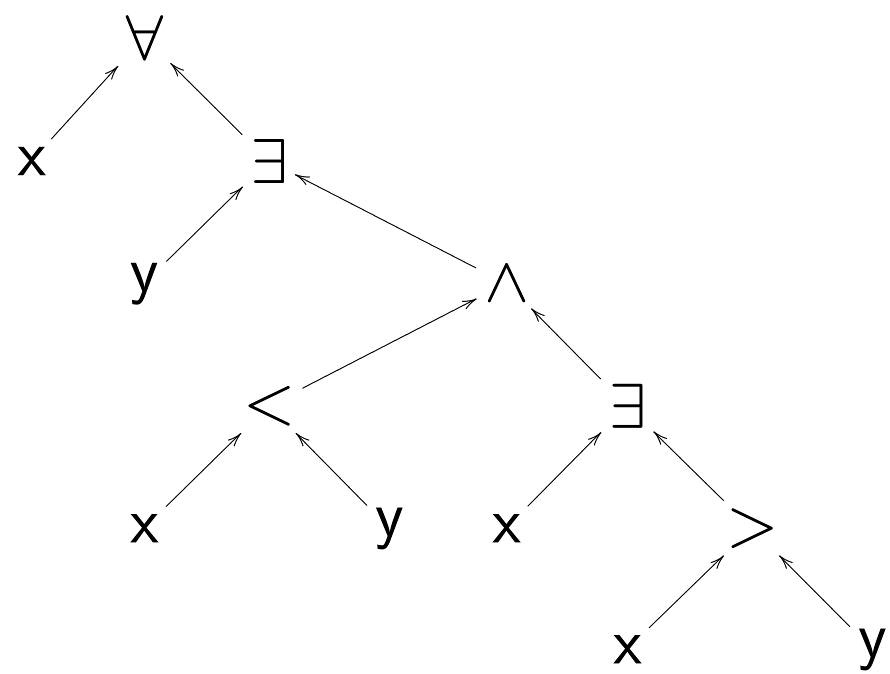
there exists a finite set of inference rules

from which can be constructed formal proofs of all assertions in $\vdash$.

No first-order theory of arithmetic is complete and axiomatizable.

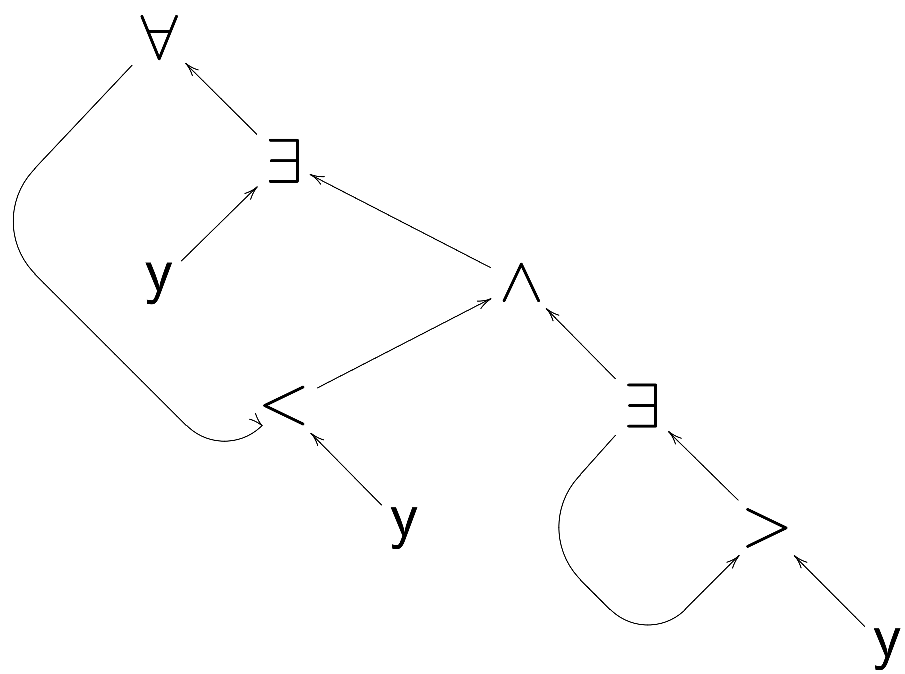
Variable Binding

$$\forall x.\ \exists y.\ x < y \wedge \exists x.\ x > y$$



Variable Binding

$$\forall x.\ \ \exists y.\ \ x < y \ \wedge\ \exists x.\ \ x > y$$



Bound and Free Variables

In $\forall v.\, p$, v is the binding occurrence (binder) and p is its scope.

If a non-binding occurrence of v is within the scope of a binder for v, then it is a bound occurrence; otherwise it's a free one.

$$\begin{array}{rcl} FV_{intexp}(0) & = & \{\} \\ FV(v) & = & \{v\} \\ FV(-e) & = & FV(e) \\ FV(e_0 + e_1) & = & FV(e_0) \cup FV(e_1) \end{array}
\qquad
\begin{array}{rcl} FV_{assert}(\mathbf{true}) & = & \{\} \\ FV(e_0 = e_1) & = & FV(e_0) \cup FV(e_1) \\ FV(\neg p) & = & FV(p) \\ FV(p_0 \wedge p_1) & = & FV(p_0) \cup FV(p_1) \\ FV(\forall v.\, p) & = & FV(p) - \{v\} \end{array}$$

Example:

$$FV(\exists \mathsf{y}.\, \mathsf{x} < \mathsf{y} \wedge \exists \mathsf{x}.\, \mathsf{x} > \mathsf{y}) = \{\mathsf{x}\}$$

Only Assignment of Free Variables Matters

Coincidence Theorem:

If $\sigma v = \sigma' v$ for all $v \in FV_\theta(p)$, then $[\![p]\!]_\theta \sigma = [\![p]\!]_\theta \sigma'$

(where p is a phrase of type θ).

Proof: By structural induction.

Inductive hypothesis:

The statement of the theorem holds

for all phrases of depth less than that of the phrase p'.

Base cases:

$p' = \mathsf{0} \;\Rightarrow\; [\![\mathsf{0}]\!]_{intexp}\sigma = 0 = [\![\mathsf{0}]\!]_{intexp}\sigma'$

$p' = v \;\Rightarrow\; [\![v]\!]_{intexp}\sigma = \sigma\, v = \sigma' v = [\![v]\!]_{intexp}\sigma'$, since $FV(v) = \{v\}$.

Proof of Concidence Theorem, cont'd

Coincidence Theorem:

If $\sigma v = \sigma' v$ for all $v \in FV_\theta(p)$, then $[\![p]\!]_\theta \sigma = [\![p]\!]_\theta \sigma'$.

Inductive cases:

$p' = e_0 + e_1$: by IH $[\![e_i]\!]_{intexp}\sigma = [\![e_i]\!]_{intexp}\sigma', i \in \{1, 2\}$.

$$\begin{aligned} [\![p']\!]_{intexp}\sigma &= [\![e_0]\!]_{intexp}\sigma + [\![e_1]\!]_{intexp}\sigma \\ &= [\![e_0]\!]_{intexp}\sigma' + [\![e_1]\!]_{intexp}\sigma' = [\![p']\!]_{intexp}\sigma' \end{aligned}$$

$p' = \forall u.\, q$: $\sigma v = \sigma' v$, $\forall v \in FV(p') = FV(q) - \{u\}$

then $[\sigma|u : n]v = [\sigma'|u : n]v$, $\forall v \in FV(q)$, $n \in \mathbf{Z}$

Then by IH $[\![q]\!]_{assert}[\sigma|u : n] = [\![q]\!]_{assert}[\sigma'|u : n]$ for all $n \in \mathbf{Z}$,

hence $\forall n \in \mathbf{Z}.\, [\![q]\!]_{assert}[\sigma|u : n] = \forall n \in \mathbf{Z}.\, [\![q]\!]_{assert}[\sigma'|u : n]$

$$[\![\forall u.\, q]\!]_{assert}\sigma = [\![\forall u.\, q]\!]_{assert}\sigma'.$$

Substitution

$$\left.\begin{array}{l} -/\delta \in \mathit{intexp} \to \mathit{intexp} \\ -/\delta \in \mathit{assert} \to \mathit{assert} \end{array}\right\} \text{when } \delta \in \mathit{var} \to \mathit{intexp}$$

$$\begin{array}{rcl} \mathsf{0}/\delta & = & \mathsf{0} \\ (\mathsf{-}e)/\delta & = & \mathsf{-}(e/\delta) \\ (e_0\mathsf{+}e_1)/\delta & = & (e_0/\delta)\mathsf{+}(e_1/\delta) \\ & \ldots & \end{array}
\qquad
\begin{array}{rcl} v/\delta & = & \delta v \\ (p_0 \wedge p_1)/\delta & = & (p_0/\delta) \wedge (p_1/\delta) \\ (\forall v.\, p)/\delta & = & \forall v'.\, (p/[\delta | v : v']), \\ \text{where } v' \notin & \displaystyle\bigcup_{u \in FV(p) - \{v\}} & FV(\delta u) \end{array}$$

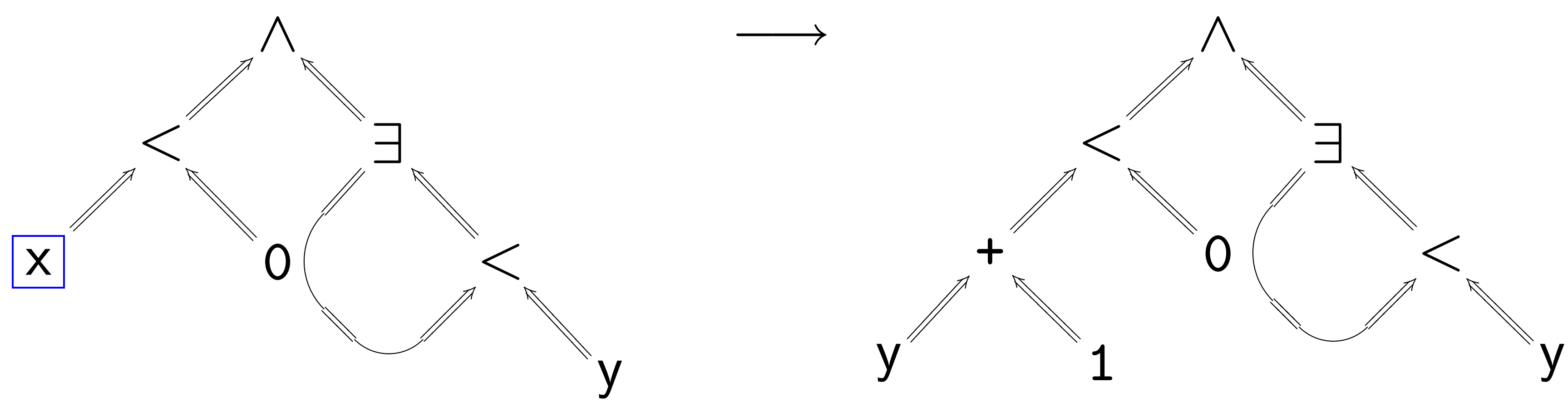
Examples:

$$(\mathsf{x} < \mathsf{0} \wedge \exists \mathsf{x}.\, \mathsf{x} \leq \mathsf{y})/[\mathsf{x} : \mathsf{y+1}] = \mathsf{y+1} < \mathsf{0} \wedge \exists \mathsf{x}.\, \mathsf{x} \leq \mathsf{y}$$

$$(\mathsf{x} < \mathsf{0} \wedge \exists \mathsf{x}.\, \mathsf{x} \leq \mathsf{y})/[\mathsf{y} : \mathsf{x+1}] = \mathsf{x} < \mathsf{0} \wedge \exists \mathsf{z}.\, \mathsf{z} \leq \mathsf{x+1}$$

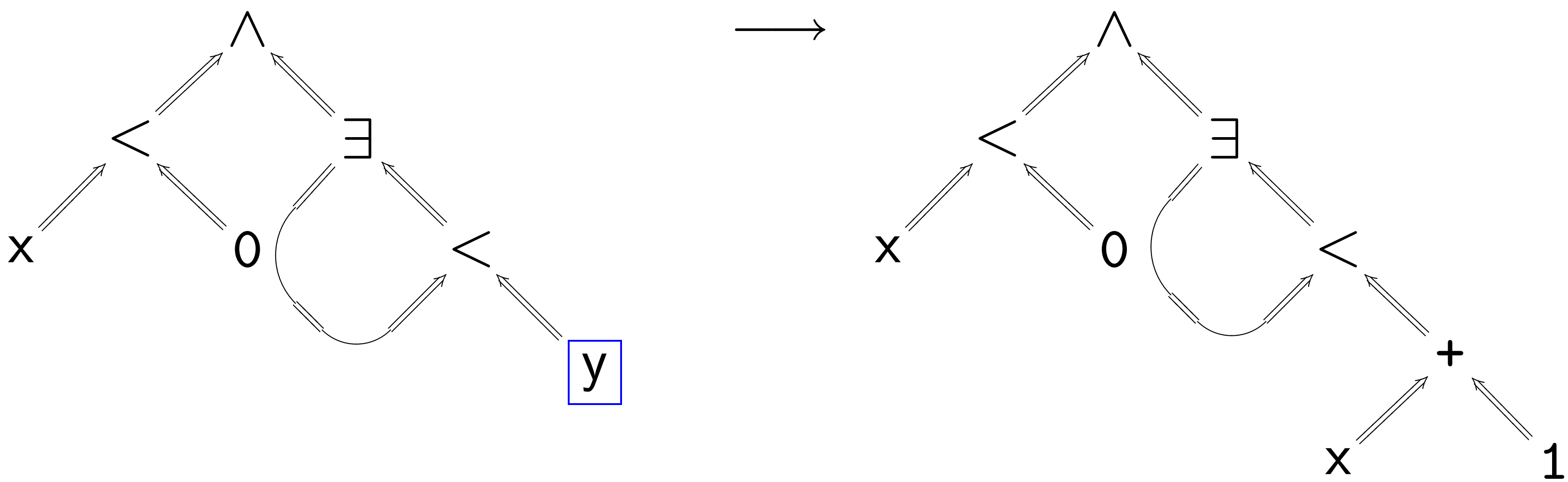
Preserving Binding Structure

$$(\mathsf{x} < \mathsf{0} \wedge \exists \mathsf{x}.\, \mathsf{x} \leq \mathsf{y})/[\boxed{\mathsf{x}} : \mathsf{y+1}] \quad = \quad \mathsf{y+1} < \mathsf{0} \wedge \exists \mathsf{x}.\, \mathsf{x} \leq \mathsf{y}$$



Avoiding Variable Capture

$$(\mathsf{x} < 0 \wedge \exists \mathsf{x}.\, \mathsf{x} \leq \mathsf{y})/[\boxed{\mathsf{y}} : \mathsf{x}+1] \quad = \quad \mathsf{x} < 0 \wedge \exists \mathsf{z}.\, \mathsf{z} \leq \mathsf{x}+1$$



Substitution Theorems

Substitution Theorem:

If $\sigma = [\![-]\!]_{intexp}\sigma' \cdot \delta$ on $FV(p)$, then $([\![-]\!]\sigma)p = ([\![-]\!]\sigma' \cdot (-/\delta))p$.

Finite Substitution Theorem:

$$[\![p/v_0 \rightarrow e_0, \ldots v_{n-1} \rightarrow e_{n-1}]\!]\sigma = [\![p]\!][\sigma | v_0 : [\![e_0]\!]\sigma, \ldots].$$

where

$$p/v_0 \rightarrow e_0, \ldots v_{n-1} \rightarrow e_{n-1} \stackrel{\text{def}}{=} p/[c\text{var} | v_0 : e_0 | \ldots | v_{n-1} : e_{n-1}].$$

Renaming:

If $u \notin FV(q) - \{v\}$, then $[\![\forall u.\, (q/v \rightarrow u)]\!]_{boolexp} = [\![\forall v.\, q]\!]_{boolexp}$.