BIANALYTIC MAPS BETWEEN FREE SPECTRAHEDRA

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ABSTRACT. Linear matrix inequalities (LMIs) $I_d + \sum_{j=1}^g A_j x_j + \sum_{j=1}^g A_j^* x_j^* \succeq 0$ play a role in many areas of applications and the set of solutions to one is called a spectrahedron. LMIs in (dimension–free) matrix variables model most problems in linear systems engineering, and their solution sets $\mathcal{D}_A$ are called free spectrahedra. These are exactly the free semialgebraic convex sets.

This paper studies free analytic maps between free spectrahedra and, under certain irreducibility assumptions, classifies all those that are bianalytic. The foundation of such maps turns out to be a very small class of birational maps we call convexotonic. The convexotonic maps in g variables sit in correspondence with g-dimensional algebras. If two bounded free spectrahedra $\mathcal{D}_A$ and $\mathcal{D}_B$ meeting our irreducibility assumptions are free bianalytic with map denoted p, then p must (after possibly an affine linear transform) extend to a convexotonic map corresponding to a g-dimensional algebra spanned by $(U-I)A_1, \dots, (U-I)A_g$ for some unitary U. Furthermore, B and UA are unitarily equivalent.

The article also establishes a Positivstellensatz for free analytic functions whose real part is positive semidefinite on a free spectrahedron and proves a representation for a free analytic map from $\mathcal{D}_A$ to $\mathcal{D}_B$ (not necessarily bianalytic). Another result shows that a function analytic on any radial expansion of a free spectrahedron is approximable by polynomials uniformly on the spectrahedron. These theorems are needed for classifying free bianalytic maps.

1. Introduction

Given a tuple $A = (A_1, \dots, A_g)$ of complex $d \times d$ matrices and indeterminates $x = (x_1, \dots, x_g)$, the expression

$$L_A(x) = I_d + \sum_{j=1}^g A_j x_j + \sum_{j=1}^g A_j^* x_j^*$$

is a monic linear pencil. The set

$$\mathcal{D}_A(1) = \{z \in \mathbb{C}^g : L_A(z) \text{ is positive semidefinite}\}$$

is known as a **spectrahedron** (synonymously **LMI domain**). Spectrahedra play a central role in semidefinite programming, convex optimization and in real algebraic geometry [BPR13]. They also figure prominently in the study of determinantal representations [Brä11, GK-VVW, NT12, Vin93] and the solution of the Lax conjecture [HV07], in the solution of the Kadison-Singer paving conjecture [MSS15], and in systems engineering [BGFB94, SIG96]. The monic linear pencil L_A is

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naturally evaluated at a tuple $X = (X_1, \dots, X_g)$ of $g \times g$ matrices by

$$L_A(X) = I_d \otimes I_n + \sum_{j=1}^{g} A_j \otimes X_j + \sum_{j=1}^{g} A_j^* \otimes X_j^*$$

with output a $dn \times dn$ self-adjoint matrix. Let $M_n(\mathbb{C})^g$ denote the set of g-tuples of $n \times n$ matrices. We call the sequence $(\mathcal{D}_A(n))_n$, where

$$\mathcal{D}_A(n) = \{X \in M_n(\mathbb{C})^g : L_A(X) \text{ is positive semidefinite}\}$$

a **free spectrahedron** (or a **free LMI domain**). Free spectrahedra arise naturally in many systems engineering problems described by a signal flow diagram [dOHMP09]. They are also canonical examples of matrix convex sets [EW97, HKM] and thus are intimately connected to the theory of completely positive maps and operator systems and spaces [Pau02].

In this article we study bianalytic maps p between free spectrahedra. Our belief, supported by the results in this paper and our experience with free spectrahedra (see for instance [HM12] and [HKM12b]), is that the existence of bianalytic maps imposes rigid, but elegant, structure on both the free spectrahedra as well as the map p. Motivation for this study comes from several sources. Free analysis, including free analytic functions is a recent development [KVV14, Tay72, Pas14, AM15, BGM06, Pop06, Pop10, KŠ, HKM12b, BKP16] with close ties to free probability [Voi04, Voi10] and quantum information theory [NC11, HKM]. In engineering systems theory, certain model problems can be described by a system of matrix inequalities. For optimization and design purposes, it is hoped that these inequalities have a convex solution set. In this case, under a boundedness hypothesis, the solution set is a free spectrahedron [HM12]. If the domain is not convex one might replace it by its matrix convex hull [HKM16] or map it bianalytically to a free spectrahedron. Two such maps then lead to a bianalytic map between free spectrahedra.

Studying bianalytic maps between free spectrahedra is a free analog of rigidity problems in several complex variables [DAn93, For89, For93, HJ01, HJY14, Kra01]. Indeed, there is a large literature on bianalytic maps on convex sets. For instance, Faran [Far86] showed that any proper analytic map from the unit ball in $\mathbb{C}^n$ to the unit ball in $\mathbb{C}^N$ with $N \le 2n-2$ that is real analytic up to the boundary, is (up to automorphisms of the domain and codomain) the standard linear embedding $z \mapsto (z, 0)$. However, when $N = 2n-1$, Huang and Ji [HJ01] proved this map and the Whitney map $z = (z', z_n) \mapsto (z', z_n z)$ are the only such maps. Forstnerič [For93] showed that any proper analytic map between balls with sufficient regularity at the boundary must be rational. We refer to [HJ01, HJY14] for further recent developments.

The remainder of this introduction is organized as follows. Basic terminology and background appear in Subsection 1.1. A novel family of maps we call convexotonic maps and which we believe comprise, up to affine linear equivalence, exactly the bianalytic maps between free spectrahedra, is described in Subsection 1.2. This subsection also contains the main result of the article, Theorem 1.4 on bianalytic mappings between free spectrahedra. Subsections 1.3 and 1.4 describe Positivstellensätze and results related to recent free Oka-Weil theorems [AM14, BMV] on (uniform) polynomial approximation of free spectrahedra and functions analytic in a suitable neighborhood of a spectrahedron. Both are ingredients in the proof of Theorem 1.4.

1.1. **Basic definitions.** Notations, definitions and background needed to describe the results in this paper are collected in this section.

1.1.1. *Free polynomials.* Let $\langle x \rangle$ denote the set of all **words** in x. This includes the empty word denoted by 1. The **length of a word** $w \in \langle x \rangle$ is denoted by $|w|$. Let $\mathbb{C}\langle x \rangle = \mathbb{C}\langle x_1, \dots, x_g \rangle$ denote the $\mathbb{C}$-algebra freely generated by g freely noncommuting letters $x = (x_1, \dots, x_g)$. Its elements are linear combinations of words in x and are called **analytic free polynomials**. We shall also consider the **free polynomials** $\mathbb{C}\langle x, x^*\rangle$ in both the variables $x = (x_1, \dots, x_g)$ and their formal adjoints, $x^* = (x_1^*, \dots, x_g^*)$. For instance, $x_1x_2 + x_2x_1 + 5x_1^3$ is analytic, but $x_1^*x_2 + 3x_2x_1^5$ is not. A polynomial is **hereditary** provided all the x^* variables, if any, always appear on the left and is thus a finite linear combination of terms v^*w where v and w are words in x. A particular example of these are polynomials which are of the form analytic plus anti-analytic, i.e., $f + g^*$ for some $f, g \in \mathbb{C}\langle x\rangle$. These definitions naturally extend to matrices over polynomials.

1.1.2. *Free domains, matrix convex sets and spectrahedra.* For positive integers n, let $M_n(\mathbb{C})^g$ denote the set of g-tuples of $n \times n$ matrices (with complex entries). Let $M(\mathbb{C})^g$ denote the sequence $(M_n(\mathbb{C})^g)_n$. A **subset** Γ of $M(\mathbb{C})^g$ is a sequence $(\Gamma(n))_n$ where $\Gamma(n) \subseteq M_n(\mathbb{C})^g$. The subset Γ is a **free set** if it is closed under direct sums and simultaneous conjugation by a unitary matrix; i.e., if $X \in \Gamma(n)$ and $Y \in \Gamma(m)$, then

$$X \oplus Y = \left(\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix}, \dots, \begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix} \right) \in \Gamma(n+m)$$

and if U is an $n \times n$ unitary matrix, then

$$U^*XU = \big(U^*X_1U, \dots, U^*X_gU\big) \in \Gamma(n).$$

The free set Γ is a **matrix convex set** (alternately **free convex set**) if it is also closed under simultaneous conjugation by isometries; i.e., if $X \in \Gamma(n)$ and V is an $n \times m$ isometry, then $V^*XV \in \Gamma(m)$. In the case that $0 \in \Gamma(1)$, Γ is a matrix convex set if and only if it is closed under direct sums and simultaneous conjugation by contractions. It is straightforward to see that a matrix convex set is **levelwise** convex; i.e., each $\Gamma(n)$ is a convex set in $M_n(\mathbb{C})^g$.

A distinguished class of matrix convex domains are those described by a linear matrix inequality. Given a positive integer d and $A_1, \dots, A_g \in M_d(\mathbb{C})$, the linear matrix-valued free polynomial

$$\Lambda_A(x) = \sum_{j=1}^{g} A_j x_j \in M_d(\mathbb{C}) \otimes \mathbb{C}\langle x_1, \dots, x_g\rangle \tag{1.1}$$

is a **(homogeneous) linear pencil**. Its adjoint is, by definition, $\Lambda_A(x)^* = \sum_{j=1}^{g} A_j^* x_j^*$. Let

$$L_A(x) = I_d + \Lambda_A(x) + \Lambda_A(x)^*. \tag{1.2}$$

If $X \in M_n(\mathbb{C})^g$, then $L_A(X)$ is defined using the Kronecker tensor product,

$$L_A(X) = I_d \otimes I_n + \sum_{j=1}^{g} A_j \otimes X_j + \sum_{j=1}^{g} A_j^* \otimes X_j^*,$$

yielding a self-adjoint $dn \times dn$ matrix. The inequality $L_A(X) \succeq 0$ for $X \in M(\mathbb{C})^g$ is a **linear matrix inequality (LMI)**. The sequence of solution sets $\mathcal{D}_A = (\mathcal{D}_A(n))_n$ defined by

$$\mathcal{D}_A(n) = \{X \in M_n(\mathbb{C})^g : L_A(X) \succeq 0\} \tag{1.3}$$

is a matrix convex domain which contains a neighborhood of 0. It is called a **free LMI domain** or **free spectrahedron**. It is essentially the only instance of a bounded convex free open semialgebraic set [HM12].

1.1.3. *Free functions.* Let $\mathcal{D} \subseteq M(\mathbb{C})^g$. A **free function** f from $\mathcal{D}$ into $M(\mathbb{C})^1$ is a sequence of functions $f[n] : \mathcal{D}(n) \to M_n(\mathbb{C})$ which **respects intertwining**; i.e., if $X \in \mathcal{D}(n)$, $Y \in \mathcal{D}(m)$, $\Gamma : \mathbb{C}^m \to \mathbb{C}^n$, and

$$X\Gamma = (X_1\Gamma, \dots, X_g\Gamma) = (\Gamma Y_1, \dots, \Gamma Y_g) = \Gamma Y,$$

then $f[n](X)\Gamma = \Gamma f[m](Y)$. Equivalently, f respects direct sums and similarity. The definition of a free function naturally extends to vector-valued functions $f : \mathcal{D} \to M(\mathbb{C})^h$, matrix-valued functions

$f : \mathcal{D} \to M_e(M(\mathbb{C})^1)$ and even operator-valued functions. We refer the reader to [KVV14, Voi10] for a comprehensive study of free function theory.

1.1.4. *Formal power series and free analytic functions.* Given a positive integer d and Hilbert space H, an operator-valued **formal power series** f in x is an expression of the form

$$f = \sum_{m=0}^{\infty} \sum_{\substack{w \in \langle x \rangle \\ |w|=m}} f_w w = \sum_{m=0}^{\infty} f^{(m)}, \tag{1.4}$$

where $f_w : \mathbb{C}^d \to H$ are linear maps and $f^{(m)}$ is the **homogeneous component** of degree m of f, that is, the sum of all monomials in f of degree m. For $X = (X_1, \dots, X_g) \in M_n(\mathbb{C})^g$, define

$$f(X) = \sum_{m=0}^{\infty} \sum_{\substack{w \in \langle x \rangle \\ |w|=m}} f_w \otimes w(X),$$

provided the series converges (summed in the indicated order).

The formal power series f has **positive radius of convergence** if there is a $\rho > 0$ such that if $\|X_j\| < \rho$ for each j, then $f(X)$ converges. If the norms of the coefficients of f grow slowly enough, then f will have a positive radius of convergence. For instance, if

$$\tau(f) := \liminf_{m \to \infty} \sum_{|\alpha|=m} \|f_\alpha\|^{\frac{1}{m}} < \infty,$$

then choosing $\rho = \tau(f)^{-1}$ shows f has positive radius of convergence.

An important fact for us is that a formal power series with positive radius of convergence determines a free analytic function and (under a mild local boundedness assumptions) vice versa, cf. [KVV14, Chapter 7] or [HKM12b, Proposition 2.24]. Here, a free function $f = (f[n])_n$ is **free analytic** if each $f[n]$ is analytic. Very weak hypotheses (e.g. continuity [HKM11] or even local boundedness [KVV14, AM14]) on a free function imply it is analytic.

1.2. **Bianalytic maps between free spectrahedra.** A **free analytic mapping** (or simply an analytic mapping) p is, for some pair of positive integers $g, \tilde{g}$, an expression of the form

$$p = (p^1, \dots, p^{\tilde{g}}),$$

where each p^j is an analytic function in the free variables $x = (x_1, \dots, x_g)$. Given free domains $\mathcal{D}$ and $\tilde{\mathcal{D}}$, we write $p : \mathcal{D} \to \tilde{\mathcal{D}}$ to indicate $\mathcal{D}$ is a subset of the domain of p and p maps $\mathcal{D}$ into $\tilde{\mathcal{D}}$. The domains $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are **bianalytic** if there exist free analytic mappings $p : \mathcal{D} \to \tilde{\mathcal{D}}$ and $q : \tilde{\mathcal{D}} \to \mathcal{D}$ such that, $p \circ q$ and $q \circ p$ are the identity mappings on $\tilde{\mathcal{D}}$ and $\mathcal{D}$ respectively. To emphasize the role of p (and q), we say that $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are p-**bianalytic**.

In this paper we introduce a small and highly structured class of birational maps we call **convexotonic** and to each such map p describe the pairs of spectrahedra $(\mathcal{D}, \tilde{\mathcal{D}})$ bianalytic via p. We conjecture these triples $(p, \mathcal{D}, \tilde{\mathcal{D}})$ account for all bianalytic free spectrahedra and establish the result under certain irreducibility hypotheses on $\mathcal{D}$ and $\tilde{\mathcal{D}}$. We start with the definition of the convexotonic maps, since they are at the core of this structure.

1.2.1. *Convexotonic maps.* Consider a tuple $\Xi = (\Xi_1, \dots, \Xi_g) \in M_g(\mathbb{C})^g$ satisfying

$$\Xi_k \Xi_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} \Xi_s \tag{1.5}$$

for each $1 \le j,k \le g$. Observe that $\mathcal{W} = \operatorname{span}\{\Xi_1, \dots, \Xi_g\}$ is an algebra of dimension $\le g$. We say the rational functions p and q whose entries have the form

$$p_i(x) = \sum_j x_j \left(I - \Lambda_\Xi(x)\right)^{-1}_{j,i} \qquad \text{and} \qquad q_i(x) = \sum_j x_j \left(I + \Lambda_\Xi(x)\right)^{-1}_{j.i},$$

that is, in row form,

$$p(x) = x(I - \Lambda_\Xi(x))^{-1} \qquad \text{and} \qquad q = x(I + \Lambda_\Xi(x))^{-1} \tag{1.6}$$

are **convexotonic**. It turns out (see Proposition 6.2) the mappings p and q are inverses of one another, hence they are birational maps.

Each g-tuple Ξ as in (1.5) (even if linearly dependent) arises as the set of structure matrices for a g-dimensional algebra. For instance, consider block $(g+1) \times (g+1)$ matrices

$$R_j = \begin{pmatrix} 0 & e_j^* \\ 0 & \Xi_j \end{pmatrix}$$

with respect to the orthogonal decomposition $\mathbb{C} \oplus \mathbb{C}^g$. Here e_j is the j-th standard basis vector for $\mathbb{C}^g$. Then

$$R_k R_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} R_s, \tag{1.7}$$

hence $\mathcal{R} = \operatorname{span}\{R_1, \dots, R_g\}$ is a g-dimensional algebra with structure matrices Ξ.

Convexotonic maps are fundamental objects and to each are attached pairs of bianalytic spectrahedra. Let $\mathcal{R} = \operatorname{span}\{R_1, \dots, R_g\} \subseteq M_d(\mathbb{C})$ be a g-dimensional algebra with structure matrices Ξ, and suppose that C a $d \times d$ is a unitary matrix and $A \in M_d(\mathbb{C})^g$, with the property that $R_j = (C - I)A_j$ for $1 \le j \le g$, and

$$A_k R_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} A_s. \tag{1.8}$$

In particular, the span $\mathcal{A}$ of the A_j is a right $\mathcal{R}$-module. Note that if $C - I$ is invertible then (1.8) holds automatically. We call the so constructed $(\mathcal{D}_A, \mathcal{D}_{CA})$ a **spectrahedral pair** associated to the algebra $\mathcal{R}$.

1.2.2. *Overview of free bianalytic maps between free spectrahedra.*

Theorem 1.1. *Any spectrahedral pair $(\mathcal{D}_A, \mathcal{D}_{CA})$ with C unitary, which is associated to a g-dimensional algebra $\mathcal{R}$, has $\mathcal{D}_A$ bianalytic to $\mathcal{D}_{CA}$ under the convexotonic map p whose structure matrices Ξ are associated to the algebra $\mathcal{R}$.*

Proof. See Theorem 6.4. ∎

We conjecture that convexotonic maps are the only bianalytic maps between free spectrahedra.

Conjecture 1.2. *Up to conjugation with affine linear maps, the only bounded free spectrahedra $\mathcal{D}_A$, $\mathcal{D}_B$ which are p-bianalytic arise as spectrahedral pairs associated to an algebra $\mathcal{R}$, and p is the corresponding convexotonic map.*

Soon in Theorem 1.4 we see that the conjecture is true in a generic sense. An unusual feature of the conjecture from the viewpoint of traditional several complex variables is that typical bianalytic mapping results would be stated up to conjugation with automorphisms of $\mathcal{D}_A$ and $\mathcal{D}_B$. Here we are actually asserting conjugation up to affine linear equivalence. See also Subsection 9.3.

We emphasize there are few g-dimensional complex algebras. To give a clear picture we have calculated the convexotonic maps explicitly for $g = 2$ and $g = 3$ (in Mathematica using NCAlgebra [HOMS16]).

Proposition 1.3. *We list a basis R_1, R_2 for each of the four 2-dimensional indecomposable algebras over $\mathbb{C}$. Then we give the associated "indecomposable" convexotonic map and its (convexotonic) inverse.*

(1) *R_1 is nilpotent of order 3 and $R_2 = R_1^2$*

$$p(x_1, x_2) = \begin{pmatrix} x_1 & x_2 + x_1^2 \end{pmatrix} \qquad q(x_1, x_2) = \begin{pmatrix} x_1 & x_2 - x_1^2 \end{pmatrix}.$$

(2) $R_1^2 = R_1,\ R_1R_2 = R_2$

$$p(x) = \begin{pmatrix} (1-x_1)^{-1}x_1 & (1-x_1)^{-1}x_2 \end{pmatrix} \qquad q(x) = \begin{pmatrix} (1+x_1)^{-1}x_1 & (1+x_1)^{-1}x_2 \end{pmatrix}.$$

(3) $R_1^2 = R_1,\ R_2R_1 = R_2$

$$p(x) = \begin{pmatrix} x_1(1-x_1)^{-1} & x_2(1-x_1)^{-1} \end{pmatrix} \qquad q(x) = \begin{pmatrix} x_1(1+x_1)^{-1} & x_2(1+x_1)^{-1} \end{pmatrix}.$$

(4) $R_1^2 = R_1,\ R_1R_2 = R_2,\ R_2R_1 = R_2$

$$\begin{aligned} p(x) &= \begin{pmatrix} x_1(1-x_1)^{-1} & (1-x_1)^{-1}x_2(1-x_1)^{-1} \end{pmatrix} \\ q(x) &= \begin{pmatrix} x_1(1+x_1)^{-1} & (1+x_1)^{-1}x_2(1+x_1)^{-1} \end{pmatrix}. \end{aligned}$$

For $g = 3$ there are exactly ten plus a one parameter family of indecomposable convexotonic maps, since there are exactly this many corresponding indecomposable 3-dimensional algebras, see Appendix A to the arXiv posting `https://arxiv.org/abs/1604.04952` *of this paper.*

Proof. See Section 9. ∎

We note that all g variable convexotonic maps are linear transforms of indecomposable ones or direct sums of lower dimensional convexotonic maps; see Section 9.

We note that the composition of two convexotonic maps may not be convexotonic (see Subsection 8.4), a further indication of the very restrictive nature of convexotonic maps.

1.2.3. *Results on free bianalytic maps under a genericity assumption.* The main result of this paper supporting Conjecture 1.2 is Theorem 1.4 below, which says, under certain irreducibility conditions on A and B, if $\mathcal{D}_A$ and $\mathcal{D}_B$ are p-bianalytic, then p and its inverse q are in fact convexotonic.

A set $\{u^1, \dots, u^{g+1}\}$ is a **hyperbasis** for $\mathbb{C}^g$ if each g element subset is a basis. For the moment, the tuple $A \in M_d(\mathbb{C})^g$ is **sv-generic** if there exists $\alpha^1, \dots, \alpha^{g+1}$ and $\beta^1, \dots, \beta^g$ in $\mathbb{C}^g$ such that $I - \Lambda_A(\alpha^j)^*\Lambda_A(\alpha^j)$ is positive semidefinite and has a one-dimensional kernel spanned by u^j and the set $\{u^1, \dots, u^{g+1}\}$ is a hyperbasis for $\mathbb{C}^g$; and $I - \Lambda_A(\beta^k)\Lambda_A(\beta^k)^*$ is positive semidefinite, its kernel spanned by v^k and the set $\{v^1, \dots, v^g\}$ is a basis for $\mathbb{C}^g$. Numerical experiments suggest random tuples of A satisfy this property. Weaker (but still sufficient) versions of the sv-generic condition are given in the body of the paper, see Subsection 7.1.2.

Theorem 1.4. *Suppose $A \in M_d(\mathbb{C})^g$ and $B \in M_e(\mathbb{C})^g$ are sv-generic. If $\mathcal{D}_A$ and $\mathcal{D}_B$ are p-bianalytic with $p(0) = 0$, then $d = e$, there exists a $d \times d$ matrix C such that B is unitarily equivalent to CA, the tuple $R = (C - I)A$ spans an algebra, the span of A is a right R-module and, letting Ξ denote the structure matrices for this module, p has the form of equation (1.6).*

Proof. See Theorem 7.8. ∎

We point out the normalization condition $p(0) = 0$ can be enforced e.g. by an affine linear change of variables on the range of p, see Section 8 for details.

The proof of Theorem 1.4 is based on several intermediate results of independent interest. The first is a Positivstellensatz for a (matrix-valued) free analytic function with positive real part on a free spectrahedron.

1.3. **Positivstellensätze and representations for analytic functions.** We begin this section with a Positivstellensätze and then turn to representations they imply.

Theorem 1.5 (Analytic convex Positivstellensatz)**.** *Let $A \in M_d(\mathbb{C})^g$ assume that $\mathcal{D}_A$ is bounded and G is a matrix-valued free function $\mathcal{D}_A \to M_e(M(\mathbb{C})^1)$ analytic on a radial expansion $(1+\varepsilon)\mathcal{D}_A$ of the free spectrahedron $\mathcal{D}_A$. If $G(0) = 0$ and $I + G + G^*$ is nonnegative on $\mathcal{D}_A$, then there exists a Hilbert space H and a formal power series $W = \sum_{\alpha\in\langle x\rangle} W_\alpha \alpha$ with coefficients $W_\alpha : \mathbb{C}^e \to H \otimes \mathbb{C}^d$ such that*

$$I + G(x) + G(x)^* = W(x)^* L_{I_H \otimes A}(x) W(x)$$

holds in the ring of $e \times e$ matrices over formal power series in x, x^*.

Proof. See Section 5. ∎

For hereditary polynomials positive on a free spectrahedron, the conclusion of Theorem 1.5 is stronger. The weight(s) W in the positivity certificate (1.9) are polynomial, still analytic and we get optimal degree bounds.

Theorem 1.6 (Hereditary Convex Positivstellensatz)**.** *Let $A \in M(\mathbb{C})^g$, and let $h \in \mathbb{C}^{\nu\times\nu}\langle x, x^*\rangle$ be an hereditary matrix polynomial of degree d. Then $h|_{\mathcal{D}_A} \succeq 0$ if and only if*

$$h = \sum_{k}^{\text{finite}} h_k^* h_k + \sum_{j}^{\text{finite}} f_j^* L f_j \tag{1.9}$$

for some analytic polynomials $h_j \in \mathbb{R}^{\ell\times\nu}\langle x\rangle_{d+1}$, $f_j \in \mathbb{R}^{\ell\times\nu}\langle x\rangle_d$. Moreover, if $\mathcal{D}_A$ is bounded, then we can omit the pure sum of squares term in (1.9) provided the f_j are allowed to have degree $\le d+1$.

Proof. See Section 3. ∎

Theorem 1.7 shows, under the assumption of a **one term Positivstellensatz certificate** (1.10), a bianalytic map between free spectrahedra is convexotonic.

Theorem 1.7. *Suppose $A, B \in M_d(\mathbb{C})^g$, the set $\{A_1, \dots, A_g\}$ is linearly independent and p is a free formal power series in x (no x_j^*) with $p(0) = 0$ and $p'(0) = I$. If there exists a $d \times d$ free formal power series W such that*

$$L_B(p(x)) = W(x)^* L_A(x) W(x), \tag{1.10}$$

then p is a convexotonic map

$$p(x) = x(I - \Lambda_\Xi(x))^{-1}$$

as in (1.6), determined by a module spanned by the set $\{A_1, \dots, A_g\}$ over an algebra of dimension at most g with structure matrices Ξ.

Proof. See Theorem 6.4. ∎

In the context of Theorem 1.4, the sv-generic condition is used to show the one term Positivstellensatz hypothesis of Theorem 1.7 holds.

Functions satisfying the conclusions of the Positivstellensatz of Theorem 1.5 have the following representation.

Proposition 1.8. *Suppose e is a positive integer and $\mathcal{E}$ is a separable Hilbert space. If*
(1) $\mathbf{A}$ *is a g-tuple of operators on $\mathcal{E}$;*
(2) W *is a formal power series with coefficients $W_\alpha : \mathbb{C}^e \to \mathcal{E}$; and*
(3) G *is a formal power series with $G(0) = 0$ and $e \times e$ matrix coefficients G_α such that*

$$I_e + G(x) + G(x)^* = W(x)^* L_{\mathbf{A}}(x) W(x),$$

then there exists an isometry $C : \mathcal{E} \to \mathcal{E}$ and an isometry $\mathscr{W} : \mathbb{C}^e \to \mathcal{E}$ such that

$$G(x) = \mathscr{W}^* C\big(\sum_{j=1}^{g} \mathbf{A}_j x_j\big) W(x), \tag{1.11}$$

where, letting $R = (C - I_{\mathcal{E}})\mathbf{A}$,

$$W(x) = \big(I_{\mathcal{E}} - \sum_{j=1}^{g} R_j x_j\big)^{-1} \mathscr{W}.$$

Proof. See Subsection 4.2. ∎

An analytic (not necessarily bianalytic) map p maps $\mathcal{D}_A$ into $\mathcal{D}_B$ if and only if $L_A(X) \succeq 0$ implies $L_B(p(X)) \succeq 0$. Proposition 1.8 thus provides a representation for $G(x) = \Lambda_B(p(x))$ with a state space realization flavor.

Combining our previous results gives a representation for free analytic maps between free spectrahedra. Related results on free maps between half-planes and balls were recently obtained in [AM14, BMV].

Corollary 1.9. *Let $A \in M_d(\mathbb{C})^g$, $B \in M_e(\mathbb{C})^h$, and assume the set $\{A_1, \dots, A_g\}$ is linearly independent. Suppose $p : \mathcal{D}_A \to \mathcal{D}_B$ is a free map analytic and bounded on a radial expansion $(1+\varepsilon)\mathcal{D}_A$ of the free spectrahedron $\mathcal{D}_A$, satisfying $p(0) = 0$. Then there exists a Hilbert space H, an isometry $C : H \otimes \mathbb{C}^d \to H \otimes \mathbb{C}^d$ and an isometry $\mathscr{W} : \mathbb{C}^e \to H \otimes \mathbb{C}^d$ such that*

$$\Lambda_B(p(x)) = \mathscr{W}^* C\big(\sum_{j=1}^{g} \mathbf{A}_j x_j\big)\big(I_{H\otimes\mathbb{C}^d} - \sum_{j=1}^{g} R_j x_j\big)^{-1} \mathscr{W}, \tag{1.12}$$

where $\mathbf{A} = I_H \otimes A$ and $R = (C - I_{H\otimes\mathbb{C}^d})\mathbf{A}$.

Proof. Combine Theorem 1.5 with Proposition 1.8. ∎

1.4. **Approximating free spectrahedra and free analytic functions.** This subsection concerns approximation of functions analytic on free spectrahedra by analytic polynomials i.e., a free Oka-Weil theorem. An example is the following remarkable theorem of Agler and McCarthy [AM14].

Given a matrix-valued free analytic polynomial Q, the set

$$\mathcal{G}_Q = \{X \in M(\mathbb{C})^g : \|Q(X)\| < 1\}$$

is a (semialgebraic) **free pseudoconvex** set.

Theorem 1.10. *If f is a bounded free analytic function on a free pseudoconvex set $\mathcal{G}_Q$, then f can be uniformly approximated by analytic free polynomials on each smaller set*

$$K_{\tau Q} = \{X : \tau\|Q(X)\| \le 1\}, \qquad \tau > 1.$$

Proof. This is proved, though not stated in this form, in Section 9 of [AM14] (cf. their proof of Corollary 9.7; see also [AM14, Corollary 8.13]). ∎

In this paper we show that free functions analytic on free spectrahedra can be uniformly approximated by polynomials. Firstly, we approximate free spectrahedra by free pseudoconvex sets.

Proposition 1.11. *If $\mathcal{D}_A$ is bounded and $t > 1$, then there exists free analytic polynomial Q such that*

$$\mathcal{D}_A \subseteq \mathcal{G}_Q \subseteq t\mathcal{D}_A.$$

Moreover, if $\mathcal{G}_Q$ is a free pseudoconvex set and $\mathcal{D}_A \subseteq \mathcal{G}_Q$, then there is an $s > 1$ such that $\mathcal{D}_A \subseteq K_{sQ}$. Finally, if p is a free rational function analytic on $\mathcal{D}_A$, then there is a $t > 0$ such that p is analytic and bounded on $t\mathcal{D}_A$.

Proof. See Subsection 2.1.1. ∎

A (matrix-valued) free analytic function p analytic on a free spectrahedron $\mathcal{D}_A$ is **uniformly approximable by polynomials** on $\mathcal{D}_A$ if for each $\epsilon > 0$ there is a free polynomial q such that $\|p(X) - q(X)\| < \epsilon$ for each $X \in \mathcal{D}_A$.

Theorem 1.12. *Suppose $A \in M_d(\mathbb{C})^g$ and $\mathcal{D}_A$ is bounded. If f is analytic and bounded on a free pseudoconvex set $\mathcal{G}_Q$ containing $\mathcal{D}_A$, then f is uniformly approximable by polynomials on $\mathcal{D}_A$.*

Proof. See Subsection 2.1.2. ∎

1.5. **Readers guide.** The paper is organized as follows. The polynomial approximation results of Subsection 1.4 are proved in Section 2. Section 3 contains a preliminary polynomial version of the Hereditary Positivstellensatz. In Section 4, key algebraic consequences of an Hereditary Positivstellensatz are collected for use in the following sections. The proof of Theorem 1.6 appearing in Section 5 uses the results of the previous three sections. Theorem 1.7 is proved in Section 6. A somewhat more general version of Theorem 1.4 is the topic of Section 7. Throughout much of the article the (bi)analytic maps are assumed to satisfy the normalization $p(0) = 0$ and $p'(0)$ a projection. Section 8 describes the consequences of relaxing this assumption. Section 9 provides examples of convexotonic maps. In the several complex variables spirit of classifying domains up to affine linear equivalence, it is natural to ask if there exist matrix convex domains that are polynomially, but not affine linearly, bianalytic. The hard won answer is yes. A class of examples appears in Section 10.

2. Approximating Free Analytic Functions by Polynomials

In this section we prove Proposition 1.11 and Theorem 1.12 approximating free spectrahedra with free pseudoconvex sets and approximating free mappings analytic on free spectrahedra by free polynomials, respectively.

2.1. Approximating free spectrahedra and free analytic functions using free polynomials.

For $C > 0$, let $\mathfrak{F}_C$ denote the free set of matrices T such that $C - (T + T^*) \succeq 0$ and for $M > 0$, let $\mathfrak{F}_{C,M}$ denote those $T \in \mathfrak{F}_C$ such that $\|T\| < M$. Let φ denote the linear fractional mapping $\varphi(z) = z(1-z)^{-1}$. In particular, φ maps the region $\{z : \operatorname{Re} z \le \frac{1}{2}\}$ in the complex plane to the set $\{z : |z| \le 1, z \ne -1\}$. The inverse of φ is $\psi(w) = w(1+w)^{-1}$. Given $\epsilon > \delta > 0$ sufficiently small, the ball $\mathbb{B}_\delta(\epsilon) = \{z : |z - \epsilon| \le 1 + \delta\}$ does not contain -1 and there exists a $K \in (1, 2)$ such that $\psi(\mathbb{B}_\delta(\epsilon)) \subseteq \{z : \operatorname{Re} z < \frac{K}{2}\}$.

Lemma 2.1. *If $T \in \mathfrak{F}_C$, then the spectrum of T lies in the set $\{z : \operatorname{Re} z \le \frac{C}{2}\}$. In particular,*

(1) *if $2 > C$ and $T \in \mathfrak{F}_C$, then $I - T$ is invertible;*
(2) *if $T \in \mathfrak{F}_1$, then $T(I-T)^{-1}$ has norm at most one;*
(3) *if T is a matrix with $\|T\| < 1$, then $(I + T)$ is invertible and $T(I+T)^{-1} \in \mathfrak{F}_1$;*
(4) *given $M > 0$ and $t > 0$, there exists $2 > C > 1$, and $t > \epsilon > \delta > 0$ such that if $T \in \mathfrak{F}_{C,M}$, then*

$$\|\varphi(T) - \epsilon\| \le 1 + \delta.$$

Proof. Suppose $T \in \mathfrak{F}_C$. If v is a unit eigenvector of T and $Tv = (a+ib)v$, then

$$C = \langle Cv, v\rangle \succeq \langle Tv, v\rangle + \langle v, Tv\rangle = 2a,$$

whence $a \le \frac{C}{2}$. This also implies (1).

For (2), let $S = T(I-T)^{-1}$. The inequality,

$$T^*T \preceq I - T - T^* + T^*T$$

implies

$$(I-T)^{-*}T^*T(I-T)^{-1} \preceq I,$$

(and vice-versa), as desired. Item (3) is proved similarly.

To prove (4), fix $M > 0$ and $t > 0$ and suppose $\min\{1,t\} > \epsilon > 0$. Choose $0 < \rho < 1$ such that both

$$\big(2(1-\rho) + \epsilon(1-\rho^2)\big)M^2 < \frac{1}{2},$$

$$1 < C := \frac{1 + 2(\rho - \frac{1}{4})\epsilon - (1-\rho^2)\epsilon^2}{1 + 2(\rho - \frac{1}{2})\epsilon - (1-\rho^2)\epsilon^2} < 2.$$

It follows that

$$\begin{aligned}
\epsilon\Big(2(1-\rho) + \epsilon(1-\rho^2)\Big)T^*T &\preceq \frac{\epsilon}{2} \\
&\preceq \frac{\epsilon}{2} + \Big(1 + 2(\rho - \frac{1}{2})\epsilon - (1-\rho^2)\epsilon^2\Big)\big(C - (T+T^*)\big) \\
&\preceq \frac{\epsilon}{2} + \Bigg(\big(1 + 2(\rho - \frac{1}{4})\epsilon - (1-\rho^2)\epsilon^2\big) - \big(1 + 2(\rho - \frac{1}{2})\epsilon - (1-\rho^2)\epsilon^2\big)(T+T^*)\Bigg).
\end{aligned} \tag{2.1}$$

Let $\delta = \rho\epsilon$ and observe,

$$\epsilon\Big(2(1-\rho) + \epsilon(1-\rho^2)\Big) = (1+\epsilon)^2 - (1+\delta)^2, \tag{2.2}$$

$$\begin{aligned}
\frac{\epsilon}{2} + 1 + 2(\rho - \frac{1}{4})\epsilon - (1-\rho^2)\epsilon^2 &= \frac{\epsilon}{2} + 1 + 2\rho\epsilon - \frac{\epsilon}{2} - \epsilon^2 + (\rho\epsilon)^2 \\
&= 1 + 2\delta + \delta^2 - \epsilon^2 = (1+\delta)^2 - \epsilon^2,
\end{aligned} \tag{2.3}$$

and

$$1 + 2(\rho - \frac{1}{2})\epsilon - (1-\rho^2)\epsilon^2 = 1 + 2\rho\epsilon - \epsilon - \epsilon^2 + (\rho\epsilon)^2 = (1+\delta)^2 - \epsilon(1+\epsilon). \tag{2.4}$$

Thus, substituting $\delta = \rho\epsilon$ into equation (2.1) and using equations (2.2), (2.3), (2.4) yields,

$$\Big((1+\epsilon)^2 - (1+\delta)^2\Big)T^*T \preceq \Big((1+\delta)^2 - \epsilon^2\Big) - \Big((1+\delta)^2 - \epsilon(1+\epsilon)\Big)(T+T^*).$$

Rearranging gives,

$$(1+\epsilon)^2T^*T - \epsilon(1+\epsilon)(T+T^*) + \epsilon^2 \preceq (1+\delta)^2\big(I - (T+T^*) + T^*T\big)$$

and hence

$$\big((1+\epsilon)T - \epsilon\big)^*\big((1+\epsilon)T - \epsilon\big) \preceq (1+\delta)^2\,(I-T)^*\,(I-T).$$

Item (4) follows from this last inequality together with $(1+\epsilon)T - \epsilon = T - \epsilon(I-T)$. ■

Proposition 2.2. *For each $M > 0$ and $\rho > 0$ there exists $2 > C_0 > 1$ such that for each $1 < C < C_0$ there exists $\rho > \epsilon > \delta > 0$ such that for each $\eta > 0$ there is an analytic polynomial q in one variable such that for all $T \in \mathfrak{F}_{C,M}$,*

(a) $\|\varphi(T) - \epsilon\| < 1 + \delta$;

(b) $\|q(T) - \varphi(T)\| < \eta$.

Proof. The linear fractional map $\varphi(z) = z(1-z)^{-1}$ maps $\mathfrak{H} = \{z \in \mathbb{C} : \operatorname{Re} z < 1\}$ to the unit disc $\mathbb{D}$. By Lemma 2.1, given $M > 0$ and $\rho > 0$ there exists a $2 > C > 1$ and $\rho > \epsilon > \delta > 0$ such that if $T \in \mathfrak{F}_{C,M}$, then $\|\varphi(T) - \epsilon\| < 1 + \delta$. Let $\varphi_*(z) = \varphi(z) - \epsilon$. Its inverse is $\psi_*(w) = \psi(w + \epsilon)$. In particular, ψ_* is analytic in a neighborhood of the ball $\mathbb{B}_\delta(0) = \{z \in \mathbb{C} : |z| \leq 1 + \delta\}$ and for $\delta > 0$ sufficiently small, $\psi_*(\mathbb{B}_\delta(0))$ is a compact subset of $\mathfrak{H}$. Thus, by Runge's Theorem, there exists a polynomial p such that

$$\|p - \varphi_*\|_{\psi_*(\mathbb{B}_\delta(0))} := \sup\{|p(z) - \varphi_*(z)| : z \in \psi_*(B)\} < \eta.$$

Hence,

$$\|p \circ \psi_* - z\|_{\mathbb{B}_\delta(0)} < \eta.$$

Now let $T \in \mathfrak{F}_{C,M}$ be given. The matrix $S = \varphi(T) - \epsilon$ has norm at most $1 + \delta$ by Lemma 2.1 (4) and hence

$$\|(p \circ \psi_*)(S) - S\| \leq \eta.$$

Equivalently,

$$\|p(T) - \varphi_*(T)\| \leq \eta.$$

Choosing $q = p + \epsilon$, completes the proof. ■

Corollary 2.3. *There exists a $2 > C_0 > 1$ such that for each $M > 0$ and $C_0 > C > 1$, the set $\{\|\varphi(T)\| : T \in \mathfrak{F}_{C,M}\}$ is bounded.*

Proof. By Proposition 2.2, there exists $\epsilon, \delta > 0$ such that $\|\varphi(T) - \epsilon\| < 1 + \delta$ for $T \in \mathfrak{F}_{C,M}$. Hence,

$$\|\varphi(T)\| \leq 1 + \epsilon + \delta$$

for $T \in \mathfrak{F}_{C,M}$. ■

Lemma 2.4. *For each $M > 0$ and $2 > C > 1$ there exists an analytic polynomial s in one variable such that*

$$\mathfrak{F}_{1,M} \subseteq \mathcal{G}_s = \{T : \|s(T)\| < 1\} \subseteq \mathfrak{F}_{C,M}.$$

Proof. Choose $\rho > 0$ such that

$$\frac{M^2 + 1}{\rho} < C - 1$$

and let $R^2 = M^2 + 1 + \rho + \rho^2$. In particular,

$$\frac{R^2 - \rho^2}{\rho} = \frac{M^2 + 1 + \rho}{\rho} < C. \tag{2.5}$$

Let $s_1 = \frac{x+\rho}{R}$, $s_2(x) = \frac{x}{M}$ and $s = s_1 \oplus s_2$. Thus, $\|s(T)\| < 1$ if and only if $\|T + \rho\| < R$ and $\|T\| < M$. Suppose $T \in \mathfrak{F}_{1,M}$. Then automatically $\|s_2(T)\| < 1$. Using $T + T^* \preceq I$, estimate

$$(T + \rho)^*(T + \rho) = T^*T + \rho(T + T^*) + \rho^2 \preceq M^2 + \rho + \rho^2 < R.$$

Thus $\|s_2(T)\| < R$ and the first inclusion of the lemma is proved.

Now suppose $\|s_1(T)\| < R$. Hence, $\|T + \rho\| < R$. In this case,

$$\begin{aligned} 0 &\preceq R^2 - (T+\rho)^*(T+\rho) = R^2 - T^*T - \rho(T+T^*) - \rho^2 \\ &\preceq \rho\Big(\frac{R^2 - \rho^2}{\rho} - (T + T^*)\Big) \preceq C - (T^* + T), \end{aligned}$$

where equation (2.5) was used to obtain the last inequality. Thus, $\|s(T)\| < 1$ implies $T \in \mathfrak{F}_{C,M}$ and the proof is complete. ■

Lemma 2.5. *If $\mathcal{D}_A$ is bounded and $t > 1$, then there exists a matrix-valued free polynomial Q such that*

$$\mathcal{D}_A \subseteq \mathcal{G}_Q \subseteq t\mathcal{D}_A.$$

Proof. Since $\mathcal{D}_A$ is bounded, there is an $M > 0$ such that $t\|\Lambda_A(X)\| \leq M$ for all $X \in \mathcal{D}_A$. By Proposition 2.2, there exists a $2 > C > 1$ and a sequence of q_k polynomials converging uniformly to $\varphi(z)$ on $\mathfrak{F}_{C,M}$. Passing to a subsequence if needed, we can assume

$$\|q_k(T) - \varphi(T)\| < \frac{1}{k}$$

for $T \in \mathfrak{F}_{C,M}$. Writing

$$\begin{aligned} &2\big(q_k(T)^* q_k(T) - \varphi(T)^*\varphi(T)\big) \\ &\qquad = \big(q_k(T) - \varphi(T)\big)^*\big(q_k(T) + \varphi(T)\big) + \big(q_k(T) + \varphi(T)\big)^*\big(q_k(T) - \varphi(T)\big), \end{aligned} \tag{2.6}$$

and using $\varphi(T)$ is uniformly bounded on $\mathfrak{F}_{C,M}$ (see Corollary 2.3), there is a constant κ (independent of k and T) such that

$$q_k(T)^* q_k(T) - \varphi(T)^*\varphi(T) \preceq \frac{\kappa}{k}.$$

Hence,

$$I + \frac{\kappa}{k} - q_k(T)^* q_k(T) \succeq I - \varphi(T)^*\varphi(T).$$

Thus, if $T \in \mathfrak{F}_{C,M}$ and $I - \varphi(T)^*\varphi(T) \succeq 0$, then $I - (1 + \frac{\kappa}{k})^{-1} q_k(T)^* q_k(T) \succeq 0$.

Now, given a monic linear pencil $L_A = I + \Lambda_A + \Lambda_A^*$, let

$$Q_k = \left(1 + \frac{\kappa}{k}\right)^{-\frac{1}{2}} q_k \circ \Lambda_A.$$

If $X \in \mathcal{D}_A$, then $T = \Lambda_A(X) \in \mathfrak{F}_{1,M}$. Hence $I - Q_k(X)^* Q_k(X) \succeq 0$; i.e., $\mathcal{D}_A \subseteq K_{Q_k}$, in the notation $K_{Q_k} := \{X : \|Q_k(X)\| \leq 1\}$ of [AM14]. Moreover, since $q_k(T)$ converges to $\varphi(T)$,

$$\mathcal{D}_A = \bigcap_k^{\infty} K_{Q_k}.$$

Choose s as in Lemma 2.4 so that $\mathfrak{F}_{1,M} \subseteq \{T : \|s(T)\| \leq 1\} \subseteq \mathfrak{F}_{C,M}$. Thus,

$$\mathcal{D}_A \subseteq \{X : \|s(\Lambda_A(X))\| < 1\}.$$

Consequently, letting

$$\hat{Q}_k = \begin{pmatrix} Q_k & 0 \\ 0 & s \circ \Lambda_A \end{pmatrix},$$

we have

$$\mathcal{D}_A \subseteq \{X : \|\hat{Q}_k(X)\| \leq 1\}.$$

We now turn to showing, given $t > 1$, there is a k such that $\{X : \|\hat{Q}_k(X)\| \leq 1\} \subseteq t\mathcal{D}_A$. The estimate (2.6) works reversing the roles of q_k and φ giving the inequality

$$\varphi(T)^*\varphi(T) - q_k(T)^* q_k(T) \preceq \frac{\kappa}{k} \tag{2.7}$$

for $T \in \mathfrak{F}_{C,M}$. Now suppose $X \in M(\mathbb{C})^g$ and $I - \hat{Q}_k(X)^*\hat{Q}_k(X) \succeq 0$. Let, as before $T = \Lambda_A(X)$. It follows that $\|s(T)\| \leq 1$ and hence $T \in \mathfrak{F}_{C,M}$. We can thus apply (2.7) to conclude

$$\begin{aligned} 0 \preceq 1 - Q_k(X)^* Q_k(X) &= 1 - \left(1 + \frac{\kappa}{k}\right)^{-1} q_k(T)^* q_k(T) \\ &\preceq 1 + \frac{\kappa}{k}\left(1 + \frac{\kappa}{k}\right)^{-1} - \left(1 + \frac{\kappa}{k}\right)^{-1} \varphi(T)^*\varphi(T). \end{aligned}$$

Let $\tau_k = 1 + \frac{2\kappa}{k}$. This last inequality implies

$$(I-T)^{-*}T^*T(I-T)^{-1} \le \tau_k.$$

A bit of algebra shows this inequality is equivalent to

$$\tau_k - \tau_k(T+T^*) + (\tau_k - 1)T^*T \succeq 0.$$

Since $\tau_k \to 1$, for all sufficiently large k we have

$$t > 1 + \frac{\tau_k - 1}{\tau_k} M^2.$$

Using $T^*T \preceq M^2$, it follows that

$$\tau_k(t-(T+T^*)) \succeq \tau_k + (\tau_k - 1)M^2 - \tau_k(T+T^*) \succeq \tau_k\big(I-(T+T^*)\big) + (\tau_k-1)T^*T \succeq 0.$$

Thus $X \in t\mathcal{D}_A$. Summarizing, for sufficiently large k,

$$\mathcal{D}_A \subseteq \{X : \|\hat{Q}_k(X)\| \le 1\} \subseteq t\mathcal{D}_A. \qquad \blacksquare$$

Lemma 2.6. *Let $A \in M_d(\mathbb{C})^g$. If $\mathcal{D}_A$ is bounded and $\mathcal{G}_Q$ is a free pseudoconvex set such that $\mathcal{D}_A \subseteq \mathcal{G}_Q$, then there is $s > 1$ such that*

$$\mathcal{D}_A \subseteq K_{sQ} \subseteq \mathcal{G}_Q. \tag{2.8}$$

Proof. By definition, $\mathcal{D}_A \subseteq \mathcal{G}_Q$ is equivalent to $1 - Q^*Q \succeq 0$ on $\mathcal{D}_A$. For $C \in \mathbb{R}_{>0}$, $C - Q^*Q \succeq 0$ on $\mathcal{D}_A$ if and only if $C - Q^*Q \succeq 0$ on $\mathcal{D}_A(N)$ for $N = N(\deg Q, g, d)$ large enough [HKM12a, Remark 1.2].

Without loss of generality, $Q \neq 0$. Let $s^{-1} < 1$ be the maximum of $\|Q(X)\|$ on the compact set $\mathcal{D}_A(N)$. Then $s^{-2} - Q^*Q \succeq 0$ on $\mathcal{D}_A(N)$ and thus on $\mathcal{D}_A$ by the above. Hence $\|Q(X)\| \le s^{-1}$ on $\mathcal{D}_A$. So (2.8) holds. ■

2.1.1. *Proof of Proposition 1.11.* The first statements are immediate from Lemmas 2.5 and 2.6. Lemma 2.7 finishes off the proof. ■

2.1.2. *Proof of Theorem 1.12.* Suppose f is analytic and bounded on some $\mathcal{G}_Q$ containing $\mathcal{D}_A$. By Proposition 1.11, there is $s > 1$ with $\mathcal{D}_A \subseteq K_{sQ} \subseteq \mathcal{G}_Q$. By the free Oka-Weil Theorem 1.10, f can be uniformly approximated by polynomials on K_{sQ} and thus on $\mathcal{D}_A$. ■

2.2. **Rational functions analytic on $\mathcal{D}_A$.** In this subsection we show a rational function p without singularities on $\mathcal{D}_A$ is analytic and bounded on $t\mathcal{D}_A$ for some $t > 1$. Hence, by Lemma 2.5, p is analytic and bounded on a free pseudoconvex set $\mathcal{G}_Q$ containing $\mathcal{D}_A$.

Lemma 2.7. *Suppose $\mathcal{D}_A$ is bounded and let r be an analytic noncommutative rational function with no singularities on $\mathcal{D}_A$. Then there is a $t > 1$ such that r is bounded with no singularities on $t\mathcal{D}_A$.*

Proof. Since r is analytic on $\mathcal{D}_A$, we can consider its minimal realization,

$$r(x) = c^t\big(I - \Lambda_B(x)\big)^{-1} b$$

for some $e \times e$ tuple $B \in M_e(\mathbb{C})^g$ and vectors $b, c \in \mathbb{C}^e$. The singularity set of r coincides with the singularity set (i.e., the free locus [KV]) $\mathcal{Z}_B$ of $I - \Lambda_B$ [KVV09].

We claim that $\mathcal{Z}_B \cap \mathcal{D}_A = \varnothing$ if and only if $\mathcal{Z}_B(e) \cap \mathcal{D}_A(e) = \varnothing$. Let $X \in \mathcal{Z}_B \cap \mathcal{D}_A(m)$. Then for some nonzero $v = \sum_{j=1}^e e_j \otimes v_j$, where the e_j are standard unit vectors in $\mathbb{C}^d$, we have

$$0 = (I - \Lambda_B)(X)v = (I \otimes I - \sum_k B_k \otimes X_k)(\sum_{j=1}^e e_j \otimes v_j) = \sum_{j=1}^e e_j \otimes v_j - \sum_{j,k} B_k e_j \otimes X_k v_j.$$

Let P denote the orthogonal projection $\mathbb{C}^m \to \mathcal{V} = \operatorname{span}\{v_1, \ldots, v_e\}$. Then $P^*XP \in \mathcal{Z}_B$. Indeed, for any $u = \sum_i e_i \otimes u_i \in \mathbb{C}^d \otimes \mathcal{V}$ we have

$$\begin{aligned}
u^*(I - \Lambda_B)(P^*XP)v &= (\sum_i e_i \otimes u_i)^*(I \otimes I - \sum_k B_k \otimes P^*X_kP)(\sum_{j=1}^e e_j \otimes v_j) \\
&= \sum_i u_i^* v_i - (\sum_i e_i \otimes u_i)^*(\sum_k B_k \otimes P^*X_kP)(\sum_{j=1}^e e_j \otimes v_j) \\
&= u^*v - \sum_{i,j,k} e_i^* B_k e_j u_i^* P^* X_k P v_j = u^*v - \sum_{i,j,k} e_i^* B_k e_j u_i^* X_k v_j \\
&= (\sum_i e_i \otimes u_i)^*(I \otimes I - \sum_k B_k \otimes X_k)(\sum_{j=1}^e e_j \otimes v_j) \\
&= u^*(I - \Lambda_B)(X)v = 0.
\end{aligned}$$

If $\dim \mathcal{V} = e' \le e$ this calculation shows $P^*XP \in \mathcal{Z}_B(e') \cap \mathcal{D}_A(e')$.

Since $\mathcal{D}_A(e)$ is compact and disjoint from the closed set $\mathcal{Z}_B(e)$, there exists $t > 1$ such that $t\mathcal{D}_A(e) \cap \mathcal{Z}_B(e) = \varnothing$. But now using $t\mathcal{D}_A = \mathcal{D}_{\frac{1}{t}A}$, the above claim proves there is a $t > 1$ such that r has no singularities on $t\mathcal{D}_A$.

We now argue that in fact r is bounded on $t\mathcal{D}_A$. First observe that if $X_n \in t\mathcal{D}_A$ and $\|r(X_n)\|$ grows without bound, then so does $\|(I - \Lambda_B(X_n))^{-1}\|$. Hence, there is a sequence γ_n of unit vectors such that $(\|(I - \Lambda_B(X_n))\gamma_n\|)_n$ tends to zero. By the argument above, we can replace X_n with $Y_n = V_n^* X_n V_n$ where V_n includes an e-dimensional space containing γ_n and assume that the $Y_n \in \mathcal{D}_A(e)$. By compactness of $\mathcal{D}_A(e)$, by passing to a subsequence if needed, without loss of generality Y_n converges to some $Y \in \mathcal{D}_A(e)$ and γ_n to some unit vector γ. It follows that $(I - \Lambda_B(Y))\gamma = 0$, and we have arrived at the contradiction that $Y \in t\mathcal{D}_A$ and Y is a singularity of $I - \Lambda_B(x)$. ■

3. Hereditary Convex Positivstellensatz

In this section we present a strengthening of the Convex Positivstellensatz [HKM12a], characterizing hereditary polynomials nonnegative on free spectrahedra. In the obtained sum of squares certificate all weights will be analytic.

Fix a symmetric $q \in \mathbb{C}^{\ell \times \ell}\langle x, x^* \rangle$ and let

$$\mathcal{D}_q(n) := \{X \in M_n(\mathbb{C})^g : q(X) \succeq 0\} \qquad \text{and} \qquad \mathcal{D}_q := \bigcup_{n \in \mathbb{N}} \mathcal{D}_q(n).$$

Given $\alpha, \beta \in \mathbb{N}$, set

(3.1)
$$M_{\alpha,\beta}^{\nu,\text{her}}(q) := \Big\{ \sum_j^{\text{finite}} h_j^* h_j + \sum_i^{\text{finite}} f_i^* q f_i : \ f_i \in \mathbb{C}^{\ell \times \nu}\langle x \rangle_\beta, \, h_j \in \mathbb{C}^{\nu \times \nu}\langle x \rangle_\alpha \Big\} \ \subseteq \ \mathbb{R}^{\nu \times \nu}\langle x, x^* \rangle_{\max\{2\alpha, 2\beta + a\}},$$

where $a = \deg(q)$. Observe that $M_{\alpha,\beta}^{\nu,\text{her}}(q)$ is a proper subset of the quadratic module $M_{\alpha,\beta}^{\nu}(q)$ as defined in [HKM12a]. We emphasize that h_j, f_i are assumed to be analytic in (3.1) defining $M_{\alpha,\beta}^{\nu,\text{her}}(q)$. Obviously, if $f \in M_{\alpha,\beta}^{\nu,\text{her}}(q)$ then $f|_{\mathcal{D}_q} \succeq 0$.

For notational convenience, let $\Sigma_\alpha^{\nu,\text{her}}$ denote the appropriate cone of sum of squares obtained from $M_{\alpha,\alpha}^{\nu,\text{her}}(q)$ with $q = 1$.

We call $M^{\nu,\text{her}}_{\alpha,\beta}(q)$ the **truncated hereditary quadratic module** defined by q. We often abbreviate $M^{\nu,\text{her}}_{\alpha,\beta}(q)$ to $M^{\nu}_{\alpha,\beta}$. If $q(0) = I$ (q is **monic**), then $\mathcal{D}_q$ contains an **nc neighborhood of** 0; i.e., there exists $\epsilon > 0$ such that for each $n \in \mathbb{N}$, if $X \in M_n(\mathbb{C})^g$ and $\|X\| < \epsilon$, then $X \in \mathcal{D}_q$. Likewise $\mathcal{D}_q$ is called **bounded** provided there is a number $N \in \mathbb{N}$ for which all $X \in \mathcal{D}_q$ satisfy $\|X\| < N$.

Theorem 3.1 (Hereditary Convex Positivstellensatz)**.** *Suppose $L \in \mathbb{C}^{\ell\times\ell}\langle x, x^*\rangle$ is a monic linear pencil and $h \in \mathbb{C}^{\nu\times\nu}\langle x, x^*\rangle$ is a symmetric hereditary matrix polynomial. If $\deg(h) = d$, then*

$$h(X) \succeq 0 \text{ for all } X \in \mathcal{D}_L \quad \iff \quad h \in M^{\nu,\text{her}}_{d+1,d}(L). \tag{3.2}$$

If, in addition, the set $\mathcal{D}_L$ is bounded, then the right-hand side of this equivalence is further equivalent to

$$h \in \Big\{ \sum_j^{\text{finite}} f_j^* L f_j : f_j \in \mathbb{R}^{\ell\times\nu}\langle x\rangle_{d+1} \Big\} =: \mathring{M}^{\nu,\text{her}}_{d+1}(L). \tag{3.3}$$

3.1. **Proof of Theorem 3.1.** The proof consists of two main steps: a separation argument together with a partial Gelfand-Naimark-Segal (GNS) construction.

3.1.1. *Step 1: Towards a separation argument.* Let $\mathbb{C}^{\nu\times\nu}\langle x, x^*\rangle^{\text{her}}$ denote the vector space of all hereditary $\nu \times \nu$ matrix polynomials.

Lemma 3.2. *$M^{\nu,\text{her}}_{\alpha,\beta}(L)$ is a closed convex cone in $\mathbb{C}^{\nu\times\nu}\langle x, x^*\rangle^{\text{her}}_{\max\{2\alpha,2\beta+1\}}$.*

Proof. The proof is the same as in the free case, cf. [HKM12a, Proposition 3.1]. ■

3.1.2. *Step 2: A GNS construction.* Proposition 3.3 below, embodies the well known connection, through the Gelfand-Naimark-Segal (GNS) construction, between operators and positive linear functionals. It is adapted here to hereditary matrix polynomials.

Given a Hilbert space $\mathscr{H}$ and a positive integer ν, let $\mathscr{H}^{\oplus\nu}$ denote the orthogonal direct sum of $\mathscr{H}$ with itself ν times. Let L be a monic $\ell \times \ell$ linear pencil and abbreviate

$$M^{\nu}_{k+1} := M^{\nu,\text{her}}_{k+1,k}(L).$$

Proposition 3.3. *If $\lambda : \mathbb{C}^{\nu\times\nu}\langle x, x^*\rangle^{\text{her}}_{2k+2} \to \mathbb{C}$ is a symmetric linear functional that is nonnegative on $\Sigma^{\nu,\text{her}}_{k+1}$ and positive on $\Sigma^{\nu,\text{her}}_{k} \setminus \{0\}$, then there exists a tuple $X = (X_1, \dots, X_g)$ of operators on a Hilbert space $\mathscr{H}$ of dimension at most $\nu\sigma_{\#}(k) = \nu \dim \mathbb{R}\langle x\rangle_k$ and a vector $\gamma \in \mathscr{H}^{\oplus\nu}$ such that*

$$\lambda(f) = \langle f(X)\gamma, \gamma\rangle \tag{3.4}$$

for all $f \in \mathbb{C}^{\nu\times\nu}\langle x, x^*\rangle^{\text{her}}_k$, where $\langle \text{\textvisiblespace}, \text{\textvisiblespace}\rangle$ is the inner product on $\mathscr{H}$. Further, if λ is nonnegative on M^{ν}_{k+1}, then $X \in \mathcal{D}_L$.

Conversely, if $X = (X_1, \dots, X_g)$ is a tuple of operators on a Hilbert space $\mathscr{H}$ of dimension N, γ is a vector in $\mathscr{H}^{\oplus\nu}$, and k is a positive integer, then the linear functional $\lambda : \mathbb{C}^{\nu\times\nu}\langle x, x^*\rangle^{\text{her}}_{2k+2} \to \mathbb{C}$ defined by

$$\lambda(f) = \langle f(X)\gamma, \gamma\rangle$$

is nonnegative on Σ^{ν}_{k+1}. Further, if $X \in \mathcal{D}_L$, then λ is nonnegative also on M^{ν}_{k+1}.

Proof. First suppose that $\lambda : \mathbb{C}^{\nu\times\nu}\langle x, x^*\rangle^{\text{her}}_{2k+2} \to \mathbb{C}$ is nonnegative on $\Sigma^{\nu,\text{her}}_{k+1}$ and positive on $\Sigma^{\nu,\text{her}}_{k} \setminus \{0\}$. Consider the symmetric bilinear form, defined on the vector space $K = \mathbb{C}^{\nu\times 1}\langle x\rangle_{k+1}$ (row vectors of length ν whose entries are analytic polynomials of degree at most $k + 1$) by,

$$\langle f, h\rangle = \lambda(h^* f). \tag{3.5}$$

From the hypotheses, this form is positive semidefinite.

A standard use of Cauchy-Schwarz inequality shows that the set of null vectors

$$\mathcal{N} := \{f \in K : \langle f, f \rangle = 0\}$$

is a vector subspace of K. Whence one can endow the quotient $\tilde{\mathscr{H}} := K/\mathcal{N}$ with the induced positive definite bilinear form making it a Hilbert space. Further, because the form (3.5) is positive definite on the subspace $\mathscr{H} = \mathbb{C}^{\nu\times 1}\langle x\rangle_k$, each equivalence class in that set has a unique representative which is a ν-row of analytic polynomials of degree at most k. Hence we can consider $\mathscr{H}$ as a subspace of $\tilde{\mathscr{H}}$ with dimension $\nu\sigma_\#(k)$.

Each x_j determines a multiplication operator on $\mathscr{H}$. For $f = \begin{pmatrix} f_1 & \cdots & f_\nu \end{pmatrix} \in \mathscr{H}$, let

$$x_j f = \begin{pmatrix} x_j f_1 & \cdots & x_j f_\nu \end{pmatrix} \in \tilde{\mathscr{H}}$$

and define $X_j : \mathscr{H} \to \mathscr{H}$ by

$$X_j f = P x_j f, \quad f \in \mathscr{H},\ 1 \le j \le g,$$

where P is the orthogonal projection from $\tilde{\mathscr{H}}$ onto $\mathscr{H}$ (which is only needed on the degree $k+1$ part of $x_j f$). From the positive definiteness of the bilinear form (3.5) on $\mathscr{H}$, one easily sees that each X_j is well defined.

Let $\gamma \in \mathscr{H}^{\oplus\nu}$ denote the vector whose j-th entry, γ_j has the empty word (the monomial 1) in the j-th entry and zeros elsewhere. Finally, given words $v_{s,t} \in \langle x\rangle_k$ and $w_{s,t} \in \langle x\rangle_k$ for $1 \le s,t \le \nu$, choose $f \in \mathbb{C}^{\nu\times\nu}\langle x, x^*\rangle_k^{\mathrm{her}}$ to have (s,t)-entry $w_{s,t}^* v_{s,t}$. In particular, with $e_1,\dots,e_\nu$ denoting the standard orthonormal basis for $\mathbb{R}^\nu$, we have

$$f = \sum_{s,t=1}^{\nu} w_{s,t}^* v_{s,t} e_s e_t^*.$$

Thus,

$$\begin{aligned}\langle f(X)\gamma,\gamma\rangle &= \sum \langle f_{s,t}(X)\gamma_t,\gamma_s\rangle = \sum\langle w_{s,t}^*(X)v_{s,t}(X)\gamma_t,\gamma_s\rangle = \sum\langle v_{s,t}(X)\gamma_t, w_{s,t}(X)\gamma_s\rangle \\ &= \sum\langle v_{s,t}e_t^*, w_{s,t}e_s^*\rangle = \sum \lambda(w_{s,t}^* v_{s,t} e_s e_t^*) = \lambda\big(\sum (w_{s,t}^* v_{s,t} e_s e_t^*)\big) = \lambda(f).\end{aligned}$$

Since any $f \in \mathbb{C}^{\nu\times\nu}\langle x,x^*\rangle_k^{\mathrm{her}}$ can be written as a linear combination of words of the form w^*v with $v,w \in \langle x\rangle_k$ as was done above, equation (3.4) is established.

To prove the further statement, suppose λ is nonnegative on M_{k+1}^ν. Write $L = I + \Lambda + \Lambda^*$, where Λ is the homogeneous linear analytic part of L. Given

$$p = \begin{pmatrix} p_1 \\ \vdots \\ p_\ell \end{pmatrix} \in \mathscr{H}^{\oplus\ell},$$

note that

$$\begin{aligned}\langle L(X)p,p\rangle &= \langle (I+\Lambda(X)+\Lambda(X)^*)p,p\rangle = \langle (I+\Lambda(X))p,p\rangle + \langle p,\Lambda(X)p\rangle \\ &= \langle p + \sum A_j P x_j p, p\rangle + \langle p, \sum A_j P x_j p\rangle = \langle p + \sum A_j x_j p, p\rangle + \langle p, \sum A_j x_j p\rangle \\ &= \langle (I+\Lambda(x))p,p\rangle + \langle p,\Lambda(x)p\rangle = \lambda\big(p^*(I+\Lambda_A(x))p\big) + \lambda(p^*\Lambda(x)^*p) \\ &= \lambda\big(p^*(I+\Lambda(x)+\Lambda(x)^*)p\big) = \lambda(p^*Lp) \ge 0.\end{aligned} \tag{3.6}$$

Hence, $L(X) \succeq 0$.

The proof of the converse is routine and is not used in the sequel. ∎

3.1.3. *Step 3: Conclusion.* Let us first prove (3.2). The implication ($\Leftarrow$) is obvious. For the forward implication ($\Rightarrow$), assume $h \notin M^{\nu,\text{her}}_{d+1,d}(L)$. By Lemma 3.2 then there is a linear functional $\lambda : \mathbb{C}^{\nu\times\nu}\langle x, x^*\rangle^{\text{her}}_{2d+2} \to \mathbb{C}$ satisfying

$$\lambda(h) < 0, \qquad \lambda(M^{\nu,\text{her}}_{d+1,d}) \subseteq \mathbb{R}_{\geq 0}.$$

By adding a small multiple of a linear functional that is strictly positive on $\Sigma^{\nu,\text{her}}_{d+1} \setminus \{0\}$ (see e.g. [HKM12a, Lemma 3.2] for its existence), we may assume moreover that

$$\lambda\big(\Sigma^{\nu,\text{her}}_{d+1} \setminus \{0\}\big) \subseteq \mathbb{R}_{>0}.$$

Now Proposition 3.3 applies: there exists a tuple $X = (X_1, \dots, X_g) \in \mathcal{D}_L$ of $\nu\sigma_\#(d) \times \nu\sigma_\#(d)$ matrices, and a vector γ such that (3.4) holds for all $f \in \mathbb{C}^{\nu\times\nu}\langle x, x^*\rangle^{\text{her}}_d$. Hence

$$0 > \lambda(h) = \langle h(X)\gamma, \gamma\rangle,$$

so $h(X) \not\succeq 0$.

Finally, (3.3) follows from the Hahn-Banach theorem. Namely, if $\mathcal{D}_L$ is bounded, then $1 = \sum_j V_j^* V_j$ for some V_j, see e.g. [HKM12a]. ∎

3.2. **Applying the Hereditary Positivstellensatz to mappings between free spectrahedra.** Suppose A is a g-tuple of $d \times d$ matrices and B is an $\tilde{g}$-tuple of $e \times e$ matrices. If $p : \mathcal{D}_A \to \mathcal{D}_B$ is a polynomial map and $X \in M(\mathbb{C})^g$, then $L_A(X) \succeq 0$ implies $L_B(p(X)) \succeq 0$. Assuming $p(0) = 0$, then $L_B(p(x)) = I + G(x) + G(x)^* \succeq 0$, with $G(x) = \Lambda_B(p(x))$. Furthermore, Theorem 3.1 produces a finite-dimensional Hilbert space H and a polynomial W with coefficients $W_\alpha : \mathbb{C}^e \to H \otimes \mathbb{C}^d$ such that

$$I + G(x) + G(x)^* = W(x)^* L_{I_H \otimes A}(x) W(x). \tag{3.7}$$

We analyze consequences of the positivity certificate (3.7) in the next section. As we shall see in Proposition 4.4 below, these identities force very strong relationships between G, W and A.

4. Positivity Certificates for Analytic Mappings

In Section 5 we shall extend the results of Section 3 to real parts of analytic functions. This section contains preliminary results needed to establish the analytic Positivstellensatz, Theorem 1.5. Namely, Proposition 4.4 and Corollary 4.5 below. They establish strong consequences of a Positivstellensatz-type certificate for the real part of an analytic function.

Suppose G is an $e \times e$ matrix-valued free function analytic near 0, there is a (not necessarily finite-dimensional) Hilbert space $\mathcal{E}$ and a formal power series W with coefficients $W_\alpha : \mathbb{C}^e \to \mathcal{E}$ and a tuple A of operators on $\mathcal{E}$ such that

$$I + G(x) + G(x)^* = W(x)^* L_A(x) W(x) \tag{4.1}$$

holds in the ring of formal power series.

Let G_α denote the coefficients of G. Expanding the right-hand side of (4.1) and comparing terms gives, for all $1 \leq j, k \leq g$ and words α, β,

$$\begin{aligned} W_\beta^* A_k^* W_{x_j\alpha} + W_{x_k\beta}^* A_j W_\alpha + W_{x_k\beta}^* W_{x_j\alpha} &= 0 \\ W_\varnothing^*(A_k W_\alpha + W_{x_k\alpha}) &= G_{x_k\alpha} \\ W_\varnothing^* W_\varnothing &= I. \end{aligned} \tag{4.2}$$

The following proposition connects positivity certificates to formal power series considerations.

Proposition 4.1. *Suppose W and G are formal power series and $G(0) = 0$. If for all nilpotent $X \in M(\mathbb{C})^g$,*

$$I + G(X) + G(X)^* = W(X)^* L_A(X) W(X), \tag{4.3}$$

then the identities of equation (4.2) hold. Conversely, if (4.2) holds, then (4.3) holds for all nilpotent $X \in M(\mathbb{C})^g$.

Before formally beginning the proof of Proposition 4.1, we first state and prove a routine lemma. Fix N a positive integer. Consider the truncated Fock Hilbert space $\mathscr{F}_N$ with orthonormal basis $\{\alpha \in \langle x \rangle : |\alpha| \le N\}$. Let S (we suppress the dependence on N) denote the tuple of shifts determined by $S_j w = x_j w$ if the length of the word w is strictly less than N and $S_j w = 0$ if the length of the word w is N. In particular, S is nilpotent of order N.

Lemma 4.2. *Given Hilbert spaces H and K and operators $G_{\alpha,\beta} : H \to K$ parameterized over words α, β, if for each N*

$$\sum_{|\alpha|,|\beta| \le N} G_{\alpha,\beta} \otimes S^\beta S^{*\alpha} = 0,$$

then $G_{\alpha,\beta} = 0$ for all α, β.

Proof. In the case $N = 1$ evaluating at vectors of the form $x \otimes \varnothing$ with $x \in H$ gives

$$0 = \sum_{|\beta| \le 1} G_{\varnothing,\beta} x \otimes \beta.$$

Hence $G_{\varnothing,\beta} = 0$ for $|\beta| \le 1$. Evaluating at vectors of the form $x \otimes x_k$ gives

$$0 = \sum_{|\beta| \le 1} G_{x_k,\beta} x \otimes \beta.$$

Hence, for each $k = 1, \dots, g$ and $|\beta| \le 1$, we have $G_{x_k,\beta} = 0$. In conclusion, $G_{\alpha,\beta} = 0$ for $|\alpha|, |\beta| \le 1$.

Arguing by induction, now suppose $G_{\alpha,\beta} = 0$ for all $|\alpha|, |\beta| \le N$. Let S denote the shifts for $\mathscr{F}_{N+1}$. Given a word γ of length at most $N+1$, evaluation at $x \otimes \gamma$ and the induction hypothesis gives,

$$0 = \sum_{\beta} G_{\gamma,\beta} x \otimes \beta.$$

Hence $G_{\gamma,\beta} = 0$ for all $|\gamma|, |\beta| \le N + 1$. ∎

Proof of Proposition 4.1. Suppose (4.3) holds. In this case Lemma 4.2 implies the coefficients of the expression on the right hand side corresponding to words γ, δ where neither is $\varnothing$ are zero. These coefficients are $W_\alpha^* A_k^* W_{x_j\alpha} + W_{x_k\beta}^* A_j W_\alpha + W_{x_k\beta}^* W_{x_j\alpha}$ for words α, β and $1 \le j, k \le g$ with $\gamma = x_j\alpha$ and $\delta = x_k\beta$.

The converse is routine. ∎

For the purposes of this article, the **formal radius of convergence** $\tau(f)$ of a formal power series $f(x) = \sum f_\alpha \alpha$ with operator coefficients is

$$\tau(f) = \frac{1}{\limsup_N \left(\sum_{|\alpha|=N} \|f_\alpha\| \right)^{\frac{1}{N}}},$$

with the obvious interpretations in the cases that the limit superior is either zero or infinity. Similarly, the **spectral radius** of a tuple $X \in M_n(\mathbb{C})^g$ is

$$\rho(X) = \limsup_N \max\{\|X^\alpha\| : |\alpha| = N\}.$$

If $X \in M(\mathbb{C})^g$ and $\rho(X) < \tau(f)$, then the series

$$f(X) = \sum_{N=1}^{\infty} \sum_{|\alpha|=N} f_\alpha \otimes X^\alpha$$

converges. Let $\mathbb{D}_\tau = \{X \in M(\mathbb{C})^g : \rho(X) < \tau\}$.

For completeness, we record the following result. It is not needed elsewhere in the paper.

Proposition 4.3. *Suppose $\tau > 0$, W and G are formal power series with positive formal radii of convergence at least τ and $G(0) = 0$. If the identities of equation (4.2) hold, then*

$$I + G(X) + G(X)^* = W(X)^* L_A(X) W(X), \tag{4.4}$$

for all $X \in \mathbb{D}_\tau$.

Proof. Let R and L denote the values of the right and left hand side of (4.4) respectively and let $G^{(N)}$ and $W^{(N)}$ denote the N-th partial sums of the respective series $G(X)$ and $W(X)$. Given $\epsilon > 0$ there is an N such that

$$\|I + G^{(N)}(X) + G^{(N)}(X)^* - R\| < \epsilon,$$
$$\|W^{(N)}(X)^* L_A(X) W^{(N)}(X) - L\| < \epsilon.$$

Use (4.2) to compute

$$\begin{aligned} &I + G^{(N)}(X) + G^{(N)}(X)^* - W^{(N)}(X)^* L_A(X) W^{(N)}(X) \\ &\quad = -\sum_{k=1}^{g} \sum_{|\beta|=N} \sum_{|\alpha|\le N} W_\beta^* A_k^* W_\alpha \otimes X^{*\beta} X_k^* X^\alpha - \sum_{j=1}^{g} \sum_{|\beta|\le N} \sum_{|\alpha|=N} W_\beta^* A_j W_\alpha \otimes X^{*\beta} X_j X^\alpha \\ &= -\Big(\sum_{|\beta|=N} W_\beta \otimes X^\beta\Big)^* \Lambda_A(X)^* \Big(\sum_{|\alpha|\le N} W_\alpha \otimes X^\alpha\Big) - \Big(\sum_{|\beta|\le N} W_\beta \otimes X^\beta\Big)^* \Lambda_A(X) \Big(\sum_{|\alpha|=N} W_\alpha \otimes X^\alpha\Big). \end{aligned} \tag{4.5}$$

The norm of each of the two summands in the last line of (4.5) is at most

$$\sum_{k=1}^{g} \|A_k \otimes X_k\| \big(\sum_{|\beta|=N} \|W_\beta\| \, \|X^\beta\|\big) \big(\sum_{|\alpha|\le N} \|W_\alpha\| \, \|X^\alpha\|\big). \tag{4.6}$$

By hypothesis the second factor in (4.6) tends to 0 with N and the first and third factor are uniformly bounded on $\mathbb{D}_\tau$. Thus the left hand side of (4.5) tends to zero with N and the proof is complete. ■

4.1. **Formulas forced by the positivity certificates.** Define the **range** of a tuple of operators A on a Hilbert space $\mathcal{E}$ by

$$\mathrm{rg}(A) = \mathrm{span}(\{A_j h : 1 \le j \le g, \, h \in \mathcal{E}\}).$$

The next proposition completely characterizes all G, W solving (4.1).

Proposition 4.4. *Suppose e is a positive integer and $\mathcal{E}$ is a separable Hilbert space. If*

1. *A is a g-tuple of operators on $\mathcal{E}$;*
2. *W is a formal power series with coefficients $W_\alpha : \mathbb{C}^e \to \mathcal{E}$; and*
3. *G is a formal power series with $G(0) = 0$ and $e \times e$ matrix coefficients G_α such that the identities of equation (4.2) hold,*

then there exists an isometry $C : \mathcal{E} \to \mathcal{E}$ and an isometry $\mathscr{W} : \mathbb{C}^e \to \mathcal{E}$ such that

$$G(x) = \mathscr{W}^* C\big(\sum_{j=1}^{g} A_j x_j\big)\,\big(I - \sum_{\ell=1}^{g} R_\ell x_\ell\big)^{-1}\mathscr{W} = \mathscr{W}^* C\big(\sum_{j=1}^{g} A_j x_j\big)\,W(x), \tag{4.7}$$

where $R = (C - I)A$ and W is the rational function

$$W(x) = \big(I - \sum_{j=1}^{g} R_j x_j\big)^{-1}\mathscr{W}. \tag{4.8}$$

In particular,

(a) *the following recursion formula holds,*

$$W_{x_j\alpha} = (C - I)A_j W_\alpha;$$

(b) *the x_j coefficient of $G(x)$ in equation (4.7) is*

$$G_{x_j} = \mathscr{W}^* C A_j \mathscr{W};$$

(c) *more generally, $G_{x_j\alpha}$, the $x_j\alpha$ coefficient of G, is*

$$G_{x_j\alpha} = \mathscr{W}^* C A_j R^\alpha \mathscr{W}.$$

Conversely, given a tuple A, an operator C isometric on $\mathrm{rg}(A)$, and an isometry $\mathscr{W} : \mathbb{C}^d \to \mathcal{E}$, defining $R = (C - I)A$, W as in (4.8), and G as in equation (4.7), the identities of (4.2) hold.

The proof of Proposition 4.4 is given in Subsection 4.2 below. The next corollary specializes Proposition 4.4 to the case $G(x) = \Lambda_B(p(x))$ for a formal power series p.

Corollary 4.5. *Suppose $p = (p^1, \dots, p^{\tilde g})$ and each p^t is a formal power series. Further suppose $p(0) = 0$ and $p(x) = (x, 0) + h(x)$, where h consists of higher (two and larger) degree terms. Write*

$$p^t(x) = \sum_j \sum_\alpha p^t_{x_j\alpha} x_j \alpha.$$

Suppose the hypotheses of Proposition 4.4 are in force and $G(x) = \Lambda_B(p(x))$. Then

(a) *$B_j = \mathscr{W}^* C A_j \mathscr{W}$ for $1 \le j \le g$;*

(b) *$\Lambda_B(p(x)) = \sum_t B_t p^t(x) = \sum_{k=1}^{g} \sum_\alpha \mathscr{W}^* C A_k R^\alpha \mathscr{W} x_k \alpha$;*

(c) *for all words ω,*

$$\mathscr{W}^* C A_k R^\omega \mathscr{W} = \sum_{j=1}^{\tilde g} p^j_{x_k\omega}\, B_j = \sum_{j=1}^{\tilde g} p^j_{x_k\omega}\, \mathscr{W}^* C A_j \mathscr{W}. \tag{4.9}$$

4.2. **Proof of Proposition 4.4.** Completing the square in the first equation of (4.2) gives,

$$(A_k W_\beta + W_{x_k\beta})^* \,(A_j W_\alpha + W_{x_j\alpha}) = W_\beta^* A_k^* A_j W_\alpha. \tag{4.10}$$

Fix, for the moment, a positive integer N. Recall $W_\alpha : \mathbb{C}^e \to \mathcal{E}$ and $A_j : \mathcal{E} \to \mathcal{E}$. Let $\mathcal{K}_N = \oplus_{|\alpha| \le N} \mathbb{C}^e$, the Hilbert space direct sum of $\mathbb{C}^e$ over the set of words of length at most N in the variables $x = (x_1, \dots, x_g)$. Finally, let $\mathcal{L}_N := \oplus_{j=1}^{g} \mathcal{K}_N$ and note that $h \in \mathcal{L}_N$ takes the form $h = \oplus_j \oplus_{|\alpha| \le N} h_{j,\alpha} = \oplus h_{j,\alpha}$. Let

$$\mathcal{E}_N = \Big\{ \sum_{j,|\alpha| \le N} A_j W_\alpha\, h_{j,\alpha} : h_{j,\alpha} \in \mathbb{C}^e \text{ for } 1 \le j \le g, \ \ |\alpha| \le N \Big\} \subseteq \mathrm{rg}(A) \subseteq \mathcal{E}.$$

The subspaces $\mathcal{E}_N$ are nested increasing.

Define $X_N, Y_N : \mathcal{L}_N \to \mathcal{E}_N$,

$$X_N(\oplus_{j=1}^g \oplus_{|\alpha|\le N} h_{j,\alpha}) = \sum_{j,|\alpha|\le N} (A_j W_\alpha + W_{x_j\alpha}) h_{j,\alpha}$$

$$Y_N(\oplus_{j=1}^g \oplus_{|\alpha|\le N} h_{j,\alpha}) = \sum_{j,|\alpha|\le N} (A_j W_\alpha) h_{j,\alpha}.$$

Note that the range of Y_N lies in $\operatorname{rg}(A)$. Equation (4.10) implies that

$$X_N^* X_N = Y_N^* Y_N : \mathcal{L}_N \to \mathcal{L}_N. \tag{4.11}$$

In particular, if $Y_N h = 0$, then $X_N h = 0$. Hence, $C_N : \mathcal{E}_N \to \mathcal{E}$ defined by $C_N Y_N h = X_N h$ is well defined. Further, equation (4.11) implies that C_N is an isometry. Since $\mathcal{E}_N$ is finite-dimensional, C_N can be extended to a unitary $C_N : \mathcal{E} \to \mathcal{E}$. Thus, there is a unitary mapping $C_N : \mathcal{E} \to \mathcal{E}$ such that

$$X_N = C_N Y_N.$$

Moreover, for $N \ge M$,

$$X_M = C_N Y_M.$$

Since $(C_N)_N$ is a sequence of unitaries on $\mathcal{E}$, a subsequence $(C_{N_j})_j$ converges in the weak operator topology (WOT) to a contraction operator C. Fix M. For $N_j \ge M$, $C_{N_j} Y_M = X_M$. Hence, for a vector $h \in \mathcal{L}_M$ and a vector $e \in \mathcal{E}$,

$$\langle C Y_M h, e\rangle = \lim_j \langle C_{N_j} Y_M h, e\rangle = \langle X_M h, e\rangle.$$

Thus, $C Y_M = X_M$ for all M. In particular, for each j, α

$$A_j W_\alpha + W_{x_j\alpha} = C A_j W_\alpha \tag{4.12}$$

and C is an isometry on the span of the ranges of the Y_M. Hence, C can, without loss of generality, be assumed isometric on all of $\mathcal{E}$ and still satisfy equation (4.12) which in turn can be expressed as

$$W_{x_j\alpha} = (C - I) A_j W_\alpha. \tag{4.13}$$

Now suppose $C : \mathcal{E} \to \mathcal{E}$ is isometric on $\operatorname{rg}(A)$ and satisfies equation (4.12). In particular, equation (4.13) also holds. For notational ease, if not consistency, let $\mathscr{W} = W_\varnothing$ and $W_\ell = W_{x_\ell}$. It follows from equation (4.13) with $\alpha = \varnothing$, that

$$W_j = (C - I) A_j \mathscr{W} \tag{4.14}$$

for each j. Let

$$R_j = (C - I) A_j.$$

Thus, $W_j = R_j \mathscr{W}$.

For each k an application of equation (4.13) with $\alpha = x_j$ yields

$$W_{x_k x_j} = (C - I) A_k W_j = R_k R_j \mathscr{W},$$

for $j, k = 1, \dots g$. Induction on the length of words gives,

$$W_\alpha = R^\alpha \mathscr{W}$$

where $R = (R_1, \dots, R_g)$. Hence,

$$W(x) = (I - \sum R_j x_j)^{-1} \mathscr{W}. \tag{4.15}$$

Now using the second and third equations of (4.2) together with (4.15) gives

$$\mathscr{W}^* (I + \sum A_k x_k)(I - \sum R_\ell x_\ell)^{-1} \mathscr{W} = I + G(x).$$

Hence,

$$\mathscr{W}^*[\Big(I+\sum A_k x_k\Big)\Big(I-\sum_{\ell=1}^{g} R_\ell x_\ell\Big)^{-1}-I]\mathscr{W}=G(x)$$

and therefore,

$$G(x)=\mathscr{W}^*\big(\sum_{k=1}^{g}(A_k+R_k)x_k\big)\big(I-\sum_{\ell=1}^{g} R_\ell x_\ell\big)^{-1}\mathscr{W}.$$

Finally,

$$\begin{aligned}G(x)&=\mathscr{W}^* C\big(\sum A_k x_k\big)\big(I-(C-I)\sum A_\ell x_\ell\big)^{-1}\mathscr{W}\\ &=\mathscr{W}^* C\big(\sum A_k x_k\big)\big(I-\sum R_\ell x_\ell\big)^{-1}\mathscr{W}.\end{aligned} \tag{4.16}$$

At this point we have proved that if W and G solve equation (4.1), then there exists an isometry C on $\mathcal{E}$ such that W and G have the claimed form.

To prove the converse, given a tuple $A=(A_1,\dots,A_g)$ of operators on the Hilbert space $\mathcal{E}$, a contraction $C:\mathcal{E}\to\mathcal{E}$ that is isometric on $\operatorname{rg}(A)$, and a one-one operator $\mathscr{W}:\mathbb{C}^d\to\mathcal{E}$ let $R=(C-I)A$ and $W(x)=(I-\Lambda_R(x))^{-1}\mathscr{W}$ and define G by equation (4.7),

$$G(x)=\mathscr{W}^* C\Lambda_A(x)(I-\Lambda_R(x))^{-1}\mathscr{W}.$$

By construction, $W_\alpha=R^\alpha\mathscr{W}$. Moreover, for each word α and $1\le j\le g$,

$$(C-I)A_jW_\alpha=R_jR^\alpha\mathscr{W}=W_{x_j\alpha}.$$

Hence,

$$CA_jW_\alpha=A_jW_\alpha+W_{x_j\alpha}.$$

Since C is isometric on the range of A, given $1\le k\le g$ and a word β,

$$W_\beta^*A_k^*A_jW_\alpha=W_\beta^*A_k^*A_jW_\alpha+W_\beta^*A_k^*W_\alpha+W_{x_k\beta}^*A_jW_\alpha+W_{x_\beta}^*W_{x_j\alpha}.$$

Thus the first of the identities of equation (4.2) hold. The second identity holds by the choice of G and the proof is complete.

Remark 4.6. We note that the proof of the converse of Proposition 4.4 would, under some convergence assumptions, follow from the following formal calculation starting from the formula for G of equation (4.7). Using $R_j=(C-I)A_j$ gives

$$I-\Lambda_R(x)=I+\Lambda_A(x)-C\Lambda_A(x).$$

Let

$$H(x)=C\Lambda_A(x)(I-\Lambda_R(x))^{-1}=\Lambda_{CA}(x)(I-\Lambda_R(x))^{-1}$$

and note that

$$\mathscr{W}^*\big(I+H(x)+H^*(x)\big)\mathscr{W}=I+G(x)+G^*(x).$$

Now,

$$\begin{aligned}&I+H(x)+H(x)^*\\ &=(I-\Lambda_R(x))^{-*}[(I-\Lambda_R(x))^*(I-\Lambda_R(x))+(I-\Lambda_R(x))^*C\Lambda_A(x)+\Lambda_A(x)^*C^*(I-\Lambda_R(x))](I-\Lambda_R(x))^{-1}\\ &=(I-\Lambda_R(x))^{-*}[(I-\Lambda_R(x)+C\Lambda_A(x))^*(I-\Lambda_R(x)+C\Lambda_A(x))-\Lambda_A(x)^*C^*C\Lambda_A(x)](I-\Lambda_R(x))^{-1}\\ &=(I-\Lambda_R(x))^{-*}[\Psi(x)^*\Psi(x)-\Lambda_A(x)^*\Lambda_A(x)](I-\Lambda_R(x))^{-1}\\ &=(I-\Lambda_R(x))^{-*}[I+\Lambda_A(x)^*+\Lambda_A(x)](I-\Lambda_R(x))^{-1},\end{aligned}$$

from which it follows that

$$\mathscr{W}^*L_R(x)^{-*}\big(I+\Lambda_A(x)+\Lambda_A^*(x)\big)L_R(x)^{-1}\mathscr{W}=I+G(x)+G^*(x). \qquad \lozenge$$

4.3. **Polynomials correspond to nilpotent** R**.** A tuple of matrices $E \in (\mathbb{C}^{n\times n})^g$ is (jointly) **nilpotent** if there exists an N such that $E^w = 0$ for all words w of length $|w| \geq N$. The smallest such N is the **order of nilpotence** of E.

Corollary 4.7. *Suppose, in the context of Proposition 4.4, that $W_\varnothing$ is a $d \times D$ matrix. If*

(i) *G is a polynomial;*
(ii) $\operatorname{span}\{R^\omega W_\varnothing h : h \in \mathbb{C}^d, \omega \in \langle x\rangle\} = \mathbb{C}^D$*; and*
(iii) $\bigcap\{\ker(W_\varnothing^* C A_j R^w) : w \in \langle x\rangle,\ 1 \leq j \leq g\} = (0)$,

then the tuple R is nilpotent. In particular, if $D = d$ and G is a polynomial, then R is nilpotent.

In the language of systems theory, the system $(R, W_\varnothing, \{W_\varnothing^* C A_j\})$ is **controllable** (item (ii)) and **observable** (item (iii)).

Proof. Since $W_\varnothing^* C\Lambda_A(x)(I - \sum R_j x_j)^{-1} W_\varnothing$ is a polynomial, there exists a positive integer N such that

$$W_\varnothing^* C A_i R^\omega W_\varnothing = 0$$

for all words ω for which $|\omega| \geq N$. Hence, if $|\xi| \geq N$, then for any words α, β,

$$0 = W_\varnothing^* C A_i R^\omega W_\varnothing = W_\varnothing^* C A_i R^\alpha R^\xi R^\beta W_\varnothing$$

Conditions (ii) and (iii) now imply that $R^\xi = 0$. ■

Remark 4.8. Unfortunately, W can be a polynomial without $R = (R_1, \dots, R_g)$ being jointly nilpotent. One just needs $R^\alpha W_\varnothing = 0$ for α large enough. Of course if $W_\varnothing$ is square, then it is invertible so the R_j are jointly nilpotent if and only if W is a polynomial. ◊

5. Extending the Hereditary Positivstellensatz to Analytic Functions

In this section we extend the Hereditary Convex Positivstellensatz (Theorem 3.1) to analytic maps and rational functions between free spectrahedra using Theorem 3.1 and Proposition 4.4. Our main technique combines an approximation result (Theorem 1.12) to approximate functions analytic on free spectrahedra using polynomials with the Positivstellensatz 3.1. Here is the main result of this section.

Theorem 5.1 (Analytic convex Positivstellensatz)**.** *Let $A \in M_d(\mathbb{C})^g$, $B \in M_e(\mathbb{C})^{\tilde g}$, assume that $\mathcal{D}_A$ is bounded and let p be a free function analytic and bounded on a free pseudoconvex set $\mathcal{G}_Q$ containing $\mathcal{D}_A$. If $p : \mathcal{D}_A \to \mathcal{D}_B$ and $p(0) = 0$, then there exists a Hilbert space H and a formal power series $W = \sum_{\alpha\in\langle x\rangle} W_\alpha \alpha$ with coefficients $W_\alpha : \mathbb{C}^e \to H \otimes \mathbb{C}^d$ such that*

$$L_B(p(x)) = W(x)^* L_{I_H \otimes A}(x) W(x) \tag{5.1}$$

holds in the ring of $e \times e$ matrices over formal power series in x, x^*. Moreover,

1. *with $G(x) = \Lambda_B(p(x))$, equations (4.2), and thus the conclusions of Proposition 4.4, hold;*
2. *for any nilpotent $X \in M(\mathbb{C})^g$, (5.1) holds at X;*
3. *W has positive formal radius of convergence and thus (5.1) holds for all $X \in \mathbb{D}_\tau$ for τ sufficiently small.*

Proof. The statement follows by combining Proposition 5.2 below with the approximation result Theorem 1.12. ■

5.1. **From polynomial to analytic functions.** Fix $A \in M_d(\mathbb{C})^g$ with bounded $\mathcal{D}_A$, $B \in M_e(\mathbb{C})^{\tilde{g}}$ and a free analytic function p defined and bounded on a free pseudoconvex set $\mathcal{G}_Q$ containing $\mathcal{D}_A$ such that $p : \mathcal{D}_A \to \mathcal{D}_B$ and $p(0) = 0$.

Proposition 5.2 (Approximate Positivstellensatz)**.** *Assume the hypotheses of Theorem 5.1 hold. If there is a sequence of polynomials $(p_\ell)_\ell$ converging uniformly to p on $\mathcal{D}_A$, then there exists a Hilbert space H and a formal power series W with coefficients $W_\alpha : \mathbb{C}^e \to H$ such that*

$$L_B(p(x)) = W(x)^* L_{I_H \otimes A}(x) W(x)$$

holds in the ring of formal power series in x, x^*. Moreover,
(1) W has positive formal radius of convergence;
(2) with $G(x) = \Lambda_B(p(x))$, equations (4.2) hold.

Lemma 5.3. *If (p_ℓ) is a sequence of polynomials converging uniformly to p on $\mathcal{D}_A$, then there exists a sequence of polynomials (q_ℓ) converging uniformly to p such that $q_\ell(0) = 0$ and $q_\ell(\mathcal{D}_A) \subseteq \mathcal{D}_B$.*

Proof. Note that $(p_\ell(0))_\ell$ converges to 0 since $0 \in \mathcal{D}_A$ and $p(0) = 0$. Let $r_\ell = p_\ell - p_\ell(0)$. In particular, r_ℓ converges uniformly to p and $r_\ell(0) = 0$. Choose a sequence $(t_k)_k$ such that $0 < t_k < 1$ and $\lim t_k = 1$. Note that, for $X \in \mathcal{D}_A$,

$$L_B(t_k p(X)) \succeq (1 - t_k) I$$

For each k there is an ℓ_k such that $r_{\ell_k}(X)$ is uniformly sufficiently close to p so that

$$L_B(t_k r_{\ell_k}(X)) \succeq L_B(t_k p(X)) - (1 - t_k) I \succeq 0.$$

Hence $t_k r_{\ell_k}$ maps $\mathcal{D}_A$ into $\mathcal{D}_B$. The sequence $q_k = t_k r_{\ell_k}$ thus converges uniformly to p and satisfies $q_k(0) = 0$ and $q_k : \mathcal{D}_A \to \mathcal{D}_B$. ■

Proof of Proposition 5.2. By Lemma 5.3, without loss of generality there is a sequence $(p_\ell)_\ell$ of polynomials such that $p_\ell(0) = 0$, $p_\ell : \mathcal{D}_A \to \mathcal{D}_B$ and $(p_\ell)_\ell$ converges uniformly to p on $\mathcal{D}_A$. By Theorem 3.1, there is a separable infinite-dimensional Hilbert space H such that for each ℓ there exists a polynomial W_ℓ with coefficients $W_{\ell,\alpha} : \mathbb{C}^e \to H \otimes \mathbb{C}^d$ such that

$$L_B(p_\ell(x)) = W_\ell{}^*(x) L_{I_H \otimes A}(x) W_\ell(x). \tag{5.2}$$

Applying Proposition 4.4, there exists a unitary C_ℓ on $H \otimes \mathbb{C}^d$ such that, with $R_\ell = (C_\ell - I)[I_H \otimes A]$, we have

$$W_\ell(x) = (I - \Lambda_{R_\ell}(x))^{-1} \mathscr{W}_\ell.$$

where $\mathscr{W}_\ell : \mathbb{C}^e \to H \otimes \mathbb{C}^d$. Moreover, from the identity $W_{\ell, x_j \alpha} = (C_\ell - I)[I_H \otimes A_j] W_{\ell,\alpha}$ of item (a) of Proposition 4.4,

$$\|W_{\ell, x_j \alpha}\| \leq 2 \max\{\|A_1\|, \dots, \|A_g\|\} \, \|W_{\ell,\alpha}\|.$$

Thus, using the fact that $\mathscr{W}_\ell = W_{\ell,\varnothing}$ is an isometry and hence has norm one, $\|W_{\ell,\alpha}\|$ has a uniform bound depending only on the length of the word α (independent of ℓ).

Observe that for each α, the dimension of the range of $W_{\ell,\alpha}$ is at most e. Hence, for a fixed N, there is a constant D_N such that for each ℓ the dimension of the span of

$$H_{N,\ell} = \bigvee_{|\alpha| \leq N} \operatorname{rg}(W_{\ell,\alpha})$$

is at most D_N. (Indeed one can take D_N to be de times the number of words of length at most N.) It follows that, given N, for each ℓ there exists a subspace $H_{N,\ell}$ of H of dimension D_N such that the ranges of $W_{\ell,\alpha}$ all lie in $H_{N,\ell} \otimes \mathbb{C}^d$. For technical reasons that will soon be apparent, choose a basis $\{e_1, e_2, \dots\}$ for H and inductively construct subspaces $\mathcal{H}_{N,\ell}$ of H of dimension $2D_N$ such that $\mathcal{H}_{N,\ell}$ contains both $H_{N,\ell}$ and $\operatorname{span}(\{e_1, \dots, e_{D_N}\})$ and such that $\mathcal{H}_{N,\ell} \subseteq \mathcal{H}_{N+1,\ell}$. In particular,

$H = \oplus_{N=-1}^{\infty}(\mathcal{H}_{N+1,\ell} \ominus \mathcal{H}_{N,\ell})$, where $\mathcal{H}_{-1,\ell} = \{0\}$. Set $D_{-1} = 0$ and let $E_m = 2(D_m - D_{m-1})$. Letting $K_m = \mathbb{C}^{E_m}$ and K denote the Hilbert space $\oplus_{m=0}^{\infty} K_m$, it follows that for each ℓ there is a unitary mapping $\rho_\ell : H \to K$ such that $\rho_\ell(\mathcal{H}_{N,\ell}) = \oplus_{m=0}^{N} K_m$. We have,

$$W_\ell(x)^*(\rho_\ell \otimes I_d)^*[I_K \otimes L_A(x)](\rho_\ell \otimes I_d)W_\ell(x) = W_\ell(x)^*[I_H \otimes L_A(x)]W_\ell(x).$$

Hence, we can replace $W_\ell(x)$ with $(\rho_\ell \otimes I)W_\ell(x)$ in (5.2) and thus, given a word α of length N, assume that $W_{\ell,\alpha}$ maps into $\oplus_{m=0}^{N} K_m$ independent of ℓ.

For a fixed word α, the set $\{W_{\ell,\alpha} : \ell\}$ maps into a common finite-dimensional Hilbert space and is, in norm, uniformly bounded. Hence, by passing to a subsequence, we can assume for each word α the sequence $W_{\ell,\alpha}$ converges to some W_α in norm. Let W denote the corresponding formal power series. We will argue that

$$L_B(p(x)) = W(x)^* L_{I_H \otimes A}(x) W(x)$$

in the sense explained as follows. By construction, given α and j, for every ℓ,

$$W_{\ell,\varnothing}^*(I_H \otimes A_j)W_{\ell,\alpha} + W_{\ell,x_j\alpha}$$

is the coefficient of the $x_j\alpha$ term of $\Lambda_B(p_\ell(x))$. Thus as $p_\ell(x)$ converges uniformly to $p(x)$, this sequence converges to the $x_j\alpha$ coefficient of $\Lambda_B(p(x))$. On the other hand, from what has been proven, this sequence converges to $(I_H \otimes A_j)W_\alpha + W_{x_j\alpha}$ too. Moreover, also by construction, for each α, β and j, k,

$$W_{\ell,\beta}^*(I_H \otimes A_k)^* W_{\ell,x_j\alpha} + W_{\ell,x_k\beta}^*(I_H \otimes A_j)W_{\ell,\alpha} + W_{\ell,x_k\beta}^* W_{\ell,x_j\alpha} = 0.$$

Hence,

$$W_\beta^*(I_H \otimes A_k)^* W_{x_j\alpha} + W_{x_k\beta}^*(I_H \otimes A_j)W_\alpha + W_{x_k\beta}^* W_{x_j\alpha} = 0$$

and the claim is established. ■

Corollary 5.4 (Rational convex Positivstellensatz)**.** *Let $A \in M_d(\mathbb{C})^g$, $B \in M_e(\mathbb{C})^{\tilde{g}}$, assume that $\mathcal{D}_A$ is bounded and let r be a rational function with no singularities on $\mathcal{D}_A$. If $r : \mathcal{D}_A \to \mathcal{D}_B$ and $r(0) = 0$, then there exists a Hilbert space H and a formal power series $W = \sum_{\alpha \in \langle x \rangle} W_\alpha \alpha$ with coefficients $W_\alpha : \mathbb{C}^e \to H \otimes \mathbb{C}^d$ such that*

$$L_B(r(x)) = W(x)^* L_{I_H \otimes A}(x) W(x) \tag{5.3}$$

holds in the ring of $e \times e$ matrices over formal power series in x, x^*. Moreover,

1. *with $G(x) = \Lambda_B(r(x))$, equations (4.2), and thus the conclusions of Proposition 4.4, hold;*
2. *for any nilpotent $X \in M(\mathbb{C})^g$, (5.3) holds at X;*
3. *W has positive formal radius of convergence and thus (5.3) holds for all $X \in \mathbb{D}_\tau$ for τ sufficiently small.*

Proof. Follows from Theorem 5.1 and Lemma 2.7. ■

6. Consequences of a One Term Positivstellensatz

In this section, we consider the consequences of a one term Positivstellensatz. In particular, a one term Positivstellensatz produces a convexotonic map. Accordingly, suppose $p = (p^1, \ldots, p^g)$ where each p^j is a free formal power series in $x = (x_1, \ldots, x_g)$ such that $p(0) = 0$ and $p'(0) = I$, and tuples $A, B \in M_d(\mathbb{C}^g)$ and a formal power series W in $M_d(\mathbb{C})$ (square matrices) are given such that

$$L_B(p(x)) = W(x)^* L_A(x) W(x) \tag{6.1}$$

in the sense that the relations of equation (4.2) hold with $G(x) = \Lambda_B(p(x))$. Thus the sizes of A and B are the same and both L_A and L_B are pencils in g variables. As we will see, under the assumption that W is square, equation (6.1) implies p is a convexotonic map and imposes rigid structure on the tuple A.

Proposition 4.4 produces $d \times d$ unitary matrices C and $\mathscr{W}$ such that, with $R = (C - I)A$,

$$W(x) = (I - \Lambda_R(x))^{-1}\mathscr{W}.$$

Corollary 4.5 implies $B = \mathscr{W}^* CA\mathscr{W}$, where $\mathscr{W} = W(0)$ is unitary (in this case). Since $\mathscr{W}$ is unitary, equation (4.9) of Corollary 4.5 gives $A_k(C - I)A_j$ is in the span of $A_1, \dots, A_g$ for all j, k, i.e., for each $1 \le j \le g$ there is a $g \times g$ matrix Ξ_j (described explicitly in terms of the coefficients of p) such that for all $1 \le k \le g$,

$$A_k(C - I)A_j = \sum_{s=1}^{g} (\Xi_j)_{k,s} A_s. \tag{6.2}$$

The structure inherent in equation (6.2) is analyzed in the next subsection.

6.1. **Lurking algebras.**

Lemma 6.1. *Suppose R and E are g-tuples of matrices of the same size d and let $\mathscr{B}$ denote the span of $\{E_1, \dots, E_g\}$. If the set $\{E_1, \dots, E_g\}$ is linearly independent and $E_j R^\alpha \in \mathscr{B}$ for each $1 \le j \le g$ and word α, then there exists a g-tuple Ξ of $g \times g$ matrices such that*

$$E_j R^\alpha = \sum_{k=1}^{\tilde{g}} \Xi^\alpha_{j,k} E_k$$

for each $1 \le j \le g$ and word α.

Proof. By assumption, for each word α and $1 \le t \le g$ the matrix $E_t R^\alpha$ has a unique representation of the form

$$E_t R^\alpha = \sum_{k=1}^{g} (\Xi_\alpha)_{t,k} E_k, \tag{6.3}$$

for some $g \times g$ matrix Ξ_α. For $1 \le s \le g$, let $\Xi_s = \Xi_{x_s}$. With this notation,

$$E_t R_s = \sum_{k=1}^{g} (\Xi_s)_{t,k} E_k.$$

Applying R_u on the right to equation (6.3) gives

$$\sum_{s=1}^{g} (\Xi_{\alpha x_u})_{t,s} E_s = E_t R^\alpha R_u = \sum_{k=1}^{g} (\Xi_\alpha)_{t,k} E_k R_u = \sum_{k=1}^{g} (\Xi_\alpha)_{t,k} \sum_{\ell=1}^{g} (\Xi_u)_{k,\ell} E_\ell = \sum_{\ell=1}^{g} (\Xi_\alpha \Xi_u)_{t,\ell} E_\ell.$$

By linear independence of $\{E_1, \dots, E_g\}$, it follows that $\Xi_\alpha \Xi_u = \Xi^{\alpha x_u}$ and the proof is complete. ■

An algebra $\mathscr{A}$ has **order of nilpotence** $N \in \mathbb{N}$ if the product of any N elements of $\mathscr{A}$ is 0 and N is the smallest natural number with this property. Proposition 6.2 below explains how convexotonic maps naturally arise from the algebra-module structure of equation (6.2).

Proposition 6.2. *Let $A = (A_1, \dots, A_g) \in M_d(\mathbb{C})^g$ be given and assume that $\{A_1, \dots, A_g\}$ is linearly independent. Suppose C is a $d \times d$ matrix such that, for each $1 \le j \le g$ there exists a matrix Ξ_j such that for each $1 \le k \le g$ equation (6.2) holds. Let $R = (C - I)A$ and let $\Xi = (\Xi_1, \dots, \Xi_g)$. Then:*

(i) *the span $\mathcal{R}$ of $\{R_1, \dots, R_g\}$ is an algebra;*
(ii) *the span $\mathcal{M}$ of $\{A_1, \dots, A_g\}$ is a right $\mathcal{R}$-module and*

$$A_k R^\alpha = \sum_t (\Xi^\alpha)_{k,t} A_t; \tag{6.4}$$

(iii) *the set* $\{\Xi_1, \ldots, \Xi_g\}$ *spans an algebra* $\mathcal{X}$ *for which the matrices* Ξ *are the structure matrices; i.e.,*

$$\Xi_k \Xi_j = \sum_s^g (\Xi_j)_{k,s} \Xi_s;$$

(iv) *for all* $1 \le k \le g$ *and words* α,

$$\Xi_k \Xi^\alpha = \sum_s (\Xi^\alpha)_{k,s} \Xi_s;$$

(v) *the rational functions* p *and* q *whose entries are*

$$p^t(x) = \sum_j x_j \big(I - \Lambda_\Xi(x)\big)^{-1}_{j.t} \qquad \textit{and} \qquad q^t(x) = \sum_j x_j \big(I + \Lambda_\Xi(x)\big)^{-1}_{j.t};$$

expressed in row form,

$$p(x) = x(I - \Lambda_\Xi(x))^{-1} \qquad \textit{and} \qquad q = x(I + \Lambda_\Xi(x))^{-1},$$

are inverses of one another;

(vi) *if* $R^\alpha = 0$*, then* $\Xi^\alpha = 0$ *and conversely, if* $\Xi^\alpha = 0$*, then* $R_j R^\alpha = 0$ *for all* $1 \le j \le g$;

(vii) *if, moreover, the order of nilpotence of* $\mathcal{R}$ *is* ν*, then* $\nu \le \dim(\mathcal{R}) + 1$*, the order of nilpotence* μ *of* $\mathcal{X}$ *is at most* ν*, and, in any case,* $\nu \le g$;

(viii) $\mathcal{X}$ *is nilpotent if and only if* p *and* q *are polynomials and in this case the order of nilpotence of* $\mathcal{X}$ *is the same as the degree of* p *and* q*; and hence the degree of* p *is at most* g*. Finally, there are examples where the degree of* p *and* q *are* g*.*

Proof. Let $\Xi_j = \Xi_{x_j}$ and $\Xi = (\Xi_1, \ldots, \Xi_g)$. Letting $R = (C - I)A$, equation (6.2) and choosing $E = A$ in Lemma 6.1, it follows that $\Xi_\alpha = \Xi^\alpha$. In particular, for $1 \le j \le g$ and words α,

$$A_k R^\alpha = \sum_{j=1}^g \Xi^\alpha_{k,j} A_j. \tag{6.5}$$

Multiplying (6.5) on the left by $(C - I)$ gives,

$$R_k R^\alpha = \sum_{j=1}^g \Xi^\alpha_{k,j} R_j. \tag{6.6}$$

Thus the set $\{R_1, \ldots, R_g\}$ spans an algebra $\mathcal{R}$ and equation (6.5) says the span $\mathcal{M}$ of the set $\{A_1, \ldots, A_g\}$ is a module over $\mathcal{R}$. At this point items (i) and (ii) have been established.

To establish item (iii), compute the product $A_k R_j R_\ell$ in two different ways. On the one hand, using equation (6.6) with $\alpha = x_\ell$,

$$A_k R_j R_\ell = \sum_{t=1}^g (\Xi_\ell)_{j,t} A_k R_t = \sum_{t=1}^g (\Xi_\ell)_{j,t} \sum_{s=1}^g (\Xi_t)_{k,s} A_s = \sum_{s=1}^g \sum_{t=1}^g (\Xi_\ell)_{j,t} (\Xi_t)_{k,s} A_s.$$

On the other hand, from what has already been proved,

$$A_k R_j R_\ell = \sum_s (\Xi_j \Xi_\ell)_{k,s} A_s. \tag{6.7}$$

For a fixed k, the independence of the set $\{A_1, \ldots, A_g\}$ implies

$$\sum_t (\Xi_\ell)_{j,t} (\Xi_t)_{k,s} = (\Xi_j \Xi_\ell)_{k,s}$$

for each s and thus,

$$\sum_t (\Xi_\ell)_{j,t} \Xi_t = \Xi_j \Xi_\ell.$$

A simple induction on the length of α establishes (iv).

Next we prove (v). Using item (iv),

$$q_t(x) = \sum_j x_j \big(I + \Lambda_\Xi(x)\big)^{-1}_{j.t} = \sum_{\alpha\in\langle x\rangle} (-1)^{|\alpha|}\, \Xi^\alpha x_t \alpha$$

Hence,

$$\begin{aligned} I - \Lambda_\Xi(q(x)) &= I - \sum_{t=1}^g \Xi_t q_t = I - \sum_t \Xi_t \sum_{j,\alpha} (-1)^{|\alpha|} (\Xi^\alpha)_{j,t} x_j \alpha \\ &= I - \sum_{j,\alpha} \big(\sum_t (-1)^{|\alpha|}(\Xi^\alpha)_{j,t}\Xi_t\big) x_j\alpha = \sum_\beta (-1)^{|\beta|}\Xi^\beta \beta = (I + \Lambda_\Xi(x))^{-1}. \end{aligned}$$

Thus,

$$\begin{aligned} p\circ q(x) &= q(x)\,[I - \Lambda_\Xi(q(x))]^{-1} = q(x)\big((I+\Lambda_\Xi(x))^{-1}\big)^{-1} \\ &= x\ (I+\Lambda_\Xi(x))^{-1}\,(I+\Lambda_\Xi(x)) = x \end{aligned}$$

and it follows that q is a right inverse for p. By symmetry, it is also a left inverse establishing item (v).

To prove (vi) observe, if $R^\alpha = 0$, then $A_k R^\alpha = 0$ and hence, by equation (6.4) and the assumed linear independence of the set $\{A_1, \dots, A_g\}$, it follows that $\Xi^\alpha = 0$. Conversely, if $\Xi^\alpha = 0$, then $A_k R^\alpha = 0$ for each k and multiplying on the left by $C - I$ shows $R_k R^\alpha = 0$. Note in passing that $\mathcal{R}$ is nilpotent if and only if $\mathcal{X}$ is nilpotent and in this case, letting μ and ν denote the orders of nilpotence of $\mathcal{X}$ and $\mathcal{R}$ respectively,

$$\mu \le \nu \le \mu + 1.$$

To prove item (vii), let $\mathcal{R}^k$ denote the algebra generated by $\{R^\alpha : |\alpha| = k\}$. Here, as usual, $|\alpha|$ denotes the length of the word α. Observe that $\mathcal{R} = \mathcal{R}^1$ and $\mathcal{R}^j \supseteq \mathcal{R}^{j+1}$. Assuming the algebra $\mathcal{R}$ is nilpotent of order ν, for each j either $\mathcal{R}^j = (0)$ or $\mathcal{R}^j \supsetneq \mathcal{R}^{j+1}$. It follows that the dimension of $\mathcal{R}^j$ is at most $\dim(\mathcal{R}) + 1 - j$ and hence ν, the order of nilpotence of $\mathcal{R}$, is at most $\dim(\mathcal{R}) + 1$. That the order of nilpotence of Ξ is at most ν, the order of nilpotence of $\mathcal{R}$, follows immediately from equation item (ii) and the assumption that $\{A_1, \dots, A_g\}$ is linearly independent. Finally, if $\dim(\mathcal{R}) < g$, then the order of nilpotence of Ξ is at most $\dim(\mathcal{R}) + 1 \le g$. If $\dim(\mathcal{R}) = g$, then $\{R_1, \dots, R_g\}$ is linearly independent. Suppose $|\alpha| = \nu - 1 \le g$. Multiplying equation (6.4) on the left by $(C - I)$ gives,

$$0 = R_k R^\alpha = \sum_{s=1}^g (\Xi^\alpha)_{k,s} R_s.$$

It follows that $\Xi^\alpha = 0$.

The first part of item (viii) follows immediately from item (v). Example 6.3 immediately below exhibits a p (and q) of degree g. ■

Example 6.3. Given g, let S denote a (square) matrix nilpotent of order $g+1$ and let $R_j = S^j$. Let R denote the tuple $(R_1, \dots, R_g)$. On $\mathbb{C}^g$ with its standard orthonormal basis $\{e_1, \dots, e_g\}$, define $\Xi e_j = e_{j-1}$ for $j \ge 2$ and $\Xi e_1 = 0$. Thus Ξ is the truncated backward shift. The structure matrices Ξ_j for the algebra generated by R are then $\Xi_j = \Xi^j$. In this case the resulting polynomial p is

$$p = x(I - \Lambda_\Xi(x))^{-1} = (p^1, \dots, p^g),$$

where

$$p^m = \sum \prod_{\sum j_k = m} x_{j_k}.$$

In particular, p^m has degree m and hence p has degree g. ◊

6.2. **The convexotonic map p and its inverse q.** The following Theorem is the main result of this section. Its proof relies on Proposition 6.2.

Theorem 6.4. *Suppose A, B are g-tuples of matrices of the same size d and $p = (p^1, \dots, p^g)$ where each p^j is a formal power series, $p(0) = 0$ and $p'(0) = I$. If there exists a $d \times d$ matrix-valued free formal power series W such that equation (6.1) and the identities of equation (4.2) with $G(x) = \Lambda_B(p(x))$ hold, then*

(i) *there exists a uniquely determined $d\times d$ unitary matrix $\mathscr{W}$ and a $d\times d$ matrix C that is unitary on $\mathrm{rg}(A)$ such that*

$$W(x) = \big(I - \Lambda_{(C-I)A}(x)\big)^{-1}\mathscr{W}, \tag{6.8}$$

and $B = \mathscr{W}^* CA\mathscr{W}$ (meaning $B_j = \mathscr{W}^* CA_j\mathscr{W}$ for $j = 1, 2, \dots, g$);

(ii) *there are $g \times g$ matrices Ξ_j as in (6.2) satisfying (6.4). In particular, the set of matrices $\{R_j = (C - I)A_j : j = 1, \dots, g\}$ spans an algebra $\mathcal{R}$;*

(iii) *the rational function*

$$p(x) = x(I - \Lambda_\Xi(x))^{-1} \tag{6.9}$$

is bianalytic, both as a mapping from $\mathcal{D}_A$ to $\mathcal{D}_B$ and as a mapping on all of $M(\mathbb{C})^g$ (excluding poles);

(iv) *the inverse of p is q, given by*

$$q(x) = x(I + \Lambda_\Xi(x))^{-1}; \tag{6.10}$$

(v) *p is a polynomial if and only if the algebra Ξ spanned by $\{\Xi_j : 1 \le j \le g\}$ is nilpotent and in this case q is also a polynomial and the degrees of p and q and the order of nilpotence of Ξ are all the same and at most g, and there are examples where this degree is g.*

Conversely, if $A = (A_1, \dots, A_g)$ is a linearly independent tuple of $d \times d$ matrices and C is a $d \times d$ matrix that is unitary on the span of the ranges of the A_j such that for each j, k the matrix $A_k(C - I)A_j$ is in the $\mathcal{M}$, the span of $\{A_1, \dots, A_g\}$, then $\mathcal{R}$ equal the span of $R_j = (C - I)A_j$ is an algebra, and $\mathcal{M}$ is a right module over the algebra $\mathcal{R}$. Let $\Xi = (\Xi_1, \dots, \Xi_g)$ denote the structure matrices for the module $\mathcal{M}$. Given another unitary $\mathscr{W}$, with $B = CA$ and W given by equation (6.8), the rational function $p(x) = x(I - \Lambda_\Xi(x))^{-1}$ satisfies equation (6.1) and hence items (i) through (v).

Proof. To prove the first part of the theorem, apply Proposition 4.4 to equation (6.1) to obtain a matrix C, isometric on $\mathrm{rg}(A)$, such that, with $R = (C - I)A$ and $\mathscr{W} = W_\varnothing$, W has the form given in equation (6.8). By Corollary 4.5, $B = \mathscr{W}^* CA\mathscr{W}$ and equation (6.2) holds. In fact from equation (4.9) of Corollary 4.5,

$$A_k R^\omega = \sum_{j=1}^{\tilde{g}} p^j_{x_k\omega}\, A_j. \tag{6.11}$$

Thus, by Proposition 6.2, items (ii) and (iv) follow. Item (v) is Proposition 6.2 (viii).

To see item (iii) holds, let $(\Xi_\omega)_{k,j} = p^j_{x_k\omega}$ and let $\Xi_j = \Xi_{x_j}$. Applying Lemma 6.1 to equation (6.11) gives $\Xi_\omega = \Xi^\omega$ and hence

$$\begin{aligned} x(I - \Lambda_\Xi(x))^{-1} &= \big(\textstyle\sum_{j,\alpha}(\Xi^\alpha)_{1,j}x_j\alpha \quad \dots \quad \sum_{j,\alpha}(\Xi^\alpha)_{g,j}x_j\alpha\big) \\ &= \Big(\textstyle\sum_{j,\alpha} p^j_{x_1\alpha}x_j\alpha \quad \dots \quad \sum_{j,\alpha} p^j_{x_g\alpha}x_j\alpha\Big) = p(x). \end{aligned}$$

The converse statements of the theorem are established by verifying that, with the choices of $A, B, \mathscr{W}, C$ and W and finally p, equation (6.1) holds. ∎

6.3. **Proper analytic mappings.** In this section we apply Theorem 6.4 to the case of a mapping $p = (p^1, \dots, p^{\tilde{g}})$ in g ($g < \tilde{g}$) variables x where each p^j is a formal power series. Accordingly suppose $A \in M_d(\mathbb{C})^g$ and $B \in M_d(\mathbb{C})^{\tilde{g}}$.

Proposition 6.5. *Suppose $p(0) = 0$ and $p'(0) = \begin{pmatrix} I & 0 \end{pmatrix}$. Suppose further the set $\{B_1, \dots, B_{\tilde{g}}\}$ is linearly independent. If there exists a matrix-valued formal power series W with coefficients from $M_d(\mathbb{C})$ such that*

$$L_B(p(x)) = W(x)^* L_A(x) W(x),$$

and the identities of equation (4.2) with $G(x) = \Lambda_B(p(x))$ hold, then there exists a $\hat{g}$ and a $\hat{g}$-tuple Ξ of $\hat{g} \times \hat{g}$ matrices such that the span $\mathcal{X}$ of $\{\Xi_1, \dots, \Xi_{\hat{g}}\}$ is an algebra and $p(x) = P(x, 0)$, where $P(x, y)$ is the rational function in the variables $(x_1, \dots, x_g, y_1, \dots, y_\tau)$ (where $\tau = \hat{g} - g$) given by

$$P(x, y) = \begin{pmatrix} x & y \end{pmatrix} (I - \Lambda_\Xi(x, y))^{-1}.$$

Moreover, P is bianalytic between free spectrahedra determined by A, B, p and W.

Proof. From $L_B(p(x)) = W(x)^* L_A(x) W(x)$ and Proposition 4.4, it follows that there exists $d \times d$ unitary matrices C and $\mathscr{W}$ such that, with $R = (C - I)A$, the formal power series W is the rational function $W(x) = (I - \Lambda_R(x))^{-1} \mathscr{W}$. Further, by Corollary 4.5, for $1 \le j \le g$,

$$B_j = \mathscr{W}^* C A_j \mathscr{W},$$

and $\mathscr{W}^* C A_j R^\omega \mathscr{W}$ is a linear combination of $\{B_1, \dots, B_{\tilde{g}}\}$. Thus,

$$A R^\omega \in \mathfrak{B},$$

where $\mathfrak{B}$ denotes the span of $\{C^* \mathscr{W} B_1 \mathscr{W}^*, \dots, C^* \mathscr{W} B_{\tilde{g}} \mathscr{W}^*\}$. (In particular, $C^* \mathscr{W} B_j \mathscr{W}^* = A_j$ for $1 \le j \le g$.) Let $\mathcal{E} = \{E_1, \dots, E_\rho\}$ be any linearly independent set such that $E_j = C^* \mathscr{W} B_j \mathscr{W}^*$ for $1 \le j \le \tilde{g}$ and

$$E_k (C - I) E_j \in \mathfrak{E},$$

where $\mathfrak{E}$ is the span of the set $\mathcal{E}$. Let $F = \mathscr{W}^* C E \mathscr{W}$ and choose $\Xi = (C - I)E$ and $Y(x, y) = (I - \Lambda_\Xi(x, y))^{-1} \mathscr{W}$. By the converse portion of Proposition 4.4,

$$Y(x, y)^* L_E(x, y) Y(x, y) = L_F(P(x, y)),$$

for some power series P. By Theorem 6.4, P is a rational function with the form as claimed. Observe that $Y(x, 0) = W(x)$, $L_E(x, 0) = L_A(x)$. Hence

$$L_F(P(x, 0)) = Y(x, 0)^* L_E(x, 0) Y(x, 0) = W(x)^* L_A(x) W(x) = L_B(p(x)).$$

Since $F_j = B_j$ for $1 \le j \le \tilde{g}$, the linear independence assumption implies $P(x, 0) = p(x)$. ∎

7. Bianalytic Maps

Suppose $A \in M_d(\mathbb{C})^g$ and $B \in M_e(\mathbb{C})^g$, the domains $\mathcal{D}_A$ and $\mathcal{D}_B$ are bounded and $p : \mathcal{D}_A \to \mathcal{D}_B$ is an analytic mapping such that $p(0) = 0$, $p'(0) = I$ and p maps the boundary of $\mathcal{D}_A$ into the boundary of $\mathcal{D}_B$. Equivalently, p is proper and thus [HKM11] bianalytic. In this section we will see, up to mild assumptions on A and B, it turns out that $d = e$ and the hypothesis of Theorem 6.4 are met and hence p is convexotonic.

7.1. **An irreducibility condition.** In this subsection we introduce irreducibility conditions on tuples A and B that ultimately allow the application of Theorem 6.4.

7.1.1. *Singular vectors.* The following is an elementary linear algebra fact.

Lemma 7.1. *Suppose T is an $M \times N$ matrix of norm one and let $\mathscr{E}$ and $\mathscr{E}_*$ denote the eigenspaces corresponding to the largest eigenvalues of T^*T and TT^* respectively. Thus, for instance,*

$$\mathscr{E} = \{x \in \mathbb{C}^N : T^*Tx = x\}.$$

(i) The dimensions of $\mathscr{E}$ and $\mathscr{E}_*$ are the same.
(ii) The mapping $x \mapsto Tx$ is a unitary map from $\mathscr{E}$ to $\mathscr{E}_*$ with inverse $y \mapsto T^*y$.
(iii) Letting

$$J = \begin{pmatrix} I & T \\ T^* & I \end{pmatrix},$$

the kernel of J is the set $\{-Tu \oplus u : u \in \mathscr{E}\}$.

Proof. Simply note, if $T^*Tx = x$, then $TT^*(Tx) = Tx$ and conversely if $TT^*y = y$, then $T^*T(T^*y) = T^*y$ to prove the first two items. To prove the last item, observe that vectors of the form $-Tu \oplus u$ are in the kernel of J. On the other hand, if $v \oplus w$ is in the kernel of J, then $v + Tw = 0$ and $T^*v + w = 0$. From the first equation $T^*v + T^*Tw = 0$ and from the second $T^*Tw = w$. Thus $w \in \mathscr{E}$ and $v \oplus w = -Tw \oplus w$. ∎

Lemma 7.2. *Suppose*

(i) *$A \in M_d(\mathbb{C})^g$, $B \in M_e(\mathbb{C})^{\tilde{g}}$;*
(ii) *H is a Hilbert space;*
(iii) *C is a bounded linear operator on $H \otimes \mathbb{C}^d$;*
(iv) *$\mathscr{W} : \mathbb{C}^e \to H \otimes \mathbb{C}^d$ is an isometry;*
(v) *$p = (p^1, \dots, p^{\tilde{g}})$ is a free analytic mapping $\mathcal{D}_A \to \mathcal{D}_B$ with linear term ℓ such that $p(0) = 0$ and*

$$L_B(p(x)) = W(x)^* L_{I_H \otimes A}(x) W(x), \tag{7.1}$$

where

$$W(x) = (I - \Lambda_R(x))^{-1} \mathscr{W} \tag{7.2}$$

and $R = (C - I)A$; and

(vi) *$\alpha \in (\mathbb{C}^{n \times n})^g$ and the largest eigenvalue of $\Lambda_A(\alpha)\Lambda_A(\alpha)^*$ and $\Lambda_A(\alpha)^*\Lambda_A(\alpha)$ is 1; the eigenspaces of $\Lambda_A(\alpha)\Lambda_A(\alpha)^*$ and $\Lambda_A(\alpha)^*\Lambda_A(\alpha)$ corresponding to the eigenvalue 1 are one-dimensional, spanned by the unit vectors u_1, u_2 in $\mathbb{C}^{nd}$ respectively; and*
(vii) *$v_1 \in \mathbb{C}^{ne}$ and $\Lambda_B(\ell(\alpha))\Lambda_B(\ell(\alpha))^* v_1 = v_1$.*

Let $v_2 = -\Lambda_B(\ell(\alpha))^* v_1$ and write, for $j = 1, 2$ and $\{e_1, \dots, e_n\}$ a basis for $\mathbb{C}^n$,

$$u_j = \sum_{k=1}^n u_{j,k} \otimes e_k \in \mathbb{C}^d \otimes \mathbb{C}^n = \mathbb{C}^{nd}$$

and similarly for v_j. Then there is a vector $\lambda \in H$ (depending on α, u_j and v_j) such that,

(a) *$\Lambda_A(\alpha)u_2 = -u_1$ and $\Lambda_A(\alpha)^*u_1 = -u_2$;*
(b) *$\Lambda_B(\ell(\alpha))v_2 = -v_1$ and $\Lambda_B(\ell(\alpha))^*v_1 = -v_2$;*
(c) *$\mathscr{W} v_{2,k} = \lambda \otimes u_{2,k}$ for each $1 \le k \le n$;*
(d) *$\mathscr{W} v_{1,k} = C(\lambda \otimes u_{1,k})$ for each $1 \le k \le n$; and*
(e) *if $A = B$ and $\ell(x) = x$, then, without loss of generality, $v_1 = u_1$ and $v_2 = u_2$.*

Note that if $X \in M(\mathbb{C})^g$ is of sufficiently small norm or nilpotent, then we may substitute X for x in equation (7.1) by Theorem 5.1. Moreover, in this case we can evaluate $W(x)$ from (7.2) at X as

$$\begin{aligned} W(X) &= \big(I \otimes I_n - [(C-I)\otimes I_n]\Lambda_{I_H \otimes A}(X)\big)^{-1}(\mathscr{W} \otimes I_n) \\ &= \big(I \otimes I_n - [(C-I)\otimes I_n][I_H \otimes \Lambda_A(X)]\big)^{-1}(\mathscr{W} \otimes I_n) \end{aligned}$$

rather than appealing to convergence of a series expansion for W.

Proof of Lemma 7.2. Let

$$S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and let $X = S \otimes \alpha$. Thus X has size $2n$. The hypotheses imply that $\Lambda_A(\alpha)$ has norm exactly one. Conjugating $L_A(X)$ by the permutation matrix that implements the unitary equivalence of $A \otimes S \otimes \alpha$ with $S \otimes A \otimes \alpha$ shows, up to unitary equivalence,

$$L_A(X) = \begin{pmatrix} I & \Lambda_A(\alpha) \\ \Lambda_A(\alpha)^* & I \end{pmatrix}.$$

Thus the assumptions on $\Lambda_A(\alpha)$ imply that $L_A(X)$ is positive semidefinite with a nontrivial kernel spanned by

$$u = \sum_{j=1}^{2} e_j \otimes u_j = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{C}^2 \otimes (\mathbb{C}^n \otimes \mathbb{C}^d).$$

In particular, if z is in the kernel of $I_H \otimes L_A(X)$, then there is a vector $\lambda \in H$ such that $z = \lambda \otimes u$. Also note, $\|u_1\| = \|u_2\|$ and we assume both are unit vectors.

Since, by assumption $p(x) = \ell(x) + h(x)$, where ℓ is linear and h consists of higher order terms and X is (jointly) nilpotent (of order 2),

$$p(X) = \begin{pmatrix} 0 & \ell(x) \\ 0 & 0 \end{pmatrix}.$$

Thus,

$$L_B(p(X)) = \begin{pmatrix} I & \Lambda_B(\ell(\alpha)) \\ \Lambda_B(\ell(\alpha))^* & I \end{pmatrix}.$$

Since $L_A(X)$ is positive semidefinite, equation (7.1) implies $L_B(p(X))$ is positive semidefinite. Moreover, the hypotheses imply that the vector $v = \sum_{j=1}^2 e_j \otimes v_j$ satisfies $L_B(p(X))v = 0$. Another application of equation (7.1) shows $W(X)v$ is in the kernel of $L_{I_H \otimes A}(X)$; i.e., $W(X)v = \lambda \otimes u$, for some vector $\lambda \in H$ with $\|\lambda\| = \|W(X)v\|$. Hence,

$$\lambda \otimes (\sum_{j=1}^{2} u_j \otimes e_j) = \lambda \otimes u = W(X)v = (I - \sum_{i=1}^{g} R_i \otimes X_i)^{-1}(\mathscr{W} \otimes I_n \otimes I_2)v.$$

Multiplying by $(I - \sum_{i=1}^g R_i \otimes X_i)$ on the left yields

$$\begin{aligned} \sum_{j=1}^{2}([\mathscr{W} \otimes I_n]v_j) \otimes e_j &= [I - [(C-I)\otimes I_{2n}](I \otimes \Lambda_{I_H \otimes A}(X))](\lambda \otimes u) \\ &= \big(\lambda \otimes u_1 - [(C-I)\otimes I_n](\lambda \otimes \Lambda_A(\alpha)u_2)\big) \otimes e_1 + \lambda \otimes u_2 \otimes e_2. \end{aligned} \tag{7.3}$$

It follows that $(\mathscr{W} \otimes I_n)v_2 = \lambda \otimes u_2$ and, since $\mathscr{W}$ is an isometry, $\|\lambda\| = \|v_2\|$. Further,

$$(\mathscr{W} \otimes I_n)v_1 = \lambda \otimes u_1 - [(C-I)\otimes I_n]\big(\lambda \otimes \Lambda_A(\alpha)u_2\big).$$

Using $\Lambda_A(\alpha)u_2 = -u_1$ gives $[\mathscr{W} \otimes I_n]v_1 = [C \otimes I_n](\lambda \otimes u_1)$.

To complete the proof observe that

$$(\mathscr{W} \otimes I_n)v_2 = \sum_{k=1}^{n} \mathscr{W} v_{2,k} \otimes e_k.$$

Thus, $\mathscr{W} v_{2,k} = \lambda \otimes u_{2,k}$. Similarly,

$$(C \otimes I_n)(\lambda \otimes u_1) = (C \otimes I_n)(\sum_{k=1}^{n} \lambda \otimes u_{1,k} \otimes e_k) = \sum_{k=1}^{n} C(\lambda \otimes u_{1,k}) \otimes e_k.$$

Thus, $\mathscr{W} v_{1,k} = C(\lambda \otimes u_{1,k})$ for each $1 \leq k \leq n$. ∎

7.1.2. *The Eig-generic condition.* We now introduce some refinements of the notion of sv-generic we saw in the introduction. A subset $\{b_1, \dots, b_{\ell+1}\}$ of a finite-dimensional vector space V is a **hyperbasis** if each subset of ℓ vectors is a basis. In particular, if $\{b_1, \dots, b_\ell\}$ is a basis for V and $b_{\ell+1} = \sum_{j=1}^{\tilde{g}} c_j b_j$ and $c_j \neq 0$ for each j, then $\{b_1, \dots, b_{\ell+1}\}$ is a hyperbasis and conversely each hyperbasis has this form. Given a tuple $A \in M_d(\mathbb{C})^g$, let

$$\ker(A) = \bigcap_{j=1}^{g} \ker(A_j).$$

Given a positive integer m, let $\{e_j : 1 \leq j \leq m\}$ denote the standard basis for $\mathbb{C}^m$.

Definition 7.3. The tuple $A \in M_d(\mathbb{C})^g$ is **weakly eig-generic** if there exists an $\ell \leq d+1$ and, for $1 \leq j \leq \ell$, positive integers n_j and tuples $\alpha^j \in (\mathbb{C}^{n_j \times n_j})^g$ such that

(a) for each $1 \leq j \leq \ell$, the eigenspace corresponding to the largest eigenvalue of $\Lambda_A(\alpha^j)^* \Lambda_A(\alpha^j)$ has dimension one and hence is spanned by a vector $u^j = \sum_{a=1}^{n_j} u_a^j \otimes e_a$; and
(b) the set $\mathscr{U} = \{u_a^j : 1 \leq j \leq \ell,\ 1 \leq a \leq n_j\}$ contains a hyperbasis for $\ker(A)^\perp = \operatorname{rg}(A^*)$.

The tuple is **eig-generic** if it is weakly eig-generic and $\ker(A) = (0)$.

Finally, a tuple A is **$*$-generic** (resp. **weakly $*$-generic**) if there exists an $\ell \leq d$ and tuples β^j such that the kernels of $I - \Lambda_A(\beta^j)\Lambda_A(\beta^j)^*$ have dimension one and are spanned by vectors $\mu^j = \sum \mu_a^j \otimes e_a$ for which the set $\{\mu_a^j : j, a\}$ spans $\mathbb{C}^d$ (resp. $\operatorname{rg}(A) = \ker(A^*)^\perp$).

Remark 7.4. It is illustrative to consider two special cases of the weak eig-generic condition. First suppose $n_j = 1$ for all $1 \leq j \leq \ell$. Hence $\ell - 1$ is the dimension of $\ker(A)^\perp$, the kernel of $I - \Lambda_A(\alpha^j)^* \Lambda_A(\alpha)$ is spanned by a a single (non-zero) vector $u^j \in \mathbb{C}^d$ and the set $\{u^1, \dots, u^\ell\}$ is a hyperbasis for $\ker(A)^\perp$. If we also assume $\ker(A)^\perp = (0)$ and there exists β^j for $j = 1, \dots, n$ such that $I - \Lambda_A(\beta^j)\Lambda_A(\beta^j)^*$ is positive definite with one-dimensional kernel spanned by v^j and moreover $\{v^1, \dots, v^d\}$ is a basis for $\mathbb{C}^d$, then A is sv-generic as defined in the introduction.

For the second case, suppose, for simplicity, that $\ker(A) = (0)$. If there exists an $\alpha^1 \in (\mathbb{C}^{n \times n})^g$ such that $I - \Lambda_A(\alpha^1)^* \Lambda_A(\alpha^1)$ is positive semidefinite with a one-dimensional kernel spanned by

$$u^1 = \sum_{k=1}^{n} u_k^1 \otimes e_k \in \mathbb{C}^d \otimes \mathbb{C}^n$$

and if the set $\{u_k^1 : 1 \leq k \leq n\}$ spans $\mathbb{C}^d$, then A is eig-generic. To prove this statement, suppose, without loss of generality, $\{u_k^1 : 1 \leq k \leq g\}$ is a basis for $\mathbb{C}^d$. Now take a unitary matrix T such that $T_{k1} \neq 0$ for each $1 \leq k \leq d$ and $T_{k1} = 0$ for each $d+1 \leq k \leq n$. Let $\alpha^2 = T\alpha^1 T^*$. It follows that $I - \Lambda_A(\alpha^2)^* \Lambda_A(\alpha^2)$ is positive semidefinite with a one-dimensional kernel spanned by the vector $u^2 = (I_d \otimes T)u^1$ and further

$$u^2 = \sum_{k=1}^{n} u_k^1 \otimes T e_k = \sum_{j=1}^{n} \sum_{k=1}^{n} u_k^1 \otimes T_{k,j} e_j = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} T_{k,j} u_k^1 \right) \otimes e_j.$$

Thus, in view of the assumptions on T,

$$u_1^2 = \sum_{k=1}^{n} T_{k,1} u_k^1 = \sum_{k=1}^{d} T_{k,1} u_k^1.$$

Since $T_{k,1} \neq 0$ for $1 \leq k \leq d$, the set $\{u_1^1, \dots, u_g^1, u_1^2\}$ is a hyperbasis for $\mathbb{C}^d$ and the tuple A is eig-generic. $\Diamond$

Remark 7.5. The one-dimensional kernel assumption is key for the eig-generic property, and has been successfully analyzed in the two papers [KŠV, KV]. Namely, if the tuple $A \in M_d(\mathbb{C})^g$ is minimal w.r.t. the size needed to describe the free spectrahedron $\mathcal{D}_A$, then $\dim \ker L_A(X) = 1$ for all X in an open and dense subset of the boundary $\partial\mathcal{D}_A(n)$ provided n is large enough. $\Diamond$

7.2. **The structure of bianalytic maps.** In this section our main results on bianalytic maps between free spectrahedra appear as Theorem 7.8 and Corollary 7.9. We begin by collecting consequences of the eig-generic assumptions.

Lemma 7.6. *Suppose*

(i) $A \in M_d(\mathbb{C})^g$, $B \in M_e(\mathbb{C})^h$;
(ii) H *is a Hilbert space,* C *is an isometry on* $H \otimes \mathbb{C}^d$ *and*

$$W(x) = (I - \Lambda_R(x))^{-1} \mathscr{W}, \tag{7.4}$$

where $R = (C - I)[I_H \otimes A]$ *and* $\mathscr{W} : \mathbb{C}^e \to H \otimes \mathbb{C}^d$ *is an isometry;*
(iii) $p = (p^1, \dots, p^{\tilde{g}})$ *is a free analytic mapping* $\mathcal{D}_A \to \mathcal{D}_B$ *such that* $p(0) = 0$, *and*

$$L_B(p(x)) = W(x)^* L_{I_H \otimes A}(x) W(x),$$

in the sense that

$$L_B(p(X)) = W(X)^* L_{I_H \otimes A}(X) W(X)$$

for each nilpotent $X \in M(\mathbb{C})^g$;
(iv) p *maps the boundary of* $\mathcal{D}_A$ *into the boundary of* $\mathcal{D}_B$; *and*
(v) *for* $1 \leq j \leq \ell$ *there exists tuples* α^j *in* $(\mathbb{C}^{n_j \times n_j})^g$ *such that* $I - \Lambda(\alpha^j)^* \Lambda(\alpha^j)$ *is positive definite with a one-dimensional kernel spanned by*

$$u_2^j = \sum_{k=1}^{n_j} u_{2,k}^j \otimes e_k;$$

If

(a) A *is* $*$*-generic* $\big($*resp. weakly* $*$*-generic*$\big)$, *then* $d \leq e$ $\big($*resp.* $\dim(\mathrm{rg}(A^*)) \leq \dim(\mathrm{rg}(B^*))\big)$;
(b) $e = d$ $\big($*resp.* $\dim(\mathrm{rg}(A^*)) = \dim(\mathrm{rg}(B^*))\big)$ *and the tuples* α^j *and vectors* u_2^j *validate the eig-generic* $\big($*resp. weak eig-generic*$\big)$ *assumption for* A, *then there exists a vector* $\lambda \in H$ *and vectors*

$$v_2^j = \sum_{k=1}^{n_j} v_{2,k}^j \otimes e_j$$

in the kernel of $I - \Lambda_B(p(\alpha^j))^* \Lambda_B(p(\alpha^j))$ *such that, for all* $1 \leq j \leq \ell + 1$ *and* $1 \leq k \leq n_j$,

$$\mathscr{W} v_{2,k}^j = \lambda \otimes u_{2,k}^j;$$

(c) $e = d$ *and* A *is eig-generic* $\big($*resp.* $\dim(\mathrm{rg}(A^*)) = \dim(\mathrm{rg}(B^*)$ *and* A *is weakly eig-generic*$\big)$, *then there exists a unit vector* $\lambda \in H$ *and a* $d \times d$ *unitary* M $\big($*resp. a unitary map* M *from* $\mathrm{rg}(B^*)$ *to* $\mathrm{rg}(A^*)$$)$ *such that* $\mathscr{W} = \lambda \otimes M$ $\big($*resp.* $\mathscr{W} v = \lambda \otimes M v$ *for* $v \in \mathrm{rg}(B^*)\big)$; *and*

(d) *A is eig-generic and $*$-generic and $e=d$ (resp. A is weakly eig-generic and weakly $*$-generic, $\dim(\mathrm{rg}(A^*))=\dim(\mathrm{rg}(B^*))$ and $\dim(\mathrm{rg}(A))=\dim(\mathrm{rg}(B))$), then there is a vector $\lambda\in H$ and $d\times d$ unitary matrices M and Z such that $\mathscr{W}=\lambda\otimes M$ and $C(\lambda\otimes I_d)=\lambda\otimes Z$ (resp. a unitary map M and an isometry N from $\mathrm{rg}(B^*)$ to $\mathrm{rg}(A^*)$ and from $\mathrm{rg}(B^*)\cap\mathrm{rg}(B)$ into $\mathrm{rg}(A)$ respectively such that $\mathscr{W}v=\lambda\otimes Mv$ for $v\in\mathrm{rg}(B^*)$ and $C(\lambda\otimes Nv)=\lambda\otimes Mv$ for $v\in\mathrm{rg}(B^*)\cap\mathrm{rg}(B)$).*

Proof. We begin with some calculations preliminary to proving all items claimed in the lemma. Let α^j as described in item v. Let $\mathcal{J}=\{(j,k):1\le j\le \ell,\ 1\le k\le n_j\}$ and, as in the proof of Lemma 7.2, let

$$S=\begin{pmatrix}0&1\\0&0\end{pmatrix}.$$

For $1\le j\le\ell$, let $X^j=S\otimes\alpha^j$. The hypotheses imply X^j is in the boundary of $\mathcal{D}_A$. By item iv, $p(X)$ is in the boundary of $\mathcal{D}_B$. Observe that $p(X)=\ell(X)$, where ℓ is the linear part of p. Thus (up to unitary equivalence)

$$L_B(p(X))=\begin{pmatrix}I&\Lambda_B(\ell(\alpha^j))\\\Lambda_B(\ell(\alpha^j))^*&I\end{pmatrix}$$

is positive semidefinite and there exist nonzero vectors $v_i^j\in\mathbb{C}^{ne}$ such that $v^j=e_1\otimes v_1^j+e_2\otimes v_2^j$ lies in the kernel of $L_B(p(X))$. Hence, by Lemma 7.1, $\Lambda_B(\ell(\alpha^j))\Lambda_B(\ell(\alpha^j))^*v_1^j=v_1^j$ and $v_2^j=-\Lambda_B(\alpha^j)^*v_1^j$. Consequently, Lemma 7.2 applies. In particular, writing

$$v_i^j=\sum_{k=1}^{n_j}v_{i,k}^j\otimes e_k,$$

there exist non-zero vectors $\lambda^j\in H$ such that

$$\mathscr{W}v_{2,k}^j=\lambda^j\otimes u_{2,k}^j \tag{7.5}$$

for $(j,k)\in\mathcal{J}$.

Fix $\mathcal{I}\subseteq\mathcal{J}$ and let $\mathscr{U}_{\mathcal{I}}=\{u_{2,k}^j:(j,k)\in\mathcal{I}\}\subseteq\mathbb{C}^d$. Suppose

$$0=\sum_{(j,k)\in\mathcal{I}}c_k^jv_{2,k}^j. \tag{7.6}$$

Applying $\mathscr{W}$ to equation (7.6)

$$0=\sum_{(j,k)\in\mathcal{I}}c_k^j\lambda_j\otimes u_{2,k}^j.$$

Given a vector $\gamma\in H$, applying the operator $\gamma^*\otimes I$ yields

$$0=\sum_{(j,k)\in\mathcal{I}}c_k^j\gamma^*\lambda_ju_{2,k}^j=\sum_{(j,k)\in\mathcal{I}}a_k^ju_{2,k}^j, \tag{7.7}$$

where $a_k^j=c_k^j\gamma^*\lambda_j$.

Suppose $\mathscr{U}_{\mathcal{I}}$ is linearly independent and $1\le p\le\ell$. Choosing $\gamma=\lambda_p$ in equation (7.7) gives $a_k^p=c_k^p\lambda_p^*\lambda_p=0$ for each k such $(p,k)\in\mathcal{I}$. Thus $c_k^p=0$ for each k such that $(p,k)\in\mathcal{I}$ and finally $c_k^p=0$ for each $(p,k)\in\mathcal{I}$. Hence $\{v_{2,k}^j:(j,k)\in\mathcal{I}\}\subseteq\mathbb{C}^e$ (resp. $\mathrm{rg}(B^*)$) is linearly independent. Thus the cardinality of $\mathcal{I}$ is at most e (resp. $\dim(\mathrm{rg}(B^*))$).

To prove item (a), now suppose that A is $*$-generic and, without loss of generality, $\{u_{2,k}^j:(j,k)\in\mathcal{J}\}$ spans $\mathbb{C}^d$. It then contains a linearly independent subset $\mathscr{U}_{\mathcal{I}}$ and hence $d\le e$ (resp. $\dim(\mathrm{rg}(A^*))=\dim(\mathrm{rg}(B^*))$). Moreover, if $e=d$ (resp. $\dim(\mathrm{rg}(A^*))=\dim(\mathrm{rg}(B^*))$), then $\{v_{2,k}^j:(j,k)\in\mathcal{J}\}$ spans $\mathbb{C}^d$.

To prove item (b), suppose the tuples α^j and vectors u^j above satisfy the (resp. weakly) eig-generic assumption. Thus, there is an $\mathcal{I} \subseteq \mathcal{J}$ such that $\mathscr{U}_{\mathcal{I}}$ is a hyperbasis for $\mathbb{C}^d$ (resp. $\operatorname{rg}(A^*)$).

Since a hyperbasis for $\mathbb{C}^d$ contains $d+1$ (resp. $\dim(\operatorname{rg}(A^*))+1$) elements, the cardinality of $\mathcal{I}$ is $d+1$ (resp. $\dim(\operatorname{rg}(A^*))+1$). Assuming $d=e$ (resp. $\dim(\operatorname{rg}(A^*)) = \dim(\operatorname{rg}(B^*))$), the set $\{v_k^j : (j,k) \in \mathcal{I}\}$ is then a set of $e+1$ (resp. $\dim(\operatorname{rg}(B^*))+1$) elements in $\mathbb{C}^e$ (resp. $\operatorname{rg}(B^*)$), so is linearly dependent. Hence, there exists, for $(j,k) \in \mathcal{I}$, scalars c_k^j not all of which are zero such that equation (7.6) holds. Suppose $(p,m) \in \mathcal{I}$ and $c_m^p \neq 0$. In equation (7.7) with $\gamma = \lambda_p$, the coefficient a_m^p is non-zero. Thus, the hyperbasis property of $\mathcal{U}_I$ implies $a_k^j \neq 0$ for each $(j,k) \in \mathcal{I}$ and hence $c_k^j \neq 0$ for each $(j,k) \in \mathcal{I}$. Another application of equation (7.7), but this time with a (nonzero) vector γ orthogonal to λ_p gives $a_m^p = 0$ and hence, again by the hyperbasis property, $a_k^j = 0$ for all $(j,k) \in \mathcal{I}$. Since $c_k^j \neq 0$, it follows that γ is orthogonal to each λ_j and consequently the λ_j are all colinear.

By scaling λ_j and v^j as needed, it may be assumed that there is a unit vector $\lambda \in H$ such that $\lambda_j = \lambda$ for all j. With this re-normalization,

$$\mathscr{W} v_k^j = \lambda \otimes u_{2,k}^j,$$

completing the proof of item (b).

To prove item (c), observe, since $\{v_{2,k}^j : (j,k) \in \mathcal{J}\}$ spans $\mathbb{C}^d$ (resp. $\operatorname{rg}(B^*)$), it follows that for each $v \in \mathbb{C}^d$ (resp. $v \in \operatorname{rg}(B^*)$) there is a $u \in \mathbb{C}^d$ (resp. $u \in \operatorname{rg}(A^*)$) such that

$$\mathscr{W} v = \lambda \otimes u. \tag{7.8}$$

Hence, by linearity and since $\mathscr{W}$ is an isometry, there is a unitary mapping

$$M : \operatorname{rg}(B^*) \to \operatorname{rg}(A^*)$$

such that $\mathscr{W}|_{\operatorname{rg}(B^*)} = \lambda \otimes M$. In particular, if $\operatorname{rg}(B^*) = \mathbb{C}^d$ as is the case if A is eig-generic, then $\mathscr{W} = \lambda \otimes M$.

Turning to the proof of item (d), for $1 \le j \le \ell$, assuming A is $*$-generic (resp. weakly $*$-generic), there exists tuples β^j of sizes n_j and vectors

$$\mathrm{u}_1^j = \sum_{k=1}^{n_j} \mathrm{u}_{1k}^j \otimes e_k$$

satisfying the $*$-generic (resp. weak $*$-generic) condition for A. That is, $I - \Lambda_A(\beta^j)\Lambda_A(\beta^j)^*$ is positive semidefinite with one-dimensional kernel spanned by u_1^j and the set of vectors $\{\mathrm{u}_{1k}^j : 1 \le j \le \ell,\ 1 \le k \le n_j\}$ spans $\mathbb{C}^d$ (resp. $\operatorname{rg}(A)$). By Lemma 7.2, there exist vectors

$$\mathrm{u}_2^j = \sum_{k=1}^{n_j} \mathrm{u}_{2k}^j \otimes e_k$$

such that $I - \Lambda_A(\beta^j)^*\Lambda_A(\beta^j)$ has a one-dimensional kernel spanned by u_2^j. On the other hand, the tuples

$$X^j = \begin{pmatrix} 0 & \beta^j \\ 0 & 0 \end{pmatrix}$$

lie in the boundary of $\mathcal{D}_A$. Hence, as before $p(X^j)$ lies in the boundary of $\mathcal{D}_B$. Thus

$$L_B(p(X^j)) = \begin{pmatrix} I & \Lambda_B(\ell(\beta^j)) \\ \Lambda_B(\ell(\beta^j))^* & I \end{pmatrix}$$

is positive semidefinite and has a kernel. Hence, there exists vectors $\mathrm{v}^j = \mathrm{v}_1^j \oplus \mathrm{v}_2^j$ such that $L_B(p(X^j))\mathrm{v}^j = 0$. By Lemma 7.1 these vectors are related by

$$\begin{aligned} \Lambda_A(\beta^j)^* \mathrm{u}_1^j &= -\mathrm{u}_2^j, & \Lambda_B(\ell(\beta^j))^* \mathrm{v}_1^j &= -\mathrm{v}_2^j, \\ \Lambda_A(\beta^j) \mathrm{u}_2^j &= -\mathrm{u}_1^j, & \Lambda_B(\ell(\beta^j)) \mathrm{v}_2^j &= -\mathrm{v}_1^j. \end{aligned} \tag{7.9}$$

Write

$$\mathrm{v}_i^j = \sum_{k=1}^{n_j} \mathrm{v}_{i,k}^j \otimes e_k.$$

By Lemma 7.2, for each j there exists a vector $\tau^j \in H$ such that

$$\mathscr{W} \mathrm{v}_{2,k}^j = \tau^j \otimes \mathrm{u}_{2,k}^j \qquad \text{and} \qquad \mathscr{W} \mathrm{v}_{1,k}^j = C\tau^j \otimes \mathrm{u}_{1,k}^j$$

for each $1 \le k \le n_j$. Now suppose A is either eig-generic and $e = d$ or A is weakly eig-generic and $\dim(\mathrm{rg}(B^*)) = \dim(\mathrm{rg}(A^*))$. In this case there is unit vector λ and unitary mapping M such that $\mathscr{W}\mathrm{v} = \lambda \otimes M\mathrm{v}$ on $\mathrm{rg}(B^*)$. Since $\mathrm{v}_{2k}^j \in \mathrm{rg}(B^*)$ and $\mathscr{W}$ and C are isometries, it follows that $\tau^j = \rho_j \lambda$ for some scalar $\rho_j \neq 0$. Hence,

$$\mathscr{W} \mathrm{v}_{1,k}^j = C\lambda \otimes \rho_j \mathrm{u}_{1,k}^j \tag{7.10}$$

for each $1 \le j \le \ell$ and $1 \le k \le n_j$. Since $\{\mathrm{u}_{1,k}^j\}$ spans $\mathrm{rg}(A)$, equation (7.10) implies there is an isometry $Z : \mathrm{rg}(A) \to \mathrm{rg}(B)$ such that

$$C(\lambda \otimes \mathrm{u}) = \mathscr{W} Z\mathrm{u} \tag{7.11}$$

for $\mathrm{u} \in \mathrm{rg}(A)$. In particular, $\dim(\mathrm{rg}(A)) \le \dim(\mathrm{rg}(B))$. Hence, if $\mathrm{rg}(A) = \mathbb{C}^d$ (as is the case if A is $*$-generic) and $e = d$, then $\mathrm{rg}(B) = \mathbb{C}^e$ and Z is onto. In the case that A is only weakly $*$-generic, we have assumed the dimensions of $\mathrm{rg}(A)$ and $\mathrm{rg}(B)$ are the same and so again Z is onto. So in either case, Z is unitary. In particular, given $\mathrm{v} \in \mathrm{rg}(B) \cap \mathrm{rg}(B^*)$, there is a $\mathrm{u} \in \mathrm{rg}(A)$ such that $Z\mathrm{u} = \mathrm{v}$ and

$$C(\lambda \otimes Z^*\mathrm{v}) = \mathscr{W}\mathrm{v}.$$

On the other hand, as $\mathrm{v} \in \mathrm{rg}(B^*)$, we have $\mathscr{W}\mathrm{v} = \lambda \otimes M\mathrm{v}$. Hence,

$$C(\lambda \otimes Z^*\mathrm{v}) = \lambda \otimes M\mathrm{v}. \tag{7.12}$$

Hence, letting N denote the restriction of Z^* to $\mathrm{rg}(B) \cap \mathrm{rg}(B^*)$ the desired conclusion follows.

We now take up the case A is eig-generic and $*$-generic and $e = d$. In this case, M and N are $d \times d$ unitary matrices and letting $\mathrm{u} = Z^*\mathrm{v}$ in equation (7.12) gives,

$$C(\lambda \otimes \mathrm{u}) = \lambda \otimes MN\mathrm{u}$$

and the proof of item (d) is complete. ∎

Remark 7.7. In the context of item (d), the dimension of $\mathrm{rg}(B) \cap \mathrm{rg}(B^*)$ is at most the dimension of $\mathrm{rg}(A) \cap \mathrm{rg}(A^*)$. In the case that these dimensions coincide, the identity $C^*(\lambda \otimes Mv) = \lambda \otimes Nv$ for $v \in \mathrm{rg}(B) \cap \mathrm{rg}(B^*)$ implies there is a unitary mapping Z of $\mathrm{rg}(A) \cap \mathrm{rg}(A^*)$ such that $C^*(\lambda \otimes z) = \lambda \otimes Zz$ for $z \in \mathrm{rg}(A) \cap \mathrm{rg}(A^*)$; i.e., $C^* = I \otimes Z$ on $\mathbb{C}\lambda \otimes \mathrm{rg}(A) \cap \mathrm{rg}(A^*)$. ◊

Theorem 7.8. *If*

(i) $A \in M_d(\mathbb{C})^g$ *and* $B \in M_e(\mathbb{C})^h$ *and* $h \ge g$;
(ii) $\mathcal{D}_A$ *is bounded;*
(iii) p *is a mapping from* $\mathcal{D}_A$ *into* $\mathcal{D}_B$ *that is analytic and bounded on a free pseudoconvex set* $\mathcal{G}_Q$ *containing* $\mathcal{D}_A$, *with* $p(0) = 0$;
(iv) p *maps the boundary of* $\mathcal{D}_A$ *into the boundary of* $\mathcal{D}_B$,

then

(a) *if A is $*$-generic, then $d \le e$;*
(b) *if A is eig-generic, A^* is $*$-generic, $d = e$ and $p(x) = (x, 0) + f(x)$, where f consists of terms of degree two and higher, then the conclusion of Proposition 6.5 holds; and*
(c) *if A is eig-generic and $*$-generic, $g = h$ and $d = e$ and $p(x) = x + f(x)$, where f consists of terms of degree two and higher, then the conclusion of Theorem 6.4 holds. In particular, p is convexotonic and is a bianalytic map between $\mathcal{D}_A$ and $\mathcal{D}_B$.*

Corollary 7.9. *Suppose*
(i) *$A \in M_d(\mathbb{C})^g$ and $B \in M_e(\mathbb{C})^g$;*
(ii) *$\mathcal{D}_A$ is bounded;*
(iii) *A is eig-generic and $*$-generic;*
(iv) *p is an analytic mapping $\mathcal{D}_A \to \mathcal{D}_B$ analytic and bounded on a free pseudoconvex set $\mathcal{G}_Q$ containing $\mathcal{D}_A$ with $p(x) = x + h(x)$ where h consists of terms of degree two and higher;*
(v) *r is an analytic mapping $\mathcal{D}_B \to \mathcal{D}_A$ analytic and bounded on a free pseudoconvex set $\mathcal{G}_Q$ containing $\mathcal{D}_B$, with $r(x) = x + f(x)$ where f consists of terms of degree two and higher;*
(vi) *p maps the boundary of $\mathcal{D}_A$ into the boundary of $\mathcal{D}_B$ and r maps the boundary of $\mathcal{D}_B$ to the boundary of $\mathcal{D}_A$.*

If either $d = e$ or B is $*$-generic, then p and r are bianalytic maps between $\mathcal{D}_A$ and $\mathcal{D}_B$ and for each the conclusions of Theorem 6.4 hold.

Proof. By Theorem 7.8 (a), the assumption A is $*$-generic implies $d \le e$. If B is assumed $*$-generic, then reversing the roles of A and B and using r in place of p, implies $e \le d$. Hence equality holds and item (c) of Theorem 7.8 applies to complete the proof. ∎

Proof of Theorem 7.8. Since p maps $\mathcal{D}_A$ into $\mathcal{D}_B$ and $p(0) = 0$, by the Analytic Positivstellensatz (here is where the hypothesis $\mathcal{D}_A$ is bounded is used), Theorem 5.1, there exists a Hilbert space H, a unitary mapping C on $H \otimes \mathbb{C}^d$ and an isometry $\mathscr{W} : \mathbb{C}^e \to H \otimes \mathbb{C}^d$ such that

$$L_B(p(x)) = W(x)^* \big(I_H \otimes L_A(x)\big) W(x),$$

where

$$W(x) = (I - \Lambda_R(x))^{-1} \mathscr{W}$$

and $R = (C - I)(I_H \otimes A)$. Moreover,

$$L_B(p(X)) = W(X)^* L_{I_H \otimes A}(X) W(X)$$

holds for nilpotent $X \in M(\mathbb{C})^g$ and the identities of equation (4.2) hold with $G(x) = \Lambda_B(p(x))$.

To prove items (b), (a), note if A is $*$-generic, then $d \le e$ by Lemma 7.6 (a). To prove item (c), observe if it is also assumed that $d = e$ and A is eig-generic, then Lemma 7.6 (d) implies there is a vector $\lambda \in H$ and $d \times d$ unitary matrices M and N such that $\mathscr{W} = \lambda \otimes M$ and $C(\lambda \otimes I) = \lambda \otimes N$. To complete the proof, let

$$\mathtt{W}(x) = [\lambda^* \otimes I] W(x).$$

Importantly, $\mathtt{W}$ is square ($d \times d$) matrix-valued analytic function. Further,

$$\begin{aligned} L_B(p(x)) &= W(x)^* (I_H \otimes L_A(x)) W(x) = \mathtt{W}(x)^* [\lambda^* \otimes I](I_H \otimes L_A(x))[\lambda \otimes I]\mathtt{W}(x) \\ &= \mathtt{W}(x)^* L_A(x) \mathtt{W}(x) \end{aligned}$$

In particular, for nilpotent X, $L_B(p(X)) = \mathtt{W}(X)^* L_A(X) \mathtt{W}(X)$. Thus, using Proposition 4.1 and $\tilde{g} = g$, Theorem 6.4 applies giving the conclusion of item (c). In the case $\tilde{g} > g$, using Proposition 4.1, Proposition 6.5 applies giving the conclusion of item (b). ∎

8. Affine Linear Change of Variables

This section describes the effects of change of variables by way of pre and post composition with an affine linear map on an analytic mapping between free spectrahedra.

Suppose $A = (A_1, \dots, A_g) \in M_d(\mathbb{C})^g$ determines a bounded LMI domain $\mathcal{D}_A$, $B = (B_1, \dots, B_{\tilde{g}}) \in M_e(\mathbb{C})^{\tilde{g}}$ and $p : \mathcal{D}_A \to \mathcal{D}_B$ is analytic with $p(0) = b$.

We first, in Subsection 8.1, turn our attention to conditions on A and B which guarantee that $p'(0)$ is one to one. Next, in Subsection 8.2, assuming $p'(0)$ is one to one, we apply a linear transforms on the range of p placing it into the canonical form $p(x) = (x, 0) + h(x)$, where $h(x)$ consists of higher order terms ($h(0) = 0$ and $h'(0) = 0$). In Subsection 8.3 we consider an affine linear change of variables on the domain of p.

8.1. Conditions guaranteeing $p'(0)$ is one to one.

Natural hypotheses on a mapping $\mathcal{D}_A$ to $\mathcal{D}_B$ via p lead to the conclusion that $p'(0)$ is one-one.

Lemma 8.1. *Suppose $A \in M_d(\mathbb{C})^g$ and $B \in M_e(\mathbb{C})^{\tilde{g}}$ and $p : \mathcal{D}_A \to \mathcal{D}_B$ is analytic and $p(0)$ is in the interior of $\mathcal{D}_B$. If*

- (i) *p is proper;*
- (ii) *$\mathcal{D}_A$ is bounded,*

then $p'(0)$ is one-one.

Proof. Let $b = p(0)$ and $\mathfrak{B} = L_B(p(0))$ and observe $\mathfrak{B}$ is positive definite. Let $\mathcal{H}$ denote the positive square root of $\mathfrak{B}$ and define $F = \mathcal{H}^{-1}B\mathcal{H}^{-1}$. Thus,

$$\mathcal{H}^{-1}L_B(p(x))\mathcal{H}^{-1} = L_F(q(x)),$$

where $q(x) = p(x) - b$. In particular, $L_B(p(X)) \succeq 0$ if and only if $L_F(q(X)) \succeq 0$ and p is proper if and only if q is. If $p'(0) = q'(0)$ is not one-one, there exists an $a \in \mathbb{C}^g$ such that $q'(0)a = 0$. Let S denote a matrix nilpotent of order two. It follows that

$$q(tX) = t(q'(0)a) \otimes S = 0.$$

Since $\mathcal{D}_A$ is bounded and contains 0 in its interior, if $X \neq 0$, there exists a t such that tX is in the boundary of $\mathcal{D}_A$. On the other hand, since $q(tX) = 0$, the tuple tX is not in the boundary of $\mathcal{D}_F$, contradicting the assumption that p is proper. Hence, $X = 0$ and thus $a = 0$ and the proof is complete. ∎

8.2. Affine linear change of variables for the range of p.

In this section we compute explicitly the effect of an affine linear change of variables in the range space of p. This can be used to produce a new map $\tilde{p}$ with $\tilde{p}'(0) = I$, which has later use in the proof of Theorem 10.3. Given a $g \times g$ matrix M and an analytic mapping $q = \begin{pmatrix} q^1 & \cdots & q^g \end{pmatrix}$, let qM denote the analytic mapping,

$$q(x)M = \begin{pmatrix} q^1(x) & \dots q^g(x) \end{pmatrix} M = \left(\sum q^j(x)M_{j,1}, \dots, \sum q^j(x)M_{j,g} \right)$$

On the other hand, for $B \in M_n(\mathbb{C})^g$, we often write MB for $(M \otimes I)B$ where B is treated as a column vector. Thus,

$$MB = \begin{pmatrix} \sum M_{1,j}B_j \\ \vdots \\ \sum M_{g,j}B_j \end{pmatrix}.$$

Since we are viewing x and $p(x)$ as row vectors in the case p has $\tilde{g}$ entries, $p'(0)$ is a $g \times \tilde{g}$ matrix.

Proposition 8.2. *Suppose $A \in M_d(\mathbb{C})^g$ and $B \in M_e(\mathbb{C})^{\tilde{g}}$ and $p : \mathcal{D}_A \to \mathcal{D}_B$ is an analytic map with $p(0) = b \in \mathbb{C}^{\tilde{g}}$. Let $\mathcal{H}$ denote the positive square root of*

$$\mathfrak{B} = L_B(b) = I + \sum_{j=1}^{g} b_j B_j + \sum_{j=1}^{g} (b_j B_j)^*,$$

and let $F = p'(0)\,\mathcal{H}^{-1} B \mathcal{H}^{-1} = \mathcal{H}^{-1}\, p'(0) B\, \mathcal{H}^{-1}$.

Suppose $p'(0)$ is one-one and choose any invertible $\tilde{g} \times \tilde{g}$ matrix M whose first g rows are those of of $p'(0)$ and let and let ℓ denote the affine linear polynomial $\ell(x) = (-b+x)M^{-1}$. The analytic map $\tilde{p} = \ell \circ p$ maps $\mathcal{D}_A$ into $\mathcal{D}_F$ and satisfies $\tilde{p}(0) = 0$ and $\tilde{p}'(0) = \begin{pmatrix} I_d & 0 \end{pmatrix}$. Thus $p(x) = (x \ \ 0) + h(x)$ where $h(0) = 0$ and $h'(0) = 0$. In particular, if p maps the boundary of $\mathcal{D}_A$ into the boundary of $\mathcal{D}_B$, then so does $\tilde{p}$; and if p is bianalytic, then so is $\tilde{p}$.

Written in more expansive notation, for each $1 \le i \le g$,

$$F_i = \mathcal{H}^{-1}(MB)_i \mathcal{H}^{-1} = \sum_{j=1}^{g} p'(0)_{i,j} \mathcal{H}^{-1} B_j \mathcal{H}^{-1},$$

$$B_i = \mathcal{H}(M^{-1}F)_i \mathcal{H} = \sum_{j=1}^{g} \left(M^{-1}\right)_{i,j} \mathcal{H} F_j \mathcal{H}.$$

Proof. For notational ease, let $L = M^{-1}$ and consider,

$$\begin{aligned}
\sum_{k=1}^{\tilde{g}} (MB)_k \otimes \tilde{p}(x)_k &= \sum_{k=1}^{\tilde{g}} (MB)_k \otimes ((-b + p(x))L)_k \\
&= \sum_{k=1}^{\tilde{g}} \left[\left(\sum_{j=1}^{\tilde{g}} M_{k,j} B_j \right) \otimes \left(\sum_{i=1}^{\tilde{g}} (-b_i + p_i(x))\,(L)_{i,k} \right) \right] \\
&= \sum_{j=1}^{\tilde{g}} \sum_{i=1}^{\tilde{g}} \left(\sum_{k=1}^{\tilde{g}} (L)_{i,k} M_{k,j} \right) \Big[B_j \otimes \big(-b_i + p_i(x) \big) \Big] \\
&= \sum_{j=1}^{\tilde{g}} \sum_{i=1}^{\tilde{g}} (I_{i,j}) \Big[B_j \otimes \big(-b_i + p_i(x) \big) \Big] = \sum_{j=1}^{\tilde{g}} B_j \otimes p_j(x) - \sum_{j=1}^{\tilde{g}} B_j b_j.
\end{aligned}$$

Given a tuple X, it follows that

$$\begin{aligned}
L_F(\tilde{p}(X)) &= I + \sum_{j=1}^{\tilde{g}} F_j \otimes \tilde{p}_j(X) + \sum_{j=1}^{\tilde{g}} {F_j}^* \otimes \tilde{p}_j(X)^* \\
&= I + \sum_{j=1}^{\tilde{g}} \left(\mathcal{H}^{-1}(MB\mathcal{H}^{-1} \right)_j \otimes \tilde{p}_j(X) + \sum_{j=1}^{\tilde{g}} \left(\mathcal{H}^{-1}(MB)\mathcal{H}^{-1} \right)_j^* \otimes \tilde{p}_j(X)^* \\
&= (\mathcal{H}^{-1} \otimes I) \left(\mathfrak{B} + \sum_{j=1}^{\tilde{g}} (MB)_j \otimes \tilde{p}_j(X) + \sum_{j=1}^{\tilde{g}} (MB)_j^* \otimes \tilde{p}_j(X)^* \right) (\mathcal{H}^{-1} \otimes I) \\
&= (\mathcal{H}^{-1} \otimes I) \left(I + \sum_{j=1}^{\tilde{g}} B_j \otimes p_j(X) + \sum_{j=1}^{\tilde{g}} B_j^* \otimes p(X)_j^* \right) (\mathcal{H}^{-1} \otimes I) \\
&= (\mathcal{H}^{-1} \otimes I) L_B(p(X)) (\mathcal{H}^{-1} \otimes I).
\end{aligned}$$

Since $\mathcal{H}^{-1} \otimes I$ is invertible, $L_F(\tilde{p}(X)) \succeq 0$ if and only if $L_B(p(X)) \succeq 0$. Assuming $p : \mathcal{D}_A \to \mathcal{D}_B$ and $X \in \mathcal{D}_A$, it follows that $\tilde{p}(X) \in \mathcal{D}_F$.

Next we note that $\tilde{p}(0) = (-b + p(0))L = 0$ and that $\ell'(x) = L$. Hence, with $P = \begin{pmatrix} I & 0 \end{pmatrix}$,

$$\tilde{p}'(x) = \ell'\big(p(x)\big)p'(x) = p(x)L$$

and thus, $\tilde{p}'(0) = p(0)L = PML = P$. ∎

8.3. **Change of basis in the $\mathcal{R}$ module generated by $\mathcal{A}$.** Here we consider change of the basis A for the right R-module it generates. Had we used a different basis would that yield a much different convexotonic map? More specifically, our formula for the mapping p depends (only) upon the structure matrices Ξ for the module generated by the tuple A over the algebra generated by the tuple $R = (U - I)A$. We now see that a linear change of the A variables produces a simple linear "similarity" transform $\tilde{p}$ of the mapping p.

Consider a linear change of variables determined by an invertible matrix $M \in \mathbb{C}^{g\times g}$. That is, $\tilde{A} = MA$ where A is regarded as the column of matrices $\begin{pmatrix} A_1 \\ \vdots \\ A_g \end{pmatrix}$, so $\tilde{A}_i = \sum_{j=1}^g M_{ij}A_j$ for $i = 1, \dots, g$.

The matrix M implements a change of basis on the span of $\{A_1, \dots, A_g\}$. We emphasize that the vectors of variables and maps are row vectors. Observe

$$\begin{aligned} \tilde{A}(C - I)\tilde{A}_j &= MA(C - I)(\sum_{k=1}^g M_{jk}A_k) = M(\sum_{k=1}^g M_{jk}(A(C - I)A_k)) \\ &= M(\sum_{k=1}^g M_{jk}(\Xi_k A)) = (M(\sum_{k=1}^g M_{jk}\Xi_k)M^{-1})\tilde{A}. \end{aligned}$$

Thus, $(\tilde{A}, C)$ satisfy the hypotheses of Proposition 6.2 with structure matrices

$$\widetilde{\Xi}_j = M(\sum_{k=1}^g M_{jk}\Xi_k)M^{-1}. \tag{8.1}$$

Concretely,

$$(\widetilde{\Xi_j})_{s,q} = \sum_{t,k,p} M_{s,t}M_{j,k}(\Xi_k)_{t,p}(M^{-1})_{p,q}.$$

8.3.1. *Computation of the mappings after linear change of coordinates.* Recall the rational function $p(y) = y(1 - \sum_{i=1}^g y_i\, \Xi_i)^{-1}$ associated to the Ξ. We now look at the effect of the linear change of variable via the invertible matrix M. The rational function $\tilde{p}$ determined by the new $\widetilde{\Xi}_k$, is

$$\begin{aligned} \tilde{p}(y) = y(1 - \sum_{i=1}^g y_i\widetilde{\Xi}_i)^{-1} &= y(1 - \sum_{i=1}^g y_i M(\sum_{k=1}^g M_{ik}\Xi_k)M^{-1})^{-1} \\ &= yM(1 - \sum_{k=1}^g(\sum_{i=1}^g y_i M_{ik})\Xi_k)^{-1}M^{-1}. \end{aligned}$$

Note that $\sum_{i=1}^g y_iM_{ik}$ is the k-th column of yM, thus $\tilde{p}(y) = p(yM)M^{-1}$. Similarly for $\tilde{p}$ inverse, denoted $\tilde{q}$, so we can summarize this as

$$\tilde{p}(y) = p(yM)M^{-1}, \qquad \tilde{q}(y) = q(yM)M^{-1}.$$

(For an example see (8.2) below.)

The following proposition summarizes the mapping implications of this change of variable.

Proposition 8.3. *If $p : \mathcal{D}_A \to \mathcal{D}_B$, then $\tilde{p} : \mathcal{D}_{\tilde{A}} \to \mathcal{D}_{\tilde{B}}$ where*

$$\mathcal{D}_{\tilde{A}} := \{y : yM \in \mathcal{D}_A\}, \qquad \mathcal{D}_{\tilde{B}} := \{z : zM \in \mathcal{D}_B\}$$

Proof. Given $y \in \mathcal{D}_{\tilde{A}}$, set $yM = x$, which by definition is in $\mathcal{D}_A$. By the formula above $\tilde{p}(y) = p(x)M^{-1} =: z$. Thus $zM = p(x) \in \mathcal{D}_B$, hence by definition of $\mathcal{D}_{\tilde{B}}$, we have $\tilde{p}(y) = z \in \mathcal{D}_{\tilde{B}}$. ∎

8.4. **Composition of convexotonic maps is not necessarily convexotonic.** Let $g = 2$ and let p be the indecomposable convexotonic map of Type I from Section 9, and consider the change of basis matrix

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

as in Subsection 8.3. The corresponding convexotonic map is

$$\tilde{p}(x,y) = p(y,x)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x+y^2 & y \end{pmatrix}. \tag{8.2}$$

Then

$$p(\tilde{p}(x,y)) = \begin{pmatrix} x+y^2 & y+x^2+xy^2+y^2x+y^4 \end{pmatrix}$$

is not convexotonic since it is a polynomial of degree > 2.

9. Constructing all Convexotonic Maps

To construct all convexotonic maps in g variables first one lists the indecomposable ones, i.e., those associated with an indecomposable algebra. Then build general convexotonic maps as direct sums of these. We illustrate this by giving all convexotonic maps in dimension 2 in Subsections 9.1 and 9.2. Finally, in Subsection 9.3 we show how the automorphisms of the complex wild ball are, after affine linear changes of variables, convexotonic.

9.1. **Convexotonic maps for $g = 2$.** In very small dimensions $g \le 5$ indecomposable algebras are classified [Maz79]. We work out the corresponding convexotonic maps for $g = 2$. The following is the list of indecomposable two-dimensional algebras over $\mathbb{C}$ (with basis R_1, R_2).

notation	nonzero products			properties	
I	$R_1^2 = R_2$			commutative	nilpotent
II	$R_1^2 = R_1$	$R_1R_2 = R_2$			
III	$R_1^2 = R_1$	$R_2R_1 = R_2$			
IV	$R_1^2 = R_1$	$R_1R_2 = R_2$	$R_2R_1 = R_2$	commutative	with identity

Accordingly we refer to these as algebras of type I – IV.

9.1.1. *Type I.* If R_1 is nilpotent of order 3, then $\Xi_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \Xi_2 = 0$. These structure matrices produce the convexotonic maps

$$p(x_1,x_2) = \begin{pmatrix} x_1 & x_2 + x_1^2 \end{pmatrix} \qquad q(x_1,x_2) = \begin{pmatrix} x_1 & x_2 - x_1^2 \end{pmatrix}. \tag{9.1}$$

Note $p(0) = 0$, $p'(0) = I$ and likewise for q.

9.1.2. *Type II.* Let $R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The corresponding structure matrices Ξ are $\Xi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \Xi_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. So

$$p(x) = \begin{pmatrix} (1-x_1)^{-1}x_1 & (1-x_1)^{-1}x_2 \end{pmatrix} \qquad q(x) = \begin{pmatrix} (1+x_1)^{-1}x_1 & (1+x_1)^{-1}x_2 \end{pmatrix}.$$

9.1.3. *Type III.* Let $R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. The structure matrices are $\Xi_1 = I_2$, $\Xi_2 = 0$. So

$$p(x) = \begin{pmatrix} x_1(1-x_1)^{-1} & x_2(1-x_1)^{-1} \end{pmatrix} \qquad q(x) = \begin{pmatrix} x_1(1+x_1)^{-1} & x_2(1+x_1)^{-1} \end{pmatrix}.$$

9.1.4. *Type IV.* Take $R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The corresponding structure matrices are $\Xi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\Xi_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. So

$$\begin{aligned} p(x) &= \begin{pmatrix} x_1(1-x_1)^{-1} & (1-x_1)^{-1}x_2(1-x_1)^{-1} \end{pmatrix} \\ q(x) &= \begin{pmatrix} x_1(1+x_1)^{-1} & (1+x_1)^{-1}x_2(1+x_1)^{-1} \end{pmatrix}. \end{aligned}$$

The other indecomposable convexotonic maps correspond to these after a linear change of basis, cf. Section 8.3. If the change of basis corresponds to an invertible 2×2 matrix M, then the corresponding convexotonic map is

$$\tilde{p}(x) = p(xM)M^{-1}.$$

9.2. **Convexotonic maps associated to decomposable algebras.** Here we explain which convexotonic maps arise from decomposable algebras. Suppose $\mathcal{R} = \mathcal{R}' \oplus \mathcal{R}''$ and $\mathcal{R}', \mathcal{R}''$ are indecomposable finite-dimensional algebras. Let $\{R_1, \dots, R_g\}$ be a basis for $\mathcal{R}'$ and let $\{R_{g+1}, \dots, R_h\}$ be a basis for $\mathcal{R}''$. Then $\{R_1 \oplus 0, \dots, R_g \oplus 0, 0 \oplus R_{g+1}, \dots, 0 \oplus R_h\}$ is a basis for $\mathcal{R}$ with the corresponding structure matrices

$$\Xi_j = \begin{cases} \Xi_j' \oplus 0 & : j \le g \\ 0 \oplus \Xi_j'' & : j > g, \end{cases}$$

where Ξ_j' and Ξ_j'' denote the structure matrices for $\mathcal{R}'$ and $\mathcal{R}''$, respectively. The convexotonic map corresponding to $\mathcal{R}$ is

$$\begin{aligned} p_{\mathcal{R}}(x) &= \begin{pmatrix} x_1 & \cdots & x_g & x_{g+1} & \cdots & x_h \end{pmatrix} \Big(I - \sum_{j=1}^{h} \Xi_j x_j \Big)^{-1} \\ &= \begin{pmatrix} x_1 & \cdots & x_g & x_{g+1} & \cdots & x_h \end{pmatrix} \begin{pmatrix} I - \sum_{j=1}^{g} \Xi_j' x_j & 0 \\ 0 & I - \sum_{j=g+1}^{h} \Xi_j'' x_j \end{pmatrix}^{-1} \\ &= \begin{pmatrix} p_{\mathcal{R}'}(x_1, \dots, x_g) & p_{\mathcal{R}''}(x_{g+1}, \dots, x_h) \end{pmatrix}. \end{aligned}$$

9.3. **Biholomorphisms of balls.** In this subsection we show how (linear fractional) biholomorphisms of balls in $\mathbb{C}^g$ can be presented using convexotonic maps. Let $\{\hat{e}_1, \dots, \hat{e}_{g+1}\}$ denote the standard basis of row vectors for $\mathbb{C}^{g+1}$ and let $A_j = \hat{e}_1^* \hat{e}_{j+1}$ for $j = 1, \dots, g$. Since $\mathcal{D}_A(1) = \{z \in \mathbb{C}^g : \sum_j |z_j|^2 \le 1\}$ is the unit ball in $\mathbb{C}^g$, the free spectrahedron $\mathcal{D}_A$ is a free version of the ball. Fix a (row) vector $v \in \mathbb{C}^g$ with $\|v\| < 1$ and let $xv^* = \sum \overline{v}_j x_j$, where $x = (x_1, \dots, x_g)$ is a row vector of free variables. By [Pop10], up to rotation, automorphisms of $\mathcal{D}_A$ have the form,

$$\mathcal{F}_v(x) = v - \big(1 - vv^*\big)^{\frac{1}{2}} \big(1 - xv^*\big)^{-1} x \big(I - v^*v\big)^{\frac{1}{2}}.$$

Modulo affine linear transformations, $\mathcal{F}_v$ is of the form

$$\big(1 - xv^*\big)^{-1} x = \begin{pmatrix} (1 - xv^*)^{-1} x_1 & \cdots & (1 - xv^*)^{-1} x_g \end{pmatrix}$$

since $\big(1 - vv^*\big)^{\frac{1}{2}}$ is a number and $\big(I - v^*v\big)^{\frac{1}{2}}$ is a matrix independent of x. Further,

$$\big(1 - xv^*\big)^{-1} x = x\big(I - v^*x\big)^{-1}.$$

Now let Ξ denote the g-tuple of $g\times g$ matrices $\Xi_j = e_j \otimes v^*$, where $\{e_1,\dots,e_g\}$ is the standard basis of row vectors for $\mathbb{C}^g$. Then $\Xi_j\Xi_k = \overline{v}_j\Xi_k = \sum_s (\Xi_k)_{j,s}\Xi_s$, so the tuple Ξ satisfies (1.5). Moreover,

$$x(I-\lambda_\Xi(x))^{-1} = x(I-v^*x)^{-1} = (1-xv^*)^{-1}x.$$

Thus $\mathcal{F}_v$ is a convexotonic map.

10. Bianalytic Spectrahedra which are not Affine Linearly Equivalent

In this section we present bounded free spectrahedra that are polynomially equivalent, but not affine linearly equivalent (over $\mathbb{C}$).

Suppose A and B are eig-generic tuples, $\mathcal{D}_A$ and $\mathcal{D}_B$ are bounded and there is a polynomial bianalytic map $p:\mathcal{D}_A\to\mathcal{D}_B$ with $p(x) = x+h(x)$ (h for higher order terms). In particular, by Theorem 6.4, A and B have the same size and $B = W^*VAW$ for unitaries V and W. Further, there is a representation for p in terms of the g-tuple Ξ of $g\times g$ matrices determined by

$$A_k(V-I)A_j = \sum_{s=1}^{g}(\Xi_j)_{ks}A_s. \tag{10.1}$$

10.1. A class of examples. Let Q be an invertible 2×2 matrix, so that

$$\mathcal{D}_Q(1) = \big\{c\in\mathbb{C} : I_2 + \big((cQ)^* + cQ\big)\succeq 0\big\}$$

is bounded. Choose P_{12}, P_{21}, P_{22} invertible, same size as Q with $P_{21}P_{12} = -2Q$. Now let

$$A_1 = \begin{pmatrix} 0 & P_{12}\\ P_{21} & P_{22}\end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0\\ 0 & Q\end{pmatrix}.$$

Given γ unimodular, let

$$V_\gamma = \begin{pmatrix}\gamma I_2 & 0\\ 0 & I_2\end{pmatrix}.$$

Proposition 10.1. *With notation as above,*

$$A_k(V_\gamma - I)A_j = \sum_{s=1}^{2}(\Xi_j)_{ks}A_s, \tag{10.2}$$

where $\Xi = (\Xi_1,\Xi_2)$ is the tuple defined by

$$\Xi_1 = \begin{pmatrix} 0 & -2(\gamma-1)\\ 0 & 0\end{pmatrix}$$

and $\Xi_2 = 0$. Thus the polynomial mapping

$$p_\gamma(x_1,x_2) = x(I-\Lambda_\Xi(x))^{-1} = (x_1, x_2 + 2(1-\gamma)x_1^2)$$

is a bianalytic $p:\mathcal{D}_A\to\mathcal{D}_B$ with $B = V_\gamma A$.

Moreover, if $P_{22} = \alpha_1 Q + \alpha_3(P_{12}^*P_{12} + P_{21}P_{21}^*)$, then for each unimodular φ,

$$\begin{aligned} s_\varphi = (\varphi x_1,\ &-(1-\varphi)\,(4\overline{\alpha_3}\varphi - \alpha_1)\,x_1 + x_2 + 2(1-\varphi^2)x_1^2)\\ &+ (\overline{\alpha_3}(1-\varphi),\ -\overline{\alpha_3}(1-\varphi)\,(2\overline{\alpha_3}(1-\varphi)+\alpha_1)).\end{aligned}$$

is a polynomial automorphism of $\mathcal{D}_A$.

Proof. Equation (10.2) follows from the computations, $(V_\gamma - I)A_2 = 0 = A_2(V_\gamma - I)$ and

$$A_1(V_\gamma - I)A_1 = -2(\gamma-1)A_2.$$

The converse portion of Theorem 6.4 now implies that

$$\begin{aligned} p =& x(I-\Lambda_\Xi(x))^{-1} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 2(\gamma-1)x_1 \\ 0 & 1 \end{pmatrix}^{-1} \\ =& \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 2(1-\gamma)x_1 \\ 0 & 1 \end{pmatrix} = (x_1, 2(1-\gamma)x_1^2 + x_2) \end{aligned}$$

is bianalytic between $\mathcal{D}_A$ and $\mathcal{D}_{V_\gamma A}$ as claimed.

To prove the second part of the proposition, suppose φ is unimodular. Let $\delta = \overline{\alpha_3}(1-\varphi)$ and $\eta = -4\varphi\delta + (1-\varphi)\alpha_1$ and let ρ denote the affine linear polynomial,

$$\rho(x_1,x_2) = (\varphi x_1,\, x_2 + \eta x_1) + \delta(1,\, 2\delta - \alpha_1).$$

With these notations,

$$L_A(\rho(x_1,x_2)) = L_A(\rho(0,0)) + (\varphi A_1 + \eta A_2)x_1 + A_2 x_2 + (\overline{\varphi}A_1^* + \overline{\eta}A_2^*)x_1^* + A_2^* x_2^*$$

and, using $P_{22} - \alpha_1 Q = \alpha_3(P_{12}^*P_{12} + P_{21}P_{21}^*)$,

$$\begin{aligned} L_A(\rho(0,0)) &= I + \delta A_1 + \delta^* A_1^* + (2\delta^2 - \delta\alpha_1)A_2 + \overline{(2\delta^2 - \delta\alpha_1)}A_2^* \\ &= \begin{pmatrix} I & \delta P_{12} + \overline{\delta}P_{21}^* \\ \overline{\delta}P_{12}^* + \delta P_{21} & \delta\alpha_3(P_{12}^*P_{12} + P_{21}P_{21}^*) + \overline{\delta\alpha_3}(P_{12}^*P_{12} + P_{21}P_{21}^*) - 2\delta^2 Q - 2(\overline{\delta})^2 Q^* \end{pmatrix} \\ &= \mathscr{Y}^*\mathscr{Y}, \end{aligned}$$

where

$$\mathscr{Y} = \begin{pmatrix} \varphi I & \varphi\big(\delta P_{12} + \overline{\delta}P_{21}^*\big) \\ 0 & I \end{pmatrix}.$$

Indeed, the only entry of this equality that is not immediate occurs in the $(2,2)$ entry. Since φ is unimodular, $|\delta|^2 = \alpha_3\delta + \overline{\delta\alpha_3}$ and thus the $(2,2)$ entry of $\mathscr{Y}^*\mathscr{Y}$ is

$$\begin{aligned} I + \big(\delta P_{12} + \overline{\delta}P_{21}^*\big)^*\big(\delta P_{12} + \overline{\delta}P_{21}^*\big) &= I + |\delta|^2\big(P_{12}^*P_{12} + P_{21}P_{21}^*\big) + \delta^2 P_{21}P_{12} + \overline{\delta}^2 P_{12}^*P_{21}^* \\ &= I + \delta\alpha_3(P_{12}^*P_{12} + P_{21}P_{21}^*) + \overline{\delta\alpha_3}(P_{12}^*P_{12} + P_{21}P_{21}^*) - 2\delta^2 Q - 2(\overline{\delta})^2 Q^*, \end{aligned}$$

where $P_{21}P_{12} = -2Q$ was also used.

Next, let $B = V_\gamma A$, where $\gamma = \varphi^2$. For notational ease let $Y = \delta P_{12} + \overline{\delta}P_{21}^*$ and verify

$$\begin{aligned} \varphi[P_{21}Y + Y^*P_{12}] + P_{22} &= \varphi[\delta(P_{21}P_{12} + P_{21}P_{12}) + \delta^*(P_{12}P_{12}^* + P_{21}P_{21}^*)] + P_{22} \\ &= \varphi[-4\delta Q + (1-\overline{\varphi})\alpha_3(P_{12}P_{12}^* + P_{21}P_{21}^*)] + P_{22} \\ &= -4\varphi\delta Q + (\varphi - 1)(P_{22} - \alpha_1 Q) + P_{22} \\ &= ((1-\varphi)\alpha_1 - 4\varphi\delta)Q + \varphi P_{22} \\ &= \eta Q + \varphi P_{22}. \end{aligned}$$

Hence,

$$\mathscr{Y}^* B_1 \mathscr{Y} = \begin{pmatrix} 0 & \varphi P_{12} \\ P_{21} & \varphi[P_{21}Y + Y^*P_{12}] + P_{22} \end{pmatrix} = \varphi A_1 + \eta A_2.$$

Likewise,

$$\mathscr{Y}^* B_2 \mathscr{Y} = \mathscr{Y}^* A_2 \mathscr{Y} = A_2.$$

It follows that

$$L_A(\rho(x)) = \mathscr{Y}^* L_B(x)\mathscr{Y}$$

and thus, as $\mathscr{Y}$ is invertible, $\rho = \rho_\varphi$ is a bianalytic affine linear map from $\mathcal{D}_B$ to $\mathcal{D}_A$. Thus, $\rho_\varphi \circ p_\gamma$ is a polynomial automorphism of $\mathcal{D}_A$. Finally, since

$$\rho_\varphi \circ p_\gamma(x) = s_\varphi(x),$$

the proof of the proposition is complete. ■

The next objective is to establish a converse of Proposition 10.1 under some mild additional assumptions on P_{ij} and Q. As a corollary, we produce examples of tuples A and B such that $\mathcal{D}_A$ and $\mathcal{D}_B$ are polynomially, but not linearly, bianalytic.

Lemma 10.2. *If $\{P_{12}^*P_{12}, P_{21}P_{21}^*\}$ is linearly independent and $\mathcal{D}_Q$ is bounded, then $\mathcal{D}_A$ is also bounded.*

Proof. First observe that independence of $\{P_{12}^*P_{12}, P_{21}P_{21}^*\}$ implies independence of $\{P_{12}, P_{21}^*\}$ since $P_{12} = tP_{21}^*$ implies $P_{12}^*P_{12} = |t|^2 P_{21}P_{21}^*$. Let

$$Z = xA_1 + \overline{x}A_1^* + yA_2 + \overline{y}A_2^* = \begin{pmatrix} 0 & M \\ M^* & N \end{pmatrix}.$$

We claim Z has both positive and negative eigenvalues, provided not both x and y are 0.

In the case $M \neq 0$, the matrix Z has both positive and negative eigenvalues. Note $M = xP_{12} + \overline{x}P_{21}^*$ and by independence $M = 0$ if and only if $x = 0$. In the case $x = 0$, (and thus $y \neq 0$), $N = yQ + (yQ)^*$. Since, by hypothesis, $\mathcal{D}_Q$ is bounded, Corollary 10.8 implies N has both positive and negative eigenvalues. Therefore, once again by Corollary 10.8, $\mathcal{D}_A$ is a bounded domain. ■

Theorem 10.3. *Suppose $\{P_{12}^*P_{12}, P_{21}P_{21}^*\}$ is linearly independent and $\mathcal{D}_Q$ is bounded. Suppose further that A is eig-generic and $*$-generic and either B is eig-generic or has size 4 (the same size as A).*

(1) *If $p : \mathcal{D}_A \to \mathcal{D}_B$ is a polynomial bianalytic map with $p(x) = x + h(x)$, then there is a unimodular γ such that, up to unitary equivalence, $B = V_\gamma A$ and*

$$p = (x_1, x_2 + 2(1-\gamma)x_1^2).$$

Now suppose further that $\{Q, Q^*, P_{12}^*P_{12}, P_{21}P_{21}^*\}$ is linearly independent, there is a $c \neq 0$ so that $P_{21}^* + cP_{12}$ is not invertible but $P_{21} - cP_{12}$ is invertible and $\mathcal{D}_Q$ is bounded.

(2) *If $\{Q, P_{22}, P_{12}^*P_{12}, P_{21}P_{21}^*\}$ is linearly independent, then $\mathcal{D}_A$ has no non-trivial polynomial automorphisms. In particular, if $q : \mathcal{D}_A \to \mathcal{D}_A$ is a bianalytic polynomial, then $q(0) = 0$, $q'(0) = I$ and $q(x_1, x_2) = (x_1, x_2 + 2(1-\gamma)x_1^2)$ for some unimodular γ.*

(3) *If $P_{22} = \alpha_1 Q + \alpha_2 Q^* + \alpha_3 P_{12}^*P_{12} + \alpha_4 P_{21}P_{21}^*$, then either*
   (a) *$\alpha_2 \neq 0$ and conclusion of item (2) holds; or*
   (b) *$\alpha_2 = 0$ in which case $\alpha_3 = \alpha_4$ and a polynomial automorphism s of $\mathcal{D}_A$ must have the form*

$$\begin{aligned} s = s_\varphi = {} & (\varphi x_1,\ -(1-\varphi)\left(4\overline{\alpha_3}\varphi - \alpha_1\right) x_1 + x_2 + 2(1-\varphi^2)x_1^2) \\ & + (\overline{\alpha_3}(1-\varphi),\ -\overline{\alpha_3}(1-\varphi)\left(2\overline{\alpha_3}(1-\varphi) + \alpha_1\right)) \end{aligned}$$

   for some unimodular φ.

Remark 10.4. Numerical experiments show, choosing P_{12} and P_{21} invertible with enough care so that, with $Q = -\frac{1}{2}P_{21}P_{12}$, the domain $\mathcal{D}_Q$ is bounded and the resulting tuple A is irreducible, the tuple A is eig-generic and $*$-generic.

Of course the polynomial automorphisms of $\mathcal{D}_A$ form a group under composition. In fact, as is straightforward to verify, $s_\varphi \circ s_\psi = s_{\varphi\psi}$. Further, combining items (3) and (1), produces a parameterization of all bianalytic polynomials $\mathcal{D}_A$ to $\mathcal{D}_B$ (under the prevailing assumptions on A and B). ◊

Example 10.5. As a concrete example, choose

$$Q = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

We note that $xQ + \overline{x}Q$ has both positive and negative eigenvalues for $x \neq 0$, so $\mathcal{D}_Q$ is bounded. Let

$$P_{12} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_{21} = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}, \quad P_{22} = I_2,$$

and writing A as was done above we claim A is eig-generic. Furthermore $\{Q, Q^*, P_{12}^*P_{12}, P_{21}P_{21}^*\}$ and $\{Q, P_{22}, P_{12}^*P_{12}, P_{21}P_{21}^*\}$ are linearly independent, so Theorem 10.3 applies, thus $p(x) = (x_1, x_2 + 4x_1^2)$ is the unique bianalytic map between $\mathcal{D}_A$ and $\mathcal{D}_B$, and they are not affine linearly equivalent.

Alternatively, let

$$P_{22} = \begin{pmatrix} 10 & -1 \\ -1 & 2 \end{pmatrix} = P_{21}P_{21}^* + P_{12}^*P_{12},$$

then we have a form for $q(x)$ and a class of affine linear automorphisms of $\mathcal{D}_A$.

Finally, letting $P_{22} = 0$, we get our family of automorphisms of $\mathcal{D}_A$ parameterized by the unimodular complex numbers. $\diamondsuit$

10.2. **The proof of Theorem 10.3.** Before turning to the proof of Theorem 10.3 proper, we record a simple lemma and other preliminary results.

Lemma 10.6. *Suppose $A \in M_n$. If $A^*XA = X$ for all $X \in M_n$, then A is a unimodular multiple of the identity.*

Proof. Choosing $X = I$ gives $A^*A = I$ and hence $A^* = A^{-1}$. Thus $XA = AX$ for all X, so A is a unimodular multiple of the identity. ∎

Proposition 10.7. *Let L be a linear pencil. If $\mathcal{D}_L$ is bounded, then $\mathcal{D}_L(1)$ is bounded. Conversely, if $\mathcal{D}_L$ is not bounded, then there exists $\alpha \in \mathbb{C}^g$ such that $t\alpha \in \mathcal{D}_L(1)$ for all $t \in \mathbb{R}_{>0}$.*

Proof. This result is the complex version of the full strength of [HKM13, Proposition 2.4]. Unfortunately, the statement of the result there is weaker than what is actually proved. Simply note that over the complex numbers, if T is a matrix and $\langle T\gamma, \gamma\rangle = 0$ for all vectors γ, then, by polarization, $\langle T\gamma, \delta\rangle = 0$ for all vectors γ, δ and hence $T = 0$. (By comparison, over the real numbers the same conclusion holds provided T is self-adjoint.) ∎

Corollary 10.8. *Let L be a monic linear pencil with truly linear part Λ. Thus, $L(x) = I + \Lambda(x) + \Lambda(x)^*$. The domain $\mathcal{D}_L$ is bounded if and only if $\Lambda(\alpha)$ has both positive and negative eigenvalues for each $\alpha \in \mathbb{C}^g \setminus \{0\}$.*

Proof of Theorem 10.3. To prove item (1), observe the hypotheses allow the application of Theorem 6.4. In particular, there exists a unitary V satisfying equation (10.1) and, in terms of the tuple Ξ of structure matrices,

$$p(x) = x(I - \Lambda_\Xi(x))^{-1}.$$

Write $V = (V_{jk})$ as a 2×2 matrix to match the 2×2 block structure of A. Straightforward computation gives,

$$(V - I)A_2 = \begin{pmatrix} 0 & V_{12}Q \\ 0 & (V_{22} - I)Q \end{pmatrix}.$$

Hence

$$A_1(V - I)A_2 = \begin{pmatrix} 0 & P_{12}(V_{22} - I)Q \\ 0 & P_{21}V_{21}Q \end{pmatrix}.$$

Since P_{12} and Q are invertible and, by equation (10.1), $A_1(V-I)A_2$ lies in the span of $\{A_1, A_2\}$, it follows that $V_{22}-I=0$. Since V is unitary, $VV^*=I$. Thus

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & I \end{pmatrix}\begin{pmatrix} V_{11}^* & V_{21}^* \\ V_{12}^* & I \end{pmatrix} = \begin{pmatrix} V_{11}V_{11}^*+V_{12}V_{12}^* & V_{11}V_{21}^*+V_{12} \\ V_{21}V_{11}^*+V_{12}^* & V_{21}V_{21}^*+I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

It follows that $V_{21}=0$ and thus $V_{12}=0$ as well. Finally,

$$A_1(V-I)A_1 = \begin{pmatrix} 0 & 0 \\ 0 & P_{21}(V_{11}-I)P_{12} \end{pmatrix}.$$

Hence equation (10.1) holds in this case ($j=1=k$) if and only if there is a λ such that $P_{21}(V_{11}-I)P_{12}=\lambda Q$ (note that $A_1(V-I)A_1=\delta A_1+\lambda A_2$, but $\delta=0$). Since $P_{21}P_{12}=-2Q$, it follows that

$$V_{11}-I=\lambda P_{21}^{-1}QP_{12}^{-1}=-\frac{1}{2}\lambda I.$$

Thus $V_{11}=(1-\frac{1}{2}\lambda)I$ and $|1-\frac{1}{2}\lambda|=1$. Hence,

$$V=V_\gamma=\begin{pmatrix} \gamma I & 0 \\ 0 & I \end{pmatrix}$$

for some unimodular γ. The tuple Ξ of structure matrices and polynomial p are thus described in Proposition 10.1.

Turning to the second part of the theorem, suppose $q:\mathcal{D}_A\to\mathcal{D}_A$ is a polynomial automorphism. Let $b=q(0)$ and let $\mathcal{H}$ denote the positive square root of $L_A(b)$. By Proposition 8.2, there exist F and a bianalytic polynomial $\tilde q:\mathcal{D}_A\to\mathcal{D}_F$ with $\tilde q(x)=\big(-b+p(x),p'(0)^{-1}\big)$, such that $\tilde q(0)=0$, $\tilde q'(0)=I$,

$$A_j=(q'(0)^{-1})_{j,1}\mathcal{H}F_1\mathcal{H}+\cdots+(q'(0)^{-1})_{j,g}\mathcal{H}F_g\mathcal{H}$$

and

$$F_j=q'(0)_{j,1}\mathcal{H}^{-1}A_1\mathcal{H}^{-1}+\cdots+q'(0)_{j,g}\mathcal{H}^{-1}A_g\mathcal{H}^{-1}.$$

Now F is the same size as A and A is eig-generic and A^* is $*$-generic, hence we can apply item (1) to the bianalytic polynomial $\tilde q:\mathcal{D}_A\to\mathcal{D}_F$. In particular, there is a unimodular γ such that $F=V_\gamma A$ and $\tilde q=(x_1,x_2+2(1-\gamma)x_1^2)$. By Proposition 8.2,

$$A_i=\sum_{j=1}^{g}\big(q'(0)^{-1}\big)_{i,j}\,\mathcal{H}F_j\mathcal{H}.$$

Since $F_j=\mathcal{V}^*VA_j\mathcal{V}$,

$$\mathscr{H}^{-*}A_i\mathscr{H}^{-1}=\sum_{j=1}^{g}\big(q'(0)^{-1}\big)_{i,j}\,VA_j,$$

where $\mathscr{H}=\mathcal{V}\mathcal{H}$. Setting

$$q'(0)^{-1}=\begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix},$$

gives

$$\mathscr{H}^{-*}A_1\mathscr{H}^{-1}=\mathscr{H}^{-*}\begin{pmatrix} 0 & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\mathscr{H}^{-1}=\begin{pmatrix} 0 & \lambda_1\gamma P_{12} \\ \lambda_1 P_{21} & \mu_1 Q+\lambda_1 P_{22} \end{pmatrix} \tag{10.3}$$

and likewise

$$\mathscr{H}^{-*}A_2\mathscr{H}^{-1}=\mathscr{H}^{-*}\begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}\mathscr{H}^{-1}=\begin{pmatrix} 0 & \lambda_2\gamma P_{12} \\ \lambda_2 P_{21} & \mu_2 Q+\lambda_2 P_{22} \end{pmatrix}. \tag{10.4}$$

By equation (10.3), $\lambda_1\neq 0$ since A_1 is invertible.

Let $Y = \mathscr{H}^{-1}$ and write $Y = (Y_{j,k})$ in the obvious way as a 2×2 matrix with 2×2 block entries. From equation (10.4),

$$\begin{pmatrix} Y_{21}^*QY_{21} & Y_{21}^*QY_{12} \\ * & Y_{22}^*QY_{22}\end{pmatrix} = \begin{pmatrix} 0 & \lambda_2\gamma P_{12} \\ \lambda_2 P_{21} & \mu_2 Q + \lambda_2 P_{22}\end{pmatrix}.$$

Thus, as Q is invertible, $Y_{21} = 0$ and therefore $\lambda_2 = 0$. We also record $Y_{22}^*QY_{22} = \mu_2 Q$ or equivalently $Y_{22}^*P_{21}P_{12}Y_{22} = \mu_2 P_{21}P_{12}$. Turning to equation (10.3),

$$\begin{pmatrix} 0 & Y_{11}^*P_{12}Y_{22} \\ Y_{22}^*P_{21}Y_{11} & Y_{22}^*P_{21}Y_{12} + Y_{12}^*P_{12}Y_{22} + Y_{22}^*P_{22}Y_{22}\end{pmatrix} = \begin{pmatrix} 0 & \lambda_1\gamma P \\ \lambda_1 P & \mu_1 Q + \lambda_1 P_{22}\end{pmatrix}.$$

Hence,

$$\begin{aligned} \lambda_2 &= 0 \\ Y_{22}^*P_{21}P_{12}Y_{22} &= \mu_2 P_{21}P_{12} \\ Y_{11}^*P_{12}Y_{22} &= \lambda_1\gamma P_{12} \\ Y_{22}^*P_{21}Y_{11} &= \lambda_1 P_{21} \\ Y_{22}^*P_{21}Y_{12} + Y_{12}^*P_{12}Y_{22} + Y_{22}^*P_{22}Y_{22} &= \mu_1 Q + \lambda_1 P_{22}. \end{aligned} \tag{10.5}$$

Taking determinants in the second of these equations gives $|\det(Y_{22})|^2 = \mu_2^2$ and therefore μ_2 is real. Taking determinants in the third and fourth equations gives $\overline{\lambda_1^2} = \lambda_1^2\gamma^2$. Thus,

$$\overline{\lambda_1} = \pm\lambda_1\gamma.$$

In particular,

$$|\overline{\lambda_1}|^2 = \pm\lambda_1^2\gamma. \tag{10.6}$$

Multiplying the third equation on the left by the fourth equation gives

$$Y_{22}^*P_{21}Y_{11}Y_{11}^*P_{12}Y_{22} = \lambda_1^2\gamma P_{21}P_{12}.$$

Using the second equation

$$Y_{22}^*P_{21}Y_{11}Y_{11}^*P_{12}Y_{22} = \frac{\lambda_1^2\gamma}{\mu_2} Y_{22}^*P_{21}P_{12}Y_{22}.$$

Since $Y_{22}^*P_{21}$ and $P_{12}Y_{22}$ are invertible,

$$Y_{11}Y_{11}^* = \frac{\lambda_1^2\gamma}{\mu_2} > 0.$$

In particular, Y_{11} is a multiple of a unitary.

Next multiply the fourth equation by its adjoint on the right to obtain

$$|\lambda_1|^2 P_{21}P_{21}^* = Y_{22}^*P_{21}Y_{11}(Y_{22}^*P_{21}Y_{11})^* = \frac{\lambda_1^2\gamma}{\mu_2} Y_{22}^*P_{21}P_{21}^*Y_{22}. \tag{10.7}$$

Multiplying the third equation by its adjoint on the left gives (as $|\gamma| = 1$)

$$|\lambda_1|^2 P_{12}^*P_{12} = (Y_{11}^*P_{12}Y_{22})^*Y_{11}^*P_{12}Y_{22} = \frac{\lambda_1^2\gamma}{\mu_2} Y_{22}^*P_{12}^*P_{12}Y_{22}. \tag{10.8}$$

In view of equation (10.6), if $\mu_2 > 0$, then $\lambda_1^2\gamma = |\lambda_1|^2$ and if $\mu_2 < 0$, then $-\lambda_1^2\gamma = |\lambda_1|^2$. Hence, with $|\kappa|^2 = |\mu_2|$ and $Z = \kappa Y_{22}$ either $\mu_2 > 0$ and

$$\begin{aligned} Z^*P_{21}P_{12}Z &= P_{21}P_{12} & \qquad Z^*P_{21}P_{21}^*Z &= P_{21}P_{21}^* \\ Z^*P_{12}^*P_{21}^*Z &= P_{12}^*P_{21}^* & \qquad Z^*P_{12}^*P_{12}Z &= P_{12}^*P_{12} \end{aligned} \tag{10.9}$$

or $\mu_2 < 0$ and

$$\begin{aligned} Z^* P_{21} P_{12} Z &= -P_{21} P_{12} & Z^* P_{21} P_{21}^* Z &= P_{21} P_{21}^* \\ Z^* P_{12}^* P_{21}^* Z &= -P_{12}^* P_{21}^* & Z^* P_{12}^* P_{12} Z &= P_{12}^* P_{12}. \end{aligned}$$

We will argue that this second case does not occur. Recall we are assuming P_{21} and P_{12} are both invertible. This implies Z is invertible. Observe that, assuming this second set of equations, for complex numbers c,

$$Z^*((P_{12} + cP_{21}^*)^*(P_{12} + cP_{21}^*))Z = (P_{12} - cP_{21}^*)^*(P_{12} - cP_{21}^*). \tag{10.10}$$

By assumption there is a $c \neq 0$ such that $P_{21}^* + cP_{12}$ is not invertible but $P_{21}^* - cP_{12}$ is invertible which leads to the contradiction that the left hand side of equation (10.10) is invertible, but the right hand side is not. It follows that $\mu_2 > 0$ and $\lambda_1^2 \gamma = |\lambda_1|^2$.

Assuming $\{Q, Q^*, P_{21}P_{21}^*, P_{12}^*P_{12}\}$ is linearly independent, this set spans the 2×2 matrices. Hence, by Lemma 10.6, $Y_{22} = \kappa I$ for some κ with $|\kappa|^2 = \mu_2$. Hence many of the identities in equation (10.5) now imply that Y_{11} is also a multiple of the identity. For instance, using the third equality,

$$\lambda_1 P_{21} = Y_{22}^* P_{21} Y_{11} = \overline{\kappa} P_{21} Y_{11}$$

and hence $Y_{11} = \frac{\lambda_1}{\overline{\kappa}} I$.

Thus,

$$\mathscr{H}^{-1} = Y = \begin{pmatrix} \frac{\lambda_1}{\overline{\kappa}} I & Y_{12} \\ 0 & \kappa I \end{pmatrix}$$

and consequently

$$\mathcal{V}\mathcal{H} = \mathscr{H} = \begin{pmatrix} \frac{\overline{\kappa}}{\lambda_1} & -\frac{\overline{\kappa}}{\kappa\lambda_1} Y_{12} \\ 0 & \frac{1}{\kappa} \end{pmatrix}.$$

It follows that

$$\mathcal{H}^2 = \mathscr{H}^* \mathscr{H} = \begin{pmatrix} \frac{|\kappa|^2}{|\lambda_1|^2} I & -\frac{\bar{\kappa}}{|\lambda_1|^2} Y_{12} \\ -\frac{\kappa}{|\lambda_1|^2} Y_{12}^* & \frac{1}{|\kappa|^2} I + \frac{1}{|\lambda_1|^2} Y_{12}^* Y_{12} \end{pmatrix}$$

On the other hand,

$$\begin{aligned} \mathcal{H}^2 = L_A(b) &= (I + \sum b_j A_j + \sum (b_j A_j)^*) \\ &= \begin{pmatrix} I & b_1 P_{12} + \overline{b_1} P_{21}^* \\ b_1 P_{21} + \overline{b_1} P_{12}^* & I + b_2 Q + \overline{b_2} Q^* + b_1 P_{22} + \overline{b_1} P_{22}^* \end{pmatrix}. \end{aligned}$$

It follows that,

$$\begin{aligned} \frac{|\kappa|^2}{|\lambda_1|^2} &= 1 \\ -\frac{\bar{\kappa}}{|\lambda_1|^2} Y_{12} &= b_1 P_{12} + \overline{b_1} P_{21}^* \\ \frac{1}{|\kappa|^2} I + \frac{1}{|\lambda_1|^2} Y_{12}^* Y_{12} &= I + b_2 Q + \overline{b_2} Q^* + b_1 P_{22} + \overline{b_1} P_{22}^*. \end{aligned} \tag{10.11}$$

Note that combining the first two of these equations gives,

$$Y_{12} = -\kappa(b_1 P_{12} + \overline{b_1} P_{21}^*). \tag{10.12}$$

Since $Y_{22} = \kappa I$, the last equality in equation (10.5) gives

$$\overline{\kappa} P_{21} Y_{12} + \kappa Y_{12}^* P_{12} + |\kappa|^2 P_{22} = \mu_1 Q + \lambda_1 P_{22}.$$

It follows, using the second equality in equation (10.11),

$$\begin{aligned}\mu_1 Q + (\lambda_1 - |\kappa|^2)P_{22} &= -\,|\lambda_1|^2 P_{21}(b_1 P_{12} + \overline{b_1} P_{21}^*) - |\lambda_1|^2(\overline{b_1} P_{12}^* + b_1 P_{21})P_{12} \\ &= -\,|\lambda_1|^2 \left(b_1 P_{21}P_{12} + \overline{b_1} P_{21}P_{21}^* + \overline{b_1} P_{12}^* P_{12} + b_1 P_{21}P_{12}\right).\end{aligned}$$

Simplifying with $P_{21}P_{12} = -2Q$ and bringing to one side gives

$$0 = (\mu_1 + 4b_1|\lambda_1|^2)Q + \overline{b_1}|\lambda_1|^2(P_{21}P_{21}^* + P_{12}^*P_{12}) + (\lambda_1 - |\kappa|^2)P_{22}. \tag{10.13}$$

We now proceed to prove item (2). Assuming $\{Q, P_{12}^*P_{12}, P_{21}P_{21}^*, P_{22}\}$ is linearly independent, equation (10.13) and the fact that $\lambda_1 \neq 0$ yields $b_1 = 0$. So $\mu_1 = 0$ and $\lambda_1 = |\kappa|^2$. Furthermore, $|\kappa| = |\lambda_1|$, implies $\lambda_1 = |\lambda_1|$. Hence $\lambda_1 = 1$, $|\kappa| = 1$ and $\gamma = 1$. It also follows that $Y_{12} = 0$ by the third equation in (10.11). Furthermore,

$$I = I + b_2 Q + \overline{b_2} Q^*,$$

so $b_2 = 0$, as $\{Q, Q^*\}$ is linearly independent. Hence $L_A(b) = I = \mathcal{H}$ and

$$\mathcal{V} = \overline{\kappa} I_4.$$

Finally, $Y_{12} = 0$ also implies $\mu_2 = 1$. Thus, $F = A$ and $V = V_\gamma = I$, $q(0) = 0$ and finally, $q'(0) = I$ too. Hence, $q(x) = x$ and the proof of item (2) is complete.

Moving on to item (3), assume now

$$P_{22} = \alpha_1 Q + \alpha_2 Q^* + \alpha_3 P_{12}^*P_{12} + \alpha_4 P_{21}P_{21}^*.$$

If $\alpha_2 \neq 0$ then $\{Q, P_{12}^*P_{12}, P_{21}P_{21}^*, P_{22}\}$ must also be linearly independent, and hence the conclusions of item (2) hold.

To complete the proof of the theorem, suppose $\alpha_2 = 0$ and recall equation (10.13),

$$\begin{aligned}0 = {}&(\mu_1 + 4b_1|\lambda_1|^2 + \alpha_1(\lambda_1 - |\kappa|^2))Q \\ &+ (\overline{b_1}|\lambda_1|^2 + \alpha_3(\lambda_1 - |\kappa|^2))P_{12}^*P_{12} + (\overline{b_1}|\lambda_1|^2 + \alpha_4(\lambda_1 - |\kappa|^2))P_{21}P_{21}^*.\end{aligned}$$

Since $\{Q, P_{12}^*P_{12}, P_{21}P_{21}^*\}$ is linearly independent,

$$\begin{aligned}\mu_1 + 4b_1|\lambda_1|^2 + \alpha_1(\lambda_1 - |\lambda_1|^2) &= 0 \\ \overline{b_1}|\lambda_1|^2 + \alpha_3(\lambda_1 - |\lambda_1|^2) &= 0 \\ \overline{b_1}|\lambda_1|^2 + \alpha_4(\lambda_1 - |\lambda_1|^2) &= 0.\end{aligned} \tag{10.14}$$

It follows that $\alpha_3 = \alpha_4$ and

$$b_1 = \frac{\overline{\alpha_3}(\lambda_1 - 1)}{\lambda_1}. \tag{10.15}$$

Now, using equation (10.12) and looking at the third equation in equation (10.11),

$$\begin{aligned}&\frac{1}{|\kappa|^2} I + \frac{|\kappa|^2}{|\lambda_1|^2}(|b_1|^2 P_{12}^*P_{12} + 2b_1^2 Q + 2\overline{b_1}^2 Q^* + |b_1|^2 P_{21}P_{21}^*) \\ &\quad = I + b_2 Q + \overline{b_2} Q^* + b_1 P_{22} + \overline{b_1} P_{22}^*.\end{aligned}$$

Using $P_{22} = \alpha_1 Q + \alpha_3(P_{12}^*P_{12} + P_{21}P_{21}^*)$,

$$\begin{aligned}0 = {}&(1 - \frac{1}{|\kappa|^2})I + (b_2 - 2b_1^2 + \alpha_1 b_1)Q + (\overline{b_2} - 2\overline{b_1}^2 + \overline{\alpha_1}\overline{b_1})Q^* \\ &+ (b_1\alpha_3 + \overline{b_1}\overline{\alpha_3} - |b_1|^2)(P_{12}^*P_{12} + P_{21}P_{21}^*).\end{aligned} \tag{10.16}$$

Using equation (10.15),

$$\begin{aligned} b_1\alpha_3+\overline{b_1}\overline{\alpha_3}-|b_1|^2 &= |\alpha_3|^2\left(\frac{\lambda_1-1}{\lambda_1}+\frac{\overline{\lambda_1}-1}{\overline{\lambda_1}}-\frac{(\lambda_1-1)(\overline{\lambda_1}-1)}{\lambda_1\overline{\lambda_1}}\right)\\ &= |\alpha_3|^2\left(1-\frac{1}{|\lambda_1|^2}\right)=|\alpha_3|^2\left(1-\frac{1}{|\kappa|^2}\right). \end{aligned}$$

Let $z=(b_2-2b_1^2+\alpha_1 b_1)$, solving for the Q and Q^* terms, equation (10.16) becomes

$$zQ+\overline{z}Q^*=\left(1-\frac{1}{|\kappa|^2}\right)\left(I+|\alpha_3|^2(P_{12}^*P_{12}+P_{21}P_{21}^*)\right)$$

Write $C=(1-\frac{1}{|\kappa|^2})$, let $t\in\mathbb{R}$ with $tC>0$ and consider

$$L_Q(tz)=I+tzQ+t\overline{z}Q^*=(1+tC)I+|\alpha_3|^2tC(P_{12}^*P_{12}+P_{21}P_{21}^*).$$

But $P_{12}^*P_{12},P_{21}P_{21}^*\succeq 0$, so $L_Q(tz)\succeq 0$ for all t with $tC>0$ which contradicts $\mathcal{D}_Q$ being bounded. Hence both $z=0$ and $C(I+|\alpha_3|^2(P_{12}^*P_{12}+P_{21}P_{21}^*))=0$. So either $C=0$ or $I=-|\alpha_3|^2(P_{12}^*P_{12}+P_{21}P_{21}^*)$. However $I\succeq 0$ while $-|\alpha_3|^2(P_{12}^*P_{12}+P_{21}P_{21}^*)\preceq 0$, hence this second equality never holds. Thus $C=0$.

It follows that $|\kappa|^2=|\lambda_1|^2=\mu_2=1$, so

$$\begin{aligned} b_1&=\overline{\alpha_3}(1-\overline{\lambda_1}) & \mu_1&=-\lambda_1(1-\overline{\lambda_1})\left(4\overline{\alpha_3}\overline{\lambda_1}+\alpha_1\right)\\ b_2&=\overline{\alpha_3}(1-\overline{\lambda_1})\left(2\overline{\alpha_3}(1-\overline{\lambda_1})-\alpha_1\right) & Y_{12}&=\kappa(\overline{\alpha_3}(1-\overline{\lambda_1})P_{12}-\alpha_3(1-\lambda_1)P_{21}^*), \end{aligned}\tag{10.17}$$

and

$$F_1=\mathcal{H}^{-1}\left(\overline{\lambda_1}B_1+(1-\overline{\lambda_1})(4\overline{\alpha_3}\overline{\lambda_1}+\alpha_1)B_2\right)\mathcal{H}^{-1},\qquad F_2=\mathcal{H}^{-1}B_2\mathcal{H}^{-1}.$$

Recall,

$$q(0)=(b_1,b_2),\quad q'(0)=\begin{pmatrix}\lambda_1&\mu_1\\0&1\end{pmatrix}^{-1}=\begin{pmatrix}\overline{\lambda_1}&-\overline{\lambda_1}\mu_1\\0&1\end{pmatrix},$$

so plugging in;

$$q(0)=\left(\overline{\alpha_3}\left(1-\overline{\lambda_1}\right),\quad \overline{\alpha_3}\left(1-\overline{\lambda_1}\right)\left(2\overline{\alpha_3}(1-\overline{\lambda_1})-\alpha_1\right)\right),$$

and

$$q'(0)=\begin{pmatrix}\overline{\lambda_1}&(1-\overline{\lambda_1})(4\overline{\alpha_3}\overline{\lambda_1}+\alpha_1)\\0&1\end{pmatrix}.$$

Next, we know that $\ell(x)=\left(-b+x,q'(0)^{-1}\right)$ and $\ell^{-1}(x)=(x,q'(0))+b$, so yet again plugging in;

$$\begin{aligned} \ell(x)&=(-\lambda_1q(0)_1+\lambda_1x_1,\lambda_1q'(0)_{1,2}-\lambda_1q'(0)_{1,2}x_1+x_2),\\ \ell^{-1}(x)&=\left(q(0)_1+\overline{\lambda_1}x_1,q(0)_2+q'(0)_{1,2}x_1+x_2\right). \end{aligned}$$

Using the fact that $q=\ell^{-1}\circ\tilde{q}$,

$$q(x)=\ell^{-1}\left(\tilde{q}(x)\right)=\left(q(0)_1+\overline{\lambda_1}x_1,\quad q(0)_2+q'(0)_{1,2}x_1+x_2+(1+\overline{\lambda_1}^2)x_1^2\right).$$

Recall however, that $q=q_{\overline{\lambda_1}}$, so taking a unimodular ϕ and then setting $s_\phi=p^{-1}\circ\ell^{-1}\circ\tilde{q}$,

$$s_\phi^1(x)=\overline{\alpha_3}(1-\phi)+\phi x_1$$

$$s_\phi^2(x)=-\overline{\alpha_3}(1-\phi)\left(2\overline{\alpha_3}(1-\phi)+\alpha_1\right)-(1-\phi)\left(4\overline{\alpha_3}\phi-\alpha_1\right)x_1+x_2+2(1-\phi^2)x_1^2,$$

which by construction is an automorphism of $\mathcal{D}_A$. Moreover, if ψ is another unimodular, then

$$s_\phi\circ s_\psi=s_{\phi\psi}$$

These automorphisms must be the only automorphisms of $\mathcal{D}_A$, since if there were some other form of automorphism then by composing with q we would get a different form for a bianalytic polynomial from $\mathcal{D}_A$ to $\mathcal{D}_A$ which cannot happen. ■

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Appendix A. Convexotonic maps in 3 variables

This is a list of all convexotonic maps coming from indecomposable 3 dimensional algebras over $\mathbb{C}$. These maps were generated in a Mathematica Notebook running under the package NCAlgebra. The notebook, given defining relations on a basis for a g-dimensional algebra, produces the associated convexotonic maps p and their inverses p^{-1}. The notebook was prepared by Eric Evert with help from Zongling Jiang (graduate students at UCSD) and Shiyuan Huang and Ashwin Trisal (undergraduates) in consultation with this papers' authors.

Algebra 1. Relations on a basis R_1, R_2, R_3:

$$R_1R_1 = 0, \quad R_2R_2 = 0, \quad R_3R_3 = 0,$$

$$R_1R_2 = 0, \quad R_2R_1 = 0, \quad R_1R_3 = R_2, \quad R_3R_1 = R_2, \quad R_2R_3 = 0, \quad R_3R_2 = 0.$$

Structure matrices

$$\Xi = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

$$p(x) = (x_1,\ x_2 + x_1x_3 + x_3x_1,\ x_3) \qquad p^{-1}(y) = (y_1,\ y_2 - y_1y_3 - y_3y_1,\ y_3)$$

Algebra 2. Let α be a real number. Note the case $\alpha = 1$ is Algebra 1.
Relations on a basis R_1, R_2, R_3:

$$R_1R_1 = 0, \quad R_2R_2 = 0, \quad R_3R_3 = 0,$$

$$R_1R_2 = 0, \quad R_2R_1 = 0, \quad R_1R_3 = R_2, \quad R_3R_1 = \alpha R_2, \quad R_2R_3 = R_2, \quad R_3R_2 = 0.$$

Structure matrices

$$\Xi = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

$$p(x) = (x_1,\ x_2 + x_1x_3 + \alpha x_3x_1,\ x_3) \qquad p^{-1}(y) = (y_1,\ y_2 - y_1y_3 - \alpha y_3y_1,\ y_3)$$

Algebra 3. Relations on a basis R_1, R_2, R_3:

$$R_1R_1 = R_2, \quad R_2R_2 = 0, \quad R_3R_3 = 0,$$

$$R_1R_2 = R_3, \quad R_2R_1 = R_3, \quad R_1R_3 = 0, \quad R_3R_1 = 0, \quad R_2R_3 = 0, \quad R_3R_2 = 0.$$

Structure matrices

$$\Xi = \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

$$p(x) = (x_1,\ x_2 + x_1x_1,\ x_3 + x_1(x_1^2 + x_2) + x_2x_1)$$
$$p^{-1}(y) = (y_1,\ y_2 - y_1^2,\ y_3 + y_1(y_1^2 - y_2) - y_2y_1)$$

Algebra 4. Relations on a basis R_1, R_2, R_3:

$$R_1R_1 = 0, \quad R_2R_2 = 0, \quad R_3R_3 = R_3,$$

$$R_1R_2 = 0, \quad R_2R_1 = 0, \quad R_1R_3 = R_2, \quad R_3R_1 = 0, \quad R_2R_3 = R_2, \quad R_3R_2 = 0.$$

Structure matrices

$$\Xi = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$p(x) = (x_1,\ (x_2 + x_1x_3)(1 - x_3)^{-1},\ x_3(1 - x_3)^{-1})$$
$$p^{-1}(y) = (y_1,\ (y_2 - y_1y_3)(1 + y_3)^{-1},\ y_3(1 + y_3)^{-1})$$

Algebra 5. Relations on a basis R_1, R_2, R_3:

$$R_1R_1 = 0, \quad R_2R_2 = 0, \quad R_3R_3 = R_3,$$

$$R_1R_2 = 0, \quad R_2R_1 = 0, \quad R_1R_3 = 0, \quad R_3R_1 = R_1, \quad R_2R_3 = R_2, \quad R_3R_2 = 0.$$

Structure matrices

$$\Xi = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$p(x) = ((1 - x_3)^{-1}x_1,\ x_2(1 - x_3)^{-1}, x_3(1 - x_3)^{-1})$$
$$p^{-1}(y) = ((1 + y_3)^{-1}y_1,\ y_2(1 + y_3)^{-1},\ y_3(1 + y_3)^{-1})$$

Algebra 6. Relations on a basis R_1, R_2, R_3:

$$R_1R_1 = 0, \quad R_2R_2 = 0, \quad R_3R_3 = R_3,$$

$$R_1R_2 = 0, \quad R_2R_1 = 0, \quad R_1R_3 = 0, \quad R_3R_1 = R_2, \quad R_2R_3 = 0, \quad R_3R_2 = R_2.$$

Structure matrices

$$\Xi = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$p(x) = (x_1,\ (1 - x_3)^{-1}(x_2 + x_3x_1),\ x_3(1 - x_3)^{-1})$$
$$p^{-1}(y) = (y_1,\ (1 + y_3)^{-1}(y_2 - y_3y_1),\ y_3(1 + y_3)^{-1})$$

Algebra 7. Relations on a basis R_1, R_2, R_3:

$$R_1R_1 = 0, \quad R_2R_2 = R_2, \quad R_3R_3 = R_3,$$

$$R_1R_2 = R_1, \quad R_2R_1 = 0, \quad R_1R_3 = 0, \quad R_3R_1 = R_1, \quad R_2R_3 = 0, \quad R_3R_2 = 0.$$

Structure matrices

$$\Xi = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$p(x) = ((1 - x_3)^{-1}x_1,\ x_2,\ x_3(1 - x_3)^{-1})$$
$$p^{-1}(y) = (1 + y_3)^{-1}y_1(1 + y_2)^{-1}, 1 - (1 + y_2)^{-1}, y_3(1 + y_3)^{-1}$$

Algebra 8. Relations on a basis R_1, R_2, R_3:

$$R_1R_1 = 0, \quad R_2R_2 = 0, \quad R_3R_3 = R_3, \quad R_1R_2 = 0, \quad R_2R_1 = 0, \quad R_1R_3 = R_1,$$

$$R_3R_1 = R_1, \quad R_2R_3 = R_2, \quad R_3R_2 = 0.$$

Structure matrices

$$\Xi = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$p(x) = ((1-x_3)^{-1}x_1(1-x_3)^{-1}, x_2(1-x_3)^{-1}, (1-x_3)^{-1}x_3)$$
$$p^{-1}(y) = ((1+y_3)^{-1}y_1(1+y_3)^{-1},\ y_2(1+y_3)^{-1},\ y_3(1+y_3)^{-1})$$

Algebra 9. Relations on a basis R_1, R_2, R_3:

$$R_1R_1 = 0, \quad R_2R_2 = 0, \quad R_3R_3 = R_3,$$

$$R_1R_2 = 0, \quad R_2R_1 = 0, \quad R_1R_3 = 0, \quad R_3R_1 = R_1, \quad R_2R_3 = R_2, \quad R_3R_2 = R_2.$$

Structure matrices

$$\Xi = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$p(x) = ((1-x_3)^{-1}x_1,\ (1-x_3)^{-1}x_2(1-x_3)^{-1},\ x_3(1-x_3)^{-1})$$
$$p^{-1}(y) = ((1+y_3)^{-1}y_1,\ (1+y_3)^{-1}y_2(1+y_3)^{-1},\ y_3(1+y_3)^{-1})$$

Algebra 10. Relations on a basis R_1, R_2, R_3:

$$R_1R_1 = 0, \quad R_2R_2 = 0, \quad R_3R_3 = R_3,$$

$$R_1R_2 = 0, \quad R_2R_1 = 0, \quad R_1R_3 = R_1, \quad R_3R_1 = R_1, \quad R_2R_3 = R_2, \quad R_3R_2 = R_2.$$

Structure matrices

$$\Xi = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$p(x) = ((1-x_3)^{-1}x_1(1-x_3)^{-1}, (1-x_3)^{-1}x_2(1-x_3)^{-1}, x_3(1-x_3)^{-1})$$
$$p^{-1}(y) = ((1+y_3)^{-1}y_1(1+y_3)^{-1},\ (1+y_3)^{-1}y_2(1+y_3)^{-1},\ y_3(1+y_3)^{-1})$$

Algebra 11. Relations on a basis R_1, R_2, R_3:

$$R_1R_1 = 0, \quad R_2R_2 = 0, \quad R_3R_3 = R_3,$$

$$R_1R_2 = 0, \quad R_2R_1 = 0, \quad R_1R_3 = R_2, \quad R_3R_1 = R_2, \quad R_2R_3 = R_2, \quad R_3R_2 = R_2.$$

Structure matrices

$$\Xi = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$p(x) = (x_1, (1-x_3)^{-1}(x_2 + x_1x_3 + x_3x_1 - x_3x_1x_3)(1-x_3)^{-1}, x_3(1-x_3)^{-1})$$
$$p^{-1}(y) = (y_1,\ (1+y_3)^{-1}(y_2 + y_1y_3 + y_3y_1 - y_3y_1y_3)(1+y_3)^{-1},\ y_3(1+y_3)^{-1})$$

Algebra 12. Relations on a basis R_1, R_2, R_3:

$$R_1R_1 = R_2, \quad R_2R_2 = 0, \quad R_3R_3 = R_3,$$

$$R_1R_2 = 0, \quad R_2R_1 = 0, \quad R_1R_3 = R_1, \quad R_3R_1 = R_1, \quad R_2R_3 = R_2, \quad R_3R_2 = R_2.$$

Structure matrices

$$\Xi = \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$p(x) = ((1-x_3)^{-1}x_1(1-x_3)^{-1},\ (1-x_3)^{-1}(\ x_1(1-x_3)^{-1}x_1 + x_2\ )(1-x_3)^{-1},\ x_3(1-x_3)^{-1})$$

$$p^{-1}(y) = ((1+y_3)^{-1}y_1(1+y_3)^{-1}, -(1+y_3)^{-1}(\ y_1(1+y_3)^{-1}y_1 - y_2\ )(1+y_3)^{-1}, y_3(1+y_3)^{-1})$$

An appealing way to present the map p in this case is to write it as

$$\pi(x) = (x_1,\ x_2 + x_1(1-x_3)^{-1}x_1,\ x_3 - x_3^2)$$

with each entry multiplied on the right and the left by $(1-x_3)^{-1}$.

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