KURANISHI STRUCTURE, PSEUDO-HOLOMORPHIC CURVE, AND VIRTUAL FUNDAMENTAL CHAIN: PART 2

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ABSTRACT. This article consists the second parts of the article we promised at the end of [FOOO15, Section 1]. We discuss the foundation of the virtual fundamental chain and cycle technique, especially its version appeared in [FOn] and also in [FOOO4, Section A1, Section 7.5], [FOOO7, Section 12], [Fu2].

This article is independent of our earlier writing [FOOO15]. We also do not assume that the readers have any knowledge on the pseudo-holomorphic curve.

In this second part, we consider a system of spaces with Kuranishi structures (abbreviated as K-system) and its simultaneous perturbations.

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Part 2. System of K-spaces and smooth correspondences

15. Introduction to Part 2

In Part 1, we have described the foundation of the theory of Kuranishi structure. good coordinate system, CF-perturbation (also multivalued perturbation), and defined the integration along the fiber (push out) of a strongly submersive map with respect to a CF-perturbation and also prove Stokes' formula. Using these ingredients, we have established the notion of smooth correspondence and proved the composition formula. Especially, this provides us a way to obtain a virtual fundamental chain for each K-space (space with Kuranishi structure). Thus Part 1 is the story for each single K-space. On the other hand, in Part 2 we are going to study a system of K-spaces. In actual geometric applications, there are the cases for which it is not enough to study each single K-space individually, but is necessary to study a system of K-spaces satisfying certain *compatibility conditions*, especially, compatibility conditions at boundary and corner. The compatibility conditions we describe depend on the situation we consider in their detail. Here we have two geometric examples in mind. One is the Floer cohomology for periodic Hamiltonian system which is established by [FOn], [LiuTi] for general closed symplectic manifold, and the other is the A_{∞} algebra associated to a Lagrangian submanifold and the Floer cohomology for the Lagrangian intersection established by [FOOO3] [FOOO4]. Since this article intends to provide a 'package' of the statements appearing in the actual argument above, we begin with axiomatizing the properties and conditions in a purely abstract setting, motivated by these two geometric examples. In this way we discuss two kinds of systems of K-spaces in Part 2: One is a linear K-system containing the Floer theory for periodic Hamiltonian system as a typical example, and the other is a tree-like K-system containing the theory of the A_{∞} algebra associated to a Lagrangian submanifold.

We emphasize that we discuss those two cases as a prototype of the applications of the results of this article. In fact if we are interested only in defining Floer cohomology of periodic Hamiltonian system and proving its basic properties, certainly there is a shorter proof than those given in Sections 16-20. In this article we give a general proof which can be used in similar situations with minimal change. For this reason we tried to avoid using special feature of the particular situation we work with but use only the arguments which are general enough. A similar argument works in most of the other cases where the method of pseudo-holomorphic curves is applied. (We do not know the case this method does not work.) We confirm that it also works at least in the following situations.

- (1) Constructing and proving basic properties of the Gromov-Witten invariant.
- (2) Studying several Lagrangian submanifolds and constructing an A_{∞} category (Fukaya catgory).
- (3) The family version of (2) and equivariant versions of (1)(2).
- (4) Including immersed Lagrangian submanifolds.
- (5) Using the Lagrangian correspondence to construct an A_{∞} functor.
- (6) Including bulk deformations into the Lagrangian Floer theory to define open-closed, closed-open maps, and proving their basic properties.
- (7) Studying the moduli space of psuedo-holomorphic maps from a bordered Riemann surface of arbitrary genus with Lagrangian boundary condition to construct an IBL-infinity structure.
- (8) Including the non-compact case in defining and studying symplectic homology and wrapped Floer homology.
- (9) In the case of symplectic manifolds with contact type boundary, using closed Reeb orbits or Reeb chords to establish the foundation of symplectic field theory and its version with Legendrian submanifolds.

The purpose of Part 2 is summarized as follows: If we are given a system of K-spaces satisfying the axiom we describe in this article as an input, then we prove that we can derive certain algebraic structures from the system of K-spaces as an output. This is a 'package' producing an algebraic structure from a geometric input. The problem is that the resulting algebraic structure itself depends on the various choices made in the course of the construction, in general. We will specify in which sense the algebraic structure is invariant and prove the invariance in this article. Although these typical examples of K-systems arise from moduli spaces of pseudo-holomorphic curves, the authors expect that this kind of axiomatization and framework will be available for other problems arising from other situations in future

- 15.1. Outline of the story of linear K-system. In Sections 16-20 we study systems of K-spaces, which axiomatize the situation appearing during the construction of Floer cohomology of periodic Hamiltonian systems.
- 15.1.1. Floer cohomology of periodic Hamiltonian systems. We first review the outline of the construction of Floer cohomology of periodic Hamiltonian systems in [Fl2].

Let $H: S^1 \times M \to \mathbb{R}$ be a real valued smooth function on the product of S^1 and a symplectic manifold M. For each $t \in S^1$ we obtain a function $H_t: M \to \mathbb{R}$ by

$$H_t(x) = H(t, x).$$

We denote its Hamiltonian vector field by X_{H_t} defined by $dH_t = \omega(X_{H_t}, \cdot)$. A periodic solution of the periodic Hamiltonian system generated by H is by definition a smooth map $\ell: S^1 \to M$ that satisfies the equation:

$$\frac{d\ell}{dt} = X_{H_t} \circ \ell. \tag{15.1}$$

Let Per(H) be the set of all solutions of (15.1). We will work in the Bott-Morse situation so we assume that Per(H) is a smooth manifold satisfying certain nondegeneracy condition. (See [FOOO12, Subsection 1.2 (2)] for example.)

Remark 15.1. To define Floer cohomology, it is enough to discuss the Morse case, that is, the case when H satisfies certain nondegeneracy condition so that Per(H) is discrete. However to calculate Floer cohomology, it is useful to include the Bott-Morse case especially the case with $H \equiv 0$.

We decompose Per(H) into connected components and denote

$$\operatorname{Per}(H) = \bigcup_{\overline{\alpha} \in \overline{\mathfrak{A}}} \overline{R}_{\overline{\alpha}}.$$

For $\overline{\alpha}_-, \overline{\alpha}_+ \in \overline{\mathfrak{A}}$ we consider the set of solutions of the following equation (Floer's equation) for a map $u : \mathbb{R} \times S^1 \to M$

$$\frac{\partial u}{\partial \tau} + J \left(\frac{\partial u}{\partial t} - X_{H_t}(u) \right) = 0 \tag{15.2}$$

together with the following asymptotic boundary condition.

Condition 15.2. There exist $\gamma_{-\infty} \in \overline{R}_{\overline{\alpha}_{-}}$ and $\gamma_{+\infty} \in \overline{R}_{\overline{\alpha}_{+}}$ such that

$$\lim_{\tau \to -\infty} u(\tau, t) = \gamma_{-\infty}(t),$$

$$\lim_{\tau \to +\infty} u(\tau, t) = \gamma_{+\infty}(t).$$
(15.3)

We denote by $\widetilde{\mathcal{M}}(\overline{R}_{\overline{\alpha}_{-}}, \overline{R}_{\overline{\alpha}_{+}})$ the set of solutions of (15.2) satisfying Condition 15.2. We can define an \mathbb{R} action on $\widetilde{\mathcal{M}}(\overline{R}_{\overline{\alpha}_{-}}, \overline{R}_{\overline{\alpha}_{+}})$ by $(\tau_{0} \cdot u)(\tau, t) = u(\tau + \tau_{0}, t)$, and denote the associated quotient space by $\widetilde{\mathcal{M}}(\overline{R}_{\overline{\alpha}_{-}}, \overline{R}_{\overline{\alpha}_{+}})$. We decompose it into

$$\overset{\circ}{\mathcal{M}}(\overline{R}_{\overline{\alpha}_{-}},\overline{R}_{\overline{\alpha}_{+}})=\bigcup_{\beta}\overset{\circ}{\mathcal{M}}(\overline{R}_{\overline{\alpha}_{-}},\overline{R}_{\overline{\alpha}_{+}};\beta),$$

according to the homology class β of u. In place of this decomposition we proceed as follows. We denote by $\mathfrak A$ the set of pairs $(\overline{\alpha}, [w])$ where $\overline{\alpha} \in \overline{\mathfrak A}$ and [w] is the homology class of a disc map w bounding $\ell \in \overline{R_{\alpha}}$. For each $\alpha \in \mathfrak A$ we define R_{α} as the set of pairs consisting of an element ℓ of $\overline{R_{\alpha}}$ and an equivalence class [w] of the homology class of disc w bounding ℓ .

Then let $\mathcal{M}(R_{\alpha_{-}}, R_{\alpha_{+}})$ be the union of $\mathcal{M}(\overline{R}_{\overline{\alpha}_{-}}, \overline{R}_{\overline{\alpha}_{+}}; \beta)$ over β with $[w^{-}] \# [\beta] = [w^{+}]$ (where $\alpha_{\pm} = (\overline{\alpha}_{\pm}, [w^{\pm}])$ and denote the union by $\mathcal{M}^{\text{reg}}(H; \alpha_{-}, \alpha_{+})$. Using the notion of stable map, we can compactify each of $\mathcal{M}^{\text{reg}}(H; \alpha_{-}, \alpha_{+})$ and denote the compactification by $\mathcal{M}(H; \alpha_{-}, \alpha_{+})$. (See [FOn, Definition 19.9].)

¹The equivalence class is defined by using the symplectic area and the Maslov index.

We define asymptotic evaluation maps $\mathcal{M}^{\text{reg}}(H; \alpha_-, \alpha_+) \to \overline{R}_{\alpha_+}$ by

$$\operatorname{ev}_{-}([u]) = \gamma_{-\infty}, \quad \operatorname{ev}_{+}([u]) = \gamma_{+\infty},$$

where $\gamma_{-\infty}$, $\gamma_{+\infty}$ are as in (15.3). They induce maps

$$\operatorname{ev}_{\pm}: \mathcal{M}(H; \alpha_{-}, \alpha_{+}) \to R_{\alpha_{+}},$$

which we call the evaluation maps at infinity. Using the non degeneracy condition of R_{α} , we can show that $\mathcal{M}(H; \alpha_{-}, \alpha_{+})$ carries a Kuranishi structure with corners and the evaluation map

$$(ev_-, ev_+): \mathcal{M}(H; \alpha_-, \alpha_+) \to R_{\alpha_-} \times R_{\alpha_+}$$
 (15.4)

is a strongly smooth and weakly submersive map. Moreover we can find the following isomorphism of K-spaces:

$$\partial \mathcal{M}(H; \alpha_{-}, \alpha_{+}) = \bigcup_{\alpha} \mathcal{M}(H; \alpha_{-}, \alpha) \,_{\text{ev}_{+}} \times_{\text{ev}_{-}} \mathcal{M}(H; \alpha, \alpha_{+}). \tag{15.5}$$

When we regard the left hand side as the *normalized* boundary, then the right hand side becomes the *disjoint* union.

See [FOOO15, Part 5] for the construction of such a K-system.

15.1.2. Periodic Hamiltonian system and axiom of linear K-system. As we explained in the previous subsubsection, when we are given a time dependent Hamiltonian H, we obtain a system consisting of a set of smooth manifolds R_{α} and a set of K-spaces $\mathcal{M}(H; \alpha_{-}, \alpha_{+})$, together with evaluation maps (15.4). The axiom of linear K-systems, which we present in Section 16 Condition 16.1, spells out properties of such a system which we need to define Floer cohomology.

Condition 16.1 (III)(IV) require the existence of a set of manifolds $\{R_{\alpha} \mid \alpha \in \mathfrak{A}\}$, a set of K-spaces $\{\mathcal{M}(H; \alpha_{-}, \alpha_{+}) \mid \alpha_{-}, \alpha_{+} \in \mathfrak{A}\}$ and evaluation maps (15.4) indexed by a countable set \mathfrak{A} . In this abstract situation we call $\mathcal{M}(H; \alpha_{-}, \alpha_{+})$ the *space of connecting orbits*.

Condition 16.1 (VI) (and (I)) requires that we can associate the Maslov index $\mu(\alpha)$ to each R_{α} which determines the dimension of $\mathcal{M}(H; \alpha_{-}, \alpha_{+})$.

It is well-known that the energy

$$\int_{\mathbb{R}\times S^1} \left\| \frac{\partial u}{\partial \tau} \right\|^2 + \left\| \frac{\partial u}{\partial t} - X_{H_t}(u) \right\|^2 d\tau dt \tag{15.6}$$

of the solution u of (15.2) is a difference of the value of certain action functional at the asymptotic boundary values α_- , α_+ . Moreover the energy is nonnegative and is zero only when $\partial u/\partial \tau = 0$. Condition 16.1 (V) (and (I)) is an axiomatization of this property.

We note that, in our Bott-Morse situation, we need to introduce an appropriate O(1)-principal bundle $o_{R_{\alpha}}$ to our critical submanifold to define Floer cohomology. Namely the contribution of R_{α} to the Floer cohomology is the cohomology group of R_{α} with the coefficients twisted by this local system. (See [FOOO4, Subsection 8.8].) Then the orientation local system of the moduli space $\mathcal{M}(H; \alpha_{-}, \alpha_{+})$ is related to the orientations of $R_{\alpha_{\pm}}$ and to $o_{R_{\alpha_{\pm}}}$ in an appropriate way. Condition 16.1 (VII) is an axiomatization of this property.

Note that an element of R_{α} is a pair $(\ell, [w])$ where ℓ is a periodic orbit of our Hamiltonian system and [w] is a homology class of disks bounding ℓ . For an element $\beta \in H_2(M; \mathbb{Z})$ represented by a sphere, we can glue w with a representative of β to

obtain another disk. By this operation we obtain another $R_{\beta\#\alpha}$. It is easy to see $\mathcal{M}(H;\alpha_-,\alpha_+)\cong\mathcal{M}(H;\beta\#\alpha_-,\beta\#\alpha_+)$. Condition 16.1 (VIII) is an axiomatization of this property.

One important property of the moduli space of pseudo-holomorphic curve or the solution of Floer's equation, is Gromov compactness. It claims the compactness of the union of the moduli spaces whose elements have energy smaller than a fixed number. Condition 16.1 (IX) is an axiomatization of this property.

The boundary of our moudli space $\mathcal{M}(H; \alpha_-, \alpha_+)$ is described as (15.5). Moreover this isomorphism is not only one as topological spaces but also one as spaces with oriented Kuranishi structure. Condition 16.1 (X) is an axiomatization of this property.

Our moduli space $\mathcal{M}(H; \alpha_-, \alpha_+)$ has not only boundary but also corners in general. Its codimension k (normalized) corner $\widehat{S}_k \mathcal{M}(H; \alpha_-, \alpha_+)$ is described as the disjoint union of the fiber products

$$\mathcal{M}(H; \alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \mathcal{M}(H; \alpha_{1}, \alpha_{2}) \times_{R_{\alpha_{2}}} \dots \times_{R_{\alpha_{k-1}}} \mathcal{M}(H; \alpha_{k-1}, \alpha_{k}) \times_{R_{\alpha_{k}}} \mathcal{M}(H; \alpha_{k}, \alpha_{+}),$$

where $\alpha_1, \ldots, \alpha_k \in \mathfrak{A}$. Condition 16.1 (XI) is an axiomatization of this property. We explain Condition 16.1 (XII) in Subsubsection 15.1.4.

A system satisfying Condition 16.1 is called a *linear K-system* (Definition 16.6 (2)). Now the main result of Sections 16-20 is as follows. Suppose we are given a linear K-system. We consider a direct sum of \mathbb{R} vector spaces

$$\bigoplus_{\alpha \in \mathfrak{A}} \Omega(R_{\alpha}; o_{R_{\alpha}}). \tag{15.7}$$

Using (15.7) and the energy filtration we define (in Definitions 16.8 and 16.11) a module

$$CF(\mathcal{C}; \Lambda_{0,\text{nov}})$$

over the universal Novikov ring $\Lambda_{0,\text{nov}}$. (See Definition 16.10 for the definition of $\Lambda_{0,\text{nov}}$.) Here \mathcal{C} denotes the totality of the part of data of our linear K-system which is related to $\{R_{\alpha}\}$. We call it critical submanifold data. (See Definition 16.6 (1).)

Theorem 15.3. To each linear K-system, we can associate a cochain complex, Floer cochain complex, which we denote by $(CF(\mathcal{C}; \Lambda_{0,nov}), d)$. This complex is independent of the choices up to cochain homotopy equivalence.

This is a slightly simplified version of Theorem 16.9.

15.1.3. Construction of Floer cochain complex. We will prove Theorem 15.3 (or Theorem 16.9) in detail in Section 19. The proof we present there is written in a way so that it is a prototype of the proof of various similar results and can be adapted easily to the proof of similar results.

The coboundary operator d in Theorem 15.3 is a sum of the exterior differential $d_0: \Omega(R_\alpha; o_{R_\alpha}) \to \Omega(R_\alpha; o_{R_\alpha})$ and the operator $d_{\alpha_-, \alpha_+}: \Omega(R_{\alpha_-}; o_{R_{\alpha_-}}) \to \Omega(R_{\alpha_+}; o_{R_{\alpha_+}})$ obtained by the smooth correspondence

$$R_{\alpha_{-}} \xleftarrow{\operatorname{ev}_{-}} \mathcal{M}(H; \alpha_{-}, \alpha_{+}) \xrightarrow{\operatorname{ev}_{+}} R_{\alpha_{+}}.$$
 (15.8)

Then (15.5) together with Stokes' formula ([Part I, Theorem 9.26]) and Composition formula ([Part I, Theorem 10.20]) 'imply'

$$d_0 \circ d_{\alpha_-,\alpha_+} + d_{\alpha_-,\alpha_+} \circ d_0 + \sum_{\alpha} d_{\alpha_-,\alpha} \circ d_{\alpha,\alpha_+} = 0.$$

This will imply that $d = d_0 + \sum d_{\alpha_-,\alpha_+}$ satisfies $d \circ d = 0$. Thus we obtain Floer cohomology.

More precisely speaking, to define the operator $d_{\alpha_{-},\alpha_{+}}$ from (15.8) as smooth correspondence we need to take and fix a CF-perturbation on $\mathcal{M}(H;\alpha_{-},\alpha_{+})$.

To apply Stokes' formula, our CF-perturbations of various moduli spaces must be compatible with the isomorphism (15.5). Namely we need to show the next statement.

Proposition 15.4. (slightly imprecise statement) For each given linear K-system and sufficiently small $\epsilon > 0$, there exists a system of CF-perturbations $\widehat{\mathfrak{S}_{\alpha_{-},\alpha_{+}}^{\epsilon}}$ on $\mathcal{M}(H;\alpha_{-},\alpha_{+})$ such that:

- (1) $\mathfrak{S}_{\alpha_{-},\alpha_{+}}^{\epsilon}$ is transversal to 0 and ev_{+} is strongly submersive with respect to this CF-perturbation.
- (2) The restriction of $\widehat{\mathfrak{S}_{\alpha_{-},\alpha_{+}}^{\epsilon}}$ to the boundary is equivalent to the fiber product of $\widehat{\mathfrak{S}_{\alpha_{-},\alpha}^{\epsilon}}$ and $\widehat{\mathfrak{S}_{\alpha_{-},\alpha_{+}}^{\epsilon}}$ via the isomorphism (15.5).

This is slightly imprecise statement and is *not* the statement we will prove. The precise statement we will prove is Proposition 19.1. The main difference between Proposition 15.4 and Proposition 19.1 is the following.

- (a) The CF-perturbation $\mathfrak{S}_{\alpha_{-},\alpha_{+}}^{\epsilon}$ is not defined on the Kuranishi structure of $\mathcal{M}(H;\alpha_{-},\alpha_{+})$ itself, which is given by the axiom of linear K-system, but defined on its thickening.
- (b) We replace $\mathcal{M}(H; \alpha_{-}, \alpha_{+})$ by $\mathcal{M}(H; \alpha_{-}, \alpha_{+})^{\boxplus \tau_{0}}$, which is a K-space obtained by putting the collar to the space $\mathcal{M}(H; \alpha_{-}, \alpha_{+})$ outside.
- (c) We fix E_0 and construct CF-perturbations for only finitely many moduli spaces, that is, the moduli spaces consisting of the elements of energy (15.6) $\leq E_0$.

The reason of Item (a) is that, to construct a CF-perturbation we need first to construct a good coordinate system and then go back to the Kuranishi structure. We explained this point already in [Part I, Subsection 1.2].

The reason of Item (b) is more technical. We perform various operations in a neighborhood of the boundary and corner of our K-space. Those constructions are easier to carry out if the charts of our K-space have collars. We will use it to extend the Kuranishi structure on the boundary which is a thickening of the given one, to the interior. (See however Remark 15.5 (1).) Existence of collar on a given cornered manifold or orbifold is fairly standard fact in differential topology. In case of Kuranishi structure or good coordinate system, to put the collar to the all charts so that coordinate changes preserve it is rather cumbersome. (This is because the way to put collar to a given cornered orbifold is not unique.) We take a short cut and put the collar 'outside' rather than 'inside'. The process to put the collar outside is describe in detail in Section 17. See Subsection 17.1 for more detailed explanation on this point.

The reason of Item (c) is that it is difficult to perturb infinitely moduli spaces simultaneourly. We go back to this point in Subsubsection 15.1.7.

Remark 15.5. (1) Recall that in this article we start from a purely abstract setting of a K-space with boundary and corner, which is not necessarily arising from a particular geometric situation like the moduli space of pseudo-holomorphic curves. In the geometric setting studied in [FOn],[FOOO3],[FOOO4], an extension of the Kuranishi structure to a small neighborhood of ∂X in X is given from its construction. In fact, we start from a Kuranishi structure $\partial \widehat{\mathcal{U}}$ on the boundary (which we obtain from geometry or analysis) and construct a good coordinate system $\widehat{\mathcal{U}}_{\partial}$ and use it to find $\widehat{\mathcal{U}}_{\partial}^+$ and its perturbation. We need to extend $\widehat{\mathcal{U}}_{\partial}^+$ and its perturbation to a neighborhood of ∂X . In this situation, the Kuranishi charts of $\widehat{\mathcal{U}}_{\partial}^+$ are obtained as open subcharts of certain Kuranishi charts of $\partial \widehat{\mathcal{U}}$. (See the proof of [Part I, Theorem 3.30, Proposition 6.44].) Therefore it can be indeed extended using the extension of $\widehat{\partial \mathcal{U}}$ directly.

(2) In the proof of well-definedness of the virtual fundamental chain, that corresponds to the well defined-ness of the Gromov-Witten invariant, which is given in Part 1 of this article, 'trivialization of corner' is not necessary. This is because we only need to apply Stokes' formula and do not need the chain level argument. See [Part I, Propositions 8.15,8.16] and their proofs.

15.1.4. Corner compatibility conditions. The proof of Proposition 15.4 (or Proposition 19.1) is by induction on energy. We consider the isomorphism (15.5):

$$\partial \mathcal{M}(H; \alpha_{-}, \alpha_{+}) = \bigcup_{\alpha} \mathcal{M}(H; \alpha_{-}, \alpha) \ _{\text{ev}_{+}} \times_{\text{ev}_{-}} \mathcal{M}(H; \alpha, \alpha_{+}).$$

We observe the energy of the moduli space appearing in the right hand side is strictly smaller than one appearing in the left hand side. So by induction hypothesis the CF-perturbation of the right hand side is already given. Therefore the statement we need to work out this induction is something like the following (*).

(*) Let $(X,\widehat{\mathcal{U}})$ be a K-space with corner. Suppose a CF-perturbation $\widehat{\mathfrak{S}}_{\partial}^{\epsilon}$ is given on the normalized boundary $\partial(X,\widehat{\mathcal{U}})$, satisfying certain transversality properties. Then we can find a CF-perturbation $\widehat{\mathfrak{S}}^{\epsilon}$ on $(X,\widehat{\mathcal{U}})$ which has the same transversality property and whose restriction to the boundary coincides with $\widehat{\mathfrak{S}}_{\partial}^{\epsilon}$.

However, we note that the statement (*), as it is, does *not* hold. In fact, since $(X, \widehat{\mathcal{U}})$ has not only boundary but also corners, we need to assume certain compatibility conditions for $\widehat{\mathfrak{S}}_{\underline{\partial}}^{\epsilon}$ at the corner. Let us elaborate this point below.

We remark that we use the *normalized* corner of an orbifold (or Kuranishi structure) with corners. Typically a point in the corner \hat{S}_2U of an orbifold U corresponds to two points in the *normalized* boundary. In other words we have a double cover

$$\pi: \partial \partial U \to \widehat{S}_2 U.$$
 (15.9)

Suppose we are given a CF-perturbation $\widehat{\mathfrak{S}}_{\partial}^{\epsilon}$ on the normalized corner ∂U . The compatibility condition we need to assume for $\widehat{\mathfrak{S}}_{\partial}^{\epsilon}$ is that if $\pi(x) = \pi(y)$ then the perturbation $\widehat{\mathfrak{S}}_{\partial}^{\epsilon}$ at x coincides with $\widehat{\mathfrak{S}}_{\partial}^{\epsilon}$ at y. Namely we need to require the next condition:

(*) There exists a CF-perturbation $\widehat{\mathfrak{S}_{S_2U}^{\epsilon}}$ on \widehat{S}_2U , whose pull-back to $\partial\partial U$ is equivalent to the restriction of $\widehat{\mathfrak{S}_{\delta}^{\epsilon}}$ to $\partial\partial U$

We can state a similar condition for Kuranishi structure on X. We note that to state the condition (\star) precisely we first need to clarify the relationship between the Kuranishi structure on $\partial \partial X$ and one on \widehat{S}_2X .

In the situation of our application where $X = \mathcal{M}(H; \alpha_-, \alpha_+)$, we have isomorphisms

$$\begin{split} &\partial \mathcal{M}(H;\alpha_{-},\alpha_{+}) \\ &\cong \partial \left(\bigcup_{\alpha} \mathcal{M}(H;\alpha_{-},\alpha) \,_{\operatorname{ev}_{+}} \times_{\operatorname{ev}_{-}} \mathcal{M}(H;\alpha,\alpha_{+}) \right) \\ &\cong \bigcup_{\alpha} \partial (\mathcal{M}(H;\alpha_{-},\alpha)) \,_{\operatorname{ev}_{+}} \times_{\operatorname{ev}_{-}} \mathcal{M}(H;\alpha,\alpha_{+}) \\ &\cup \bigcup_{\alpha} \mathcal{M}(H;\alpha_{-},\alpha) \,_{\operatorname{ev}_{+}} \times_{\operatorname{ev}_{-}} \partial (\mathcal{M}(H;\alpha,\alpha_{+})) \\ &\cong \bigcup_{\alpha_{1},\alpha_{2}} (\mathcal{M}(H;\alpha_{-},\alpha_{1}) \,_{\operatorname{ev}_{+}} \times_{\operatorname{ev}_{-}} (\mathcal{M}(H;\alpha_{1},\alpha_{2})) \,_{\operatorname{ev}_{+}} \times_{\operatorname{ev}_{-}} \mathcal{M}(H;\alpha_{2},\alpha_{+}) \\ &\cup \bigcup_{\alpha_{1},\alpha_{2}} \mathcal{M}(H;\alpha_{-},\alpha_{1}) \,_{\operatorname{ev}_{+}} \times_{\operatorname{ev}_{-}} ((\mathcal{M}(H;\alpha_{1},\alpha_{2}) \,_{\operatorname{ev}_{+}} \times_{\operatorname{ev}_{-}} \mathcal{M}(H;\alpha_{2},\alpha_{+})). \end{split}$$

On the other hand, by Condition 16.1 (XI) we assumed:

$$\widehat{S}_{2}\mathcal{M}(H;\alpha_{-},\alpha_{+})$$

$$\cong \bigcup_{\alpha_{1},\alpha_{2}} \mathcal{M}(H;\alpha_{-},\alpha_{1}) _{\text{ev}_{+}} \times_{\text{ev}_{-}} \mathcal{M}(H;\alpha_{1},\alpha_{2}) _{\text{ev}_{+}} \times_{\text{ev}_{-}} \mathcal{M}(H;\alpha_{2},\alpha_{+})$$

By these isomorphisms we obtain a double cover

$$\pi': \partial \partial \mathcal{M}(H; \alpha_-, \alpha_+) \to \widehat{S}_2 \mathcal{M}(H; \alpha_-, \alpha_+).$$

The Condition 16.1 (XII) (the second of corner compatibility condition) requires that this double cover π' coincides with the double cover π in (15.9).

Remark 15.6. (1) The condition $\pi = \pi'$ (Condition 16.1 (XII)) is not automatic and we need to assume it as a part of the axiom of linear K-system. In fact, we can define the covering map π in a canonical way for an arbitrary K-space X. On the other hand, the covering map π' depends on the choice of the isomorphism (15.5) and similar isomorphisms for the corner. (Condition 16.1 (XI)). Note that in our axiomatization only the existence of the isomorphism (15.5) is required. The isomorphism such as (15.5) is not unique. In fact, we can change it by composing any automorphism of $\partial \mathcal{M}(H; \alpha_-, \alpha_+)$. If we change the isomorphism (15.5) then the identity $\pi = \pi'$ will no longer hold.

In other words, Condition 16.1 (XII) is one on the consistency between various choices of the isomorphisms (15.5) and similar isomorphisms for the corner.

(2) In our geometric situation, we define the isomorphism (15.5) using geometric description of the boundary of our moduli space $\mathcal{M}(H; \alpha_-, \alpha_+)$. Then the condition $\pi = \pi'$ is fairly obvious. In this article, we need to state this condition explicitly because our purpose here is to formulate the precise

conditions for our system of K-spaces under which we can define the Floer cohomology in the way independent of the geometric origin of such a system of K-spaces.

Actually we need to require consistency at the corner of arbitrary codimension using the covering space

$$\pi_{m,\ell}: \widehat{S}_m(\widehat{S}_\ell X) \to \widehat{S}_{m+\ell} X,$$
 (15.10)

which exists for any K-space X with corner. (See Proposition 24.16.) If we assume the corner compatibility condition, the property (\star) and its analogue for higher codimensional corner can be shown inductively. The inductive step of this induction can be stated as follows.

Proposition 15.7. (slightly imprecise statement) Let $(X, \widehat{\mathcal{U}})$ be a K-space with corners. Suppose for each k we have a CF-pertubation $\widehat{\mathfrak{S}}_k$ on $\widehat{S}_k(X, \widehat{\mathcal{U}})$ with the following properties.

For each m and ℓ , the following two CF-perturbations on $\widehat{S}_m(\widehat{S}_{\ell}X)$ are equivalent each other.

- (1) The restriction of $\widehat{\mathfrak{S}}_{\ell}$ to $\widehat{S}_m(\widehat{S}_{\ell}X)$.
- (2) The pull-back of $\widehat{\mathfrak{S}_{m+\ell}}$ by the covering map (15.10).

Then there exists a CF-perturbation $\widehat{\mathfrak{S}}$ on $(X,\widehat{\mathcal{U}})$ such that its restriction to $\widehat{S}_k(X,\widehat{\mathcal{U}})$ coincides with $\widehat{\mathfrak{S}}_k$ for each k.

This is a simplified statement and is *not* the statement we will prove in Section 17. The statement we will prove is Proposition 17.65. The difference between Proposition 15.7 and Proposition 17.65 is the following.

- (a) The CF-perturbation we start with is not given on $\widehat{S}_k(X,\widehat{\mathcal{U}})$ itself but is given on a thickening of $\widehat{S}_k(X,\widehat{\mathcal{U}})$. The CF-perturbation we obtain is defined on a thickening of $(X,\widehat{\mathcal{U}})$.
- (b) We replace X by $X^{\coprod \tau_0}$, which is a K-space obtained from X by putting the collar outside.
- (c) We assume that $\widehat{\mathfrak{S}}_k$ satisfies an appropriate transversality property and will find a CF-perturbation $\widehat{\mathfrak{S}}$ satisfying the same transversality property.

The reason for (a) is that we need to go once to a good coordinate system and come back to construct a CF-perturbation. The reason for (b) is explained in detail in Subsection 17.1. We actually need to construct a system of CF-perturbations satisfying certain transversality properties. This is the reason for (c).

15.1.5. Well-defined-ness of Floer cohomology and morphism of linear K-system. The most important property of Floer cohomology of periodic Hamiltonian system is its invariance under the choice of Hamiltonians. Our story contains axiomatization of this equivalence. For this purpose we introduce the notion of morphisms between two linear K-systems. To explain the relevant axiom we consider the case of linear K-system arising from the periodic Hamiltonian system and the associated Floer equation. Let $H^i: S^1 \times M \to \mathbb{R}$ be a periodic Hamiltonian function for i = 1, 2. Using the set of critical points we obtain a set of manifolds $\{R^i_\alpha \mid \alpha \in \mathfrak{A}_i\}$, i = 1, 2 and we obtain a set, the compactified moduli space, $\mathcal{M}(H^i; \alpha_-, \alpha)$ of solutions of Floer's equation (15.2) for $H = H^i$, $\alpha_\pm \in \mathfrak{A}_i$. They define cochain complexes $(CF(\mathcal{C}^i; \Lambda_{0,\text{nov}}), d^i)$ and its Floer cohomologies by Theorem 15.3 for i = 1, 2.

The well established method to prove the independence of the Floer cohomology of periodic Hamiltonian system under the choice of Hamiltonian is to use the moduli space of the next equation (15.12). (This method was invented by Floer [Fl2].) We take a function $\mathcal{H}: \mathbb{R} \times S^1 \times M \to \mathbb{R}$ such that

$$\mathcal{H}(\tau, t, x) = \begin{cases} H^1(t, x) & \text{if } \tau < -C \\ H^2(t, x) & \text{if } \tau > C \end{cases}$$
 (15.11)

where C is a sufficiently large fixed number. We put $H_{\tau,t}(x)=\mathcal{H}(\tau,t,x)$ and consider the equation

$$\frac{\partial u}{\partial \tau} + J \left(\frac{\partial u}{\partial t} - X_{H_{\tau,t}}(u) \right) = 0 \tag{15.12}$$

with the asymptotic boundary condition for $\tau \to \pm \infty$ given by $R_{\alpha_-}^1$, $R_{\alpha'_+}^2$, respectively. We denote the compactified moduli space of the solution of (15.12) with this boundary condition by $\mathcal{M}(H_{\tau,t};\alpha_-,\alpha'_+)$. We will define the notion of morphism of linear K-systems in Definition 16.18 and Condition 16.16. The notion of interpolation space $\mathcal{N}(\alpha_-,\alpha'_+)$ appearing in the definition of morphisms is the axiomatization of the properties of this moduli space $\mathcal{M}(H_{\tau,t};\alpha_-,\alpha'_+)$. For example, (16.25) corresponds to the property of the boundary of $\mathcal{M}(H_{\tau,t};\alpha_-,\alpha'_+)$, that is,

$$\partial \mathcal{M}(H_{\tau,t}; \alpha_{-}, \alpha'_{+}) \cong \bigcup_{\alpha \in \mathfrak{A}_{1}} \mathcal{M}(H^{1}; \alpha_{-}, \alpha) \times_{R^{1}_{\alpha}} \mathcal{M}(H_{\tau,t}; \alpha, \alpha'_{+})$$

$$\cup \bigcup_{\alpha' \in \mathfrak{A}_{2}} \mathcal{M}(H_{\tau,t}; \alpha_{-}, \alpha') \times_{R^{2}_{\alpha'}} \mathcal{M}(H^{2}; \alpha', \alpha'_{+}). \tag{15.13}$$

Thus the set of K-spaces $\{\mathcal{M}(H_{\tau,t};\alpha_-,\alpha'_+) \mid \alpha_- \in \mathfrak{A}_1, \alpha'_+ \in \mathfrak{A}_2\}$ together with various other data defines a morphism from the linear K-system associated to H^1 to the linear K-system associated to H^2 .

In the study of Floer cohomology of periodic Hamiltonian system the moduli space $\mathcal{M}(H_{\tau,t};\alpha_-,\alpha'_+)$ is used to define a cochain map from Floer's cochain complex associated to H^1 to Floer's cochain complex associated to H^2 . We can carry out this construction by using the properties spelled out in Definition 16.18 and Condition 16.16 only and prove the next result.

Theorem 15.8. If \mathfrak{N} is a morphism from one linear K-system \mathcal{F}_1 to another linear K-system \mathcal{F}_2 , then \mathfrak{N} induces a cochain map

$$\mathfrak{N}_*: (CF(\mathcal{C}^1; \Lambda_{\mathrm{nov}}), d^1) \to (CF(\mathcal{C}^2; \Lambda_{\mathrm{nov}}), d^2).$$

Here $(CF(\mathcal{C}^i; \Lambda_{nov}), d^i)$ is the cochain complex associated to \mathcal{F}_i by Theorem 15.3. The cochain map \mathfrak{N}_* depends on various choices. However it is independent of the choices up to cochain homotopy.

Theorem 15.8 is Theorem 16.31 (1). The proof is similar to the proof of Theorem 15.3 and is given in Subsection 19.6.

We define the notion of composition of morphisms in Section 18 and show that $\mathfrak{N} \mapsto \mathfrak{N}_*$ is functorial with respect to the composition of the morphisms in Subsection 19.4. When the interpolation spaces of the morphism $\mathfrak{N}_{i+1i}: \mathcal{F}_i \to \mathcal{F}_{i+1}$ is given by $\mathcal{N}_{ii+1}(\alpha^i, \alpha^{i+1})$ for i = 1, 2, the interpolation space of the composition $\mathfrak{N}_{32} \circ \mathfrak{N}_{21}: \mathcal{F}_1 \to \mathcal{F}_3$ is the K-space $\mathcal{N}_{13}(\alpha^1, \alpha^3)$ obtained, roughly speaking, by gluing the K-spaces

$$\mathcal{N}_{12}(\alpha^1, \alpha^2) \times_{R_{\alpha_2}^2} \mathcal{N}_{23}(\alpha^2, \alpha^3) \tag{15.14}$$

for various α^2 along the boundaries and corners. We need to smooth a part of corners of this fiber product to glue. For this purpose, we need the definition of smoothing corners of K-spaces with corners. We will discuss it in Section 18. See Subsection 18.1 for an issue of smoothing corners of K-spaces. To define smoothing corners in a canonical way we use the collar (which was put outside). So more precisely we use

$$\bigcup_{\alpha_2 \in \mathfrak{A}_2} \mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R_{\alpha_2}^2}^{\boxplus \tau} \mathcal{N}_{23}(\alpha_2, \alpha_3)$$
 (15.15)

in place of (15.14). See Definition 18.37 for the definition of (15.15).

We also define the notion of homotopy and homotopy of homotopies etc. of morphisms and show that homotopy between morphisms \mathfrak{N} and \mathfrak{N}' induces a cochain homotopy between \mathfrak{N}_* and \mathfrak{N}'_* .

15.1.6. *Identity morphism*. To make the assignment $\mathfrak{N} \mapsto \mathfrak{N}_*$ functorial, we need the notion of the identity morphisms. In the second half of Section 18 we define and prove a basic property of the identity morphism of linear K-system.

In the geometric situation of the linear K-system arising from periodic Hamiltonian system, the morphism among such linear K-systems are defined by using the moduli space of the solutions of equation (15.12), where $\mathcal{H}: \mathbb{R} \times S^1 \times M \to \mathbb{R}$. To obtain the identity morphism of linear K-system associated to $H: S^1 \times M \to \mathbb{R}$, we consider the case of \mathcal{H} such that $\mathcal{H}(\tau,t,x)=H(t,x)$. In other words, we use τ independent \mathcal{H} . However, note that the moduli space of solutions of (15.12) for this τ independent \mathcal{H} is different from the moduli space of solutions of Floer's equation (15.1). Namely

$$\mathcal{M}(H_{\tau,t}; \alpha_-, \alpha_+) \neq \mathcal{M}(H; \alpha_-, \alpha_+)$$

in case $H_{\tau,t} = H_t$ for all τ . Indeed, the dimensions are different. To define $\mathcal{M}(H; \alpha_-, \alpha_+)$ we divide our space by the \mathbb{R} action given by translation on $\tau \in \mathbb{R}$ direction. Since \mathcal{H} is happen to be τ independent, our equation (15.12) is invariant under this \mathbb{R} action, too. However, by definition, $\mathcal{M}(H_{\tau,t}; \alpha_-, \alpha_+)$ is a special case of general \mathcal{H} . For general \mathcal{H} , (15.12) is not invariant under \mathbb{R} action. Before compactifiation we can identity

$$\overset{\circ}{\mathcal{M}}(H;\alpha_{-},\alpha_{+})\times\mathbb{R}=\overset{\circ}{\mathcal{M}}(H_{\tau,t};\alpha_{-},\alpha_{+}),$$

when $H_{\tau,t} = H_t$. However the relationship between compactified moduli spaces $\mathcal{M}(H; \alpha_-, \alpha_+)$ and $\mathcal{M}(H_{\tau,t}; \alpha_-, \alpha_+)$ is not so simple.

We describe in Subsection 18.9 a way to obtain $\mathcal{M}(H_{\tau,t};\alpha_-,\alpha_+)$ (the case $H_{\tau,t}$ is τ independent) from $\mathcal{M}(H;\alpha_-,\alpha_+)$, in an abstract setting. In other words, we start with the spaces of connecting orbits $\mathcal{M}(\alpha_-,\alpha_+) = \mathcal{M}(H;\alpha_-,\alpha_+)$ of a liner K-system \mathcal{F} and define the interpolation spaces of the identity morphism $\mathcal{ID}: \mathcal{F} \to \mathcal{F}$. We also show that identity morphism is a 'homotopy unit'. Namely we show in Subsection 18.9 that the composition of the identity morphism \mathcal{ID} with other morphism \mathfrak{N} is homotopic to \mathfrak{N} . (Proposition 18.63.)

To construct the identity morphism and prove its homotopy-unitality, we imitate the proof of the corresponding results in the case of periodic Hamiltonian system, and rewrite it so that it works in the purely abstract setting of linear K-system without any specific geometric origin. Although we explain the geometric origin of the construction of Subsection 18.9 in Subsection 18.10, the discussion of Subsection 18.10 is *not* used in Subsection 18.9 or any other part to prove main results of

this article. We expect that those explanation is useful for readers who know Floer cohomology of periodic Hamiltonian system to understand the contents of Subsection 18.9.

In Section 19, we use the identity morphism to prove the second half of Theorem 15.3, that is, independence of the Floer cochain complex $(CF(\mathcal{C}; \Lambda_{0,\text{nov}}), d)$ of the choices up to cochain homotopy equivalence, as follows. We first consider the case when our linear K-system is obtained from a Hamiltonian $H: S^1 \times M \to \mathbb{R}$ by using Floer's equation (15.2). We fix a choice of an almost complex structure and the Kuranishi structure on $\mathcal{M}(H; \alpha_-, \alpha_+)$. Namely we fix choices which determine a linear K-system. We then take two different systems of CF-perturbations on it, which we denote by $\widehat{\mathfrak{C}}_{\alpha_-,\alpha_+}^{i,\epsilon}$, i=1,2. We then obtain two different cochain complexes which we denote by $(CF(\mathcal{C}; \Lambda_{0,\text{nov}}), d^i)$, i=1,2. We want to prove that they are cochain homotopy equivalent.

In this case the interpolation spaces of the identity morphism are $\mathcal{M}(H_{\tau,t}; \alpha_-, \alpha_+)$ (for various α_{\pm}) with $H_{\tau,t} = H_t$ for all τ . Its boundary is described by (15.13). In our situation it becomes:

$$\partial \mathcal{M}(H_{\tau,t}; \alpha_{-}, \alpha_{+}) \cong \bigcup_{\alpha \in \mathfrak{A}} \mathcal{M}(H; \alpha_{-}, \alpha) \times_{R_{\alpha}} \mathcal{M}(H_{\tau,t}; \alpha, \alpha_{+})$$

$$\bigcup_{\alpha' \in \mathfrak{A}} \mathcal{M}(H_{\tau,t}; \alpha_{-}, \alpha') \times_{R_{\alpha'}} \mathcal{M}(H; \alpha', \alpha_{+}).$$
(15.16)

Now we consider a CF-perturbation $\widehat{\mathfrak{S}}_{\alpha_-,\alpha}^{1,\epsilon}$ on $\mathcal{M}(H;\alpha_-,\alpha)$, which is the first factor of the first term of the right hand side, and another CF-perturbation $\widehat{\mathfrak{S}}_{\alpha',\alpha_+}^{2,\epsilon}$ on $\mathcal{M}(H;\alpha',\alpha_+)$, which is the second factor of second term of the right hand side. We then take a system of CF-perturbations on various $\mathcal{M}(H_{\tau,t};\alpha_-,\alpha_+)$ so that these CF-perturbations together with $\widehat{\mathfrak{S}}_{\alpha_-,\alpha}^{1,\epsilon}$, $\widehat{\mathfrak{S}}_{\alpha',\alpha_+}^{2,\epsilon}$ are compatible with the isomorphism (15.16). (To show the existence of such a system of CF-perturbations, we need to examine all the corners of arbitrary codimension and check the compatibility at the corners. We can do so by induction using a similar argument as explained in Subsubsection 15.1.4.) Then the correspondence by $\mathcal{M}(H_{\tau,t};\alpha_-,\alpha_+)$ together with this system of CF-perturbations defines a cochain map from $(CF(\mathcal{C};\Lambda_{0,\text{nov}}),d^1)$ to $(CF(\mathcal{C};\Lambda_{0,\text{nov}}),d^2)$. This is a consequence of Stokes' formula ([Part I, Proposition 9.16]) and Composition formula ([Part I, Theorem 10.20]).

This cochain map is actually an isomorphism since it is the identity map modulo T^{ϵ} for some $\epsilon > 0$.

In the case of linear K-system, which may not come from a particular geometric construction, we can proceed in the same way using the identity morphism, to prove that $(CF(\mathcal{C}; \Lambda_{0,\text{nov}}), d^1)$ is cochain homotopy equivalent to $(CF(\mathcal{C}; \Lambda_{0,\text{nov}}), d^2)$.

15.1.7. Homotopy limit. We note that to construct a system of CF-perturbations for all the spaces of connecting orbits appearing in a linear K-system, we need to find infinitely many CF-perturbations simultaneously. There is an issue to do so. We explained this issue in detail in [FOOO4, Subsection 7.2.3]. The method to resolve it is the same as [FOOO4, Section 7.2]. The algebraic part of this method is summarized as follows. For E > 0 a pair (C, d) of a free Λ_0 module C and $d: C \to C$ is said to be a partial cochain complex of energy cut level E if $d \circ d \equiv 0$ mod T^E . Let $(C_1, d), (C_2, d)$ be partial cochain complexes of energy cut level E.

A Λ_0 module homomorphism $\varphi: C_1 \to C_2$ is said to be a partial cochain map of energy cut level E if $\varphi \circ d \equiv d \circ \varphi \mod T^{\tilde{E}}$. We also note that if (C,d) is a partial cochain complex of energy cut level E' and if E < E' then (C, d) is a partial cochain complex of energy cut level E.

Lemma 15.9. Let (C_i,d) be gapped partial cochain complex of energy cut level E_i for i=1,2 with $E_1 < E_2$. Let $\varphi: C_1 \to C_2$ be a gapped partial cochain map of energy cut level T^{E_1} . We assume $\overline{\varphi}: C_1/\Lambda_{+,nov}C_1 \to C_2/\Lambda_{+,nov}C_2$ is an isomorphism.

Then there exist $d^+: C_1 \to C_1$ and $\varphi^+: C_1 \to C_2$ such that:

- (1) (C_1, d^+) is a partial cochain complex of energy cut level E_2 .
- (2) $\varphi^+:(C_1,d^+)\to(C_2,d)$ is a partial cochain map of energy cut level E_2 .
- (3) $d^+ \equiv d \mod T^{E_1}$ and $\varphi^+ \equiv \varphi \mod T^{E_1}$.

See Definition 16.11 for the definition of gapped-ness. Lemma 15.9 is Lemma 19.13. We use Lemma 15.9 to construct the cochain complex $(CF(\mathcal{C}; \Lambda_{0,\text{nov}}), d)$ appearing in Theorem 15.3 as follows. We take $0 < E_1 < E_2 < \dots$ with $E_i \to \infty$. We use the argument outlined in Subsubsections 15.1.3 -15.1.4 using the finitely many moduli spaces (consisting of elements of energy $\langle E_i \rangle$ to construct cochain complex $(CF(\mathcal{C}; \Lambda_{0,\text{nov}}), d^i)$ modulo T^{E_i} for each i. We next use the argument outlined in Subsubsections 15.1.5 -15.1.6 to find a cochain map $\varphi_i: (CF(\mathcal{C}; \Lambda_{0,\text{nov}}), d^i) \to$ $(CF(\mathcal{C}; \Lambda_{0,\text{nov}}), d^{i+1})$ modulo T^{E_i} for each i. Now we use Lemma 15.9 inductively and to obtain $d_k^i: CF(\mathcal{C}) \to CF(\mathcal{C})$ for k > i and $\varphi_{i,k}: (CF(\mathcal{C}), d_k^i) \to$ $(CF(\mathcal{C}), d_k^{i+1})$ such that:

- (1) $(CF(\mathcal{C}), d_k^i)$ is a cochain complex modulo T^{E_k} .
- (2) φ_{i,k} is a cochain map module T^{E_k}.
 (3) dⁱ_k ≡ dⁱ_{k+1} mod T^{E_k}, φ_{i,k} ≡ φ_{i,k+1} mod T^{E_k}.

Then $\lim_{k\to\infty} d_k^1: CF(\mathcal{C}) \to CF(\mathcal{C})$ becomes the required coboundary oprator.

To construct a cochain map we use a similar argument using homotopy modulo T^E instead of cochain map modulo T^E . To construct a cochain homotopy between cochain maps, we also use a similar argument using homotopy of homotopies modulo T^E . Algebraic lemmas we use in place of Lemma 15.9 or Lemma 19.13 are Propositions 19.33 and 19.39.

- 15.1.8. Story over rational coefficient. In Section 20 we consider the case when all the spaces R_{α} are 0-dimensional and prove that we can use Novikov ring whose ground ring is \mathbb{Q} in that case. The proof is based on the results of [Part I, Sections 13 and 14].
- 15.2. Outline of the story of tree-like K-system. In Sections 21-22 we study systems of K-spaces, which axiomatize the situation appearing during the construction of the filtered A_{∞} algebra associated to a Lagrangian submanifold. ([FOOO3, FOOO4].)
- 15.2.1. Moduli space of pseudo-holomorphic disks: review. In this subsubsection, we review basic properties of the moduli space of pseudo-holomorphic disks to motivate the definitions in later subsubsections.

Let M be a symplectic manifold and L its Lagrangian submanifold. We assume that M is compact or tame (i.e., carrying a tame almost complex structure) and L is compact, oriented and relatively spin. We have the Maslov index group homomorphism $\mu: H_2(M,L;\mathbb{Z}) \to 2\mathbb{Z}$ and the energy group homomorphism $E: H_2(M,L;\mathbb{Z}) \to \mathbb{R}$ defined by

$$E(\beta) = \int_{D^2} u^* \omega$$

with $[u] = \beta$. For $\beta \in H_2(M, L; \mathbb{Z})$ we consider the moduli space $\mathcal{M}_{k+1}(\beta)^2$ consisting of $((D^2, \vec{z}), u)$ such that:

- (1) $u:(D^2,\partial D^2)\to (M,L)$ is pseudo-holomorphic.
- (2) $\vec{z} = (z_0, \dots, z_k)$ are k+1 marked points of the boundary ∂D^2 .
- (3) $z_i \neq z_j$ if $i \neq j$.
- (4) (z_0, \ldots, z_k) respects the counter clockwise cyclic order of ∂D^2 .

We define evaluation maps

$$\operatorname{ev} = (\operatorname{ev}_0, \dots, \operatorname{ev}_k) : \stackrel{\circ}{\mathcal{M}}_{k+1}(\beta) \longrightarrow L^{k+1}$$

by $\operatorname{ev}_i((D^2, \vec{z}), u) = u(z_i)$. Then $\mathcal{M}_{k+1}(\beta)$ has a compactification $\mathcal{M}_{k+1}(\beta)$ to which ev_i is extended. Moreover $\mathcal{M}_{k+1}(\beta)$ has an oriented Kuranish structure with corners of dimension

$$\dim \mathcal{M}_{k+1}(\beta) = \mu(\beta) + k - 2.$$

The normalized boundary of $\mathcal{M}_{k+1}(\beta)$ is a disjoint union of the fiber products:

$$\mathcal{M}_{k_1+1}(\beta_1) \,_{\operatorname{ev}_0} \times_{\operatorname{ev}_i} \mathcal{M}_{k_2+1}(\beta_2)$$

where $\beta_1 + \beta_2 = \beta$, $k_1 + k_2 = k + 1$, $i = 1, \dots, k_2$.

These facts are proved in [FOOO7, Subsection 7-1].

15.2.2. Axiom of tree-like K-system and main theorem constructing the filtered A_{∞} algebra. Axioms of the tree-like K-system over L or the A_{∞} correspondence over L are given as Conditions 21.7 and Definition 21.9 and is obtained by axiomatizing the properties of the system of the moduli spaces $\mathcal{M}_{k+1}(\beta)$ and the evaluation maps ev_i , which are described in the previous subsubsection. The way to axiomatize various structures are parallel to the case of linear K-system and we do not repeat it. The main result we obtain is the next theorem: For a closed oriented manifold L we denote by $\Omega(L)$ the de Rham complex of L. We put

$$\Omega(L;\Lambda_0) = \Omega(L)\widehat{\otimes}\Lambda_0.$$

See Definition 16.10 for the coefficient ring Λ_0 . Here $\widehat{\otimes}$ denotes the completion of the algebraic tensor product. Namely, an element of $\Omega(L; \Lambda_0)$ is a formal sum

$$\sum_{i=0}^{\infty} T^{\lambda_i} h_i$$

where $\lambda_i \in \mathbb{R}_{>0}$ with $\lambda_1 < \lambda_2 < \dots$, $\lim_{i \to \infty} \lambda_i = +\infty$ and $h_i \in \Omega(L)$.

Theorem 15.10. Suppose $(\mathcal{M}_{k+1}(\beta), \text{ev}, \mu, E)$ is a tree-like K-system over L. Then we can associate a filtered A_{∞} structure $\{\mathfrak{m}_k \mid k=0,1,2\ldots\}$ on $\Omega(L;\Lambda_0)$.

The filtered A_{∞} algebra $(\Omega(L; \Lambda_0), \{\mathfrak{m}_k \mid k = 0, 1, 2...\})$ is independent of the various choices up to homotopy equivalence.

²Though it is better to write it as $\mathcal{M}_{k+1}(L;\beta)$, we omit L for the simplicity of notation.

This is the de Rham version of the half of [FOOO3, Theorem A]. ([FOOO3, Theorem A] also contains the part of constructing tree-like K-system arising from a geometric situation described in the previous subsubsection.) It is a consequence of Theorem 21.35 (1) by putting

$$\mathfrak{m}_k = \sum_{\beta} T^{E(\beta)} \mathfrak{m}_{k,\beta}. \tag{15.17}$$

We recall that the filtered A_{∞} structure assigns maps

$$\mathfrak{m}_{k}: \underbrace{\Omega(L; \Lambda_{0})[1] \widehat{\otimes} \dots \widehat{\otimes} \Omega(L; \Lambda_{0})[1]}_{k \text{ times}} \to \Omega(L; \Lambda_{0})[1]$$
 (15.18)

 $k = 0, 1, 2, \ldots$, that satisfy the A_{∞} relations

$$\sum_{k_1+k_2=k+1} \sum_{i=1}^{k-k_2+1} (-1)^* \mathfrak{m}_{k_1}(x_1, \dots, \mathfrak{m}_{k_2}(x_i, \dots, x_{i+k_2-1}), \dots, x_k) = 0.$$
 (15.19)

Here [1] is the degree +1 shift functor. When we define \mathfrak{m}_k by (15.17), the formula (15.19) follows from (21.25). See [FOOO3, Definition 3.2.20] for the definition of filtered A_{∞} algebra and [FOOO3, Definition 4.2.42] for the definition of homotopy equivalence of filtered A_{∞} algebras. Roughly speaking, the A_{∞} operation

$$\mathfrak{m}_{k,\beta} : \underbrace{\Omega(L)[1] \otimes \cdots \otimes \Omega(L)[1]}_{k \text{ times}} \longrightarrow \Omega(L)[1]$$

is defined by

$$\mathfrak{m}_{k,\beta}(h_1,\ldots,h_k) = \operatorname{ev_0}!(\operatorname{ev}_1^* h_1 \wedge \cdots \wedge \operatorname{ev}_k^* h_k)$$
(15.20)

using the correspondence

$$\mathcal{M}_{k+1}(\beta) \tag{15.21}$$

$$(\text{ev}_1, \dots, \text{ev}_k) \qquad \text{ev}_0$$

See (22.13) for the precise definition.³ The integration along the fiber ev₀! in the formula (15.20) is defined by using an appropriate system of CF-perturbations $\widehat{\mathfrak{S}}_{k+1}(\beta)$ on the K-space $\mathcal{M}_{k+1}(\beta)$, which is the main part of the data defining tree-like K-system.

The formula (15.19) (or (21.25)) is obtained from Stokes' formula ([Part I, Theorem 9.26]) and Composition formula ([Part I, Theorem 10.20]) via the isomorphism

$$\partial \mathcal{M}_{k+1}(\beta) = \bigcup_{k_1 + k_2 = k} \bigcup_{i=1,\dots,k_2} \bigcup_{\beta_1 + \beta_2 = \beta} \mathcal{M}_{k_1 + 1}(\beta_1) \,_{\text{ev}_0} \times_{\text{ev}_i} \mathcal{M}_{k_2 + 1}(\beta_2), \quad (15.22)$$

which is a part of the axiom of a tree-like K-system, Condition 21.7 (IX).

For this argument to work, we need to choose a system of CF-perturbations $\widehat{\mathfrak{S}}_{k+1}(\beta)$ so that it is compatible with the isomorphism (15.22). Proposition 22.3 is the precise statement which claims the existence of such a system.

Construction of such a system of CF-perturbations is parallel to that of a linear K-system. We uses an induction over k and $E(\beta)$ to construct $\widehat{\mathfrak{S}}_{k+1}(\beta)$. The

 $^{^3}$ Strictly speaking, in (22.13) we define a partial A_{∞} algebra structure (see Definition 21.22) which depends on a parameter $\epsilon > 0$. We then use a 'homotopy limit' in the way similar to that of the construction of A_{∞} algebra explained in Subsubsection 15.1.7.

inductive step of this construction uses Proposition 15.7 (or its precise version Proposition 17.65.) To inductively verify the assumptions of Proposition 15.7 we need to construct our system $\widehat{\mathfrak{S}}_{k+1}(\beta)$ to be compatible not only along the boundary but also at the corners. Therefore we need to assume the compatibility of Kuranishi structures on $\mathcal{M}_{k+1}(\beta)$ at the corners. This is the corner compatible conditions Condition 21.7 (X) and (XI).

15.2.3. Bifurcation method and pseudo-isotopy. As explained in the previous subsubsection, the construction of a filtered A_{∞} structure from the tree-like K-system given in this article is mostly similar to the construction of Floer's cochain complex from a linear K-system.

The difference between two constructions lies in the morphism part of the construction. In the case of linear K-system we defined a morphism between two such K-systems and associated to the morphism a cochain map between their Floer's cochain complexes. (In particular, using the identity morphism we proved independence of the resulting cochain complex under the various choices we make, modulo cochain homotopy equivalence.) In the situation of tree-like K-system we define the notion of pseudo-isotopy between two tree-like K-systems and of filtered A_{∞} algebras. Then we show that the resulting pseudo-isotopy between two tree-like K-systems induces a pseudo-isotopy between the A_{∞} algebras. It is easy to show that two filtered A_{∞} algebras are homotopy equivalent if they are pseudo-isotopic. (See [Fu2, Theorem 8.2].)

In our geometric situation of Lagrangian Floer theory, a pseudo-isotopy of the tree-like K-system is obtained as follows. In the situation of Subsubsection 15.2.1, we consider two compatible almost complex structures J_1, J_2 on M. Then we obtain the moduli spaces of pseudo-holomorhic disks $\mathcal{M}_{k+1}(\beta; J_i)$ for i = 1, 2. For each i = 1, 2 we fix some choices to define a system of Kuranishi structures on $\mathcal{M}_{k+1}(\beta; J_i)$ so that it defines a tree-like K-system. We denote these choices by Ξ_i and the K-space obtained via these choices $\mathcal{M}_{k+1}(\beta; J_i; \Xi_i)$.

Now we consider a one parameter family of compatible almost complex structures $\{J_t \mid t \in [1,2]\}$ which joins J_1 to J_2 . We consider the moduli space

$$\mathcal{M}_{k+1}(\beta; [1, 2]) = \bigcup_{t \in [1, 2]} \mathcal{M}_{k+1}(\beta; J_t) \times \{t\}.$$
 (15.23)

Here $\mathcal{M}_{k+1}(\beta; J_t)$ is the moduli space of J_t holomorphic discs with boundary condition L, homology class β , and k+1 marked points. We can find a system of Kuranishi structures on it such that its restriction to the part t=1 (resp. t=2) coincides with Ξ_1 (resp. Ξ_2 .) We have the evaluation maps

$$ev = (ev_0, ..., ev_k) : \mathcal{M}_{k+1}(\beta; [1, 2]) \to L^{k+1}$$

and

$$ev_{[1,2]}: \mathcal{M}_{k+1}(\beta; [1,2]) \to [1,2].$$

We axiomatize the properties of the system consisting of $\mathcal{M}_{k+1}(\beta; [1,2])$, the evaluation maps, etc.. and define the notion of [1,2]-parametrized family of A_{∞} correspondences. (See Condition 21.11 and Definition 21.15. We define more general notion of P-parametrized A_{∞} correspondence in Definition 21.13.)

We now make choices of CF-perturbations etc. on $\mathcal{M}_{k+1}(\beta; J_i; \Xi_i)$ i=1,2 and we use them to construct the filtered A_{∞} structures.

In the same way as in (15.20) we use the evaluation maps ev and $ev_{[1,2]}$ together with our CF-perturbations to define operators

$$\mathfrak{m}_{k,\beta}: \Omega(L\times[1,2])^{\otimes k} \to \Omega(L\times[1,2])$$

Then $\mathfrak{m}_k = \sum_{k,\beta} T^{E(\beta)} \mathfrak{m}_{k,\beta}$ satisfies the A_{∞} relation (15.19). The system of operators $\mathfrak{m}_{k,\beta}$ on $\Omega(L \times [1,2])$ which satisfies the A_{∞} relation and some additional properties is called a *pseudo-isotopy of filtered* A_{∞} *algebras*. See Definition 21.26.

Thus we will prove the following:

Theorem 15.11. A pseudo-isotopy of A_{∞} correspondences induce a pseudo-isotopy of filtered A_{∞} algebras.

Theorem 15.11 is Theorem 21.35 (3).

We note that Theorem 15.11 implies the second half of Theorem 15.10 as follows. Let $(\mathcal{M}_{k+1}(\beta), \text{ev}, \mu, E)$ be an A_{∞} correspondence. We can define a pseudo-isotopy of this A_{∞} correspondence with itself by taking

$$\mathcal{M}_{k+1}(\beta; [1,2]) = \mathcal{M}_{k+1}(\beta) \times [1,2]$$

etc.. Then we apply Theorem 15.11 to show that the filtered A_{∞} algebras obtained by two different CF-perturbations from $(\mathcal{M}_{k+1}(\beta), \text{ev}, \mu, E)$ are pseudo-isotopic to each other.

Remark 15.12. Here we use the construction of a pseudo-isotopy from an A_{∞} correspondence to itself for the construction of a pseudo-isotopy of A_{∞} algebras in the way similar as we use the identity morphism in Subsubsection 15.1.6 for the construction of a cochain map between Floer cochain complexes. We like to mention that the construction of a pseudo-isotopy from an A_{∞} correspondence to itself is much easier than the construction of the identity morphism.

For the actual proof of Theorems 15.10 and 15.11 we need to use 'homotopy limit' argument similar to those in Subsubsection 15.1.7. We need the notion of pseudo-isotopy of pseudo-isotopies etc. for this purpose. The algebraic lemma corresponding to Lemma 15.9 is Propositions 22.9 and 22.14.

15.2.4. Bifurcation method and self-gluing. In this article we use morphism of K-systems to prove independence of the Floer's cochain complex associated to a given linear K-system of the choices. On the other hand, we use pseudo-isotopy to prove independence of the filtered A_{∞} structure associated to a tree-like K-system of the choices. Actually we can also use morphism for the tree-like K-system and pseudo-isotopy for the linear K-system. We use two different methods in order to demonstrate both of these two methods. We may call 'cobordism method' instead of 'the method using morphism', and 'bifurcation method' instead of 'the method using pseudo-isotopy'. The difference of those two methods is explained also in [FOOO4, Subsection 7.2.14].

The cobordism method is used in the Lagrangian Floer theory in [FOOO3]. An axiomatization of morphism of tree-like K-system is given in [Fu3]. The bifurcation method is used in Lagrangian Floer theory in, for example, [AFOOO, AJ, Fu2]. Each of these two methods has certain advantage and disadvantage.

One advantage of the bifurcation method is that usually it is shorter and simpler to use the bifurcation method than the cobordism method. See for example, Remark 15.12.

On the other hand, we cannot prove independence of Floer cohomology of periodic Hamiltonian system under the change of Hamiltonian function, by bifurcation method. This is because we need to study the situation where the sets of critical points are different.

Micheal Hutchings [H2]⁴ and Paul Seidel [Se, Remark 10.14] mentioned some issue to prove invariance of Floer cohomology by using the bifurcation method. We explain below how those issue was resolved in our previous writings.

We discuss the case of Morse-Novikov cohomology. Let M be a compact Riemannian manifold and h a closed 1-form given by the exterior derivative of a Morse function locally. Let R(h) be the set of the critical points of h. For $p,q \in R(h)$ we consider the compactified moduli space of gradient lines of h joining p to q and denote it by $\mathcal{M}(h;p,q)$. (We identify two gradient lines ℓ,ℓ' as an element of $\mathcal{M}(h;p,q)$ if $\ell(\tau)=\ell'(\tau+\tau_0)$ for some $\tau_0\in\mathbb{R}$.) We take its subset $\mathcal{M}(h;p,q;E)$ such that an integration of ℓ along the gradient line is ℓ . The matrix element of the coboundary oprator of Morse-Novikov complex is the sum of signed counts of the order of $\mathcal{M}(h;p,q;E)$ together with the weight T^E . (We assume that the gradient vector field of ℓ is Morse-Smale.) The proof that it defines a cochain complex is the same as the case of Morse complex, which is similar to one we explained in Subsubsection 15.1.5, as was observed by Novikov.

The issue is the way how to prove the independence of the cohomology of the cochain complex associated to h when we move h. Suppose we have two closed 1-forms h and h' whose de Rham cohomology classes in $H^1(M)$ coincide. For simplicity we assume R(h) = R(h'). We take a one parameter family h_t such that $R(h) = R(h_t)$ and $h_0 = h$, $h_1 = h'$. We consider

$$\mathcal{M}(h_*; p, q; E) = \bigcup_{t \in [0, 1]} \mathcal{M}(h_t; p, q; E) \times \{t\}$$
 (15.24)

and try to use it to show this independence. ((15.24) is similar to (15.23). So the method we explain below is a bifurcation method.)

Note that the virtual dimension of $\mathcal{M}(h; p, p; E)$ is -1. Therefore the virtual dimension of $\mathcal{M}(h_*; p, p; E)$ is 0. So there may be a discrete subset $\{t_i\} \subset [0, 1]$ where $\mathcal{M}(h_t; p, p; E)$ is nonempty. Now since $\mathcal{M}(h_{t_1}; p, p; E)$ is nonempty, $\mathcal{M}(h_{t_1}; p, p; 2E)$ contains an object obtained by concatenating an element of $\mathcal{M}(h_{t_1}; p, p; E)$ with itself. (In other words if $[\ell] \in \mathcal{M}(h_{t_1}; p, p; E)$ then

$$([\ell], [\ell]) \in \mathcal{M}(h_{t_1}; p, p; E) \times \mathcal{M}(h_{t_1}; p, p; E) \subset \mathcal{M}(h_{t_1}; p, p; 2E).)$$

By continuing this process we obtain an element of $\mathcal{M}(h_{t_1}; p, p; kE)$ for any k. So we have an element of the strata of arbitrary negative dimension. We explain three different ways to resolve this issue.

1. Instead of the moduli spaces (15.24) we can use a different moduli space below. We take a non-decreasing function $\chi: \mathbb{R} \to [0,1]$ such that $\chi(\tau) = 0$ for small τ and $\chi(\tau) = 1$ for large τ . We consider the 'nonautonomous' equation

$$\frac{d\ell}{d\tau}(\tau) = \operatorname{grad} h_{\chi(\tau)} \tag{15.25}$$

⁴Actually Hutchings' concern is not so much on the proof of independence of Morse-Novikov cohomology of the choices but rather the explicit form of the cochain homotopy equivalence, between two Morse-Novikov complexes before and after wall crossing. The discussion below clarifies the way to prove the independence of Morse-Novikov cohomology, but to find the explicit form of of the cochain homotopy equivalence we need to study more. See [H1].

such that $\ell(+\infty) = q$ and $\ell(-\infty) = p$. Let $\mathcal{M}(h_{\chi(\tau)}; p, q; E)$ be the set of solutions of this equation with energy E. Contrary to the definition of coboundary oprator there is no translational symmetry. By counting the order of $\mathcal{M}(h_{\chi(\tau)}; p, q; E)$ we obtain a cochain map from the Morse-Novikov complex of h to one of h'. The standard argument shows that it becomes a cochain homotopy equivalence. [Fl2].

This is the standard approach to show the well-definedness of Morse-Novikov cohomology. We note that there is no self-gluing issue in this approach since the equation (15.25) is not invariant under this self gluing construction.

In other words, when using the cobordism method the issue of self gluing never occurs.

2. We next explain how the usage of the bifurcation method together with the de Rham model resolves the issue of 'self-gluing'.

We consider the moduli space $\mathcal{M}(h_*; p, q; E)$ in (15.24). We define 'evaluation maps' to the interval [0, 1]. Namely we send $\mathcal{M}(h_t; p, q; E) \times \{t\}$ to $t \in [0, 1]$. We usually consider the situation where both projections pr_s and pr_t exist, where the former is the source projection and the latter is the target projection. In our setting they are both the same map defined above. We have a diagram:

$$[0,1] \xleftarrow{\operatorname{pr}_s} \mathcal{M}(h_*; p, q; E) \xrightarrow{\operatorname{pr}_t} [0,1]$$

The 'pseudo-isotopy' of Morse-Novikov complex is a cochain complex defined on

$$\left(\bigoplus_{p\in R(h)}\Omega([0,1])\otimes [p]\right)\widehat{\otimes}\Lambda_0$$

(where $\Omega([0,1])$ is the de Rham complex of the interval) or its completion by using the Novikov ring. We take the interval for each $p \in R(h)$ and denote it by $[0,1]_p$. So 'pseudo-isotopy' of Morse-Novikov complex is defined on

$$CF(h_*) = \Omega\left(\coprod_{p \in R(h)} [0, 1]_p\right) \widehat{\otimes} \Lambda_0.$$
 (15.26)

The above diagram is regarded as

$$[0,1]_p \stackrel{\operatorname{pr}_s}{\longleftarrow} \mathcal{M}(h_*;p,q;E) \stackrel{\operatorname{pr}_t}{\longrightarrow} [0,1]_q.$$
 (15.27)

It 'defines' a map $d_{p,q;E}: \Omega([0,1]_p) \to \Omega([0,1]_q)$ by

$$d_{p,q:E}(u) = (pr_t)!(pr_s^*(u)). \tag{15.28}$$

Note that the pull back of differential form is defined under rather mild assumption. However the push out or integration along the fiber $(pr_t)!$ is harder to define. This point is indeed related to the self-gluing issue as follows.

Suppose $\mathcal{M}(h_*; p, p; E)$ is transversal. It consists of finitely many points. Let us assume that it consists of a single point \mathfrak{p} and $t_0 = \operatorname{pr}_s(\mathfrak{p}) = \operatorname{pr}_t(\mathfrak{p})$. We take $1 \in \Omega^0([0, 1])$. Then

$$d_{p,p;E}(1) = (pr_t)!(pr_s^*(1))$$

is the delta form $\delta_{t_0}dt$ supported at t_0 . Now we remark that the pull back

$$\operatorname{pr}_{s}^{*}(\delta_{t_{0}}dt)$$

is not defined. The standard condition for distribution to be pulled back is *not* satisfied in this case. This point is related to the fact the fiber product

$$\mathcal{M}(h_*; p, p; E)_{\mathrm{pr}_*} \times_{\mathrm{pr}_*} \mathcal{M}(h_*; p, p; E)$$

is not transversal.

This discussion clarifies that the reason why the problem of self-gluing occurs lies in the fact that p_t is not a submersion. We can not expect submersivity because of the dimensional reason. We note that we are somehow in the Bott-Morse situation here even in the case when the set of critical points of h_t is a discrete set for each h_t , in case we study a one parameter family of Morse forms. Therefore the way to resolve this issue is similar to the way to study the Bott-Morse situation. This issue can be taken care of both in de Rham and singular homology models. Let us first explain the case of de Rham model.

The problem here is that pr_t is not a submersion. The solution to this problem is to use a CF-perturbation. As a simplified version of the CF perturbation, we take a family of perturbations (globally) parameterized by a finite dimensional space, say W. For $w \in W$ we have perturbed moduli space $\mathcal{M}(h_*; p, p; E; w)$. We put

$$\mathcal{M}(h_*; p, p; E; W) = \bigcup_{w \in W} \mathcal{M}(h_*; p, p; E; w) \times \{w\}.$$

By taking the dimension of W sufficiently large we may assume that the map

$$\operatorname{pr}_{t}: \mathcal{M}(h_{*}; p, p; E; W) \to [0, 1]$$
 (15.29)

is a submersion. (The space W depends on p, E.) Let $\operatorname{pr}_W: \mathcal{M}(h_*; p, p; E; W) \to W$ be the projection. We take a differential form χ_W of degree $\dim W$ and with compact support such that $\int_W \chi_W = 1$. Now we define

$$d_{p,p;E}(u) = (\operatorname{pr}_t)_!(\operatorname{pr}_s^*(u) \wedge \operatorname{pr}_W^* \chi_W).$$

Since (15.29) is a submersion this is always well defined. In this way we can define a cochain complex on (15.26) by

$$\delta = d + \sum T^E d_{p,p;E}$$

where T is a formal parameter. (Novikov parameter.) We can also show $\delta \circ \delta = 0$. We use $(CF(h_*), \delta)$ to prove that CF(h) is cochain homotopy equivalent to CF(h') as follows.

By considering embeddings $\{0\} \rightarrow [0,1]$ and $\{1\} \rightarrow [0,1]$ we have a map

$$CF(h_*) \to CF(h), \qquad CF(h_*) \to CF(h').$$

We can show that they are cochain maps. Moreover using the fact that de Rham cohomology of [0,1] is \mathbb{R} we can show that they are cochain homotopy equivalence. Thus we find that CF(h) is cochain homotopic to CF(h'). This is a baby version of the proof of independence of filtered A_{∞} structure of the almost complex structure etc. using the pseudo-isotopy, which we present in Sections 21-22.

In this formulation, we have an equality

$$d_{p,p;E_1} \circ d_{p,p;E_2} = 0.$$

This is because $d_{p,p;E}$ increase degree of differential form by 1 and de Rham complex of [0,1] has elements in only 0-th and 1-st degree. So self-gluing problem does not occur.

3. We finally prove the independence of Morse-Novikov homology using the singular homology model by the bifurcation method.

Let P be a smooth singular chain of $[0,1]_p$. That is, it is a pair of a simplex and a smooth map from it to $[0,1]_p$.

The analogue of (15.28) in singular homology is as follows. We take fiber product

$$P \times_{\operatorname{pr}_{s}} \mathcal{M}(h_{*}; p, q; E) \tag{15.30}$$

and take its triangulation. Using the map pr_t we regard it as an element of singular chain complex of $[0,1]_q$. So we consider

$$CF(h_*)^s = \bigoplus_{p \in R(h)} S([0,1]_p) \widehat{\otimes} \Lambda_0.$$

Here $S([0,1]_p)$ is the smooth singular chain complex of the interval $[0,1]_p$. By (15.30), we 'obtain'

$$d_{p,q;E}: S([0,1]_p) \to S([0,1]_q).$$

and boundary operator on $CF(h_*)^s$.

The issue is the fiber product (15.30) may not be transversal. Also there is no way to perturb $\mathcal{M}(h_*; p, q; E)$ so that (15.30) is transversal for all P.

The idea to resolve this issue, which appeared in [FOOO4, Proposition 7.2.35 and etc.], is the following: We first take fiber product (15.30) and then perturb it. In other words, our perturbation depends not only on $\mathcal{M}(h_*; p, q; E)$ but also on the singular chain P. ⁵

Since we can make pr_s a submersion on each Kuranishi chart, (this is the definition of weak submersivity!), we can consider the fiber product (15.30) which carries a Kuranishi structure. Then we define

$$\mathcal{M}(h_*; p, q; P) := P \times_{\operatorname{pr}_s} \mathcal{M}(h_*; p, q; E),$$

which is a space with Kuranishi structure. We take a system of perturbations of all of them and triangulations of their zero sets such that the following holds.

- (1) (See [FOOO4, Compatibility Condition 7.2.38].) On $\mathcal{M}(h_*; p, q; \partial P) \subset \partial \mathcal{M}(h_*; p, q; P)$ the perturbations and triangulations are compatible.
- (2) (See [FOOO4, Compatibility Condition 7.2.44].) On

$$P \times_{\operatorname{pr}_{s}} \mathcal{M}(h_{*}; p, r; E_{1}) \operatorname{pr}_{t} \times_{\operatorname{pr}_{s}} \mathcal{M}(h_{*}; r, q; E_{2})$$

$$\subset \partial \mathcal{M}(h_{*}; p, q; E_{1} + E_{2}; P)$$
(15.31)

the perturbations and triangulations are compatible.

The meaning of (1) is clear. Let us explain the meaning of (2). We consider the fiber product $\mathcal{M}(h_*; p, r; P) = P \times_{\operatorname{pr}_s} \mathcal{M}(h_*; p, r; E)$. The perturbation (multisection) and a triangulation of the perturbed space (the zero set of the multisection) are given for this space. We regard the triangulated space of the perturbed moduli space as a singular chain $\sum Q_i$ of $[0, 1]_r$. Then the left hand side is the union of

$$Q_i \times_{\operatorname{pr}_s} \mathcal{M}(h_*; r, q; E) = \mathcal{M}(h_*; r, q; E; Q_i).$$

⁵The way we explain below is a slightly improved version of the one appeared in [FOO013]. In [FOO04] we took a countably generated subcomplex of smooth singular chain complex. (We use Baire's category theorem uncountably many times in [FOO013] then we do not need to take countably generated subcomplex as we did in [FOO04].) Here we take singular chain complex itself, that is the way of [FOO013].

The perturbation and a triangulation of its zero set are also given. We require that the restriction of the perturbation of $\mathcal{M}(h_*; p, q; E; P)$ and the triangulation of its zero set coincide with the ones which are combination of $\mathcal{M}(h_*; r, q; E; Q_i)$ and of $\mathcal{M}(h_*; p, r; P)$.

Let us elaborate the last point more. At each point in (15.31) the obstruction bundle is a direct sum of ones of $\mathcal{M}(h_*; p, r; E_1)$ and of $\mathcal{M}(h_*; r, q; E_2)$. We require that the first component of the perturbation is one for $\mathcal{M}(h_*; p, r; P)$ and the second component of the perturbation is one for $\mathcal{M}(h_*; r, q; E; Q_i)$.

This is the meaning of the compatibility (2). Construction of the perturbation and a triangulation satisfying (1)(2) are give by an induction over E and dim P. This is the way of obtaining $d_{p,q;E}: S([0,1]_p) \to S([0,1]_q)$ and the way taken in [FOOO4].

We elaborate this construction a bit more explicitly and show how it resolves the issue of self-gluing. We consider the case of $\mathcal{M}(h_*; p, p; E)$ that is zero dimensional. Suppose for simplicity that it consists of one point and its t coordinate is t_0 . We consider a 0-chain $P(t_1) = \{t_1\} \in [0,1]_p$. If $t_1 \neq t_0$ then $P(t_1) \times_{\operatorname{pr}_s} \mathcal{M}(h_*; p, p; E)$ is transversal and is the empty set. If $t_1 = t_0$ then $P(t_0) \times_{\operatorname{pr}_s} \mathcal{M}(h_*; p, p; E)$ is not transversal and we need to perturb it. After perturbation it becomes empty again.

Next we consider a 1-chain $P(a,b) = [a,b] \subset [0,1]_p$. If $a,b \neq t_0$, then $P(a,b) \times_{\operatorname{pr}_s} \mathcal{M}(h_*;p,p;E)$ is transversal. It is an empty set if $t_0 \notin [a,b]$ and is one point if $t_0 \in [a,b]$. If $a=t_0$, then $P(t_0,b) \times_{\operatorname{pr}_s} \mathcal{M}(h_*;p,p;E)$ is not transversal. We already fixed perturbation of $P(t_0) \times_{\operatorname{pr}_s} \mathcal{M}(h_*;p,p;E) \subset \partial P(t_0,b) \times_{\operatorname{pr}_s} \mathcal{M}(h_*;p,p;E)$. We extend it to obtain a perturbation of $P(t_0,b) \times_{\operatorname{pr}_s} \mathcal{M}(h_*;p,p;E)$. Whether it becomes an empty set or a one-point set depends on the choice of the perturbation of $P(t_0) \times_{\operatorname{pr}_s} \mathcal{M}(h_*;p,p;E)$.

Now we consider the self-gluing. We take the fiber product

$$[0,1]_p \times_{\operatorname{pr}_s} \mathcal{M}(h_*; p, p; E)_{\operatorname{pr}_t} \times_{\operatorname{pr}_s} \mathcal{M}(h_*; p, p; E). \tag{15.32}$$

We do not perturb $[0,1]_p \times_{\operatorname{pr}_s} \mathcal{M}(h_*;p,p;E)$ and this consists of a single point which is mapped to t_0 by pr_t . So the second fiber product is not transversal. However we have already fixed the perturbation of $P(t_0) \times_{\operatorname{pr}_s} \mathcal{M}(h_*;p,p;E)$ and by this perturbation (15.32) becomes the empty set. In other words, by this perturbation the perturbed zero set does *not* hit the corner. This is the way how the self-gluing issue is resolved. This argument is a version of the way we handled the Bott-Morse situation in [FOOO4].

Note that Akaho-Joyce [AJ] used the bifurcation method to show the well-defined-ness of the A_{∞} structure using the singular homology. We think the way they adopt is basically the same as we described here.

15.3. Outline of the appendices.

15.3.1. Orbifolds and covering space of orbifolds/K-spaces. Section 23 is a review of the notion of orbifolds and vector bundles on them. We consider effective orbifolds only and use embeddings only as morphisms. In this way we can avoid several delicate issues arising in the discussion of orbifolds. If we go beyond those cases, we need to work with the framework of the 2-category to have a proper notion of morphisms. We also use the language of chart and coordinate transformation, which is closer to the standard definition of manifold. It is well-know that there is an alternative way using the language of groupoid. (See for example [ALR].) Using the groupoid language is somewhat similar to the way taken in algebraic geometry to

define the notion of stacks. One advantage of using the groupoid language is that the discussion then becomes closer to the 'coordinate free' exposition. We remark that in our definition of Kuranishi structure using the coordinates and the coordinate transformations is inevitable. No 'coordinate free' definition of Kuranishi structure is known. So we think that the coordinate description of orbifolds is more natural for the study of Kuranishi structure. In other approach to Kuranishi-like structure such as Joyce's, which is closer to that of algebraic geometry, the groupoid description of orbifolds seems to be more natural.

To define the bundle extension data (See [Part I, Definition 12.24]) which we used in [Part I, Sections 12 and 13] we use some basic results of vector bundle in its orbifold version. They are well known and have been established long time ago. Since its proof is rather a straight forward modification of the case of manifolds, it seems that it is hard to find a reference which proves them in the literature. We provide the proof of those facts (Lemma 23.38, Corollary 23.40, Propositions 23.43 and 23.49, etc.) by this reason.

In Section 24 we discuss the covering space of an orbifold and a K-space, and define the covering space

$$\widehat{S}_m(\widehat{S}_\ell X) \to \widehat{S}_{m+\ell} X \tag{15.33}$$

for the formulation of the corner compatibility condition. See Subsubsection 15.1.4. We define the notion of covering space of orbifolds in Subsections 24.1 and generalize it to the case of K-spaces in Subsection 24.1. Then the covering space (15.33) is defined in Subsection 24.3.

15.3.2. Admissibility of orbifolds and of Kuranishi structures. In Section 25, we discuss the notion of admissible orbifolds and admissible Kuranishi structures. Admissibility we study here is the property of the coordinate change etc. with respect to the coordinate normal to the boundary or corner. Admissibility is used in the discussion of Section 17 to put the collar 'outside'. We explain how it is used there briefly below.

We consider the case of an n dimensional manifold X with boundary ∂X . (For the simplicity of exposition we assume X has a boundary but no corner.) Let $p \in \partial X$. We take its coordinate chart and so we have a diffeomorphism ψ_p from $\overline{V}_p \times [0,1)$ to a neighborhood U_p of p in X. Here \overline{V}_p is an open subset of \mathbb{R}^{n-1} . The space $X^{\boxplus 1}$ is obtained by taking $\overline{V}_p \times [-1,1)$ for each p. We glue them as follows. Let $q \in \partial X$ and we take ψ_q , $\overline{V}_q \times [0,1)$, U_q as above. Let $V_{pq} = \psi_q^{-1}(U_p)$ which is an open subset of $\overline{V}_q \times [0,1)$. The coordinate change is

$$\varphi_{pq} = \psi_p^{-1} \circ \psi_q : V_{pq} \to \overline{V}_p \times [0,1).$$

We extend it to

$$\varphi_{pq}^{\boxplus 1}: V_{pq}^{\boxplus 1} \to \overline{V}_p \times [0,1)$$

as follows. We put $\overline{V}_{pq} = V_{pq} \cap (\overline{V}_q \times \{0\})$. The restriction of φ_{pq} to \overline{V}_{pq} defines a map $\overline{\varphi}_{pq} : \overline{V}_{pq} \to \overline{V}_p$. (Here we identify $\overline{V}_p = \overline{V}_p \times \{0\}$.) We put

$$V_{pq}^{\boxplus 1} = V_{pq} \cup (\overline{V}_{pq} \times [-1, 0])$$

where we glue two spaces in the right hand side at $\overline{V}_{pq} \times \{0\}$. The map $\varphi_{pq}^{\boxplus 1}$ is defined by

$$\varphi_{pq}^{\boxplus 1}(x,t) = \begin{cases} \varphi_{pq}(x,t) & \text{if } (x,t) \in V_{pq}, \\ (\overline{\varphi}_{pq}(x),t) & \text{if } (x,t) \in \overline{V}_{pq} \times [-1,0]. \end{cases}$$

It is easy to see that these maps $\varphi_{pq}^{\boxplus 1}$ satisfy an appropriate cocycle condition. Then we can glue various charts by these maps $\varphi_{pq}^{\boxplus 1}$ to obtain a space $X^{\boxplus 1}$. It is also clear that $X^{\boxplus 1}$ is a topological manifold since $\varphi_{pq}^{\boxplus 1}$ is a homeomorphism to its image. However in general, $\varphi_{pq}^{\boxplus 1}$ is not differentiable at $\overline{V}_{pq} \times \{0\}$. So in general there

is no obvious smooth structure on $X^{\boxplus 1}$.

We introduce the notion of the admissibility of manifolds (orbifolds, K-spaces) so that if we start from an adimissible manifold then the coordinate change $\varphi_{pq}^{\boxplus 1}$ becomes smooth. Roughly speaking, the admissibility means that we are given distinguished system of coordinates so that the coordinate change φ_{pq} among those coordinates has the following additional properties.

(*) We put
$$\varphi_{pq}(x,t) = (y(x,t),s(x,t))$$
 then

$$s(x,t) - t, \qquad \frac{\partial}{\partial t} y(x,t)$$

together with all of their derivatives go to zero as $t \to 0$.

(More precisely, we assume certain exponential decay.) It is easy to see that (*) implies that $\varphi_{pq}^{\boxplus 1}$ is smooth. So $X^{\boxplus 1}$ becomes a smooth manifold in case X is an admissible manifold and we use admissible coordinate to define $X^{\boxplus 1}$.

We can generalize the admissibility to the case of orbifold with corners. See Definitions 25.11 and 25.13. Then we can define admissibility of various notions on an admissbile orbifold. For example, admissibility of a vector bundle, a section of it, and a smooth map to another manifold (without boundary). We can also define admissibility of an embedding of an admissible orbifold to another admissible orbifold. We can use them to define admissibility of Kuranishi charts and coordinate changes. Then we can define the notion of admissible Kuranishi structure. (Definition 25.36.) On an admissible Kuranishi structure we can define the notion of admissible CF-perturbation. It is mostly obvious that the story of Part 1 can be worked out in the admissible category. One slightly nontrivial point to check is the existence of bundle extension data (see [Part I, Definition 12.24]) in the admissible category. We used this notion in [Part I, Sections 12 and 13]. So we need to establish the existence of admissible bundle extension data to prove the existence of CF-perturbation etc. in the admissible category. As we mentioned in Subsubsection 15.3.1, the existence of bundle extension data is proved by using certain standard construction of vector bundle etc. (eg. existence of tubular neighborhood). The proof of admissible version of those standard results are nothing more than obvious adaptations of the standard proofs. Nevertheless for completeness' sake we provide those proofs in Section 25.

Using admissibility we can extend vector bundle E on X to a vector bundle $E^{\boxplus 1}$ on $X^{\boxplus 1}$. Also an admissible section of E is canonically extended to a smooth section of $E^{\boxplus 1}$.

This is the way how we put a collar to the outside of a K-space X in Section 17 and obtain $X^{\boxplus 1}$. We can also extend various admissible object of a K-space X to a collared object in $X^{\oplus 1}$. Thus the operation $X \mapsto X^{\oplus 1}$ from admissible objects to collared objects is completely canonical and functorial. We write those constructions in Section 17 in detail for completeness. However we emphasize that this construction is indeed straightfoward.

We note that for any orbifold X with corner there exists a system of charts by which X becomes an admissible orbifold. In the case of manifold with boundary, there exists a coordinate system such that the coordinate change $\varphi_{pq} = \psi_p^{-1} \circ \psi_q$: $V_{pq} \to \overline{V}_p \times [0,1)$ preserves the second factor [0,1] and the first factor (\overline{V}_p) factor) of $\varphi_{pq}(x,t)$ depends only on x. This statement is nothing but the existence of the collar 'inside' of a manifold X with boundary. The existence of the collar of any manifold or orbifold with corner is a classical fact which is easy to prove. So there is not much reason to put a collar 'outside' in the case of an orbifold. However, in the case of Kuranishi structure, there is some cumbersome issue to give a detailed proof of an existence of Kuranishi structure so that all the coordinate changes preserve the collar. (As we mentioned before, this is because the way to put collar to a manifold is not canonical.) The short cut we take is to use the admissible structure to put the collar outside which is canonical.

To apply this story to our geometric situation such as the case of the moduli space of pseudo-holomorphic curves, we need to establish existence of the admissible structure for such moduli spaces. This point is related to the exponential decay estimate of the gluing analysis in the following way.

The boundary or corner of the moduli space of pseudo-holomorphic curves appears typically at the infinity of the moduli space and the coordinate normal to the boundary or the corner is the gluing parameter. Let us consider the case of moduli space of pseudo-holomorphic disks and consider the configuration of the three disks as in Figure 1 below. This curve has two boundary nodes written p and q in the

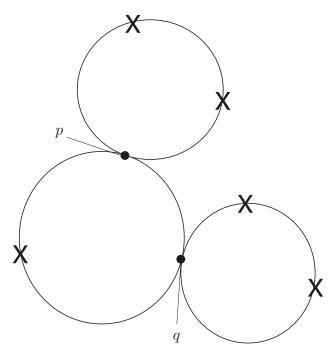


FIGURE 1. Bordered curve consisting of three disks

figure. (We add five boundary marked points so that this configuration is stable.) The parameter space to resolve these singularities involve two real numbers. We write them as T_p and T_q . They are the length of the neck $[0, T_p] \times [0, 1]$ (resp. $[0, T_q] \times [0, 1]$) so $T_p, T_q \in (C, \infty]$ for some large positive number C.

Let us resolve these two singularities. We can do so in one of the following three different ways.

- (1) We first resolve the singularity at p and then at q.
- (2) We first resolve the singularity at q and then at p.
- (3) We resolve the two singularities at the same time.

Let $\mathcal{M}_5(\beta)$ be the compactified moduli space of pseudo-holomorphic discs with 5 marked points, of homology class $\beta \in H_2(M, L)$ and boundary condition given by a certain Lagrangian submanifold $L \subset M$. The configuration in Figure 1 together with appropriate pseudo-holomorphic maps, gives an element of this compactified moduli space. Suppose this element is Fredholm regular for simplicity. Let \overline{V} be the intersection of its neighborhood in $\mathcal{M}_5(\beta)$ and the stratum consisting of elements whose source curve is still singular with 2 boundary node. (In other words \overline{V} is a neighborhood of this element in the codimension two stratum of $\mathcal{M}_5(\beta)$.)

Then any one of the above three gluing constructions gives a map from an open subset of $\overline{V} \times (C, \infty]^2$ onto an open subset of $\mathcal{M}_5(\beta)$. Let us write them as ψ_1, ψ_2, ψ_3 , respectively. The issue is whether T_p and T_q coordinates are preserved by the coordinate change $\psi_3^{-1} \circ \psi_1$ etc. In fact, certainly it is *not* preserved by this coordinate change.

This is related to the construction of the collar of the resulting Kuranishi structure. Namely if T_p , T_q coordinates happen to be preserved by the coordinate change then we can use this geometric coordinate itself as a collaring of the corner. In other words, the neighborhood of the corner (which we denote by $S_2\mathcal{M}_5(\alpha)$) in $\mathcal{M}_5(\beta)$ is diffeomorphic to $S_2\mathcal{M}_5(\beta) \times [0,\epsilon)^2$ and the coordinates of the factor $[0,\epsilon)^2$ can be taken, for example, as $(1/T_p,1/T_q)$.

However it seems not so easy to find a gluing construction such that $\psi_3^{-1} \circ \psi_1$ etc. preserves T_p and T_q coordinates.

On the other hand, there is no need at all to obtain the collar of the corner directly by the analytic construction of the chart. In the case of single orbifold existence of the collar can be proved by an easy standard argument. In the situation of Kuranishi structure things are a bit more complicated, since we need to find collars for various Kuranishi charts which are preserved by the coordinate change. Though we can find such a system of collars for good coordinate system after appropriate shrinking, its proof is a bit cumbersome to write down in detail.

Our short cut is to put collar outside, and for this purpose we need to find an admissible coordinate system.

For this purpose it suffices to show that $\psi_3^{-1} \circ \psi_1$ preserves gluing parameter $([0,\epsilon)^2)$ modulo an error term which is exponentially small in T_p, T_q . This is easier than proving that $\psi_3^{-1} \circ \psi_1$ exactly preserves the gluing parameter. We can prove this property as follows. We first observe that though the compatibility of the three different ways of gluing (1)(2)(3) above does not hold it is easy to construct pre-gluing for which (1)(2)(3) above are compatible. This is because pre-gluing is a simple process by using partition of unity and pre-gluing on one neck region does not affect the other neck region. The exponential decay estimate of the gluing construction (see [FOOO17] for the detail of the proof) then implies that actual gluing map is close to pre-gluing modulo an error which is exponentially small in T_p, T_q . Therefore the coordinate change has the required properties.

Remark 15.13. To find a collar related to the gluing parameter is very much different type of matter from a similar issue to put a collar appearing in the study of homotopy of almost complex structures etc.. The later appears, for example, when one proves independence of the Gromov-Witten invariant of the choice of almost complex structure. In the later problem we can take a homotopy J_t between two almost complex structures J and J' so that $J_t = J$ for $t \in [0, \epsilon]$ and $J_t = J'$ for $t \in [1-\epsilon, 1]$. So existence of collar is trivial to prove in that situation. The situation is different for the gluing parameter. (We also note that to prove independence of the Gromov-Witten invariant of the choice of almost complex structure we do not need to use the collar. See the proof of [Part I, Proposition 8.16].)

15.3.3. Stratified submersion. When we consider a map f from a manifold (an orbifold, a K-space) with corner X to another manifold M without boundary or corner, we say that f is a submersion if its restriction to all the corners S_kX are submersions. It implies that the push out $f_!h$ of all the smooth forms h on X by f is a smooth form on M.

When we study a family of K-systems parametrized by a manifold with corner P, we need to discuss the submersivity of a map from a manifold (an orbifold, a K-space) with corner X to a manifold with corner P.

We use the notion of stratified submersion for such a purpose. In Section 26 we define such a notion and discuss push out of a differential form to a manifold with corners.

15.3.4. Integration along the fiber and local system. As we mentioned in certain situation (for example when we study the Floer cohomology of periodic Hamiltonian system in the Bott-Morse situation) we need to introduce certain O(1) principal bundle on the space R_{α} . In Part 1 the integration along the fiber is defined in the situation when our K-space is oriented. We need to extend it slightly to include the case when the target and source spaces come with O(1) principal bundle, the K-space (which gives smooth correspondence) may not be oriented, and the target and source spaces R_{α} may not be oriented, but their orientation local systems and the O(1) principal bundles we put on R_{α} are related by some particular way. In Section 27 we discuss such generalization.

Convention on the way to use several notations.

- ^ and ^ We use 'hat' such as $\widehat{\mathcal{U}}$, \widehat{f} , $\widehat{\mathfrak{S}}$, \widehat{h} of an object defined on a Kuranishi structure $\widehat{\mathcal{U}}$. We use 'triangle' such as $\widehat{\mathcal{U}}$, \widehat{f} , $\widehat{\mathfrak{S}}$, \widehat{h} of an object defined on a good coordinate system $\widehat{\mathcal{U}}$.
- p and \mathfrak{p} For a Kuranishi structure $\widehat{\mathcal{U}}$ on $Z \subseteq X$ we write \mathcal{U}_p for its Kuranishi chart, where $p \in Z$. (We use an italic letter p.) For a good coordinate system $\widehat{\mathcal{U}}$ on $Z \subseteq X$ we write $\mathcal{U}_{\mathfrak{p}}$ for its Kuranishi chart, where $\mathfrak{p} \in \mathfrak{P}$. (We use a German character \mathfrak{p} .) Here \mathfrak{P} is a partial ordered set.
 - The mark indicates the end of Situation. See [Part I, Situation 6.3], for example.
- M and X Usually we denote by M a smooth manifold and by X a K-space, or an orbifold unless otherwise mentioned.

List of Notations in Part 1:

 \circ Int A, \mathring{A} : Interior of a subset A of a topological space.

- $\circ \overline{A}$: Closure of a subset A of a topological space.
- \circ Perm(k): The permutation group of order k!.
- \circ Supp(h), Supp(f): The support of a differential form h, a function f, etc..
- $\circ \varphi^{\star} \mathscr{F}$: Pull-back of a sheaf \mathscr{F} by a map φ .
- \circ X: A paracompact metrizable space. (Part I).
- \circ Z: A compact subspace of X. (Part I).
- o $\mathcal{U} = (U, \mathcal{E}, \psi, s)$: A Kuranishi chart, [Part I, Definition 3.1].
- $\circ \mathcal{U}|_{U_0} = (U_0, \mathcal{E}|_{U_0}, \psi|_{U_0 \cap s^{-1}(0)}, s|_{U_0})$: open subchart of $\mathcal{U} = (U, \mathcal{E}, \psi, s)$, [Part I, Definition 3.1].
- o $\Phi = (\varphi, \widehat{\varphi})$: Embedding of Kuranishi charts, [Part I, Definition 3.2].
- o $o_p, o_p(q)$: Points in a Kuranishi neighborhood U_p of p. [Part I, Definition 3.4].
- $\Phi_{21} = (U_{21}, \varphi_{21}, \widehat{\varphi}_{21})$: Coordinate change of Kuranishi charts from \mathcal{U}_1 to \mathcal{U}_2 , [Part I, Definition 3.5].
- o $\widehat{\mathcal{U}} = (\{\mathcal{U}_p\}, \{\Phi_{pq}\})$: Kuranishi structure, [Part I, Definition 3.8].
- \circ $(X, \hat{\mathcal{U}}), (X, Z; \hat{\mathcal{U}})$: K-space, relative K-space, [Part I, Definition 3.11].
- o $\widehat{\mathcal{U}} = ((\mathfrak{P}, \leq), \{\mathcal{U}_{\mathfrak{p}}\}, \{\Phi_{\mathfrak{pq}}\})$: Good coordinate system, [Part I, Definition 3.14].
- \circ $|\widehat{\mathcal{U}}|$: [Part I, Definition 3.15].
- o $\widehat{\Phi}:\widehat{\mathcal{U}}\to\widehat{\mathcal{U}}'$: KK-embedding. An embedding of Kuranishi structures, [Part I, Definition 3.20].
- o $\widehat{\Phi}:\widehat{\mathcal{U}}\to\widehat{\mathcal{U}'}:$ GG-embedding. An embedding of good coordinate systems, [Part I, Definition 3.24].
- o $\widehat{\Phi}:\widehat{\mathcal{U}}\to\widehat{\mathcal{U}}$: KG-embedding, An embedding of a Kuranishi structure to a good coordinate system, [Part I, Definition 3.29].
- o $\widehat{\Phi}:\widehat{\mathcal{U}}\to\widehat{\mathcal{U}}$: GK-embedding. An embedding of good coordinate system to a Kuranishi structure, [Part I, Definition 5.9].
- o $\hat{f}:(X,Z;\hat{\mathcal{U}})\to Y$ and $\hat{f}:(X,Z;\hat{\mathcal{U}})\to Y$: Strongly continuous map, [Part I, Definitions 3.35 and 3.38].
- o $(X, Z; \widehat{\mathcal{U}}) \times_N M$, $(X_1, Z_1; \widehat{\mathcal{U}}_1) \times_M (X_2, Z_2; \widehat{\mathcal{U}}_2)$: Fiber product of Kuranishi structures, [Part I, Definition 4.9].
- o $S_k(X, Z; \hat{\mathcal{U}})$, $S_k(X, Z; \hat{\mathcal{U}})$: Corner structure stratification, [Part I, Definition 4.15].
- $\circ S_{\mathfrak{d}}(X,Z;\widehat{\mathcal{U}}), S_{\mathfrak{d}}(X,Z;\widehat{\mathcal{U}})$: Dimension stratification, [Part I, Definition 5.1].
- $\circ \widehat{\mathcal{U}} < \widehat{\mathcal{U}^+} : \widehat{\mathcal{U}^+}$ is a thickening of $\widehat{\mathcal{U}}$. [Part I, Definition 5.3].
- $\circ \mathcal{S}_{\mathfrak{p}}(X, Z; \overline{\mathcal{U}}; \mathcal{K})$: [Part I, Definition 5.6 (4)].
- $\circ \ \mathcal{K} = \{\mathcal{K}_{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{P}\}: \ A \ \text{support system. Part I, Definition 5.6 (1)}].$
- \circ $(\mathcal{K}^1, \mathcal{K}^2)$ or $(\mathcal{K}^-, \mathcal{K}^+)$: A support pair, [Part I, Definition 5.6 (2)].
- $\circ \mathcal{K}^1 < \mathcal{K}^2$: [Part I, Definition 5.6].
- \circ $|\mathcal{K}|$: [Part I, Definition 5.6].
- $\circ B_{\delta}(A)$: Metric open ball, [Part I, (6.20)].
- o $S_x = (W_x, \omega_x, \{\mathfrak{s}_x^{\epsilon}\})$: CF-perturbation (=continuous family perturbation) on one orbifold chart. [Part I, Definition 7.3].
- $\circ \mathcal{S}_x^{\epsilon} = (W_x, \omega_x, \mathfrak{s}_x^{\epsilon}) \text{ for each } \epsilon > 0$: [Part I, Definition 7.3].
- $\circ \mathfrak{S} = \{(\mathfrak{V}_{\mathfrak{r}}, \mathcal{S}_{\mathfrak{r}}) \mid \mathfrak{r} \in \mathfrak{R}\}:$ Representative of a CF-perturbation on Kuranishi chart \mathcal{U} . [Part I, Definition 7.15].

Here $\mathfrak{V}_{\mathfrak{r}} = (V_{\mathfrak{r}}, E_{\mathfrak{r}}, \Gamma_{\mathfrak{r}}, \phi_{\mathfrak{r}}, \widehat{\phi}_{\mathfrak{r}})$ is an orbifold chart of (U, \mathcal{E}) and $\mathcal{S}_{\mathfrak{r}} = (W_{\mathfrak{r}}, \omega_{\mathfrak{r}}, \{\mathfrak{s}_{\mathfrak{r}}^{\epsilon}\})$ is a CF-perturbation of \mathcal{U} on \mathfrak{V}_r .

- $\circ \mathfrak{S}^{\epsilon} = \{(\mathfrak{V}_{\mathfrak{r}}, \mathcal{S}^{\epsilon}_{\mathfrak{r}}) \mid \mathfrak{r} \in \mathfrak{R}\} \text{ for each } \epsilon > 0. \text{ [Part I, Definition 7.15].}$
- $\circ \widehat{\mathfrak{S}} = {\mathfrak{S}_{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{P}}$: CF-perturbation of good coordinate system. [Part I, Definition 7.47].
- S: CF-perturbation of Kuranishi structure. [Part I, Definition 9.1].
- o \mathcal{S} : Sheaf of CF-perturbations. [Part I, Proposition 7.21].
- $\circ \mathscr{S}_{h0}, \mathscr{S}_{fh}, \mathscr{S}_{fhg}$: Subsheaves of \mathscr{S} . [Part I, Definition 7.25].
- \circ $\widehat{f}!(\widehat{h};\widehat{\mathfrak{S}}^{\epsilon})$: push out or integration along the fiber of \widehat{h} by $(\widehat{f},\widehat{\mathfrak{S}}^{\epsilon})$ on good coordinate system. [Part I, Definition 7.78].
- $\circ \widehat{f}!(\widehat{h};\widehat{\mathfrak{S}^{\epsilon}})$: push out or integration along the fiber of \widehat{h} by $(\widehat{f},\widehat{\mathfrak{S}^{\epsilon}})$ on Kuranishi structure. [Part I, Definition 9.13].
- \circ Corr_($\mathfrak{X},\widehat{\mathfrak{S}^{\epsilon}}$): Smooth correspondence associated to good coordinate system. [Part I, Definition 7.85].
- o $\mathrm{Corr}_{(\mathfrak{X},\widehat{\mathfrak{S}^c})}\colon \mathrm{Smooth}$ correspondence of Kuranishi structure [Part I, Definition 9.23].
- $\circ (\mathfrak{s}_{\mathfrak{n}}^n)^{-1}(0)$: The zero set of multisection.
- $\circ \Pi((\mathfrak{S}^{\epsilon})^{-1}(0))$: Support set of a CF-perturbation \mathfrak{S}^{ϵ} . [Part I, Definition
- \circ (V, Γ, ϕ) : Orbifold chart, [Part I, Definitions 23.1 and 23.6].
- \circ $(V, E, \Gamma, \phi, \phi)$: Orbifold chart of a vector bundle, [Part I, Definitions 23.17] and 23.22].
- \circ (X, \mathcal{E}) : Orbibundle, [Part I, Definition 23.20].

List of Notations in Part 2:

- $\circ \mathcal{M}(\alpha_{-}, \alpha_{+})$: Space of connecting orbits. Condition 16.1.
- $\circ \ \mathcal{C} = (\mathfrak{A}, \mathfrak{G}, \{R_{\alpha}\}_{{\alpha} \in \mathfrak{A}}, \{o_{R_{\alpha}}\}_{{\alpha} \in \mathfrak{A}}, E, \mu, \{\operatorname{PI}_{\beta, \alpha}\}_{\beta \in \mathfrak{G}, {\alpha} \in \mathfrak{A}}): \text{ A critical subman-}$ ifold data. Definition 16.6.
- $\circ \mathcal{F} = \left(\mathcal{C}, \{ \mathcal{M}(\alpha_{-}, \alpha_{+}) \}_{\alpha_{\pm} \in \mathfrak{A}}, (ev_{-}, ev_{+}), \{ OI_{\alpha_{-}, \alpha_{+}} \}_{\alpha_{\pm} \in \mathfrak{A}}, \{ PI_{\beta; \alpha_{-}, \alpha_{+}} \}_{\beta \in \mathfrak{G}, \alpha_{\pm} \in \mathfrak{A}} \right) :$ A linear K-system. Definition 16.6.
- $\circ \Lambda_{\text{nov}}^R$, Λ_{nov}^R , $\Lambda_{+,\text{nov}}^R$: Universal Novikov ring, and its ideal. Definition 16.10. When $R = \mathbb{R}$, we drop R from these notations.
- $\circ~\Lambda^R,\,\Lambda^R_0,\,\Lambda^R_+$: Universal Novikov ring, and its ideal. (The version without e.) Definition 16.10. When $R = \mathbb{R}$, we drop R from these notations.
- $\circ \mathcal{N}(\alpha_1, \alpha_2)$: Interpolation space. Condition 16.16.
- $\circ \mathfrak{N}_{ii+1} : \mathcal{F}_i \to \mathcal{F}_{i+1}$: Morphism of linear K-systems.
- $\circ \mathcal{N}_{i+1}$: Interpolation space of the morphism \mathfrak{N}_{i+1} . Lemma-Definition 16.35. See also Remark 18.36.
- o $\mathcal{FF} = (\{E^i\}, \{\mathcal{F}^i\}, \{\mathfrak{N}^i\})$: An inductive system of partial linear K-systems. Definition 16.36.
- $\begin{array}{l} \circ \ V_x^{\boxplus \tau} \colon \text{Definition 17.4.} \\ \circ \ \mathcal{U}_x^{\boxplus \tau} = (U_x^{\boxplus \tau}, \mathcal{E}_x^{\boxplus \tau}, \psi_x^{\boxplus \tau}, s_x^{\boxplus \tau}) \colon \text{Trivialization of one Kuranishi chart } \mathcal{U}_x \ \text{at} \end{array}$ $x \in S_k(U)$. Lemma-Definition 17.10. See also Lemma 17.20.
- $\circ \mathcal{S}_{x}^{\boxplus \tau}$: Lemma-Definition 17.11.
- $\circ \mathcal{U}^{\stackrel{\iota}{\boxtimes} \tau}$: τ -collaring, or τ -corner trivialization of \mathcal{U} . Lemma-Definition 17.21.
- $\circ X^{\boxplus \tau}$: τ -collaring, or τ -corner trivialization of X. Definition 17.26.

- o $(X,\widehat{\mathcal{U}})^{\boxplus \tau} = (X^{\boxplus \tau},\widehat{\mathcal{U}^{\boxplus \tau}})$: τ -collaring, or τ -corner trivialization of K-space $(X,\widehat{\mathcal{U}})$. Lemma-Definition 17.35.
- $\circ (X, \widehat{\mathcal{U}})^{\boxminus \tau} = (X^{\boxminus \tau}, \widehat{\mathcal{U}^{\boxminus \tau}})$: Inward τ -collaring of (X, \mathcal{U}) . Definition 17.45.
- o $(X,\widehat{\mathcal{U}})^{\mathfrak{C} \boxplus \tau} = (X^{\mathfrak{C} \boxplus \tau},\widehat{\mathcal{U}^{\mathfrak{C} \boxplus \tau}})$: τ -C-corner trivialization, or partial trivialization of corners, of $(X,\widehat{\mathcal{U}})$. Definition 18.10.
- $\circ \mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R_{\alpha_2}^2}^{\boxplus \tau} \mathcal{N}_{23}(\alpha_2, \alpha_3)$: Partially trivialized fiber product. Definition 18.37.
- $\circ \partial_{\mathfrak{C}}U$: Normalized \mathfrak{C} -partial boundary of U. When we denote by \mathfrak{C} a decomposition of the normalized boundary $\partial U = \partial^0 U \cup \partial^1 U$ into two disjoint unions, we write $\partial_{\mathfrak{C}}U = \partial^0 U$. Situation 18.1.
- $\circ S_k^{\mathfrak{C}}(U)$: Definition 18.2.
- $\circ \widehat{\mathcal{U}_1}^{\stackrel{\sim}{\mathbb{H}}_{\tau_1}} < \widehat{\mathcal{U}_2}^{\boxplus_{\tau_2}}$ as collared Kuranishi structures: Proposition 19.1.
- $\mathcal{G}(k+1,\beta)$: The set of all decorated ribbon trees $(\mathcal{T},\beta(\cdot))$ with (k+1) exterior vertices and $\sum_{\mathbf{v}\in C_{0,\mathrm{int}}(\mathcal{T})}(\beta(\mathbf{v}))=\beta$. Definition 21.2.
- \circ $G(\mathcal{AC})$ (resp. $G(\mathcal{AC}_P)$): The discrete submonoid associated to an (resp. a P-parametrized) A_{∞} correspondence \mathcal{AC} (resp. \mathcal{AC}_P). Definition 22.1.

Throughout Part 2, an orbifold with corner means an admissible orbifold with corner in the sense of Subsection 25.1. So all notions related to the orbifold with corner are ones in the admissible category.

16. Linear K-system: Floer Cohomology I: Statement

16.1. **Axiom of linear K-system.** We axiomatize the properties which are satisfied by the system of moduli spaces of solutions of Floer's equation.

Condition 16.1. We consider the following objects.

- (I) \mathfrak{G} is an additive group, and $E:\mathfrak{G}\to\mathbb{R}$ and $\mu:\mathfrak{G}\to\mathbb{Z}$ are group homomorphisms. We call $E(\beta)$ the energy of β and $\mu(\beta)$ the Maslov index of β .
- (II) $\mathfrak A$ is a set on which $\mathfrak G$ acts freely. We assume that the quotient set $\mathfrak A/\mathfrak G$ is a finite set. $E:\mathfrak A\to\mathbb R$ and $\mu:\mathfrak A\to\mathbb Z$ are maps such that

$$E(\beta \cdot \alpha) = E(\alpha) + E(\beta), \quad \mu(\beta \cdot \alpha) = \mu(\alpha) + \mu(\beta)$$

for any $\alpha \in \mathfrak{A}$, $\beta \in \mathfrak{G}$. We also call E the energy and μ the Maslov index.

- (III) (Critical submanifold) For any $\alpha \in \mathfrak{A}$ we have a finite dimensional compact manifold R_{α} (without boundary), which we call a *critical submanifold*.
- (IV) (Connecting orbit) For any $\alpha_-, \alpha_+ \in \mathfrak{A}$, we have a K-space with corners $\mathcal{M}(\alpha_-, \alpha_+)$ and strongly smooth maps

$$(ev_-, ev_+) : \mathcal{M}(\alpha_-, \alpha_+) \to R_{\alpha_-} \times R_{\alpha_+}.$$

We assume that ev_{+} is weakly submersive. We call $\mathcal{M}(\alpha_{-}, \alpha_{+})$ the space of connecting orbits and ev_{\pm} the evaluation maps at infinity.

- (V) (Positivity of energy) We assume $\mathcal{M}(\alpha_-, \alpha_+) = \emptyset$ if $E(\alpha_-) \geq E(\alpha_+)$.
- (VI) (Dimension) The dimension of the space of connecting orbits is given by

$$\dim \mathcal{M}(\alpha_{-}, \alpha_{+}) = \mu(\alpha_{+}) - \mu(\alpha_{-}) - 1 + \dim R_{\alpha_{+}}. \tag{16.1}$$

(VII) (Orientation)

⁶We note that $\mathcal{M}(\alpha, \alpha) = \emptyset$ in particular.

- (i) For any $\alpha \in \mathfrak{A}$, a principal O(1) bundle $o_{R_{\alpha}}$ on R_{α} is given. We call it an orientation system of the critical submanifold.
- (ii) For any $\alpha_1, \alpha_2 \in \mathfrak{A}$, an isomorphism

$$OI_{\alpha_{-},\alpha_{+}} : ev_{-}^{*}(o_{R_{\alpha_{-}}}) \cong ev_{+}^{*}(o_{R_{\alpha_{+}}}) \otimes ev_{+}^{*}(\det TR_{\alpha_{+}}) \otimes o_{\mathcal{M}(\alpha_{-},\alpha_{+})}$$
(16.2)

of principal O(1) bundles is given⁷. Here $o_{\mathcal{M}(R_{\alpha_-},R_{\alpha_+})}$ is an orientation bundle of the K-space $\mathcal{M}(\alpha_-,\alpha_+)$ in the sense of [Part I, Definition 3.10]. We call the isomorphism (16.2) the *orientation isomorphism*⁸. (See Section 27. Otherwise the reader may consider only the case when all the spaces R_{α} and $\mathcal{M}(\alpha_-,\alpha_+)$ are oriented and $o_{R_{\alpha_+}}$ are trivial.) ⁹ More precisely, we fix a choice of homotopy class of the isomorphism (16.2).

(VIII) (Periodicity)

(i) For any $\beta \in \mathfrak{G}$ a diffeomorphism

$$\operatorname{PI}_{\beta;\alpha}: R_{\alpha} \to R_{\beta\alpha}$$
 (16.3)

is given such that the equality

$$\mathrm{PI}_{\beta_2;\beta_1\alpha} \circ \mathrm{PI}_{\beta_1;\alpha} = \mathrm{PI}_{\beta_2\beta_1;\alpha}$$

holds.

(ii) Moreover, an isomorphism

$$\operatorname{PI}_{\beta:\alpha_{-},\alpha_{+}}: \mathcal{M}(\alpha_{-},\alpha_{+}) \to \mathcal{M}(\beta\alpha_{-},\beta\alpha_{+})$$
 (16.4)

of K-spaces in the sense of [Part I, Definition 4.22] is given such that the equality

$$\operatorname{PI}_{\beta_2;\beta_1\alpha_-,\beta_1\alpha_+} \circ \operatorname{PI}_{\beta_1;\alpha_-,\alpha_+} = \operatorname{PI}_{\beta_2\beta_1;\alpha_-,\alpha_+}$$

holds. The diagram below commutes. $\,$

$$\mathcal{M}(\alpha_{-}, \alpha_{+}) \xrightarrow{\mathrm{PI}_{\beta;\alpha_{-},\alpha_{+}}} \mathcal{M}(\beta\alpha_{-}, \beta\alpha_{+})
(\mathrm{ev}_{-}, \mathrm{ev}_{+}) \downarrow \qquad \qquad \downarrow (\mathrm{ev}_{-}, \mathrm{ev}_{+})
R_{\alpha_{-}} \times R_{\alpha_{+}} \xrightarrow{(\mathrm{PI}_{\beta;\alpha_{-}}, \mathrm{PI}_{\beta;\alpha_{+}})} R_{\beta\alpha_{-}} \times R_{\beta\alpha_{+}}$$
(16.5)

We call $\text{PI}_{\beta;\alpha}$, $\text{PI}_{\beta;\alpha_-,\alpha_+}$ the *periodicity isomorphisms*. The periodicity isomorphism preserves $o_{R_{\alpha}}$ and commutes with the orientation isomorphism.

(IX) (Gromov compactness) For any $E_0 \geq 0$ and $\alpha_- \in \mathfrak{A}$ the set

$$\{\alpha_{+} \in \mathfrak{A} \mid \mathcal{M}(\alpha_{-}, \alpha_{+}) \neq \emptyset, \ E(\alpha_{+}) \leq E_{0} + E(\alpha_{-})\}$$

$$(16.6)$$

is a finite set.

⁷ In the case of the linear K-system obtained from periodic Hamiltonian system, we take $o_R = \Theta_R^-$ for each critical submanifold R, where Θ_R^- is defined as the determinant of the index bundle of certain family of elliptic operators. See [FOOO4, Definition 8.8.2] for the precise definition. Then [FOOO4, Proposition 8.8.6] yields the isomorphism (16.2). On the other hand, in [FOOO4, Proposition 8.8.7] we take $o_R = \det TR \otimes \Theta_R^-$. Note that we use singular chains in [FOOO4], while we use differential forms in the current manuscript. So the choices of o_R are slightly different.

⁸See [FOOO4, Section 8.8].

⁹Note that $o_{R_{\alpha}}$ may not coincide with the principal O(1) bundle giving an orientation of R_{α} . For example, $o_{R_{\alpha}}$ may be nontrivial even in the case when R_{α} is orientable. On the other hand, if $\mathcal{M}(\alpha_{-},\alpha_{+})$ is orientable then $o_{\mathcal{M}(\alpha_{-},\alpha_{+})}$ is trivial.

(X) (Compatibility at the boundary) The normalized boundary of the space of connecting orbits is decomposed into a disjoint union of fiber products:¹⁰

$$\partial \mathcal{M}(\alpha_{-}, \alpha_{+}) \cong \coprod_{\alpha} \left(\mathcal{M}(\alpha_{-}, \alpha) \,_{\text{ev}_{+}} \times_{\text{ev}_{-}} \, \mathcal{M}(\alpha, \alpha_{+}) \right).$$
 (16.7)

Here \cong means an isomorphism of K-spaces. When we compare the orientations between the two sides, we swap the order of the factors in the fiber product in the right hand side of (16.7) and put the sign as follows:

$$\partial \mathcal{M}(\alpha_{-}, \alpha_{+}) \cong \coprod_{\alpha} (-1)^{\dim \mathcal{M}(\alpha, \alpha_{+})} \left(\mathcal{M}(\alpha, \alpha_{+}) \text{ ev}_{-} \times_{\text{ev}_{+}} \mathcal{M}(\alpha_{-}, \alpha) \right). \tag{16.8}$$

See Remark 16.2 for this isomorphism. This isomorphism commutes with the periodicity and orientation isomorphisms and is compatible with various evaluation maps. Namely the restriction of ev_ (resp. ev_+) of the left hand side of (16.12) coincides with ev_ (resp. ev_+) of the factor $\mathcal{M}(\alpha_-, \alpha_1)$ (resp. $\mathcal{M}(\alpha_k, \alpha_+)$) of the right hand side.

Remark 16.2. (1) The sign in (16.8) is consistent with that of [FOOO4, p.728] (the line 6 of the proof of Theorem 8.8.10 (3)). Here we should take into account the following two points about the notation and convention. The first point is about notation. In [FOOO4, Chapter 8] we write $\mathcal{M}(\alpha, \beta)$ for $\lim_{\tau \to -\infty} u(\tau, t) \in R_{\beta}$, $\lim_{\tau \to +\infty} u(\tau, t) \in R_{\alpha}$ for the discussion on orientations of moduli spaces of connecting orbits. Actually, this notation is different from those used in other chapters of [FOOO3, FOOO4], as we note at the beginning of Section 8.7 and p.723 in [FOOO4]. The order of the position of α, β in the notation $\mathcal{M}(\alpha, \beta)$ used in this article is opposite to the one used in [FOOO4, Chapter 8]. This is just a difference of notations.

The second point is the order of factors in the fiber product. This is not only a difference in conventions but also really affects the orientation on the fiber product. In [FOOO4, Chapter 8], e.g. Section 8.3, we use the evaluation map at the 0-th marked point of the second factor when taking the fiber product, while we are using the evaluation map at the 0-th marked point of the first factor in this book. See (16.7), for example. (As we note in Convention 8.3.1 and in the third line of [FOOO4, p.699], we regard the 0-th marked point z_0 as +1 in the unit disk in \mathbb{C} which corresponds to $+\infty$ in the strip $\mathbb{R} \times [0,1]$.) In (16.8) we follow the convention used in [FOOO4, Chapter 8]. That is why we swap the factors in the right hand side in (16.7). Now by taking the difference of notation mentioned here into account, the formula in the line 6 of the proof of [FOOO4, Theorem 8.8.10 (3), p.728] can be rewritten with the notation in this article as

$$\partial \mathcal{M}_{k+\ell+2}(R_{h_1}, R_{h_3}) \supset (-1)^{d_0} \mathcal{M}_{k+2}(R_{h_2}, R_{h_3}) \times_{R_{h_2}} \mathcal{M}_{\ell+2}(R_{h_1}, R_{h_2}),$$

$$d_0 = k\ell + k(\mu(h_2) - \mu(h_1)) + (\mu(h_3) - \mu(h_2)) + k - 1 + \dim R_{h_3}.$$
(16.9)

Applying this formula for the case $\alpha_{-} = h_1, \alpha = h_2, \alpha_{+} = h_3$ and $k = \ell = 0$, we obtain (16.8) by the dimension formula (16.1).

Whenever we have to explore the orientation, we swap the factors in the fiber product to follow the convention used in [FOOO4, Chapter 8]. We will mention it where such places appear later. On the other hand, when we do not need to study orientations, e.g. at (XI) (XII) compatibility at the corner below, we do not swap the order of factors in fiber products. See the footnote therein.

¹⁰ The right hand side is a finite union by Condition (IX).

(2) In general, since we find

$$X_1 \times_Y X_2 = (-1)^{(\dim X_1 - \dim Y)(\dim X_2 - \dim Y)} X_2 \times_Y X_1,$$
 (16.10)

we can rewrite (16.8) as

$$\partial \mathcal{M}(\alpha_{-}, \alpha_{+}) \cong \coprod_{\alpha} (-1)^{\epsilon} \left(\mathcal{M}(\alpha_{-}, \alpha) _{\text{ev}_{+}} \times_{\text{ev}_{-}} \mathcal{M}(\alpha, \alpha_{+}) \right)$$
 (16.11)

where

$$\epsilon = \dim \mathcal{M}(\alpha, \alpha_{+}) + (\dim \mathcal{M}(\alpha_{-}, \alpha) - \dim R_{\alpha})(\dim \mathcal{M}(\alpha, \alpha_{+}) - \dim R_{\alpha})$$

$$= (\mu(\alpha) - \mu(\alpha_{-})) (\mu(\alpha_{+}) - \mu(\alpha) - 1 + \dim R_{\alpha_{+}} - \dim R_{\alpha}) + \dim R_{\alpha}$$

$$= (\mu(\alpha) - \mu(\alpha_{-})) \dim \mathcal{M}(\alpha, \alpha_{+}) - (\mu(\alpha) - \mu(\alpha_{-}) - 1) \dim R_{\alpha}.$$

(XI) (Compatibility at the corner I) Let $\widehat{S}_k(\mathcal{M}(\alpha_-, \alpha_+))$ be the normalized corner of the K-space $\mathcal{M}(\alpha_-, \alpha_+)$ in the sense of Definition 24.17. Then we have:

$$\widehat{S}_{k}(\mathcal{M}(\alpha_{-}, \alpha_{+})) \cong \coprod_{\alpha_{1}, \dots, \alpha_{k} \in \mathfrak{A}} \left(\mathcal{M}(\alpha_{-}, \alpha_{1}) \,_{\text{ev}_{+}} \times_{R_{\alpha_{1}}} \cdots \,_{R_{\alpha_{k}}} \times_{\text{ev}_{-}} \mathcal{M}(\alpha_{k}, \alpha_{+}) \right).$$

$$(16.12)$$

Here the right hand side is a disjoint union. This is an isomorphism of K-spaces. It preserves the periodicity isomorphisms.¹¹ It is compatible with various evaluation maps. Namely the restriction of ev_ (resp. ev₊) of the left hand side of (16.12) coincides with ev_ (resp. ev₊) of the factor $\mathcal{M}(\alpha_-, \alpha_1)$ (resp. $\mathcal{M}(\alpha_k, \alpha_+)$) of the right hand side.

(XII) (Compatibility at the corner II) The isomorphism in (XI) satisfies the compatibility condition given in Condition 16.3 below.

To describe Condition 16.3 we need some notation. By (24.5) and (16.12) we have

$$\widehat{S}_{\ell}(\widehat{S}_{k}(\mathcal{M}(\alpha_{-},\alpha_{+}))) \cong \coprod_{\alpha_{1},\ldots,\alpha_{k} \in \mathfrak{A}} \coprod_{\ell_{0}+\cdots+\ell_{k}=\ell} \left(\widehat{S}_{\ell_{0}}\mathcal{M}(\alpha_{-},\alpha_{1}) \underset{\mathrm{ev}_{+}}{\mathrm{ev}_{+}} \times_{R_{\alpha_{1}}} \cdots \underset{R_{\alpha_{k}}}{R_{\alpha_{k}}} \times_{\mathrm{ev}_{-}} \widehat{S}_{\ell_{k}}\mathcal{M}(\alpha_{k},\alpha_{+})\right).$$

We apply (16.12) to each of the fiber product factors of the right hand side and obtain

$$\widehat{S}_{\ell}(\widehat{S}_{k}(\mathcal{M}(\alpha_{-},\alpha_{+}))) \\
\cong \coprod_{\alpha_{1},\dots,\alpha_{k}\in\mathfrak{A}} \coprod_{\ell_{0}+\dots+\ell_{k}=\ell} \coprod_{\alpha_{0,1},\dots,\alpha_{0,\ell_{0}}} \dots \coprod_{\alpha_{k,1},\dots,\alpha_{k,\ell_{k}}} \\
\left(\mathcal{M}(\alpha_{-},\alpha_{0,1}) \underset{\text{ev}_{+}}{\text{ev}_{+}} \times_{R_{\alpha_{0,1}}} \dots \underset{R_{\alpha_{0,\ell_{0}}}}{R_{\alpha_{0,\ell_{0}}}} \times_{\text{ev}_{-}} \mathcal{M}(\alpha_{0,\ell_{0}},\alpha_{1})\right) \\
= \underset{\text{ev}_{+}}{\text{ev}_{+}} \times_{R_{\alpha_{1}}} \left(\mathcal{M}(\alpha_{1},\alpha_{1,1}) \underset{\text{ev}_{+}}{\text{ev}_{+}} \times_{R_{\alpha_{1,1}}} \dots \underset{R_{1,\alpha_{\ell_{k}}}}{R_{1,\alpha_{\ell_{k}}}} \times_{\text{ev}_{-}} \mathcal{M}(\alpha_{1,\ell_{k}},\alpha_{2})\right) \\
\dots \\
= \underset{\text{ev}_{+}}{\text{ev}_{+}} \times_{R_{\alpha_{k}}} \left(\mathcal{M}(\alpha_{k},\alpha_{k,1}) \underset{\text{ev}_{+}}{\text{ev}_{+}} \times_{R_{\alpha_{k,1}}} \dots \underset{R_{k,\alpha_{\ell_{k}}}}{R_{k,\alpha_{\ell_{k}}}} \times_{\text{ev}_{-}} \mathcal{M}(\alpha_{k,\ell_{k}},\alpha_{+})\right).$$
(16.13)

¹¹We do not assume any compatibility of the orientation isomorphism *at the corner*, because what we will use is Stokes' formula where boundary but not corner appears.

By applying (16.12) to $k + \ell$ in place of k we obtain

$$\widehat{S}_{k+\ell}(\mathcal{M}(\alpha_{-}, \alpha_{+})) \cong \coprod_{\alpha_{1}, \dots, \alpha_{k+\ell} \in \mathfrak{A}} \left(\mathcal{M}(\alpha_{-}, \alpha_{1}) \underset{\text{ev}_{+}}{\text{ev}_{+}} \times_{R_{\alpha_{1}}} \cdots \underset{R_{\alpha_{k+\ell}}}{R_{\alpha_{k+\ell}}} \times_{\text{ev}_{-}} \mathcal{M}(\alpha_{k+\ell}, \alpha_{+}) \right).$$
(16.14)

It is easy to observe that each summand of (16.13) appears in (16.14) and vice versa.

Condition 16.3. The covering map $\pi_{\ell,k} : \widehat{S}_{\ell}(\widehat{S}_k(\mathcal{M}(\alpha_-, \alpha_+))) \to \widehat{S}_{k+\ell}(\mathcal{M}(\alpha_-, \alpha_+))$ in Proposition 24.16 restricts to the identity map on each summand of (16.13).

Remark 16.4. It is easy to see that each summand in (16.14) appears exactly $(k + \ell)!/k!\ell!$ times in (16.13). This is the covering index of the map $\pi_{\ell,k}$.

Remark 16.5. In general, in the case of orbifold, the map $\overset{\circ}{S}_{k-1}(\partial U) \to \overset{\circ}{S}_k(U)$ is not k to one map set-theoretically. (See [Part I, Remark 8.6 (3)].) However it is so in our case since the isotropy group acts trivially on the part $[0,1)^k$ (which is the normal direction to the stratum) in our case.

Definition 16.6. (1) Let \mathfrak{A} , \mathfrak{G} , R_{α} , $o_{R_{\alpha}}$, E, μ , $\operatorname{PI}_{\beta,\alpha}$ be the objects as in Conditions 16.1 (I), (III), (III), (VII)-(i), (VIII)-(i). We denote them by

$$\mathcal{C} = \left(\mathfrak{A}, \mathfrak{G}, \{R_{\alpha}\}_{\alpha \in \mathfrak{A}}, \{o_{R_{\alpha}}\}_{\alpha \in \mathfrak{A}}, E, \mu, \{\operatorname{PI}_{\beta, \alpha}\}_{\beta \in \mathfrak{G}, \alpha \in \mathfrak{A}}\right)$$

and call \mathcal{C} a critical submanifold data .

(2) Together with a critical submanifold data C, a linear system of spaces with Kuranishi structures, abbreviated as a linear K-system, \mathcal{F}^{13} consists of objects

$$\begin{split} \mathcal{F} &= \left(\mathcal{C}, \{\mathcal{M}(\alpha_-, \alpha_+)\}_{\alpha_{\pm} \in \mathfrak{A}}, (ev_-, ev_+), \{OI_{\alpha_-, \alpha_+}\}_{\alpha_{\pm} \in \mathfrak{A}}, \{PI_{\beta; \alpha_-, \alpha_+}\}_{\beta \in \mathfrak{G}, \alpha_{\pm} \in \mathfrak{A}}\right) \\ &\text{which satisfy Condition 16.1.} \end{split}$$

- (3) Let $E_0 > 0$. A partial linear K-system consists of the same objects as in Condition 16.1 except the following points. We call E_0 its energy cut level.
 - (a) The K-space (the space of connecting orbits) $\mathcal{M}(\alpha_-, \alpha_+)$ is defined only when $E(\alpha_+) E(\alpha_-) \leq E_0$.
 - (b) Periodicity and orientation isomorphisms of the space of connecting orbits are defined only among those satisfying $E(\alpha_+) E(\alpha_-) \leq E_0$.
 - (c) Compatibility isomorphisms of Kuranishi structures at the boundary given in Condition 16.1 (X) are defined only when the left hand side of (16.8) is defined.
 - (d) Compatibility isomorphism of Kuranishi structures at the corners in Condition 16.1 (XI) (XII) are defined only for $\mathcal{M}(\alpha_-, \alpha_+)$ with $E(\alpha_+) E(\alpha_-) \leq E_0$.

Remark 16.7. (1) We define the notion of partial linear K-system to take care of the 'running out problem' which we discussed in [FOOO4, Section 7.2.3]. (See Section 19 for the way how we will use it.) We can use a similar

 $^{^{12}}$ Namely, we collect the same data as in Conditions 16.1 as far as critical submanifolds concern.

 $^{^{13}}$ We use \mathcal{F} to denote a linear K-system. Here F stands for Floer.

- argument also to define symplectic homology (see [FH], [CFH], [BO]) in the case X is noncompact but convex at infinity, in complete generality. One difference between the current context and the context of symplectic homology lies in the finiteness statement given in Condition 16.1 (II).
- (2) To construct Floer cohomology in the situation where we have only partial linear K-systems, we need to define the notion of *inductive systems of partial linear K-systems*. To define the notion of such an inductive system, we need the notion of morphism of linear K-systems. We will define it in Definition 16.18.
- 16.2. Floer cohomology associated to a linear K-system. To state our main theorem on linear K-systems, we need to prepare some notations.

Definition 16.8. Let \mathcal{C} be a critical submanifold data as in Definition 16.6.

(1) We put

$$CF(\mathcal{C})^0 = \bigoplus_{\alpha \in \mathcal{A}} \Omega(R_\alpha; o_{R_\alpha}),$$
 (16.15)

where $\Omega(R_{\alpha}; o_{R_{\alpha}})$ is the de Rham complex of R_{α} twisted by the principal O(1) bundle $o_{R_{\alpha}}$. (See Section 27.)

(2) $CF(\mathcal{C})^0$ is a graded filtered \mathbb{R} vector space. Its grading is given so that the degree d part $CF^d(\mathcal{C})$ is to be

$$CF^{d}(\mathcal{C})^{0} = \bigoplus_{\alpha \in \mathfrak{A}} \Omega^{d-\mu(\alpha)}(R_{\alpha}; o_{R_{\alpha}}),$$

and its filtration $\mathfrak{F}CF^d(\mathcal{C})^0$ is given by

$$\mathfrak{F}^{\lambda}CF(\mathcal{C})^{0} = \bigoplus_{\alpha \in \mathfrak{A} \atop E(\alpha) \geq \lambda} \Omega(R_{\alpha}; o_{R_{\alpha}}).$$

Here $\lambda \in \mathbb{R}$.

- (3) The completion of $CF(\mathcal{C})^0$ with respect to the filtration $\mathfrak{F}^{\lambda}CF(\mathcal{C})^0$ is denoted by $CF(\mathcal{C})$. It is a graded filtered \mathbb{R} vector space which is complete. Its element corresponds to an infinite formal sum $\sum_{i=1}^{\infty} h_i$, where
 - (a) $h_i \in \Omega(R_{\alpha_i}; o_{R_{\alpha_i}}),$
 - (b) $\alpha_i \in \mathfrak{A}$,
 - (c) If $i \leq j$ then $E(\alpha_i) \leq E(\alpha_i)$,
 - (d) $\lim_{i\to\infty} E(\alpha_i) = \infty$ unless $\{\alpha_i\}$ is a finite set.
- (4) For each $\beta \in \mathfrak{G}$, the inverse of the periodicity diffeomorphism $\operatorname{PI}_{\beta;\alpha}$ induces an isomorphism $\Omega(R_{\alpha}; o_{R_{\alpha}}) \to \Omega(R_{\beta\alpha}; o_{R_{\beta\alpha}})$. Its direct sum extends to the above mentioned completion which we call *periodicity isomorphism* and write

$$\operatorname{PI}_{\beta}^*: CF(\mathcal{C}) \to CF(\mathcal{C}).$$

It is of degree $\mu(\beta)$ and satisfies

$$PI_{\beta}^{*}(\mathfrak{F}^{\lambda}CF(\mathcal{C})) \subset \mathfrak{F}^{\lambda+E(\beta)}CF(\mathcal{C}). \tag{16.16}$$

(5) We define $d_0: CF(\mathcal{C}) \to CF(\mathcal{C})$ by

$$d_0\left(\sum_{i=1}^{\infty} h_i\right) = \sum_{i=1}^{\infty} (-1)^{\dim R_{\alpha_i} + \mu(\alpha_i) + 1 + \deg h_i} d_{dR} h_i$$
 (16.17)

where $h_i \in \Omega^{\deg h_i}(R_{\alpha_i}; o_{R_{\alpha_i}})$ and d_{dR} on the right hand side is the de Rham differential. (See [FOOO3, p.150] for the exponent dim $R_{\alpha_i} + \mu(\alpha_i)$ of the sign and [FOOO3, Remark 3.5.8] for the exponent $1 + \deg h_i$.) The map d_0 has degree 1 and preserves the filtration.

We call $CF(\mathcal{C})$ together with differential d given in Theorem 16.9, the Floer cochain complex associated to \mathcal{C} .

Theorem 16.9. Suppose we are in the situation of Definition 16.8.

- (1) For any linear K-system we can define a map $d: CF(\mathcal{C}) \to CF(\mathcal{C})$ such that
 - (a) $d \circ d = 0$.
 - (b) d has degree 1 and preserves the filtration.
 - (c) d commutes with the periodicity isomorphism PI_{β}^* .
 - (d) There exists $\epsilon > 0$ such that:

$$(d-d_0)(\mathfrak{F}^{\lambda}CF(\mathcal{C})) \subset \mathfrak{F}^{\lambda+\epsilon}CF(\mathcal{C}).$$

(2) The definition of the map d in (1) involves various choices related to the associated Kuranishi structure and d depends on them. However it is independent of such choices in the following sense.

If d_1 , d_2 are obtained from two different choices, there exists $\psi : CF(\mathcal{C}) \to CF(\mathcal{C})$ with the following properties.

- (a) $d_2 \circ \psi = \psi \circ d_1$.
- (b) ψ has degree 0 and preserves the filtration.
- (c) ψ commutes with the periodicity isomorphism PI_{β}^* .
- (d) There exists $\epsilon > 0$ such that:

$$(\psi - \mathrm{id})(\mathfrak{F}^{\lambda}CF(\mathcal{C})) \subset \mathfrak{F}^{\lambda + \epsilon}CF(\mathcal{C})$$

where id is the identity map.

(e) In particular, ψ induces an isomorphism in cohomologies:

$$H(CF(\mathcal{C}), d_1) \to H(CF(\mathcal{C}), d_2).$$

(f) The cochain map ψ itself depends on various choices. However it is independent of the choices up to cochain homotopy. Namely if ψ_i i = 1,2 are obtained from the different choices, there exists a map H of degree -1 such that

$$\psi_1 - \psi_2 = d_2 \circ H + H \circ d_1.$$

Moreover there exists $\epsilon > 0$ such that:

$$H(\mathfrak{F}^{\lambda}CF(\mathcal{C})) \subset \mathfrak{F}^{\lambda+\epsilon}CF(\mathcal{C}).$$

The proof will be given in Section 19.

In the independence statement such as Theorem 16.9 (2) the critical submanifolds R_{α} are fixed. For most of the important applications of Floer cohomology we need to prove certain independence statement in the situation where the critical submanifolds vary. In that case we need to take an appropriate Novikov ring as the relevant coefficient ring. We introduce the following universal Novikov ring for this purpose.

Definition 16.10. Let R be a commutative ring with unity:

(1) The set Λ_{nov}^R consists of the formal sums

$$\mathfrak{x} = \sum_{i=1}^{\infty} P_i(e) T^{\lambda_i} \tag{16.18}$$

where

- (a) $P_i(e) \in R[e^{1/2}, e^{-1/2}].$ 14
- (b) T is a formal parameter.
- (c) $\lambda_i \in \mathbb{R}$ and $\lambda_i < \lambda_j$ for i < j.
- (d) $\lim_{i\to\infty} \lambda_i = \infty$ unless (16.18) is a finite sum.
- (2) We define the norm of the element (16.18) by

$$\|\mathfrak{x}\| = \exp\left(-\inf\{\lambda_i \mid P_i(e) \neq 0\}\right).$$

We put ||0|| = 0. We define a filtration on Λ_{nov}^R by

$$\mathfrak{F}^{\lambda}\Lambda_{\text{nov}}^{R} = \{\mathfrak{x} \in \Lambda_{\text{nov}}^{R} \mid ||\mathfrak{x}|| \le e^{-\lambda}\}.$$

- (3) Let us consider the subset of Λ_{nov}^R such that $P_i(e) = 0$ except finitely many indices i. We define a ring structure on it in an obvious way. It is also an \mathbb{R} -vector space and $\|\mathbf{r}\|$ is a norm of it. Λ_{nov}^R is its completion with respect to this norm. The ring structure extends to Λ_{nov}^R and Λ_{nov}^R becomes a complete normed ring. We call Λ_{nov}^R the universal Novikov ring with ground ring R.
- (4) The subset consisting of elements $\mathfrak{x} \in \Lambda_{\text{nov}}^R$ with $\|\mathfrak{x}\| \leq 1$ is a subring, which we write $\Lambda_{0,\text{nov}}^R$ and call it also the *universal Novikov ring*. Moreover, we put $\Lambda_{+,\text{nov}}^R = \{\mathfrak{x} \in \Lambda_{\text{nov}}^R \mid \|\mathfrak{x}\| < 1\}$.
- (5) We put $\deg T = 0$, $\deg e = 2$ and the degree of elements of R to be 0. Then $\Lambda_{0,\text{nov}}^R$ and Λ_{nov}^R are graded rings.
- (6) In case $R = \mathbb{R}$ we omit \mathbb{R} and write Λ_{nov} , $\Lambda_{0,\text{nov}}$ and $\Lambda_{+,\text{nov}}$.
- (7) When we do not include the indeterminate e, we write Λ^R , Λ^R_0 and Λ^R_+ in place of Λ^R_{nov} , $\Lambda^R_{0,\text{nov}}$ and $\Lambda^R_{+,\text{nov}}$. When $R = \mathbb{R}$, we also drop \mathbb{R} from these notations.

We now start from the cochain complex obtained in Theorem 16.9 and obtain a cochain complex over the universal Novikov ring $\Lambda_{0,\text{nov}}$.

Definition 16.11. Let \mathcal{C} be a critical submanifold data as in Definition 16.6.

(1) We consider $CF(\mathcal{C})^0$ as in (16.15) and take an algebraic tensor product with Λ_{nov} over \mathbb{R} , that is, $CF(\mathcal{C})^0 \otimes_{\mathbb{R}} \Lambda_{\text{nov}}$. It is a Λ_{nov} module. The Λ_{nov} module $CF(\mathcal{C})^0 \otimes_{\mathbb{R}} \Lambda_{\text{nov}}$ is graded by $\deg(x \otimes \mathfrak{x}) = \deg x + \deg \mathfrak{x}$, where $x \in CF(\mathcal{C})^0$ and $\mathfrak{x} \in \Lambda_{\text{nov}}$ are elements of pure degree. The Λ_{nov} module $CF(\mathcal{C})^0 \otimes_{\mathbb{R}} \Lambda_{\text{nov}}$ is filtered by

$$\mathfrak{F}^{\lambda}(CF(\mathcal{C})^0\otimes_{\mathbb{R}}\Lambda_{\mathrm{nov}})=\bigcup_{\lambda_1+\lambda_2\geq \lambda}\mathfrak{F}^{\lambda_1}CF(\mathcal{C})^0\otimes_{\mathbb{R}}\mathfrak{F}^{\lambda_2}\Lambda_{\mathrm{nov}}.$$

(2) For $\beta \in \mathfrak{G}$ we define a Λ_{nov} module homomorphism $\widehat{\operatorname{PI}}_{\beta}^* : CF(\mathcal{C})^0 \otimes_{\mathbb{R}} \Lambda_{\text{nov}} \to CF(\mathcal{C})^0 \otimes_{\mathbb{R}} \Lambda_{\text{nov}}$ by

$$\widehat{\operatorname{PI}}_{\beta}^*(x \otimes \mathfrak{x}) = \operatorname{PI}_{\beta}^*(x) \otimes T^{-E(\beta)} e^{-\mu(\beta)/2} \mathfrak{x}.$$

¹⁴In Item (5) we put deg e=2. The Novikov ring Λ_{nov}^R here is the same as one introduced in [FOOO3], where the indeterminate e has degree 2.

It preserves the degree and the filtration.

- (3) We consider the closed Λ_{nov} submodule of $CF(\mathcal{C})^0 \otimes_{\mathbb{R}} \Lambda_{\text{nov}}$ generated by elements of the form $\widehat{\text{PI}}^*_{\beta}(x \otimes \mathfrak{x}) x \otimes \mathfrak{x}$ for $x \otimes \mathfrak{x} \in CF(\mathcal{C})^0 \otimes_{\mathbb{R}} \Lambda_{\text{nov}}$ and $\beta \in \mathfrak{G}$. We denote by $(CF(\mathcal{C})^0 \otimes_{\mathbb{R}} \Lambda_{\text{nov}})/\sim$ the quotient module by this submodule. It is filtered and graded.
- (4) We denote the completion of $(CF(\mathcal{C})^0 \otimes_{\mathbb{R}} \Lambda_{\text{nov}})/\sim$ with respect to the filtration by $CF(\mathcal{C}; \Lambda_{\text{nov}})$. It is a filtered and graded Λ_{nov} module. We write its filtration by $\mathfrak{F}^{\lambda}CF(\mathcal{C}; \Lambda_{\text{nov}})$.
- (5) We put $CF(\mathcal{C}; \Lambda_{0,\text{nov}}) = \mathfrak{F}^0 CF(\mathcal{C}; \Lambda_{\text{nov}})$. It is a $\Lambda_{0,\text{nov}}$ module.
- (6) A subset $G \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}$ is called a *discrete submonoid* if the following holds. Denote by $E: G \to \mathbb{R}_{\geq 0}$ and $\mu: G \to \mathbb{Z}$ the natural projections.
 - (a) If $\beta_1, \beta_2 \in G$, then $\beta_1 + \beta_2 \in G$. $(0,0) \in G$.
 - (b) The image $E(G) \subset \mathbb{R}_{>0}$ is discrete.
 - (c) For each $E_0 \in \mathbb{R}_{\geq 0}$ the inverse image $G \cap E^{-1}([0, E_0])$ is a finite set.
- (7) A $\Lambda_{0,\text{nov}}$ -module homomorphism $\psi : CF(\mathcal{C}_1; \Lambda_{0,\text{nov}}) \to CF(\mathcal{C}_2; \Lambda_{0,\text{nov}})^{-15}$ is called G-gapped, if ψ is decomposed into the following form:

$$\psi = \sum_{\beta \in G} \psi_{\beta} T^{E(\beta)} e^{\mu(\beta)/2}, \tag{16.19}$$

where $\psi_{\beta}: CF(\mathcal{C}_1) \to \mathcal{C}F(\mathcal{C}_2)$ is an \mathbb{R} -linear map. In addition, if ψ is a cochain map, it is called a G-gapped cochain map. We simply say that ψ is gapped if it is G-gapped for some discrete submonoid G.

(8) If an operator $d: CF(\mathcal{C}; \Lambda_{0,\text{nov}}) \to CF(\mathcal{C}; \Lambda_{0,\text{nov}})$ satisfying $d \circ d = 0$ is gapped, we call $(CF(\mathcal{C}; \Lambda_{0,\text{nov}}), d)$ a gapped cochain complex.

Lemma 16.12. $CF(C; \Lambda_{0,nov})$ is a free $\Lambda_{0,nov}$ module.

Proof. This is a consequence of Condition 16.1 (II) and its definition. (See also [FOOO12, Lemma 2.3].)

Corollary 16.13. Let C be the critical submanifold data given in Definition 16.6.

- (1) In the situation of Theorem 16.9 (1), the operator d on CF(C) induces an operator : $CF(C; \Lambda_{0,nov}) \to CF(C; \Lambda_{0,nov})$, which we also denote by d. It has the same properties as in Theorem 16.9 (1) (a)-(d). Moreover, it is gapped in the sense of Definition 16.11.
- (2) In the situation of Theorem 16.9(2), the cochain map ψ induces a cochain map ψ : $(CF(\mathcal{C}, \Lambda_{0,nov}), d_1) \rightarrow (CF(\mathcal{C}, \Lambda_{0,nov}), d_2)$. It has the same properties as in Theorem 16.9 (2) (a)-(d). Moreover, it is a gapped cochain map.

Proof. (1) This is immediate from Theorem 16.9 (1) (b)(c)(d). The gapped-ness follows from the compactness axiom (IX) in Condition 16.1.

(2) This is immediate from Theorem 16.9 (2) (b)(c)(d). From the the construction of ψ given in Subsection 19.6, we can see the gapped-ness. Here the compactness axiom Condition 16.16 (IX) for morphisms of linear K-systems is used.

Definition 16.14. In the situation of Theorem 16.9 (1), we call the cohomology of $(CF(\mathcal{C}; \Lambda_{nov}), d)$ the *Floer cohomology* of linear K-system \mathcal{F} . It is independent of the choices by Corollary 16.13. We call the cochain complex $(CF(\mathcal{C}; \Lambda_{nov}), d)$ the *Floer cochain complex over the universal Novikov ring.*

¹⁵Here the case $C_1 = C_2$ is also included.

16.3. **Morphism of linear K-systems.** We next define morphisms between linear K-systems.

Situation 16.15. (1) For each i = 1, 2, let

$$C_i = \left(\mathfrak{A}_i, \mathfrak{G}_i, \{R_{\alpha_i}^i\}_{\alpha \in \mathfrak{A}_i}, \{o_{R_{\alpha_i}^i}\}_{\alpha \in \mathfrak{A}_i}, E, \mu, \{\operatorname{PI}_{\beta_i, \alpha_i}^i\}_{\beta_i \in \mathfrak{G}_i, \alpha_i \in \mathfrak{A}_i}\right)$$

be a critical submanifold data and

$$\mathcal{F}_i = \left(\mathcal{C}_i, \{ \mathcal{M}^i(\alpha_{i-}, \alpha_{i+}) \}_{\alpha_{i\pm} \in \mathfrak{A}_i}, (ev_-, ev_+), \{ OI^i_{\alpha_{i-}, \alpha_{i+}} \}_{\alpha_{i\pm} \in \mathfrak{A}_i}, \{ PI^i_{\beta_i; \alpha_{i-}, \alpha_{i+}} \}_{\beta_i \in \mathfrak{G}_i, \alpha_{i\pm} \in \mathfrak{A}_i} \right)$$

a linear K-system. We assume $\mathfrak{G}_1 = \mathfrak{G}_2$ (together with energy E and the Maslov index μ on it) and denote it by \mathfrak{G} .

(2) We also consider the same objects as in (1) except we only suppose that they consist of partial linear K-systems of energy cut level E_0 .

Condition 16.16. In Situation 16.15 (1) we consider the following objects.

(I)(II)(III) Nothing to add to those from Condition 16.1.

(IV) (Interpolation space) For any $\alpha_1 \in \mathfrak{A}_1$ and $\alpha_2 \in \mathfrak{A}_2$, we have a K-space with corners $\mathcal{N}(\alpha_1, \alpha_2)$ and strongly smooth maps

$$(\mathrm{ev}_-,\mathrm{ev}_+): \mathcal{N}(\alpha_1,\alpha_2) \to R^1_{\alpha_1} \times R^2_{\alpha_2}.$$

We assume that ev_+ is weakly submersive. We call $\mathcal{N}(\alpha_1, \alpha_2)$ the interpolation space and ev_{\pm} the evaluation maps at infinity.

(V) (Energy loss) We assume $\mathcal{N}(\alpha_1, \alpha_2) = \emptyset$ if $E(\alpha_1) \geq E(\alpha_2) + c$ for some $c \geq 0$. We call c the energy loss.

There is an exception in case c = 0. See Definition 16.19.

(VI) (Dimension) The dimension of the interpolation space is given by

$$\dim \mathcal{N}(\alpha_1, \alpha_2) = \mu(\alpha_2) - \mu(\alpha_1) + \dim R_{\alpha_2}. \tag{16.20}$$

(VII) (Orientation) For any $\alpha_1 \in \mathfrak{A}_1$ and $\alpha_2 \in \mathfrak{A}_2$, an isomorphism

$$\operatorname{OI}_{\alpha_1,\alpha_2} : \operatorname{ev}_-^*(o_{R_{\alpha_1}^1}) \cong \operatorname{ev}_+^*(o_{R_{\alpha_2}^2}) \otimes \operatorname{ev}_+^*(\det TR_{\alpha_2}^2) \otimes o_{\mathcal{N}(\alpha_1,\alpha_2)}$$
 (16.21)

of principal O(1) bundles is given. Here $o_{\mathcal{N}(\alpha_1,\alpha_2)}$ is the orientation bundle which gives an orientation of K-space $\mathcal{N}(\alpha_1,\alpha_2)$. We call OI_{α_1,α_2} an orientation isomorphism.

(VIII) (Periodicity) For any $\beta \in \mathfrak{G}$ an isomorphism

$$\operatorname{PI}_{\beta:\alpha_1,\alpha_2}: \mathcal{N}(\alpha_1,\alpha_2) \to \mathcal{N}(\beta\alpha_1,\beta\alpha_2)$$
 (16.22)

of K-spaces is given such that the equality

$$\mathrm{PI}_{\beta_2;\beta_1\alpha_1,\beta_1\alpha_2} \circ \mathrm{PI}_{\beta_1;\alpha_1,\alpha_2} = \mathrm{PI}_{\beta_2\beta_1;\alpha_1,\alpha_2}$$

holds and the following diagram commutes.

$$\mathcal{N}(\alpha_{1}, \alpha_{2}) \xrightarrow{\mathrm{PI}_{\beta;\alpha_{1},\alpha_{2}}} \mathcal{N}(\beta\alpha_{1}, \beta\alpha_{2})$$

$$(\mathrm{ev}_{1}, \mathrm{ev}_{2}) \downarrow \qquad \qquad \downarrow (\mathrm{ev}_{1}, \mathrm{ev}_{2})$$

$$R_{\alpha_{1}}^{1} \times R_{\alpha_{2}}^{2} \xrightarrow{(\mathrm{PI}_{\beta;\alpha_{1}}, \mathrm{PI}_{\beta;\alpha_{2}})} R_{\beta\alpha_{1}}^{1} \times R_{\beta\alpha_{2}}^{2}$$

$$(16.23)$$

We call $PI_{\beta;\alpha_1,\alpha_2}$ the *periodicity isomorphism*. The periodicity isomorphism commutes with the orientation isomorphism.

(IX) (Gromov compactness) For any $E_0 \ge 0$ and $\alpha_1 \in \mathfrak{A}_1$ the set

$$\{\alpha_2 \in \mathfrak{A}_2 \mid \mathcal{N}(\alpha_1, \alpha_2) \neq \emptyset, \ E(\alpha_2) \leq E_0 + E(\alpha_1)\}$$
 (16.24)

is a finite set.

(X) (Compatibility at the boundary) The normalized boundary of the interpolation space is decomposed into the fiber products as follows:¹⁶

$$\begin{aligned}
&\mathcal{O}\mathcal{N}\left(\alpha_{1}, \alpha_{2}\right) \\
&\cong \coprod_{\alpha'_{1} \in \mathfrak{A}_{1}} \left(-1\right)^{\dim \mathcal{N}\left(\alpha'_{1}, \alpha_{2}\right)} \left(\mathcal{N}\left(\alpha'_{1}, \alpha_{2}\right)_{\text{ev}_{-}} \times_{\text{ev}_{+}} \mathcal{M}^{1}\left(\alpha_{1}, \alpha'_{1}\right)\right) \\
&\sqcup \coprod_{\alpha'_{2} \in \mathfrak{A}_{2}} \left(-1\right)^{\dim \mathcal{M}^{2}\left(\alpha'_{2}, \alpha_{2}\right)} \left(\mathcal{M}^{2}\left(\alpha'_{2}, \alpha_{2}\right)_{\text{ev}_{-}} \times_{\text{ev}_{+}} \mathcal{N}\left(\alpha_{1}, \alpha'_{2}\right)\right),
\end{aligned} (16.25)$$

where the right hand side is the disjoint union.¹⁷ Here \cong means the isomorphism of K-spaces. This isomorphism commutes with the periodicity and orientation isomorphisms. It is compatible with various evaluation maps.

(XI) (Compatibility at the corner I) The normalized corner $\widehat{S}_k(\mathcal{N}(\alpha_1, \alpha_2))$ is decomposed into a disjoint union of

$$\mathcal{M}^{1}(\alpha_{1}, \alpha_{1,1}) \times_{R_{\alpha_{1,1}}^{1}} \cdots \times_{R_{\alpha_{1,k_{1}-1}}^{1}} \mathcal{M}^{1}(\alpha_{1,k_{1}-1}, \alpha_{1,k_{1}}) \times_{R_{\alpha_{1,k_{1}}}^{1}} \mathcal{N}(\alpha_{1,k_{1}}, \alpha_{2,1}) \times_{R_{\alpha_{2,1}}^{1}} \mathcal{M}^{2}(\alpha_{2,1}, \alpha_{2,2}) \times_{R_{\alpha_{2,2}}^{2}} \cdots \times_{R_{\alpha_{2,k_{2}}}^{2}} \mathcal{M}^{2}(\alpha_{2,k_{2}}, \alpha_{2})$$

$$(16.26)$$

where $k_1 + k_2 = k$, $\alpha_{1,i} \in \mathfrak{A}_1$, $\alpha_{2,i} \in \mathfrak{A}_2$. This is an isomorphism of K-spaces. The isomorphism preserves the periodicity isomorphisms and is compatible with various evaluation maps.

(XII) (Compatibility at the corner II) The isomorphism (XI) satisfies the compatibility condition given by Condition 16.17 below.

Condition 16.17. Condition 16.16 (XI) and Condition 16.1 (XI) (together with (24.5)) define an isomorphism from $\widehat{S}_{\ell}(\widehat{S}_k(\mathcal{N}(\alpha_1, \alpha_2)))$ to the disjoint union of the summand (16.26) with k replaced by $k+\ell$. Then the map $\pi_{\ell,k}:\widehat{S}_{\ell}(\widehat{S}_k(\mathcal{N}(\alpha_1, \alpha_2))) \to \widehat{S}_{\ell+k}(\mathcal{N}(\alpha_1, \alpha_2))$ in Proposition 24.16 becomes the identity map under those isomorphisms.

Definition 16.18. Let us consider Situation 16.15.

- (1) A morphism of linear K-systems from \mathcal{F}_1 to \mathcal{F}_2 consists of the objects as in Condition 16.16. We say c the energy loss of our morphism.
- (2) Let $E_i > 0$ and let \mathcal{F}_i be a partial linear K-system with energy cut levels E_i for each i = 1, 2. Suppose $E_1 \geq E_2 + c$ with $c \geq 0$. A morphism of partial linear K-systems from \mathcal{F}_1 to \mathcal{F}_2 consists of the same objects as those in Condition 16.16 except the following:
 - (a) The K-space, the interpolation space, $\mathcal{N}(\alpha_1, \alpha_2)$ is defined only when $-c \leq E(\alpha_2) E(\alpha_1) \leq E_2$.
 - (b) The periodicity isomorphisms of the interpolation spaces are defined only among those satisfying $-c \le E(\alpha_2) E(\alpha_1) \le E_2$.

 $^{^{16}}$ See Remark 16.2 for the sign and the order of the factors in the fiber products.

 $^{^{17}}$ The right hand side is a finite sum by Condition (IX).

- (c) The compatibility of the isomorphism of K-spaces at the boundary in Condition 16.16 (X) is defined only when the left hand side of (16.25) is defined.
- (d) The compatibility of the isomorphism of K-spaces at the corners in Condition 16.16 (XI) is required only for $\mathcal{N}(\alpha_1, \alpha_2)$ with $-c \leq E(\alpha_2) E(\alpha_1) \leq E_2$.

Definition 16.19. Suppose that the index set \mathfrak{A}_1 of the critical submanifold of \mathcal{F}_1 is identified with the index set \mathfrak{A}_2 of the critical submanifold of \mathcal{F}_2 . We define the notion of a morphism with energy loss 0 and congruent to an isomorphism by requiring the following:

For $\alpha = \alpha_1 = \alpha_2 \in \mathfrak{A}_1 = \mathfrak{A}_2$, we require $\mathcal{N}(\alpha, \alpha) = R_\alpha$ instead of requiring it to be an empty set. We also require the maps $\operatorname{ev}_{\pm} : R_\alpha \to R_\alpha$ to be identity maps. Moreover for $\alpha_1 \neq \alpha_2$ we require $\mathcal{N}(\alpha_1, \alpha_2) = \emptyset$ if $E(\alpha_1) = E(\alpha_2)$.

If $\mathcal{F}_1 = \mathcal{F}_2$ in addition, we call the morphism a morphism with energy loss 0 and congruent to the identity morphism, instead of a morphism with energy loss 0 and congruent to an isomorphism.

16.4. Homotopy and higher homotopy of morphisms of linear K-systems. To establish basic properties of Floer cohomology associated to a linear K-system or an inductive system of partial linear K-systems, we will use the notion of homotopy between the morphisms and also higher homotopy such as homotopy of homotopies etc.. To include the most general case, we define the notion of morphisms parametrized by a manifold with corners P.

Situation 16.20. (1) Let C_i , \mathcal{F}_i (i=1,2) be critical submanifold data and linear K-systems as in Situation 16.15, respectively. We assume $\mathfrak{G}_1 = \mathfrak{G}_2$ and denote the common group by \mathfrak{G} .

(2) Let P be a smooth manifold with boundary or corner.

Condition 16.21. In Situation 16.20 we consider the following objects.

- (I), (II), (III) Nothing to add.
- (IV) (Interpolation space) For any $\alpha_1 \in \mathfrak{A}_1$ and $\alpha_2 \in \mathfrak{A}_2$, we have a K-space with corners $\mathcal{N}(\alpha_1, \alpha_2; P)$ and a strongly smooth map

$$(\mathrm{ev}_P,\mathrm{ev}_-,\mathrm{ev}_+): \mathcal{N}(\alpha_1,\alpha_2;P) \to P \times R^1_{\alpha_1} \times R^2_{\alpha_2}.$$

We assume that

$$(ev_P, ev_+): \mathcal{N}(\alpha_1, \alpha_2; P) \to P \times R^2_{\alpha_2}$$

is a corner stratified weak submersion. (See Definition 26.6 for the notion of corner stratified submersivity.) We call $\mathcal{N}(\alpha_1, \alpha_2; P)$ a P-parametrized interpolation space.

(V) (Energy loss) We assume $\mathcal{N}(\alpha_1, \alpha_2; P) = \emptyset$ if $E(\alpha_1) \geq E(\alpha_2) + c$. We call $c \geq 0$ the energy loss.

Exception: In case c=0 and $\alpha=\alpha_1=\alpha_2$, we require $\mathcal{N}(\alpha,\alpha;P)=P\times R_\alpha$ instead of requiring it to be an empty set. We also require that $\operatorname{ev}_\pm:P\times R_\alpha\to R_\alpha$ and $\operatorname{ev}_P:P\times R_\alpha\to P$ are projections.

(VI) (Dimension) The dimension is given by

$$\mathcal{N}(\alpha_1, \alpha_2; P) = \mu(\alpha_2) - \mu(\alpha_1) + \dim R_{\alpha_2}^2 + \dim P. \tag{16.27}$$

(VII) (Orientation) P is oriented. For any $\alpha_1 \in \mathfrak{A}_1$ and $\alpha_2 \in \mathfrak{A}_2$, we are given an *orientation isomorphism*

$$\operatorname{OI}_{\alpha_1,\alpha_2;P} : \operatorname{ev}_{-}^*(o_{R_{\alpha_1}^1}) \cong \operatorname{ev}_{+}^*(o_{R_{\alpha_2}^2}) \otimes \operatorname{ev}_{+}^*(\det TR_{\alpha_2}^2) \otimes o_{\mathcal{N}(\alpha_1,\alpha_2;P)}.$$
 (16.28)

(VIII) (Periodicity) For any $\beta \in \mathfrak{G}$ an isomorphism

$$\operatorname{PI}_{\beta;\alpha_1,\alpha_2;P}: \mathcal{N}(\alpha_1,\alpha_2;P) \to \mathcal{N}(\beta\alpha_1,\beta\alpha_2;P)$$
 (16.29)

of K-spaces is given. The equality

$$\mathrm{PI}_{\beta_2;\beta_1\alpha_1,\beta_1\alpha_2;P} \circ \mathrm{PI}_{\beta_1;\alpha_1,\alpha_2;P} = \mathrm{PI}_{\beta_2\beta_1;\alpha_1,\alpha_2;P}$$

holds. The isomorphism $\operatorname{PI}_{\beta;\alpha_1,\alpha_2;P}$ is compatible with $(\operatorname{ev}_-,\operatorname{ev}_+,\operatorname{ev}_P)$ and preserves the orientation isomorphism.

(IX) (Gromov compactness) For any $E_0 \ge 0$ and $\alpha_1 \in \mathfrak{A}_1$ the set

$$\{\alpha_2 \in \mathfrak{A}_2 \mid \mathcal{N}(\alpha_1, \alpha_2; P) \neq \emptyset, \ E(\alpha_2) \le E_0 + E(\alpha_1)\}$$
 (16.30)

is a finite set.

To state the next condition we note that Lemma 26.5 implies the following.

Lemma 16.22. We assume $\mathcal{N}(\alpha_1, \alpha_2; P)$ satisfies Condition 16.21 (IV)-(IX).

(1) We can define a K-space $\mathcal{N}(\alpha_1, \alpha_2; \partial P)$ by the fiber product:

$$\mathcal{N}(\alpha_1, \alpha_2; \partial P) := \partial P_P \times_{\text{ev}_P} \mathcal{N}(\alpha_1, \alpha_2; P).$$

- (2) The periodicity and orientation isomorphisms of $\mathcal{N}(\alpha_1, \alpha_2; P)$ induce those of $\mathcal{N}(\alpha_1, \alpha_2; \partial P)$.
- (3) (ev_-, ev_+, ev_P) of $\mathcal{N}(\alpha_1, \alpha_2; P)$ induces $(ev_-, ev_+, ev_{\partial P})$ of $\mathcal{N}(\alpha_1, \alpha_2; \partial P)$.
- (4) Objects defined in (1)-(3) above satisfy Condition 16.21 (I)-(IX).

Condition 16.23. (X) (Compatibility at the boundary) The normalized boundary of the P-parametrized interpolation space is decomposed into the fiber products as follows. ¹⁸

$$\partial \mathcal{N}(\alpha_{1}, \alpha_{2}; P)$$

$$\cong \coprod_{\alpha \in \mathfrak{A}_{2}} (-1)^{\dim \mathcal{M}^{2}(\alpha, \alpha_{2})} \mathcal{M}^{2}(\alpha, \alpha_{2}) \,_{\text{ev}_{-}} \times_{\text{ev}_{+}} \mathcal{N}(\alpha_{1}, \alpha; P)$$

$$\sqcup \coprod_{\alpha \in \mathfrak{A}_{1}} (-1)^{\dim \mathcal{N}(\alpha, \alpha_{2}; P)} \mathcal{N}(\alpha, \alpha_{2}; P) \,_{\text{ev}_{-}} \times_{\text{ev}_{+}} \mathcal{M}^{1}(\alpha_{1}, \alpha)$$

$$\sqcup \mathcal{N}(\alpha_{1}, \alpha_{2}; \partial P), \tag{16.31}$$

where the right hand side is the disjoint union. Here \cong means the isomorphism of K-spaces. This isomorphism preserves the orientation isomorphism and the periodicity isomorphism, which in the right hand side are obtained by taking fiber products thereof respectively. It is also compatible with various evaluation maps.

To state the next condition we note that Lemma 26.5 also implies the following:

Lemma 16.24. We assume $\mathcal{N}(\alpha_1, \alpha_2; P)$ satisfies Condition 16.21 (IV)-(IX).

(1) We can define a K-space by the fiber product

$$\widehat{S}_k(P)_P \times_{\text{ev}_P} \mathcal{N}(\alpha_1, \alpha_2; P).$$

We denote it by $\mathcal{N}(\alpha_1, \alpha_2; \widehat{S}_k(P))$.

¹⁸See Remark **16.2**.

- (2) The periodicity isomorphism of $\mathcal{N}(\alpha_1, \alpha_2; P)$ induces one of $\mathcal{N}(\alpha_1, \alpha_2; \widehat{S}_k(P))$.
- (3) (ev_-, ev_+, ev_P) of $\mathcal{N}(\alpha_1, \alpha_2; P)$ induces $(ev_-, ev_+, ev_{\widehat{S}_k(P)})$ of $\mathcal{N}(\alpha_1, \alpha_2; \widehat{S}_k(P))$.
- (4) Objects defined in (1)-(3) above satisfy Condition 16.21 (I)-(IX).

The next lemma is also a conclusion of Lemma 26.5.

Lemma 16.25. Suppose we are in the situation of Lemma 16.24.

- (1) The $(k + \ell)/k!\ell!$ fold covering map $\widehat{S}_k(\widehat{S}_\ell(P)) \to \widehat{S}_{k+\ell}(P)$ induces a $(k + \ell)!/k!\ell!$ fold covering map : $\mathcal{N}(\alpha_1, \alpha_2; \widehat{S}_k(\widehat{S}_\ell(P))) \to \mathcal{N}(\alpha_1, \alpha_2; \widehat{S}_{k+\ell}(P))$.
- (2) The covering map (1) commutes with the periodicity isomorphisms. It is also compatible with various evaluation maps. Especially the following diagram commutes.

$$\mathcal{N}(\alpha_{1}, \alpha_{2}; \widehat{S}_{k}(\widehat{S}_{\ell}(P))) \longrightarrow \mathcal{N}(\alpha_{1}, \alpha_{2}; \widehat{S}_{k+\ell}(P))$$

$$\stackrel{\text{ev}_{\widehat{S}_{k}}(\widehat{S}_{\ell}(P))}{\widehat{S}_{k}(\widehat{S}_{\ell}(P))} \downarrow \qquad \qquad \downarrow^{\text{ev}_{\widehat{S}_{k+\ell}(P)}}$$

$$\widehat{S}_{k}(\widehat{S}_{\ell}(P)) \longrightarrow \widehat{S}_{k+\ell}(P)$$
(16.32)

Condition 16.26. (XI) (Compatibility at the corner) The normalized corner $\hat{S}_k \mathcal{N}((\alpha_-, \alpha_+); P)$ is decomposed into a disjoint union of fiber products:

$$\mathcal{M}^{1}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \cdots \times_{R_{\alpha_{k_{1}-1}}} \mathcal{M}^{1}(\alpha_{k_{1}-1}, \alpha_{k_{1}})$$

$$\times_{R_{\alpha_{k_{1}}}} \mathcal{N}(\alpha_{k_{1}}, \alpha_{k_{1}+1}; \widehat{S}_{k_{3}}(P))$$

$$\times_{R_{\alpha_{k_{1}+1}}} \mathcal{M}^{2}(\alpha_{k_{1}+1}, \alpha_{k_{1}+2}) \times_{R_{\alpha_{k_{1}+2}}} \cdots \times_{R_{\alpha_{k_{1}+k_{2}}}} \mathcal{M}^{2}(\alpha_{k_{1}+k_{2}}, \alpha_{+}).$$
(16.33)

Here $k_1 + k_2 + k_3 = k$. This is an isomorphism of K-spaces. This isomorphism respects the periodicity isomorphisms. It is also compatible with various evaluation maps. Moreover Condition 16.28 below holds.

To state Condition 16.28 we make a digression. We consider $\hat{S}_{\ell}((16.33))$, where (16.33) stands for the K-space defined by (16.33). Applying (24.5) to (16.33), we get an isomorphism from the normalized corner $\hat{S}_{\ell}(\hat{S}_k(\mathcal{N}(\alpha_-, \alpha_+; P)))$ to the disjoint union of fiber products of the normalized corners of the factors of (16.33). We can identify the normalized corners of various factors of (16.33) by using (16.12) and Condition 16.26. Thus we obtain the next lemma.

Lemma 16.27. The isomorphisms in Condition 16.26 and in (16.12) canonically induce an isomorphism from $\widehat{S}_{\ell}(\widehat{S}_k(\mathcal{N}(\alpha_-, \alpha_+; P)))$ to a disjoint union of the fiber products of the following form:

$$\begin{split} &\mathcal{M}^{1}(\alpha_{-},\alpha'_{1})\times_{R_{\alpha'_{1}}}\cdots\times_{R_{\alpha'_{k'_{1}-1}}}\mathcal{M}^{1}(\alpha'_{k'_{1}-1},\alpha'_{k'_{1}})\\ &\times_{R_{\alpha'_{k'_{1}}}}\mathcal{N}(\alpha'_{k'_{1}},\alpha'_{k'_{1}+1};\widehat{S}_{\ell'}(\widehat{S}_{k'_{3}}(P)))\\ &\times_{R_{\alpha'_{k'_{1}+1}}}\mathcal{M}^{2}(\alpha'_{k'_{1}+1},\alpha'_{k'_{1}+2})\times_{R_{\alpha'_{k'_{1}+2}}}\cdots\times_{R_{\alpha'_{k'_{1}+k'_{2}}}}\mathcal{M}^{2}(\alpha'_{k'_{1}+k'_{2}},\alpha_{+}). \end{split} \tag{16.34}$$

Here $k'_1 + k'_2 + k'_3 + \ell' = \ell + k$. (Note that the same copies of the form (16.34) appear several times in $\widehat{S}_{\ell}(\widehat{S}_{k}(\mathcal{N}(\alpha_{-}, \alpha_{+}; P)))$.)

Now we state the last condition.

Condition 16.28. (Compatibility at the corner continued) Under the isomorphism between (16.34) and (16.33) (with k replaced by $k+\ell$), the $(k+\ell)!/k!\ell!$ fold covering map $\hat{S}_{\ell}(\hat{S}_k(\mathcal{N}(\alpha_-,\alpha_+;P))) \to \hat{S}_{\ell+k}(\mathcal{N}(\alpha_-,\alpha_+;P))$ obtained in Proposition 24.16 is identified with the fiber product of the identity maps and of

$$\mathcal{N}(\alpha'_{k'_1}, \alpha'_{k'_1+1}; \widehat{S}_{\ell'}(\widehat{S}_{k'_2}(P))) \to \mathcal{N}(\alpha'_{k'_1}, \alpha'_{k'_1+1}; \widehat{S}_{\ell'+k'_2}(P)),$$

which is the map given in Lemma 16.25.

Definition 16.29. (1) A *P*-parametrized family of morphisms from \mathcal{F}_1 to \mathcal{F}_2 consists of objects satisfying Conditions 16.21, 16.23, 16.26, 16.28.

- (2) If a P-parametrized family of morphisms \mathfrak{N}_P from \mathcal{F}_1 to \mathcal{F}_2 and a ∂P -parametrized family of morphisms $\mathfrak{N}_{\partial P}$ from \mathcal{F}_1 to \mathcal{F}_2 are related as in Condition 16.23, we call $\mathfrak{N}_{\partial P}$ the boundary of \mathfrak{N}_P and write $\partial \mathfrak{N}_P$.
- (3) Let $E_i > 0$ and \mathcal{F}_i be partial linear K-systems with energy cut levels E_i (i = 1, 2). Suppose $E_1 \geq E_2 + c$ for some $c \geq 0$. A *P-parametrized family of morphisms of partial linear K-systems* from \mathcal{F}_1 to \mathcal{F}_2 consists of the same objects as in Condition 16.21 except the following:
 - (a) The K-space, the P parametrized interpolation space, $\mathcal{N}(\alpha_1, \alpha_2; P)$ is defined only when $-c \leq E(\alpha_2) E(\alpha_1) \leq E_2$.
 - (b) The periodicity and orientation isomorphisms of the P-parametrized interpolation spaces are defined only among those which satisfy $-c \le E(\alpha_2) E(\alpha_1) \le E_2$.
 - (c) The compatibility of the isomorphism of K-spaces at the boundary in Condition 16.23 (X) is defined only when the left hand side of (16.31) is defined.
 - (d) We require Condition 16.26, 16.28 (XI) only when $\mathcal{N}(\alpha_-, \alpha_+; P)$ is defined.

Definition 16.30. Let \mathcal{F}_1 , \mathcal{F}_2 be linear K-systems and \mathfrak{N}_1 , \mathfrak{N}_2 two morphisms between them.

- (1) A homotopy from \mathfrak{N}_1 to \mathfrak{N}_2 is a P = [1,2] parametrized family of morphisms from \mathfrak{N}_1 to \mathfrak{N}_2 such that its boundary is $\mathfrak{N}_1 \sqcup -\mathfrak{N}_2$. Here $-\mathfrak{N}_2$ is \mathfrak{N}_2 with the orientation systems of the critical submanifolds and orientation isomorphisms inverted and \sqcup denotes the disjoint union.
- (2) We say that \mathfrak{N}_1 is homotopic to \mathfrak{N}_2 if there exists a homotopy between them.
- (3) The case of morphisms between partial linear K-systems is defined in the same way.

Theorem 16.31. Let \mathcal{F}_1 , \mathcal{F}_2 , \mathcal{F}_3 be linear K-systems. We make choices mentioned in Theorem 16.9 (2) and obtain cochain complexes $CF(\mathcal{F}_i; \Lambda_{nov})$, i = 1, 2, 3.

(1) A morphism $\mathfrak{N}: \mathcal{F}_1 \to \mathcal{F}_2$ induces a $\Lambda_{\rm nov}$ module homomorphism

$$\psi_{\mathfrak{N}}: CF(\mathcal{F}_1; \Lambda_{\text{nov}}) \to CF(\mathcal{F}_2; \Lambda_{\text{nov}})$$
(16.35)

with the following properties.

- (a) $\psi_{\mathfrak{N}} \circ d = d \circ \psi_{\mathfrak{N}}$, where d in the left hand side (resp. right hand side) is the coboundary operator of $CF(\mathcal{F}_1; \Lambda_{nov})$ (resp. $CF(\mathcal{F}_2; \Lambda_{nov})$.)
- (b) The homomorphism $\psi_{\mathfrak{N}}$ preserves degree and satisfies

$$\psi_{\mathfrak{N}}\left(\mathfrak{F}^{\lambda}CF(\mathcal{F}_1;\Lambda_{\text{nov}})\right)\subset\mathfrak{F}^{\lambda-c}CF(\mathcal{F}_1;\Lambda_{\text{nov}}),$$

where c is the energy loss of \mathfrak{N} . In particular, if c = 0 then $\psi_{\mathfrak{N}}$ induces a $\Lambda_{0,\text{nov}}$ module homomorphism:

$$\psi_{\mathfrak{N}}: CF(\mathcal{F}_1; \Lambda_{0,\text{nov}}) \to CF(\mathcal{F}_2; \Lambda_{0,\text{nov}}).$$

(2) Let $\mathfrak{N}_1, \mathfrak{N}_2 : \mathcal{F}_1 \to \mathcal{F}_2$ be morphisms. If \mathfrak{H} is a homotopy from \mathfrak{N}_1 to \mathfrak{N}_2 , then it induces a Λ_{nov} module homomorphism:

$$\psi_{\mathfrak{H}}: CF(\mathcal{F}_1; \Lambda_{\text{nov}}) \to CF(\mathcal{F}_2; \Lambda_{\text{nov}})$$
 (16.36)

with the following properties.

- (a) $\psi_{\mathfrak{H}} \circ d + d \circ \psi_{\mathfrak{H}} = \psi_{\mathfrak{N}_1} \psi_{\mathfrak{N}_2}$.
- (b) The homomorphism $\psi_{\mathfrak{H}}$ decreases degree by 1 and satisfies

$$\psi_{\mathfrak{H}}\left(\mathfrak{F}^{\lambda}CF(\mathcal{F}_1;\Lambda_{\text{nov}})\right)\subset\mathfrak{F}^{\lambda-c}CF(\mathcal{F}_1;\Lambda_{\text{nov}})$$

where c is the energy loss of \mathfrak{H} . In particular, if c = 0 then $\psi_{\mathfrak{H}}$ induces a $\Lambda_{0,\text{nov}}$ module homomorphism:

$$\psi_{\mathfrak{H}}: CF(\mathcal{F}_1; \Lambda_{0,\text{nov}}) \to CF(\mathcal{F}_2; \Lambda_{0,\text{nov}}).$$

- (3) Let $\mathfrak{N}_{i+1i}: \mathcal{F}_i \to \mathcal{F}_{i+1}$ be morphisms of linear K-systems of for i=1,2. Let $\mathfrak{N}_{31}: \mathcal{F}_1 \to \mathcal{F}_3$ be the composition $\mathfrak{N}_{32} \circ \mathfrak{N}_{21}$. Then the composition $\psi_{\mathfrak{N}_{23}} \circ \psi_{\mathfrak{N}_{12}}$ is cochain homotopic to $\psi_{\mathfrak{N}_{13}}$.
- (4) If $\mathcal{I}D_{\mathcal{F}_i}: \overline{\mathcal{F}}_i \to \mathcal{F}_i$ is the identity morphism, the cochain map $\psi_{\mathcal{I}D_{\mathcal{F}_i}}$ induced by it is cochain homotopic to the identity.

Remark 16.32. The morphism $\psi_{\mathfrak{N}}$ in Item (1) depends on the choices made for its construction. More precisely, we first make choices mentioned in Theorem 16.9 Item (1) to define coboundary operators of $CF(\mathcal{F}_i; \Lambda_{\text{nov}})$, i = 1, 2. Then we make a choice (compatible with the first choices) to define the map $\psi_{\mathfrak{N}}$ which is a cochain map with respect to the coboundary operators obtained from the choices we made. Note that we can use Item (2) to show that up to cochain homotopy the cochain map $\psi_{\mathfrak{H}}$ is independent of the choices we made to define it.

We will define the notion of the identity morphism later in Lemma-Definition 18.60 of Subsection 18.9. The definition of composition of morphisms is given in Subsection 16.5 and the precise definition of the notation used there is postponed until Section 18 where we will discuss 'smoothing corners' systematically. The proof of Theorem 16.31 is given in Section 19.

Definition 16.33. We say two morphisms \mathfrak{N}_1 and \mathfrak{N}_2 are congruent modulo T^E if the following holds.

- (1) $\mathfrak{N}_1(\alpha_1, \alpha_2) \cong \mathfrak{N}_2(\alpha_1, \alpha_2)$ for any α_1, α_2 with $E(\alpha_2) E(\alpha_1) \leq E$.
- (2) The above isomorphism is one between K-spaces. It preserves the orientation, the periodicity isomorphism, and the isomorphisms describing the compatibilities at their boundary and corners.

16.5. Composition of morphisms of linear K-systems. In this subsection we discuss composition of morphisms.

Situation 16.34. Suppose we are in Situation 16.20 for i = 1, 2, 3 and let \mathcal{F}_i be linear K-systems for i = 1, 2, 3.

Lemma-Definition 16.35. Suppose we are in Situation 16.34.

(1) Let $\mathfrak{N}_{i+1i}: \mathcal{F}_i \to \mathcal{F}_{i+1}$ be morphisms and $\mathcal{N}_{ii+1}(\alpha_i, \alpha_{i+1})$ their interpolation spaces. Then we can define the composition $\mathfrak{N}_{32} \circ \mathfrak{N}_{21}: \mathcal{F}_1 \to \mathcal{F}_3$. The interpolation space¹⁹ of $\mathfrak{N}_{32} \circ \mathfrak{N}_{21}$ is the union of the fiber products

$$\bigcup_{\alpha_2 \in \mathfrak{A}_2} \mathcal{N}_{12}(\alpha_1, \alpha_2)_{\text{ev}_+} \times_{\text{ev}_-}^{\boxplus 1} \mathcal{N}_{23}(\alpha_2, \alpha_3). \tag{16.37}$$

The precise meaning of the union \cup in (16.37) is defined during the proof in this subsection and in Section 18. The symbol $\times^{\boxplus 1}$ will be defined in Definition 18.37.

(2) Let $\mathfrak{N}_{i+1i}^{P_i}: \mathcal{F}_i \to \mathcal{F}_{i+1}$ be P_i -parametrized morphisms and $\mathcal{N}_{ii+1}(\alpha_i, \alpha_{i+1}; P_i)$ their interpolation spaces. We can define the composition $\mathfrak{N}_{32}^{P_2} \circ \mathfrak{N}_{21}^{P_1}: \mathcal{F}_1 \to \mathcal{F}_3$ that is a $P_1 \times P_2$ -parametrized morphism. Its interpolation space is

$$\bigcup_{\alpha_2 \in \mathfrak{A}_2} \mathcal{N}_{12}(\alpha_1, \alpha_2; P_1)_{\text{ev}_+} \times_{\text{ev}_-}^{\boxplus 1} \mathcal{N}_{23}(\alpha_2, \alpha_3; P_2). \tag{16.38}$$

The precise meaning of the union \cup in (16.38) is defined during the proof in this subsection and in Section 18. The symbol $\times^{\boxplus 1}$ will be also defined in Definition 18.37.

- (3) The energy loss of the morphism $\mathfrak{N}_{32} \circ \mathfrak{N}_{21}$ is the sum of the energy loss of \mathfrak{N}_{32} and of \mathfrak{N}_{21} . The same holds for the P_i parametrized version.
- (4) We have $\mathfrak{N}_{43} \circ (\mathfrak{N}_{32} \circ \mathfrak{N}_{21}) = (\mathfrak{N}_{43} \circ \mathfrak{N}_{32}) \circ \mathfrak{N}_{21}$. The same holds for the P_i parametrized version.
- (5) We have

$$\partial(\mathfrak{N}_{32}^{P_2}\circ\mathfrak{N}_{21}^{P_1}) = (\mathfrak{N}_{32}^{\partial P_2}\circ\mathfrak{N}_{21}^{P_1}) \cup (\mathfrak{N}_{32}^{P_2}\circ\mathfrak{N}_{21}^{\partial P_1}) \tag{16.39}$$

The boundary in the left hand side is taken in the sense of Definition 16.29 (2). The precise meaning of the union \cup in the right hand side is defined during the proof in this subsection and by Definition-Lemma 18.56

(6) We can generalize (1)-(5) to the case of partial linear K-systems.

Idea of the proof. Here we explain only the basic geometric idea of the proof. To give a rigorous proof, there is an issue about providing a precise definition of 'smoothing corner' in the definition of the union in (16.37) and (16.38). We postpone this point until Section 18.

(1) We first explain the meaning of the union in (16.37). Note that 20

$$\bigcup_{\alpha_2 \in \mathfrak{A}_2} \partial \left(\mathcal{N}_{23}(\alpha_2, \alpha_3)_{\text{ev}_-} \times_{\text{ev}_+} \mathcal{N}_{12}(\alpha_1, \alpha_2) \right)$$

$$\supseteq \bigcup_{\alpha_2 \in \mathfrak{A}_2} \partial \mathcal{N}_{23}(\alpha_2, \alpha_3)_{\text{ev}_-} \times_{\text{ev}_+} \mathcal{N}_{12}(\alpha_1, \alpha_2)$$

$$(16.40)$$

$$\bigcup_{\alpha_2 \in \mathfrak{A}_2} (-1)^{\dim \mathcal{N}_{23}(\alpha_2,\alpha_3) + \dim R_{\alpha_2}} \mathcal{N}_{23}(\alpha_2,\alpha_3)_{\operatorname{ev}_-} \times_{\operatorname{ev}_+} \partial \mathcal{N}_{12}(\alpha_1,\alpha_2).$$

See [FOOO4, Lemma 8.2.3 (1)] for the sign. We also note that the fiber product

$$\mathcal{N}_{12}(\alpha_1, \alpha_2)_{\text{ev}_+} \times_{\text{ev}_-} \mathcal{M}^2(\alpha_2, \alpha_2')_{\text{ev}_+} \times_{\text{ev}_-} \mathcal{N}_{23}(\alpha_2', \alpha_3)$$
 (16.41)

appears in both of the second and third lines of (16.40). To construct the union in (16.37), we glue several components along the codimension one boundaries such as

¹⁹We denote by \mathcal{N}_{12} the interpolation space of the morphism \mathfrak{N}_{21} . See Remark 18.36.

²⁰See Remark 16.2 for the order of factors and signs.

(16.41). We note that those parts we glue contain certain (higher codimensional) corners.²¹ See Figure 2. For this purpose we use the notion of smoothing corners of

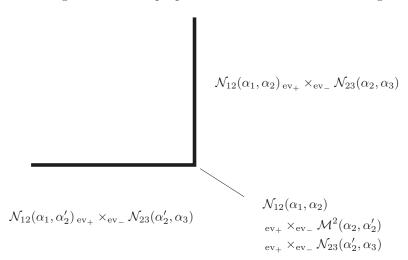


FIGURE 2. (16.41) looks like a corner

Kuranishi structure. We will discuss the smoothing in detail in Subsections 18.3-18.5. We also observe that there exists a certain K-space such that the disjoint union of the summands of (16.37) appears in its boundary. (See Proposition 18.38.) Combining them, we can put the structure of K-space on the union (16.37). See Lemma-Definition 18.40. Thus we obtain the interpolation space of the composition

$$\mathfrak{N}_{32} \circ \mathfrak{N}_{21}$$
.

Then it is straightforward to check that this interpolation space satisfies the defining conditions of morphism.

The proof of (2) is entirely smilar to the proof of (1). (See Subsection 18.8.) (3) is immediate from the definition.

We next discuss the proof of (4). Note that both sides of the equality are a union

$$\bigcup_{\alpha_2,\alpha_3} \mathcal{N}_{12}(\alpha_1,\alpha_2)_{\operatorname{ev}_+} \times_{\operatorname{ev}_-} \mathcal{N}_{23}(\alpha_2,\alpha_3)_{\operatorname{ev}_+} \times_{\operatorname{ev}_-} \mathcal{N}_{34}(\alpha_3,\alpha_4).$$

While we define this union from the disjoint union, we perform the process of smoothing corner and gluing K-spaces along the boundary twice which we described during the proof of (1). Once at the boundary components of the form

$$\mathcal{N}_{12}(\alpha_1, \alpha_2')_{\text{ev}_+} \times_{\text{ev}_-} \mathcal{M}^2(\alpha_2', \alpha_2)_{\text{ev}_+} \times_{\text{ev}_-} \mathcal{N}_{23}(\alpha_2, \alpha_3)_{\text{ev}_+} \times_{\text{ev}_-} \mathcal{N}_{34}(\alpha_3, \alpha_4)$$
 and once at the boundary components of the form

$$\mathcal{N}_{12}(\alpha_1, \alpha_2)_{\text{ev}_+} \times_{\text{ev}_-} \mathcal{N}_{23}(\alpha_2, \alpha_3')_{\text{ev}_+} \times_{\text{ev}_-} \mathcal{M}^3(\alpha_3', \alpha_3)_{\text{ev}_+} \times_{\text{ev}_-} \mathcal{N}_{34}(\alpha_3, \alpha_4).$$

Since these two components do not intersect at the interior, we can perform these two process independently and can exchange the order of them. (4) follows. (There is an issue of showing that this isomorphism is compatible with the smooth structure we gave during the proof of (1). See Subsection 18.7.)

²¹So the order of factors in (16.41) does not matter as we note in Remark 16.2.

We next prove (5). By (16.31) applied to $\mathcal{N}_{12}(\alpha_1, \alpha_2; P_1)$ and to $\mathcal{N}_{23}(\alpha_2, \alpha_3; P_2)$, the normalized boundary of the interpolation space of $\mathfrak{N}_{32}^{P_2} \circ \mathfrak{N}_{21}^{P_1}$ is a disjoint union of the following four kinds of components:

- (A) $\mathcal{N}_{12}(\alpha_1, \alpha_2; \partial P_1) \times_{R_{\alpha_2}} \mathcal{N}_{23}(\alpha_2, \alpha_3; P_2)$.
- (B) $\mathcal{N}_{12}(\alpha_1, \alpha_2; P_1) \times_{R_{\alpha_2}} \mathcal{N}_{23}(\alpha_2, \alpha_3; \partial P_2).$
- (C) $\mathcal{M}^1(\alpha_1, \alpha_1') \times_{R_{\alpha_1'}} \tilde{\mathcal{N}}_{12}(\alpha_1', \alpha_2; P_1) \times_{R_{\alpha_2}} \mathcal{N}_{23}(\alpha_2, \alpha_3; P_2).$
- (D) $\mathcal{N}_{12}(\alpha_1, \alpha_2; P_1) \stackrel{1}{\times}_{R_{\alpha_2}} \mathcal{N}_{23}(\alpha_2, \alpha_3'; P_2) \times_{R_{\alpha_2'}} \mathcal{M}^3(\alpha_3', \alpha_3')$.

Note that potentially there are components of the form

$$\mathcal{N}_{12}(\alpha_1, \alpha_2'; P_1) \times_{R_{\alpha_2'}} \mathcal{M}^2(\alpha_2', \alpha_2) \times_{R_{\alpha_2}} \mathcal{N}_{23}(\alpha_2, \alpha_3; P_2)$$

$$(16.42)$$

in the boundary of the interpolation space of $\mathfrak{N}_{32}^{P_2} \circ \mathfrak{N}_{21}^{P_1}$. However (16.42) appears twice in the boundary of the interpolation space of $\mathfrak{N}_{32}^{P_2} \circ \mathfrak{N}_{21}^{P_1}$. Once in

$$\partial \mathcal{N}_{12}(\alpha_1, \alpha_2; P_1) \times_{R_{\alpha_2}} \mathcal{N}_{23}(\alpha_2, \alpha_3; P_2)$$

and once in

$$\mathcal{N}_{12}(\alpha_1, \alpha_2; P_1) \times_{R_{\alpha_2}} \partial \mathcal{N}_{23}(\alpha_2, \alpha_3; P_2).$$

When we define the interpolation space of $\mathfrak{N}_{32}^{P_2} \circ \mathfrak{N}_{21}^{P_1}$, we glue

$$\mathcal{N}_{12}(\alpha_1, \alpha_2; P_1) \times_{R_{\alpha_2}} \mathcal{N}_{23}(\alpha_2, \alpha_3; P_2)$$

for various α_2 along (16.42). (See Definition-Lemma 18.56 for detail.) Therefore those components of the form (16.42) do not appear in the boundary of the interpolation space of $\mathfrak{N}_{32}^{P_2} \circ \mathfrak{N}_{21}^{P_1}$. Hence the disjoint union of (A)-(D) is the normalized boundary of the interpolation space of $\mathfrak{N}_{32}^{P_2} \circ \mathfrak{N}_{21}^{P_1}$.

On the other hand, the right hand side of (16.39) is the union of the components of types (A),(B). Note that by Definition 16.29 (2) and (16.31), we have

(The interpolation space of
$$\partial(\mathfrak{N}_{32}^{P_2}\circ\mathfrak{N}_{21}^{P_1}))\cup(\mathbf{C})\cup(\mathbf{D})$$

$$= \partial \left(\text{The interpolation space of } \left(\mathfrak{N}^{P_2}_{32} \circ \mathfrak{N}^{P_1}_{21} \right) \right).$$

Then (5) follows from these facts.

The proof of
$$(6)$$
 is the same as the proofs of (1) - (5) .

16.6. Inductive system of linear K-systems.

Definition 16.36. Let C be a critical submanifold data as in Definition 16.6.

- (1) Let $E_0 < E_1$. A partial linear K-system with energy cut level E_1 induces a partial linear K-system with energy cut level E_0 by forgetting all the structures of the former which do not appear in the definition of the latter. The same applies to the morphism, homotopy etc. We call these processes the energy cut at E_0 .
- (2) An inductive system of partial linear K-systems

$$\mathcal{FF} = (\{E^i\}, \{\mathcal{F}^i\}, \{\mathfrak{N}^i\})$$

consists of the following objects.²²

(a) We are given an increasing sequence E^i of positive numbers such that $\lim_{i\to\infty}E^i=\infty$.

 $^{^{22}}$ Hereafter we use the superscript i as the suffix of the inductive system.

- (b) For each i we are given a partial linear K-system with energy cut level E^i , which we denote by \mathcal{F}^i .
- (c) The critical submanifold data

$$\mathcal{C} = \left(\mathfrak{A}, \mathfrak{G}, \{R_{\alpha}\}_{{\alpha} \in \mathfrak{A}}, \{o_{R_{\alpha}}\}_{{\alpha} \in \mathfrak{A}}, E, \mu, \{\operatorname{PI}_{\beta, \alpha}\}_{\beta \in \mathfrak{G}, {\alpha} \in \mathfrak{A}}\right)$$

that is a part of data of \mathcal{F}^i , is independent of i and is given at the beginning.

- (d) For each i we are given a morphism $\mathfrak{N}^i: \mathcal{F}^i \to \mathcal{F}^{i+1}|_{E^i}$, where $\mathcal{F}^{i+1}|_{E^i}$ is the partial linear K-system of energy cut level E^i that is induced from \mathcal{F}^{i+1} by the energy cut.
- (e) The energy loss of \mathfrak{N}^i is 0.
- (f) \mathfrak{N}^i is congruent to the identity morphism modulo ϵ_i for some $\epsilon_i > 0$. See Definition 16.33.
- (g) We assume the following uniform Gromov compactness. For any $E_0 \geq$ 0 and $\alpha_1 \in \mathfrak{A}_1$ the set

$$\{\alpha_2 \in \mathfrak{A}_2 \mid \exists i \, \mathcal{N}^i(\alpha_1, \alpha_2) \neq \emptyset, \, E(\alpha_2) \leq E_0 + E(\alpha_1)\}$$
 (16.43)

is a finite set. Moreover

$$\{\alpha_2 \in \mathfrak{A}_2 \mid \exists i \ \mathcal{M}^i(\alpha_1, \alpha_2) \neq \emptyset, \ E(\alpha_2) \leq E_0 + E(\alpha_1)\}$$
 (16.44)

is a finite set. Here $\mathcal{N}^i(\alpha_1,\alpha_2)$ is the interpolation space of the morphism \mathfrak{N}^i and $\mathcal{M}^i(\alpha_1, \alpha_2)$ is the space of connecting orbits of \mathcal{F}^i .

(3) Let $\mathcal{FF}_j = (\{E_i^i\}, \{\mathcal{F}_i^i\}, \{\mathfrak{N}_i^i\})$ (j=1,2) be two inductive systems of partial linear K-systems. We assume $E_1^i \geq E_2^i - c$ for some $c \geq 0$. A morphism

$$(\{\mathfrak{N}_{21}^i\}, \{\mathfrak{H}^i\}): \mathcal{FF}_1 \to \mathcal{FF}_2$$

with energy loss c is a pair of $\{\mathfrak{N}_{21}^i\}$ and $\{\mathfrak{H}^i\}$ with the following properties:

- (a) $\mathfrak{N}_{21}^i: \mathcal{F}_1^i \to \mathcal{F}_2^i$ is a morphism of energy loss c.
- (b) \mathfrak{H}^i is a homotopy between $\mathfrak{N}_2^i \circ \mathfrak{N}_{21}^i$ and $\mathfrak{N}_{21}^{i+1} \circ \mathfrak{N}_1^i$. Here we regard them as morphisms with energy cut level E_2^i and of energy loss c.
- (4) In the situation of (3), let $(\{\mathfrak{N}^{i}_{(k)21}\}, \{\mathfrak{H}^{i}_{(k)}\}): \mathcal{FF}_1 \to \mathcal{FF}_2$ be morphisms for k = a, b. A homotopy from $(\{\mathfrak{N}_{(a)21}^i\}, \{\mathfrak{H}_{(a)}^i\})$ to $(\{\mathfrak{N}_{(b)21}^i\}, \{\mathfrak{H}_{(b)}^i\})$ is the pair $(\{\mathfrak{H}^i_{(ab)}\}, \{\mathcal{H}^i\})$ with the following properties:
 - (a) $\mathfrak{H}^{i}_{(ab)}$ is a homotopy from $\mathfrak{N}^{i}_{(a)21}$ to $\mathfrak{N}^{i}_{(b)21}$.
 - (b) \mathcal{H}^i is a $[0,1]^2$ parametrized morphism from \mathcal{F}_1^i to \mathcal{F}_2^{i+1} . ²³
 - (c) The normalized boundary $\partial \mathcal{H}^i$ is a disjoint union of the following 4 homotopies, see Figure 3:
 - (i) $\mathfrak{H}^{i}_{(a)}$
 - (ii) $\mathfrak{N}_{2}^{i} \circ \mathfrak{H}_{(ab)}^{i}$

 - (iii) $\mathfrak{H}^{i}_{(b)}$ (iv) $\mathfrak{H}^{i+1}_{(ab)} \circ \mathfrak{N}^{i}_{1}$
 - (d) For any $E_0 \geq 0$ and $\alpha_1 \in \mathfrak{A}_1$ the set

$$\{\alpha_2 \in \mathfrak{A}_2 \mid \exists i \ \mathcal{N}(\alpha_1, \alpha_2; \mathfrak{H}^i_{(ab)}) \neq \emptyset, \ E(\alpha_2) \leq E_0 + E(\alpha_1)\}$$
 (16.45)

²³In other words it is a homotopy of homotopies.

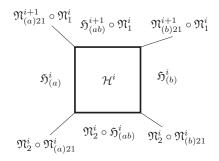


FIGURE 3. homotopy of homotopies \mathcal{H}^i

is a finite set. Moreover

$$\{\alpha_2 \in \mathfrak{A}_2 \mid \exists i \ \mathcal{N}(\alpha_1, \alpha_2; \mathcal{H}^i) \neq \emptyset, \ E(\alpha_2) \leq E_0 + E(\alpha_1)\}$$
 (16.46)

is a finite set. Here $\mathcal{N}(\alpha_1, \alpha_2; \mathfrak{H}^i_{(ab)})$ (resp. $\mathcal{N}(\alpha_1, \alpha_2; \mathcal{H}^i)$) is the interpolation space which we used to define $\mathfrak{H}^i_{(ab)}$ (resp. \mathcal{H}^i .).

Here all the (parametrized) morphisms have energy cut level E_2^i .

(5) Two morphisms of inductive systems of partial linear K-systems are said to be *homotopic* if there exists a homotopy between them.

Remark 16.37. In certain cases especially in the study of symplectic homology (see [FH], [CFH], [BO]) and wrapped Floer homology (see [AS]), we need to study the case when the critical submanifold data $\mathcal C$ varies. We can actually study such a situation in a similar way. However we do not try to work it out in this article because this is an expository article providing the technical detail of the results which had been established in the previously published literatures.

Lemma-Definition 16.38. (1) We can compose morphisms of inductive system of partial linear K-systems.

- (2) Composition of homotopic morphisms are homotopic.
- (3) Homotopy between morphisms is an equivalence relation.

Proof. (1) Let $(\{\mathfrak{N}^i_{j+1j}\}, \{\mathfrak{S}^i_{(j+1j)}\}): \mathcal{FF}_j \to \mathcal{FF}_{j+1}$ be morphisms for j=1,2. We put

$$\mathfrak{N}_{31}^i = \mathfrak{N}_{32}^i \circ \mathfrak{N}_{21}^i.$$

Then $\mathfrak{H}^i_{(32)} \circ \mathfrak{N}^i_{21}$ is a homotopy from $\mathfrak{N}^i_3 \circ \mathfrak{N}^i_{31} = \mathfrak{N}^i_3 \circ \mathfrak{N}^i_{32} \circ \mathfrak{N}^i_{21}$ to $\mathfrak{N}^{i+1}_{32} \circ \mathfrak{N}^i_2 \circ \mathfrak{N}^i_{21}$ and $\mathfrak{N}^{i+1}_{32} \circ \mathfrak{N}^i_{(21)}$ is a homotopy from $\mathfrak{N}^{i+1}_{32} \circ \mathfrak{N}^i_2 \circ \mathfrak{N}^i_{21}$ to $\mathfrak{N}^{i+1}_{31} \circ \mathfrak{N}^i_1 = \mathfrak{N}^{i+1}_{32} \circ \mathfrak{N}^{i+1}_{21} \circ \mathfrak{N}^i_1$. Therefore we can construct $\mathfrak{H}^i_{(31)}$ by gluing $\mathfrak{H}^i_{(32)} \circ \mathfrak{N}^i_{21}$ and $\mathfrak{N}^{i+1}_{32} \circ \mathfrak{N}^i_{(21)}$. (See Subsubsection 18.8.2 for gluing process. .) See Figure 4.

(2) Let $(\{\mathfrak{N}^i_{k,21}\}, \{\mathfrak{S}^i_{(k,21)}\}): \mathcal{FF}_1 \to \mathcal{FF}_2$ be morphisms for k=a,b and let $(\{\mathfrak{N}^i_{32}\}, \{\mathfrak{S}^i_{(32)}\}): \mathcal{FF}_2 \to \mathcal{FF}_3$ be a morphism. Let $(\{\mathfrak{S}^i_{(ab)}\}, \{\mathcal{H}^i\})$ be a homotopy from $(\{\mathfrak{N}^i_{a,21}\}, \{\mathfrak{S}^i_{(a,21)}\})$ to $(\{\mathfrak{N}^i_{b,21}\}, \{\mathfrak{S}^i_{(b,21)}\})$. Then we have

$$\partial(\mathfrak{N}_{32}^{i+1} \circ \mathcal{H}^{i}) = \mathfrak{N}_{32}^{i+1} \circ \mathfrak{H}_{(ab)}^{i+1} \circ \mathfrak{N}_{1}^{i} \cup \mathfrak{N}_{32}^{i+1} \circ \mathfrak{H}_{(b,21)}^{i} \\ \cup \mathfrak{N}_{32}^{i+1} \circ \mathfrak{N}_{2}^{i} \circ \mathfrak{H}_{(ab)}^{i} \cup \mathfrak{N}_{32}^{i+1} \circ \mathfrak{H}_{(a,21)}^{i}$$

$$(16.47)$$

FIGURE 4. composition of homotopies

and

$$\begin{array}{c} \partial(\mathfrak{H}^{i}_{(32)}\circ\mathfrak{H}^{i}_{(ab)}) = & \mathfrak{M}^{i+1}_{32}\circ\mathfrak{M}^{i}_{2}\circ\mathfrak{H}^{i}_{(ab)}\cup\mathfrak{H}^{i}_{32}\circ\mathfrak{M}^{i}_{b,21} \\ & \cup \mathfrak{M}^{i}_{3}\circ\mathfrak{M}^{i}_{32}\circ\mathfrak{H}^{i}_{(ab)}\cup\mathfrak{H}^{i}_{32}\circ\mathfrak{M}^{i}_{a,21}. \end{array} \tag{16.48}$$

We note that both are $[0,1]^2$ -parametrized families of morphisms. By the definition of composition we have

$$\begin{split} \mathfrak{N}_{32}^{i+1} \circ \mathfrak{H}_{(a,21)}^{i} & \cup \mathfrak{H}_{32}^{i} \circ \mathfrak{N}_{a,21}^{i} = \mathfrak{H}_{(a,31)}^{i}, \\ \mathfrak{N}_{32}^{i+1} \circ \mathfrak{H}_{(b,21)}^{i} & \cup \mathfrak{H}_{32}^{i} \circ \mathfrak{N}_{b,21}^{i} = \mathfrak{H}_{(b,31)}^{i}. \end{split}$$

We put

$$\begin{split} \mathfrak{H}^{i\prime}_{(ab)} &= \mathfrak{N}^i_{32} \circ \mathfrak{H}^i_{(ab)} \\ \mathcal{H}^{i\prime} &= (\mathfrak{N}^{i+1}_{32} \circ \mathcal{H}^i) \cup_{\mathfrak{N}^{i+1}_{32} \circ \mathfrak{N}^i_{2} \circ \mathfrak{H}^i_{(ab)}} (\mathfrak{H}^i_{(32)} \circ \mathfrak{H}^i_{(ab)}). \end{split}$$

See Figure 5. Here in the right hand side of the second formula, we glue two $[0, 1]^2$ -parametrized morphisms along one of the components of their boundaries. We can do it in the same way as the proof of Lemma-Definition 16.35. See Subsubsection 18.8.2 for detail. We can then easily see that

$$\partial(\mathcal{H}^{i\prime})=\mathfrak{H}^{i}_{(a,31)}\cup\mathfrak{H}^{i}_{(b,31)}\cup(\mathfrak{N}^{i}_{3}\circ\mathfrak{H}^{i}_{(ab)})\cup(\mathfrak{H}^{i+1}_{(ab)}\circ\mathfrak{N}^{i}_{1}).$$

Namely $(\{\mathfrak{N}_{32}^i\}, \{\mathfrak{H}_{(32)}^i\}) \circ (\{\mathfrak{N}_{a,21}^i\}, \{\mathfrak{H}_{(a,21)}^i\})$ is homotopic to $(\{\mathfrak{N}_{32}^i\}, \{\mathfrak{H}_{(32)}^i\}) \circ (\{\mathfrak{N}_{b,21}^i\}, \{\mathfrak{H}_{(b,21)}^i\})$. The case of the composition of homotopies and morphisms in

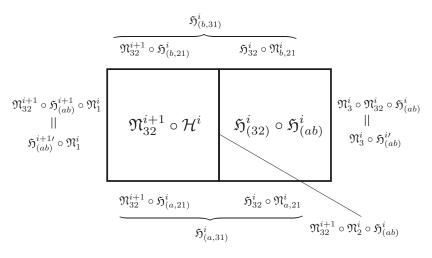


Figure 5. $\mathcal{H}^{i\prime}$

the opposite direction is similar.

(3) will be proved in Subsubsection 18.8.2.

Theorem 16.39. Let $\mathcal{FF} = (\{E^i\}, \{\mathcal{F}^i\}, \{\mathfrak{N}^i\})$ be an inductive system of partial linear K-systems. Note that the $\Lambda_{0,nov}$ module $CF(\mathcal{F}^i; \Lambda_{0,nov})$ is independent of i, which we denote by $CF(\mathcal{FF}; \Lambda_{0,nov})$.

- (1) To our inductive system of partial linear K-systems \mathcal{FF} , we can associate a map $d: CF(\mathcal{FF}; \Lambda_{0,nov}) \to CF(\mathcal{FF}; \Lambda_{0,nov})$ such that:
 - (a) $d \circ d = 0$.
 - (b) There exists $\epsilon > 0$ such that:

$$(d-d_0)(\mathfrak{F}^{\lambda}CF(\mathcal{F}\mathcal{F};\Lambda_{0,\text{nov}}))\subset \mathfrak{F}^{\lambda+\epsilon}CF(\mathcal{F}\mathcal{F};\Lambda_{0,\text{nov}}).$$

See (16.17) for the definition of d_0 .

- (2) The definition of the map d in (1) involves various choices and d depends on them. However it is independent of such choices in the following sense: Suppose d_1 , d_2 are obtained by two different choices. Then there exists a map $\psi: CF(\mathcal{FF}; \Lambda_{0,nov}) \to CF(\mathcal{FF}; \Lambda_{0,nov})$ with the following properties.
 - (a) $d_2 \circ \psi = \psi \circ d_1$.
 - (b) ψ is degree 0 and preserves the filtration.
 - (c) There exists $\epsilon > 0$ such that:

$$(\psi - \mathrm{id})(\mathfrak{F}^{\lambda}CF(\mathcal{F}\mathcal{F}; \Lambda_{0,\mathrm{nov}})) \subset \mathfrak{F}^{\lambda + \epsilon}CF(\mathcal{F}\mathcal{F}; \Lambda_{0,\mathrm{nov}})$$

where id is the identity map.

(d) In particular, ψ induces an isomorphism on cohomologies:

$$H(CF(\mathcal{FF}; \Lambda_{0 \text{ nov}}), d_1) \to H(CF(\mathcal{FF}; \Lambda_{0 \text{ nov}}), d_2).$$

- (e) ψ depends on various choices but it is independent of the choices up to cochain homotopy.
- (3) A morphism $\mathfrak{NN}: \mathcal{FF}_1 \to \mathcal{FF}_2$ induces a Λ_{nov} module homomorphism

$$\psi_{\mathfrak{MM}}: CF(\mathcal{FF}_1; \Lambda_{0,\text{nov}}) \to CF(\mathcal{FF}_2; \Lambda_{0,\text{nov}})$$
 (16.49)

with the following properties.

- (a) $\psi_{\mathfrak{NN}} \circ d = d \circ \psi_{\mathfrak{NN}}$, where d in the left hand side (resp. right hand side) is the coboundary oprator of $CF(\mathcal{FF}_1; \Lambda_{0,nov})$ (resp. $CF(\mathcal{FF}_2; \Lambda_{0,nov})$.)
- (b) $\psi_{\mathfrak{N}\mathfrak{N}}$ preserves degree and

$$\psi_{\mathfrak{M}\mathfrak{N}}\left(\mathfrak{F}^{\lambda}CF(\mathcal{F}\mathcal{F}_1;\Lambda_{\mathrm{nov}})\right)\subset\mathfrak{F}^{\lambda-c}CF(\mathcal{F}\mathcal{F}_1;\Lambda_{\mathrm{nov}})$$

where c is the energy loss of \mathfrak{NN} . In particular, if c = 0 then $\psi_{\mathfrak{NN}}$ induces a $\Lambda_{0,nov}$ module homomorphism:

$$\psi_{\mathfrak{MM}}: CF(\mathcal{FF}_1; \Lambda_{0,\text{nov}}) \to CF(\mathcal{FF}_2; \Lambda_{0,\text{nov}}).$$

(4) Let \mathfrak{NN}_a , \mathfrak{NN}_b : $\mathcal{FF}_1 \to \mathcal{FF}_2$ be morphisms. If \mathfrak{HH} is a homotopy from \mathfrak{NN}_a to \mathfrak{NN}_b , it induces a Λ_{nov} module homomorphism:

$$\psi_{55}: CF(\mathcal{FF}_1; \Lambda_{\text{nov}}) \to CF(\mathcal{FF}_2; \Lambda_{\text{nov}})$$
 (16.50)

with the following properties.

(a) $\psi_{\mathfrak{H}\mathfrak{H}} \circ d + d \circ \psi_{\mathfrak{H}} = \psi_{\mathfrak{M}\mathfrak{N}_1} - \psi_{\mathfrak{M}\mathfrak{N}_2}$.

(b) ψ_{55} decreases degree by 1 and

$$\psi_{\mathfrak{H}\mathfrak{H}}\left(\mathfrak{F}^{\lambda}CF(\mathcal{F}\mathcal{F}_{1};\Lambda_{\mathrm{nov}})\right)\subset\mathfrak{F}^{\lambda-c}CF(\mathcal{F}\mathcal{F}_{1};\Lambda_{\mathrm{nov}})$$

where c is the energy loss of \mathfrak{HH} . In particular, if c=0 then $\psi_{\mathfrak{HH}}$ induces a $\Lambda_{0,\mathrm{nov}}$ module homomorphism :

$$\psi_{\mathfrak{H}\mathfrak{H}}: CF(\mathcal{F}\mathcal{F}_1; \Lambda_{0,\text{nov}}) \to CF(\mathcal{F}\mathcal{F}_2; \Lambda_{0,\text{nov}}).$$

(5) Let \mathcal{FF}_a , \mathcal{FF}_b , \mathcal{FF}_c be inductive systems of partial linear K-systems and $\mathfrak{MM}_{ba}: \mathcal{FF}_a \to \mathcal{FF}_b$, $\mathfrak{MM}_{cb}: \mathcal{FF}_b \to \mathcal{FF}_c$ be morphisms. Then we have

$$\psi_{\mathfrak{N}\mathfrak{N}^{cb} \circ \mathfrak{N}\mathfrak{N}^{ba}} \sim \psi_{\mathfrak{N}\mathfrak{N}^{cb}} \circ \psi_{\mathfrak{N}\mathfrak{N}^{ba}}$$

where \sim means cochain homotopic.

(6) The identity morphism induces a cochain map which is cochain homotopic to the identity map.

The proof will be given in Section 19. The notion of identity morphism which appears in Item (6) will be defined in Section 18.9.

Remark 16.40. The morphism $\psi_{\mathfrak{NM}}$ in Item (3) depends on the choices made for its construction. More precisely, we first make choices mentioned in Item (2) to define coboundary operators of $CF(\mathcal{FF}_i; \Lambda_{\text{nov}})$, i=1,2. Then we make a choice (compatible with the first choices) to define the map $\psi_{\mathfrak{NM}}$ which is a cochain map with respect to the coboundary operators obtained from the choices we made. Note that we can use Item (4) to show that up to cochain homotopy the cochain map $\psi_{\mathfrak{NM}}$ is independent of the choices we need to make to define it.

Definition 16.41. We call the cohomology group of $(CF(\mathcal{FF}; \Lambda_{0,nov}), d)$ the *Floer cohomology* of an inductive system of partial linear K-systems \mathcal{FF} and denote the cohomology by $HF(\mathcal{FF}; \Lambda_{0,nov})$.

- 17. Extension of Kuranishi structure and its perturbation from boundary to its neighborhood
- 17.1. Introduction to Section 17. In Section 16 we formulated various versions of corner compatibility conditions. To prove the results stated in Section 16 (and which also appear in other places of this article and will appear in future) we need to extend the Kuranishi structure given on the boundary ∂X satisfying corner compatibility conditions to one on X. More precisely, we will start with the situation where we have a Kuranishi structure $\widehat{\mathcal{U}}$ on X and a Kuranishi structure $\widehat{\mathcal{U}}^+$ on ∂X such that

$$\partial \widehat{\mathcal{U}} < \widehat{\mathcal{U}_{\partial}^+}$$

and want to find a Kuranishi structure $\widehat{\mathcal{U}^+}$ on X such that

$$\partial \widehat{\mathcal{U}^+} = \widehat{\mathcal{U}^+_{\partial}}, \quad \widehat{\mathcal{U}} < \widehat{\mathcal{U}^+}.$$

We also need a similar statement for CF-perturbations.

It is rather easy to show that if a Kuranishi structure is defined on a neighborhood Ω of a compact set K then we can extend the Kuranishi structure without changing it in a neighborhood of K. (See Lemma 17.59.) This statement however is not enough to prove the existence of extension in the above situation, since there we are given a Kuranishi structure on ∂X only and not on its neighborhood. In this

section, we discuss the problem of extending a Kuranishi structure and a CF-perturbations on ∂X to its neighborhood.

Remark 17.1. If we carefully examine the whole proofs of the geometric applications appeared in previous literatures such as [FOn],[FOOO3],[FOOO4], we will find out that, for the Kuranishi structure $\widehat{\mathcal{U}_{\partial}^+}$ we actually use, an extension to its small neighborhood of ∂X in X is given from its construction. (This is the reason why in our earlier papers we did not elaborate on the statement and its proof given in this section.) In fact, in the actual situation of applications, we start from a Kuranishi structure $\partial \widehat{\mathcal{U}}$ on the boundary (which we obtain from geometry or analysis) and construct a good coordinate system $\widehat{\mathcal{U}_{\partial}}$ and use it to find $\widehat{\mathcal{U}_{\partial}^+}$ and its perturbation. We need to extend $\widehat{\mathcal{U}_{\partial}^+}$ and its perturbation to a neighborhood of ∂X . In this situation, the Kuranishi charts of $\widehat{\mathcal{U}_{\partial}^+}$ are obtained as open subcharts of certain Kuranishi charts of $\partial \widehat{\mathcal{U}}$. (See the proof of [Part I, Theorem 3.30, Proposition 6.44].) Therefore it can indeed be extended using the extension of $\partial \widehat{\mathcal{U}}$, directly.

Nevertheless, the reason why we prove these extension results in this article is as follows. In this article we want to present various parts of the proofs of the whole story into a 'package' as much as possible. In other words, we want to decompose whole proofs of the geometric results into pieces, so that each piece can be stated and proved rigorously and independently from other pieces. Namely, for each divided 'package', we want to state the precise assumption and conclusion together with the proof. In this way, one can use and quote each 'package' without referring their proofs. We want to do so because the whole story has now grown huge which becomes harder to follow all at once.

For this purpose, we want to specify and restrict the information which we 'remember' at each step of the inductive construction of the K-system and its perturbations. When we construct a system of virtual fundamental chains of, for example, $\mathcal{M}(\alpha_-,\alpha_+)$, we use, as an induction hypothesis, a certain structure on $\mathcal{M}(\alpha,\alpha')$ for $E(\alpha_-) \leq E(\alpha) < E(\alpha') \leq E(\alpha_+)$ (and $E(\alpha') - E(\alpha) < E(\alpha_+) - E(\alpha_-)$) and use only those structures during our construction. We need to make a careful choice of the structures we 'remember' during the inductive steps, when we proceed from one step of the induction to the next, so that the induction works. Our choice in this article is that we 'remember' Kuranishi structure and its perturbation but forget the good coordinate system (which we use to construct the perturbation) and objects on it. We also forget the way how those Kuranishi structures and various objects on them are constructed, but explicitly list up all the properties we use in the next step of the induction. Therefore the relation between $\widehat{\mathcal{U}}_{\partial}^+$ and $\partial \widehat{\mathcal{U}}$ (except $\partial \widehat{\mathcal{U}} < \widehat{\mathcal{U}}_{\partial}^+$) is among the data which we forget when we proceed to the next step of the induction.

The simplest version of the extension result we mentioned above is the following.

(*) Suppose we are given a continuous function f on $\partial([0,1)^n)$. We assume that the restriction of f to $[0,1)^k \times \{0\} \times [0,1)^{n-k-1}$ is smooth for any k. Then f is extended to a smooth function on $[0,1)^n$.

The statement (*) is, of course, classical. (See [FOOO7, Lemma 7.2.121] for its proof, for example.) We can use this statement together with various technique of manifold theory (such as partition of unity and induction on the number of

coordinate charts) to prove the existence of an extension of CF-perturbation if we include enough structure into the assumptions. (However the proof is rather cumbersome to work out in detail. In fact, it uses triple induction. Two of the triple are indexed by the partially ordered set appearing in the definition of good coordinate system. Another is the codimension of the starta of corner structure stratifiation.)

In this section we take a short cut in the following way. Suppose M is a manifold with boundary (but has no corner). Then it is well-known that a neighborhood of ∂M is identified with $\partial M \times [0,\epsilon)$. If M has corners, we can identify a neighborhood of $\overset{\circ}{S}_k(M)$ in M with a twisted product

$$\overset{\circ}{S}_k(M)\tilde{\times}[0,1)^{n-k}$$

that is a fiber bundle over $\overset{\circ}{S}_k(M) = S_k(M) \setminus S_{k+1}(M)$ (see [Part I, Definition 4.13]), whose structure group is a finite group of permutations of (n-k) factors. We also require a compatibility of these structures for various k. Such a structure may be regarded as a special case of the 'system of tubular neighborhoods' of a stratified space introduced by Mather [Ma]. Its existence is claimed and can be proved in the same way as in [Ma]. (See also [FOOO13, Proposition 8.1].)

We can include certain 'topological' objects such as vector bundle (especially obstruction bundle) and generalize the notion of trivialization of the structure in a neighborhood of boundary and corner, and prove its existence without assuming extra conditions. However, we can *not* expect the Kuranishi map respects this trivialization. Namely the Kuranishi map in general may not be constant in the $[0,1)^k$ factor. So we need some discussion to use it on the way how we extend perturbation to a neighborhood of ∂X .

The main idea we use in this section for the short cut is summarized as follows. In case when M is a manifold with boundary, we can attach $\partial M \times [-1,0]$ to M by identifying $\partial M \times \{0\}$ with $\partial M \subset M$, and enhance M to a manifold $M^{\boxplus 1}$ with boundary so that a neighborhood of its boundary is canonically identified with $\partial M \times [-1,0)$. Then we can extend a vector bundle E on M to $M^{\boxplus 1}$ so that its restriction to this neighborhood of the boundary is canonically identified with $E|_{\partial M} \times [-1,0)$. Then its section such as Kuranishi map can be extended to $M^{\boxplus 1}$ so that on $\partial M \times [-1,0)$ it is constant in the [-1,0) direction. In other words, we attach the collar 'outside' of M in place of constructing it 'inside'.

We can perform a similar construction in the case when M has a corner, the case of orbifold, and the case of K-space. Then we replace the moduli spaces such as $\mathcal{M}(\alpha_-, \alpha_+)$ by $\mathcal{M}(\alpha_-, \alpha_+)^{\boxplus 1}$ so that the fiber product description of their boundaries remains the same and their Kuranishi structures have canonical trivializations near the boundary. We can use them to extend the Kuranishi structures and their perturbations given on the boundary to a neighborhood of the boundary.

There is a slight issue about the smoothness of the structure at the point $\partial M \times \{0\}$. The shortest and simplest way to resolve the issue is to use *exponential decay* estimate of various objects appearing there in geometric situations, which we proved in [FOOO4, Lemma A1.58], [FOOO15, Theorems 13.2 and 19.5], [FOOO17]. In the abstract setting, we impose certain admissibility on orbifold, vector bundle, etc., to incorporate such exponential decay property. See Section 25.

Remark 17.2. In fact, we can work in the piecewise smooth category for the purpose of putting the collar outside. However, working in the smooth category rather than the piecewise smooth category reduces the amount of checks we need during various constructions.

We will carry out this idea in detail in this section. It is rather lengthy since we need to write down and check various compatibilities of many objects with our trivialization. However, the compatibilities in almost all cases are fairly obvious. In other words, the proofs are nontrivial only because they are lengthy.

17.2. **Trivialization of corners on one chart.** In this section, for a Kuranishi chart $\mathcal{U} = (U, \mathcal{E}, \psi, s)$, we will define in Lemma-Definition 17.10 a certain Kuranishi chart $\mathcal{U}^{\boxplus \tau}$ at a point x of corners of U, which is called *trivialization of corners*. Geometrically speaking, 'trivialization' means 'trivialization of normal bundle of corners', which gives us certain coordinates on the corners. We consider the following situation in this subsection.

Situation 17.3. Let $\mathcal{U} = (U, \mathcal{E}, \psi, s)$ be a Kuranishi neighborhood of X and

$$x \in \overset{\circ}{S}_k(U).$$

Let $\mathfrak{V}_x = (V_x, E_x, \Gamma_x, \phi_x, \widehat{\phi}_x)$ be an orbifold chart of (U, \mathcal{E}) at x. Recall from Definition 23.22 (also Definition 23.17) that V_x is a manifold and

$$\phi_x: V_x \to X, \quad \widehat{\phi}_x: V_x \times E \to \mathcal{E}.$$

Let s_x be a representative of the Kuranishi map s on \mathfrak{V}_x . We assume that V_x is an open subset of the direct product $\overline{V_x} \times [0,1)^k$ where $\overline{V_x}$ is a manifold without boundary and that $o_x \in V_x$ corresponds to $(\overline{o}_x, (0, \dots, 0)) \in \overline{V_x} \times [0,1)^k$. For $y \in V_x \subset \overline{V_x} \times [0,1)^k$ we denote by $\overline{y} \in \overline{V_x}$ the $\overline{V_x}$ -component of y. See Lemma 23.13 for this notation. \blacksquare

Definition 17.4. Under Situation 17.3, we define the retraction map

$$\mathcal{R}_x: \overline{V_x} \times (-\infty, 1)^k \to \overline{V_x} \times [0, 1)^k$$

by

$$\mathcal{R}_x(\overline{y},(t_1,\ldots,t_k))=(\overline{y},(t_1',\ldots,t_k')),$$

where

$$t_i' = \begin{cases} t_i & \text{if } t_i \ge 0, \\ 0 & \text{if } t_i \le 0. \end{cases}$$

For $\tau > 0$, we define an open subset $V_x^{\boxplus \tau}$ of $\overline{V_x} \times [-\tau, 1)^k$ by $\mathcal{R}_x^{-1}(V_x) \cap (\overline{V_x} \times [-\tau, 1)^k)$. Here the retraction map \mathcal{R}_x naturally induces a map

$$\mathcal{R}_x: V_x^{\boxplus \tau} \to V_x \subseteq \overline{V_x} \times [0,1)^k.$$

Next, we define a Γ_x action on $V_r^{\boxplus \tau}$.

Definition 17.5. Let $\gamma \in \Gamma_x$. By the definition of admissible orbifold (Definition 25.7 (1)(b)), there exists $\sigma_{\gamma} \in \text{Perm}(k)$ with the following properties. We define maps $\varphi_0^{\gamma} : \overline{V}_x \times [0,1)^k \to \overline{V}_x$ and $\varphi_i^{\gamma} : \overline{V}_x \times [0,1)^k \to [0,1)$ requiring the components of $\gamma \cdot (\overline{y}, (t_1, \ldots, t_k))$ to be

$$\gamma \cdot (\overline{y}, (t_1, \dots, t_k)) = \left(\varphi_0^{\gamma}(\overline{y}, (t_1, \dots, t_k)), \left(\varphi_{\sigma_{\gamma}^{-1}(i)}^{\gamma}(\overline{y}, (t_1, \dots, t_k))\right)_{i=1}^k\right).$$

Here the left hand side is the given Γ_x action on V_x . Then we find the following:

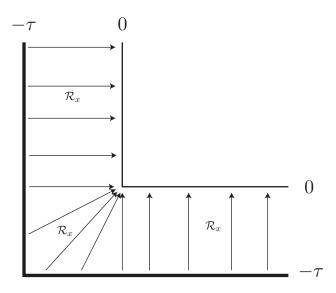


FIGURE 6. the retraction \mathcal{R}^x

- (1) The map φ_0^{γ} is admissible.
- (2) The function $(\overline{y}, (t_1, \ldots, t_k)) \mapsto \varphi_i^{\gamma}(\overline{y}, (t_1, \ldots, t_k)) t_i$ is exponentially small at the boundary.

The existence of such φ_0^{γ} , φ_i^{γ} is nothing but the definition of an admissible action. Then we define $\widehat{\varphi}_0^{\gamma}: \overline{V}_x \times [-\tau,1)^k \to \overline{V}_x$ and $\widehat{\varphi}_i^{\gamma}: \overline{V}_x \times [-\tau,1)^k \to [-\tau,1)$ as follows.

(A)
$$\widehat{\varphi}_0^{\gamma} = \varphi_0^{\gamma} \circ \mathcal{R}_x$$
.

(R)

$$\widehat{\varphi}_{i}^{\gamma}(\overline{y},(t_{1},\ldots,t_{k})) = \begin{cases} t_{i} & \text{if } t_{i} \leq 0, \\ (\varphi_{i}^{\gamma} \circ \mathcal{R}_{x})(\overline{y},(t_{1},\ldots,t_{k})) & \text{if } t_{i} \geq 0. \end{cases}$$

We now put

$$\gamma \cdot (\overline{y}, (t_1, \dots, t_k)) = \left(\widehat{\varphi}_0^{\gamma}(\overline{y}, (t_1, \dots, t_k)), \left(\widehat{\varphi}_{\sigma_{\gamma}^{-1}(i)}^{\gamma}(\overline{y}, (t_1, \dots, t_k))\right)_{i=1}^k\right).$$

We note that $y \mapsto \gamma \cdot y$ coincides with the given γ action if $y \in V_x$.

We now prove that this defines a Γ_x -action on $V_x^{\boxplus \tau}$. We first show:

Lemma 17.6. We have $\widehat{\varphi}_i^{\gamma}(\overline{y},(t_1,\ldots,t_k)) > 0$ if and only if $t_i > 0$. Moreover $\widehat{\varphi}_i^{\gamma}(\overline{y},(t_1,\ldots,t_k)) = 0$ if and only if $t_i = 0$.

Proof. If $t_i \leq 0$ then $\widehat{\varphi}_i^{\gamma}(\overline{y}, (t_1, \dots, t_k)) = t_i \leq 0$ by (B). On the other hand, if $t_i > 0$ then $\widehat{\varphi}_i^{\gamma}(\overline{y}, (t_1, \dots, t_k)) = (\varphi_i^{\gamma} \circ \mathcal{R}_x)(\overline{y}, (t_1, \dots, t_k)) > 0$ by (B). The lemma follows.

Lemma 17.7. We have $\gamma \cdot (\mathcal{R}_x(y)) = \mathcal{R}_x(\gamma \cdot y)$ for any $y \in V_x^{\boxplus \tau}$.

Proof. The coincidence of \overline{V}_x coordinates is an immediate consequence of the definition. We will check the coincidence of the $\sigma_{\gamma}(i)$ -th coordinates of $[-\tau, 1)^k$ factors. If $t_i \leq 0$, then this coordinate is 0 for both $\gamma \cdot (\mathcal{R}_x(y))$ and $\mathcal{R}_x(\gamma \cdot y)$.

If $t_i > 0$, they both are the $\sigma_{\gamma}(i)$ -th coordinates of the $[-\tau, 1)^k$ factor of γ . $(\mathcal{R}_x(y)).$

In fact, this is obvious for $\gamma \cdot (\mathcal{R}_x(y))$. To show this claim for $\mathcal{R}_x(\gamma \cdot y)$, we first observe that the $\sigma_{\gamma}(i)$ -th coordinate of the $[-\tau,1)^k$ factor of $\gamma \cdot y$ is the $\sigma_{\gamma}(i)$ -th coordinate of the $[-\tau,1)^k$ factor of $\gamma \cdot (\mathcal{R}_x(y))$ by (B). Then we use Lemma 17.6 to show that \mathcal{R}_x does not change this coordinate.

Lemma 17.8. We have $\gamma \cdot (\mu \cdot y) = (\gamma \mu) \cdot y$ for any $y \in V_x^{\boxplus \tau}$ and $\gamma, \mu \in \Gamma_x$.

Proof. We calculate

$$\widehat{\varphi}_0^{\gamma}(\mu \cdot y) = \varphi_0^{\gamma}(\mathcal{R}_x(\mu \cdot y)) = \varphi_0^{\gamma}(\mu \cdot \mathcal{R}_x(y)) = \widehat{\varphi}_0^{\gamma\mu}(y).$$

Therefore the \overline{V}_x factor coincides.

Let $y = (\overline{y}, (t_1, \dots, t_k))$. Suppose $t_i > 0$. Then $\widehat{\varphi}_i^{\mu}(\overline{y}, (t_1, \dots, t_k))) > 0$ by Lemma 17.6. Therefore we obtain

$$\widehat{\varphi}_{\sigma_{\mu}(i)}^{\gamma}(\mu \cdot y) = \varphi_{\sigma_{\mu}(i)}^{\gamma}(\mathcal{R}_{x}(\mu \cdot y)) = \varphi_{\sigma_{\mu}(i)}^{\gamma}(\mu \cdot \mathcal{R}_{x}(y)) = \widehat{\varphi}_{i}^{\gamma\mu}(y).$$

Suppose $t_i \leq 0$. Then $\widehat{\varphi}_i^{\mu}(\overline{y},(t_1,\ldots,t_k))) = t_i \leq 0$ by Lemma 17.6. Thus we get

$$\widehat{\varphi}_{\sigma_{\mu}(i)}^{\gamma}(\mu \cdot y) = t_i = \widehat{\varphi}_i^{\gamma \mu}(y).$$

The proof of Lemma 17.8 is complete.

We have thus defined a Γ_x action on $V_r^{\boxplus \tau}$.

a 17.9. (1) For any admissible map $f: V_x \to M$ the composition $f \circ \mathcal{R}_x: V_x^{\boxplus \tau} \to M$ is smooth.

- (2) If f is a submersion, so is f ∘ R_x.
 (3) The Γ_x action on V_x^{⊞τ} is smooth and R_x is Γ_x equivariant.

Proof. (1). Since the problem is local, it suffices to prove the case when $M = \mathbb{R}$. We use Lemma 25.5 to obtain f_I such that $f = \sum_I f_I$. Here $I \subset \{1, \dots, k\}$ and $f_I: \overline{V}_x \times [0,1)^I \to \mathbb{R}$ is exponentially small near the boundary. We define a function $\widehat{f}_I: V_r^{\boxplus \tau} \to \mathbb{R}$ by

$$\widehat{f}_{I}(\overline{y},(t_{1},\ldots,t_{k})) = \begin{cases} \widehat{f}_{I}(\overline{y},t_{I}) & \text{if } t_{i} \geq 0 \text{ for all } i \in I, \\ 0 & \text{if } t_{i} \leq 0 \text{ for some } i \in I. \end{cases}$$

Since $f_I: \overline{V}_x \times [0,1)^I \to \mathbb{R}$ is exponentially small near the boundary, \widehat{f}_I is smooth. It is easy to see that $f \circ \mathcal{R}_x = \sum_I \widehat{f}_I$. Therefore $f \circ \mathcal{R}_x$ is smooth.

We note that the submersivity of f by definition implies the submersivity of the restriction of f to $S_k V_x$. (2) is immediate from this fact and the construction.

The first half of (3) follows from definition. The second half of (3) is Lemma 17.7.

We define $U_x^{\boxplus \tau}$ to be the quotient $V_x^{\boxplus \tau}/\Gamma_x$ under the Γ_x action on $V_x^{\boxplus \tau}$ described above. Then Lemma 17.8 yields the retraction map

$$\mathcal{R}_x: U_x^{\boxplus \tau} \to U_x$$

induced by the one on $V_x^{\boxplus \tau}$. We denote

$$\mathcal{E}_x = (E_x \times V_x)/\Gamma_x,$$

which is a vector bundle on an orbifold $U_x = V_x/\Gamma_x$. Then we define $\mathcal{E}_x^{\boxplus \tau}$ to be the pull-back

$$\mathcal{E}_x^{\boxplus \tau} = \mathcal{R}_x^*(\mathcal{E}_x) = (E_x \times V_x^{\boxplus \tau})/\Gamma_x, \tag{17.1}$$

which is a smooth vector bundle on an orbifold $U_x^{\boxplus \tau}$. The Kuranishi map s_x which is a section of \mathcal{E}_x induces a section $s_x^{\boxplus \tau}$ of $\mathcal{E}_x^{\boxplus \tau}$. In the same way as in the proof of Lemma 17.9 (1) we can show that $s_x^{\boxplus \tau}$ defines a smooth section.

$$(X \cap U_x)^{\boxplus \tau} := (s_x^{\boxplus \tau})^{-1}(0)/\Gamma_x,$$
 (17.2)

which is a paracompact Hausdorff space. (We note that $X \cap U_x$ in the notation $(X \cap U_x)$ $(U_x)^{\boxplus \tau}$ does NOT stand for the set-theoretical intersection, but is just a notation.) We have a map $\psi_x^{\boxplus \tau}: (s_x^{\boxplus \tau})^{-1}(0)/\Gamma_x \to (X \cap U_x)^{\boxplus \tau}$ that is the identity map. Then from the definition we find

Lemma-Definition 17.10. Let $\mathcal{U} = (U, \mathcal{E}, \psi, s)$ be a Kuranishi chart of X and $x \in$ $\overset{\circ}{S}_k(U)$ as in Situation 17.3. Then $(U_x^{\boxplus \tau} = V_x^{\boxplus \tau}/\Gamma_x, \mathcal{E}_x^{\boxplus \tau}, \psi_x^{\boxplus \tau}, s_x^{\boxplus \tau})$ is a Kuranishi chart of $(X \cap U_x)^{\boxplus \tau}$. We denote

$$\mathcal{U}_{x}^{\boxplus \tau} = (U_{x}^{\boxplus \tau}, \mathcal{E}_{x}^{\boxplus \tau}, \psi_{x}^{\boxplus \tau}, s_{x}^{\boxplus \tau})$$

and call it trivialization of corners of \mathcal{U}_x .

Let $S_x = (W_x, \omega_x, \mathfrak{s}_x)$ be a CF-perturbation of \mathcal{U} on \mathfrak{V}_x . ([Part I, Definition 7.5].) We define $\mathfrak{s}_x^{\boxplus \tau} : V_x^{\boxplus \tau} \times W_x \to E_x$ by

$$\mathfrak{s}_{x}^{\boxplus \tau}(y,\xi) = \mathfrak{s}_{x}(\mathcal{R}_{x}(y),\xi). \tag{17.3}$$

mma-Definition 17.11. (1) The boundary $\partial(U_x^{\boxplus \tau})$ (resp. $\partial(\mathcal{S}_x^{\boxplus \tau})$) is canonically diffeomorphic to $(\partial U_x)^{\boxplus \tau}$ (resp. $(\partial \mathcal{S}_x)^{\boxplus \tau}$). (2) The triple $(W_x, \omega_x, \mathfrak{s}_x^{\boxplus \tau})$ is a CF-perturbation of $(V_x^{\boxplus \tau}/\Gamma_x, \mathcal{E}_x^{\boxplus \tau}, \mathfrak{s}_x^{\boxplus \tau}, \psi_x^{\boxplus \tau})$. Lemma-Definition 17.11.

- We denote it by $S_x^{\boxplus \tau}$.
- (3) If S_x is equivalent to S'_x , then $S_x^{\boxplus \tau}$ is equivalent to $S'^{\boxplus \tau}_x$.
- (4) If $f: U \to M$ is a smooth map and strongly submersive on \mathfrak{V}_x with respect to S_x , then $f \circ R_x$ is strongly submersive with respect to $S_x^{\boxplus \tau}$. We denote it by $f_x^{\boxplus \tau}: U_x^{\boxplus \tau} \to M$.
- (5) The strong transversality to $N \to M$ is also preserved.
- (6) The versions of (2)-(5) where 'CF-perturbation' is replaced by 'multivalued perturbation' also hold.
- (7) If h_x is a differential form on U_x , that is, h_x is a Γ_x -invariant differential form on V_x , then $\widetilde{h}_x^{\boxplus \tau} := \mathcal{R}_x^* \widetilde{h}_x$ defines a differential form on $U_x^{\boxplus \tau}$. We denote it by $h_x^{\boxplus \tau}$.
- (8) In the situation of (2)(4)(7), we assume that h_x is compactly supported. Then we have:

$$(f_x^{\boxplus \tau})!(h_x^{\boxplus \tau}; \mathcal{S}_x^{\boxplus \tau}) = f_x!(h_x; \mathcal{S}_x). \tag{17.4}$$

Proof. The proofs are mostly immediate from the definition. We only prove (8) for completeness' sake. To prove (8), we recall the definition of the push out from [Part I, Definition 7.10], which results in

$$\int_{M} f_{x}^{\boxplus \tau}!(h_{x}^{\boxplus \tau}; \mathcal{S}_{x}^{\boxplus \tau}) \wedge \rho = (-1)^{|\rho||\omega|} \int_{(s^{\boxplus \tau})^{-1}(0)} \pi_{1}^{*} h_{x}^{\boxplus \tau} \wedge \pi_{1}^{*} (f_{x}^{\boxplus \tau})^{*} \rho \wedge \pi_{2}^{*} \omega_{x} \quad (17.5)$$

$$\int_{M} f_{x}!(h_{x}; \mathcal{S}_{x}) \wedge \rho = (-1)^{|\rho||\omega|} \int_{(s_{x})^{-1}(0)} \pi_{1}^{*} \widetilde{h}_{x} \wedge \pi_{1}^{*} f_{x}^{*} \rho \wedge \pi_{2}^{*} \omega_{x}$$
 (17.6)

for any differential form ρ on M. Here $\pi_1: V_x^{\boxplus \tau} \times W_x \to V_x^{\boxplus \tau}, \ \pi_2: V_x^{\boxplus \tau} \times W_x \to W_x$ are projections, $\widetilde{h}^{\boxplus \tau}$ is a Γ_x -invariant differential form on $V_x^{\boxplus \tau}$ defined by $\mathcal{R}_x^* \widetilde{h}_x$ and $f^{\boxplus \tau} = f_x \circ \mathcal{R}_x: V_x^{\boxplus \tau} \to M$ is a submersion defined by Lemma 17.9.

We compare the right hand sides of the above two integrals. First we note $s_x^{\boxplus \tau} = s_x \circ (\mathcal{R}_x \times \mathrm{id}_{W_x})$ and $\mathcal{R}_x|_{V_x} = \mathrm{id}_{V_x}$ on $\partial V_x \subset V_x \subset V_x^{\boxplus \tau}$. We decompose

$$(s_x^{\boxplus \tau})^{-1}(0) = \left((s_x^{\boxplus \tau})^{-1}(0) \cap (V_x \times W_x) \right) \cup \left((s_x^{\boxplus \tau})^{-1}(0) \setminus (V_x \times W_x) \right),$$

and its associated integral and note $s_x^{\boxplus \tau} \equiv s_x$ on $V_x \subset V_x^{\boxplus \tau}$. Then contribution of the integral (17.5) over the first part of the domain becomes (17.6).

It remains to check the contribution of (17.5) on the region $(s_x^{\boxplus \tau})^{-1}(0) \setminus (V_x \times W_x)$. In the rest of the proof, we may assume that ρ is chosen so that the degree of

 $\pi_1^* \widetilde{h}_x \wedge \pi_1^* f_x^* \rho \wedge \pi_2^* \omega_x$ matches the dimension of $(s_x)^{-1}(0)$. We note that the retraction $\mathcal{R}_x \times \operatorname{id}_{W_x} : V_x^{\boxplus \tau} \setminus V_x \times W_x \to \partial V_x \times W_x$ also induces a retraction of $(s_x^{\boxplus \tau})^{-1}(0) \setminus (V_x^{\boxplus \tau} \times W_x)$ to $s_x^{-1}(0) \cap (\partial V_x \times W_x)$ with one-dimensional fiber by definition of $s_x^{\boxplus \tau}$. We also note

$$\widetilde{h}^{\boxplus \tau} \wedge f^{\boxplus \tau} \rho = (\mathcal{R}_x \times \mathrm{id}_{W_x})^* \eta$$

for some form η defined on ∂V_x by the definitions of $\widetilde{h}^{\boxplus \tau}$ and $\wedge f^{\boxplus \tau}$ given above. We derive

$$\int_{(s_x^{\boxplus \tau})^{-1}(0)\setminus(V_x\times W_x)} \pi_1^* h^{\boxplus \tau} \wedge \pi_1^* f^{\boxplus \tau} \rho \wedge \pi_2^* \omega_x$$

$$= \int_{(s_x^{\boxplus \tau})^{-1}(0)\setminus(V_x\times W_x)} \pi_1^* (\mathcal{R}_x \times \mathrm{id}_{W_x})^* \eta \wedge \pi_2^* \omega_x$$

$$= \int_{(s_x^{\boxplus \tau})^{-1}(0)\setminus(V_x\times W_x)} (\mathcal{R}_x \times \mathrm{id}_{W_x})^* (\pi_1^* \eta \wedge \pi_2^* \omega_x)$$

$$= \int_{s_x^{-1}(0)\cap(\partial V_x\times W_x)} (\pi_1^* \eta \wedge \pi_2^* \omega_x).$$

For the last equality we use

$$(s_x^{\boxplus \tau})^{-1}(0) \setminus (V_x \times W_x) = (\mathcal{R}_x \times \mathrm{id}_{W_x})^{-1}(s_x^{-1}(0) \cap (\partial V_x \times W_x))$$

by definition of $s_x^{\boxplus \tau}$. By submersion property of s_x , we have

$$\dim s_x^{-1}(0) \cap (\partial V_x \times W_x) = \dim s_x^{-1}(0) - 1.$$

Therefore by the degree assumption made on ρ above, the last integral vanishes. Now the proof of (17.4) is complete.

Lemma 17.12. We put $\overset{\circ}{S}_{k}(U_{\tau}^{\boxplus \tau}) = S_{k}(U_{\tau}^{\boxplus \tau}) \cap \mathcal{R}_{\tau}^{-1}(\overset{\circ}{S}_{k}(U_{x})).$

- (1) The closure of $\overset{\circ}{S}_k(U_n^{\boxplus \tau})$ in $\overset{\circ}{S}_k(U_n^{\boxplus \tau})$ is an orbifold with corners.
- (2) The map \mathcal{R}_x induces an orbifold diffeomorphism from $\operatorname{Clos}(\overset{\circ}{S}_k(U_x^{\boxplus \tau}))$ to

Proof. It suffices to prove the lemma for the case when $V = [0,1)^k$, which is obvious. (See Figure 7.)

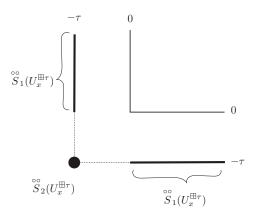


FIGURE 7. $\overset{\circ\circ}{S}_k(U_x^{\boxplus \tau})$

17.3. Trivialization of corners and embedding. To shorten the discussion we study trivialization of the corner for the case of coordinate change and of a local representative of an embedding simultaneously. In this subsection we consider the following situation.

Situation 17.13. Suppose we are in Situation 17.3. Let $x' \in S_k(U)$ and $\mathfrak{V}_{x'} =$ $(V_{x'}, E_{x'}, \Gamma_{x'}, \phi_{x'}, \widehat{\phi}_{x'})$ be an orbifold chart of (U', \mathcal{E}') at x. (See Definition 23.17.) Let $s_{x'}$ be a representative of the Kuranishi map s on $\mathfrak{V}_{x'}$. Let $(U', \mathcal{E}') \to (U, \mathcal{E})$ be an embedding. We take its local representative $(h_{xx'}, \varphi_{xx'}, \widehat{\varphi}_{xx'})$.

This includes the case when $(h_{xx'}, \varphi_{xx'}, \widehat{\varphi}_{xx'})$ represents an isomorphism, which is nothing but the case of coordinate change of orbifold.

Lemma 17.14. In Situation 17.13, there exists a unique injective map $j: \{1, \ldots, k'\} \rightarrow$ $\{1,\ldots,k\}$ with the following properties.

- (1) For $\gamma' \in \Gamma_{x'}$, we have: $\sigma(\gamma')(\mathfrak{j}(i)) = \mathfrak{j}(\sigma'(\gamma')(i))$. (2) If $\varphi_{xx'}(\overline{y}', (t'_1, \ldots, t'_{k'})) = (\overline{y}, (t_1, \ldots, t_k))$ then $t_{\mathfrak{j}(i)} = 0$ if and only if $t'_i = 0$.

Proof. The existence of j satisfying (2) is immediate from the fact that $\varphi_{xx'}$ preserves stratification $S_n(V_{x'})$, $S_n(V_x)$. Such j is necessarily unique. Then (1) follows from this uniqueness.

For $A \subset \{1, \ldots, k\}$ we put

$$V(A) = \{ (\overline{y}, (t_1, \dots, t_k)) \in V_x \mid \text{If } i \in A \text{ then } t_i = 0 \}.$$
 (17.7)

Definition 17.15. We define $\varphi_{xx'}^{\boxplus \tau}: V_{x'}^{\boxplus \tau} \to V_x^{\boxplus \tau}$ as follows.

- (1) If $y' \in V_{x'}$ then $\varphi_{xx'}^{\boxplus \tau}(y') = \varphi_{xx'}(y')$.
- (2) Let $y' = (\overline{y}', (t'_1, \dots, t'_{k'}))$ and $\mathcal{R}_{x'}(y') \in V_{x'}(A')$, where $A' \subset \{1, \dots, k'\}$. We define $y_0 = \varphi_{xx'}(\mathcal{R}_{x'}(y'))$ and write $y_0 = (\overline{y}_0, (t_{0,1}, \dots, t_{0,k}))$. Then we define

$$\varphi_{xx'}^{\boxplus \tau}(y') = (\overline{y}_0, (t_1, \dots, t_k))$$

where

$$t_i = \begin{cases} t'_{i'} & \text{if } i = \mathfrak{i}(i'), \ i' \in A', \\ t_{0,i} & \text{if } i \notin \mathfrak{i}(A'). \end{cases}$$

(1) The map $\varphi_{rr'}^{\boxplus \tau}: V_{r'}^{\boxplus \tau} \to V_{r}^{\boxplus \tau}$ is a smooth embedding of Lemma 17.16. manifolds.

- (2) The map $\varphi_{xx'}^{\boxplus \tau}: V_{x'}^{\boxplus \tau} \to V_{x}^{\boxplus \tau}$ is $h_{xx'}$ equivariant. (3) We have $\varphi_{xx'} \circ \mathcal{R}_{x'} = \mathcal{R}_{x} \circ \varphi_{xx'}^{\boxplus \tau}$.

Proof. Statements (2) and (3) are obvious from the definition. We prove (1). For simplicity of notation we consider the case i(i) = i. We study the smoothness at the point $y' = (\overline{y}', (t'_1, \dots, t'_{k'}))$. We may assume without loss of generality that

$$t'_1 = \dots = t'_{\ell} = 0, \quad t'_{\ell+1}, \dots, t'_m < 0, \quad t'_{m+1}, \dots, t'_{k'} > 0.$$

Let $z' = (\overline{z}', (s'_1, \dots, s'_{k'})) \in V_{x'}^{\boxplus \tau}$ be a point in a neighborhood of y'. By taking the neighborhood sufficiently small, we may assume that $s'_{\ell+1},\ldots,s'_m<0$ and $s'_{m+1}, \ldots, s'_{k'} > 0$. We put

$$\mathcal{R}_{x'}(z') = (\overline{z}', (s_1'', \dots, s_{k'}'')).$$

Then we have $s''_{\ell+1}, \ldots, s''_m = 0, s''_{m+1} = s'_{m+1}, \ldots, s''_{k'} = s'_{k'}$. Moreover for $i \leq \ell$,

$$s_i'' = \begin{cases} 0 & \text{if } s_i' \le 0\\ s_i' & \text{if } s_i' \ge 0. \end{cases}$$

We denote $\varphi_{xx'}^{\boxplus \tau}(z') = z$ and $z = (\overline{z}, (s_1, \ldots, s_k))$. By definition we have

$$\overline{z} = \pi_0(\varphi_{xx'}(\overline{z}', (s_1'', \dots, s_{k'}'')))$$

where $\pi_0: V_x \times [0,1)^k \to V_x$ is the projection. Therefore, the smooth dependence of \overline{z} on z' can be proved in the same way as the proof of Lemma 17.9 (1). (Namely we use Lemma 25.9(2).)

We also have

$$(s_{m+1},\ldots,s_k) = \pi_{m+1,\ldots,k}(\varphi_{xx'}(\overline{z}',(s_1'',\ldots,s_{k'}'')))$$

where $\pi_{m+1,...,k}: V_x \times [0,1)^k \to [0,1)^{k-m}$ is the projection to the last k-m factors. Therefore, the smooth dependence of $(s_{k'+1},\ldots,s_k)$ on z' can be proved in the same way as in the proof of Lemma 17.9. Moreover $s_i = s_i'$ for $i = \ell + 1, \ldots, m$. So the smooth dependence of s_i on z' is obvious. Finally for $i = 1, ..., \ell$ we have

$$s_i = \begin{cases} \pi_i(\varphi_{xx'}(\overline{z}', (s_1'', \dots, s_{k'}'))) & \text{if } s_i' \ge 0\\ s_i' & \text{if } s_i' \le 0. \end{cases}$$

Here $\pi_i: V_x \times [0,1)^k \to [0,1)$ is the projection to the *i*-th factor of $[0,1)^k$. Then the smooth dependence of (s_1, \ldots, s_m) on z' follows from Lemma 25.5.

The injectivity of $\varphi_{xx'}^{\boxplus \tau}$ can be proved easily. The smoothness of local inverse of $\varphi_{xx'}^{\boxplus \tau}$ can be proved in the same way as in the proof of smoothness of $\varphi_{xx'}^{\boxplus \tau}$.

By Lemma 17.16 (3) the embedding of bundles $\widehat{\varphi}_{xx'}:\mathcal{E}_{x'}\to\mathcal{E}_x$ over $\varphi_{xx'}$ induces

$$\mathcal{R}_{x'}^*\mathcal{E}_{x'} o \mathcal{R}_x^*\mathcal{E}_x$$

over $\varphi_{xx'}^{\boxplus \tau}$. Since $\mathcal{E}_{x'}^{\boxplus \tau} = \mathcal{R}_{x'}^* \mathcal{E}_{x'}$ and $\mathcal{E}_{x}^{\boxplus \tau} = \mathcal{R}_{x}^* \mathcal{E}_{x}$ by definition, we obtain

$$\widehat{\varphi}_{xx'}^{\boxplus \tau}: \mathcal{E}_{x'}^{\boxplus \tau} \to \mathcal{E}_{x}^{\boxplus \tau}.$$

In the same way as in the proof of Lemma 17.16 (1), we can show that $\widehat{\varphi}_{xx'}^{\boxplus \tau}$ is a smooth embedding of vector bundles. We have thus proved the next lemma.

Lemma 17.17. Under the situation above, $(h_{xx'}, \varphi_{xx'}^{\boxplus \tau}, \widehat{\varphi}_{xx'}^{\boxplus \tau})$ is an embedding of orbifold charts. If $(h_{xx'}, \varphi_{xx'}, \widehat{\varphi}_{xx'})$ is a coordinate change, $(h_{xx'}, \varphi_{xx'}^{\boxplus \tau}, \widehat{\varphi}_{xx'}^{\boxplus \tau})$ is also a coordinate change.

We also have the following:

Lemma 17.18. Let $(h_{xx''}, \varphi_{xx''}, \widehat{\varphi}_{xx''})$ (resp. $(h_{x'x''}, \varphi_{x'x''}, \widehat{\varphi}_{x'x''})$) be as in Situation 17.13, where x, x' in Situation 17.13 is replaced by x, x'' (resp. x', x''). We define $(h_{xx''}, \varphi_{xx''}^{\boxplus \tau}, \widehat{\varphi}_{xx''}^{\boxplus \tau})$ (resp. $(h_{x'x''}, \varphi_{x'x''}^{\boxplus \tau}, \widehat{\varphi}_{x'x''}^{\boxtimes \tau})$) from $(h_{xx''}, \varphi_{xxx''}, \widehat{\varphi}_{xx''})$ (resp. $(h_{x'x''}, \varphi_{x'x''}, \widehat{\varphi}_{x'x''})$) in the same way as in the proof of Lemma 17.17. Then we have

$$\varphi_{xx''}^{\boxplus \tau} = \varphi_{xx'}^{\boxplus \tau} \circ \varphi_{x'x''}^{\boxplus \tau}, \quad \widehat{\varphi}_{xx''}^{\boxplus \tau} = \widehat{\varphi}_{xx'}^{\boxplus \tau} \circ \widehat{\varphi}_{x'x''}^{\boxplus \tau}, \quad s_x^{\boxplus \tau} \circ \varphi_{xx'}^{\boxplus \tau} = \widehat{\varphi}_{xx'}^{\boxplus \tau} \circ s_{x'}^{\boxplus \tau}.$$

Definition 17.19. In the Situation 17.13 we define

$$(X \cap U)^{\boxplus \tau} = \bigcup_{x \in \psi(s^{-1}(0))} \left((s_x^{\boxplus \tau})^{-1}(0) / \Gamma_x \right) / \sim.$$
 (17.8)

Here the equivalence relation \sim in the right hand is defined as follows. Let $y_i \in V_{x_i}^{\boxplus \tau}/\Gamma_{x_i}$ with $s_{x_i}(\tilde{y}_i) = 0$, $[\tilde{y}_i] = y_i$ for i = 1, 2. Then $y_1 \sim y_2$ if and only if there exist \mathfrak{V}_x , $\tilde{y} \in V_x$, and $(h_{x_ix}, \varphi_{x_ix}, \widehat{\varphi}_{x_ix})$ as in Situation 17.13 such that $s(\tilde{y}) = 0$ and

$$[\varphi_{x_i x}(\tilde{y})] = y_i$$

in $V_{x_i}^{\boxplus \tau}/\Gamma_{x_i}$ for i=1,2. ²⁴

The maps $\mathcal{R}_x: (s_x^{\boxplus \tau})^{-1}(0) \to V_x$ for various x induce a map $(X \cap U)^{\boxplus \tau} \to X$, which we denote by

$$\mathcal{R}: (X \cap U)^{\boxplus \tau} \to X. \tag{17.9}$$

Lemma 17.20. In Situation 17.3 we have a Kuranishi chart

$$\mathcal{U}^{\boxplus \tau} = (U^{\boxplus \tau}, \mathcal{E}^{\boxplus \tau}, \psi^{\boxplus \tau}, s^{\boxplus \tau})$$

of $(X \cap U)^{\boxplus \tau}$ such that $(V_x^{\boxplus \tau}/\Gamma_x, \mathcal{E}_x^{\boxplus \tau}, \psi_x^{\boxplus \tau}, s_x^{\boxplus \tau})$ becomes its orbifold chart.

Proof. This is a consequence of Lemmas 17.16, 17.17, 17.18. \Box

Lemma-Definition 17.21. In the situation of Lemma 17.20, we call $\mathcal{U}^{\boxplus \tau}$ the τ -collaring, or τ -corner trivialization of \mathcal{U} .

- (1) $\partial(U^{\boxplus \tau})$ is canonically diffeomorphic to $(\partial U)^{\boxplus \tau}$.
- (2) Let $\mathfrak{S} = \{(\mathfrak{V}_{\mathfrak{r}}, \mathcal{S}_{\mathfrak{r}}) \mid \mathfrak{r} \in \mathfrak{R}\}$ be a CF-perturbation of \mathcal{U} . Then $\{(\mathfrak{V}_{\mathfrak{r}}^{\boxplus \tau}, \mathcal{S}_{\mathfrak{r}}^{\boxplus \tau}) \mid \mathfrak{r} \in \mathfrak{R}\}$ is a CF-perturbation of $\mathcal{U}^{\boxplus \tau}$. We denote it by $\mathfrak{S}^{\boxplus \tau}$ and call it the τ -collaring of \mathfrak{S} .
- (3) If \mathfrak{S} is equivalent to \mathfrak{S}' , then $\mathfrak{S}^{\boxplus \tau}$ is equivalent to $\mathfrak{S}'^{\boxplus \tau}$.
- (4) If $f: U \to M$ is a smooth map and strongly submersive on K with respect to \mathfrak{S} , then $f \circ \mathcal{R}$ is strongly submersive with respect to $\mathfrak{S}^{\boxplus \tau}$. We denote it by $f^{\boxplus \tau}$ and call it τ -collaring of f.
- (5) The strong transversality to a map $N \to M$ is also preserved.
- (6) The versions of (2)-(5) where 'CF-perturbation' is replaced by 'multivalued perturbation' also hold.
- (7) If h is a differential form on U, then h_x for various x are glued to define a differential form on $U^{\boxplus \tau}$. We denote it by $h^{\boxplus \tau}$ and call it the τ -collaring of h.

 $^{^{24}}$ Lemma 17.18 implies that this is an equivalence relation.

(8) In the situation of (2)(3)(4)(7), we assume that h is compactly supported. Then we have

$$f_!^{\boxplus \tau}(h^{\boxplus \tau}; \mathfrak{S}^{\boxplus \tau}) = f_!(h; \mathfrak{S}). \tag{17.10}$$

This follows immediately from Lemma-Definition 17.11. Also the next lemma is a straightforward generalization of Lemma 17.12 to the case of Kuranishi chart.

Lemma 17.22. We put

$$\overset{\circ\circ}{S}_k(U^{\boxplus\tau}) = S_k(U^{\boxplus\tau}) \cap \mathcal{R}^{-1}(\overset{\circ}{S}_k(U)).$$

- (1) The closure of $\overset{\circ\circ}{S}_k(U^{\boxplus \tau})$ in $\overset{\circ}{S}_k(U^{\boxplus \tau})$ is an orbifold with corners. We call $\overset{\circ\circ}{S}_k(U^{\boxplus \tau})$ a small corner of codimension k.
- (2) The retraction map $\mathcal R$ induces an orbifold diffeomorphism from $\mathrm{Close}(\overset{\circ}{S}_k(U^{\boxplus \tau}))$ onto $\widehat{S}_k(U)$.

17.4. Trivialization of corners of Kuranishi structure. In this subsection and the next, we study trivialization of corners of Kuranishi structure and good coordinate system. For a K-space $(X, \widehat{\mathcal{U}})$ we firstly describe the underlying topological space $X^{\boxplus \tau}$ of the trivialization of corners of X in Definition 17.26. In the next subsection, we define the trivialization of corners $(X^{\boxplus \tau}, \widehat{\mathcal{U}}^{\boxplus \tau})$ (or sometimes called τ -collaring) of the K-space $(X, \widehat{\mathcal{U}})$. We first consider the following situation.

Situation 17.23. Let $\mathcal{U}_i = (U_i, \mathcal{E}_i, s_i, \psi_i)$ be Kuranishi charts of X and $\Phi_{21} =$ $(\varphi_{21},\widehat{\varphi}_{21})$ an embedding of Kuranishi charts. We may decorate \mathcal{U}_i by some of the following in addition:

- (1) We are given CF-perturbations \mathfrak{S}^i of \mathcal{U}_i (i=1,2) such that \mathfrak{S}^1 , \mathfrak{S}^2 are compatible with Φ_{21} .
- (2) We are given differential forms h_i on U_i (i = 1, 2) such that $h_1 = \varphi_{21}^* h_2$.
- (3) We are given smooth maps $f_i: U_i \to M \ (i=1,2)$ such that $f_1 = f_2 \circ \varphi_{21}$.
- (4) We are given multivalued perturbations \mathfrak{s}^i of \mathcal{U}_i (i=1,2) such that \mathfrak{s}^1 , \mathfrak{s}^2 are compatible with Φ_{21} .
- (5) We have another Kuranishi chart \mathcal{U}_3 and an embedding $\Phi_{32}:\mathcal{U}_2\to\mathcal{U}_3$. We put $\Phi_{31} = \Phi_{32} \circ \Phi_{21}$.

Lemma 17.24. In Situation 17.23 we have an embedding of Kuranishi charts $\Phi_{21}^{\boxplus \tau}$: $\mathcal{U}_1^{\boxplus \tau} \to \mathcal{U}_2^{\boxplus \tau}$, whose restriction to \mathcal{U}_1 coincides with Φ_{21} . Moreover we have the

- (1) In case of Situation 17.23 (1), $\mathfrak{S}^{1\boxplus \tau}$, $\mathfrak{S}^{2\boxplus \tau}$ are compatible with $\Phi_{21}^{\boxplus \tau}$.

- (2) In case of Situation 17.23 (2), $h_1^{\boxplus \tau} = (\varphi_{21}^{\boxplus \tau})^* h_2^{\boxplus \tau}$. (3) In case of Situation 17.23 (3), $f_1^{\boxplus \tau} = f_2^{\boxplus \tau} \circ \varphi_{21}^{\boxplus \tau}$. (4) In case of Situation 17.23 (4), $\mathfrak{s}^{1\boxplus \tau}$ and $\mathfrak{s}^{2\boxplus \tau}$ are compatible with $\Phi_{21}^{\boxplus \tau}$. (5) In case of Situation 17.23 (5), we have $\Phi_{31}^{\boxplus \tau} = \Phi_{32}^{\boxplus \tau} \circ \Phi_{21}^{\boxplus \tau}$. (6) $\varphi_{21}^{\boxplus \tau} \circ \mathcal{R}_1 = \mathcal{R}_2 \circ \varphi_{21}^{\boxplus \tau}$.

Remark 17.25. Both of $\mathcal{U}_1^{\boxplus \tau}$ and $\mathcal{U}_2^{\boxplus \tau}$ are Kuranishi charts of the topological space $(X \cap U)_2^{\coprod \tau}$.

Proof of Lemma 17.24. We use Lemma 23.13 to obtain objects $\{\mathfrak{V}_r^i \mid \mathfrak{r} \in \mathfrak{R}_i\}$ and $(h_{\mathfrak{r},21},\varphi_{\mathfrak{r},21},\hat{\varphi}_{\mathfrak{r},21})$ which have the properties spelled out there. Let $\mathfrak{r}\in\mathfrak{R}_1$. Then the map $\varphi_{\mathfrak{r},21}: V^1_{\mathfrak{r}} \to V^2_{\mathfrak{r}}$ is extended to $\varphi_{\mathfrak{r},21}^{\boxplus \tau}: V^{1\boxplus \tau}_{\mathfrak{r}} \to V^{2\boxplus \tau}_{\mathfrak{r}}$ as follows. Let $(\overline{y}',(t_1',\ldots,t_{d(\mathfrak{r})}'))\in V^1_{\mathfrak{r}}$ and

$$(\overline{y}'',(t_1'',\ldots,t_{d(\mathfrak{r})}''))=\varphi_{\mathfrak{r},21}(\mathcal{R}_{\mathfrak{r}}(\overline{y}',(t_1',\ldots,t_{d(\mathfrak{r})}'))).$$

Then we put

$$\varphi_{\mathfrak{r},21}^{\boxplus \tau}(\overline{y}',(t_1',\ldots,t_{d(\mathfrak{r})}'))=(\overline{y},(t_1,\ldots,t_{d(\mathfrak{r})}))$$

where $\overline{y}' = \overline{y}''$ and

$$t_i = \begin{cases} t_i' & \text{if } t_i' \le 0, \\ t_i'' & \text{if } t_i' \ge 0. \end{cases}$$

We define $\widehat{\varphi}_{\mathfrak{r},21}^{\boxplus \tau}$ in a similar way. Using Lemma 23.13 (4) and Lemma 23.25, it is easy to see that $(h_{\mathfrak{r},21}, \varphi_{\mathfrak{r},21}^{\boxplus \tau}, \widehat{\varphi}_{\mathfrak{r},21}^{\boxplus \tau})$ is a representative of the required embedding. It is straightforward to check (1)-(6).

We will glue \mathcal{U}_p for various p using Lemma 17.24 to obtain a Kuranishi structure $\widehat{\mathcal{U}^{\boxplus \tau}}$. (See Lemma-Definition 17.35.) Its underlying topological space is obtained as follows.

Definition 17.26. (1) Let $(X, \widehat{\mathcal{U}})$ be a K-space. We define a topological space $X^{\boxplus \tau}$ as follows. We take a disjoint union

$$\coprod_{p \in X} (s_p^{\boxplus \tau})^{-1}(0)/\Gamma_p$$

and define an equivalence relation \sim as follows: Let $x_p \in (s_p^{\boxplus \tau})^{-1}(0)$ and $x_q \in (s_q^{\boxplus \tau})^{-1}(0)$. We define $[x_p] \sim [x_q]$ if there exist $r \in X$ and $x_r \in (s_p^{\boxplus \tau})^{-1}(0) \cap U_{pr}^{\boxplus \tau} \cap U_{qr}^{\boxplus \tau}$ such that

$$[x_p] = \varphi_{pr}^{\boxplus \tau}([x_r]), \quad [x_q] = \varphi_{qr}^{\boxplus \tau}([x_r]).$$
 (17.11)

As we will see in Lemma 17.27, \sim is an equivalence relation. We define $X^{\boxplus \tau}$ as the set of the equivalence classes of this equivalence relation \sim

$$X^{\boxplus \tau} := \left(\prod_{p \in X} (s_p^{\boxplus \tau})^{-1}(0) / \Gamma_p \right) / \sim . \tag{17.12}$$

(2) For $k = 1, 2, \ldots$ we define

$$S_k(X^{\boxplus \tau}) := \left(\coprod_{p \in X} S_k(U_p^{\boxplus \tau}) \cap \left((s_p^{\boxplus \tau})^{-1}(0) / \Gamma_p \right) \right) / \sim .$$
 (17.13)

The relation \sim is defined on the sets $s_p^{-1}(0)/\Gamma_p$ or the enhanced sets $(s_p^{\boxplus \tau})^{-1}(0)/\Gamma_p$. So the messy process to shrink the domain to ensure the consistency of coordinate changes are not necessary here. In fact, we show the following.

Lemma 17.27. The relation \sim in Definition 17.26 is an equivalence relation.

Proof. (1) We just check transitivity. Other properties are easier to prove. Suppose $[x_p] \sim [x_q]$ and $[x_q] \sim [x_r]$. By Definition 17.26, there exist $u,v \in X$ and $x_u \in U_u^{\boxplus \tau}, x_v \in U_v^{\boxplus \tau}$ such that

$$[x_p] = \varphi_{pu}^{\boxplus \tau}([x_u]), \ [x_q] = \varphi_{qu}^{\boxplus \tau}([x_u])$$

$$[x_q] = \varphi_{qv}^{\boxplus \tau}([x_v]), \ [x_r] = \varphi_{rv}^{\boxplus \tau}([x_v]).$$

By Lemma 17.24 (6), we obtain $\psi_p(\mathcal{R}_p([x_p])) = \psi_q(\mathcal{R}_q([x_q])) = \psi_r(\mathcal{R}_r(x_r))$ from (17.11). Denote this common point by $t \in X$. We may take $U_t = \overline{U_t} \times [0,1)^d$ where the $[0,1)^d$ component of o_t is zero and \overline{U}_t has no boundary. We note $U_t^{\boxplus \tau} = \overline{U_t} \times [-\tau,1)^d$. Since $t \in s_p^{-1}(0) \cap s_q^{-1}(0) \cap s_r^{-1}(0) \cap s_v^{-1}(0)$, we have coordinate changes $\varphi_{pt}, \varphi_{qt}, \varphi_{rt}, \varphi_{ut}, \varphi_{vt}$ so that

$$\mathcal{R}_p([x_p]) = \varphi_{pu} \circ \varphi_{ut}(o_t) = \varphi_{pt}(o_t),$$

$$\mathcal{R}_q([x_q]) = \varphi_{qu} \circ \varphi_{ut}(o_t) = \varphi_{qv} \circ \varphi_{vt}(o_t) = \varphi_{qt}(o_t),$$

and

$$\mathcal{R}_r([x_r]) = \varphi_{rv} \circ \varphi_{vt}(o_t) = \varphi_{rt}(o_t).$$

Also note that the restrictions of $\varphi_{pt}^{\boxplus \tau}$, $\varphi_{qt}^{\boxplus \tau}$, $\varphi_{qt}^{\boxplus \tau}$, $\varphi_{ut}^{\boxplus \tau}$ and $\varphi_{vt}^{\boxplus \tau}$ to $\mathcal{R}_t^{-1}(o_t)$ are bijections to $\mathcal{R}_p^{-1}(\mathcal{R}_p([x_p]))$, $\mathcal{R}_q^{-1}(\mathcal{R}_q([x_q]))$, $\mathcal{R}_r^{-1}(\mathcal{R}_r([x_r]))$, $\mathcal{R}_u^{-1}(\mathcal{R}_u([x_u]))$ and $\mathcal{R}_v^{-1}(\mathcal{R}_v([x_v]))$, respectively. Hence there exists $x_t \in U_t^{\boxplus \tau}$ such that $[x_q] = \varphi_{qt}^{\boxplus \tau}([x_t])$, $[x_u] = \varphi_{ut}^{\boxplus \tau}([x_t])$ and $[x_v] = \varphi_{vt}^{\boxplus \tau}([x_t])$. Therefore we have

$$[x_p] = \varphi_{pu}^{\boxplus}([x_u]) = \varphi_{pt}^{\boxplus \tau}([x_t]) \text{ and } [x_r] = \varphi_{rv}^{\boxplus \tau}([x_v]) = \varphi_{rt}^{\boxplus \tau}([[x_t]),$$
 which imply that $[x_p] \sim [x_r]$.

Remark 17.28. In this section we consider the case of a Kuranishi structure on a compact metrizable space X. We may also consider the case of Kuranishi structure of a pair $Z \subseteq X$ of a metrizable space X and its compact subspace Z. There is actually nothing new to do so, except the following point. To define a topological space $X^{\boxplus \tau}$ we used a Kuranishi structure on X. If we are given a Kuranishi structure of $Z \subseteq X$, we can still define $Z^{\boxplus \tau}$. However we can define $X^{\boxplus \tau}$ only in a neighborhood of Z. The situation here is similar to the situation we met in defining the boundary $\partial(X, Z; \widehat{\mathcal{U}})$. See [Part I, Remark 8.9 (2)]. It seems unlikely that this point becomes an important issue in the application. In fact, in all the cases we know so far appearing in the actual applications, it is enough to define $X^{\boxplus \tau}$ in a neighborhood of Z, or there is an obvious way to define $X^{\boxplus \tau}$ in the particular situations.

Definition 17.29. Using a good coordinate system $\widehat{\mathcal{U}}$ of X, we can define $X^{\boxplus \tau}$ in a similar way as the case of Kuranishi structure.

17.5. Collared Kuranishi structure. For a point $p \in X$ we can define its Kuranishi neighborhood as $\mathcal{U}_p^{\boxplus \tau}$. There is a slight issue in defining a Kuranishi neighborhood compatible with the collar structure.

Example 17.30. We consider an orbifold $X = [0, \infty)^2/\mathbb{Z}_2$, where \mathbb{Z}_2 acts by exchanging the factors. We want to regard it as a '1-collared orbifold'. If p = [0, 0] we can take an obvious choice $[0, 1)^2/\mathbb{Z}_2$ as its 'collared neighborhood'. There is an issue in case p = [(0, 0.5)]. We might try to take its neighborhood such as $[0, 1) \times (0.3, 0.7)$. This however does not work. In fact $(0.4, 0.6) \sim (0.6, 0.4)$ but \mathbb{Z}_2 is not contained in the isotropy group of (0, 0.5). It seems impossible to find a good 'collared neighborhood' of [(0, 0.5)] such that the 'length' of the collar is 1. This is a technical problem and certainly we should regard $[0, \infty)^2/\mathbb{Z}_2$ to have a collar of length ≥ 1 .

It seems to the authors that the best way to define the appropriate notion of τ -collared cornered orbifold is as follows: We do not define an orbifold chart at the points in $[0,1)^2 \setminus \{(0,0)\}$. The points of $[0,1)^2 \setminus \{(0,0)\}$ are contained in the chart

at (0,0) so we do not need an orbifold chart at those points. We will define the notion of τ -collared Kuranishi structure along this line below.

Remark 17.31. The above mentioned trouble occurs only when the action of the isotropy group on the normal factor $[0,1)^k$ is nontrivial. So it does not occur in the situation of our applications in Sections 16-22. However, we present the formulation which works in more general cases. It actually appears when we will study the moduli space of pseudo-holomorphic curves from a bordered Riemann surface of arbitrary genus with arbitrary number of boundary components.

There occurs no similar issue for the definition of the τ -collared good coordinate system.

Definition 17.32. Given $\tau > 0$ let

$$X' = X^{\boxplus \tau}$$

be the τ -collaring of certain Kuranishi structure $\widehat{\mathcal{U}}$ on X. We put

$$\overset{\circ}{S}_{k}(X',\widehat{\mathcal{U}}) = S_{k}(X') \cap \mathcal{R}^{-1}(\overset{\circ}{S}_{k}(X,\widehat{\mathcal{U}})), \tag{17.14}$$

where $S_k(X')$ is defined by (17.13), and define a subset $B_{\tau}(\overset{\circ}{S}_k(X',\widehat{\mathcal{U}})) \subset X'$ as the union of

$$\psi_p^{\boxplus \tau} \left((s_p^{\boxplus \tau})^{-1}(0) \cap \{ (\overline{y}, (t_1, \dots, t_k)) \mid t_i \le 0, \ i = 1, \dots, k \} \right)$$
 (17.15)

for $p \in \overset{\circ}{S}_k(X, \widehat{\mathcal{U}})$.

We note that if $p' \in \overset{\circ}{S}_k(X', \widehat{\mathcal{U}})$ then $p' = \psi_p^{\boxplus \tau}(\overline{y}, (-\tau, \dots, -\tau))$ for $p = \mathcal{R}(p')$. Therefore $\overset{\circ}{S}_k(X', \widehat{\mathcal{U}}) \subset B_{\tau}(\overset{\circ}{S}_k(X', \widehat{\mathcal{U}}))$. We also note that

$$B_{\tau}(\overset{\circ}{S}_{0}(X',\widehat{\mathcal{U}})) = \overset{\circ}{S}_{0}(X',\widehat{\mathcal{U}}) = \overset{\circ}{S}_{0}(X,\widehat{\mathcal{U}}),$$

$$B_{\tau}(\overset{\circ}{S}_{k}(X',\widehat{\mathcal{U}})) \cap X = \overset{\circ}{S}_{k}(X,\widehat{\mathcal{U}}).$$

$$(17.16)$$

See Figure 8.

Lemma 17.33. Let $X' = X^{\boxplus \tau}$ be as above. Then it has the following decomposition:

$$X' = \coprod_{k} B_{\tau}(\overset{\circ\circ}{S}_{k}(X',\widehat{\mathcal{U}}))$$

where the right hand side is the disjoint union.

Proof. Let $p' \in X'$ and put $p = \mathcal{R}(p')$. We can write as $p' = \psi_p^{\boxplus \tau}(\overline{y}, (t_1, \dots, t_k))$. Without loss of generality, we may assume that $t_1, \dots, t_\ell \leq 0 < t_{\ell+1}, \dots, t_k$ for some ℓ . We put $q = \psi_p^{\boxplus \tau}(\overline{y}, (0, \dots, 0, t_{\ell+1}, \dots, t_k)) \in \mathring{S}_{\ell}(X, \widehat{\mathcal{U}})$. We may choose the coordinate of q so that the map $j : \{1, \dots, \ell\} \to \{1, \dots, k\}$ appearing in the coordinate change from an orbifold chart at q to an orbifold chart at p, which appeared in Lemma 17.14, is j(i) = i. Then we can take \overline{y}' such that

$$\psi_p^{\boxplus \tau}(\overline{y}, (0, \dots, 0, t_{\ell+1}, \dots, t_k)) = \psi_q^{\boxplus \tau}(\overline{y}', (0, \dots, 0)).$$

Then $p' = \psi_q^{\boxplus \tau}(\overline{y}', (t_1, \dots, t_\ell)) \in B_{\tau}(\overset{\circ}{S}_{\ell}(X', \widehat{\mathcal{U}}))$. Moreover,

$$B_{\tau}(\overset{\circ}{S}_{k}(X',\widehat{\mathcal{U}}))\cap B_{\tau}(\overset{\circ}{S}_{\ell}(X',\widehat{\mathcal{U}}))=\emptyset$$

for $k \neq \ell$ is obvious from definition.

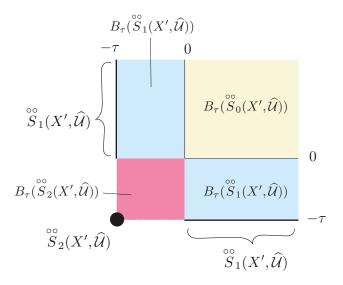


FIGURE 8. $B_{\tau}(\overset{\circ\circ}{S}_{k}(X',\widehat{\mathcal{U}}))$

Definition 17.34. Suppose we are in the situation of Definition 17.32. In particular, X' is a compact metrizable space homeomorphic to $X^{\boxplus \tau}$ for a certain K-space $(X, \widehat{\mathcal{U}})$.

- (1) Let $p' \in \overset{\circ}{S}_k(X', \widehat{\mathcal{U}})$. A τ -collared Kuranishi neighborhood at p' is a Kuranishi chart $\mathcal{U}_{p'}$ of X' such that $\mathcal{U}_{p'} = (\mathcal{U}_p)^{\boxplus \tau}$ for a certain Kuranishi neighborhood \mathcal{U}_p of $p = \mathcal{R}(p')$. (See Figure 9.)

 (2) For $p' \in \overset{\circ}{S}_k(X', \widehat{\mathcal{U}})$ and $q' \in \overset{\circ}{S}_\ell(X')$, let $\mathcal{U}_{p'} = \widehat{\mathcal{U}_p^{\boxplus \tau}} = (\mathcal{U}_p)^{\boxplus \tau}$ and $\mathcal{U}_{q'} = \widehat{\mathcal{U}_{p'}} = \widehat{\mathcal{U}_{p'}}$
- (2) For $p' \in \overset{\circ}{S}_k(X', \widehat{\mathcal{U}})$ and $q' \in \overset{\circ}{S}_\ell(X')$, let $\mathcal{U}_{p'} = \widehat{\mathcal{U}_p^{\boxplus \tau}} = (\mathcal{U}_p)^{\boxplus \tau}$ and $\mathcal{U}_{q'} = \widehat{\mathcal{U}_q^{\boxplus \tau}}$ be their τ -collared Kuranishi neighborhoods, respectively. Suppose $q' \in \psi_{p'}(s_{p'}^{-1}(0))$. A τ -collared coordinate change $\Phi_{p'q'}$ from $\mathcal{U}_{q'}$ to $\mathcal{U}_{p'}$ is $\Phi_{pq}^{\boxplus \tau}$ defined by Lemma 17.24, where Φ_{pq} is a coordinate change from \mathcal{U}_q to \mathcal{U}_p .
- (3) A τ -collared Kuranishi structure $\widehat{\mathcal{U}}'$ on X' consists of the following objects:
 - (a) For each $p' \in \overset{\circ\circ}{S}_k(X',\widehat{\mathcal{U}}), \, \widehat{\mathcal{U}'}$ assignes a τ -collared Kuranishi neighborhood $\mathcal{U}_{p'}$.
 - (b) For each $p' \in \overset{\circ}{S}_k(X',\widehat{\mathcal{U}})$ and $q' \in \overset{\circ}{S}_\ell(X',\widehat{\mathcal{U}})$ with $q' \in \psi_{p'}(s_{p'}^{-1}(0)), \widehat{\mathcal{U}}'$ assignes a τ -collared coordinate change $\Phi_{p'p'}$.
 - assignes a τ -collared coordinate change $\Phi_{p'q'}$. (c) If $p' \in \overset{\circ}{S}_k(X', \widehat{\mathcal{U}}), q' \in \overset{\circ}{S}_\ell(X', \widehat{\mathcal{U}}), r' \in \overset{\circ}{S}_m(X', \widehat{\mathcal{U}})$ with $q' \in \psi_{p'}(s_{p'}^{-1}(0))$ and $r' \in \psi_{q'}(s_{q'}^{-1}(0))$, then we require

$$\Phi_{p'q'} \circ \Phi_{q'r'}|_{U_{p'q'r'}} = \Phi_{p'r'}|_{U_{p'q'r'}}$$

where $U_{p'q'r'} = U_{p'r'} \cap \varphi_{q'r'}^{-1}(U_{p'q'})$.

(4) A τ -collared K-space is a pair of a compact metrizable space and its τ -collared Kuranishi structure. It is obtained from a K-space $(X, \hat{\mathcal{U}})$ as in Lemma-Definition 17.35 below.

- (5) We can define the notion of τ -collared CF-perturbation, τ -collared multivalued perturbation, τ -collared good coordinate system, τ -collared Kuranishi chart, τ -collared vector bundle, τ -collared smooth section, τ -collared embedding of various kinds, etc. in the same way. Actually those objects on $(X^{\boxplus \tau}, \widehat{\mathcal{U}^{\boxplus \tau}})$ are obtained from the corresponding objects on $(X, \widehat{\mathcal{U}})$ by applying the process of τ -collaring on each chart as in Lemma 17.37 below.
- (6) An \mathscr{A} -parametrized family of τ -collared CF-perturbations is said to be uniform if it is of the form $\{\widehat{\mathfrak{S}_{\sigma}^{\boxplus \tau}} \mid \sigma \in \mathscr{A}\}$ for a certain uniform family $\{\widehat{\mathfrak{S}_{\sigma}} \mid \sigma \in \mathscr{A}\}$ of CF-perturbations on $(X,\widehat{\mathcal{U}})$. The definition of uniform family of τ -collared multivalued perturbations is the same.

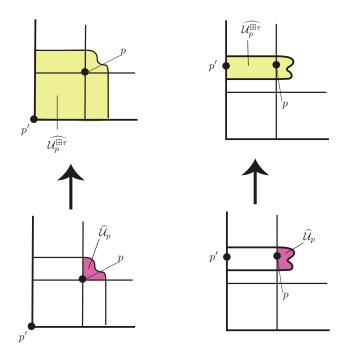


FIGURE 9. $\widehat{\mathcal{U}_p^{\boxplus \tau}}$

Lemma-Definition 17.35. For any K-space $(X,\widehat{\mathcal{U}})$ we can assign a τ -collared K-space $(X^{\boxplus \tau},\widehat{\mathcal{U}^{\boxplus \tau}})$ such that:

- (1) Its underlying topological space $X^{\boxplus \tau}$ is as in Definition 17.26.
- (2) If $p \in \overset{\circ\circ}{S}_k(X^{\boxplus \tau})$, its Kuranishi neighborhood is $\mathcal{U}_{\mathcal{R}(p)}^{\boxplus \tau}$, where $\mathcal{U}_{\mathcal{R}(p)}^{\boxplus \tau}$ is defined in Lemma 17.20.
- (3) The coordinate changes are $\Phi_{\mathcal{R}(p)\mathcal{R}(q)}^{\boxplus \tau}$, where $\Phi_{\mathcal{R}(p)\mathcal{R}(q)}^{\boxplus \tau}$ is defined in Lemma 17.24.

We call $(X^{\boxplus \tau}, \widehat{\mathcal{U}}^{\boxplus \tau})$ the τ -collaring (or trivialization of corners) of $(X, \widehat{\mathcal{U}})$. We sometimes write $(X, \widehat{\mathcal{U}})^{\boxplus \tau}$ in place of $(X^{\boxplus \tau}, \widehat{\mathcal{U}}^{\boxplus \tau})$.

Proof. This is immediate from the definition.

For a τ -collared Kuranishi structure $\widehat{\mathcal{U}^{\boxplus \tau}}$ on $X' = X^{\boxplus \tau}$ we sometimes write $\overset{\circ \circ}{S}_k((X,\widehat{\mathcal{U}})^{\boxplus \tau})$ etc. in place of $\overset{\circ \circ}{S}_k(X^{\boxplus \tau},\widehat{\mathcal{U}^{\boxplus \tau}})$ etc..

Definition 17.36. Let $\widehat{\mathcal{U}} = ((\mathfrak{P}, \leq), \{\mathcal{U}_{\mathfrak{p}} \mid \mathfrak{p} \in \mathfrak{P}\}, \{\Phi_{\mathfrak{pq}} \mid \mathfrak{q} \leq \mathfrak{p}\})$ be a good coordinate system of X. We define a good coordinate system $\widehat{\mathcal{U}^{\boxplus_{\tau}}}$ of $X^{\boxplus_{\tau}}$ as

$$\widehat{\mathcal{U}^{\boxplus \tau}} = ((\mathfrak{P}, \leq), \{\mathcal{U}_{\mathfrak{p}}^{\boxplus \tau} \mid \mathfrak{p} \in \mathfrak{P}\}, \{\Phi_{\mathfrak{p}\mathfrak{q}}^{\boxplus \tau} \mid \mathfrak{q} \leq \mathfrak{p}\}).$$

We call $(X^{\boxplus \tau}, \widehat{\mathcal{U}^{\boxplus \tau}})$ the τ -collaring (or trivialization of corners) of $(X, \widehat{\mathcal{U}})$.

We call a good coordinate system to be τ -collared if it is isomorphic to $\widehat{\mathcal{U}}^{\boxplus \tau}$ for some $\widehat{\mathcal{U}}$.

The next lemma says that many objects defined on $(X, \widehat{\mathcal{U}})$ have corresponding collared objects on $(X^{\boxplus \tau}, \widehat{\mathcal{U}^{\boxplus \tau}})$. This is actually a trivial statement to prove.

Lemma 17.37. We consider the situation of Lemma-Definition 17.35.

- (1) $(\partial(X,\widehat{\mathcal{U}}))^{\boxplus \tau}$ is canonically isomorphic to $\partial(X^{\boxplus \tau},\widehat{\mathcal{U}}^{\boxplus \tau})$.
- (2) A CF-perturbation $\widehat{\mathfrak{S}}$ on $(X,\widehat{\mathcal{U}})$ induces a τ -collared CF-perturbation $\widehat{\mathfrak{S}}^{\boxplus \tau}$ on $(X^{\boxplus \tau},\widehat{\mathcal{U}}^{\boxplus \tau})$.
- (3) A strongly continuous map \widehat{f} from $(X,\widehat{\mathcal{U}})$ induces a τ -collared strongly continuous map $\widehat{f}^{\boxplus \tau}$ from $(X^{\boxplus \tau},\widehat{\mathcal{U}^{\boxplus \tau}})$. Strong smoothness and weak submersivity are preserved.
- (4) In the situation of (2)(3), if \hat{f} is strongly submersive with respect to $\widehat{\mathfrak{S}}$, then $\widehat{f^{\boxplus \tau}}$ is a τ -collared strongly submersive map with respect to $\widehat{\mathfrak{S}^{\boxplus \tau}}$.
- (5) Transversality to a map $N \to M$ is also preserved.
- (6) The versions of (2)(4) where 'CF-perturbation' is replaced by 'multivalued perturbation' also hold.
- (7) A differential form \widehat{h} on $(X,\widehat{\mathcal{U}})$ induces a τ -collared differential form $\widehat{h^{\boxplus \tau}}$ on $(X^{\boxplus \tau},\widehat{\mathcal{U}^{\boxplus \tau}})$.
- (8) In the situation of (2)(4)(7), if $\widehat{f}:(X,\widehat{\mathcal{U}})\to M$ is strongly submersive with respect to $\widehat{\mathfrak{S}}$, then we have

$$\widehat{f!}(\widehat{h};\widehat{\mathfrak{S}}) = \widehat{f^{\boxplus \tau}}!(\widehat{h^{\boxplus \tau}};\widehat{\mathfrak{S}^{\boxplus \tau}}). \tag{17.17}$$

- (9) We put $\overset{\circ}{S}_{k}(X^{\boxplus \tau}, \widehat{\mathcal{U}}^{\boxplus \tau}) = S_{k}(X^{\boxplus \tau}, \widehat{\mathcal{U}}^{\boxplus \tau}) \cap \mathcal{R}^{-1}(\overset{\circ}{S}_{k}(X, \widehat{\mathcal{U}}))$ and call it the small codimension k corner. (We note that $\overset{\circ}{S}_{k}(X^{\boxplus \tau}, \widehat{\mathcal{U}}^{\boxplus \tau}) = \overset{\circ}{S}_{k}(X^{\boxplus \tau}, \widehat{\mathcal{U}})$ in (17.14).) Then the Kuranishi structure of $S_{k}(X^{\boxplus \tau}, \widehat{\mathcal{U}}^{\boxplus \tau})$ induces a Kuranishi structure on $\operatorname{Clos}(\overset{\circ}{S}_{k}(X^{\boxplus \tau}, \widehat{\mathcal{U}}^{\boxplus \tau}))$.
- (10) The restriction of the retraction map \mathcal{R} is an underlying homeomorphism of an isomorphism between the K-spaces $\operatorname{Clos}(\overset{\circ}{S}_k(X^{\boxplus \tau}, \widehat{\mathcal{U}}^{\boxplus \tau}))$ and $\widehat{S}_k(X, \widehat{\mathcal{U}})$.
- (11) If there exists an embedding $\widehat{\mathcal{U}} \to \widehat{\mathcal{U}^+}$ of Kuranishi structures, then the space $X^{\boxplus \tau}$ defined by $\widehat{\mathcal{U}}$ is canonically homeomorphic to the one defined by $\widehat{\mathcal{U}^+}$. The same holds for various types of embeddings between Kuranishi structures $\widehat{\mathcal{U}}$ and/or good coordinate systems $\widehat{\mathcal{U}}$.
- (12) Various types of embeddings between Kuranishi structures $\widehat{\mathcal{U}}$ and/or good coordinate systems $\widehat{\mathcal{U}}$ induce τ -collared embeddings between $\widehat{\mathcal{U}}^{\boxplus \tau}$ and/or $\widehat{\mathcal{U}}^{\boxplus \tau}$.

Compatibility among various objects on them (such as CF-perturbation) is preserved under the operation \boxplus .

- (13) If $\widehat{\mathcal{U}^+}$ is a thickening of $\widehat{\mathcal{U}}$, then $\widehat{\mathcal{U}^{+\boxplus \tau}}$ is a thickening of $\widehat{\mathcal{U}^{\boxplus \tau}}$.
- (14) If $(\widetilde{X}, \widehat{\widetilde{\mathcal{U}}})$ is a k-fold covering of $(X, \widehat{\mathcal{U}})$, then $(\widetilde{X}^{\boxplus \tau}, \widehat{\mathcal{U}}^{\boxplus \tau})$ is a k-fold covering of $(X^{\boxplus \tau}, \mathcal{U}^{\boxplus \tau})$.
- (15) The same results as (1)-(14) hold when we replace Kuranishi structure by good coordinate system.

Proof. (1)-(8) are consequence of Lemma-Definition 17.21 (1)-(8), respectively. (9), (10) are consequences of Lemma 17.22 (1), (2), respectively. The proof of (11) is similar to the proof of Lemma 17.27. (12) follows from Lemma 17.24. We can prove (13) by putting $O_p^{\boxplus \tau} = \mathcal{R}_{\mathcal{R}(p)}^{-1}(O_{\mathcal{R}(p)})$ and $W_p(q)^{\boxplus \tau} = \mathcal{R}_{\mathcal{R}(p)}^{-1}(W_{\mathcal{R}(p)}(\mathcal{R}(q)))$, where the notations for $O_{\mathcal{R}(p)}, W_{\mathcal{R}(p)}$ are as in [Part I, Definition 5.3 (2)]. (14) is obvious from the definition. The proof of (15) is the same as the proof of (1)-(14).

Lemma 17.38. If $(X',\widehat{\mathcal{U}'})$ is τ -collared, then for any $0 < \tau' < \tau$, X' has a τ' -collared Kuranishi structure which is determined in a canonical way from the τ -collared Kuranishi structure $(X', \widehat{\mathcal{U}'})$. The same holds for CF-perturbation, multivalued perturbation, good coordinate system and various other objects.

Proof. The lemma follows from, roughly speaking.

$$((X,\widehat{\mathcal{U}})^{\boxplus \tau_1})^{\boxplus \tau_2} \cong (X,\widehat{\mathcal{U}})^{\boxplus \tau} \tag{17.18}$$

where $\tau = \tau_1 + \tau_2$. (Here τ_2 corresponds to τ' in Lemma 17.38.) To be precise, we will define a τ_2 -collared Kuranishi structure of $(X, \widehat{\mathcal{U}})^{\boxplus \tau}$ below. In other words, we will define a Kuranishi structure on $X^{\boxplus \tau_1}$. We first note that the equality

$$((\mathcal{U}_p)^{\boxplus \tau_1})^{\boxplus \tau_2} = \mathcal{U}_p^{\boxplus (\tau_1 + \tau_2)}$$

$$\tag{17.19}$$

literally holds for a Kuranishi chart \mathcal{U}_p . We put $X' = X^{\boxplus \tau}$, $X'' = X^{\boxplus \tau_1}$. Then we have $X' = X''^{\boxplus \tau_2}$. We note that the set $S_k(X', \widehat{\mathcal{U}^{\boxplus \tau}})$ depends on whether we regard $X' = X^{\boxplus \tau}$ or $X' = X''^{\boxplus \tau_2}$. So we write $\overset{\circ\circ}{S}_k(X';\tau)$ when we regard $X'=X^{\boxplus\tau}$, and $\overset{\circ\circ}{S}_k(X';\tau_2)$ when we regard $X' = X''^{\boxplus \tau_2}$.

We will define a τ_2 -collared Kuranishi chart \mathcal{U}'_p for each $p\in \overset{\circ\circ}{S}_k(X';\tau_2)$. Let $p\in$ $\overset{\circ}{S}_k(X';\tau_2)$. By Lemma 17.33 there exists a unique n such that $p \in B_{\tau}(\overset{\circ}{S}_n(X';\tau))$. Therefore there exists $p' \in \overset{\circ}{S}_n(X,\widehat{\mathcal{U}})$ such that

$$p = \psi_{p'}(\overline{y}, (t_1, \dots, t_n))$$

and $s_{p'}(\overline{y},(t_1,\ldots,t_n))=0$. Such p' is unique, if we require

$$p' = \psi_{p'}(\overline{y}, (0, \dots, 0))$$

in addition. In fact, $p' = \mathcal{R}(p)$. We will take this choice.

 $^{^{25} \}text{Strictly speaking, we defined the space } X''^{\boxplus \tau_2}$ only when X'' has a Kuranishi structure. We have defined a τ_1 -collared Kuranishi structure on X'' but not a Kuranishi structure on it yet. But it is straightforward to define a space $X''^{\boxplus \tau_2}$ when a collared Kuranishi structure on X'' is given. On the other hand, we will also define a Kuranishi structure on X''. So $X''^{\boxplus \tau_2}$ makes sense in either way. (Indeed, the two ways to define $X''^{\boxplus \tau_2}$ coincide.)

We may take our coordinates $\overline{V}_{p'} \times [0,c)^n$ of $V_{p'}$ so that $\Gamma_{p'}$ acts by permutation of the $[0,c)^n$ factors. ²⁶ By changing the enumeration of the t_i 's, we may assume, without loss of generality, that

$$t_1 = \dots = t_k = -\tau < -\tau_1 < t_{k+1} \le \dots \le t_n.$$

(It suffices to consider the case $k \geq 1$.) In fact, we have $t_i \notin (-\tau, -\tau_1]$ by the

assumption $p \in \overset{\circ}{S}_k(X'; \tau_2)$. We take $k = a_0 < a_1 < \dots < a_m \le n$ such that $\{t_1, \dots, t_n\} = \{-\tau, t_{a_1}, \dots, t_{a_m}\}$ and $i \ne j \Rightarrow t_{a_i} \ne t_{a_j}$. Then for given t'_1, \dots, t'_n , we define $s_0, s'_1, s_1, s'_2, \dots, s'_m, s_m$

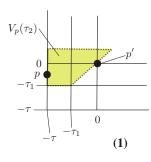
$$s_0 = \max\{t'_1, \dots, t'_k, -\tau_1\},\$$

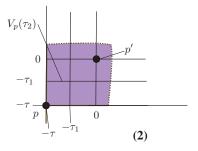
and

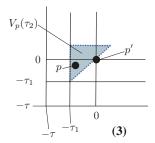
$$s'_{j} = \min\{t'_{i} \mid t_{i} = t_{a_{j}}\}, \qquad s_{j} = \max\{t'_{i} \mid t_{i} = t_{a_{j}}\}$$

for j = 1, 2, ..., m. Now we define

$$V_p(\tau_2) = \{ (\overline{y}, (t'_1, \dots, t'_n)) \in V_{p'}^{\boxplus \tau} \mid s_0 < s'_1 \le s_1 < s'_2 \le \dots < s_{m-1} < s'_m \}$$
 and put $\Gamma_p = \{ \gamma \in \Gamma_{p'} \mid \gamma p = p \}$.







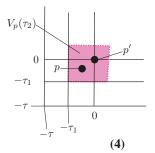


FIGURE 10. $V_p(\tau_2)$

Sublemma 17.39. (1) If
$$\gamma \in \Gamma_p$$
, then $\gamma V_p(\tau_2) = V_p(\tau_2)$. (2) If $\gamma \in \Gamma_{p'}$ and $\gamma V_p(\tau_2) \cap V_p(\tau_2) \neq \emptyset$, then $\gamma \in \Gamma_p$.

 $^{^{26}}$ Finding such a choice so that it is compatible with various coordinate changes is nontrivial. However it is easy to make such a choice at each point. See Remark 17.55 for the ambiguity caused by the choice of such coordinates.

Proof. Let $A_j = \{i \in \{1, \ldots, n\} \mid t_i = t_{a_j}\}$. Then $\Gamma_{\hat{p}}$ induces a permutation of $\{1,\ldots,n\}$ by Definition 25.7 (1)(b). It is easy to see that

$$\Gamma_p = \{ \gamma \in \Gamma_{\hat{p}} \mid \gamma A_j = A_j \text{ for all } j \}.$$

The sublemma follows from this fact and the definition.

We restrict $(\mathcal{U}_{p'})^{\boxplus \tau}$ to $V_p(\tau_2)/\Gamma_p$ to obtain a Kuranishi chart \mathcal{U}'_p . We denote $U_p(\tau_2) = V_p(\tau_2)/\Gamma_p$

We observe the following fact.

(*) If
$$(\overline{y}, (t'_1, \dots, t'_n)) \in V_p(\tau_2)$$
 and $t''_1, \dots, t''_k \in [-\tau, -\tau_1)$, then $(\overline{y}, (t''_1, \dots, t''_k, t'_{\ell+1}, \dots, t'_n)) \in V_p(\tau_2)$.

Using (*), we can prove

$$(V_p(\tau_2) \cap V_p^{\boxplus \tau_1})^{\boxplus \tau_2} = V_p(\tau_2).$$

Therefore \mathcal{U}'_n is τ_2 -collared. It is easy to construct coordinate change to obtain a au_2 -collared Kuranishi structure.

The second half of the lemma follows easily from the construction of the Kuranishi chart \mathcal{U}'_n .

Remark 17.40. In the situation of Lemma 17.38 we define a Kuranishi structure $\widehat{\mathcal{U}''}$ on $X^{\boxplus \tau_1}$ such that $(X^{\boxplus \tau_1}, \widehat{\mathcal{U}''})^{\boxplus \tau_2} = (X', \widehat{\mathcal{U}'})$ as follows. We first consider the case when $p \in \operatorname{Int} X^{\boxplus \tau_1}$. We put

$$\mathcal{U}_p^{"} = \mathcal{U}_{\mathcal{R}(p)}^{\boxplus \tau}|_{U_{p'}^{\boxplus \tau_1} \cap U_p(\tau_2)}.$$
(17.20)

Here $\mathcal{R}: X' \to X$ is the retraction map as in (17.9).

Let us elaborate the right hand side of (17.20). If $p \in \text{Int}X$, then $\mathcal{U}'_p = \mathcal{U}_p$ and

 $U_{\mathcal{R}(p)}^{\boxplus \tau_1} = U_p$. Therefore $\mathcal{U}_p'' = \mathcal{U}_p' = \mathcal{U}_p$. Suppose $p \notin \operatorname{Int} X$. (This case corresponds to (3) and (4) in Figure 10.) Then $\mathcal{R}(p) = p' \in \overset{\circ}{S}_k(X,\widehat{\mathcal{U}})$ for $k \geq 1$. Using the parametrization map $\psi_{p'}^{\boxplus \tau}$ of the Kuranishi chart $\mathcal{U}_{n'}^{\boxplus \tau}$ we can write

$$p = \psi_{p'}^{\boxplus \tau}(\overline{y}, (t_1, \dots, t_k)).$$

Since $p \in \text{Int } X'$, we have $t_i > -\tau_1$. Therefore, by definition,

$$U_{p'}^{\boxplus \tau_1} \cap U_p(\tau_2) = \{ (\overline{y}', (t'_1, \dots, t'_k)) \in U_{p'}^{\boxplus \tau_1} \mid t'_i > -\tau_1 \}.$$

Then \mathcal{U}_p'' is the restriction of $\mathcal{U}_{p'}^{\boxplus \tau}$ to this set.

We note that if $p \notin \text{Int } X$ and $p \in \text{Int } X^{\boxplus \tau_1}$, then $p \notin \overset{\circ}{S}_k(X, \tau_1)$ for any k.

We now consider the case when $p \in \overset{\circ}{S}_k(X^{\oplus \tau_1}, \widehat{\mathcal{U}}), k \geq 1$. Consider the map $X^{\boxplus \tau} \to X^{\boxplus \tau_1}$ defined as in (17.9). This is the retraction map when we regard $X^{\boxplus \tau} = (X^{\boxplus \tau_1})^{\boxplus \tau_2}$. We denote it by \mathcal{R}' , which is different from the retraction map $\mathcal{R}: X^{\boxplus \tau} \to X$. Then there exists a unique $\widehat{p} \in \overset{\circ}{S}_k(X^{\boxplus \tau})$ such that $\mathcal{R}'(\widehat{p}) = p$. (See Lemma 17.41 below.) We put

$$\mathcal{U}_p^{\prime\prime} = \mathcal{U}_{\widehat{p}}^{\prime}|_{U_{p^{\prime}}^{\boxplus \tau_1} \cap U_{\widehat{p}}(\tau_2)}.$$

See Figure 11 and compare it with Figure 10 (1)(2).

We used the next lemma in the above remark.

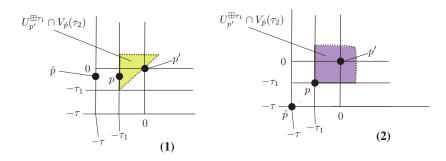


FIGURE 11. \mathcal{U}_p''

Lemma 17.41. Let $p \in X^{\boxplus \tau}$ and $\mathcal{R}(p) \in \overset{\circ}{S}_k(X,\widehat{\mathcal{U}})$. Then there exists a unique $\widehat{p} \in \overset{\circ}{S}_k(X^{\boxplus \tau},\widehat{\mathcal{U}^{\boxplus \tau}})$ such that $\mathcal{R}'(\widehat{p}) = p$, $\mathcal{R}(\widehat{p}) = \mathcal{R}(p)$.

Proof. We use the parametrization map
$$\psi_{\mathcal{R}(p)}^{\boxplus \tau}$$
 to write $p = \psi_{\mathcal{R}(p)}^{\boxplus \tau}(\overline{y}, (t_1, \dots, t_k)), t_i \leq 0$. Then $\widehat{p} = \psi_{\mathcal{R}(p)}^{\boxplus \tau}(\overline{y}, (-\tau, \dots, -\tau)).$

In the construction of this or the next section, we need to replace a τ -collared structure by a τ' -collared structure with $\tau' < \tau$ several times.

17.6. Extension of collared Kuranishi structure. The tiresome routine works to repeat the definitions in earlier sections are mostly over. What we gain from these routine works is Propositions 17.46 and 17.58 below which are extension theorems of a τ -collared Kuranishi structure and a τ -collared CF-perturbation, respectively. In this subsection we prove Proposition 17.46 and in the next subsection we prove Proposition 17.58.

Remark 17.42. Let $S_k(X; \widehat{\mathcal{U}})$ be a codimension k stratum of a K-space $(X, \widehat{\mathcal{U}})$ and $\widehat{S}_k(X; \widehat{\mathcal{U}})$ a normalized codimension k corner of $(X, \widehat{\mathcal{U}})$. (See [Part I, Definition 4.15] for $S_k(X; \widehat{\mathcal{U}})$ and Definition 24.17 for $\widehat{S}_k(X; \widehat{\mathcal{U}})$, respectively.) If $\widehat{\mathcal{U}} \to \widehat{\mathcal{U}}^+$ is a KK-embedding (embedding of Kuranishi structures) of X, the underlying topological space of $S_k(X; \widehat{\mathcal{U}})$ (resp. $\widehat{S}_k(X; \widehat{\mathcal{U}})$) is canonically homeomorphic to the underlying topological space of $S_k(X; \widehat{\mathcal{U}}^+)$ (resp. $\widehat{S}_k(X; \widehat{\mathcal{U}}^+)$.)

Hereafter we write $S_k(X)$ or $\widehat{S}_k(X)$ in place of $S_k(X; \widehat{\mathcal{U}})$, $\widehat{S}_k(X; \widehat{\mathcal{U}})$. They stand for the underlying topological spaces.

17.6.1. Statement. To state the extension theorem (Proposition 17.46) of a τ -collared Kuranishi structure we consider the following situation.

Situation 17.43. Let X be a paracompact Hausdorff space with τ -collared Kuranishi structure $\widehat{\mathcal{U}}$. Let ∂X be the normalized boundary of X.

For given $\tau > 0$, we are given a τ -collared Kuranishi structure $\widehat{\mathcal{U}_{\partial}^+}$ of ∂X such that

$$\partial \widehat{\mathcal{U}} < \widehat{\mathcal{U}_{\partial}^{+}}. \tag{17.21}$$

We assume that $\widehat{\mathcal{U}_{\partial}^+}$ satisfies the following conditions:

(1) For each $k \geq 1$ there exists a τ -collared Kuranishi structure $\widehat{\mathcal{U}_{S_k}^+}$ on $\widehat{S}_k(X)$ such that $\widehat{\mathcal{U}_{S_1}^+} = \widehat{\mathcal{U}_{\partial}^+}$.

- (2) The τ -collared K-space $\widehat{S}_k(\widehat{S}_\ell(X), \widehat{\mathcal{U}_{S_\ell}^+})$ is isomorphic to the $(k+\ell)!/k!\ell!$ fold covering space of $(\widehat{S}_{k+\ell}(X), \widehat{\mathcal{U}_{S_{k+\ell}}^+})$.
- (3) The following diagram of K-spaces commutes.

$$\widehat{S}_{k_{1}}(\widehat{S}_{k_{2}}(\widehat{S}_{k_{3}}(X),\widehat{\mathcal{U}_{S_{k_{3}}}^{+}}))) \xrightarrow{\pi_{k_{1},k_{2}}} \widehat{S}_{k_{1}+k_{2}}(\widehat{S}_{k_{3}}(X),\widehat{\mathcal{U}_{S_{k_{3}}}^{+}}))
\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad (17.22)$$

$$\widehat{S}_{k_{1}}(\widehat{S}_{k_{2}+k_{3}}(X),\widehat{\mathcal{U}_{S_{k_{2}+k_{3}}}^{+}})) \longrightarrow (\widehat{S}_{k_{1}+k_{2}+k_{3}}(X),\widehat{\mathcal{U}_{S_{k_{1}+k_{2}+k_{3}}}^{+}})$$

Here π_{k_1,k_2} is the covering map in Proposition 24.16. The right vertical and lower horizontal arrows are covering maps in (2). The left vertical arrow is induced by the covering map $\widehat{S}_{k_2}(\widehat{S}_{k_3}(X),\widehat{\mathcal{U}_{S_{k_3}}^+})) \to (\widehat{S}_{k_2+k_3}(X),\widehat{\mathcal{U}_{S_{k_2+k_3}}^+})$ in (2).

- (4) There exists a τ -collared embedding $\widehat{S}_k(X,\widehat{\mathcal{U}}) \to \widehat{\mathcal{U}_{S_k}^+}$.
- (5) The following diagram of K-spaces commutes.

$$\widehat{S}_{k}(\widehat{S}_{\ell}(X,\widehat{\mathcal{U}})) \longrightarrow \widehat{S}_{k}(\widehat{S}_{\ell}(X),\widehat{\mathcal{U}_{S_{\ell}}^{+}})
\downarrow \qquad \qquad \downarrow
\widehat{S}_{k+\ell}(X,\widehat{\mathcal{U}}) \longrightarrow (\widehat{S}_{k+\ell}(X),\widehat{\mathcal{U}_{S_{k+\ell}}^{+}})$$
(17.23)

Here the map in the first horizontal line is induced by the embedding $\widehat{S}_{\ell}(X,\widehat{\mathcal{U}}) \to \widehat{\mathcal{U}}_{S_{\ell}}^+$. The map in the second horizontal line is give by (4). The map in the first vertical column is given by Proposition 24.16. The map in the second vertical column is given by (2).

- Remark 17.44. (1) Here we used the notion of covering space of K-space we discuss in Section 24 to formulate the compatibility condition in Situation 17.43 at the corner of general codimension. In our application in Sections 16-22, the stratum $\widehat{S}_k(\widehat{S}_\ell(X),\widehat{\mathcal{U}_{S_k}^+})$ is a disjoint union of $(k+\ell)!/k!\ell!$ copies of $\widehat{S}_{k+\ell}(X,\widehat{\mathcal{U}_{S_{k+\ell}}^+})$. So the notion of covering space of K-space is not necessary, there. The result in the generality stated here will become necessary to study the case of higher genus Lagrangian Floer theory and or symplectic field theory.
 - (2) If $\widehat{\mathcal{U}_{S_{k+\ell}}^+}$ is a restriction of Kuranishi structure $\widehat{\mathcal{U}}^+$ such that $\widehat{\mathcal{U}} < \widehat{\mathcal{U}}^+$, then Condition (1)-(5) above follows from Proposition 24.16. Proposition 17.46 below may be regarded as a converse of this statement.

Definition 17.45. In Situation 17.43, we define

$$X_0 := X \setminus X^{\boxminus \tau}. \tag{17.24}$$

Here $X^{\boxminus \tau}$ is a subset of X such that $(X^{\boxminus \tau}, \widehat{\mathcal{U}^{\boxminus \tau}})^{\boxminus \tau} = (X, \widehat{\mathcal{U}})$. We note that X_0 is an open neighborhood of $S_1(X)$ in X. We call $(X, \widehat{\mathcal{U}})^{\boxminus \tau} := (X^{\boxminus \tau}, \widehat{\mathcal{U}^{\boxminus \tau}})$ inward τ -collaring of $(X, \widehat{\mathcal{U}})$.

Now the next proposition is our main result of this subsection. We complete the proof at the end of this subsection.

Proposition 17.46. Under Situation 17.43, for any $0 < \tau' < \tau$, there exists a τ' -collared Kuranishi structure $\widehat{\mathcal{U}^+}$ on X_0 with the following properties.

- (1) The restriction of $\widehat{\mathcal{U}^+}$ to $\widehat{S}_k(X)$ is isomorphic to $\widehat{\mathcal{U}^+_{S_k}}$ as τ' -collared Kuranishi structures.
- (2) There exists an embedding of τ' -collared Kuranishi structres $\widehat{\mathcal{U}}|_{X_0} \to \widehat{\mathcal{U}}^+$.
- (3) The restriction of (2) to $\widehat{S}_k X$ coincides with the one induced from (17.21) under the identification (1).
- (4) There exists an isomorphism between the K-spaces $\widehat{S}_k(X,\widehat{\mathcal{U}^+})$ and $(\widehat{S}_k(X),\widehat{\mathcal{U}_{S_k}^+})$ such that the following diagram of K-spaces commutes.

$$\widehat{S}_{k}(\widehat{S}_{\ell}(X,\widehat{\mathcal{U}^{+}})) \stackrel{\cong}{\longrightarrow} \widehat{S}_{k}(\widehat{S}_{\ell}(X),\widehat{\mathcal{U}_{S_{\ell}}^{+}})
\downarrow \qquad \qquad \downarrow
\widehat{S}_{k+\ell}(X,\widehat{\mathcal{U}^{+}}) \stackrel{\cong}{\longrightarrow} (\widehat{S}_{k+\ell}(X),\widehat{\mathcal{U}_{S_{k+\ell}}^{+}})$$
(17.25)

Here the first horizontal arrow is (1), the second horizontal arrow is one claimed in (4), the left vertical arrow is given by Proposition 24.16 and the right vertical arrow is given in Situation 17.43 (2).

- (5) The two embeddings $\widehat{\mathcal{U}}|_{\widehat{S}_k(X,\widehat{\mathcal{U}^+})} \to \widehat{\mathcal{U}^+}|_{\widehat{S}_k(X,\widehat{\mathcal{U}^+})}$ and $\widehat{\mathcal{U}}|_{\widehat{S}_k(X,\widehat{\mathcal{U}^+})} \to \widehat{\mathcal{U}^+_{S_k}}$ coincide via the isomorphism in (4). Here the first embedding is induced by the embeddings $\widehat{\mathcal{U}} \to \widehat{\mathcal{U}^+}$ and the second embedding is as in Situation 17.43 (4).
- 17.6.2. Extension theorem for a single collared Kuranishi chart. The main part of the proof of Proposition 17.46 is to prove the corresponding result for one Kuranishi chart. For this purpose we consider the following situation.

Situation 17.47. Let \mathcal{U} be a τ -collared Kuranishi chart of X and \mathcal{U}_{∂}^+ a τ -collared Kuranishi chart of ∂X . We assume that there exists an embedding

$$\partial \mathcal{U} \to \mathcal{U}_{2}^{+}$$
 (17.26)

of τ -collared Kuranishi charts and the following conditions are satisfied:

- (1) For each $k \geq 1$ there exists a τ -collared Kuranishi chart $\mathcal{U}_{S_k}^+$ on $\widehat{S}_k(X)$ such that $\mathcal{U}_{S_1}^+ = \mathcal{U}_{\partial}^+$.
- (2) The orbifold $\widehat{S}_k(U_{S_\ell}^+)$ is isomorphic to the $(k+\ell)!/k!\ell!$ fold covering space of $U_{S_{k+\ell}}^+$. The restriction of the obstruction bundle and Kuranishi map of $U_{S_\ell}^+$ to $\widehat{S}_k(U_{S_\ell}^+)$ are pull-backs of ones of $U_{S_{k+\ell}}^+$.
- (3) There exists an embedding $\mathcal{U}|_{\widehat{S}_k(X,\mathcal{U})} \to \mathcal{U}_{S_k}^+$ of τ -collared Kuranishi charts.
- (4) The following diagram commutes.

$$\widehat{S}_{k_1}(\widehat{S}_{k_2}(\mathcal{U}_{S_{k_3}}^+)) \xrightarrow{\pi_{k_1,k_2}} \widehat{S}_{k_1+k_2}(\mathcal{U}_{S_{k_3}}^+)
\downarrow \qquad \qquad \downarrow
\widehat{S}_{k_1}(\mathcal{U}_{S_{k_2+k_3}}^+) \longrightarrow \mathcal{U}_{S_{k_1+k_2+k_3}}^+$$
(17.27)

Here π_{k_1,k_2} is the covering map in Proposition 24.16. The right vertical and lower horizontal arrows are the covering maps given in (2). The left vertical arrow is induced by the covering map $\widehat{S}_{k_2}(\mathcal{U}_{S_{k_3}}^+) \to \mathcal{U}_{S_{k_2+k_3}}^+$ of (2).

- (5) There exists an embedding $\widehat{S}_k(\mathcal{U}) \to \mathcal{U}_{S_k}^+$.
- (6) The following diagram commutes.

$$\widehat{S}_{k}(\widehat{S}_{\ell}(\mathcal{U})) \longrightarrow \widehat{S}_{k}(\mathcal{U}_{S_{\ell}}^{+})
\downarrow \qquad \qquad \downarrow
\widehat{S}_{k+\ell}(\mathcal{U}) \longrightarrow \mathcal{U}_{S_{k+\ell}}^{+}$$
(17.28)

The maps are as in the case of Diagram (17.23).

The following is the extension theorem for a single collared Kuranishi chart.

Lemma 17.48. Under Situation 17.47, we put

$$U_0 := U \setminus U^{\boxminus \tau}.$$

Then for any $0 < \tau' < \tau$ there exists a τ' -collared Kuranishi chart \mathcal{U}^+ of $X_0 = X \setminus X^{\boxminus \tau}$ with the following properties.

- (1) The restriction of \mathcal{U}^+ to $\widehat{S}_k(U)$ is isomorphic to $\mathcal{U}_{S_k}^+$ as τ' -collared Kuranishi charts.
- (2) There exists an embedding of τ' -collared Kuranishi charts $\mathcal{U}|_{U_0} \to \mathcal{U}^+$.
- (3) The restriction of (2) to $\hat{S}_k(U)$ coincides with one induced from Situation 17.43 (4) under the identification (1).
- (4) The following diagram commutes.

$$\widehat{S}_{k}(\widehat{S}_{\ell}(\mathcal{U}^{+})) \xrightarrow{\cong} \widehat{S}_{k}(\mathcal{U}_{S_{\ell}}^{+})$$

$$\downarrow \qquad \qquad \downarrow$$

$$\widehat{S}_{k+\ell}(\mathcal{U}^{+}) \xrightarrow{\cong} \mathcal{U}_{S_{k+\ell}}^{+}$$
(17.29)

Here the first horizontal arrow is induced from (1). The second horizontal arrow is (1). The left vertical arrow is given by Proposition 24.16. The right vertical arrow is induced by a map given in Situation 17.47 (2).

(5) The embeddings $\widehat{S}_k(\mathcal{U})|_{U_0} \to \widehat{S}_k(\mathcal{U}^+)$ and $\widehat{S}_k(\mathcal{U})|_{U_0} \to \mathcal{U}_{S_k}^+$ coincide via the isomorphism in Situation 17.47 (3). Here the first embedding is induced by the embedding $\mathcal{U}|_{U_0} \to \mathcal{U}^+$ and the second embedding is as in Situation 17.47 (3).

The proof of Lemma 17.48 occupies Subsubsections 17.6.3–17.6.4.

17.6.3. Construction of Kuranishi chart \mathcal{U}^+ . Let $p' \in S_1(U)$. Firstly we will construct a Kuranishi chart $\mathcal{U}^+_{p'}$ mentioned in Lemma 17.48. We use Kuranishi neighborhoods of various $\tilde{p}' \in \widehat{S}_k(U)$ with $\pi(\tilde{p}') = p'$. The Kuranishi neighborhoods we use are those of the Kuranishi chart $\mathcal{U}^+_{S_k}$ given in Situation 17.47 (1). We will modify and glue them in a canonical way to obtain $\mathcal{U}^+_{p'}$. The detail is in order.

modify and glue them in a canonical way to obtain $\mathcal{U}_{p'}^+$. The detail is in order. We first begin with describing the situation of the Kuranishi chart \mathcal{U} in Situation 17.47 in more detail. Suppose $p \in \mathring{S}_k(U)$ for some k. Let $\mathfrak{V}_{\mathfrak{r}} = (V_{\mathfrak{r}}, \Gamma_{\mathfrak{r}}, \phi_{\mathfrak{r}})$ be an orbifold chart of U, which is the underlying orbifold of our Kuranishi chart \mathcal{U} . (Here \mathfrak{r} stands for the index of orbifold charts.) Let $p \in U_{\mathfrak{r}} = V_{\mathfrak{r}}/\Gamma_{\mathfrak{r}}$ such that the base point $o_{\mathfrak{r}}$ of the chart goes to p, (Definition 23.1) that is, $p = \phi_{\mathfrak{r}}(o_{\mathfrak{r}}) \in \mathring{S}_k(U)$. Then we may assume that $V_{\mathfrak{r}} \subset \overline{V}_{\mathfrak{r}} \times [0,1)^k$ and $o_{\mathfrak{r}} = (\overline{o}_{\mathfrak{r}}, (0,\ldots,0))$. Here $\overline{V}_{\mathfrak{r}}$ is

a manifold without boundary. We have a representation $\sigma: \Gamma_{\mathfrak{r}} \to \operatorname{Perm}(k)$ by the definition of admissible orbifold (See Definition 25.7 (1)(b)), where Perm(k) is the group of permutations of $\{1, \ldots, k\}$.

Let $A \subset \{1, \ldots, k\}$. We put

$$\overset{\circ}{S}_{A}([0,1)^{k}) = \{(t_{1},\ldots,t_{k}) \in [0,1)^{k} \mid i \in A \Rightarrow t_{i} = 0, i \notin A \Rightarrow t_{i} > 0\},$$

$$\overset{\circ}{S}_{A}(V_{\mathfrak{r}}) = V_{\mathfrak{r}} \cap (\overline{V}_{\mathfrak{r}} \times \overset{\circ}{S}_{A}([0,1)^{k})),$$

$$\Gamma_{\mathfrak{r}}^{A} = \{\gamma \in \Gamma_{\mathfrak{r}} \mid \gamma A = A\}.$$
(17.30)

The subset A determines a point p(A) of $\widehat{S}_{\ell}(U)$ which goes to p by the map $\widehat{S}_{\ell}(U) \to$ U. Here $\ell = \#A$. An orbifold chart of $\widehat{S}_{\ell}(U)$ at p(A) is given by $S_A(V_r)$ which is the closure of $\overset{\circ}{S}_A(V_{\mathfrak{r}})$, the isotropy group $\Gamma^A_{\mathfrak{r}}$ and ψ_A which is a lift of the restriction of ψ to $S_A(V_r)$. We put

$$V_{\mathfrak{r}}(p;A) = (S_A(V_{\mathfrak{r}}))^{\boxplus \tau} \times [-\tau, 0)^A.$$
 (17.31)

Let us elaborate (17.31). We first note

$$S_A(V_{\mathfrak{r}}) \subset \overline{V}_{\mathfrak{r}} \times [0,1)^{A^c}$$

where $A^c = \{1, \ldots, k\} \setminus A$. (In fact, if $i \in A$ then $t_i = 0$ for an element of $S_A(V_{\mathfrak{r}})$.) We have a retraction map

$$\mathcal{R}_{A^c}: \overline{V}_{\mathfrak{r}} \times [-\tau, 1)^{A^c} \to \overline{V}_{\mathfrak{r}} \times [0, 1)^{A^c},$$

which changes $t_i < 0$ to $t_i = 0$. Then we find

$$(S_A(V_{\mathfrak{r}}))^{\boxplus \tau} = (\mathcal{R}_{A^c})^{-1}(S_A(V_{\mathfrak{r}})) \subset \overline{V}_{\mathfrak{r}} \times [-\tau, 1)^{A^c}.$$

Let $\Pi_{A^c}: \overline{V}_r \times [-\tau, 1)^k \to \overline{V}_r \times [-\tau, 1)^{A^c}$ be the projection. Then we can write

$$V_{\mathfrak{r}}(p;A) = \{ y = (\overline{y}, (t_1, \dots, t_k)) \in \overline{V}_{\mathfrak{r}} \times [-\tau, 1)^k \mid \Pi_{A^c}(y) \in (S_A(V_{\mathfrak{r}}))^{\boxplus \tau}, i \in A \Rightarrow t_i \in [-\tau, 0) \}.$$

$$(17.32)$$

Remark 17.49. We observe that the space $V_{\mathbf{r}}(p,A)$, when it is written as (17.31), is defined by the data of $S_A(V_{\mathfrak{r}})$ only. In particular, it is independent of $\mathring{S}_0(V_{\mathfrak{r}})$. Note, for our extension $V_{\mathfrak{r}}^+$, we are given only data on $S_1(V_{\mathfrak{r}}^+)$. We will use this remark to construct $V_{\mathfrak{r}}^+(p,A)$ in this situation.

We next suppose that $B \supset A$ with $\#B = \ell + m$. Then the triple (p, A, B)determines a point $p(A, B) \in \widehat{S}_m(\widehat{S}_\ell(U))$. We consider the maps $\pi_m : \widehat{S}_m(\widehat{S}_\ell(U)) \to$ S_m($\widehat{S}_{\ell}(U)$) and $\pi_{m,\ell}: \widehat{S}_m(\widehat{S}_{\ell}(U)) \to \widehat{S}_{m+\ell}(U)$ defined in Proposition 24.16. We have $\pi_m(p(A,B)) = p(A)$ and $\pi_{m,\ell}(p(A,B)) = p(B)$.

We put $\Gamma_{\mathfrak{r}}^{A,B} = \Gamma_{\mathfrak{r}}^A \cap \Gamma_{\mathfrak{r}}^B$. The map $S_A(V_{\mathfrak{r}})/\Gamma_{\mathfrak{r}}^{A,B} \to S_A(V_{\mathfrak{r}})/\Gamma_{\mathfrak{r}}^A$ is a restriction of π_m and the map $S_A(V_{\mathfrak{r}})/\Gamma_{\mathfrak{r}}^A$ is a restriction of $\pi_{m,\ell}$. We note

$$V_{\mathfrak{r}}(p;B) = \{ (\overline{y}, (t_1, \dots, t_k)) \in V_{\mathfrak{r}}(p;A) \mid i \in B \Rightarrow t_i \in [-\tau, 0) \}.$$

Therefore we get $V_{\mathfrak{r}}(p;B) \subset V_{\mathfrak{r}}(p;A)$.

Using the expression (17.31), we can rewrite the embedding $V_{\mathfrak{r}}(p;B) \subset V_{\mathfrak{r}}(p;A)$ as follows. We have

$$S_{B \setminus A}(S_A(V_{\mathfrak{r}})) \times [0, \epsilon)^{B \setminus A} \subset S_A(V_{\mathfrak{r}}).$$
 (17.33)

Here $S_{B\setminus A}(S_A(V_{\mathfrak{r}}))$ is a subset of $\widehat{S}_m(S_A(V_{\mathfrak{r}}))$. Note $\pi_0(\widehat{S}_m(S_A(V_{\mathfrak{r}})))$ corresponds one to one to the set of B satisfying $\{1,\ldots,k\}\supset B\supset A$ with $\#B=\ell+m$. Then $S_{B\setminus A}(S_A(V_{\mathfrak{r}}))$ is the connected component corresponding to B under this one to one correspondence.

(17.33) implies

$$(S_{B\setminus A}(S_A(V_{\mathfrak{r}})))^{\boxplus \tau} \times [-\tau, 0)^{B-A} \subset (S_A(V_{\mathfrak{r}}))^{\boxplus \tau}. \tag{17.34}$$

The map $\pi_{m,\ell}$ induces a map

$$\pi_{B \setminus A,A} : S_{B \setminus A}(S_A(V_{\mathfrak{r}})) \to S_B(V_{\mathfrak{r}}),$$
 (17.35)

which is an isomorphism. (The $(m+\ell)!/m!\ell!$ different choices of the points in $\pi_{m,\ell}^{-1}$ (one point) corresponds to $(m+\ell)!/m!\ell!$ different choices of A in the given B.) Therefore composing the inverse of (17.35) and the inclusion (17.34), we obtain

$$(S_{B}(V_{\mathfrak{r}}))^{\boxplus \tau} \times [-\tau, 0)^{B}$$

$$\xrightarrow{(\pi_{B \setminus A, A}^{\boxplus \tau})^{-1} \times \mathrm{id}} (S_{B \setminus A}(S_{A}V_{\mathfrak{r}}))^{\boxplus \tau} \times [-\tau, 0)^{B \setminus A} \times [-\tau, 0)^{A}$$

$$\longrightarrow (S_{A}(V_{\mathfrak{r}}))^{\boxplus \tau} \times [-\tau, 0)^{A}.$$

$$(17.36)$$

It is easy to see that (17.36) coincides with the inclusion $V_{\mathfrak{r}}(p;B) \subset V_{\mathfrak{r}}(p;A)$.

For $A \subset B \subset C$ we have $V_{\mathfrak{r}}(p;C) \subset V_{\mathfrak{r}}(p;B) \subset V_{\mathfrak{r}}(p;A)$. The composition of the two embeddings $V_{\mathfrak{r}}(p;C) \subset V_{\mathfrak{r}}(p;B)$ and $V_{\mathfrak{r}}(p;B) \subset V_{\mathfrak{r}}(p;A)$ coincides with $V_{\mathfrak{r}}(p;C) \subset V_{\mathfrak{r}}(p;A)$. This is equivalent to the commutativity of Diagram (17.22). (See Sublemma 17.50.) We put

$$V_{\mathfrak{r}}(p) = \bigcup_{A \subseteq \{1,\dots,k\}} V_{\mathfrak{r}}(p;A).$$

This is a $\Gamma_{\mathfrak{r}}$ equivariant open subset of $\mathcal{R}^{-1}(\{p\}) \setminus U$. We may take $V_{\mathfrak{r}}(p)/\Gamma_{\mathfrak{r}}$ (together with other data) as a Kuranishi neighborhood of the point in $\mathcal{R}^{-1}(\{p\}) \setminus U$.

To construct $V_{\mathfrak{r}}^+(p)$ we imitate the above description using only the data given on the boundary as follows.

The Kuranishi chart $\mathcal{U}_{S_{\ell}}^+$ given in Situation 17.47 induces $V_{\mathfrak{r},S_A}^+/\Gamma_{\mathfrak{r}}^A$. (It is an open subset of $U_{S_{\ell}}^+$. Also recall $\#A = \ell$.) We define

$$V_{\mathfrak{r}}^{+}(p;A) = (V_{\mathfrak{r},S_{A}}^{+})^{\boxplus \tau} \times [-\tau,0)^{A}.$$
 (17.37)

For $B \supset A$ we have an embedding denoted by $h_{A,B}$

$$h_{A,B}: (S_{B\setminus A}(V_{\mathfrak{r},S_A}^+))^{\boxplus \tau} \times [-\tau,0)^{B\setminus A} \hookrightarrow (V_{\mathfrak{r},S_A}^+)^{\boxplus \tau}. \tag{17.38}$$

The covering map $\widehat{S}_m(\mathcal{U}_{S_\ell}^+) \to \mathcal{U}_{S_{\ell+m}}^+$ given in Situation 17.47 (2) induces the map

$$\pi'_{B\backslash A,A}: S_{B\backslash A}(V_{\mathfrak{r},S_A}^+) \to V_{\mathfrak{r},S_B}^+ \tag{17.39}$$

which is an isomorphism. We define

$$\phi_{AB}: V_{\mathfrak{r}}^{+}(p;B) \to V_{\mathfrak{r}}^{+}(p;A),$$

by

$$(V_{\mathfrak{r},S_B}^+)^{\boxplus \tau} \times [-\tau,0)^B$$

$$(\pi_{B\backslash A,A}^{\prime \boxplus \tau})^{-1} \times \mathrm{id} (S_{B\backslash A}(V_{\mathfrak{r},S_A}^+))^{\boxplus \tau} \times [-\tau,0)^{B\backslash A} \times [-\tau,0)^A \qquad (17.40)$$

$$\xrightarrow{h_{A,B} \times \mathrm{id}} (V_{\mathfrak{r},S_A}^+)^{\boxplus \tau} \times [-\tau,0)^A.$$

Sublemma 17.50. *If* $C \supset B \supset A$, then $\phi_{AB} \circ \phi_{BC} = \phi_{AC}$.

Proof. The sublemma follows from the commutativity of Diagram (17.27) as follows. Recall that $h_{A,B}$ is the inclusion map in (17.38). Then the following diagram commutes.

$$(V_{\mathfrak{r},S_{B}}^{+})^{\boxplus \tau} \stackrel{\pi'^{\boxplus \tau}_{B \setminus A,A}}{\longleftarrow} (S_{B \setminus A}(V_{\mathfrak{r},S_{A}}^{+}))^{\boxplus \tau}$$

$$\uparrow^{h_{B,C}} \qquad \uparrow^{h_{B,C}}$$

$$(S_{C \setminus B}(V_{\mathfrak{r},S_{B}}^{+}))^{\boxplus \tau} \times [-\tau,0)^{C \setminus B} \stackrel{(S_{C \setminus B}(\pi'_{B \setminus A,A}))^{\boxplus \tau} \times \mathrm{id}}{\longleftarrow} (S_{C \setminus B}(S_{B \setminus A}(V_{\mathfrak{r},S_{A}}^{+})))^{\boxplus \tau} \times [-\tau,0)^{C \setminus B}$$

$$(17.41)$$

The commutativity of Diagram (17.41) is a consequence of the commutativity of Diagram (17.27) and the fact that $\pi'_{B\setminus A,A}$ is a diffeomorphism of cornered manifolds and the definition of $h_{*,*}$. We also have the following commutative diagram.

$$(V_{\mathfrak{r},S_A}^+)^{\boxplus \tau} \times [-\tau,0)^A \qquad \stackrel{h_{A,B} \times \mathrm{id}_A}{\longleftarrow} \qquad (S_{B \setminus A}(V_{\mathfrak{r},S_A}^+))^{\boxplus \tau} \times [-\tau,0)^B$$

$$\uparrow^{h_{A,C} \times \mathrm{id}_A} \qquad \qquad \uparrow^{h_{B,C} \times \mathrm{id}_B}$$

$$(S_{C \setminus A}(V_{\mathfrak{r},S_A}^+))^{\boxplus \tau} \times [-\tau,0)^C \stackrel{(\pi_{C \setminus B,B \setminus A})^{\boxplus \tau} \times \mathrm{id}_C}{\longleftarrow} \qquad (S_{C \setminus B}(S_{B \setminus A}(V_{\mathfrak{r},S_A}^+)))^{\boxplus \tau} \times [-\tau,0)^C$$

Note that $\pi_{C\setminus B,B\setminus A}$ in Diagram (17.42) is the map in Proposition 24.16. (The map $\pi'_{B\setminus A,A}$ in Diagram 17.41 is one in Situation 17.47 (2).) The commutativity of Diagram 17.42 is an immediate consequence of the definition of $\pi_{C\setminus B,B\setminus A}$ and $h_{*,*}$. Therefore we have

$$\begin{aligned} \phi_{AB} \circ \phi_{BC} \\ &= (h_{A,B} \times \mathrm{id}) \circ ((\pi'^{\boxplus \tau}_{B \setminus A,A})^{-1} \times \mathrm{id}) \circ (h_{B,C} \times \mathrm{id}) \circ ((\pi'^{\boxplus \tau}_{C \setminus B,B})^{-1} \times \mathrm{id}) \\ &= (h_{A,B} \times \mathrm{id}) \circ (h_{B,C} \times \mathrm{id}) \circ ((S_{C \setminus B}(\pi'_{B \setminus A,A}))^{\boxplus \tau})^{-1} \times \mathrm{id}) \circ ((\pi'^{\boxplus \tau}_{C \setminus B,B})^{-1} \times \mathrm{id}) \\ &= (h_{A,C} \times \mathrm{id}) \circ ((\pi_{C \setminus B,B \setminus A})^{\boxplus \tau} \times \mathrm{id}) \\ &\qquad \qquad \circ ((S_{C \setminus B}(\pi'_{B \setminus A,A}))^{\boxplus \tau})^{-1} \times \mathrm{id}) \circ ((\pi'^{\boxplus \tau}_{C \setminus B,B})^{-1} \times \mathrm{id}) \\ &= (h_{A,C} \times \mathrm{id}) \circ ((\pi'^{\boxplus \tau}_{C \setminus A,A})^{-1} \times \mathrm{id}) \\ &= \phi_{AC}. \end{aligned}$$

Here the first equality is the definition. The second equality is the commutativity of Diagram (17.41). The third equality is the commutativity of Diagram (17.42). The fourth equality is the commutativity of Diagram (17.27).

We consider the disjoint union

$$\coprod_{A} V_{\mathfrak{r}}^{+}(p;A)$$

and define \sim on it as follows. For $x \in V_{\mathfrak{r}}^+(p;A)$ and $y \in V_{\mathfrak{r}}^+(p;B)$ we say $x \sim y$ if and only if there exist C and $z \in V_{\mathfrak{r}}^+(p;C)$ such that $x = \phi_{AC}(z)$ and $y = \phi_{BC}(z)$.

Sublemma 17.51. \sim is an equivalence relation.

Proof. It suffices to prove the transitivity. Let $x = (x', (t_i)_{i \in A})$ where $x' \in (V_{\mathfrak{r}, S_A}^+)^{\boxplus \tau}$ and $t_i \in [-\tau, 0)$ for $i \in A$. We furthermore write $x' = (\overline{x}, (t_i)_{i \in A^c})$. We observe

that x' is in the image of $(S_{C\setminus A}(V_{\mathfrak{r},S_A}^+))^{\boxplus \tau} \times [-\tau,0)^{C\setminus A}$ if and only if $t_i < 0$ for all $i \in C$. Therefore for each x there exists unique C such that

- (1) $x \in \operatorname{Im}(\phi_{AC})$.
- (2) If $x \in \text{Im}(\phi_{AD})$ then $D \subseteq C$.

Transitivity follows from this fact, Sublemma 17.50 and the fact that ϕ_{AB} is injective.

We define

$$V_{\mathfrak{r}}^+(p) = \coprod_A V_{\mathfrak{r}}^+(p;A)/\sim.$$

Sublemma 17.52. The quotient space $V_{\mathfrak{r}}^+(p)$ is Hausdorff with respect to the quotient topology.

Proof. Let $x, y \in V_{\mathfrak{r}}^+(p)$ such that $x \neq y$. We take C_x and C_y as in (1)(2) above and take the representatives $\tilde{x} \in V_{\mathfrak{r}}^+(p; C_x)$ and $\tilde{y} \in V_{\mathfrak{r}}^+(p; C_y)$, respectively. If $C_x = C_y = C$, we can find an open set U_x, U_y in $V_{\mathfrak{r}}^+(p; C)$ such that $\tilde{x} \in U_x, \tilde{y} \in U_y$ and $U_x \cap U_y = \emptyset$. The images of U_x and U_y in $V_{\mathfrak{r}}^+(p)$ separate x, y.

Suppose $C_x \neq C_y$. We may assume that there exists $j \in C_x \setminus C_y$. We write $\tilde{x} = (x', (t_i^0)_{i \in C_x}), x' = (\overline{x}, (t_i^0)_{i \in C_x^c})$. We also write $\tilde{y} = (y', (s_i^0)_{i \in C_y}), y' = (\overline{y}, (s_i^0)_{i \in C_y^c})$. Then $t_j^0 < 0$ and $s_j^0 \geq 0$. Let U_x be the set of all points in $V_{\mathfrak{r}}^+(p; C_x)$ such that $t_j < t_j^0/2$ and U_y be the set of all points in $V_{\mathfrak{r}}^+(p; C_y)$ such that $s_j > t_j^0/2$. They induce disjoint open sets in $V_{\mathfrak{r}}^+(p)$ containing x and y respectively.

Since ϕ_{AB} 's are open embeddings of manifolds, Sublemma 17.52 implies that $V_{\rm r}^+(p)$ is a smooth manifold.

We next define an obstruction bundle and a Kuranishi map on it. (17.37) shows that each $V_{\mathfrak{r}}^+(p;A)$ comes with an obstruction bundle and a Kuranishi map on it. We denote them by $\mathcal{V}_{\mathfrak{r},p;A}^+$. Moreover (17.40) implies that each ϕ_{AB} is covered by the bundle isomorphism and Kuranishi map is compatible with it. Moreover the identity $\phi_{AB} \circ \phi_{BC} = \phi_{AC}$ is promoted to the identity among bundle maps. Therefore we obtain an obstruction bundle and a Kuranishi map on $V_{\mathfrak{r}}^+(p)$. (This is nothing but [Part I, Lemma 3.17].) We denote them by $\mathcal{E}_{\mathfrak{r},p}$ and $s_{\mathfrak{r},p}^+$. We can also define $\psi_{\mathfrak{r},p}^+: (s_{\mathfrak{r},p}^+)^{-1}(0) \to X_0$ in an obvious way. The following is immediate from the construction.

Sublemma 17.53. For each $\gamma \in \Gamma_p$ and A there exists $\varphi_{\gamma,A} : \mathcal{V}_{p;A}^+ \to \mathcal{V}_{p;\gamma A}^+$. Moreover $\varphi_{\gamma,A} \circ \varphi_{AB} = \varphi_{(\gamma A)(\gamma B)} \circ \varphi_{\gamma,B}$.

Then Sublemma 17.53 implies that

$$(V_{\mathfrak{r}}^+(p), \Gamma_{\mathfrak{r},p}, \mathcal{E}_{\mathfrak{r},p}, s_{\mathfrak{r},p}^+, \psi_{\mathfrak{r},p}^+)$$

is a Kuranishi chart at each point of $\mathcal{R}^{-1}(p)$.

Next we define coordinate change. Let $p' \in \mathcal{R}^{-1}(p) \cap U_0$, $q' \in \mathcal{R}^{-1}(q) \cap U_0$ and $q' \in \psi_p^+((s_p^+)^{-1}(0))$. (We assume $q \in S_1(U)$.) Then we have $q \in \psi_{A_0}(s_{A_0}^{-1}(0))$ for some A_0 . Here s_{A_0} is a Kuranishi map of a Kuranishi chart $\mathcal{U}_{S_{k'}}$ at p. We put $k = \#A_0$.

Let $p \in \mathring{S}_k(X)$. We will use the same notations as those used in the construction of the Kuranishi chart $(V_{\mathfrak{r}}^+(p), \Gamma_{\mathfrak{r},p}, \mathcal{E}_{\mathfrak{r},p}, s_{\mathfrak{r},p}^+, \psi_{\mathfrak{r},p}^+)$.

We take $k' \leq k$ such that $q \in \overset{\circ}{S}_{k'}(X)$ and we may assume $q \in \psi_{A_q}(s_{A_q}^{-1}(0))$ with $\#A_q=k'$. We have $\tilde{q}\in S_{A_q}(V_{\mathfrak{r}})$ which parametrizes q. Moreover for each $A\subseteq A_q$ there exists $\tilde{q}_A \in S_A(V_{\mathfrak{r}})$ which parametrizes q.

For $A \subset A_q$ with $\#A = \ell$, the Kuranishi chart $\mathcal{U}_{S_\ell}^+$ gives an orbifold chart $\mathfrak{V}_{\mathfrak{o},A}^+ = (V_{\mathfrak{o},A}^+, \Gamma_{\mathfrak{o}}^A, \psi_{\mathfrak{o},A}^+)$ at \tilde{q}_A . The coordinate change of the underlying orbifold $U_{S_\ell}^+$ of $\mathcal{U}_{S_\ell}^+$ induces a group homomorphism $h_{\mathfrak{ro}}^A:\Gamma_{\mathfrak{o}}^A\to\Gamma_{\mathfrak{r}}^A$ and an $h_{\mathfrak{ro}}^A$ equivariant

$$\varphi_{\mathfrak{ro}}^{A+}: V_{\mathfrak{o},A}^{+} \to V_{\mathfrak{r},A}^{+}.$$
 (17.43)

Moreover the admissibility of our orbifolds implies that we have

$$V_{\mathfrak{o},A}^{+} \subset \overline{V}_{\mathfrak{o},A}^{+} \times [0,1)^{A_{q} \setminus A},$$

$$V_{\mathfrak{r},A}^{+} \subset \overline{V}_{\mathfrak{r},A}^{+} \times [0,1)^{A_{p} \setminus A}$$

$$(17.44)$$

and

$$\varphi_{\mathfrak{ro}}^{A+}(\overline{y},(t_i)_{i\in A_q\backslash A}) = \left(\varphi_{\mathfrak{ro},0}^{A+}(\overline{y},(t_i)),\ (\varphi_{\mathfrak{ro},j}^{A+}(\overline{y},(t_i))_{j\in A_p\backslash A}\right)$$

such that:

- (1) $\varphi_{\mathfrak{ro},0}^{A+}$ is admissible.
- (2) For $j \in A_q \setminus A$, $\varphi_{\mathfrak{ro},j}^{A+} t_j$ is exponentially small near the boundary. (3) For $j \in A_p \setminus A_q$, $\varphi_{\mathfrak{ro},j}^{A+}$ is admissible.

Below we will extend $\varphi_{\mathfrak{ro}}^{A+}$ to

$$\varphi_{\mathfrak{ro}}^{A+\boxplus \tau}:V^+(q,A)\to V^+(p,A).$$

Note

$$V_o^+(q, A) = (V_{o, A}^+)^{\boxplus \tau} \times [-\tau, 0)^A,$$

$$V_{\mathbf{r}}^+(p, A) = (V_{\mathbf{r}, A}^+)^{\boxplus \tau} \times [-\tau, 0)^A.$$

Let $y = (y', (t_i)_{i \in A}) \in V_0^+(q, A)$. We define

$$\varphi_{r_0}^{A+\boxplus \tau}(y) = ((\varphi_{r_0}^{A+})^{\boxplus \tau}(y'), (t_i)_{i \in A}) \in V_r^+(p, A). \tag{17.45}$$

Sublemma 17.54. If $A \subseteq B \subseteq A_q$ then

$$\varphi_{\mathfrak{ro}}^{A+\boxplus \tau}\circ\phi_{AB}=\phi_{AB}\circ\varphi_{\mathfrak{ro}}^{B+\boxplus \tau}.$$

Proof. This is a consequence of the fact that two maps appearing in (17.40) is compatible with the coordinate change. П

By Sublemma 17.54 and [Part I, Lemma 3.18] we can glue $\varphi_{\mathfrak{ro}}^{A+\boxplus \tau}$ for various Ato obtain a map

$$\varphi_{\mathfrak{ro}}^{+ \boxplus \tau} : V_{\mathfrak{o}}^+(q) \to V_{\mathfrak{r}}^+(p)$$

and a bundle map

$$\widehat{\varphi}_{\mathfrak{ro}}^{+\boxplus \tau}: \mathcal{E}_{\mathfrak{o},q}^+ \to \mathcal{E}_{\mathfrak{r},p}^+.$$

They are $h_{pq}: \Gamma_q \to \Gamma_p$ equivariant by construction. Moreover the Kuranishi maps and parametrizations ψ_p^+, ψ_q^+ are compatible to it. We have thus constructed the coordinate change.

The cocycle condition among the coordinate changes follows from the cocycle condition among the coordinate changes of various $\mathcal{U}_{S_c}^+$. ²⁷

 $^{^{27}}$ Since we are constructing the space U together with its orbifold structure, we need to check the cocycle condition. It is easy to check however.

17.6.4. Completion of the proof of Lemma 17.48. To complete the proof of Lemma 17.48 it suffices to check that the Kuranishi chart \mathcal{U}^+ we obtained has the required properties.

Proof of the τ' -collaredness of $\widehat{\mathcal{U}^+}$. Each $V^+(p,A) = (V^+_{\mathfrak{r},A})^{\boxplus \tau} \times [-\tau,0)^A$ is τ' -collared. The open embedding ϕ_{AB} which we used to glue them are τ' -collared. The coordinate change is obtained by gluing $\varphi_{\mathfrak{ro}}^{A+\boxplus \tau}$, which is τ' -collared. Therefore, we can construct a τ' -collared Kuranishi structure in the same way as in the proof of Lemma 17.38.

Remark 17.55. We note that the definition of τ' -collared Kuranishi structure produced in Lemma 17.38 has slight ambiguity. Namely it depends on the choice of the coordinate $\overline{V}_{\widehat{n}} \times [-\tau,c)^k$ on which isotropy group acts by permutation on the $[-\tau,c)^k$ factor. We mentioned this point in the footnote in the proof of Lemma 17.38. Note that on the region $[-\tau, 0]$ there is no ambiguity like that at all. Actually we observe that the proof of the τ' -collared-ness of $\widehat{\mathcal{U}}^+$ is the only place where Lemma 17.38 is used in the application. In the situation of Lemma 17.38 the coordinate $\overline{V}_{\widehat{p}} \times [-\tau, c)^k$ for which the action of isotropy group is given by exchanging the factor on $[-\tau,c)^k$ is given. Namely in our situation $[-\tau,c)$ corresponds to $[-\tau,0]$ where we shift the parameter so that $0 \in [-\tau, c)$ corresponds $\tau' \in [-\tau, 0]$. Thus the ambiguity mentioned in the footnote in the proof of Lemma 17.38 is not at all an issue here.

Note that Lemma 17.38 is literally correct with the proof given. The concern of this remark is the precise meaning of the word 'canonical' in Lemma 17.38.

Proof of Lemma 17.48 (1). Let $p' \in S_{\ell}(U^{\boxplus \tau} \setminus U)$ and $\mathcal{R}(p') = p \in S_{\ell}(U)$. We take $k \geq \ell$ such that $p \in \overset{\circ}{S}_k(U)$. We use the same notations used in the construction of $V^+(p)$. Take p_A' a point in the underlying topological space of $\widehat{S}_\ell(U^{\boxplus \tau} \setminus U)$ which goes to p'. We have a corresponding point $p_A \in \widehat{S}_{\ell}(U)$ which goes to p. The point p_A corresponds to a certain subset $A \subset \{1, \ldots, k\}$ with $\#A = \ell$. An orbifold neighborhood of p_A in $U_{S_\ell}^+$ is $V_{\mathfrak{r},S_A}^+/\Gamma_{\mathfrak{r}}^A$ by definition. Note that $V_{\mathfrak{r}}^+(p;A) = V_{\mathfrak{r},S_A}^+ \times [-\tau,0)^A$ and a neighborhood of p_A' in $S_A(V_{\mathfrak{r}}^+(p;A))$

is $V_{\mathfrak{r},S_A}^+ \times \{(0,\ldots,0)\}$ in $V_{\mathfrak{r}}^+(p;A)$.

Thus we have shown that $\widehat{S}_{\ell}(\mathcal{U}^+)$ and $\mathcal{U}_{S_{\ell}}^+$ are locally diffeomorphic each other as orbifolds. This diffeomorphism is compatible with the gluing by ϕ_{AB} and by coordinate changes. So the underlying orbifolds of $\widehat{S}_{\ell}(\mathcal{U}^+)$ and $\mathcal{U}_{S_{\ell}}^+$ are diffeomorphic. Moreover it is covered by the bundle isomorphism of obstruction bundles which is compatible with coordinate change and Kuranishi map.

Proof of Lemma 17.48 (2). By assumption there exists an embedding $\mathcal{U}|_{S_k(U)} \to$ $\mathcal{U}_{S_{L}}^{+}$. (Situation 17.47 (3).) By comparing (17.31) and (17.37) it induces an embedding $V_{\mathfrak{r}}(p;A) \to V_{\mathfrak{r}}^+(p;A)$.

By the commutativity of (17.28) we have the following commutative diagram

$$V_{\mathfrak{r}}(p;B) \longrightarrow V_{\mathfrak{r}}^{+}(p;B)$$

$$\downarrow^{\phi_{AB}} \qquad \qquad \downarrow^{\phi_{AB}}$$

$$V_{\mathfrak{r}}(p;A) \longrightarrow V_{\mathfrak{r}}^{+}(p;A)$$

for $B \supset A$. Therefore we have an embedding $V_{\mathfrak{r}}(p) \to V_{\mathfrak{r}}^+(p)$. It is covered by a bundle map and is Γ_p equivariant. Moreover it is compatible with Kuranishi map. Thus this embedding $V_{\mathfrak{r}}(p) \to V_{\mathfrak{r}}^+(p)$ is promoted to an embedding of Kuranishi charts.

On the other hand, the embeddings $V_*(p;A) \to V_*^+(p;A)$ commute with the embeddings $\varphi_{\mathfrak{ro}}^{A+}$ and $\varphi_{\mathfrak{ro}}^{A}$. Therefore we can glue the embeddings $V_*(p) \to V_*^+(p)$ to obtain the required embedding.

Proof of Lemma 17.48 (3). This follows from the proof of Lemma 17.48 (1),(2). \Box

Proof of Lemma 17.48 (4). This follows from the proof of Lemma 17.48 (1).

Proof of Lemma 17.48 (5). This follows from the proof of Lemma 17.48 (2). \Box

Therefore the proof of Lemma 17.48 is now complete.

17.6.5. Proof of Proposition 17.46.

Proof. It suffices to construct the coordinate change between Kuranishi charts produced in Lemma 17.48 and show that the coordinate changes are compatible with various embeddings and isomorphisms appearing in the statement of Proposition 17.46 and of Lemma 17.48. This is indeed straightforward. In fact, the Kuranishi structure in Lemma 17.48 is constructed from $\mathcal{U}_{S_k}^+$, which are Kuranishi charts of $\widehat{\mathcal{U}_{S_k}^+}$. They are compatible with the coordinate change by definition. The process to construct our Kuranishi chart from $\mathcal{U}_{S_k}^+$ is by trivialization of the corner, $*\mapsto *^{\boxplus \tau}$, and gluing by the map in Situation 17.47 (2). The former is compatible with coordinate change as we proved in the first half of this section. The latter is compatible since it is induced by the corresponding map (Situation 17.43 (2)) of Kuranishi structures.

Remark 17.56. What is written as \mathcal{U} in the notation of Proposition 17.46 corresponds to $\mathcal{U}^{\boxplus \tau}$ in the notation of Lemma 17.48.

17.7. Extension of collared CF-perturbation. In this section we prove Proposition 17.58.

Situation 17.57. In Situation 17.43, let $\widehat{\mathfrak{S}_{\partial}^+}$ be a τ -collared CF-perturbation of $\widehat{\mathcal{U}_{\partial}^+}$. We assume the following:

- (1) For each $k \geq 1$ there exists a τ -collared CF-perturbation $\widehat{\mathfrak{S}}_{S_k}^+$ of $\widehat{\mathcal{U}}_{S_k}^+$ such that $\widehat{\mathfrak{S}}_{S_1}^+ = \widehat{\mathfrak{S}}_{\partial}^+$.
- (2) The pull-back of $\widehat{\mathfrak{S}}_{S_{k+\ell}}^+$ by $\pi_{k,\ell}: \widehat{S}_k(\widehat{S}_\ell(X),\widehat{\mathcal{U}}_{S_\ell}^+) \to (\widehat{S}_{k+\ell}(X),\widehat{\mathcal{U}}_{S_{k+\ell}}^+)$ is equivalent to the restriction of $\widehat{\mathfrak{S}}_\ell^+$.

Proposition 17.58. Suppose we are in Situation 17.57. Then for any $0 < \tau' < \tau$ there exists a τ' -collared CF-perturbation $\widehat{\mathfrak{S}^+}$ of the Kuranishi structure $\widehat{\mathcal{U}^+}$ obtained in Proposition 17.46 such that the restriction of $(\widehat{S}_k(X), \widehat{\mathcal{U}_{S_k}^+})$ is equivalent to $\widehat{\mathfrak{S}_{S_k}^+}$. When $\widehat{\mathfrak{S}_{S_k}^+}$ varies in a uniform family, we may take $\widehat{\mathfrak{S}^+}$ to be uniform.

Proof. We first consider the situation of one chart. We use the same notations used in the construction of the chart $V^+(p;A)$ during the proof of Lemma 17.48. By assumption (Situation 17.57 (1)) we are given a CF-perturbation $\mathcal{S}_{\mathfrak{r},A}^+$ on $V_{\mathfrak{r},S_A}^+/\Gamma_{\mathfrak{r}}^A$. It induces $\mathcal{S}_{\mathfrak{r},A}^{+\boxplus \tau}$ on $(V_{\mathfrak{r},S_A}^+)^{\boxplus \tau}/\Gamma_{\mathfrak{r}}^A$. We extend it by constant on the $[-\tau,0)^A$ factor to obtain $\mathcal{S}_{p,A}$ on $V^+(p;A)/\Gamma_{\mathfrak{r}}^A$.

If $A' = \gamma A$ for $\gamma \in \Gamma_{\mathfrak{r}}$, then $\mathcal{S}_{p,A}$ is isomorphic to $\mathcal{S}_{p,\gamma A}$. Therefore we obtain $\mathcal{S}_{p,\ell}^+$ on

$$\left(\bigcup_{A;\#A=\ell} V_{\mathfrak{r}}^{+}(p;A)\right)/\Gamma_{\mathfrak{r}}.$$
(17.46)

Note that (17.46) is diffeomorphic to $V_{\mathfrak{r}}^+(p,A)/\Gamma_{\mathfrak{r}}^A$. The open sets (17.46) for various ℓ cover $V_{\mathfrak{r}}^+(p)$.

Let $B \supset A$. Situation 17.57 (2) implies that the restriction of $\mathcal{S}_{p,A}$ by the map ϕ_{AB} is equivalent to $\mathcal{S}_{p,B}$. Therefore $\mathcal{S}_{p,\ell}^+$ is equivalent to $\mathcal{S}_{p,m}^+$ on the intersection of the domains (17.46). Thus we get a CF-perturbation \mathfrak{S}_p^+ on $V_{\mathfrak{r}}^+(p)/\Gamma_{\mathfrak{r}}$.

If $q \in \psi_p^+((s_p^+)^{-1}(0))$, then we have $\varphi_{\mathfrak{ro}}^{A+}: V_{\mathfrak{o},A}^+ \to V_{\mathfrak{r},A}^+$, that is (17.43). Since $\widehat{\mathfrak{S}}_{s_\ell}^+$ is a CF-perturbation, the pull-back of $\mathcal{S}_{\mathfrak{r},A}^+$ by $\varphi_{\mathfrak{ro}}^{A+}$ is equivalent to $\mathcal{S}_{\mathfrak{o},A}^+$. Therefore \mathfrak{S}_p^+ and \mathfrak{S}_q^+ are glued to define a CF-perturbation on the union of domains. We have thus constructed a CF-perturbation on each of the Kuranishi charts. The compatibility with the coordinate change follows from the corresponding compatibility of $\widehat{\mathcal{U}_{S_k}^+}$'s. Thus we have obtained a CF-perturbation $\widehat{\mathfrak{S}}^+$.

The equivalence of the restriction of $\widehat{\mathfrak{S}^+}$ to $(\widehat{S}_k(X), \widehat{\mathcal{U}_{S_k}^+})$ and $\widehat{\mathfrak{S}_k^+}$ is obvious from the construction. \square

17.8. Extension of Kuranishi structure and CF-perturbation from a neighborhood of a compact set. In this subsection we prove extension lemmas of Kuranishi structure and of CF-perturbation defined on a neighborhood of a compact set. These lemmas are used in the next subsection. Note that they are results about Kuranishi structure and CF-perturbation, not about τ -collared ones.

Lemma 17.59. Let K be a compact set of X and $Z \subseteq X$ a compact neighborhood of K such that $K \subset \operatorname{Int} Z$. Suppose we are given a Kuranishi structure $\widehat{\mathcal{U}}$ on X and $\widehat{\mathcal{U}}_{+}^{L}$ on Z. We assume

$$\widehat{\mathcal{U}}|_{Z} < \widehat{\mathcal{U}_{Z}^{+}}.$$

Let Ω be a relatively compact neighborhood of K in Z such that

$$K \subset \Omega \subset \overline{\Omega} \subset \operatorname{Int} Z$$
.

We also assume the following:

(i) Let $p \in K$ and $\mathcal{U}_{Z,p}^+ = (U_{Z,p}^+, \mathcal{E}_{Z,p}^+, s_{Z,p}^+, \psi_{Z,p}^+)$ be the Kuranishi neighborhood of $\widehat{\mathcal{U}_{Z}^+}$ at p. We assume

$$\psi_{Z,p}^+((s_{Z,p}^+)^{-1}(0))\subset\Omega.$$

(ii) Let $p \in K$ and $\mathcal{U}_p = (U_p, \mathcal{E}_p, s_p, \psi_p)$ be the Kuranishi neighborhood of $\widehat{\mathcal{U}}$ at p. We assume

$$\psi_p((s_p)^{-1}(0)) \subset \Omega.$$

Then there exist a Kuranishi structure $\widehat{\mathcal{U}^+}$ on X and an embedding $\widehat{\mathcal{U}} \to \widehat{\mathcal{U}^+}$ with the following properties:

- (1) For any $p \in K$ the Kuranishi neighborhood \mathcal{U}_p^+ of $\widehat{\mathcal{U}^+}$ at p is isomorphic to the Kuranishi neighborhood $\mathcal{U}_{Z,p}^+$ of $\widehat{\mathcal{U}_{Z}^+}$ at p. For any $p,q\in K$ the coordinate change between \mathcal{U}_{n}^{+} and \mathcal{U}_{a}^{+} coincides with the coordinate change between $\mathcal{U}_{Z,p}^+$ and $\mathcal{U}_{Z,q}^+$.
- (2) $\widehat{\mathcal{U}}^+|_{\Omega}$ is an open substructure of $\widehat{\mathcal{U}}_{Z}^+|_{\Omega}$.
- (3) $\widehat{\mathcal{U}} < \widehat{\mathcal{U}}^+$.
- (4) The next diagram commutes.

$$\widehat{\mathcal{U}}|_{\Omega} \xrightarrow{\text{embedding}} \widehat{\mathcal{U}}^{+}|_{\Omega}$$
embedding
$$\widehat{\mathcal{U}}^{+}_{Z}|_{\Omega}$$
(17.47)

Here the right down arrow is the open embedding given by (2).

(5) The embedding $\widehat{\mathcal{U}}|_K \to \widehat{\mathcal{U}}_Z^+|_K$ coincides with the embedding $\widehat{\mathcal{U}}|_K \to \widehat{\mathcal{U}}^+|_K$ via the isomorphism (1).

Proof. We take an open set $\Omega_1 \subset X$ such that

$$\overline{\Omega} \subset \Omega_1 \subset \overline{\Omega}_1 \subset \operatorname{Int} Z$$
.

We next take an open set $W_1 \subset X$ such that

$$\overline{W}_1 \cap \overline{\Omega} = \emptyset, \quad \Omega_1 \cup W_1 = X.$$

We replace $\widehat{\mathcal{U}}$ and $\widehat{\mathcal{U}_Z^+}$ by their open substructures (but without changing the Kuranishi neighborhoods of the point $p \in K$) if necessary, and may assume that the following holds.

- (1) If $\psi_{Z,p}^+(U_{Z,p}^+ \cap (s_{Z,p}^+)^{-1}(0)) \cap \overline{W}_1 \neq \emptyset$, then $\psi_{Z,p}^+(U_{Z,p}^+ \cap (s_{Z,p}^+)^{-1}(0)) \cap \overline{\Omega} = \emptyset$. (2) If $\psi_p(U_p \cap s_p^{-1}(0)) \cap \overline{W}_1 \neq \emptyset$, then $\psi_p(U_p \cap s_p^{-1}(0)) \cap \overline{\Omega} = \emptyset$.

We define a Kuranishi structure $\widehat{\mathcal{U}}'$ on X as follows.

- (a) If $p \in \overline{\Omega}_1$, we put $\mathcal{U}'_p = \mathcal{U}^+_{Z,p}$.
- (b) If $p \notin \overline{\Omega}_1$, we put $\mathcal{U}'_p = \mathcal{U}_p|_{U_p \setminus \psi_p^{-1}(\overline{\Omega}_1)}$.

The coordinate change is defined as follows. Let $q \in \psi'_p((s'_p)^{-1}(0))$. If $p, q \in \overline{\Omega}_1$, then we define $\Phi'_{pq} = \Phi^+_{Z,pq}$. If $p,q \notin \overline{\Omega}_1$, then we define $\Phi'_{pq} = \Phi_{pq}|_{U_{pq} \setminus \psi_q^{-1}(\overline{\Omega}_1)}$. Among the other two cases $q \in \overline{\Omega}_1$, $p \notin \overline{\Omega}_1$ cannot occur (by (b)). We consider the remaining case $q \notin \overline{\Omega}_1$, $p \in \overline{\Omega}_1$. We have an embedding $\Phi_q : \mathcal{U}_q \to \mathcal{U}_{Z,q}^+$. We compose it with the embedding of Kuranishi structure $\widehat{\mathcal{U}_Z^+}$ to obtain

$$\Phi_{Z,pq}^+ \circ \Phi_q : \mathcal{U}_q|_{(\varphi_q)^{-1}(U_{Z,pq}^+)} \to \mathcal{U}_{Z,q}^+|_{U_{Z,pq}^+} \to \mathcal{U}_{Z,p}^+.$$

The composition gives the coordinate change Φ'_{pq} in this case.

Note $\Phi_{Z,pq}^+ \circ \Phi_q = \Phi_p \circ \Phi_{pq}$ on $(\varphi_q)^{-1}(U_{Z,pq}^+)$, by the definition of embedding of Kuranishi structures. Using this fact, it is easy to see that they define a Kuranishi structure on X.

The Kuranishi structure $\widehat{\mathcal{U}}'$ has all the properties we need, except the property (3). We will further modify $\widehat{\mathcal{U}}'$ to $\widehat{\mathcal{U}}^+$ by this reason. Firstly, we use [Part I, Propositions 6.44 and 6.49] to find a Kuranishi structure $\widehat{\mathcal{U}''}$ such that

$$\widehat{\mathcal{U}'} < \widehat{\mathcal{U}''}$$
.

Though there are various choices of such $\widehat{\mathcal{U}''}$, we choose one of them in the proof of Lemma 17.59. For the later purpose, we will take more specific $\widehat{\mathcal{U}''}$ in the proof of

Next, we modify $\widehat{\mathcal{U}''}$ to $\widehat{\mathcal{U}^+}$ which has the required properties as follows. We take an open set $W_2 \subset X$ such that

$$\overline{W}_2 \cap \overline{\Omega}_1 = \emptyset, \quad Z \cup W_2 = X.$$

We replace various Kuranishi structures involved by their open substructures (but without changing the Kuranishi neighborhoods of the point $p \in K$) and may assume

- (I) If $\psi_p(U_{Z,p}^+ \cap (s_{Z,p}^+)^{-1}(0)) \cap \overline{W}_2 \neq \emptyset$, then $\psi_{Z,p}^+(U_{Z,p}^+ \cap (s_{Z,p}^+)^{-1}(0)) \cap \overline{\Omega}_1 = \emptyset$. (II) If $\psi_p(U_p \cap s_p^{-1}(0)) \cap \overline{W}_2 \neq \emptyset$, then $\psi_p(U_p \cap s_p^{-1}(0)) \cap \overline{\Omega}_1 = \emptyset$.

Now we define a Kuranishi structure $\widehat{\mathcal{U}^+}$ on X as follows.

- (A) If $p \in \overline{W}_2$, we put $\mathcal{U}_p^+ = \mathcal{U}_p''$.
- (B) If $p \notin \overline{W}_2$, we put $\mathcal{U}_p^+ = \mathcal{U}_p'|_{U_p' \setminus \psi_p^{-1}(\overline{W}_2)}$.

The coordinate change is defined as follows. Let $q \in \psi_p^+((s_p^+)^{-1}(0))$. If $p, q \in \overline{W}_2$, then we define $\Phi_{pq}^+ = \Phi_{pq}^{"}$. If $p, q \notin \overline{W}_2$, then we define $\Phi_{pq}^+ = \Phi_{pq}^{'}|_{U_{pq}^+ \setminus \psi_q^{-1}(\overline{W}_2)}$. Among the other two cases $q \in \overline{W}_2$, $p \notin \overline{W}_2$ cannot occur. Suppose $p \in \overline{W}_2$, $q \notin \overline{W}_2$. Then there is an embedding $\Phi_q : \mathcal{U}'_q \to \mathcal{U}''_q$. The coordinate change of $\widehat{\mathcal{U}}^+$ is given by the composition

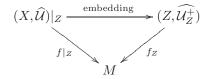
$$\Phi_{pq}'' \circ \Phi_q : \mathcal{U}_q'|_{(\varphi_q)^{-1}(U_{pq}'')} \to \mathcal{U}_q''|_{U_{pq}''} \to \mathcal{U}_p''.$$

It is easy to see that this Kuranishi structure $\widehat{\mathcal{U}^+}$ has the required properties.

We next discuss extension of CF-perturbations.

Situation 17.60. Suppose we are in the situation of Lemma 17.59. We assume the following in addition.

- (1) We have a strongly smooth and weakly submersive map $f:(X,\widehat{\mathcal{U}})\to M$ to a manifold M. (See [Part I, Definition 3.38].)
- (2) We have a CF-perturbation \mathfrak{S}_Z^+ of \mathcal{U}_Z^+ .
- (3) We have a strongly smooth map $f_Z:(Z,\widehat{\mathcal{U}_Z^+})\to M$ which is strongly submersive with respect to $\widehat{\mathfrak{S}_{Z}^{+}}$. (See [Part I, Definition 9.2].)
- (4) The following diagram commutes:



Lemma 17.61. In Situation 17.60, we may choose the Kuranishi structure \mathcal{U}^+ in Lemma 17.59 so that the following holds in addition.

- (1) There exists a CF-perturbation $\widehat{\mathfrak{S}^+}$ of $\widehat{\mathcal{U}^+}$.
- (2) The map f extends to a strongly smooth map $f^+:(X,\widehat{\mathcal{U}^+})\to M$.
- (3) The extended map $f^+:(X,\widehat{\mathcal{U}^+})\to M$ is strongly submersive with respect to $\widehat{\mathfrak{S}^+}$.
- (4) For any $p \in K$, two CF-perturbations $\widehat{\mathfrak{S}_{Z}^{+}}$ and $\widehat{\mathfrak{S}^{+}}$ assign the same CF-perturbation on the Kuranishi chart $\mathcal{U}_{Z,p}^{+} = \mathcal{U}_{p}^{+}$.
- (5) \$\hat{\mathbb{G}_Z^+}|_{\Omega}\$ is a restriction of \$\hat{\mathbb{G}^+}|_{\Omega}\$ to the open substructure.
 (6) When \$\hat{\mathbb{G}_Z^+}|_{\Omega}\$ varies in a uniform family, we may take \$\hat{\mathbb{G}^+}\$ to be uniform.

Proof. The lemma is a consequence of combination of results in [Part I, Sections 3,6,7 and 9. Before we start the proof, we recall from [Part I, Section 7] that we used a good coordinate system to define a CF-perturbation. Thus we also need to involve an extension of good coordinate system in the course of the proof of Lemma 17.61. Indeed, we use a good coordinate system from the given Kuranishi structure to find an extension of the given CF-perturbation, and come back from the good coordinate system to a Kuranishi structure together with the CF-perturbation. This process is described in [Part I, Section 9]. This is a rough description of the structure of the proof of Lemma 17.61 given below.

Now we start the proof. In the proof of Lemma 17.59, we took a relatively compact open subset $\Omega_1 \subset \operatorname{Int} Z$ such that

$$\overline{\Omega} \subset \Omega_1 \subset \overline{\Omega}_1 \subset \operatorname{Int} Z$$
.

As in the proof, we replace $\widehat{\mathcal{U}}$ and $\widehat{\mathcal{U}_Z^+}$ by their open substructures without changing the Kuranishi neighborhoods of the point $p \in K$ if necessary, and may assume that the map $f_Z:(Z,\mathcal{U}_Z^+)\to M$ is strictly strongly submersive ([Part I, Definition 9.2]) with respect to \mathfrak{S}_Z^+ . We put

$$Z_1 = \overline{\Omega}_1, \quad \widehat{\mathcal{U}_{Z_1}^+} = \widehat{\mathcal{U}_Z^+}|_{Z_1}.$$

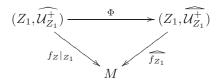
Then by the definition of the Kuranishi structure $\widehat{\mathcal{U}}$ in the proof of Lemma 17.59, we note

$$\widehat{\mathcal{U}}'|_{Z_1} = \widehat{\mathcal{U}_{Z_1}^+}.\tag{17.48}$$

We apply [Part I, Theorem 3.30] to the Kuranishi structure $\widehat{\mathcal{U}_{Z_1}^+}$ to find a good coordinate system $\overline{\mathcal{U}_{Z_1}^+}$ on Z_1 and a KG-embedding

$$\Phi : \widehat{\mathcal{U}_{Z_1}^+} \longrightarrow \widehat{\mathcal{U}_{Z_1}^+}. \tag{17.49}$$

Since we are given a CF-perturbation $\widehat{\mathfrak{S}}_{Z}^{+}|_{Z_{1}}$ of $\widehat{\mathcal{U}}_{Z_{1}}^{+}$, [Part I, Lemma 9.10] shows that there exists a CF-perturbation $\widehat{\mathfrak{S}_{Z_1}^+}$ of $\widehat{\mathcal{U}_{Z_1}^+}$ such that $\widehat{\mathfrak{S}_{Z_1}^+}$ and $\widehat{\mathfrak{S}_{Z_1}^+}|_{Z_1}$ are compatible with the KG-embedding Φ in (17.49). In addition, [Part I, Lemma 7.53 and Proposition 7.57] yield that there exists a strongly smooth map $\overline{f_{Z_1}}$: $(Z_1,\widehat{\mathcal{U}_{Z_1}^+}) \to M$ such that it is a strongly submersive with respect to $\widehat{\mathfrak{S}_{Z_1}^+}$ and the following diagram commutes.



Since $\widehat{\mathcal{U}_{Z_1}^+} = \widehat{\mathcal{U}'}|_{Z_1}$ is the restriction of the Kuranishi structure $\widehat{\mathcal{U}'}$ on X to Z_1 , we can apply [Part I, Proposition 7.52] for the case $Z_1 = Z_1$ and $Z_2 = X$ to obtain a good coordinate system $\widehat{\mathcal{U}'}$ on X such that it is an extension of $\widehat{\mathcal{U}_{Z_1}^+}$, and $\widehat{\mathcal{U}'}$ and $\widehat{\mathcal{U}'}$ are compatible in the sense of [Part I, Definition 3.32], i.e., there exists a KG-embedding

$$\widehat{\mathcal{U}'} \longrightarrow \widehat{\mathcal{U}'}$$
.

Moreover, the CF-perturbation $\widehat{\mathfrak{S}_{Z_1}^+}$ is also extended to a CF-perturbation $\widehat{\mathfrak{S}'}$ of $\widehat{\mathcal{U}'}$ by [Part I, Proposition 7.57]. In addition, by [Part I, Lemma 7.53] there exists a strongly smooth and strongly submersive map

$$\widehat{f'}: (X,\widehat{\mathcal{U}'}) \longrightarrow M$$

with respect to $\widehat{\mathfrak{S}'}$ such that the following diagram commutes.

$$(Z_{1},\widehat{\mathcal{U}'}|_{Z_{1}}) \xrightarrow{f_{Z}|_{Z_{1}}} (Z_{1},\widehat{\overline{\mathcal{U}'}}|_{Z_{1}})$$

Now we go back to Kuranishi structure from good coordinate system. We apply [Part I, Proposition 6.44] for the case $\widehat{\mathcal{U}}_0 = \widehat{\mathcal{U}} = \widehat{\mathcal{U}}'$ to find a Kuranishi structure $\widehat{\mathcal{U}''}$ together with a GK-embedding

$$\widehat{\mathcal{U}'} \longrightarrow \widehat{\mathcal{U}''},$$

and a strongly smooth map

$$\widehat{f'}:(X,\widehat{\mathcal{U''}})\longrightarrow M$$

such that $\widehat{f'}$ is a pull-back of $\widehat{f'}$. By [Part I, Lemma 5.14], $\widehat{\mathcal{U}''}$ is a thickening of $\widehat{\mathcal{U}'}$:

$$\widehat{\mathcal{U}'} < \widehat{\mathcal{U}''}$$
.

Moreover, [Part I, Lemma 9.9] shows that there exists a CF-perturbation $\widehat{\mathfrak{S}''}$ of $\widehat{\mathcal{U}''}$ such that $\widehat{f'}$ is strongly submersive with respect to $\widehat{\mathfrak{S}''}$.

Finally in the exactly same way as in the proof of Lemma 17.59, we modify the Kuranishi structure $\widehat{\mathcal{U}''}$ obtained above to $\widehat{\mathcal{U}^+}$. Accordingly, we also have the corresponding CF-perturbation $\widehat{\mathfrak{S}^+}$ of $\widehat{\mathcal{U}^+}$ and the corresponding map $f^+:(X,\widehat{\mathcal{U}^+})\to M$. Then all the assertions of Lemma 17.61 follow from the construction.

17.9. Main results of Section 17. We now combine the results of Subsections 17.6-17.8 to prove results which we will use later in our applications.

Proposition 17.62. In Situation 17.43, there exists a τ' -collared Kuranishi structure $\widehat{\mathcal{U}^{++}}$ on X for any $0 < \tau' < \tau$ such that Proposition 17.46 holds, by replacing $\widehat{\mathcal{U}^{+}}$ by $\widehat{\mathcal{U}^{++}}$.

Remark 17.63. The difference between Proposition 17.46 and Proposition 17.62 is that the τ' -collared Kuranishi structure $\widehat{\mathcal{U}^{++}}$ in Proposition 17.62 is defined on whole X, while the τ' -collared Kuranishi structure $\widehat{\mathcal{U}^{+}}$ in Proposition 17.46 is defined only on a neighborhood of the boundary.

Proof. We will use Lemma 17.59 to prove Proposition 17.62. For this purpose, we will slightly modify the τ' -collared Kuranishi structure $\widehat{\mathcal{U}^+}$ produced in Proposition 17.46 to get $\widehat{\mathcal{U}_{\Omega}^+}$ satisfying the assumption of Lemma 17.59. The detail is in order.

We take $0 < \tau' < \tau''' < \tau'''' < \tau$. Let $\widehat{\mathcal{U}^+}$ be the τ' -collared Kuranishi structure produced in Proposition 17.46. We note that $X' = X^{\boxplus \tau} = (X^{\boxplus (\tau - \tau')})^{\boxplus \tau'}$. We take the Kuranishi structure $\widehat{\mathcal{U}''}$ on $X^{\boxplus (\tau - \tau')}$ such that $(X^{\boxplus (\tau - \tau')}, \widehat{\mathcal{U}''})^{\boxplus \tau'} = (X^{\boxplus \tau}, \widehat{\mathcal{U}^+})$. (See Remark 17.40 for the description of the Kuranishi structure $\widehat{\mathcal{U}''}$.) We shrink the Kuranishi neighborhood \mathcal{U}''_p of the Kuranishi structure $\widehat{\mathcal{U}''}$ to obtain \mathcal{U}'''_p and a Kuranishi structure $\widehat{\mathcal{U}'''}$ so that the following is satisfied.

- (1) If $p \notin X^{\boxplus (\tau \tau'')}$, then $\psi''_n((s''_n)^{-1}(0)) \cap X^{\boxplus (\tau \tau''')} = \emptyset$.
- (2) If $p \in S_1(X^{\boxplus(\tau-\tau')})$, then $\partial \mathcal{U}_p''' = \partial \mathcal{U}_p''$.

Note that (2) above implies

$$(X,\widehat{\mathcal{U}'''})^{\boxplus(\tau-\tau')}|_{\overline{X\backslash X^{\boxplus(\tau-\tau')}}} = (X',\widehat{\mathcal{U}^+})|_{\overline{X\backslash X^{\boxplus(\tau-\tau')}}}.$$
 (17.50)

We put $X'' = \operatorname{Int}(X \setminus X^{\boxplus(\tau-\tau')})$, $K = \overline{X'' \setminus X^{\boxplus(\tau-\tau'')}}$ and $Z = \overline{X'' \setminus X^{\boxplus(\tau-\tau''')}}$. We now apply Lemma 17.59. Here the role of X, K, Z, $\widehat{\mathcal{U}_Z^+}$ in Lemma 17.59 is played by X'', K, Z and $\widehat{\mathcal{U}'''}$, respectively.

Remark 17.64. As we mentioned at the top of Subsection 17.8, Lemma 17.59 is about genuine Kuranishi structure and not τ -collared Kuranishi structure. So here we apply Lemma 17.59 literally, not its τ -collared version.

We thus obtain a Kuranishi structure which we wrote $\widehat{\mathcal{U}^+}$ in Lemma 17.59. We denote it here by $\widehat{\mathcal{U}'^+}$.

We put $\widehat{\mathcal{U}^{++}} = \widehat{\mathcal{U}^{'+\boxplus \tau'}}$. (17.50), Lemma 17.59 (1) and the fact that $\widehat{\mathcal{U}^{+}}$ satisfies Proposition 17.46 imply that $\widehat{\mathcal{U}^{++}}$ satisfies Proposition 17.46. The proof of Proposition 17.62 is now complete.

We next include CF-perturbations.

Proposition 17.65. In Situation 17.57 there exists a τ' -collared CF-perturbation $\widehat{\mathfrak{S}^{++}}$ on the Kuranishi structure $\widehat{\mathcal{U}^{++}}$ obtained in Proposition 17.62 such that

(1) Its restriction to $(\widehat{S}_k(X), \widehat{\mathcal{U}_{S_k}^+})$ is equivalent to $\widehat{\mathfrak{S}_k^+}$.

- (2) If $f:(X,\widehat{\mathcal{U}}) \to M$ is weakly submersive and $f|_{\widehat{\mathcal{U}_k^+}}$ is strongly submersive with respect to $\widehat{\mathfrak{S}_k^+}$, then we may take $\widehat{\mathfrak{S}^{++}}$ such that $f:(X,\widehat{\mathcal{U}^{++}}) \to M$ is strongly submersive with respect to $\widehat{\mathfrak{S}^{++}}$.
- (3) Uniformity of CF-perturbations is preserved in this construction.

Proof. We use the notation in the proof of Proposition 17.62. We apply Proposition 17.58 to obtain a CF-perturbation $\widehat{\mathfrak{S}^+}$ on $\widehat{\mathcal{U}^+}$. Since $\widehat{\mathfrak{S}^+}$ is τ' -collared, it is induced by a CF-perturbation $\widehat{\mathfrak{S}''}$ of $\widehat{\mathcal{U}''}$. Therefore by restriction we obtain a CF-perturbation $\widehat{\mathfrak{S}^+_Z}$ on $\widehat{\mathcal{U}^+_Z}$. Here Z is one in the proof of Proposition 17.62. Thus we can apply Lemma 17.61 to obtain required $\widehat{\mathfrak{S}^{++}}$ and $\widehat{\mathcal{U}^{++}}$.

One minor point remains to be explained to apply the results of this subsection. Note that a τ -collared Kuranishi structure is not a Kuranishi structure, since some points are not assigned its Kuranishi neighborhood to. Since in Part 1 the 'push out' is defined for the case of Kuranishi structure and good coordinate system but not for the case of τ -collared Kuranishi structure, we need some explanation to define the notion of the 'push out' for τ -collared Kuranishi structure.

One can define the notion of τ -collared good coordinate system, and prove the existence of τ -collared good coordinate system compatible with each τ -collared Kuranishi structure, and use it to define the 'push out'. (We note that τ -collared good coordinate system is a special case of good coordinate system. See the end of Remark 17.31.) It is certainly possible to proceed in that way.

Here we take a slightly different way which seems shorter as follows.

Lemma 17.66. Let $\widehat{\mathcal{U}}$ be a τ -collared Kuranishi structure on $X' = X^{\boxplus \tau}$.

- (1) We can associate a Kuranishi structure $\widehat{\mathcal{U}'}$ on X' such that:
 - (a) If $p \in \overset{\circ\circ}{S}_k(X')$ then $\mathcal{U}'_p = \mathcal{U}_p$.
 - (b) For each $p \in X'$ we take the (unique) point $\widehat{p} \in \overset{\circ}{S}_k(X')$ such that $\mathcal{R}(p) = \mathcal{R}(\widehat{p})$ and $\mathcal{R}(p) \in \overset{\circ}{S}_k(X)$. (See Lemma 17.41.) Then \mathcal{U}'_p is an open subchart of $\mathcal{U}_{\widehat{p}}$
- (2) If $\widehat{\mathfrak{S}}$ is a τ -collared CF-perturbation on $\widehat{\mathcal{U}}$, it induces a CF-perturbation $\widehat{\mathfrak{S}}'$ on $\widehat{\mathcal{U}}'$.
- (3) If $\widehat{\mathcal{U}'}$ and $\widehat{\mathcal{U}''}$ are two Kuranishi structures satisfying (1) (a)(b) above, then there exists a Kuranishi structure $\widehat{\mathcal{U}'''}$ such that it satisfies (1) (a)(b) above and it is an open substructure of both $\widehat{\mathcal{U}'}$ and $\widehat{\mathcal{U}''}$. Similar uniqueness statement holds for the CF-perturbation in (2).
- (4) When a τ -collared CF-perturbation $\widehat{\mathfrak{S}}$ on $\widehat{\mathcal{U}}$ varies in a uniform family, we may take the induced CF-perturbation $\widehat{\mathfrak{S}}'$ in (2) to be uniform.

The proof is obvious so omitted.

Definition 17.67. Let $f:(X',\widehat{\mathcal{U}})\to M$ be a strongly smooth map which is strongly submersive with respect to $\widehat{\mathfrak{S}}$. Let h be a differential form on $(X',\widehat{\mathcal{U}})$. We take a Kuranishi structure $\widehat{\mathcal{U}}'$ as in Lemma 17.66 and define its *push out*

$$f_!(h;\widehat{\mathfrak{S}}^{\epsilon}) = f_!(h;\widehat{\mathfrak{S}}^{\prime\epsilon}).$$
 (17.51)

Here h in the right hand side is the differential form on $(X', \widehat{\mathcal{U}'})$ which is induced from h by Lemma 17.66 (1).

Lemma 17.68. The right hand side of (17.51) is independent of the choices of $\widehat{\mathcal{U}}'$, $\widehat{\mathfrak{S}}'$. Moreover Stokes' formula ([Part I, Theorem 9.26]) and the composition formula ([Part I, Theorem 10.20]) hold for the push out in Definition 17.67.

Proof. Independence follows from [Part I, Theorem 9.14]. Stokes' formula and the composition formula are direct consequences of the corresponding results ([Part I, Theorems 9.26 and 10.20]). \Box

18. Smoothing corners and composition of morphisms

The goal of this section is to define composition of morphisms of linear K-systems (Lemma-Definition 18.40) and to show that it is associative (Proposition 18.44). There are two key ingredients for the construction of composition of morphisms. One is 'partial trivialization of corners' and the other is 'smoothing corners'. The precise definition of 'partial trivialization of corners' will be given in Definition 18.10 and 'smoothing corners' will be described in detail in Subsections 18.4 and 18.5. After that, we will define composition of morphisms in Subsection 18.6 and prove its associativity in Subsection 18.7. We also discuss identity morphism in Subsection 18.9.

18.1. **Introduction to Section 18.** In this subsection, we explain the idea of the construction of composition of morphisms and its geometric background.

Firstly, we explain the reason why we need the notion of 'partial trivialization of corners', or more generally, 'partially trivialized fiber products', instead of trivialization of corners or usual fiber products. Let $\mathcal{N}_{12}(\alpha_1, \alpha_2)$ and $\mathcal{N}_{23}(\alpha_2, \alpha_3)$ be interpolation spaces of morphisms \mathfrak{N}_{12} and \mathfrak{N}_{23} of linear K-systems, respectively. As we mentioned in Lemma-Definition 16.35, the interpolation space of the morphism $\mathfrak{N}_{32} \circ \mathfrak{N}_{21}$ is a union of fiber products

$$\bigcup_{\alpha_2} \mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R_{\alpha_2}} \mathcal{N}_{23}(\alpha_2, \alpha_3). \tag{18.1}$$

The union in (18.1) may not be a disjoint union, in general. In fact, the summands corresponding to α_2 and to α_2' may intersect at

$$\mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R_{\alpha_2}} \mathcal{M}^2(\alpha_2, \alpha_2') \times_{R_{\alpha_2'}} \mathcal{N}_{23}(\alpha_2', \alpha_3). \tag{18.2}$$

Moreover three of such summands may intersect at

$$\mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R_{\alpha_2}} \mathcal{M}^2(\alpha_2, \alpha_2') \times_{R_{\alpha_2'}} \mathcal{M}^2(\alpha_2', \alpha_2'') \times_{R_{\alpha_2''}} \mathcal{N}_{23}(\alpha_2'', \alpha_3).$$
 (18.3)

The pattern how the summands of (18.1) intersect each other is similar to the way how the components of the boundary of certain K-space (or of orbifold) intersect each other. (Namely, each of the summands of (18.1) corresponds to a codimension one boundary and (18.2), (18.3) correspond to codimension 2 and 3 corners, respectively.)

We can use this observation to apply a version of Proposition 17.46, that is, we 'put the collar' *outside* the union (18.1) to obtain a collared K-space so that its boundary is

$$\bigcup_{\alpha_2} \mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R_{\alpha_2}}^{\boxplus \tau} \mathcal{N}_{23}(\alpha_2, \alpha_3).$$

Here the notation $\times_{R_{\alpha_2}}^{\boxplus \tau}$ is defined in Definition 18.37, which is called *partially trivialized fiber product*. The notion of *partial trivialization of corners* is introduced in Subsection 18.2. See Definition 18.10. The reason why we introduce the notion of partial trivialization of corner is as follows. Note that the boundary or corner of K-space of the summand of (18.1) has different components from those appearing in (18.2), (18.3). In fact, a boundary of the form

$$\mathcal{M}^1(\alpha_1,\alpha_1') \times_{R_{\alpha_1'}} \mathcal{N}_{12}(\alpha_1',\alpha_2) \times_{R_{\alpha_2}} \mathcal{N}_{23}(\alpha_2,\alpha_3)$$

also appears. To study this kind of boundary components together, we introduce the notion of partial trivialization of corner and use it to modify Proposition 17.46 so that it can be directly applicable to our situation.

After we have done partial trivialization of corners, we will next discuss $smoothing\ corners$. Here we use the fact that a K-space X has a collar where all the objects are 'constant' in the direction of the collar. See Subsection 18.3 for the reason why this fact is useful for our construction. Then we can finally define composition of morphisms.

In the rest of this subsection, we explain a geometric origin of the idea that the union (18.1) looks like a boundary of certain K-space, by considering the situation of the Floer cohomology of periodic Hamiltonian system. (The story of linear K-system will be applied to define and study the Floer cohomology of periodic Hamiltonian system. Namely, we associate such a system to each periodic Hamiltonian function H. See Subsection 15.1.) Let H^1 , H^2 , H^3 be periodic Hamiltonian functions. To define a cochain map between Floer's cochain complexes associated to them, we use $\tau \in \mathbb{R}$ dependent Hamiltonian functions interpolating them. Namely, we take $H^{ij}: \mathbb{R} \times S^1 \times M \to \mathbb{R}$ such that

$$H^{ij}(\tau, t, x) = \begin{cases} H^j(t, x) & \text{if } \tau \le -T_0, \\ H^i(t, x) & \text{if } \tau \ge T_0. \end{cases}$$
 (18.4)

We then use the moduli space of the solutions of the equation

$$\frac{\partial u}{\partial \tau} + J \left(\frac{\partial u}{\partial t} - X_{H_{\tau,t}^{ij}}(u) \right) = 0 \tag{18.5}$$

where $H^{ij}_{\tau,t}(x) = H^{ij}(\tau,t,x)$ and $X_{H^{ij}_{\tau,t}}$ is its Hamiltonian vector field. The solution space of (18.5) with an appropriate boundary condition becomes an interpolation space $\mathcal{N}^{ji}(\alpha_i,\alpha_j)$ of the morphism \mathfrak{N}^{ji} .

To study the relation between \mathfrak{N}^{31} and the composition $\mathfrak{N}^{32} \circ \mathfrak{N}^{21}$ we use one parameter family of τ -dependent Hamiltonian functions $H^{31,T}$ where

$$H^{31,T}(\tau,t,x) = \begin{cases} H^{1}(t,x) & \text{if } \tau \leq -T_{0} - T \\ H^{21}(\tau+T,t,x) & \text{if } -T_{0} - T \leq \tau \leq T_{0} - T \\ H^{2}(t,x) & \text{if } T_{0} - T \leq \tau \leq T - T_{0} \\ H^{32}(\tau-T,t,x) & \text{if } T - T_{0} \leq \tau \leq T + T_{0} \\ H^{3}(t,x) & \text{if } T + T_{0} \leq \tau. \end{cases}$$
(18.6)

See Figure 12.

We may choose $H^{31}=H^{31,2T_0}$, for example. We consider the limit as $T\to +\infty$ and the set of solutions of (18.5) with $X_{H^{ij}_{\tau,t}}$ replaced by $X_{H^{31,T}_{\tau,t}}$. The moduli space

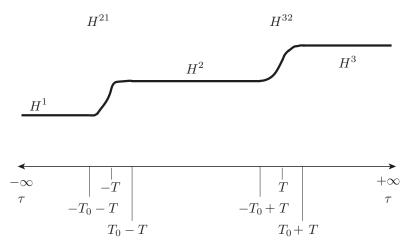


FIGURE 12. the concatenation H^{31}

of its solutions becomes the union of fiber products (18.1). Thus the union of solution spaces for $T \in [2T_0, \infty)$ and the fiber product (18.1) gives a homotopy between the morphism defined by $H^{31} = H^{31,2T_0}$ and the composition whose interpolation space is (18.1). The space (18.1) itself is a part of the boundary of this cobordism.

18.2. Partial trivialization of cornered K-space. In this subsection we explain that the story of trivialization of corner in Section 17 can be generalized to the case of partial trivialization in a quite straightforward way. Because of the nature of this book, we repeat the statements. We believe that the readers can go through those parts very quickly since there is nothing new to do.

Situation 18.1. Let U be an admissible orbifold with corners. We decompose its normalized boundary ∂U into a disjoint union

$$\partial U = \partial^0 U \cup \partial^1 U$$

where both $\partial^0 U$ and $\partial^1 U$ are open subsets in ∂U . We denote this decomposition by \mathfrak{C} . We also denote $\partial^0 U$ by $\partial_{\mathfrak{C}} U$.

Definition 18.2. In Situation 18.1 we define a closed subset $S_k^{\mathfrak{C}}(U)$ of U as follows. Let $p \in U$. We take its orbifold chart (V_p, Γ_p, ϕ_p) where $V_p \subset \overline{V}_p \times [0, 1)^{k'}$ and $p = \phi(y_0, (0, \dots, 0))$. For $i = 1, 2, \dots, k'$ we put

$$\partial_i V_p = V_p \cap (\overline{V}_p \times [0,1)^{i-1} \times \{0\} \times [0,1)^{k'-i}).$$

We require

$$\#\{i \in \{1,\dots,k'\} \mid \phi(\partial_i V_p) \subset \partial^0 U\} = k.$$
(18.7)

Then $S_k^{\circ \mathfrak{C}}(U)$ is the set of all $p \in U$ such that (18.7) is satisfied. We put

$$S_k^{\mathfrak{C}}(U) = \bigcup_{\ell \geq k} \overset{\circ}{S_{\ell}^{\mathfrak{C}}}(U).$$

Convention 18.3. In case $p \in S_k^{\mathfrak{C}}(U)$ as above, we take its orbifold chart (V_p, Γ_p, ϕ_p) as above such that $\phi(\partial_i V_p) \subset \partial^0 U$ if and only if $i = k' - k + 1, \dots, k'$.

Situation 18.4. Let $(X, \widehat{\mathcal{U}})$ be an *n*-dimensional K-space. We assume that for each $p \in \partial X$, we have a decomposition of its Kuranishi neighborhood into a disjoint union

$$\partial U_p = \partial^0 U_p \cup \partial^1 U_p \tag{18.8}$$

such that for each coordinate change Φ_{pq} with $p,q \in \partial X$ we have

$$\varphi_{pq}(U_{pq} \cap \partial^0 U_p) \subset \partial^0 U_p, \qquad \varphi_{pq}(U_{pq} \cap \partial^1 U_p) \subset \partial^1 U_p.$$
 (18.9)

We also denote this decomposition by $\mathfrak{C}.\blacksquare$

Definition 18.5. In Situation 18.4 we define $S_k^{\mathfrak{C}}(X,\widehat{\mathcal{U}})$ as follows. If $p \notin \partial X$ then $p \in S_0^{\mathfrak{C}}(X,\widehat{\mathcal{U}})$ but $p \notin S_k^{\mathfrak{C}}(X,\widehat{\mathcal{U}})$ for $k \geq 1$. If $p \in \partial X$ then $p \in S_k^{\mathfrak{C}}(X,\widehat{\mathcal{U}})$ if and only if $o_p \in S_k^{\mathfrak{C}}(U_p)$. We denote $\partial_{\mathfrak{C}}X = S_1^{\mathfrak{C}}(X,\widehat{\mathcal{U}})$. We also put

$$\overset{\circ}{S_k^{\mathfrak{C}}}(X,\widehat{\mathcal{U}}) = S_k^{\mathfrak{C}}(X,\widehat{\mathcal{U}}) \setminus \bigcup_{\ell > k} S_{\ell}^{\mathfrak{C}}(X,\widehat{\mathcal{U}}).$$

We can define a similar notion for good coordinate system by modifying the above definition in an obvious way. 28

We can generalize Proposition 24.16 without change as follows.

Proposition 18.6. In Situation 18.4, for each k there exist a compact (n-k)-dimensional K-space $\widehat{S}_k^{\mathfrak{C}}(X,\widehat{\mathcal{U}})$ with corners, a map $\pi_k:\widehat{S}_k^{\mathfrak{C}}(X,\widehat{\mathcal{U}})\to S_k^{\mathfrak{C}}(X,\widehat{\mathcal{U}})$, and a decomposition of $\partial \widehat{S}_k^{\mathfrak{C}}(X,\widehat{\mathcal{U}})$ as in Situation 18.4, (which we also denote by \mathfrak{C}), and a map $\pi_{\ell,k}:\widehat{S}_\ell^{\mathfrak{C}}(\widehat{S}_k^{\mathfrak{C}}(X,\widehat{\mathcal{U}}))\to\widehat{S}_{k+\ell}^{\mathfrak{C}}(X,\widehat{\mathcal{U}})$ for each ℓ,k such that they enjoy the following properties:

- (1) The map π_k is a continuous map of underlying topological space.
- (2) The interior of $\widehat{S}_k^{\mathfrak{C}}(X,\widehat{\mathcal{U}})$ is isomorphic to $S_k^{\mathfrak{C}}(X,\widehat{\mathcal{U}})$. The underlying homeomorphism of this isomorphism is the restriction of π_k .
- (3) The map $\pi_{\ell,k}$ is an $(\ell+k)!/\ell!k!$ fold covering map of K-spaces.
- (4) The following objects on $(X,\widehat{\mathcal{U}})$ induce those on $\widehat{S}_k^{\mathfrak{C}}(X,\widehat{\mathcal{U}})$. Moreover the induced objects are compatible with the covering maps $\pi_{\ell,k}$.
 - (a) CF-perturbation.
 - (b) Multivalued perturbation.
 - (c) Differential form.
 - (d) Strongly continuous map. Strongly smooth map.
 - (e) Covering map.
- (5) The following diagram commutes.

$$\widehat{S}_{k_{1}}^{\mathfrak{C}}(\widehat{S}_{k_{2}}^{\mathfrak{C}}(\widehat{S}_{k_{3}}^{\mathfrak{C}}(X,\widehat{\mathcal{U}}))) \xrightarrow{\pi_{k_{1},k_{2}}} \widehat{S}_{k_{1}+k_{2}}^{\mathfrak{C}}(\widehat{S}_{k_{3}}^{\mathfrak{C}}(X,\widehat{\mathcal{U}}))$$

$$\widehat{S}_{k_{1}}^{\mathfrak{C}}(\pi_{k_{2},k_{3}}) \downarrow \qquad \qquad \downarrow \pi_{k_{1}+k_{2},k_{3}}$$

$$\widehat{S}_{k_{1}}^{\mathfrak{C}}(\widehat{S}_{k_{2}+k_{3}}^{\mathfrak{C}}(X,\widehat{\mathcal{U}})) \xrightarrow{\pi_{k_{1},k_{2}+k_{3}}} \widehat{S}_{k_{1}+k_{2}+k_{3}}^{\mathfrak{C}}(X,\widehat{\mathcal{U}})$$

$$(18.10)$$

Here $\widehat{S}_{k_1}^{\mathfrak{C}}(\pi_{k_2,k_3})$ is the covering map induced from π_{k_2,k_3} .

²⁸We do not define it in detail since it is never used in this book.

(6) For i=1,2 let $f_i:(X_i,\widehat{\mathcal{U}}_i)\to M$ be a strongly smooth map. If f_1 is transversal to f_2 , then

$$\widehat{S}_k^{\mathfrak{C}}\left((X_1,\widehat{\mathcal{U}_1})\times_M(X_2,\widehat{\mathcal{U}_2})\right)\cong\coprod_{k_1+k_2=k}\widehat{S}_{k_1}^{\mathfrak{C}}(X_1,\widehat{\mathcal{U}_1})\times_M\widehat{S}_{k_2}^{\mathfrak{C}}(X_2,\widehat{\mathcal{U}_2}).$$

Here the right hand side is the disjoint union. The decomposition of the fiber product $(X_1,\widehat{\mathcal{U}_1})\times_M(X_2,\widehat{\mathcal{U}_2})$ as in Situation 18.4 is induced from those of $(X_1, \widehat{\mathcal{U}}_1)$ and $(X_2, \widehat{\mathcal{U}}_2)$ as follows.

$$\partial^0((X_1,\widehat{\mathcal{U}_1})\times_M(X_2,\widehat{\mathcal{U}_2}))$$

$$= \partial^{0}(X_{1}, \widehat{\mathcal{U}_{1}}) \times_{M} (X_{2}, \widehat{\mathcal{U}_{2}}) \cup (-1)^{\dim X_{1} + \dim M} (X_{1}, \widehat{\mathcal{U}_{1}}) \times_{M} \partial^{0}(X_{2}, \widehat{\mathcal{U}_{2}}).$$

- (7) (1)-(6) also hold when we replace 'Kuranishi structure' by 'good coordinate
- (8) Various kinds of embeddings of Kuranishi structures and/or good coordinate systems induce ones of $\widehat{S}_{k}^{\mathfrak{C}}(X,\widehat{\mathcal{U}})$.

The proof is the same as the proof of Proposition 24.16 and so is omitted.

Next we generalize the process of trivializing corner in Section 17 to partial trivialization.

Situation 18.7. Suppose that we are in Situation 17.3 and a decomposition $\partial U =$ $\partial^0 U \cup \partial^1 U$ is given as in Situation 18.1.

In Situation 18.7 We define a map

$$\mathcal{R}_x^{\mathfrak{C}}: \overline{V_x} \times [0,1)^{k'-k} \times (-\infty,1)^k \to \overline{V_x} \times [0,1)^{k'}$$

by

$$\mathcal{R}_x^{\mathfrak{C}}(\overline{y},(t_1,\ldots,t_k))=(\overline{y},(t_1',\ldots,t_k'))$$

where

$$t_i' = \begin{cases} t_i & \text{if } t_i \ge 0, \\ 0 & \text{if } t_i \le 0. \end{cases}$$

For $\tau \geq 0$, we define an open subset $V_x^{\mathfrak{C} \boxplus \tau}$ of $\overline{V_x} \times [0,1)^{k'-k} \times [-\tau,1)^k$ to be

$$V_x^{\mathfrak{C} \boxplus \tau} = (\mathcal{R}_x^{\mathfrak{C}})^{-1}(V_x) \cap (\overline{V_x} \times [0,1)^{k'-k} \times [-\tau,1)^k).$$

Then $\mathcal{R}_x^{\mathfrak{C}}$ induces a map $\mathcal{R}_x^{\mathfrak{C}}: V_x^{\mathfrak{C} \boxplus \tau} \to V_x \subset \overline{V_x} \times [0,1)^{k'}$. We can define a Γ_x action on $V_x^{\mathfrak{C} \boxplus \tau}$ in the same way as in Definition 17.5 and put $U_x^{\mathfrak{C} \boxplus \tau} = V_x^{\mathfrak{C} \boxplus \tau} / \Gamma_x$. We define $\mathcal{E}_x^{\mathfrak{C} \boxplus \tau}$ in the same way as in (17.1) by

$$\mathcal{E}_x^{\mathfrak{C} \boxplus \tau} = (\mathcal{R}_x^{\mathfrak{C}})^* (\mathcal{E}_x) = (E_x \times V_x^{\mathfrak{C} \boxplus \tau}) / \Gamma_x.$$

The section s_x of \mathcal{E}_x induces a section $s_x^{\mathfrak{C} \boxplus \tau}$ of $\mathcal{E}_x^{\mathfrak{C} \boxplus \tau}$ in an obvious way. We define

$$(X \cap V_x)^{\mathfrak{C} \boxplus \tau} = (s_x^{\mathfrak{C} \boxplus \tau})^{-1}(0)/\Gamma_x.$$

Let $\psi_x^{\mathfrak{C}\boxplus \tau}: (s_x^{\mathfrak{C}\boxplus \tau})^{-1}(0)/\Gamma_x \to (X\cap V_x)^{\mathfrak{C}\boxplus \tau}$ be the identity map. Then similarly to Lemma-Definition 17.10, we find that

$$\mathcal{U}^{\mathfrak{C} \boxplus \tau} = (V_x^{\mathfrak{C} \boxplus \tau}/\Gamma_x, \mathcal{E}_x^{\mathfrak{C} \boxplus \tau}, \psi_x^{\mathfrak{C} \boxplus \tau}, s_x^{\mathfrak{C} \boxplus \tau})$$

is a Kuranishi chart of $(X \cap V_x)^{\mathfrak{C} \boxplus \tau}$. Moreover the following objects on $\mathcal{U} =$ $(U, \mathcal{E}, s, \psi)$ induce the corresponding objects on $\mathcal{U}^{\mathfrak{C} \boxplus \tau}$. The proof is the same as that of Lemma-Definition 17.11 so is omitted.

- CF-perturbation.
- Strongly smooth map.
- Differential form.
- Multivalued perturbation.

We put $\overset{\circ\circ}{S}_k(V_x^{\mathfrak{C} \boxplus \tau}) = S_k(V_x^{\mathfrak{C} \boxplus \tau}) \cap (\mathcal{R}_x^{\mathfrak{C}})^{-1}(\overset{\circ}{S}_k(V_x^{\mathfrak{C} \boxplus \tau}))$. Then Lemma 17.12 is generalized in an obvious way. Furthermore we have

Lemma 18.8. Suppose we are in Situation 17.23 and $\partial U_i = \partial^0 U_i \cup \partial^1 U_i$ for i=1,2. We denote by $\mathfrak C$ the decomposition of the boundary. Then $\Phi_{21}=(\varphi_{21},\widehat{\varphi}_{21})$ induces an embedding $\Phi_{21}^{\mathfrak C \boxplus \tau}: \mathcal U_1^{\mathfrak C \boxplus \tau} \to \mathcal U_2^{\mathfrak C \boxplus \tau}$ of Kuranishi charts whose restriction to \mathcal{U}_1 coincides with Φ_{21} . Moreover we have

- In case of Situation 17.23 (1), S¹€[⊞]τ, S²€[⊞]τ are compatible with Φ[€]₂₁τ.
 In case of Situation 17.23 (2), (φ[€]₂₁)*(h[€]₂)*(h[€]₂)*(h[€]₁).
 In case of Situation 17.23 (3), f[€]₂ (1)*(p[⊕]₂₁)*(p[⊕]₂₁)*(p[⊕]₁).
 In case of Situation 17.23 (4), s¹€[⊞]τ, s²€^Шτ are compatible with Φ[©]₂₁τ.
 In case of Situation 17.23 (4), s¹€^Шτ, s²€^Шτ are compatible with Φ[©]₂₁τ.
 In case of Situation 17.23 (5), we have Φ[□]_{€31} = Φ[©]₃₂ · Φ[©]₂₁τ.

- (6) $\varphi_{21}^{\mathfrak{C} \boxplus \tau} \circ \mathcal{R}_{1}^{\mathfrak{C}} = \mathcal{R}_{2}^{\mathfrak{C}} \circ \varphi_{21}^{\mathfrak{C} \boxplus \tau}.$

The proof is the same as the proof of Lemma 17.24.

Definition 18.9. Suppose we are in Situation 18.4. Consider a disjoint union

$$\coprod_{p \in X} (s_p^{\mathfrak{C} \boxplus \tau})^{-1}(0) / \Gamma_p$$

and define an equivalence relation \sim on it as follows: Let $x_p \in (s_p^{\mathfrak{C} \boxplus \tau})^{-1}(0)$ and $x_q \in (s_q^{\mathfrak{C} \boxplus \tau})^{-1}(0)$. Then we define $[x_p] \sim [x_q]$ if there exist $r \in X$ and $x_r \in (s_p^{\mathfrak{C} \boxplus \tau})^{-1}(0) \cap U_{pr}^{\mathfrak{C} \boxplus \tau} \cap U_{qr}^{\mathfrak{C} \boxplus \tau}$ such that

$$[x_p] = \varphi_{pr}^{\mathfrak{C} \boxplus \tau}([x_r]), \quad [x_q] = \varphi_{qr}^{\mathfrak{C} \boxplus \tau}([x_r]).$$
 (18.11)

The same argument as in Lemma 17.27 show that \sim is an equivalence relation. Then we define a topological space $X^{\mathfrak{C} \boxplus \tau}$ by the set of the equivalence classes of this equivalence relation \sim with the quotient topology:

$$X^{\mathfrak{C} \boxplus \tau} := \left(\prod_{p \in X} (s_p^{\mathfrak{C} \boxplus \tau})^{-1}(0) / \Gamma_p \right) / \sim .$$

In the situation of Definition 18.9 we put

$$\overset{\circ}{S}_{k}(X^{\mathfrak{C} \boxplus \tau}) = S_{k}(X^{\mathfrak{C} \boxplus \tau}, \widehat{\mathcal{U}}^{\mathfrak{C} \boxplus \tau}) \cap (\mathcal{R}^{\mathfrak{C}})^{-1}(\overset{\circ}{S}_{k}(X, \widehat{\mathcal{U}})). \tag{18.12}$$

We define $B_{\tau}(\overset{\circ}{S}_{k}(X^{\mathfrak{C} \boxplus \tau})) \subset X^{\mathfrak{C} \boxplus \tau}$ as the union of

$$\psi_p^{\mathfrak{C} \boxplus \tau} \left((s_p^{\mathfrak{C} \boxplus \tau})^{-1}(0) \cap \{ (\overline{y}, (t_1, \dots, t_k)) \mid t_i \le 0, \ i = k' - k + 1, \dots, k' \} \right). \tag{18.13}$$

We can show

$$X^{\mathfrak{C} \boxplus \tau} = \coprod_{k} B_{\tau}(\overset{\circ\circ}{S}_{k}(X^{\mathfrak{C} \boxplus \tau}))$$
 (18.14)

in the same way as in Lemma 17.33.

cion 18.10. (1) Let $p' \in \overset{\circ\circ}{S}_k(X^{\mathfrak{C} \boxplus \tau})$. A τ - \mathfrak{C} -collard Kuranishi neighborhood at p' is a Kuranishi chart $\mathcal{U}_{p'}$ of $X^{\mathfrak{C} \boxplus \tau}$ which is $(\mathcal{U}_p)^{\mathfrak{C} \boxplus \tau}$ for certain Definition 18.10. Kuranishi neighborhood \mathcal{U}_p of $p = \mathcal{R}^{\mathfrak{C}}(p')$.

- (2) Let $p' \in \overset{\circ}{S}_k(X^{\mathfrak{C} \boxplus \tau}), q' \in \overset{\circ}{S}_{\ell}(X^{\mathfrak{C} \boxplus \tau})$ and let $\mathcal{U}_{p'} = (\mathcal{U}_p)^{\mathfrak{C} \boxplus \tau}, \mathcal{U}_{q'} = (\mathcal{U}_q)^{\mathfrak{C} \boxplus \tau}$ be their τ - \mathfrak{C} -collared Kuranishi neighborhoods, respectively. Suppose $q' \in$ $\psi_{p'}(s_{p'}^{-1}(0))$. A τ - \mathfrak{C} -collared coordinate change $\Phi_{p'q'}$ from $\mathcal{U}_{q'}$ to $\mathcal{U}_{p'}$ is by definition $\Phi_{pq}^{\mathfrak{C} \boxplus \tau}$ where Φ_{pq} is a coordinate change from \mathcal{U}_q to \mathcal{U}_p .
- (3) A τ - \mathfrak{C} -collared Kuranishi structure $\widehat{\mathcal{U}}'$ on $X^{\mathfrak{C} \boxplus \tau}$ is the following objects.
 - (a) For each $p' \in \overset{\circ}{S}_k(X^{\mathfrak{C} \boxplus \tau})$, $\widehat{\mathcal{U}}'$ assigns a τ - \mathfrak{C} -collared Kuranishi neigh-
 - (b) For each $p' \in \overset{\circ\circ}{S}_k(X^{\mathfrak{C} \boxplus \tau})$ and $q' \in \overset{\circ\circ}{S}_\ell(X^{\mathfrak{C} \boxplus \tau})$ with $q' \in \psi_{p'}(s_{n'}^{-1}(0)), \widehat{\mathcal{U}'}$ assigns a τ - \mathfrak{C} -collared coordinate change $\Phi_{p'q'}$.
 - (c) If $p' \in \overset{\circ}{S}_k(X^{\mathfrak{C} \boxplus \tau}), q' \in \overset{\circ}{S}_{\ell}(X^{\mathfrak{C} \boxplus \tau}), r' \in \overset{\circ}{S}_m(X^{\mathfrak{C} \boxplus \tau}) \text{ with } q' \in \psi_{p'}(s_{p'}^{-1}(0))$ and $r' \in \psi_{q'}(s_{q'}^{-1}(0))$, then we require

$$\Phi_{p'q'} \circ \Phi_{q'r'}|_{U_{p'q'r'}} = \Phi_{p'r'}|_{U_{p'q'r'}}$$

where $U_{p'q'r'} = U_{p'r'} \cap \varphi_{q'r'}^{-1}(U_{p'q'}).$

(4) A τ - \mathfrak{C} -collared K-space is a pair of paracompact Hausdorff space $X^{\mathfrak{C} \boxplus \tau}$ and its τ - \mathfrak{C} -collared Kuranishi structure $\widehat{\mathcal{U}}^{\mathfrak{C} \boxplus \widehat{\tau}}$. We call

$$(X^{\mathfrak{C} \boxplus \tau}, \widehat{\mathcal{U}^{\mathfrak{C} \boxplus \tau}})$$

the τ - \mathfrak{C} -corner trivialization, or partial trivialization of corners, of $(X,\widehat{\mathcal{U}})$. We sometimes write

$$(X,\widehat{\mathcal{U}})^{\mathfrak{C} \boxplus \tau}$$

in place of $(X^{\mathfrak{C} \boxplus \tau}, \widehat{\mathcal{U}^{\mathfrak{C} \boxplus \tau}})$.

(5) We can define the notion of τ - \mathfrak{C} -collared CF-perturbation, τ - \mathfrak{C} -collared multivalued perturbation, τ - \mathfrak{C} -collared good coordinate system, τ - \mathfrak{C} -collared Kuranishi chart, τ - \mathfrak{C} -collared vector bundle, τ - \mathfrak{C} -collared smooth section, τ - \mathfrak{C} -collared embedding of various kinds, etc. in the same way.

The decomposition \mathfrak{C} induces a decomposition of the boundary $\partial(X^{\mathfrak{C} \boxplus \tau}, \widehat{\mathcal{U}}^{\mathfrak{C} \boxplus \tau})$ in an obvious way. We also dente it by \mathfrak{C} .

Lemma 18.11. Lemma 17.37 can be generalized to a C-version in an obvious way.

Lemma 18.12. If $(X', \widehat{\mathcal{U}'})$ is τ - \mathfrak{C} -collared, then for any $0 < \tau' < \tau$, X' has a τ' - \mathfrak{C} -collared Kuranishi structure determined in a canonical way from the τ - \mathfrak{C} -collared Kuranishi structure (X', \mathcal{U}') . The same holds for CF-perturbation, multivalued perturbation and good coordinate system.

The proof is the same as the proof of Lemma 17.38.

Situation 18.13. Let $(X, \widehat{\mathcal{U}})$ be a K-space. Suppose $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}$ are decompositions of $\partial(X,\widehat{\mathcal{U}})$ as in Situation 18.4. We assume the following two conditions.

- $\begin{aligned} (1) \ \partial_{\mathfrak{C}_1} U_p \cap \partial_{\mathfrak{C}_2} U_p &= \emptyset. \\ (2) \ \partial_{\mathfrak{C}_1} U_p \cup \partial_{\mathfrak{C}_2} U_p &= \partial_{\mathfrak{C}} U_p. \blacksquare \end{aligned}$

Lemma 18.14. In Situation 18.13 we have the following canonical isomorphisms.

$$((X,\widehat{\mathcal{U}})^{\mathfrak{C}_1 \boxplus \tau})^{\mathfrak{C}_2 \boxplus \tau} \cong ((X,\widehat{\mathcal{U}})^{\mathfrak{C}_2 \boxplus \tau})^{\mathfrak{C}_1 \boxplus \tau} \cong (X,\widehat{\mathcal{U}})^{\mathfrak{C} \boxplus \tau}.$$

Remark 18.15. The decomposition \mathfrak{C}_2 of $\partial(X,\widehat{\mathcal{U}})$ induces one on $\partial((X,\widehat{\mathcal{U}})^{\mathfrak{C}_1 \boxplus \tau})$, which we denote by the same symbol. We used this fact in the statement of Lemma 18.14.

Proof of Lemma 18.14. Suppose a Kuranishi chart of $(X, \widehat{\mathcal{U}})$ is given as a quotient open subset V_p of $\overline{V}_p \times [0,1)^{k_1} \times [0,1)^{k_2} \times [0,1)^{k_3}$ where

$$\begin{split} &\partial_{\mathfrak{C}_1} V_p = V_p \cap \left(\overline{V}_p \times [0,1)^{k_1} \times \partial [0,1)^{k_2} \times [0,1)^{k_3} \right), \\ &\partial_{\mathfrak{C}_2} V_p = V_p \cap \left(\overline{V}_p \times [0,1)^{k_1} \times [0,1)^{k_2} \times \partial [0,1)^{k_3} \right). \end{split}$$

Then

$$\partial_{\mathfrak{C}} V_p = V_p \cap (\overline{V}_p \times [0,1)^{k_1} \times \partial([0,1)^{k_2} \times [0,1)^{k_3}))$$
.

Therefore we have

$$V_p^{\mathfrak{C}_1 \boxplus \tau} = \mathcal{R}^{-1}(V_p) \cap \left(\overline{V}_p \times [0,1)^{k_1} \times [-\tau,1)^{k_2} \times [0,1)^{k_3}\right)$$

and

$$V_p^{\mathfrak{C} \boxplus \tau} = \mathcal{R}^{-1}(V_p) \cap \left(\overline{V}_p \times [0,1)^{k_1} \times [-\tau,1)^{k_2} \times [-\tau,1)^{k_3}\right)\right).$$

Lemma 18.14 follows easily.

18.3. In which sense smoothing corner is canonical? We next discuss smoothing corner. Smoothing corner of manifolds is a standard process and its generalization to orbifolds is also straightforward. An issue to generalize the smoothing corner to Kuranishi structures lies in the way of how we fix a smooth structure around the corners and how much we can make the smooth structure canonical. This technicalities can be go around fairly nicely especially when we use *trivialization of corner* introduced in Section 17, that is exactly the case we use in our story. We discuss those issues in Subsections 18.3-18.5.

We begin with the following remark. Let M be a manifold (or an orbifold) with corners. We have a structure of manifold with boundary (but without corner) on the *same* underlying topological space. We write this manifold with boundary (but without corner) as M'. We denote by $i': M \to M'$ the identity map. Then it has the following properties.

(*) i' induces a smooth embedding $\widehat{S}_k(M) \to M'$.

Moreover if M is admissible, we have the following:

- (1) If $f: M \to \mathbb{R}$ is an admissible function, then $f \circ (i')^{-1}$ is smooth.
- (2) If $\mathcal{E} \to M$ is an admissible vector bundle, the underlying continuous map $\mathcal{E} \to M$ has a structure of C^{∞} -vector bundle on M'. We write it $\mathcal{E}' \to M'$.
- (3) If s is an admissible section of \mathcal{E} , the same (set-theoretical) map $M' \to \mathcal{E}'$ is a smooth section.

The proofs of (1)-(3) above are easy. Since we do not use them, we do not prove them. We next see how much the smooth structure of M' is canonical. The following statement is also standard.

Lemma 18.16. We can construct M' from M in such a way that the differential manifold M' is well-defined modulo diffeomorphism. More precisely, we have the following: Suppose we obtain another M'' from M. Let $i'': M \to M''$ be the identity map. Then we have a diffeomorphism $f: M' \to M''$ such that $f(i'(S_k(M))) = i''(S_k(M))$, for $k = 0, 1, 2, \ldots$

In other words, M' is well-defined modulo stratification preserving diffeomorphism. Here the stratification means the corner structure stratification of M. This lemma is fairly standard and its proof is omitted. It seems that it is more nontrivial to find a 'canonical' way so that the above diffeomorphism f can be taken to be the identity map. In this article, we do not try to find such a way in general situation of (admissible) orbifold, but will do so in the case of *collared orbifold*.

Before doing so, we explain a reason why the uniqueness in the sense of Lemma 18.16 is not enough for our purpose. When we generalize the process of smoothing corner to that for Kuranishi structures, we need to study the situation where we have an embedding $N \to M$ of cornered orbifolds. When we smooth the corners of N and M, we want the same map $N' \to M'$ to be a smooth embedding. This is not obvious because of the non-uniqueness of the smooth structure we put on M' and N'. However, it is still true and not difficult to prove that we can find a smooth structures of M' and N' so that $N' \to M'$ is a smooth embedding.

On the other hand, in order to smooth the corner of Kuranishi structure we need to smooth the corner of all the Kuranishi charts, simultaneously. This now becomes a nontrivial problem. If we try to use the uniqueness in Lemma 18.16, we should include the diffeomorphism f as a part of data in the construction. Then the compatibility of the coordinate change might be broken.²⁹

We go around this issue by using the collar. When we use the collar, the way how we smooth the corner still involves choices. However, we can make the choice to smooth the model $[0,1)^k$ only once and then use that particular choice to smooth all the collared orbifolds, simultaneously. This way is canonical enough so that all the embeddings of Kuranishi charts become smooth embeddings automatically after smoothing the corners.

18.4. Smoothing corner of $[0,\infty)^k$. In this subsection, we fix data which we need to smooth the corner of a partially collared orbifold and a partially collared Kuranishi structure. Namely we fix a way to smooth the local model $[0,\infty)^k$ so that it is compatible with various k and also with the symmetry exchanging the factors. The latter is important to study the case of orbifolds. Let $\operatorname{Perm}(k+1)$ be a group of permutations of $\{1,\ldots,k+1\}$.

Definition 18.17. We define a Perm(k+1) action on \mathbb{R}^k as follows. We regard

$$\mathbb{R}^k = \left\{ (t_0, \dots, t_k) \in \mathbb{R}^{k+1} \mid \sum_{i=0}^k t_i = 0 \right\}.$$

The group $\operatorname{Perm}(k+1)$ acts on \mathbb{R}^{k+1} by exchanging the factors. It restricts to an action of $\operatorname{Perm}(k+1)$ to \mathbb{R}^k . In this section, the $\operatorname{Perm}(k+1)$ action on \mathbb{R}^k always means this particular action.

Below we will show the existence of a set of homeomorphisms

$$\Phi_k: [0, \infty)^k \to \mathbb{R}^{k-1} \times [0, \infty) \tag{18.15}$$

and smooth structures \mathfrak{sm}_k on $[0,\infty)^k$, simultaneously, for any $k\in\mathbb{Z}_+$, which satisfy Condition 18.18.

²⁹It seems that we can still prove that for a given good coordinate system we can construct a smoothing corner compatible with the coordinate change for any proper open substructure of it. We need to work out rather cumbersome induction to prove it at the level of detailed-ness we intend to achieve in this article.

Condition 18.18. We require Φ_k and \mathfrak{sm}_k to satisfy the following conditions.

- (1) Φ_k is a diffeomorphism from $([0,\infty)^k,\mathfrak{sm}_k)$ to $\mathbb{R}^{k-1}\times[0,\infty)$. Here we use the standard smooth structure on $\mathbb{R}^{k-1}\times[0,\infty)$.
- (2) Let $\Phi_k(t_1, ..., t_k) = (x, t)$. Then

$$\Phi_k(ct_1,\ldots,ct_k) = (x,ct)$$

for $c \in \mathbb{R}_+$.

(3) For $\mathbf{t} \in [0,\infty)^k \setminus 0$ there exists an open neighborhood of \mathbf{t} that is isometric to $V \times [0,\epsilon)^\ell$ where $V \subset \mathbb{R}^{k-\ell}$. (Here V is an open set. When we say 'isometric', we use the Euclidian metrics on $[0,\infty)^k$, $\mathbb{R}^{k-\ell}$, $[0,\epsilon)^\ell$.) Then the map

$$id \times \Phi_{\ell} : V \times [0, \epsilon)^{\ell} \to V \times \mathbb{R}^{\ell-1} \times [0, \infty)$$

is a diffeomorphism onto its image. Here we put the restriction of the smooth structure \mathfrak{sm}_k to $V \times [0, \epsilon)^{\ell}$. (The space $V \times [0, \epsilon)^{\ell}$ is identified with an open subset of $[0, \infty)^k \setminus 0$ by the isometry.)

an open subset of $[0,\infty)^k \setminus 0$ by the isometry.)

(4) The map $\Phi_k : [0,\infty)^k \to \mathbb{R}^{k-1} \times [0,\infty)$ is $\operatorname{Perm}(k)$ equivariant. Here $\operatorname{Perm}(k)$ acts on $[0,\infty)^k$ by permutation of factors, and acts on \mathbb{R}^{k-1} by Definition 18.17. On the last factor $[0,\infty)$ the action is trivial.

Remark 18.19. In (3) above we require a neighborhood of \mathbf{t} to be isometric to $V \times [0, \epsilon)^{\ell}$. The reason why we require such a rather restrictive assumption that they are isometric is that we want to specify the diffeomorphism from a neighborhood of \mathbf{t} to $V \times [0, \epsilon)^{\ell}$. We use the Euclidian metric on \mathbb{R}^n here only to make the choice of diffeomorphism (that is isometry) as canonical as possible.

Lemma 18.20. For any $k \in \mathbb{Z}_+$ there exist Φ_k and \mathfrak{sm}_k satisfying Condition 18.18.

Proof. The proof is by induction on k. For k = 1, Φ_1 is the identity map and \mathfrak{sm}_1 is the standard smooth structure on $[0, \infty)$.

Suppose we have Φ_i , \mathfrak{sm}_i for i < k. We observe that Condition (3) determines a smooth structure \mathfrak{sm}_k on $[0,\infty)^k \setminus 0$ uniquely. Indeed, well-defined-ness of this structure can be checked by Condition (3) itself inductively. Moreover, by the definition of the smooth structure \mathfrak{sm}_k , we find that the Perm(k)-action is smooth with respect to this smooth structure. The map $(t_1, \ldots, t_k) \mapsto (ct_1, \ldots, ct_k)$ is also a diffeomorphism for this smooth structure if $c \in \mathbb{R}_+$.

Next we will construct a homeomorphism Φ_k and extend the smooth structure \mathfrak{sm}_k to $[0,\infty)^k$. We choose a compact subset $S \subset [0,\infty)^k \setminus 0$ which is a smooth (k-1)-dimensional submanifold with corners (with respect to the standard structure of manifold with corners of $[0,\infty)^{k+1}$) such that:

- (a) S is a slice of the multiplicative \mathbb{R}_+ action on $[0,\infty)^k \setminus 0$.
- (b) S is perpendicular to all the strata $\overset{\circ}{S}_{\ell}([0,\infty)^k)$. We can take a tubular neighborhood of $\overset{\circ}{S}_{\ell}([0,\infty)^k) \cap S$ in S such that the fiber of the projection to $\overset{\circ}{S}_{\ell}([0,\infty)^k) \cap S$ is flat with respect to the Euclidean metric of $[0,\infty)^k$.
- (c) S is invariant under the Perm(k)-action on $[0,\infty)^k$.

We can find such S by $\operatorname{Perm}(k)$ -equivariantly modifying the intersection of unit ball and $[0,\infty)^k$ around the boundary a bit.

By Condition (b) S is a smooth submanifold with boundary of $([0,\infty)^k \setminus 0, \mathfrak{sm}_k)$. Since we can construct S by modifying the intersection of unit ball and $[0,\infty)^k$

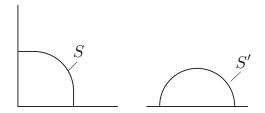


FIGURE 13. submanifolds S and S'

around the boundary a bit, there exists a $\operatorname{Perm}(k)$ equivariant diffeomorphism from S to

$$S' = \{(x, t) \in \mathbb{R}^{k-1} \times \mathbb{R}_{>0} \mid ||x||^2 + t^2 = 1\}.$$

Here we use the smooth structure of S induced from $([0,\infty)^k \setminus 0,\mathfrak{sm}_k)$ and the standard smooth structure on S'. See Figure 13. (Note that S becomes a manifold with boundary and without corner with respect to this smooth structure.) We fix this diffeomorphism. Then we can define Φ_k by extending the diffeomorphism $S \to S'$ so that Condition (2) is satisfied. By construction, Φ_k is a diffeomorphism outside the origin.

Then we extend the smooth structure \mathfrak{sm}_k to the origin so that Φ_k also becomes a diffeomorphism at the origin. The proof is now complete by induction.

Remark 18.21. During the proof we made choices of S and a diffeomorphism between S and S' for each k. The resulting smooth structure \mathfrak{sm}_k depends on these choices in the sense that the identity map is not a diffeomorphism when we use two smooth structures obtained by different choices for the source and the target. However since two different choices of S and the diffeomorphism $S \to S'$ are isotopic to each other, the resulting smooth structure \mathfrak{sm}_k is independent of the choices in the sense of diffeomorphism. (This is a proof of Lemma 18.16 in this case.)

When we apply the construction of smoothing corner of Kuranishi structure, we sometimes need to put collars to the smoothed K-space. We use Lemma 18.23 below for this purpose. Let Φ_k and \mathfrak{sm}_k be as in Condition 18.18.

Condition 18.22. For any $k \in \mathbb{Z}_+$ we consider \mathfrak{Trans}_{k-1} and Ψ_k with the following properties.

- (1) \mathfrak{Trans}_{k-1} is a smooth (k-1)-dimensional submanifold of $[0,\infty)^k$ and is contained in $(0,\infty)^k \setminus (1,\infty)^k$.
- (2) \mathfrak{Trans}_{k-1} is invariant under the Perm(k) action on $[0,\infty)^k$.
- (3) $\mathfrak{Trans}_{k-1} \cap ([0,\infty)^{k-1} \times [1,\infty)) = \mathfrak{Trans}_{k-2} \times [1,\infty)$. This is an equality as subsets of $[0,\infty)^k = [0,\infty)^{k-1} \times [0,\infty)$.

(4)

$$\Psi_k:[0,1]\times\mathfrak{Trans}_{k-1}\to[0,\infty)^k$$

is a homeomorphism onto its image. Let \mathfrak{U}_k be its image.

(5) Using the smooth structure \mathfrak{sm}_k on $[0,\infty)^k$, the subset $\mathfrak{U}_k \subset [0,\infty)^k$ is a smooth k-dimensional submanifold with boundary and Ψ_k is a diffeomorphism. Moreover

$$\partial \mathfrak{U}_k = \partial ([0,\infty)^k) \cup \mathfrak{Trans}_{k-1}$$

- and the restriction of Ψ_k to $\{0\} \times \mathfrak{Trans}_{k-1}$ is a diffeomorphism onto $\partial([0,\infty)^k)$. The restriction of Ψ_k to $\{1\} \times \mathfrak{Trans}_{k-1}$ is the identity map.
- (6) Ψ_k is equivariant under the $\operatorname{Perm}(k)$ action. (The $\operatorname{Perm}(k)$ action on \mathfrak{Trans}_{k-1} is defined in Item (2) and the action on $[0,\infty)^k$ is by permutation of factors.
- (7) If $s \ge 1$, $t \in [0, 1]$ and $(x_1, \dots, x_{k-1}) \in \mathfrak{Trans}_{k-2}$, then

$$\Psi_k(t,(x_1,\ldots,x_{k-1},s)) = (\Psi_{k-1}(t,(x_1,\ldots,x_{k-1})),s).$$

Here we use the identification in Item (3) to define the left hand side.

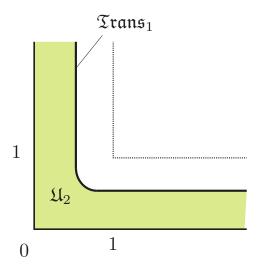


FIGURE 14. \mathfrak{Trans}_{k-1} and \mathfrak{U}_k

Lemma 18.23. For any $k \in \mathbb{Z}_+$ there exist \mathfrak{Trans}_{k-1} and Ψ_k satisfying Condition 18.22. Moreover, for each given $\delta > 0$, we may take them so that \mathfrak{U}_k contains $[0,\infty)^k \setminus [1-\delta,\infty)^k$.

Proof. The proof is by induction. If k = 1, we put $\mathfrak{Trans}_0 = \{1 - \delta/2\}$ and there is nothing to prove. Suppose we have $\mathfrak{Trans}_{k'-1}$, $\Psi_{k'}$ for k' < k. Conditions 18.22 (2) (3) determine \mathfrak{Trans}_{k-1} outside $[0,1]^k$. Conditions 18.22 (6) (7) determine Ψ_k outside $[0,1]^k$. It is easy to see that we can extend them to $[0,1]^k$ and obtain \mathfrak{Trans}_{k-1} , Ψ_k .

We note that Ψ_k defines a collar of $([0,\infty)^k,\mathfrak{sm}_k)$, which is a manifold with boundary (but without corner).

18.5. Smoothing corner of collared orbifolds and of Kuranishi structures. In this subsection we combine the story of partial trivialization of Kuranishi structure in Subsection 18.2 with the story of smoothing corner in Subsection 18.3-18.4.

Situation 18.24. Let U be an orbifold. We consider its normalized boundary ∂U . Let $\mathfrak C$ be a decomposition of ∂U as in Situation 18.1. We denote by $U^{\mathfrak C \boxplus_{\tau}}$ the partial trivialization of corners. \blacksquare

We will define smoothing corners of $U^{\mathfrak{C} \boxplus \tau}$ below.

Let $p \in \overset{\circ}{S}_k(U^{\mathfrak{C} \boxplus \tau})$ and put $\overline{p} = \mathcal{R}^{\mathfrak{C}}(p)$. The point p has an orbifold neighborhood

$$\mathfrak{V}_p = (\overline{V}_{\overline{p}} \times [-\tau, 0)^k, \Gamma_{\overline{p}}, \phi_p). \tag{18.16}$$

Here $(\overline{V}_{\overline{p}} \times [0,1)^k, \Gamma_{\overline{p}}, \phi_{\overline{p}})$ is an orbifold neighborhood of \overline{p} in U with $\overline{p} \in S_k^{\mathfrak{C}}(U)$, and $\overline{V}_{\overline{p}}$ may have boundary or corners but the boundary of $\overline{V}_{\overline{p}}$ does not correspond to a boundary component in \mathfrak{C} .

Definition 18.25. We define a smooth structure of \mathfrak{V}_p in (18.16) as follows. We identify $[-\tau,0)^k \cong [0,\tau)^k$ by the diffeomorphism $(t_1,\ldots,t_k) \mapsto (t_1+\tau,\ldots,t_k+\tau)$. We use the smooth structure \mathfrak{sm}_k on $[0,\tau)^k$ to obtain a smooth structure on $[-\tau,0)^k$. Then by taking direct product we obtain a smooth structure on $\overline{V}_{\overline{\nu}} \times [-\tau,0)^k$.

Lemma 18.26. Consider the smooth structure on $\overline{V}_{\overline{p}} \times [-\tau, 0)^k$ as in Definition 18.25. Then we have:

- (1) The $\Gamma_{\overline{p}}$ action on $\overline{V}_{\overline{p}} \times [-\tau, 0)^k$ is smooth with respect to this smooth structure.
- (2) If $q \in \psi_p(\overline{V}_{\overline{p}} \times [-\tau, 0)^k)$ and $q \in \overset{\circ}{S}_{\ell}(U^{\mathfrak{C} \boxplus \tau})$, then the coordinate change from \mathfrak{V}_q to \mathfrak{V}_p is smooth with respect to the above smooth structure.

Proof. Item (1) is a consequence of Condition 18.18 (4) and Item (2) follows from Condition 18.18 (3). \Box

Definition 18.27. We obtain an atlas of an orbifold structure on $U^{\mathfrak{C} \boxplus \tau}$ by Lemma 18.26 and Definition 18.25. We call $U^{\mathfrak{C} \boxplus \tau}$ with this smooth structure the *orbifold* with corner obtained by τ - \mathfrak{C} -partial smoothing of corners, and when no confusion can occur, we simply call the smoothing τ -partial smoothing of corners. We denote it by

$$U^{\operatorname{sm}\mathfrak{C} \boxplus \tau}$$

In case $\mathfrak C$ is the whole set of all the components of the boundary, this smooth structure has no corner.

Lemma 18.28. If
$$p \in \overset{\circ}{S}_{k+\ell}(U^{\mathfrak{C} \boxplus \tau}) \cap \overset{\circ}{S_k^{\mathfrak{C}}}(U^{\mathfrak{C} \boxplus \tau})$$
, then $p \in \overset{\circ}{S}_{\ell+1}(U^{\operatorname{sm} \mathfrak{C} \boxplus \tau})$.

The proof is obvious. We also have the following:

Lemma 18.29. If $0 < \tau' < \tau$, the orbifold obtained by τ -partial smoothing of corners is τ' -collared.

Proof. Let (V, Γ, ϕ) be an orbifold chart of U. We may chose V so that it is an open subset of $\overline{V} \times [0,1)^k \times [0,1)^{k'-k}$ and Convention 18.3 is satisfied. Then the corresponding orbifold chart of $U^{\mathfrak{C} \boxplus \tau}$ is

$$V = \mathcal{R}^{-1}(V) \cap \left(\overline{V} \times [-\tau, 1)^k \times [0, 1)^{k'-k}\right).$$

We use the smooth structure \mathfrak{sm}_k on the $[-\tau,1)^k$ factor (which we identity with $[0,1+\tau)^k$) and obtain the orbifold chart of $U^{\mathrm{sm}\mathfrak{C}\boxplus\tau}$.

Now we use \mathfrak{Trans}_{k-1} and Ψ_k produced in Lemma 18.23. The map Ψ_k is a diffeomorphism

$$\Psi_k: \mathfrak{Trans}_{k-1} \times [0,1) \to [-\tau,0)^k$$

to the image. The image is a neighborhood of $\partial [-\tau.0)^k$. Then we have a smooth embedding

$$\overline{V} \times \mathfrak{Trans}_{k-1} \times [0,1) \times [0,1)^{k'-k} \supseteq V' \to V$$

where V' is an open subset. Its image is a neighborhood of ∂V . This embedding is a diffeomorphism to its image if we use the differential structure after smoothing corners. This gives a collar of U on this chart. Using Conditions 18.22 (6)(7), we can show that this collar is compatible with the coordinate change.

We can take $\tau' > 0$ for any $\tau > 0$ by choosing δ small in Lemma 18.23.

In the next lemma we summarize the properties of (partial) smoothing corner.

Lemma 18.30. Suppose we are in Situation 18.24. Let $0 < \tau' < \tau$.

- (1) If E is an admissible vector bundle on U, $E^{\operatorname{sm}\mathfrak{C} \boxplus \tau}$ becomes an admissible vector bundle on $U^{\operatorname{sm}\mathfrak{C} \boxplus \tau}$. It is τ' -collared.
- (2) If $f: U \to M$ is an admissible map, $f^{\mathfrak{C} \boxplus \tau}: U^{\mathrm{sm}\mathfrak{C} \boxplus \tau} \to M$ is an admissible map from $U^{\mathrm{sm}\mathfrak{C} \boxplus \tau}$. When M has corner, we take the decomposition \mathfrak{C} of the normalized boundary ∂U so that $\partial_{\mathfrak{C}} U$ is contained in the horizontal boundary as in Definition 26.10. Then the same assertion also holds.
- (3) If $f: U_1 \to U_2$ is an admissible embedding and $f(\partial_{\mathfrak{C}_1} U_1) \subset \partial_{\mathfrak{C}_2} U_2 \cap f(U_1)$, then $f^{\mathfrak{C}_1 \boxplus \tau}: U_1^{\operatorname{sm} \mathfrak{C}_1 \boxplus \tau} \to U_2^{\operatorname{sm} \mathfrak{C}_2 \boxplus \tau}$ is an admissible embedding. It is τ' -collared.
- (4) In the situation of (3) suppose E₁ → U₁, E₂ → U₂ are admissible vector bundles and f is covered by an admissible embedding of vector bundles f̂: E₁ → E₂. Then f̂^{e₁⊞τ}: E₁^{sme₁⊞τ} → E₂^{sme₂⊞τ} is an admissible embedding of vector bundles which covers f^{e₁⊞τ}. It is τ'-collared.
- (5) In the situation of (1) if s is an admissible section of E, $s^{\mathfrak{C} \boxplus \tau}$ is an admissible section of $E^{\operatorname{sm} \mathfrak{C} \boxplus \tau}$. It is τ' -collared.

The proof is obvious.

Now we consider the case of Kuranishi structure. This generalization is quite straightforward.

Situation 18.31. Let $(X,\widehat{\mathcal{U}})$ be a K-space. Suppose for each $p \in X$ we have a decomposition of ∂U_p into two unions of connected components. Let \mathfrak{C}_p be the first union. We assume that for each $q \in \psi_p(s_p^{-1}(0))$ we have

$$\varphi_{pq}(\partial_{\mathfrak{C}_q}U_q\cap U_{pq})=\varphi_{pq}(U_{pq})\cap\partial_{\mathfrak{C}_p}U_p.$$

Definition 18.32. In Situation 18.31, let $(X^{\mathfrak{C} \boxplus \tau}, \widehat{\mathcal{U}^{\mathfrak{C} \boxplus \tau}})$ be a τ - \mathfrak{C} -collared K-space as in Definition 18.10. We define its partial smoothing of corners as follows:

If $p \in \overset{\circ}{S}_k(X^{\mathfrak{C} \boxplus \tau})$ and its partially τ -collared Kuranishi neighborhood is $\mathcal{U}_{\overline{p}}^{\mathfrak{C} \boxplus \tau} = (U_{\mathfrak{C}\overline{p}}^{\mathfrak{C} \boxplus \tau}, E_{\mathfrak{C}\overline{p}}^{\mathfrak{C} \boxplus \tau}, \psi_{\overline{p}}^{\mathfrak{C} \boxplus \tau})$, then we change its smooth structure by smoothing corner. We obtain a Kuranishi chart denoted by $\mathcal{U}_{\overline{p}}^{\mathrm{sm}\mathfrak{C} \boxplus \tau}$. Then the coordinate change is induced by one of $\widehat{\mathcal{U}^{\mathfrak{C} \boxplus \tau}}$ because of Lemma 18.30. It is τ' -collared for any $0 < \tau' < \tau$. We call the Kuranishi structure obtained above the Kuranishi structure obtained by partial smoothing of corners and denote it by

$$\widehat{\mathcal{U}^{\mathrm{sm}\mathfrak{C}\boxplus au}}_{ au}$$

Lemma 18.33. In the situation of Definition 18.32 the following $(\tau\text{-collared})$ object on $(X,\widehat{\mathcal{U}})$ induces the corresponding $(\tau'\text{-collared})$ object of the Kuranishi structure obtained by partial smoothing of corners.

- (1) Strongly smooth map.
- (2) CF-perturbation.
- (3) Multivalued perturbation.
- (4) Differential form.

The proof is immediate from construction.

Lemma 18.34. In the situation of Lemma 18.33, we put

$$\partial_{\mathfrak{C}}(X,\widehat{\mathcal{U}}) = \coprod_{c \in \mathfrak{C}} \partial_c(X,\widehat{\mathcal{U}}).$$

Let $\widehat{\mathcal{U}^{\mathfrak{C}}}$ be the Kuranishi structure of $\partial_{\mathfrak{C}}(X,\widehat{\mathcal{U}})$ with smoothed corners. We consider the K-space $(\partial_{\mathfrak{C}}(X),\widehat{\mathcal{U}^{\mathfrak{C}}})$. Let h be a differential form on X and $f: X \to M$ a strongly smooth map which is strongly surbmersive with respect to a CF-perturbation \mathfrak{S}^{ϵ} . The CF-perturbation \mathfrak{S}^{ϵ} induces one on $(\partial_{\mathfrak{C}}(X),\widehat{\mathcal{U}^{\mathfrak{C}}})$ and on $\partial_{c}(X,\widehat{\mathcal{U}})$. We also denote them by \mathfrak{S}^{ϵ} . Then we have

$$\sum_{c} f!(h|_{\partial_{c}(X)}; \mathfrak{S}^{\epsilon}) = f!(h|_{\partial_{\mathfrak{C}}(X)}; \mathfrak{S}^{\epsilon}).$$

The proof is obvious and so omitted.

18.6. Composition of morphisms of linear K-systems. We refer Conditions 16.1, 16.16, 16.21 and Definitions 16.6, 16.18, 16.30 for the various definitions and notations concerning linear K-system, morphisms and homotopy.

Situation 18.35. Suppose we are in Situation 16.34 and \mathfrak{N}_{i+1i} is a morphism from \mathcal{F}_i to \mathcal{F}_{i+1} . We denote by

$$\mathcal{M}^i(\alpha_-,\alpha_+)$$

the space of connecting orbits of \mathcal{F}_i and by

$$\mathcal{N}_{ii+1}(\alpha_-, \alpha_+)$$

the interpolation space of \mathfrak{N}_{i+1i} . Let $R^i_{\alpha_i}$ be a critical submanifold of \mathcal{F}_i and $\alpha_i \in \mathfrak{A}_i$.

Remark 18.36. In the above definition we denote the interpolation space of a morphism from \mathcal{F}_i to \mathcal{F}_{i+1} by $\mathcal{N}_{ii+1}(*,*)$, while we denote the corresponding morphism by \mathfrak{N}_{i+1i} to be compatible with algebraic formulas of compositions of morphisms.

Definition 18.37. In Situation 18.35 we define the partially trivialized fiber product

$$\mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R_{\alpha_2}^2}^{\boxplus \tau} \mathcal{N}_{23}(\alpha_2, \alpha_3) \tag{18.17}$$

as follows. We consider the fiber product $\mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R^2_{\alpha_2}} \mathcal{N}_{23}(\alpha_2, \alpha_3)$ equipped with fiber product Kuranishi structure. Then we consider the decomposition \mathfrak{C} of its boundary which consists of the following two kinds of components of its normalized boundary.

$$\mathcal{N}_{12}(\alpha_1, \alpha_2') \times_{R^2_{\alpha_2'}} \mathcal{M}^2(\alpha_{\alpha_2'}, \alpha_{\alpha_2}) \times_{R^2_{\alpha_2}} \mathcal{N}_{23}(\alpha_2, \alpha_3),$$

$$\mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R^2_{\alpha_2}} \mathcal{M}^2(\alpha_{\alpha_2}, \alpha_{\alpha'_2}) \times_{R^2_{\alpha'_2}} \mathcal{N}_{23}(\alpha'_2, \alpha_3).$$

The first line is contained in $\partial \mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R^2_{\alpha_2}} \mathcal{N}_{23}(\alpha_2, \alpha_3)$ and the second line is contained in $\mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R^2_{\alpha_2}} \partial \mathcal{N}_{23}(\alpha_2, \alpha_3)$. Now we define (18.17) by

$$(\mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R^2_{\alpha_2}} \mathcal{N}_{23}(\alpha_2, \alpha_3))^{\mathfrak{C} \boxplus \tau}.$$

Proposition 18.38. In Situation 18.35 there exists a τ - \mathfrak{C} -collared K-space

$$\mathcal{N}_{123}(\alpha_1,\alpha_3)$$

with the following properties:

(1) Its normalized boundary in \mathfrak{C} is isomorphic to the disjoint union of

$$\mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R_{\alpha_2}^2}^{\boxplus \tau} \mathcal{N}_{23}(\alpha_2, \alpha_3) \tag{18.18}$$

over various α_2 .

(2) Its normalized boundary which is not in $\mathfrak C$ is isomorphic to the disjoint union of

$$\mathcal{M}^{1}(\alpha_{1}, \alpha_{1}') \times_{R_{\alpha_{1}'}^{1}} \mathcal{N}_{123}(\alpha_{1}', \alpha_{3})$$

$$(18.19)$$

over various α_1' and of

$$\mathcal{N}_{123}(\alpha_1, \alpha_3') \times_{R_{\alpha_3'}} \mathcal{M}^3(\alpha_3', \alpha_3)$$

$$\tag{18.20}$$

over various α'_3 .

- (3) The isomorphisms (1)(2) satisfy the compatibility conditions, Condition 18.39, at the corners.
- (4) The evaluation maps, periodicity and orientation isomorphisms are defined on $\mathcal{N}_{123}(\alpha_1, \alpha_3)$ and commute with the isomorphisms in (1)(2)(3) above.
- (5) Similar statements of Conditions 16.16 (V)(IX) hold.

See Figure 15. We omit describing the precise ways to modify Conditions 16.16

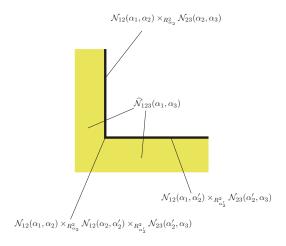


FIGURE 15. $\widehat{\mathcal{N}}_{123}(\alpha_1, \alpha_3)$

(V)(IX) to our situation. Since they are not hard, we leave them to the readers.

Condition 18.39. (1) The k dimensional normalized corner $\widehat{S}_k(\mathcal{N}_{123}(\alpha_1, \alpha_3))$ of $\mathcal{N}_{123}(\alpha_1, \alpha_3)$ is isomorphic to a disjoint union of the one of the two types (18.21) and (18.22):

$$\mathcal{M}^{1}(\alpha_{1}, \alpha_{1}^{2}) \times_{R_{\alpha_{1}^{2}}^{1}} \cdots \times_{R_{\alpha_{1}^{k_{1}-1}}^{1}} \mathcal{M}^{1}(\alpha_{1}^{k_{1}-1}, \alpha_{1}^{k_{1}})$$

$$\times_{R_{\alpha_{1}^{1}}^{1}} \widehat{S}_{m}^{\mathfrak{C}} \left(\mathcal{N}_{12}(\alpha_{1}^{k_{1}}, \alpha_{2}) \times_{R_{\alpha_{2}}^{2}}^{\mathbb{H}_{7}} \mathcal{N}_{23}(\alpha_{2}, \alpha_{3}^{1}) \right)$$

$$\times_{R_{\alpha_{3}^{3}}^{3}} \mathcal{M}^{3}(\alpha_{3}^{1}, \alpha_{3}^{2}) \times_{R_{\alpha_{3}^{2}}^{3}} \cdots \times_{R_{\alpha_{n}^{3}3}^{3}-1} \mathcal{M}^{3}(\alpha_{3}^{k_{3}-1}, \alpha_{3}),$$

$$(18.21)$$

and

$$\mathcal{M}^{1}(\alpha_{1}, \alpha_{1}^{2}) \times_{R_{\alpha_{1}^{2}}^{1}} \cdots \times_{R_{\alpha_{1}^{k_{1}-1}}^{1}} \mathcal{M}^{1}(\alpha_{1}^{k_{1}-1}, \alpha_{1}^{k_{1}})$$

$$\times_{R_{\alpha_{1}^{k_{1}}}^{1}} \mathcal{N}_{123}(\alpha_{1}^{k_{1}}, \alpha_{3}^{1})$$

$$\times_{R_{\alpha_{3}^{3}}^{3}} \mathcal{M}^{3}(\alpha_{3}^{1}, \alpha_{3}^{2}) \times_{R_{\alpha_{3}^{2}}^{3}} \cdots \times_{R_{\alpha_{3}^{k_{3}-1}}^{3}} \mathcal{M}^{3}(\alpha_{3}^{k_{3}-1}, \alpha_{3}).$$

$$(18.22)$$

Here in (18.21), $k_1 + m - 1 + k_3 = k$ and in (18.22), $k_1 + k_3 - 2 = k$. Note $k_1, k_3 \in \mathbb{Z}_{\geq 1}$. In case $k_1 = 1$ the first line of (18.21) or (18.22) is void. In case $k_3 = 1$ the third line of (18.21) or (18.22) is void. We also note that (1) (2) of Proposition 18.38 is the case k = 1 of this condition.

(2) (1) implies that $\widehat{S}_{\ell}(\widehat{S}_{k}(\mathcal{N}_{123}(\alpha_{1},\alpha_{3})))$ is isomorphic to the disjoint union of the spaces of type (18.21), (18.22) or

$$\mathcal{M}^{1}(\alpha_{1}, \alpha_{1}^{2}) \times_{R_{\alpha_{1}^{2}}^{1}} \cdots \times_{R_{\alpha_{1}^{k_{1}-1}}^{1}} \mathcal{M}^{1}(\alpha_{1}^{k_{1}-1}, \alpha_{1}^{k_{1}})$$

$$\times_{R_{\alpha_{1}^{k_{1}}}^{1}} \widehat{S}_{n}^{\mathfrak{C}}\left(\widehat{S}_{m}^{\mathfrak{C}}\left(\mathcal{N}_{12}(\alpha_{1}^{k_{1}}, \alpha_{2}) \times_{R_{\alpha_{2}}^{2}}^{\boxplus \tau} \mathcal{N}_{23}(\alpha_{2}, \alpha_{3}^{1})\right)\right)$$

$$\times_{R_{\alpha_{3}^{3}}^{3}} \mathcal{M}^{3}(\alpha_{3}^{1}, \alpha_{3}^{2}) \times_{R_{\alpha_{3}^{2}}^{3}} \cdots \times_{R_{\alpha_{n}^{k_{3}-1}}^{3}}^{3} \mathcal{M}^{3}(\alpha_{3}^{k_{3}-1}, \alpha_{3}).$$

$$(18.23)$$

The covering map $\widehat{S}_{\ell}(\widehat{S}_k(\mathcal{N}_{123}(\alpha_1, \alpha_3))) \to \widehat{S}_{\ell+k}(\mathcal{N}_{123}(\alpha_1, \alpha_3)))$ is the identity map on the component of type (18.21), (18.22) and is induced by

$$\begin{split} \widehat{S}_{n}^{\mathfrak{C}}\left(\widehat{S}_{m}^{\mathfrak{C}}\left(\mathcal{N}_{12}(\alpha_{1}^{k_{1}},\alpha_{2})\times_{R_{\alpha_{2}}^{2}}^{\boxplus\tau}\mathcal{N}_{23}(\alpha_{2},\alpha_{3}^{1})\right)\right) \\ \longrightarrow \widehat{S}_{n+m}^{\mathfrak{C}}\left(\mathcal{N}_{12}(\alpha_{1}^{k_{1}},\alpha_{2})\times_{R_{\alpha_{2}}^{2}}^{\boxplus\tau}\mathcal{N}_{23}(\alpha_{2},\alpha_{3}^{1})\right) \end{split}$$

on the component of type (18.23).

Lemma-Definition 18.40. By Proposition 18.38 we can partially smooth the corner of $\mathcal{N}_{123}(\alpha_1, \alpha_3)$. Then we obtain a K-space which is the union of $\mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R_{\alpha_2}^2}^{\boxplus \tau}$ $\mathcal{N}_{23}(\alpha_2, \alpha_3)$ over various α_2 :

$$\bigcup_{\alpha_2} \mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R_{\alpha_2}^2}^{\boxplus \tau} \mathcal{N}_{23}(\alpha_2, \alpha_3). \tag{18.24}$$

There is a morphism from \mathcal{F}_1 to \mathcal{F}_3 whose interpolation space is (18.24). We call this morphism the composition of \mathfrak{N}_{21} and \mathfrak{N}_{32} and write $\mathfrak{N}_{32} \circ \mathfrak{N}_{21}$.

This follows immediately from Proposition 18.38.

Remark 18.41. Before proving Proposition 18.38, we note the following. Using the moduli space of the solutions of the equation (18.5), which uses one parameter family of homotopies of Hamiltonians (18.6), we can find a space $\widehat{\mathcal{N}}_{123}(\alpha_1, \alpha_3)$ whose normalized boundary is the union of

$$\mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R_{\alpha_2}^2} \mathcal{N}_{23}(\alpha_2, \alpha_3), \tag{18.25}$$

and

$$\mathcal{M}^{1}(\alpha_{1}, \alpha_{1}') \times_{R_{\alpha_{1}'}^{1}} \widehat{\mathcal{N}}_{123}(\alpha_{1}', \alpha_{3})$$

$$(18.26)$$

over various $\alpha_1' \in \mathfrak{A}_1$ and

$$\widehat{\mathcal{N}}_{123}(\alpha_1, \alpha_3') \times_{R_{\alpha_3'}} \mathcal{M}^3(\alpha_3', \alpha_3)$$
(18.27)

over various $\alpha_3' \in \mathfrak{A}_3$.

We take \mathfrak{C} as the boundary of type (18.25). Then we can take

$$\mathcal{N}_{123}(\alpha_1,\alpha_3) = \widehat{\mathcal{N}}_{123}(\alpha_1,\alpha_3)^{\mathfrak{C} \boxplus \tau} \setminus \widehat{\mathcal{N}}_{123}(\alpha_1,\alpha_3).$$

In the proof of Proposition 18.38 below we will construct this space without using the whole $\mathcal{N}_{123}(\alpha_1, \alpha_3)$ but using only its 'boundary' which is (18.25), (18.19) and (18.20). This argument is very much similar to the proof of Proposition 17.46.

Proof of Proposition 18.38. We begin with proving Lemma 18.42. For $A \subset \{1, \ldots, k\}$ we regard

$$[0,1)^A = \{(t_1,\ldots,t_k) \in [0,1)^k \mid t_i = 0 \text{ for } i \notin A\}.$$

Lemma 18.42. For $A \subset B \subseteq \{1, ..., k\}$ we have a smooth embedding

$$\phi_{AB}: ([0,1)^{B^c})^{\boxplus \tau} \times [-\tau,0)^{\#B} \to ([0,1)^{A^c})^{\boxplus \tau} \times [-\tau,0)^{\#A}$$

where A^c, B^c are the complements of A, B in the set $\{1, \ldots, k\}$ respectively. Moreover, if $A \subset B \subset C \subseteq \{1, \ldots, k\}$, we have

$$\phi_{AB} \circ \phi_{BC} = \phi_{AC}. \tag{18.28}$$

Proof. Regrading $V_{\mathfrak{r},S_A}^+, V_{\mathfrak{r},S_B}^+$ in (17.40) as $[0,1)^{A^c}$, $[0,1)^{B^c}$, respectively, we can adopt (17.40) to define the map ϕ_{AB} above. (Use the obvious inclusion map and the local inverse of the covering map.) For example, if $A=\{1,\ldots,a\},\ B=\{1,\ldots,b\}$, then we have

$$\phi_{AB}((t_{b+1},\ldots,t_k),(s_1,\ldots,s_b)) = (s_{a+1},\ldots,s_b,t_{b+1},\ldots,t_k,s_1,\ldots,s_a).$$

The formula (18.28) is easy to prove from this definition.

For $\alpha_2^1, \ldots, \alpha_2^k \in \mathfrak{A}_2$, we put

$$\mathcal{N}(\alpha_1, \alpha_3; \alpha_2^1, \dots, \alpha_2^k)$$

$$= \left(\mathcal{N}_{12}(\alpha_1, \alpha_2^1) \times_{R_{\alpha_2^1}^2} \mathcal{M}^2(\alpha_2^1, \alpha_2^2) \times_{R_{\alpha_2^2}^2} \right.$$

$$\cdots \times_{R_{\alpha_2^{k-1}}^2} \mathcal{M}^2(\alpha_2^{k-1}, \alpha_2^k) \times_{R_{\alpha_2^k}^2} \mathcal{N}_{23}(\alpha_2^k, \alpha_3) \right)^{\mathfrak{C} \boxplus \tau} \times [-\tau, 0)^k.$$
(18.29)

Here \mathfrak{C} is the decomposition of the boundary such that its *complement* consists of

$$\mathcal{M}^{1}(\alpha_{1}, \alpha'_{1}) \times_{R^{1}_{\alpha'_{1}}} \mathcal{N}_{12}(\alpha'_{1}, \alpha^{1}_{2}) \times_{R^{2}_{\alpha^{1}_{2}}} \mathcal{M}^{2}(\alpha^{1}_{2}, \alpha^{2}_{2}) \times_{R^{2}_{\alpha^{2}_{2}}}$$

$$\cdots \times_{R^{2}_{\alpha^{k-1}_{2}}} \mathcal{M}^{2}(\alpha^{k-1}_{2}, \alpha^{k}_{2}) \times_{R^{2}_{\alpha^{k}_{2}}} \mathcal{N}_{23}(\alpha^{k}_{2}, \alpha_{3})$$

and

$$\mathcal{N}_{12}(\alpha_{1}, \alpha_{2}^{1}) \times_{R_{\alpha_{1}^{2}}^{2}} \mathcal{M}^{2}(\alpha_{2}^{1}, \alpha_{2}^{2}) \times_{R_{\alpha_{2}^{2}}^{2}} \\ \cdots \times_{R_{\alpha_{2}^{k-1}}^{2}} \mathcal{M}^{2}(\alpha_{2}^{k-1}, \alpha_{2}^{k}) \times_{R_{\alpha_{2}^{k}}^{2}} \mathcal{N}_{23}(\alpha_{2}^{k}, \alpha_{3}') \times_{R_{\alpha_{3}'}^{3}} \mathcal{M}^{3}(\alpha_{3}', \alpha_{3}).$$

Let $A = \{i_1, \dots, i_a\} \subset \{1, \dots, k\}$ with $i_1 < i_2 < \dots < i_a$. We define an embedding

$$\hat{\phi}_{A\{1,\ldots,k\}}: \mathcal{N}(\alpha_1,\alpha_3;\alpha_2^{i_1},\ldots,\alpha_2^{i_a}) \to \mathcal{N}(\alpha_1,\alpha_3;\alpha_2^1,\ldots,\alpha_2^k)$$

as follows. For any $p \in \mathcal{N}(\alpha_1, \alpha_3; \alpha_2^1, \dots, \alpha_2^k)$ we have a Kuranishi neighborhood U_p of $\mathcal{N}(\alpha_1, \alpha_3; \alpha_2^1, \dots, \alpha_2^k)$ such that U_p has an orbifold chart at p of the form

$$V_p^{\mathfrak{C} \boxplus \tau} \times ([-\tau, 0)^k). \tag{18.30}$$

On the other hand, p has a Kuranishi neighborhood U'_p of $\mathcal{N}(\alpha_1, \alpha_3; \alpha_2^{i_1}, \dots, \alpha_2^{i_a})$ such that U'_p has an orbifold chart at p of the form

$$V_p^{\mathfrak{C} \boxplus \tau} \times ([0,1)^{k-\#A})^{\boxplus \tau} \times [-\tau,0)^{\#A}.$$
 (18.31)

Therefore the map $\hat{\phi}_{A\{1,...,k\}}$ in Lemma 18.42 defines an embedding from (18.30) to (18.31). This is compatible with the coordinate change of Kuranishi structure and defines a required embedding. We can glue various $\mathcal{N}(\alpha_1, \alpha_3; \alpha_2^1, \ldots, \alpha_2^k)$ by the embeddings $\hat{\phi}_{A\{1,...,k\}}$. More precisely, we will glue charts of their Kuranishi structures. We can do so in the same way as the proof of Proposition 17.46 using the compatibility of interpolation spaces and (18.28).

We have thus obtained a \mathfrak{C} -collared K-space $\mathcal{N}_{123}(\alpha_1, \alpha_3)$. It is easy to see from the construction that $\mathcal{N}_{123}(\alpha_1, \alpha_3)$ has the required properties. (See Figure 15.)

By construction we have the following:

Lemma 18.43. Suppose that there exists a K-space

$$\mathcal{N}'_{123}(\alpha_1,\alpha_3)$$

and its boundary components \mathfrak{C} such that the following holds.

- $(1) \ \partial_{\mathfrak{C}} \mathcal{N}'_{123}(\alpha_1,\alpha_3) \ is \ a \ disjoint \ union \ of \ \mathcal{N}_{12}(\alpha_1,\alpha_2) \times_{R^2_{\alpha_2}} \mathcal{N}_{23}(\alpha_2,\alpha_3) \ over \ \alpha_2.$
- (2) The k-th normalized corner $\hat{S}_k^{\mathfrak{C}}(\mathcal{N}'_{123}(\alpha_1, \alpha_3))$ is identified with the disjoint union of

$$\mathcal{N}_{12}(\alpha_1, \alpha_2^1) \times_{R_{\alpha_2^1}^2} \mathcal{M}^2(\alpha_2^1, \alpha_2^2) \times_{R_{\alpha_2^2}^2} \\ \cdots \times_{R_{\alpha_2^{k-1}}^2} \mathcal{M}^2(\alpha_2^{k-1}, \alpha_2^k) \times_{R_{\alpha_2^k}^2} \mathcal{N}_{23}(\alpha_2^k, \alpha_3)$$

over $\alpha_2^1, \ldots, \alpha_2^k$.

(3) The map $\hat{S}_k(\hat{S}_{\ell}(\mathcal{N}'_{123}(\alpha_1, \alpha_2))) \to \hat{S}_{k+\ell}(\mathcal{N}'_{123}(\alpha_1, \alpha_2))$ is compatible with the isomorphisms (1)(2) in the sense similar to Condition 18.39 (2).

Then there exists an isomorphism

$$\mathcal{N}'_{123}(\alpha_1, \alpha_2)^{\mathfrak{C} \boxplus \tau} \setminus \mathcal{N}'_{123}(\alpha_1, \alpha_3) \cong \mathcal{N}_{123}(\alpha_1, \alpha_3). \tag{18.32}$$

This isomorphism is compatible with the isomorphisms in (1)(2) above and Condition 18.39 (2) at the corners.

18.7. **Associativity of the composition.** In this subsection we present the detail of the proof of the associativity of the composition.

Proposition 18.44. Suppose we are in Situation 18.35 for i = 1, 2, 3. Then we have the following identity.

$$(\mathfrak{N}_{43} \circ \mathfrak{N}_{32}) \circ \mathfrak{N}_{21} = \mathfrak{N}_{43} \circ (\mathfrak{N}_{32} \circ \mathfrak{N}_{21}).$$
 (18.33)

Proof. As we already discussed in Subsection 16.5, this equality is mostly obvious. Namely the interpolation space of the left hand and the right hand sides are isomorphic to each other by the associativity of the fiber product. The only issue is about the way how we smooth the corner. Since smoothing corners is not completely canonical, this point is non-trivial. To clarify this tiny technicality we will use the following.³⁰

Lemma 18.45. There exists a K-space

$$\mathcal{N}_{1234}(\alpha_1,\alpha_4)$$

with the following properties.

(1) Its normalized boundary is isomorphic to the union of the following four types of fiber products.

$$\mathcal{N}_{123}(\alpha_1, \alpha_3) \times_{R_{\alpha_2}^3} \mathcal{N}_{34}(\alpha_3, \alpha_4), \tag{18.34}$$

$$\mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R_{\alpha_2}^2} \mathcal{N}_{234}(\alpha_2, \alpha_4), \tag{18.35}$$

$$\mathcal{M}^{1}(\alpha_{1}, \alpha'_{1}) \times_{R_{\alpha'_{1}}^{1}} \mathcal{N}_{1234}(\alpha'_{1}, \alpha_{4}),$$
 (18.36)

$$\mathcal{N}_{1234}(\alpha_1, \alpha_4') \times_{R_{\alpha_4'}^4} \mathcal{M}^4(\alpha_4', \alpha_1').$$
 (18.37)

- (2) A similar compatibility condition as Condition 18.39 holds at the corners.
- (3) The evaluation maps, periodicity and orientation isomorphisms are defined on $\mathcal{N}_{1234}(\alpha_1, \alpha_4)$ and commute with the isomorphisms in (1)(2) above.
- (4) An appropriate version of Condition 16.16 (V)(IX) holds.

We omit the precise definition of the compatibility condition in Lemma 18.45 (2). Since it is not difficult, we leave it to the readers. We will prove Lemma 18.45 later in this subsection. We continue the proof of Proposition 18.44.

We consider two decompositions \mathfrak{C}_1 , \mathfrak{C}_2 of the boundary of $\mathcal{N}_{1234}(\alpha_1, \alpha_4)$, where \mathfrak{C}_1 consists of the boundary of type (18.34) and \mathfrak{C}_2 consists of the boundary of type (18.35). Thus we are in the situation of Lemma 18.14. Here we use the next lemma.

Lemma 18.46. Suppose we are in the situation of Lemma 18.14. Then the following constructions described in (A) and (B) give the same K-space.

- (A) (1) We first take τ - \mathfrak{C}_1 -trivialization of the corner.
 - (2) We smooth the boundaries in \mathfrak{C}_1 .
 - (3) We next take τ - \mathfrak{C}_2 -trivialization of the corner.
 - (4) We smooth the boundaries in \mathfrak{C}_2 .
- (B) (1) We first take τ - \mathfrak{C}_2 -trivialization of the corner.

³⁰The fact that the left and right hand sides induce the same cochain map in the de Rham complex by the smooth correspondence follows without comparing the smooth structures near the corner. Since the part we smooth the corner lies in the collar, it does not contribute to the integration along the fiber (see (17.17).) So we never need the part of the proof of this proposition given in this subsection for applications.

- (2) We smooth the boundaries in \mathfrak{C}_2 .
- (3) We next take τ - \mathfrak{C}_1 -trivialization of the corner.
- (4) We smooth the boundaries in \mathfrak{C}_1 .

Proof. It suffices to consider the case of $[0,1)^{k_1} \times [0,1)^{k_2}$ where \mathfrak{C}_1 corresponds to $\partial [0,1)^{k_1} \times [0,1)^{k_2}$ and \mathfrak{C}_2 corresponds to $[0,1)^{k_1} \times \partial [0,1)^{k_2}$. However, the proof for this case is straightforward from the definition.

We note that the intersection of (18.34) and (18.35) is

$$\mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R^2_{\alpha_2}} \mathcal{N}_{23}(\alpha_2, \alpha_3) \times_{R^3_{\alpha_3}} \mathcal{N}_{34}(\alpha_3, \alpha_4). \tag{18.38}$$

The interpolation spaces of both sides of (18.33) are obtained by appropriately modifying the union of (18.38) for α_2 , α_3 .

We study how the processes (A) and (B) of Lemma 18.46 affect the subspace (18.38). By (A) (1), (18.38) becomes

$$\mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R^2_{\alpha_2}} \mathcal{N}_{23}(\alpha_2, \alpha_3) \times_{R^3_{\alpha_3}}^{\boxplus \tau} \mathcal{N}_{34}(\alpha_3, \alpha_4).$$

The step (A) (2) smoothes the corners of the (union over α_3 of) the second fiber product factor. So it becomes

$$\mathcal{N}_{12}(\alpha_1,\alpha_2) \times_{R^2_{\alpha_2}} \mathcal{N}_{24}(\alpha_2,\alpha_4).$$

Here $\mathcal{N}_{24}(\alpha_2, \alpha_4)$ is the interpolation space of the composition $\mathfrak{N}_{43} \circ \mathfrak{N}_{32}$. Then (A)(3) changes it to

$$\mathcal{N}_{12}(\alpha_1, \alpha_2) \times_{R_{\alpha_2}}^{\boxplus \tau} \mathcal{N}_{24}(\alpha_2, \alpha_4).$$

Thus when (A) completed we obtain an interpolation space of

$$(\mathfrak{N}_{43} \circ \mathfrak{N}_{32}) \circ \mathfrak{N}_{21}$$
.

In the same way, (B) gives an interpolation space of

$$\mathfrak{N}_{43} \circ (\mathfrak{N}_{32} \circ \mathfrak{N}_{21}).$$

Thus Proposition 18.44 follows from Lemma 18.46.

Proof of Lemma 18.45. We fix α_1, α_4 . Let $\alpha_2^1, \ldots, \alpha_2^{k_2} \in \mathfrak{A}_2$ and $\alpha_3^1, \ldots, \alpha_3^{k_3} \in \mathfrak{A}_3$. We put

$$\begin{split} &\mathcal{N}(\alpha_{1},\alpha_{3};\alpha_{2}^{1},\ldots,\alpha_{2}^{k_{2}};\alpha_{3}^{1},\ldots,\alpha_{3}^{k_{3}}) \\ &= \left(\mathcal{N}_{12}(\alpha_{1},\alpha_{2}^{1}) \times_{R_{\alpha_{1}^{2}}^{2}} \mathcal{M}^{2}(\alpha_{2}^{1},\alpha_{2}^{2}) \times_{R_{\alpha_{2}^{2}}^{2}} \right. \\ & \left. \cdots \times_{R_{\alpha_{2}^{k_{2}-1}}^{2}} \mathcal{M}^{2}(\alpha_{2}^{k_{2}-1},\alpha_{2}^{k_{2}}) \times_{R_{\alpha_{2}^{k_{2}}}^{2}} \mathcal{N}_{23}(\alpha_{2}^{k_{2}},\alpha_{3}^{1}) \times_{R_{\alpha_{3}^{3}}^{3}} \mathcal{M}^{3}(\alpha_{3}^{1},\alpha_{3}^{2}) \times_{R_{\alpha_{3}^{3}}^{3}} \right. \\ & \left. \cdots \times_{R_{\alpha_{3}^{k_{3}-1}}^{3}} \mathcal{M}^{3}(\alpha_{3}^{k_{3}-1},\alpha_{3}^{k_{3}}) \times_{R_{\alpha_{3}^{3}}^{3}} \mathcal{N}_{34}(\alpha_{3}^{k_{3}},\alpha_{4}) \right)^{\mathfrak{C}H\tau} \times [-\tau,0)^{k_{2}+k_{3}}. \end{split}$$

Here \mathfrak{C} is defined in a similar way as in (18.29). We can glue them in a similar way as in the proof of Proposition 18.38 to obtain the required $\mathcal{N}_{1234}(\alpha_1, \alpha_4)$.

The proof of Proposition 18.44 is now complete.

We note that in the geometric situation of periodic Hamiltonian system we can prove Lemma 18.45 using the 2-parameter family of moduli space of solutions of the equation

$$\frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_{H_{\tau,t}^{ST}}(u)\right) = 0, \tag{18.39}$$

where

$$H^{ST}(\tau, t, x) = \begin{cases} H^{1}(t, x) & \text{if } \tau \leq -T - T_{0} \\ H^{21}(\tau + T, t, x) & \text{if } -T - T_{0} \leq \tau \leq -T + T_{0} \\ H^{2}(t, x) & \text{if } -T + T_{0} \leq \tau \leq -T_{0} \\ H^{32}(\tau, t, x) & \text{if } -T_{0} \leq \tau \leq T_{0} \\ H^{3}(t, x) & \text{if } T_{0} \leq \tau \leq S - T_{0} \\ H^{43}(\tau - S, t, x) & \text{if } S - T_{0} \leq \tau \leq S + T_{0} \\ H^{4}(t, x) & \text{if } S + T_{0} \leq \tau, \end{cases}$$

$$(18.40)$$

and $H^{ij}(\tau,t,x): \mathbb{R} \times S^1 \times M \to \mathbb{R}$ $(i,j=1,\ldots,4)$ is defined by (18.4).

18.8. Parametrized version of morphism: composition and gluing.

18.8.1. Compositions of parametrized morphisms.

Situation 18.47. Suppose we are in Situation 16.34 and $\mathfrak{N}_{i+1i}^{P_i}$ is a P_i -parametrized morphism from \mathcal{F}_i to \mathcal{F}_{i+1} . We denote by

$$\mathcal{M}^i(\alpha_-,\alpha_+)$$

the space of connecting orbits of \mathcal{F}_i and by

$$\mathcal{N}_{ii+1}(\alpha_-, \alpha_+; P_i)$$

the interpolation space of $\mathfrak{N}_{i+1i}^{P_i}$. Let $R_{\alpha_i}^i$ be a critical submanifold of \mathcal{F}_i and $\alpha_i \in \mathfrak{A}_i$.

Definition 18.48. In Situation 18.47 we define a K-space

$$\mathcal{N}_{12}(\alpha_1, \alpha_2; P_1) \times_{R_{\alpha_2}^2}^{\boxplus \tau} \mathcal{N}_{23}(\alpha_2, \alpha_3; P_2)$$

$$\tag{18.41}$$

as follows. We take the fiber product $\mathcal{N}_{12}(\alpha_1, \alpha_2; P_1) \times_{R_{\alpha_2}^2} \mathcal{N}_{23}(\alpha_2, \alpha_3; P_2)$ with fiber product Kuranishi structure and consider the decomposition \mathfrak{C} of its boundary consisting of the following two kinds of components of its normalized boundary:

$$\mathcal{N}_{12}(\alpha_1, \alpha_2'; P_1) \times_{R_{\alpha_2}^2} \mathcal{M}^2(\alpha_{\alpha_2'}, \alpha_{\alpha_2}) \times_{R_{\alpha_2}^2} \mathcal{N}_{23}(\alpha_2, \alpha_3; P_2),$$

$$\mathcal{N}_{12}(\alpha_1, \alpha_2; P_1) \times_{R_{\alpha_2}^2} \mathcal{M}^2(\alpha_{\alpha_2}, \alpha_{\alpha_2'}) \times_{R_{\alpha_2'}^2} \mathcal{N}_{23}(\alpha_2', \alpha_3; P_2).$$

The first line is contained in $\partial \mathcal{N}_{12}(\alpha_1, \alpha_2; P_1) \times_{R_{\alpha_2}^2} \mathcal{N}_{23}(\alpha_2, \alpha_3; P_2)$ and the second line is contained in $\mathcal{N}_{12}(\alpha_1, \alpha_2; P_1) \times_{R_{\alpha_2}^2} \partial \mathcal{N}_{23}(\alpha_2, \alpha_3; P_2)$. Now we define (18.41) by

$$(\mathcal{N}_{12}(\alpha_1,\alpha_2;P_1)\times_{R^2_{\alpha_2}}\mathcal{N}_{23}(\alpha_2,\alpha_3);P_2)^{\mathfrak{C}\boxplus \tau}.$$

Note that besides the evaluation maps ev_{-} and ev_{+} , (18.41) carries an evaluation map to $P_{1} \times P_{2}$, which is stratumwise weakly submersive.

Proposition 18.49. In Situation 18.47 there exists a τ - \mathfrak{C} -collared K-space

$$\mathcal{N}_{123}(\alpha_1, \alpha_3; P_1 \times P_2)$$

with the following properties:

(1) Its normalized boundary in $\mathfrak C$ is isomorphic to the disjoint union of

$$\mathcal{N}_{12}(\alpha_1, \alpha_2; P_1) \times_{R_{\alpha_2}}^{\boxplus \tau} \mathcal{N}_{23}(\alpha_2, \alpha_3; P_2)$$
 (18.42)

over various α_2 .

(2) Its normalized boundary not in $\mathfrak C$ is isomorphic to the disjoint union of

$$\mathcal{M}^{1}(\alpha_{1}, \alpha'_{1}) \times_{R_{\alpha'_{1}}^{1}} \mathcal{N}_{123}(\alpha'_{1}, \alpha_{3}; P_{1} \times P_{2})$$
 (18.43)

over various α'_1 and

$$\mathcal{N}_{123}(\alpha_1, \alpha_3'; P_1 \times P_2) \times_{R_{\alpha_3'}} \mathcal{M}^3(\alpha_3', \alpha_3)$$

$$\tag{18.44}$$

over various α'_3 .

- (3) Conditions similar to (3)(4)(5) of Proposition 18.38 hold.
- (4) We have an evaluation map $\mathcal{N}_{123}(\alpha_1, \alpha_3; P_1 \times P_2) \to P_1 \times P_2$ which is a stratumwise weakly submersive map. It is compatible with the boundary description given in (1)(2) above.

The proof is the same as the proof of Proposition 18.38 so omitted.

Definition-Lemma 18.50. By Proposition 18.49 we can partially smooth the corner of $\mathcal{N}_{123}(\alpha_1, \alpha_3; P_1 \times P_2)$ to obtain a K-space given by

$$\bigcup_{\alpha_2} \mathcal{N}_{12}(\alpha_1, \alpha_2; P_1) \times_{R_{\alpha_2}^2}^{\boxplus \tau} \mathcal{N}_{23}(\alpha_2, \alpha_3; P_2). \tag{18.45}$$

There is a $(P_1 \times P_2)$ -parametrized morphism from \mathcal{F}_1 to \mathcal{F}_3 whose interpolation space is given by (18.45). We call this morphism the parametrized composition of $\mathfrak{N}_{21}^{P_1}$ and $\mathfrak{N}_{32}^{P_2}$ and write $\mathfrak{N}_{32}^{P_2} \circ \mathfrak{N}_{21}^{P_2}$.

Lemma 18.51. The parametrized composition is associative in the same sense as in Proposition 18.44. Moreover the boundary of the parametrized composition $\mathfrak{N}_{32}^{P_1} \circ \mathfrak{N}_{21}^{P_2}$ is the disjoint union of

$$\mathfrak{N}_{32}^{\partial P_1} \circ \mathfrak{N}_{21}^{P_2}$$
 and $\mathfrak{N}_{32}^{P_1} \circ \mathfrak{N}_{21}^{\partial P_2}$.

Proof. The proof of the first half is similar to the proof of Proposition 18.44. The second half is obvious from the construction. \Box

18.8.2. Gluing parametrized morphisms. We first review well known obvious facts of gluing cornered manifolds along their boundaries.

Definition-Lemma 18.52. Let P_1 , P_2 be two admissible manifolds with corners, and for each i=1,2 let $\partial_{\mathfrak{C}_i}P_i\subset\partial P_i$ be an open and closed subset of the normalized boundary ∂P_i and assume $S_2^{\mathfrak{C}_i}(P_i)=\emptyset$. (See Definition 18.2 for the notation.) Let $I:\partial_{\mathfrak{C}_1}P_1\to\partial_{\mathfrak{C}_2}P_2$ be an admissible diffeomorphism of manifolds with corners.

(1) We can define a structure of admissible manifold with corners on

$$P_{1 \mathfrak{C}_{1}} \cup_{\mathfrak{C}_{2}} P_{2} = (P_{1} \cup P_{2}) / \sim.$$
 (18.46)

Here \sim is defined by $x \sim I(x)$ for $x \in \partial_{\mathfrak{C}_1} P_1$.

(2) The boundary $\partial(P_1 \,_{\mathfrak{C}_1} \cup_{\mathfrak{C}_2} P_2)$ is described as follows. Let $\partial_{\mathfrak{C}_i^c} P_i$ be the complement of $\partial_{\mathfrak{C}_i} P_i$ in ∂P_i . Then \mathfrak{C}_i induces a decomposition

$$\partial(\partial_{\mathfrak{C}_{\vec{s}}^c}P_i) = \partial_{\mathfrak{C}_i}(\partial_{\mathfrak{C}_{\vec{s}}^c}P_i) \cup \partial_{\mathfrak{C}_{\vec{s}}^c}(\partial_{\mathfrak{C}_{\vec{s}}^c}P_i).$$

Moreover I induces a diffeomorphism

$$\partial_{\mathfrak{C}_1}(\partial_{\mathfrak{C}_1^c}P_1) \cong \partial_{\mathfrak{C}_2}(\partial_{\mathfrak{C}_2^c}P_2).$$

Now we have, see Figure 16,

$$\partial(P_1_{\mathfrak{C}_1} \cup_{\mathfrak{C}_2} P_2) = \partial_{\mathfrak{C}_1^c} P_1_{\mathfrak{C}_1} \cup_{\mathfrak{C}_2} \partial_{\mathfrak{C}_2^c} P_2. \tag{18.47}$$

- (3) Suppose that we have one of the following objects on P_1 and on P_2 such that their restrictions to $\partial_{\mathfrak{C}_1} P_1$ and to $\partial_{\mathfrak{C}_2} P_2$ are isomorphic each other (or coincide with each other) when we identify $\partial_{\mathfrak{C}_1} P_1$ with $\partial_{\mathfrak{C}_2} P_2$ under the diffeomorphism I. Then we obtain a glued object on P_1 $\mathfrak{C}_1 \cup \mathfrak{C}_2 P_2$.
 - (a) Vector bundle.
 - (b) Section of vector bundle.
 - (c) Differential form.
 - (d) Smooth map to a manifold.
- (4) In addition, suppose that we have admissible manifolds with corners P'_1 , P'_2 and for each i=1,2 let $\partial_{\mathfrak{C}'_i}P'_i$ be an open and closed subset of the normalized boundary $\partial P'_i$. Let $I':\partial_{\mathfrak{C}'_1}P'_1\cong\partial_{\mathfrak{C}'_2}P'_2$ be an admissible diffemorphism as above. We also assume that for each i=1,2 we have an admissible smooth embedding $f_i:P'_i\to P_i$ of cornered orbifolds such that (a) $f_i^{-1}(\partial_{\mathfrak{C}_i}P_i)=\partial_{\mathfrak{C}'_i}P'_i$.
 - (h)

$$I \circ f_1 = f_2 \circ I'$$

on $\partial_{\mathfrak{C}'_1} P'_1$.

Then f_1, f_2 induce a smooth admissible embedding of cornered orbifolds

$$f_1 \ \mathfrak{C}_1 \cup \mathfrak{C}_2 \ f_2 : P_1' \ \mathfrak{C}_1' \cup \mathfrak{C}_2' \ P_2' \to P_1 \ \mathfrak{C}_1 \cup \mathfrak{C}_2 \ P_2.$$

(5) In the situation of (4), suppose that we are given vector bundles E_i on P_i and E_i' on P_i' . We assume $E_1|_{\partial_{\mathfrak{C}_1}P_1}\cong E_2|_{\partial_{\mathfrak{C}_2}P_2}$ and the isomorphism covers I. We assume the same condition for E_1' and E_2' . Moreover we assume that there exist embeddings of vector bundles $\hat{f}_i: E_i' \to E_i$ which cover f_i for i=1,2, and assume that \hat{f}_1 and \hat{f}_2 are compatible with the isomorphisms $E_1|_{\partial_{\mathfrak{C}_1}P_1}\cong E_2|_{\partial_{\mathfrak{C}_2}P_2}$ and $E_1'|_{\partial_{\mathfrak{C}_1'}P_1'}\cong E_2'|_{\partial_{\mathfrak{C}_2'}P_2'}$.

Then we obtain an embedding of vector bundles:

$$\hat{f}_1 \ _{\mathfrak{C}_1} \cup_{\mathfrak{C}_2} \ \hat{f}_2 : E'_1 \ _{\mathfrak{C}'_1} \cup_{\mathfrak{C}'_2} \ E'_2 \to E_1 \ _{\mathfrak{C}_1} \cup_{\mathfrak{C}_2} \ E_2.$$

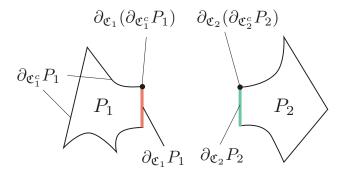
- (6) In addition, suppose that we have P'_1 , P'_2 and $\partial_{\mathfrak{C}'_i}P'_i$ (for i=1,2), I': $\partial_{\mathfrak{C}'_1}P'_1\cong \partial_{\mathfrak{C}'_2}P'_2$ as above. We also assume that for each i=1,2 we have an admissible smooth stratumwise submersion $\pi_i:P'_i\to P_i$ of cornered orbifolds such that
 - (a) $\pi_i^{-1}(\partial_{\mathfrak{C}_i} P_i) = \partial_{\mathfrak{C}_i'} P_i'$.
 - (b)

$$I \circ \pi_1 = \pi_2 \circ I'$$

on $\partial_{\mathfrak{C}'_1} P'_1$.

Then π_1, π_2 induce an admissible smooth stratumwise submersion of cornered orbifolds

$$\pi_1 \ \mathfrak{C}_1 \cup \mathfrak{C}_2 \ \pi_2 : P'_1 \ \mathfrak{C}'_1 \cup \mathfrak{C}'_2 \ P'_2 \to P_1 \ \mathfrak{C}_1 \cup \mathfrak{C}_2 \ P_2.$$



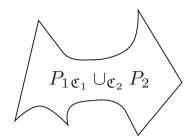


FIGURE 16. gluing cornered manifolds

Proof. The proof is mostly obvious except the smoothness of various objects (such as coordinate change) at the boundary where we glue P_1 and P_2 . The smoothness can be proved by using admissibility and Lemma 25.9 (2). In fact, the differential in the direction normal to the boundary of all the objects vanishes in infinite order. So gluing along the boundary gives smooth objects.

Situation 18.53. Let $P_1, P_2, \partial_{\mathfrak{C}_i} P_i \subset \partial P_i$ and $I: \partial_{\mathfrak{C}_1} P_1 \to \partial_{\mathfrak{C}_2} P_2$ be given as in Definition-Lemma 18.52. For i=1,2 let $(X_i,\widehat{\mathcal{U}_i})$ be a K-space and $\operatorname{ev}_{P_i}: (X_i,\widehat{\mathcal{U}_i}) \to P_i$ be a strongly smooth and stratumwise weakly submersive map. Via the map $\operatorname{ev}_{P_i}: (X_i,\widehat{\mathcal{U}_i}) \to P_i$, the decomposition $\partial P_i = \partial_{\mathfrak{C}_i} P_i \cup \partial_{\mathfrak{C}_i^c} P_i$ induces a decomposition $\partial(X_i,\widehat{\mathcal{U}_i}) = \partial_{\mathfrak{C}_i}(X_i,\widehat{\mathcal{U}_i}) \cup \partial_{\mathfrak{C}_i^c}(X_i,\widehat{\mathcal{U}_i})$ as in Situation 18.4. Namely, $\partial_{\mathfrak{C}_i}(X_i,\widehat{\mathcal{U}_i})$ is defined by taking Kuranishi charts so that they are mapped to $\partial_{\mathfrak{C}_i} P_i$ via ev_{P_i} and the coordinate change satisfies the condition as in Situation 18.4. We assume that there exists an isomorphism

$$\hat{I}: \partial_{\mathfrak{C}_1}(X_1, \widehat{\mathcal{U}_1}) \cong \partial_{\mathfrak{C}_2}(X_2, \widehat{\mathcal{U}_2})$$
 (18.48)

which is compatible with the admissible diffeomorphism $I: \partial_{\mathfrak{C}_1} P_1 \to \partial_{\mathfrak{C}_2} P_2$.

Then we glue X_1 and X_2 by the underlying homeomorphism of \hat{I} to obtain

$$X = X_1 \, \mathfrak{C}_1 \cup \mathfrak{C}_2 \, X_2$$

in the same way as (18.46).

Definition-Lemma 18.54. In Situation 18.53 we can glue $\widehat{\mathcal{U}}_1$ and $\widehat{\mathcal{U}}_2$ to obtain a Kuranishi structure $\widehat{\mathcal{U}}$ of X. We write

$$(X,\widehat{\mathcal{U}}) = (X_1,\widehat{\mathcal{U}_1})_{\mathfrak{C}_1} \cup_{\mathfrak{C}_2} (X_2,\widehat{\mathcal{U}_2}), \qquad \widehat{\mathcal{U}} = \widehat{\mathcal{U}_1}_{\mathfrak{C}_1} \cup_{\mathfrak{C}_2} \widehat{\mathcal{U}_2}.$$

It has the following properties.

(1) We have

$$\partial(X,\widehat{\mathcal{U}}) = \partial_{\mathfrak{C}_2^c}(X_1,\widehat{\mathcal{U}_1}) \ _{\mathfrak{C}_1} \cup_{\mathfrak{C}_2} \ \partial_{\mathfrak{C}_2^c}(X_2,\widehat{\mathcal{U}_2}).$$

- (2) Suppose that we have one of the following objects on $(X_1,\widehat{\mathcal{U}}_1)$ and on $(X_2,\widehat{\mathcal{U}}_2)$ such that their restrictions to $\partial_{\mathfrak{C}_1}(X_1,\widehat{\mathcal{U}}_1)$ and to $\partial_{\mathfrak{C}_2}(X_2,\widehat{\mathcal{U}}_2)$ are isomorphic each other (or coincide each other) when we identify $\partial_{\mathfrak{C}_1}(X_1,\widehat{\mathcal{U}}_1)$ with $\partial_{\mathfrak{C}_2}(X_2,\widehat{\mathcal{U}}_2)$ under the diffeomorphism \widehat{I} . Then we obtain the corresponding object on $(X,\widehat{\mathcal{U}})$.
 - (a) Differential form.
 - (b) Strongly smooth map to a manifold.
 - (c) CF-perturbation.
 - (d) Multi-valued perturbation.
- (3) Let h_i (i = 1, 2) be differential forms on $(X_i, \widehat{\mathcal{U}}_i)$, $f_i : (X_i, \widehat{\mathcal{U}}_i) \to M$ strongly smooth maps, and $\widehat{\mathfrak{S}}_i$ CF-perturbations of $(X_i, \widehat{\mathcal{U}}_i)$ such that f_i is strongly submersive with respect to $\widehat{\mathfrak{S}}_i$ respectively. Suppose we can glue them in the sense of Item (2) and obtain h, f and $\widehat{\mathfrak{S}}$. Then we have

$$f!(h;\widehat{\mathfrak{S}^{\epsilon}}) = f_1!(h_1;\widehat{\mathfrak{S}_1^{\epsilon}}) + f_2!(h_2;\widehat{\mathfrak{S}_2^{\epsilon}}).$$

(4) In addition, suppose that we have $(X_i, \widehat{\mathcal{U}}'_i)$ (i = 1, 2) and $\hat{I}' : \partial_{\mathfrak{C}_1}(X_1, \widehat{\mathcal{U}}'_1) \cong \partial_{\mathfrak{C}_2}(X_2, \widehat{\mathcal{U}}'_2)$ as above. (Note that both of $\widehat{\mathcal{U}}'_i$ and $\widehat{\mathcal{U}}_i$ are Kuranishi structures of X_i for each i = 1, 2.)

We also assume that we have embeddings $(X_i,\widehat{\mathcal{U}}_i') \to (X_i,\widehat{\mathcal{U}}_i)$ of Kuranishi structures for i=1,2 that are compatible with \widehat{I} and \widehat{I}' in an obvious sense. Then the embeddings induce an embedding of Kuranishi structures $:\widehat{\mathcal{U}}_1' \in \mathcal{U}_1 \cup \mathcal{U}_2 : \widehat{\mathcal{U}}_2' \to \widehat{\mathcal{U}}_1 \in \mathcal{U}_2 : \widehat{\mathcal{U}}_2$.

The compatibility of objects (a)(b)(c)(d) in Item (2) is preserved by this process.

(5) We have a stratumwise submersive map:

$$(X_1,\widehat{\mathcal{U}_1})_{\mathfrak{C}_1} \cup_{\mathfrak{C}_2} (X_2,\widehat{\mathcal{U}_2}) \to P_1_{\mathfrak{C}_1} \cup_{\mathfrak{C}_2} P_2.$$

The proof is immediate from Definition-Lemma 18.52.

Situation 18.55. Suppose that we are in Situation 16.20. Let P_1 , P_2 , $\partial_{\mathfrak{C}_i}P_i \subset \partial P_i$ and $I:\partial_{\mathfrak{C}_1}P_1\to\partial_{\mathfrak{C}_2}P_2$ be as in Definition-Lemma 18.52. Let \mathfrak{N}^{P_i} is a P_i -parametrized morphism from \mathcal{F}_1 to \mathcal{F}_2 . We denote by

$$\mathcal{N}_i(\alpha_-, \alpha_+; P_i)$$

the interpolation space of $\mathfrak{N}_i^{P_i}$. We obtain a $\partial_{\mathfrak{C}_i} P_i$ -morphism $\mathfrak{N}^{\partial_{\mathfrak{C}_i} P_i}$ from \mathcal{F}_1 to \mathcal{F}_2 , whose interpolation space is

$$\mathcal{N}_i(\alpha_-, \alpha_+; \partial_{\sigma} P_i) = \partial_{\sigma} P_i \times_{P} \mathcal{N}_i(\alpha_-, \alpha_+; P_i).$$

We assume that $\mathfrak{N}^{\partial_{\mathfrak{C}_1}P_1}$ is isomorphic to $\mathfrak{N}^{\partial_{\mathfrak{C}_2}P_2}$ under an isomorphism which covers $I:\partial_{\mathfrak{C}_1}P_1\to\partial_{\mathfrak{C}_2}P_2$ in an obvious sense.

Definition-Lemma 18.56. In Situation 18.55, we can glue \mathfrak{N}^{P_1} and \mathfrak{N}^{P_2} to define a P-parametrized morphism \mathfrak{N}^P for $P = P_1 \mathfrak{C}_1 \cup \mathfrak{C}_2 P_2$ from \mathcal{F}_1 to \mathcal{F}_2 with the following properties.

(1) The interpolation space of \mathfrak{N}^P is

$$\mathcal{N}_1(\alpha_-, \alpha_+; P_1)_{\mathfrak{C}_1} \cup_{\mathfrak{C}_2} \mathcal{N}_2(\alpha_-, \alpha_+; P_2).$$

(2) The boundary of \mathfrak{N}^P is

$$\mathfrak{N}^{P_1}|_{\partial_{\mathfrak{C}_1^c}P_1 \mathfrak{C}_1} \cup_{\mathfrak{C}_2} \mathfrak{N}^{P_1}|_{\partial_{\mathfrak{C}_2^c}P_2}.$$

The proof is immediate from Definition-Lemma 18.54.

Now we return to the situation of Lemma-Definition 16.38. We recall that to prove Item (1) we glued two [0,1]-parametrized morphisms $\mathfrak{H}^{i}_{(32)} \circ \mathfrak{N}^{i}_{21}$ and $\mathfrak{N}^{i}_{32} \circ \mathfrak{H}^{i}_{(21)}$ along a part of their boundaries, and to prove Item (2) we glued two [0,1]²-parametrized morphisms $\mathfrak{N}^{i+1}_{32} \circ \mathcal{H}^{i}$ and $\mathfrak{H}^{i}_{(32)} \circ \mathfrak{H}^{i}_{(ab)}$ along a part of their boundaries. Using Definition-Lemma 18.56, we can perform such gluing.

Proof of Lemma-Definition 16.38 (3). We can prove transitivity by gluing two homotopies by using Definition-Lemma 18.56. Other properties are easier to prove so omitted. \Box

18.9. **Identity morphism.** In this subsection, we define the identity morphism.

Remark 18.57. Note that we can *not* define the identity morphism by defining its interpolation space such as

$$\mathcal{N}(\alpha_{-}, \alpha_{+}) = \begin{cases} R_{\alpha_{-}} & \alpha_{-} = \alpha_{+}, \\ \emptyset & \alpha_{-} \neq \alpha_{+}. \end{cases}$$
 (18.49)

In fact, for $\alpha_{-} \neq \alpha_{+}$, (16.25) and (18.49) yield

$$\partial \mathcal{N}(\alpha_-, \alpha_+)$$

$$\supseteq (\mathcal{M}(\alpha_{-}, \alpha_{+}) \times_{R_{\alpha_{+}}} \mathcal{N}(\alpha_{+}, \alpha_{+})) \cup (\mathcal{N}(\alpha_{-}, \alpha_{-}) \times_{R_{\alpha_{-}}} \mathcal{M}(\alpha_{-}, \alpha_{+}))$$

$$= \mathcal{M}(\alpha_{-}, \alpha_{+}) \sqcup \mathcal{M}(\alpha_{-}, \alpha_{+})$$

$$(18.50)$$

which is not an empty set in general.

Suppose that we put two different systems of perturbations on $\mathcal{M}(\alpha,\beta)$. They give two different coboundary oprations of the Floer cochain complex on $CF(\mathcal{C})$. We can use the identity morphism and a system of perturbations on its interpolation spaces to define a cochain map between them. This is the way how we proceed in Section 19. While working out this proof, we put two different perturbations on the two copies on $\mathcal{M}(\alpha_-, \alpha_+)$ appearing in the right hand side of (18.50) and extend them to $\mathcal{N}(\alpha_-, \alpha_+)$. Thus the fact that $\mathcal{N}(\alpha_-, \alpha_+)$ is nonempty is important for proving that Floer cohomology is independent of perturbations by using the identity morphism.

Let \mathcal{F} be a linear K-system whose space of connecting orbits is given by $\mathcal{M}(\alpha_-, \alpha_+)$. We will define the identity morphism from \mathcal{F} to itself. The interpolation spaces are defined as follows.

Definition 18.58. (1) We define $\mathring{\mathcal{N}}(\alpha_-, \alpha_+)$ as follows.

(a) If $\alpha_{-} \neq \alpha_{+}$, then

$$\overset{\circ}{\mathcal{N}}(\alpha_{-},\alpha_{+}) = \overset{\circ}{\mathcal{M}}(\alpha_{-},\alpha_{+}) \times \mathbb{R}.$$

(b) If $\alpha = \alpha_{-} = \alpha_{+}$, then

$$\mathring{\mathcal{N}}(\alpha,\alpha) = R_{\alpha}.$$

(2) We compactify $\mathring{\mathcal{N}}(\alpha_{-}, \alpha_{+})$ as follows. In case $\alpha = \alpha_{-} = \alpha_{+}$, we put $\mathcal{N}(\alpha, \alpha) = \mathring{\mathcal{N}}(\alpha, \alpha)$.

In case $\alpha_{-} \neq \alpha_{+}$, a stratum $\overset{\circ}{S}_{k}(\mathcal{N}(\alpha_{-}, \alpha_{+}))$ of the compactification $\mathcal{N}(\alpha_{-}, \alpha_{+})$ is the union of the following two types of fiber products:

(a)

$$\mathring{\mathcal{M}}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \mathring{\mathcal{M}}(\alpha_{1}, \alpha_{2}) \times_{R_{\alpha_{2}}} \cdots \times_{R_{\alpha_{k'-1}}} \mathring{\mathcal{M}}(\alpha_{k'-1}, \alpha_{k'})$$

$$\times_{R_{\alpha_{k'}}} (\mathring{\mathcal{M}}(\alpha_{k'}, \alpha_{k'+1}) \times \mathbb{R})$$

$$\times_{R_{\alpha_{k'+1}}} \mathring{\mathcal{M}}(\alpha_{k'+1}, \alpha_{k'+2}) \times_{R_{\alpha_{k'+2}}} \cdots \times_{R_{\alpha_{k}}} \mathring{\mathcal{M}}(\alpha_{k}, \alpha_{+}).$$
(18.51)

Here we take all possible choices of k', $\alpha_1, \ldots, \alpha_k$ with $0 \le k' \le k+1$ and

$$E(\alpha_-) < E(\alpha_1) < \dots < E(\alpha_{k'}) < \dots < E(\alpha_k) < E(\alpha_+).$$

(b)

$$\overset{\circ}{\mathcal{M}}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \overset{\circ}{\mathcal{M}}(\alpha_{1}, \alpha_{2}) \times_{R_{\alpha_{2}}} \cdots \times_{R_{\alpha_{k'-1}}} \overset{\circ}{\mathcal{M}}(\alpha_{k'-1}, \alpha_{k'})
\times_{R_{\alpha_{k'}}} R_{\alpha_{k'}}
\times_{R_{\alpha_{k'}}} \overset{\circ}{\mathcal{M}}(\alpha_{k'}, \alpha_{k'+1}) \times_{R_{\alpha_{k'+1}}} \cdots \times_{R_{\alpha_{k-1}}} \overset{\circ}{\mathcal{M}}(\alpha_{k-1}, \alpha_{+}).$$
(18.52)

Here we take all possible choices of k', $\alpha_1, \ldots, \alpha_{k-1}$ with $0 \le k' \le k$ and

$$E(\alpha_{-}) < E(\alpha_{1}) < \dots < E(\alpha_{k'}) < \dots < E(\alpha_{k-1}) < E(\alpha_{+}).$$

Remark 18.59. (1) In (2)(a) above, we regard $\alpha_0 = \alpha_-$ and $\alpha_{k+1} = \alpha_+$. For example, in the case k' = 0, the stratum (18.51) becomes

$$(\mathring{\mathcal{M}}(\alpha_{-},\alpha_{1})\times\mathbb{R})\times_{R_{\alpha_{1}}}\mathring{\mathcal{M}}(\alpha_{1},\alpha_{2})\times_{R_{\alpha_{2}}}\cdots\times_{R_{\alpha_{k}}}\mathring{\mathcal{M}}(\alpha_{k},\alpha_{+}).$$

(2) Similarly in (2)(b) above, we regard $\alpha_0 = \alpha_-$ and $\alpha_k = \alpha_+$. For example, in the case k' = k, the stratum (18.52) becomes

$$\mathring{\mathcal{M}}(\alpha_{-},\alpha_{1}) \times_{R_{\alpha_{1}}} \mathring{\mathcal{M}}(\alpha_{1},\alpha_{2}) \times_{R_{\alpha_{2}}} \cdots \times_{R_{\alpha_{k-1}}} \mathring{\mathcal{M}}(\alpha_{k-1},\alpha_{+}) \times_{R_{\alpha_{+}}} R_{\alpha_{+}}.$$
 (18.53)

(3) Indeed, (18.52) is isomorphic to

$$\overset{\circ}{\mathcal{M}}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \overset{\circ}{\mathcal{M}}(\alpha_{1}, \alpha_{2}) \times_{R_{\alpha_{2}}} \cdots \times_{R_{\alpha_{k'-1}}} \overset{\circ}{\mathcal{M}}(\alpha_{k'-1}, \alpha_{k'}) \times_{R_{\alpha_{k'}}} \overset{\circ}{\mathcal{M}}(\alpha_{k'}, \alpha_{k'+1}) \times_{R_{\alpha_{k'+1}}} \cdots \times_{R_{\alpha_{k-1}}} \overset{\circ}{\mathcal{M}}(\alpha_{k-1}, \alpha_{+}).$$

We write it as in (18.52) to distinguish the various components, that is actually the same space. However, in the definition of $\overset{\circ}{S}_k(\mathcal{N}(\alpha_-, \alpha_+))$ above, we regard those various components written as (18.52) as different spaces.

(4) Similarly, the space (18.51) is independent of k' = 0, ..., k+1 up to isomorphism. However, we regard them as different spaces in the definition of $\overset{\circ}{S}_k(\mathcal{N}(\alpha_-,\alpha_+))$.

Lemma-Definition 18.60. There exists a morphism from \mathcal{F} to itself whose interpolation space has a stratification described in Definition 18.58. We call the morphism the identity morphism. We can also define the identity morphism in the case of partial linear K-systems in the same way.

Before proving the lemma we give an example.

Example 18.61. Let us consider the case when there is exactly one α such that $E(\alpha_-) < E(\alpha) < E(\alpha_+)$. Then $\mathcal{N}(\alpha_-, \alpha_+)$ is stratified as follows:

$$\mathring{S}_{0}(\mathcal{N}(\alpha_{-}, \alpha_{+})) = \mathring{\mathcal{M}}(\alpha_{-}, \alpha_{+}) \times \mathbb{R}.$$

$$\mathring{S}_{1}(\mathcal{N}(\alpha_{-}, \alpha_{+})) = (\mathring{\mathcal{M}}(\alpha_{-}, \alpha) \times \mathbb{R}) \times_{R_{\alpha}} \mathring{\mathcal{M}}(\alpha, \alpha_{+})$$

$$\bigcup \mathring{\mathcal{M}}(\alpha_{-}, \alpha) \times_{R_{\alpha}} (\mathring{\mathcal{M}}(\alpha, \alpha_{+}) \times \mathbb{R})$$

$$\bigcup R_{\alpha_{-}} \times_{R_{\alpha_{-}}} \mathring{\mathcal{M}}(\alpha_{-}, \alpha_{+})$$

$$\bigcup \mathring{\mathcal{M}}(\alpha_{-}, \alpha_{+}) \times_{R_{\alpha_{+}}} R_{\alpha_{+}}.$$
(18.54)

$$\overset{\circ}{S}_{2}(\mathcal{N}(\alpha_{-}, \alpha_{+})) = R_{\alpha_{-}} \times_{R_{\alpha_{-}}} \overset{\circ}{\mathcal{M}}(\alpha_{-}, \alpha) \times_{R_{\alpha}} \overset{\circ}{\mathcal{M}}(\alpha, \alpha_{+})
\cup \overset{\circ}{\mathcal{M}}(\alpha_{-}, \alpha) \times_{R_{\alpha}} R_{\alpha} \times_{R_{\alpha}} \overset{\circ}{\mathcal{M}}(\alpha, \alpha_{+})
\cup \overset{\circ}{\mathcal{M}}(\alpha_{-}, \alpha) \times_{R_{\alpha}} \overset{\circ}{\mathcal{M}}(\alpha, \alpha_{+}) \times_{R_{\alpha_{+}}} R_{\alpha_{+}}.$$
(18.55)

See Figure 17. Note that in this case $\mathcal{M}(\alpha_{-}, \alpha_{+})$ has a Kuranishi structure with boundary $\mathcal{M}(\alpha_{-}, \alpha) \times_{R_{\alpha}} \mathcal{M}(\alpha, \alpha_{+})$ and without corner. The K-space $\mathcal{N}(\alpha_{-}, \alpha_{+})$ may be regarded as $\mathcal{M}(\alpha_{-}, \alpha_{+}) \times [0, 1]$. However the stratification of $\mathcal{N}(\alpha_{-}, \alpha_{+})$ is different from that of $\mathcal{M}(\alpha_{-}, \alpha_{+}) \times [0, 1]$. Namely we stratify $\mathcal{M}(\alpha_{-}, \alpha_{+}) \times [0, 1]$ as follows:

Its codimension 0 stratum is

(0)
$$\mathcal{M}(\alpha_{-}, \alpha_{+}) \times [0, 1]$$
.

Its codimension 1 strata are

- (1-1) $\mathcal{M}(\alpha_{-}, \alpha_{+}) \times \{0\},\$
- $(1-2) \ \mathcal{M}(\alpha_{-}, \alpha_{+}) \times \{1\},\$
- $(1-3) \ \partial \mathcal{M}(\alpha_{-}, \alpha_{+}) \times [0, 1/2],$
- (1-4) $\partial \mathcal{M}(\alpha_-, \alpha_+) \times [1/2, 1].$

Its codimension 2 strata are

- (2-1) $\partial \mathcal{M}(\alpha_-, \alpha_+) \times \{0\},\$
- $(2-2) \partial \mathcal{M}(\alpha_-, \alpha_+) \times \{1/2\},\$
- (2-3) $\partial \mathcal{M}(\alpha_-, \alpha_+) \times \{1\}.$

The strata (1-1), (1-2), (1-3), (1-4) correspond to the 1st, 2nd, 3rd, 4th terms of (18.54), respectively. The strata (2-1), (2-2), (2-3) correspond to the 1st, 2nd, 3rd terms of (18.55), respectively. Note that $\mathcal{N}(\alpha_-, \alpha_+)$ has an \mathbb{R} -action preserving the

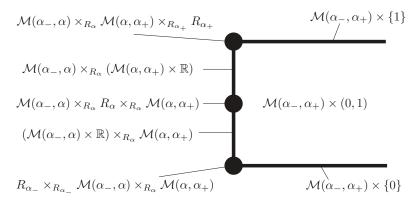


FIGURE 17. $\mathcal{N}(\alpha_-, \alpha_+)$: Case 1

stratification. Namely, on the stratum where there is a (0,1) factor, we identify it with \mathbb{R} and the \mathbb{R} action is the translation on it. The other strata are in the fixed point set of this action.

The quotient space $\mathcal{N}(\alpha_-, \alpha_+)/\mathbb{R}$ is similar to $\mathcal{M}(\alpha_-, \alpha_+)$ but is different therefrom. In the case of our example, $\overset{\circ}{S}_0(\mathcal{N}(\alpha_-, \alpha_+))/\mathbb{R} = \overset{\circ}{S}_0(\mathcal{M}(\alpha_-, \alpha_+)) = \overset{\circ}{\mathcal{M}}(\alpha_-, \alpha_+)$. However $\overset{\circ}{S}_1(\mathcal{N}(\alpha_-, \alpha_+))/\mathbb{R}$ is the union of the disjoint union of 2 copies of $\overset{\circ}{S}_0(\mathcal{M}(\alpha_-, \alpha_+))$ and the disjoint union of two copies of $\overset{\circ}{S}_1(\mathcal{M}(\alpha_-, \alpha_+))$. Moreover $\overset{\circ}{S}_2(\mathcal{N}(\alpha_-, \alpha_+))/\mathbb{R}$ is the disjoint union of three copies of $\overset{\circ}{S}_1(\mathcal{M}(\alpha_-, \alpha_+))$. Then the quotient space $\mathcal{N}(\alpha_-, \alpha_+)/\mathbb{R}$ with quotient topology is non-Hausdorff. At any rate, we do not use the quotient space in this article at all.

When there exist exactly two α and α' with $E(\alpha_{-}) < E(\alpha) < E(\alpha') < E(\alpha_{+})$, the stratification of $\mathcal{N}(\alpha_{-}, \alpha_{+})$ is drawn in Figure 18. We note that in this figure all the codimension two strata (the vertices in Figure 18) are contained in exactly three edges. So its neighborhood can be identified with a corner point.

Proof of Lemma-Definition 18.60. We first explain the way how we glue the strata and define the topology on $\mathcal{N}(\alpha_-, \alpha_+)$. (See Subsubsection 18.10.2 for a geometric origin of the identification (18.56).)

Let E be the energy homomorphism. We will identity

$$\mathcal{N}(\alpha_{-}, \alpha_{+}) \cong \mathcal{M}(\alpha_{-}, \alpha_{+}) \times [E(\alpha_{-}), E(\alpha_{+})] \tag{18.56}$$

as follows. We identify the factor \mathbb{R} in (18.51) with the open interval $(E(\alpha_{k'}), E(\alpha_{k'+1}))$. The fiber product of other factors in (18.51) is

$$\mathring{\mathcal{M}}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \mathring{\mathcal{M}}(\alpha_{1}, \alpha_{2}) \times_{R_{\alpha_{2}}} \cdots \times_{R_{\alpha_{k}}} \mathring{\mathcal{M}}(\alpha_{k}, \alpha_{+})$$
(18.57)

and is a component of $\overset{\circ}{S}_k(\mathcal{M}(\alpha_-,\alpha_+))$. Thus (18.51) is identified with a subset of

$$\overset{\circ}{S}_{k}(\mathcal{M}(\alpha_{-},\alpha_{+}))\times (E(\alpha_{k'}),E(\alpha_{k'+1}))$$

that is a subset of (18.56).

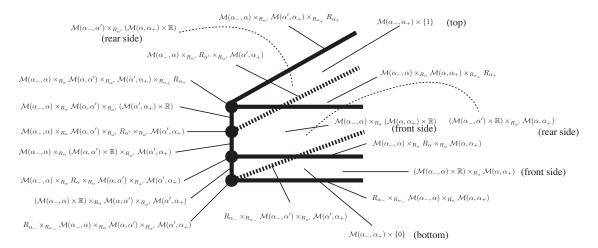


FIGURE 18. $\mathcal{N}(\alpha_-, \alpha_+)$: Case 2

We next consider the fiber product (18.52). This space is isomorphic to (18.57) and so is a component of $\overset{\circ}{S}_k(\mathcal{M}(\alpha_-,\alpha_+))$. We set its $(E(\alpha_-),E(\alpha_+))$ factor as $E(\alpha_{k'})$. Thus (18.52) is a subset of

$$\overset{\circ}{S}_k(\mathcal{M}(\alpha_-, \alpha_+)) \times \{E(\alpha_{k'})\}$$

that is a subset of (18.56).

We have thus specified the way how we embed all the strata of $\mathcal{N}(\alpha_-, \alpha_+)$ into (18.56). It is easy to see that they are disjoint from one another and their union gives (18.56). Thus we identify $\mathcal{N}(\alpha_-, \alpha_+)$ with (18.56) and define the topology on $\mathcal{N}(\alpha_-, \alpha_+)$ using this identification.

As we observed in Example 18.61, the space $\mathcal{N}(\alpha_{-}, \alpha_{+})$ is homeomorphic to $\mathcal{M}(\alpha_{-}, \alpha_{+}) \times [0, 1]$ but its corner structure stratification is different from the direct product. We next define a Kuranishi structure with corners on $\mathcal{N}(\alpha_{-}, \alpha_{+})$ compatible with the stratification by (18.51) and (18.52). We will define a Kuranishi neighborhood of each point of $\mathcal{N}(\alpha_{-}, \alpha_{+})$ below.

Firstly let \hat{p} be a point of (18.51). We write $\hat{p} = (p, s_0)$ where p is an element of (18.57) and $s_0 \in (E(\alpha_{k'}), E(\alpha_{k'+1}))$. Let \mathcal{U}_p be a Kuranishi neighborhood of p. For simplicity of the discussion, we assume that U_p consists of a single orbifold chart V_p/Γ_p . (In the general case we can perform the construction below for each orbifold chart.) Then V_p is an open set of $\overline{V}_p \times [0,1)^k$. We put $\mathcal{U}_p \times (s_0 - \epsilon, s_0 + \epsilon)$ as our Kuranishi neighborhood of \hat{p} . In fact, the parametrization map $s_{\hat{p}}^{-1}(0) \to \mathcal{N}(\alpha_-, \alpha_+)$ is defined as follows. Let $(q, s) \in s_{\hat{p}}^{-1}(0)$. Then q parametrizes certain point of $\mathcal{M}(\alpha_-, \alpha_+)$. Hence (q, s) defines a point in (18.56). Therefore we obtain $\psi_{\hat{p}}(q, s) \in \mathcal{N}(\alpha_-, \alpha_+)$ by our identification.

Next we consider the case \hat{p} is in (18.52). First we suppose k' = k. Then (18.52) coincides with (18.53) that is identified with a subset of

$$\overset{\circ}{S}_{k}(\alpha_{-},\alpha_{+}) \times \{E(\alpha_{+})\} \subset \overset{\circ}{S}_{k}(\alpha_{-},\alpha_{+}) \times \partial [E(\alpha_{-}),E(\alpha_{+})].$$

We take $\mathcal{U}_p \times (E(\alpha_+) - \epsilon, E(\alpha_+)]$ as our Kuranishi neighborhood $\mathcal{U}_{\hat{p}}$. Here \mathcal{U}_p is a Kuranishi neighborhood of p in $\mathcal{M}(\alpha_-, \alpha_+)$. The definition of the parametrization map is similar to the first case. The case k' = 0 is similar to the above case.

The case $k' \neq k$ and $k' \neq 0$ is the most involved, which we discuss now. We will construct a Kuranishi neighborhood of $\hat{p} = (p, E(\alpha_{k'}))$. Let \mathcal{U}_p be the Kuranishi neighborhood of p in (18.57). Put $p = (p_0, p_1, \ldots, p_k)$ according to this fiber product and take a Kuranishi neighborhood \mathcal{U}_{p_i} of p_i in $\mathcal{M}(\alpha_i, \alpha_{i+1})$. Then the orbifold chart of \mathcal{U}_p is given by

$$U_{p_0} \times_{R_{\alpha_1}} \times \cdots \times_{R_{\alpha_k}} U_{p_k} \times [0, \epsilon)^k$$
.

We write its element as $(x_0, \ldots, x_k; (t_1, \ldots, t_k))$. We note that the triple $(x_{k'-1}, x_{k'}, t_{k'})$ parametrizes a Kuranishi neighborhood of $(p_{k'-1}, p_{k'})$ in $\mathcal{M}(\alpha_{k'-1}, \alpha_{k'+1})$.

For $s' \in (E(\alpha_{k'}) - \epsilon, E(\alpha_{k'}) + \epsilon)$ we write $(t, s) = (t_{k'}, s' - E(\alpha_{k'}))$ and change variables from (t, s) to $(\rho, \sigma) \in [0, 1)^2$ by

$$t = \rho \sigma, \qquad s = \rho - \sigma.$$

In other words,

$$\rho = \frac{s + \sqrt{s^2 + 4t}}{2}, \qquad \sigma = \frac{-s + \sqrt{s^2 + 4t}}{2}.$$
 (18.58)

We use $x_0, \ldots, x_k, t_1, \ldots, t_{k'-1}, t_{k'+1}, \ldots, t_k$ and ρ, σ as the coordinates of

$$U_{p_0} \times_{R_{\alpha_1}} \times \dots \times_{R_{\alpha_k}} U_{p_k} \times [0, \epsilon)^k \times (E(\alpha_{k'}) - \epsilon, E(\alpha_{k'}) + \epsilon). \tag{18.59}$$

This gives a smooth structure on (18.59). In fact, the point \hat{p} lies in the codimension k+1 corner in this smooth structure which is different from the direct product smooth structure on (18.59). (We note that \hat{p} lies in the codimension k corner with respect to the direct product smooth structure.)

To complete the definition of the Kuranishi structure of $\mathcal{N}(\alpha_-, \alpha_+)$ it suffices to show that the coordinate change is admissible in this last case. (Admissibility of the coordinate change is trivial in other cases.) We will check it below.

We denote by y the totality of the coordinates $x_0, \ldots, x_k, t_1, \ldots, t_{k'-1}, t_{k'+1}, \ldots, t_k$. Then y, t, s or y, ρ, σ are the coordinates of our Kuranishi neighborhood. Let y', t', s' or y', ρ', σ' be the other coordinates. Then the coordinate change among them is given by

$$y' = \varphi(y, t), \quad t' = \psi(y, t), \quad s' = s,$$

where φ and ψ are smooth and satisfy

$$\left\| \frac{\partial y'}{\partial t} \right\|_{C^k} \le C_k e^{-c_k/t}, \quad \|t' - t\|_{C^k} \le C_k e^{-c_k/t}$$

for some $c_k > 0$, $C_k > 0$. (See Definition 25.7, Lemma 25.9 (2) and (25.6).) We note that y' and t' are independent of s. Then using (18.58) it is easy to check

$$\left\| \frac{\partial y'}{\partial \rho} \right\|_{C^k} \le C_k e^{-c_k/\rho}, \quad \left\| \frac{\partial y'}{\partial \sigma} \right\|_{C^k} \le C_k e^{-c_k/\sigma}$$

and

$$\|\rho' - \rho\|_{C^k} \le C_k e^{-c_k/\rho}, \quad \|\sigma' - \sigma\|_{C^k} \le C_k e^{-c_k/\sigma}.$$

This implies admissibility of the coordinate change with respect to our new coordinates. Thus we have constructed a Kuranishi structure on $\mathcal{N}(\alpha_-, \alpha_+)$.

We define evaluation maps ev_{\pm} on $\mathcal{N}(\alpha_{-}, \alpha_{+})$ by taking one on the factor where α_{-} or α_{+} appears. The periodicity isomorphism and orientation isomorphism are induced by ones on $\mathcal{M}(\alpha_{-}, \alpha_{+})$ in an obvious way.

We note that in (18.51) the factor $\mathring{\mathcal{M}}(\alpha_{k'}, \alpha_{k'+1}) \times \mathbb{R}$ can be identified with $\mathring{\mathcal{N}}(\alpha_{k'}, \alpha_{k'+1})$ and in (18.52) the factor $R_{\alpha_{k'}}$ can be identified with $\mathring{\mathcal{N}}(\alpha_{-}, \alpha_{+})$. The fact that $\mathcal{N}(\alpha_{-}, \alpha_{+})$ satisfies Conditions 16.23 and 16.28 is a consequence of the compatibility condition at the boundary and the corner of $\mathcal{M}(\alpha_{-}, \alpha_{+})$. The proof of Lemma-Definition 18.60 is now complete.

Remark 18.62. In the situation where we obtain a linear K-system from the Floer equation of periodic Hamiltonian system, morphisms among them are obtained by studying the two parameter family of Hamiltonians $H: \mathbb{R} \times S^1 \times M \to \mathbb{R}$, $H_{\tau,t}(x) = H(\tau,t,x)$. In that case the interpolation space is the compactified moduli space of solutions of the equation

$$\frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_{H_{\tau,t}}(u)\right) = 0. \tag{18.60}$$

See [FOOO12, Section 9]. In the case when $H_{\tau,t} = H_t$ is τ independent, the morphism we obtain becomes the identity morphism defined above. (See Subsubsection 18.10.1.)

The next proposition shows the identity morphism \mathfrak{ID} is a homotopy unit with respect to the composition of morphisms.

Proposition 18.63. For any morphism \mathfrak{N} , the compositions $\mathfrak{N} \circ \mathfrak{ID}$ and $\mathfrak{ID} \circ \mathfrak{N}$ are both homotopic to \mathfrak{N} .

Proof. We will prove $\mathfrak{N} \circ \mathfrak{ID} \sim \mathfrak{N}$. The proof of $\mathfrak{ID} \circ \mathfrak{N} \sim \mathfrak{N}$ is similar. Let $\mathcal{N}(\alpha_-, \alpha_+)$ be an interpolation space of \mathfrak{N} . By the definition of the composition $\mathfrak{N} \circ \mathfrak{ID}$, its interpolation space is decomposed into the following two types of fiber products:

(1)

$$\mathring{\mathcal{M}}^{1}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \mathring{\mathcal{M}}^{1}(\alpha_{1}, \alpha_{2}) \times_{R_{\alpha_{2}}} \cdots \times_{R_{\alpha_{k'_{1}-1}}} \mathring{\mathcal{M}}^{1}(\alpha_{k'_{1}-1}, \alpha_{k'_{1}})
\times_{R_{\alpha_{k'_{1}}}} (\mathring{\mathcal{M}}^{1}(\alpha_{k'_{1}}, \alpha_{k'_{1}+1}) \times \mathbb{R})
\times_{R_{\alpha_{k'_{1}+1}}} \mathring{\mathcal{M}}^{1}(\alpha_{k'_{1}+1}, \alpha_{k'_{1}+2}) \times_{R_{\alpha_{k'_{1}+2}}} \cdots \times_{R_{\alpha_{k_{1}-1}}} \mathring{\mathcal{M}}^{1}(\alpha_{k_{1}-1}, \alpha_{k_{1}})
\times_{R_{\alpha_{k_{1}}}} \mathring{\mathcal{N}}^{1}(\alpha_{k_{1}}, \alpha'_{1})
\times_{R_{\alpha'_{1}}} \mathring{\mathcal{N}}^{2}(\alpha'_{1}, \alpha'_{2}) \times_{R_{\alpha'_{2}}} \cdots \times_{R_{\alpha'_{k_{2}}}} \mathring{\mathcal{M}}^{2}(\alpha'_{k_{2}}, \alpha'_{+}).$$
(18.61)

(2)
$$\mathring{\mathcal{M}}^{1}(\alpha_{-},\alpha_{1}) \times_{R_{\alpha_{1}}} \mathring{\mathcal{M}}^{1}(\alpha_{1},\alpha_{2}) \times_{R_{\alpha_{2}}} \cdots \times_{R_{\alpha_{k'_{1}-1}}} \mathring{\mathcal{M}}^{1}(\alpha_{k'_{1}-1},\alpha_{k'_{1}}) \\
\times_{R_{\alpha_{k'_{1}}}} R_{\alpha_{k'_{1}}} \\
\times_{R_{\alpha_{k'_{1}}}} \mathring{\mathcal{M}}^{1}(\alpha_{k'_{1}},\alpha_{k'_{1}+1}) \times_{R_{\alpha_{k'_{1}+1}}} \cdots \times_{R_{\alpha_{k_{1}-1}}} \mathring{\mathcal{M}}^{1}(\alpha_{k_{1}-1},\alpha_{k_{1}}) \\
\times_{R_{\alpha_{k_{1}}}} \mathring{\mathcal{N}}^{1}(\alpha_{k_{1}},\alpha'_{1}) \\
\times_{R_{\alpha_{k_{1}}}} \mathring{\mathcal{N}}^{2}(\alpha'_{1},\alpha'_{2}) \times_{R_{\alpha'_{2}}} \cdots \times_{R_{\alpha'_{k_{2}}}} \mathring{\mathcal{M}}^{2}(\alpha'_{k_{2}},\alpha'_{+}). \tag{18.62}$$

Note that the process of gluing, described in the proof of Lemma-Definition 16.35, is included in this description. Namely, the fiber products (18.61), (18.62) appear only once in this decomposition. We also note that $\mathcal{N}(\alpha_-, \alpha'_+)$, that is an interpolation space of \mathfrak{N} , is decomposed into

$$\overset{\circ}{\mathcal{M}}^{1}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \cdots \times_{R_{\alpha_{k_{1}-1}}} \overset{\circ}{\mathcal{M}}^{1}(\alpha_{k_{1}-1}, \alpha_{k_{1}})$$

$$\times_{R_{\alpha_{k_{1}}}} \overset{\circ}{\mathcal{N}}(\alpha_{k_{1}}, \alpha'_{1}) \qquad (18.63)$$

$$\times_{R_{\alpha'_{1}}} \overset{\circ}{\mathcal{M}}^{2}(\alpha'_{1}, \alpha'_{2}) \times_{R_{\alpha'_{2}}} \cdots \times_{R_{\alpha'_{k_{2}}}} \overset{\circ}{\mathcal{M}}^{2}(\alpha'_{k_{2}}, \alpha'_{+}).$$

By Definition 16.30 of homotopy, it suffices to find a K-space whose boundary is a union of (18.61), (18.62), (18.63) and the obvious component $\mathcal{N}(\alpha_-, \alpha_+; \partial[0, 1])$ corresponding to the last line in (16.31). (We omit the last obvious component here.)

The interpolation space of our homotopy between $\mathfrak{N} \circ \mathfrak{ID}$ and \mathfrak{N} is stratified so that its strata are (18.61), (18.62), (18.63) and the following spaces (18.64), (18.65):

$$\mathring{\mathcal{M}}^{1}(\alpha_{-},\alpha_{1}) \times_{R_{\alpha_{1}}} \cdots \times_{R_{\alpha_{k_{1}-1}}} \mathring{\mathcal{M}}^{1}(\alpha_{k_{1}-1},\alpha_{k_{1}})$$

$$\times_{R_{\alpha_{k_{1}}}} \mathring{\mathcal{N}}(\alpha_{k_{1}},\alpha'_{1}) \times (0,1) \qquad (18.64)$$

$$\times_{R_{\alpha'_{1}}} \mathring{\mathcal{M}}^{2}(\alpha'_{1},\alpha'_{2}) \times_{R_{\alpha'_{2}}} \cdots \times_{R_{\alpha'_{k_{2}}}} \mathring{\mathcal{M}}^{2}(\alpha'_{k_{2}},\alpha'_{+}),$$

$$\mathring{\mathcal{M}}^{1}(\alpha_{-},\alpha_{1}) \times_{R_{\alpha_{1}}} \cdots \times_{R_{\alpha_{k_{1}-1}}} \mathring{\mathcal{M}}^{1}(\alpha_{k_{1}-1},\alpha_{k_{1}})$$

$$\times_{R_{\alpha_{k_{1}}}} \mathring{\mathcal{N}}(\alpha_{k_{1}},\alpha'_{1}) \times \{0,1\}$$

$$\times_{R_{\alpha'_{1}}} \mathring{\mathcal{N}}^{2}(\alpha'_{1},\alpha'_{2}) \times_{R_{\alpha'_{2}}} \cdots \times_{R_{\alpha'_{k_{2}}}} \mathring{\mathcal{M}}^{2}(\alpha'_{k_{2}},\alpha'_{+}).$$

$$(18.65)$$

We claim that the union of (18.61), (18.62), (18.63) and (18.64), (18.65) above has a Kuranishi structure with corner. The proof follows.

We first define a topology on the disjoint union $\mathcal{N}(\alpha_{-}, \alpha'_{+}; [0, 1])$ of the spaces (18.61) - (18.65). Let c be the energy loss of \mathfrak{N} . We identify the underlying topological space of $\mathcal{N}(\alpha_{-}, \alpha'_{+}; [0, 1])$ with

$$\mathcal{N}(\alpha_{-}, \alpha'_{+}) \times [E(\alpha_{-}), E(\alpha'_{+}) + c]. \tag{18.66}$$

See Subsubsection 18.10.4 for the geometric origin of this identification. We identify (18.61) - (18.65) to a subset of (18.66) as follows.

(The case of (18.61)): We identify the \mathbb{R} factor with $(E(\alpha_{k'_1}), E(\alpha_{k'_1+1}))$. The fiber product of the other factors is

$$\mathring{\mathcal{M}}^{1}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \mathring{\mathcal{M}}^{1}(\alpha_{1}, \alpha_{2}) \times_{R_{\alpha_{2}}} \cdots \times_{R_{\alpha_{k_{1}-1}}} \mathring{\mathcal{M}}^{1}(\alpha_{k'_{1}-1}, \alpha_{k'_{1}})
\times_{R_{\alpha_{k_{1}}}} \mathring{\mathcal{N}}(\alpha_{k_{1}}, \alpha'_{1})
\times_{R_{\alpha'_{1}}} \mathring{\mathcal{M}}^{2}(\alpha'_{1}, \alpha'_{2}) \times_{R_{\alpha'_{2}}} \cdots \times_{R_{\alpha'_{k_{n}}}} \mathring{\mathcal{M}}^{2}(\alpha'_{k_{2}}, \alpha'_{+}),$$
(18.67)

which is a subset of $\mathcal{N}(\alpha_-, \alpha'_+)$. Thus (18.61) is identified with the subset of (18.66). (The case of (18.62)): (18.62) is the same as (18.67) and so is a subset of $\mathcal{N}(\alpha_-, \alpha'_+)$. We take $E(\alpha'_{k_1})$ as the $[E(\alpha), E(\alpha') + c]$ factor and regard (18.62) as a subset of (18.66).

(The case of (18.63)): (18.63) is again the same as (18.67) and so is a subset of $\mathcal{N}(\alpha_-, \alpha'_+)$. We take $E(\alpha') + c$ as the $[E(\alpha), E(\alpha') + c]$ factor and regard (18.63) as a subset of (18.66).

(The case of (18.64) \cup (18.65)): We identify [0,1] with $[E(\alpha_{k_1}), E(\alpha') + c]$. The fiber product of the other factors is the same as (18.67). Thus (18.64) is identified with a subset of (18.66).

Thus we have identified the union $\mathcal{N}(\alpha_-, \alpha'_+; [0, 1])$ of (18.61) - (18.65) with (18.66). Using this identification we define the topology of $\mathcal{N}(\alpha_-, \alpha'_+; [0, 1])$.

In a way similar to the proof of Lemma-Definition 18.60 we define a Kuranishi structure on $\mathcal{N}(\alpha_-, \alpha'_+; [0, 1])$ as follows. We first observe that codimension 1 strata of $\mathcal{N}(\alpha_-, \alpha'_+; [0, 1])$ are one of the following four types of fiber products:

$$(\mathring{\mathcal{M}}^{1}(\alpha_{-},\alpha_{1}) \times \mathbb{R}) \times_{R_{\alpha_{1}}} \mathring{\mathcal{N}}(\alpha_{1},\alpha'_{+}), \tag{18.68}$$

$$\stackrel{\circ}{\mathcal{M}}^{1}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} (\stackrel{\circ}{\mathcal{N}}(\alpha_{1}, \alpha'_{+}) \times (0, 1)), \tag{18.69}$$

$$\mathring{\mathcal{N}}(\alpha_{-}, \alpha'_{+}) \times \{0\}, \tag{18.70}$$

$$\overset{\circ}{\mathcal{N}}(\alpha_{-}, \alpha'_{+}) \times \{1\}. \tag{18.71}$$

The \mathbb{R} factor in (18.68) is identified with $(E(\alpha_{-}), E(\alpha_{1}))$. The (0,1) factor in (18.69) is identified with $(E(\alpha_{1}), E(\alpha'_{+}) + c)$. They are glued where these factors are identified with $E(\alpha_{1})$. The union is identified with

$$(\mathring{\mathcal{M}}^{1}(\alpha_{-},\alpha_{1}) \times_{R_{\alpha_{1}}} \mathring{\mathcal{N}}(\alpha_{1},\alpha'_{+})) \times (E(\alpha_{-}),E(\alpha'_{+})+c)). \tag{18.72}$$

However we regard the subset

$$(\mathring{\mathcal{M}}^{1}(\alpha_{-}, \alpha_{1}) \times_{R\alpha_{1}} \mathring{\mathcal{N}}(\alpha_{1}, \alpha'_{+})) \times \{E(\alpha_{1})\}$$

$$(18.73)$$

of (18.72) not as a codimension 2 stratum, that is, a corner. In other words, we bend the space (18.72) at (18.73). This bending is performed in the same way as the argument of case $k' \neq 0$, k in the proof of Lemma-Definition 18.60.

The point $\{0\}$ in (18.70) is identified with $E(\alpha_{-})$. Therefore the closures of (18.70) and of (18.72) intersect at the codimension 2 stratum

$$(\stackrel{\circ}{\mathcal{M}}^{1}(\alpha_{-},\alpha_{1})\times_{R_{\alpha_{1}}} \stackrel{\circ}{\mathcal{N}}(\alpha_{1},\alpha'_{+}))\times \{E(\alpha_{-})\}.$$

We smooth this corner in the way we explain Subsection 18.6. In fact, this smoothing occurs during the definition of the composition $\mathfrak{N} \circ \mathfrak{ID}$. (See Definition 18.37.)

The point $\{1\}$ in (18.71) is identified with $E(\alpha'_+)$. Therefore, the closures of (18.71) and of (18.72) intersect at the codimension 2 stratum

$$(\mathring{\mathcal{M}}^1(\alpha_-, \alpha_1) \times_{R_{\alpha_1}} \mathring{\mathcal{N}}(\alpha_1, \alpha'_+)) \times \{E(\alpha'_+)\}.$$

We do not smooth this corner.

We can perform an appropriate bending or smoothing and check the consistency at the corner of higher codimension by using the compatibility conditions of $\mathcal{M}^i(\alpha_1, \alpha_2)$, (i = 1, 2) and of $\mathcal{N}(\alpha, \alpha')$.

We define a map $\mathcal{N}(\alpha_-, \alpha'_+; [0, 1]) \to [0, 1]$ appearing in Condition 16.20 (IV), so that the inverse image of $\{0\}$ is the closure of the union of the strata (18.68) and (18.70). The inverse image of $\{1\}$ is the closure of the union of the strata (18.71). We note that this map is *not* diffeomorphic to the projection

$$\mathcal{N}(\alpha_-,\alpha_+';[0,1]) \cong \mathcal{N}(\alpha_-,\alpha_+') \times [E(\alpha_-),E(\alpha_+')+c] \rightarrow [E(\alpha_-),E(\alpha_+')+c] \rightarrow [0,1],$$

where the first identification is one we used to define the topology of $\mathcal{N}(\alpha_-, \alpha'_+; [0, 1])$ by identifying it with (18.66).

We are now ready to wrap up the proof of Proposition 18.63. So far we have defined a K-space $\mathcal{N}(\alpha_-, \alpha'_+; [0, 1])$. It has most of the properties of the interpolation space of the homotopy. In other words, its boundary has mostly the required properties, except the boundary is related to the composition $\mathfrak{N} \circ \mathfrak{ID}$ before trivialization of the boundary and smoothing the corner took place. Therefore to obtain the required interpolation space of homotopy we perform partial trivialization of the corner and smoothing corner at those trivialized corners. (See Subsection 18.6.) Namely the required interpolation space of the homotopy is

$$\mathcal{N}(\alpha_-, \alpha'_+; [0, 1])^{\mathfrak{C} \boxplus \tau}$$

where \mathfrak{C} is the part of the boundary corresponding to $\{1\} \in [0,1]$ in the factor [0,1]. Thus we obtain an interpolation space of the required [0,1] parametrized morphism and complete the proof of Proposition 18.63.

Example 18.64. We consider the case when $\partial \mathcal{N}(\alpha_{-}, \alpha_{+})$ has only one component $\mathcal{M}(\alpha_{-}, \alpha) \times_{R_{\alpha}} \mathcal{N}(\alpha, \alpha_{+})$. Then the top stratum of our homotopy is $\mathcal{N}(\alpha_{-}, \alpha_{+}) \times [0, 1]$. There are 4 codimension 1 strata that are $\mathcal{N}(\alpha_{-}, \alpha_{+})$, $R_{\alpha_{-}} \times_{R_{\alpha_{-}}} \mathcal{N}(\alpha_{-}, \alpha_{+})$, $(\mathcal{M}(\alpha_{-}, \alpha) \times \mathbb{R}) \times_{R_{\alpha}} \mathcal{N}(\alpha, \alpha_{+})$, and $\mathcal{M}(\alpha_{-}, \alpha) \times_{R_{\alpha}} (\mathcal{N}(\alpha, \alpha_{+}) \times [0, 1])$. The first one corresponds to the morphism \mathfrak{N} . The union of the second and the third corresponds to the morphism $\mathfrak{N} \circ \mathfrak{D}$. (See Figure 19.)

Example 18.65. Let us consider the case when there are α_1 and α_2 such that

$$\partial \mathcal{N}(\alpha_-,\alpha_+) = \mathcal{M}(\alpha_-,\alpha_1) \times_{R_{\alpha_1}} \mathcal{N}(\alpha_1,\alpha_+) \cup \mathcal{M}(\alpha_-,\alpha_2) \times_{R_{\alpha_2}} \mathcal{N}(\alpha_2,\alpha_+)$$

and the two components in the right hand side are glued at the corner $\mathcal{M}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \times \mathcal{N}(\alpha_{1}, \alpha_{2}) \times_{R_{\alpha_{2}}} \mathcal{N}(\alpha_{2}, \alpha_{+})$. Then our homotopy has one top stratum $\mathcal{N}(\alpha, \alpha_{+}) \times$

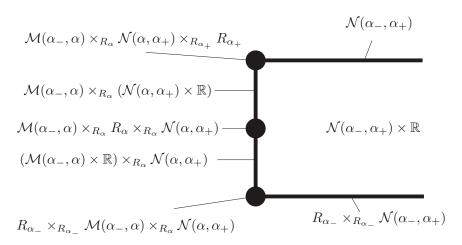


FIGURE 19. Homotopy between $\mathfrak{N} \circ \mathfrak{ID}$ and \mathfrak{N} : Case 1

(0,1) and 6 codimension 1 strata. Those 6 strata are

$$R_{\alpha_{-}} \times_{R_{\alpha_{-}}} \mathcal{N}(\alpha_{-}, \alpha_{+}),$$

$$(\mathcal{M}(\alpha_{-}, \alpha_{1}) \times \mathbb{R}) \times_{R_{\alpha_{1}}} \mathcal{N}(\alpha_{1}, \alpha_{+}),$$

$$(\mathcal{M}(\alpha_{-}, \alpha_{2}) \times \mathbb{R}) \times_{R_{\alpha_{2}}} \mathcal{N}(\alpha_{2}, \alpha_{+}),$$

$$\mathcal{M}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} (\mathcal{N}(\alpha_{1}, \alpha_{+}) \times (0, 1)),$$

$$\mathcal{M}(\alpha_{-}, \alpha_{2}) \times_{R_{\alpha_{1}}} (\mathcal{N}(\alpha_{2}, \alpha_{+}) \times (0, 1)),$$

$$\mathcal{N}(\alpha_{-}, \alpha_{+}).$$

The first 3 strata consist $\mathfrak{N} \circ \mathfrak{ID}$ and the last stratum is \mathfrak{N} . See Figure 20. We note that each vertex in the figure is contained in exactly three edges. So this configuration is one of manifold with boundary. (However, note that the boundaries in the interior of 3 strata consisting $\mathfrak{N} \circ \mathfrak{ID}$ are smooth during the definition of the composition.)

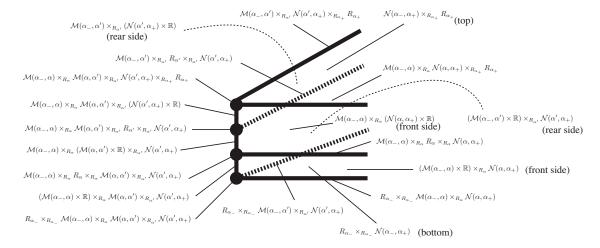


FIGURE 20. Homotopy between $\mathfrak{N} \circ \mathfrak{ID}$ and \mathfrak{N} : Case 2

Remark 18.66. The proof of Proposition 18.63 seems rather complicated. We can prove Proposition 18.63 for the case appearing in our geometric application, in a more intuitive way. See Subsection 18.10.

In the rest of this subsection, we describe an alternative way to define the identity morphism \mathfrak{ID} and a homotopy between $\mathfrak{N} \circ \mathfrak{ID}$ and \mathfrak{N} . The method we explain below is similar to that of [FOOO3, Subsection 4.6.1] in defining an A_{∞} homomorphism, where we used *time ordered fiber products*. We put

$$C_k(\mathbb{R}) = \{ (\tau_1, \dots, \tau_k) \in (\mathbb{R} \cup \{\pm \infty\})^k \mid \tau_1 \le \dots \le \tau_k \}.$$

When $\alpha_{-} \neq \alpha_{+}$, we consider the union of

$$\mathcal{M}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \mathcal{M}(\alpha_{1}, \alpha_{2}) \times_{R_{\alpha_{2}}} \cdots \times_{R_{\alpha_{k-1}}} \mathcal{M}(\alpha_{k-1}, \alpha_{+}) \times C_{k}(\mathbb{R})$$
 (18.74)

over various k. When $\alpha_- = \alpha_+ = \alpha$, we consider R_α instead of (18.74). We glue them as follows. For the point $(\tau_1, \ldots, \tau_k) \in C_k(\mathbb{R})$ with $\tau_i = \tau_{i+1}$, we put

$$(\tau_1, \ldots, \tau_i, \tau_{i+2}, \ldots, \tau_k) = (\tau'_1, \ldots, \tau'_{k-1}).$$

We also consider the embedding

$$I_{i}: \mathcal{M}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \cdots \times_{R_{\alpha_{k-1}}} \mathcal{M}(\alpha_{k-1}, \alpha_{+})$$

$$\subset \mathcal{M}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \cdots \mathcal{M}(\alpha_{i}, \alpha_{i+2}) \cdots \times_{R_{\alpha_{k-1}}} \mathcal{M}(\alpha_{k-1}, \alpha_{+})$$

which is induced by the inclusion map

$$\mathcal{M}(\alpha_i, \alpha_{i+1}) \times_{R_{\alpha_{i+1}}} \mathcal{M}(\alpha_{i+1}, \alpha_{i+2}) \subset \partial \mathcal{M}(\alpha_i, \alpha_{i+2}) \subset \mathcal{M}(\alpha_i, \alpha_{i+2}).$$

We now identify

$$(\mathfrak{x}, (t_1, \dots, t_k)) \sim (I_i(\mathfrak{x}), (t'_1, \dots, t'_{k-1})).$$

Under this identification we obtain a K-space with corners, which we take as an interpolation space $\mathcal{ID}(\alpha_-, \alpha_+)$ of our morphism \mathfrak{ID} . (We actually need to smooth the corner for this purpose.)

Note that we consider the part where $\tau_i = -\infty$, i = 1, ..., k to find

$$\widehat{S}_k(\mathcal{ID}(\alpha_-, \alpha_+))$$

$$\supset \mathcal{M}(\alpha_-, \alpha_1) \times_{R_{\alpha_1}} \cdots \times_{R_{\alpha_{k-1}}} \mathcal{M}(\alpha_{k-1}, \alpha_k) \times_{R_{\alpha_k}} \mathcal{ID}(\alpha_k, \alpha_+).$$

We also have a similar embedding in the case when $\tau_i = +\infty$ and consider both of them. Then we find that \mathfrak{ID} has the required structure at the corners.

To construct a homotopy from $\mathfrak{N}\circ\mathfrak{ID}$ to \mathfrak{N} we consider the union of

$$\mathcal{M}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \mathcal{M}(\alpha_{1}, \alpha_{2}) \times_{R_{\alpha_{2}}} \cdots \times_{R_{\alpha_{k-1}}} \mathcal{N}(\alpha_{k-1}, \alpha_{+}) \times C_{k}(\mathbb{R})$$
 (18.75)

over various k and identify them in a similar way. Here $\tau_k \in \mathbb{R} \cup \{\pm \infty\} \cong [1,2]$ defines a map to [1,2]. We can prove that this space is an interpolation space of a [1,2]-parametrized family of morphisms and gives a homotopy from \mathfrak{N} to $\mathfrak{N} \circ \mathfrak{ID}$.

Remark 18.67. The second method explained above seems to be shorter. (To work out the detail of the second map, we need to glue several K-spaces to obtain the required K-space. We omit this detail.) Here however we present the first one in detail since it is directly tied to the geometric situation appearing in the Floer theory of periodic Hamiltonian system as we mentioned in Remark 18.62 and explained in Subsection 18.10.

The identity morphism obtained by the second method also appears in geometric situation in a different case. For example, when we consider two Lagrangian

submanifolds and Floer cohomology of their intersection. Let us assume there is no pseudoholomorphic disk bounding one of those two Lagrangian submanifolds. Then we can define Lagrangian intersection Floer theory as Floer did. ([Fl1]. See also [FOOO3, Chapter 2].) When we prove the independence of the choice of compatible almost complex structures, we consider a one-parameter family of moduli spaces $\mathcal{M}(\alpha_1,\alpha_2;J_\tau)$ where α_1,α_2 are connected components of the intersections. From this we can construct a similar moduli space as (18.74) which provides a required cochain homotopy. In case if the family J_τ is the trivial family, it boils down to the identity morphism constructed by the second method.

Remark 18.68. There is an alternative way to prove well-definedness of Floer cohomology in the situation when R_{α} does not vary. We call it the bifurcation method in [FOOO4, Subsection 7.2.14]. The method to use morphisms which is taken in this section corresponds to the one which we called the cobordism method there. There is a way to translate the bifurcation method to the cobordism method, which is explained in [Fu2, Proposition 9.1, Remark 12.3]. When we start from the bifurcation method and translate it to the cobordism method, the time ordered fiber product appears. Then we end up with the same isomorphism as one we obtain by the alternative proof.

18.10. Geometric origin of the definition of the identity morphism. In this subsection we explain the geometric background of our definition of the identity morphism given in Subsection 18.9. We assume the reader is familiar with the construction of Floer cohomology of periodic Hamiltonian system such as the one given in [Fl2]. Since the content of this subsection is never used in the proof of the results of this article, the reader can skip this subsection if he/she prefers.

18.10.1. Interpolation space of the identity morphism. As we mentioned in Remark 18.62, when we study Floer cohomology of periodic Hamiltonian system on a symplectic manifold M, we define a morphism between two linear K-systems associated to $H: S^1 \times M \to \mathbb{R}$ and to $H': S^1 \times M \to \mathbb{R}$ by considering the homotopy

$$H_{\tau,t}(x) = H(\tau,t,x) : \mathbb{R} \times S^1 \times M \to \mathbb{R},$$

where $H(\tau,t,x)=H(t,x)$ for τ sufficiently small and $H(\tau,t,x)=H'(t,x)$ for τ sufficiently large, and the interpolation space of the morphism is the compactified moduli space of the solutions of the equation (18.60). Note that (18.60) is *not* invariant under translation of the $\tau \in \mathbb{R}$ direction. We denote by $\mathcal{M}(H_{\tau,t}; [\gamma_-, w_-], [\gamma_-, w_+])$ the moduli space of solutions of (18.60) with asymptotic boundary condition

$$\lim_{\tau \to -\infty} u(\tau, t) = \gamma_{-}(t), \quad \lim_{\tau \to +\infty} u(\tau, t) = \gamma_{+}(t).$$

(See Condition 15.2.) Moreover we assume $[w_-\#u] = [w_+]$. Here γ_-, γ_+ are the periodic orbits of the periodic Hamiltonian system associated to H and H' respectively, and $w_-:(D^2,\partial D^2)\to (M,\gamma_-),\ w_+:(D^2,\partial D^2)\to (M,\gamma_+)$ are disks bounding them. We denote by $[\gamma_\pm,w_\pm]$ its homology class.

We consider the Bott-Morse case. Namely the set of $[\gamma_-, w_-]$ etc. consists of (possibly positive dimensional) smooth manifolds $\{R_\alpha\}_\alpha$. We put

$$\mathcal{M}(H_{\tau,t};\alpha_{-},\alpha'_{+}) := \bigcup_{[\gamma_{-},w_{-}] \in R_{\alpha_{-}}, [\gamma_{+},w_{+}] \in R_{\alpha'_{+}}} \mathcal{M}(H_{\tau,t};[\gamma_{-},w_{-}], [\gamma_{-},w_{+}]),$$

that is the interpolation space of the morphisms. Then the codimension k corner of $\mathcal{M}(H_{\tau,t};\alpha_-,\alpha'_+)$ is described by the union of

$$\mathcal{M}(H; \alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \cdots \times_{R_{\alpha_{k_{1}-1}}} \mathcal{M}(H; \alpha_{k_{1}-1}, \alpha_{k_{1}})$$

$$\times_{R_{\alpha_{k_{1}}}} \mathcal{M}(H_{\tau, t}; \alpha_{k_{1}}, \alpha'_{1})$$

$$\times_{R_{\alpha'_{1}}} \mathcal{M}(H'; \alpha'_{1}, \alpha'_{2}) \times_{R_{\alpha'_{2}}} \cdots \times_{R_{\alpha'_{k_{\alpha}}}} \mathcal{M}(H'; \alpha'_{k_{2}}, \alpha'_{+})$$

$$(18.76)$$

with $k_1 + k_2 = k$. Here the moduli space $\mathcal{M}(H; \alpha_i, \alpha_{i+1})$ is the set of solutions of the Floer equation

$$\frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_{H_t}(u)\right) = 0 \tag{18.77}$$

with asymptotic boundary condition given by R_{α_i} , $R_{\alpha_{i+1}}$. We note that we divide the set of solutions by the \mathbb{R} action defined by translation in the τ direction to obtain $\mathcal{M}(H;\alpha_i,\alpha_{i+1})$. The space $\mathcal{M}(H';\alpha_i',\alpha_{i+1}')$ is defined in a similar way replacing H by H'.

To define the identity morphism, we consider the case when $H=H^\prime$ and take the trivial homotopy. Namely,

$$H(\tau, t, x) \equiv H(t, x).$$

Then the equation (18.60) is exactly the same as (18.77). However, there is one important difference between them: When we consider the interpolation space of the morphism $\mathcal{M}(H_{\tau,t}; \alpha_-, \alpha_+)$ with $H_{\tau,t} = H_t$, we do *not* divide the moduli space by \mathbb{R} action, even though in the case there is an \mathbb{R} action. In other words, we have an isomorphism:

$$\overset{\circ}{\mathcal{M}}(H_{\tau,t};\alpha_{-},\alpha_{+}) = \overset{\circ}{\mathcal{M}}(H;\alpha_{-},\alpha_{+}) \times \mathbb{R}.$$

Thus (18.76) becomes the closure of (18.51).

Another point we need to consider on $\mathcal{M}(H_{\tau,t}; R_{\alpha_-}, R_{\alpha_+})$ with $H_{\tau,t} = H_t$ is that the case when $R = R_{\alpha_-} = R_{\alpha_+}$ can occur. In this case, the set of solutions of (18.60) = (18.77) consists of maps each of which is constant in the $\tau \in \mathbb{R}$ direction and is an element of R in the $t \in S^1$ direction. When we consider $\mathcal{M}(H_{\tau,t}; R, R)$, we did not regard them as the elements. Hence this space is empty. This is because these elements are unstable because of the \mathbb{R} invariance.

However, when we consider $\mathcal{M}(H_{\tau,t}; R_{\alpha_-}, R_{\alpha_+})$ with $H_{\tau,t} = H_t$ and $R = R_{\alpha_-} = R_{\alpha_+}$, this element u is included. This is because we do not regard the \mathbb{R} translation as a symmetry. In other words, we have an isomorphism

$$\mathcal{M}(H_{\tau,t};R,R)=R$$

in this case. Thus (18.76) becomes the closure of (18.52).

18.10.2. Identification of interpolation space of the identity morphism with direct product. Next we explain a geometric origin of the identification

$$\mathcal{N}(\alpha_{-}, \alpha_{+}) \cong \mathcal{M}(\alpha_{-}, \alpha_{+}) \times [E(\alpha_{-}), E(\alpha_{+})], \tag{18.78}$$

that is (18.56). In our geometric situation

$$\mathcal{M}(\alpha_{-}, \alpha_{+}) = \mathcal{M}(R_{\alpha_{-}}, R_{\alpha_{+}})$$

and $E(\alpha)$ is the value of the action functional

$$\mathcal{A}_{H}([\gamma, w]) = \int w^* \omega + \int_{S^1} H(t, \gamma(t))$$

at $[\gamma, w] \in R_{\alpha}$. (Note that the sign here is opposite to one [FOOO12, (1.2)].) Then our interpolation space $\mathcal{N}(\alpha_{-}, \alpha_{+})$ is a compactification of $\mathcal{N}(\alpha_{-}, \alpha_{+}) = \mathcal{N}(R_{\alpha_{-}}, R_{\alpha_{+}}) \times \mathbb{R}$. An element of $\mathcal{N}(R_{\alpha_{-}}, R_{\alpha_{+}})$ that is the interior of $\mathcal{M}(R_{\alpha_{-}}, R_{\alpha_{+}})$ may be regarded as a map $u : \mathbb{R} \times S^{1} \to M$ satisfying (18.77) with the asymptotic boundary condition specified by $R_{\alpha_{-}}$ and $R_{\alpha_{+}}$. Since we put an \mathbb{R} factor in the definition of $\mathcal{N}(\alpha_{-}, \alpha_{+})$, its element is a map u itself but not an equivalence class by the \mathbb{R} action induced by the translation on the \mathbb{R} direction. Therefor the loop $\gamma_{\tau}: S^{1} \to M$, $\gamma_{\tau}(t) = u(\tau, t)$ is well-defined for each τ . Let $[\gamma_{-}, w_{-}]$ be an element of $R_{\alpha_{-}}$ such that $\gamma_{-}(t) = \lim_{\tau \to -\infty} u(\tau, t)$. We consider the concatenation of w_{-} and the restriction of u to $(-\infty, \tau] \times S^{1}$ and denote it by w_{τ} . We put

$$E(u) = \mathcal{A}_H([\gamma_0, w_0]). \tag{18.79}$$

Using the \mathbb{R} action $(\tau_0 \cdot u)(\tau, t) = u(\tau + \tau_0, t)$, we have

$$E(\tau_0 \cdot u) = \mathcal{A}_H([\gamma_{\tau_0}, w_{\tau_0}]).$$

Therefore $E(\tau_0 \cdot u)$ is an increasing function of $\tau_0 \in \mathbb{R}$ and

$$\lim_{\tau_0 \to -\infty} E(\tau_0 \cdot u) = E(\alpha_-), \qquad \lim_{\tau_0 \to +\infty} E(\tau_0 \cdot u) = E(\alpha_+).$$

Therefore using this function E we obtain a homeomorphism

$$\mathring{\mathcal{N}}(\alpha_{-}, \alpha_{+}) \cong \mathring{\mathcal{M}}(R_{\alpha_{-}}, R_{\alpha_{+}}) \times (E(\alpha_{-}), E(\alpha_{+})).$$

It is easy to see that this homeomorphism extends to (18.78).

18.10.3. Interpolation space of the homotopy. We next describe a geometric origin of the homotopy $\mathfrak{N} \circ \mathfrak{ID} \sim \mathfrak{N}$ given in the proof of Proposition 18.63.

We consider the case when the interpolation space of \mathfrak{N} is $\mathcal{M}(H_{\tau,t}; R_-, R_+)$ where $H_{\tau,t}$ is a homotopy from H to H'. This moduli space is the set of solutions of the equation (18.60) with asymptotic boundary condition given by R_- and R_+ . Note that $H_{\tau,t}$ is τ dependent. We write this $H_{\tau,t}$ as $H_{\tau,t}^{32}$.

On the other hand, the interpolation space of the morphism \mathfrak{ID} is by definition the set of solutions of (18.60) with $H_{\tau,t}=H_t$, which is τ independent. We write this $H_{\tau,t}=H_t$ as $H_{\tau,t}^{21}$.

Then the composition $\mathfrak{N} \circ \mathfrak{ID}$ is obtained by using the two parameter family of Hamiltonians concatenating $H_{\tau,t}$ and H_t as in (18.6). Let $H^{31,T} = H^{31,T}_{\tau,t}$ be obtained by this concatenation. More precisely, the interpolation space of $\mathfrak{N} \circ \mathfrak{ID}$ appears when we take the limit of $H^{31,T}$ as $T \to \infty$. Note that in our case $H^{31,T}$ is

$$H^{31,T}(\tau,t,x) = \begin{cases} H(t,x) & \text{if } \tau \leq -T_0 - T \\ H^{21}(\tau+T,t,x) = H(t,x) & \text{if } -T_0 - T \leq \tau \leq T_0 - T \\ H^2(t,x) = H(t,x) & \text{if } T_0 - T \leq \tau \leq T - T_0 \\ H^{32}(\tau-T,t,x) = H_{\tau-T,t}(x) & \text{if } T - T_0 \leq \tau \leq T + T_0 \\ H^3(t,x) = H'(t,x) & \text{if } T + T_0 \leq \tau. \end{cases}$$

$$(18.80)$$

See Figure 21. Thus actually we have

$$H^{31,T}(\tau,t,x) = H_{\tau-T,t}(x).$$





FIGURE 21. $H^{31,T}(\tau,t,x)$

Therefore the set of solutions of (18.5), with respect to τ , t dependent Hamiltonian $H^{31,T}$, that is

$$\frac{\partial u}{\partial \tau} + J \left(\frac{\partial u}{\partial t} - X_{H_{\tau - T, t}}(u) \right) = 0 \tag{18.81}$$

is indeed independent of T up to the canonical isomorphism. Moreover if T=0, this equation is exactly the same as one we used to define \mathfrak{N} .

We consider the union of the set of solutions of (18.81) for $T \in [0, \infty)$. Namely

$$\bigcup_{T \in [0,\infty)} \mathring{\mathcal{M}}(H^{31,T}; \alpha_-, \alpha'_+) \times \{T\}. \tag{18.82}$$

The interpolation space of the homotopy is a compactification of (18.82). Except the part $T = \infty$, the compactification is a product of $[0, \infty)$ and the compactification of the moduli space of solutions of (18.81) for fixed T = 0, which is $\mathcal{N}(\alpha_-, \alpha'_+) \times [0, \infty)$.

However there is rather a delicate issue related to the (source) \mathbb{R} action at the limit as $T \to \infty$. Recall that H^{21} is τ independent. As we already mentioned several times, we do not use this symmetry to divide our moduli space. In order to clarify this point, we take and fix a marked point and regard our moduli space as a map from a marked cylinder. (We take some marked point $(-T, 1/2) \in (-\infty, 0] \times S^1$.) The coordinate $T \in [0, \infty)$ in turn becomes the $[0, \infty)$ factor of $\mathcal{N}(\alpha_-, \alpha'_+) \times [0, \infty)$. We note that $\mathcal{N}(\alpha_-, \alpha'_+)$ is the set of solutions of

$$\frac{\partial u}{\partial \tau} + J \left(\frac{\partial u}{\partial t} - X_{H_{\tau,t}}(u) \right) = 0. \tag{18.83}$$

Then we can identify $\mathcal{N}(\alpha_-, \alpha'_+) \times [0, \infty)$ with the set of (u, (-T, 1/2)) where u solves the equation (18.83) and $T \in [0, \infty)$. In other words, putting the marked point (-T, 1/2) corresponds to shifting $H^{31,0}$ to $H^{31,T}$. Then by a standard gluing

³¹Note $\mathcal{N}(\alpha_-, \alpha'_+) = \mathcal{M}(H^{31,T}; \alpha_-, \alpha'_+).$

analysis we can describe its limit as $T \to \infty$ by the union of

$$\mathcal{M}(H; \alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \cdots \times_{R_{\alpha_{k'_{1}-2}}} \mathcal{M}(H; \alpha_{k'_{1}-2}, \alpha_{k'_{1}-1})$$

$$\times_{R_{\alpha_{k'_{1}-1}}} \mathcal{M}(H; \alpha_{k'_{1}-1}, \alpha_{k'_{1}}) \times \mathbb{R}$$

$$\times_{R_{\alpha_{k'_{1}}}} \mathcal{M}(H; \alpha_{k'_{1}}, \alpha_{k'_{1}+1}) \times_{R_{\alpha_{k'_{1}+1}}} \cdots \times_{R_{k_{1}-1}} \mathcal{M}(H; \alpha_{k_{1}-1}, \alpha_{k_{1}})$$

$$\times_{R_{\alpha_{k_{1}}}} \mathcal{M}(H; \alpha_{t,t}; \alpha_{k_{1}}, \alpha'_{1})$$

$$\times_{R_{\alpha'_{1}}} \mathcal{M}(H'; \alpha'_{1}, \alpha'_{2}) \times_{R_{\alpha'_{2}}} \cdots \times_{\alpha'_{k_{2}}} \mathcal{M}(H'; \alpha'_{k_{2}}, \alpha'_{+}).$$
(18.84)

See Figure 22, that is the case $k_1 = 1$, $k'_2 = 0$. An object in the compactification of (18.82) corresponding to the limit as $T \to \infty$ is obtained by gluing maps appearing in (18.84). In fact, since an element of $\mathcal{M}(H^{31,T};\alpha_-,\alpha'_+)$ comes with a marked point, the limit is assigned with a marked point on one of the factors. We put the \mathbb{R} factor to the factor on which the marked point lies. In (18.84) it lies on the map representing an element of $\mathcal{M}(H;\alpha_{k'_1-1},\alpha_{k'_1})$.

The space (18.84) is the closure of (18.61). By the same reason as we discussed

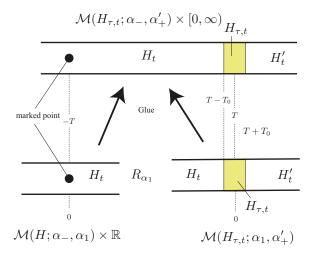


FIGURE 22. Boundary of $\mathcal{M}(H_{\tau,t};\alpha_-,\alpha'_+)\times[0,\infty)$

in Subsebsection 18.10.2, the limit as $T \to \infty$ also contains a component

$$\mathcal{M}(H; \alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \cdots \times_{\alpha_{k_{1}-1}} \mathcal{M}(H; \alpha_{k'_{1}-1}, \alpha_{k'_{1}}) \\
\times_{R_{\alpha_{k'_{1}}}} R_{\alpha_{k'_{1}}} \\
\times_{R_{\alpha_{k'_{1}}}} \mathcal{M}(H; \alpha_{k'_{1}}, \alpha_{k'_{1}+1}) \times_{\alpha_{k_{1}+1}} \cdots \times_{R_{\alpha_{k_{1}-1}}} \mathcal{M}(H; \alpha_{k_{1}-1}, \alpha_{k_{1}}) \\
\times_{R_{\alpha_{k_{1}}}} \mathcal{M}(H; \alpha_{k_{1}}, \alpha'_{1}) \\
\times_{R_{\alpha'_{1}}} \mathcal{M}(H'; \alpha'_{1}, \alpha'_{2}) \times_{R_{\alpha'_{2}}} \cdots \times_{R_{\alpha'_{k_{2}}}} \mathcal{M}(H'; \alpha'_{k_{2}}, \alpha'_{+}).$$
(18.85)

Namely, this space (18.85) corresponds to the case when the marked point lies in the component corresponding to a map $u: \mathbb{R} \times S^1 \to M$ which is constant in the \mathbb{R} direction and represents an element of $R_{\alpha_{k'_1}}$ on each $\{\tau\} \times S^1$. The space (18.85) coincides with the closure of (18.62).

Therefore the compactification of (18.82) coincides with the interpolation space of the homotopy constructed in the proof of Proposition 18.63.

18.10.4. Identification of the interpolation space of the homotopy with direct product. Finally, we explain a geometric origin of the identification

$$\mathcal{N}(\alpha_{-}, \alpha'_{+}; [0, 1]) \cong \mathcal{N}(\alpha_{-}, \alpha'_{+}) \times [E(\alpha_{-}), E(\alpha'_{+}) + c].$$
 (18.86)

The discussion is similar to that in Subsection 18.10.2 but is slightly more involved. We denote by $\mathring{\mathcal{N}}(\alpha_-, \alpha'_+; [0, 1])$ the set of pairs (u, -T) where $u : \mathbb{R} \times S^1 \to M$ is a map solving the equation (18.83) and satisfying the asymptotic boundary conditions as $\tau \to \pm \infty$ given by R_{α_-} and $R_{\alpha'_+}$, and (-T, 1/2) is the marked point in $(-\infty, 0] \times S^1$. As we discussed in Subsubsection 18.10.3, $\mathcal{N}(\alpha_-, \alpha'_+; [0, 1])$ is a compactification of this space.

We first define a map

$$E: \mathring{\mathcal{N}}(\alpha_-, \alpha'_+; [0, 1]) \to \mathbb{R}$$

and modify it to E' later. For $(u,T) \in \mathring{\mathcal{N}}(\alpha_-,\alpha'_+;[0,1])$ we put $\gamma_\tau(t) = u(\tau,t)$ and

$$\gamma_{-} = \lim_{\tau \to -\infty} \gamma_{\tau}.$$

The asymptotic boundary condition we assumed for $\overset{\circ}{\mathcal{N}}(\alpha_{-}, \alpha'_{+}; [0, 1])$ implies that there exists w_{-} such that $[\gamma_{-}, w_{-}] \in R_{\alpha_{-}}$. We denote by w_{T} the concatenation of w_{-} and the restriction of u to $(-\infty, -T) \times S^{1}$ and define

$$E(u, -T) = -\int_{D^2} w_T^* \omega - \int_{t \in S^1} H_{\tau, t}(\gamma_T(t)) \in \mathbb{R}.$$
 (18.87)

We note that

$$\lim_{T \to \infty} E(u, -T) = \mathcal{A}_H([\gamma_-, w_-]) = E(\alpha_-). \tag{18.88}$$

On the other hand, contrary to the situation of Subsection 18.10.2, the map

$$(-\infty,0] \ni -T \mapsto E(u,-T) \in \mathbb{R}$$

may not be an increasing function in general, although it is an increasing function for sufficiently large T. Moreover there is no obvious relation between E(u,0) and $E(\alpha'_+)$. Since the energy loss is c, the inequality $E(\alpha_-) < E(\alpha'_+) + c$ does hold, but $E(\alpha_-) < E(\alpha'_+)$ may not hold in general. Thus we modify E(u,T) to E'(u,T) with the following properties.

- (1) E'(u, -T) = E(u, -T) if T is sufficiently large.
- (2) $-T \mapsto E'(u, -T)$ is strictly increasing.
- (3) $E'(u,0) = E(\alpha'_+) + c$.

More explicitly, we can define E' as follows. We first take a sufficiently large number $T_1 > 0$ with the following properties:

- (a) $E(u, -T) \leq E(\alpha_+) + c$ holds for each $u \in \mathcal{N}(\alpha_-, \alpha'_+)$ and $T \geq T_1$.
- (b) $H_{\tau,t} = H_t \text{ if } \tau < -T_1.$

Then we define

$$E'(u, -T) = \begin{cases} E(u, -T) & \text{if } T \ge T_1\\ \frac{T_1 - T}{T_1} (E(\alpha_+) + c - E(u, -T_1)) + E(u, -T_1) & \text{if } T \in [0, T_1]. \end{cases}$$
(18.89)

Item (b) implies that the function $-T \mapsto E(u, -T)$ is increasing for $T > T_1$. Item (a) implies that $-T \mapsto E'(u, -T)$ is an increasing function for $T \in [0, T_1]$. We have thus verified (2). (1) and (3) are obvious from definition. We can easily extend it to $\mathcal{N}(\alpha_-, \alpha'_+; [0, 1])$. In fact, the value E' is in the interval $(E(\alpha_{k'_+, -1}), E(\alpha_{k'_+}))$ on (18.84) and is $E(\alpha_{k'})$ on (18.85). This is a consequence of Item (1). Then using E'as the second factor, we obtain the identification (18.86).

19. Linear K-system: Floer Cohomology II: Proof

The purpose of this section is to prove the theorems we claimed in Section 16.

19.1. Construction of cochain complexes. Let us start with a linear K-system or a partial linear K-system of energy cut level E_0 as in Definition 16.6. We set $E_0 = +\infty$ for the case of linear K-system and E_0 to be the energy cut level for the case of partial linear K-system. We consider the set

$$\mathfrak{E}_{\leq E_0} = \{ E(\alpha_+) - E(\alpha_-) \mid \mathcal{M}(\alpha_-, \alpha_+) \neq \emptyset, \ E(\alpha_+) - E(\alpha_-) \leq E_0 \}. \tag{19.1}$$

This is a discrete set by Condition 16.1 (IX). We put

$$\mathfrak{E}_{\leq E_0} = \{ E_{\mathfrak{E}}^1, E_{\mathfrak{E}}^2, \dots \}$$

with $E^1_{\mathfrak{E}} < E^2_{\mathfrak{E}} < \dots$ We will use the results of Section 17 to prove the next proposition by induction on k of the energy cut level $E^k_{\mathfrak{E}}$. Put $\tau_0 = 1$.

Proposition 19.1. For each $0 < \tau < 1$, there exist a τ -collared Kuranishi structure $\widehat{\mathcal{U}^+}(\alpha_-, \alpha_+)$ of $\mathcal{M}(\alpha_-, \alpha_+)^{\boxplus \tau_0}$ and a CF-perturbation $\widehat{\mathfrak{S}^+}(\alpha_-, \alpha_+)$ of $\widehat{\mathcal{U}^+}(\alpha_-, \alpha_+)$ for every α_- , α_+ with $E(\alpha_+)-E(\alpha_-) \leq E_{\mathfrak{E}}^k$ and they enjoy the following properties:

- (1) Let $\widehat{\mathcal{U}}(\alpha_-, \alpha_+)$ be the Kuranishi structure on $\mathcal{M}(\alpha_-, \alpha_+)$ given in Condition 16.1. Then $\widehat{\mathcal{U}}(\alpha_-, \alpha_+)^{\boxplus \tau_0} < \widehat{\mathcal{U}}^+(\alpha_-, \alpha_+)$ as collared Kuranishi structures.³²
- (2) The CF-perturbation $\widehat{\mathfrak{S}^+}(\alpha_-, \alpha_+)$ is transversal to 0. Moreover³³ ev₊: $\mathcal{M}(\alpha_-, \alpha_+)^{\boxplus \tau_0} \to M$ is strongly submersive with respect to $\widehat{\mathfrak{S}^+}(\alpha_-, \alpha_+)$.
- (3) We have the following isomorphism, called the periodicity isomorphism

$$\widehat{\mathcal{U}^+}(\alpha_-, \alpha_+) \longrightarrow \widehat{\mathcal{U}^+}(\beta \alpha_-, \beta \alpha_+)$$

for any $\beta \in \mathfrak{G}$. It is compatible with the periodicity isomorphism in Condition 16.1 (VIII) via the embedding in (1). The pull-back of $\mathfrak{S}^+(\beta\alpha_-,\beta\alpha_+)$ by this isomorphism is equivalent to $\widehat{\mathfrak{S}}^+(\alpha_-, \alpha_+)$.

(4) There exists an isomorphism of τ -collared K-spaces.³⁴

$$\partial(\mathcal{M}(\alpha_{-},\alpha_{+})^{\boxplus \tau_{0}},\widehat{\mathcal{U}^{+}}(\alpha_{-},\alpha_{+}))$$

$$= \coprod_{\alpha} (-1)^{\dim \mathcal{M}(\alpha,\alpha_{+})} (\mathcal{M}(\alpha,\alpha_{+})^{\boxplus \tau_{0}},\widehat{\mathcal{U}^{+}}(\alpha,\alpha_{+}))_{\text{ev}_{-}} \times_{\text{ev}_{+}} (\mathcal{M}(\alpha_{-},\alpha)^{\boxplus \tau_{0}},\widehat{\mathcal{U}^{+}}(\alpha_{-},\alpha))$$

$$(19.2)$$

 $^{^{32}}$ Let $\widehat{\mathcal{U}}_1^{\boxplus \tau_1}, \widehat{\mathcal{U}}_2^{\boxplus \tau_2}$ be two collared Kuranishi structures. We define $\widehat{\mathcal{U}}_1^{\boxplus \tau_1} < \widehat{\mathcal{U}}_2^{\boxplus \tau_2}$ as collared Kuranishi structures if there exist Kuranishi structures $\widehat{\mathcal{U}}_1', \widehat{\mathcal{U}}_2'$ such that $\widehat{\mathcal{U}}_1' < \widehat{\mathcal{U}}_2'$ and $\widehat{\mathcal{U}}_1'^{\boxplus \tau_1'} = \widehat{\mathcal{U}}_1^{\boxplus \tau_1}, \widehat{\mathcal{U}}_2'^{\boxplus \tau_2'} = \widehat{\mathcal{U}}_2^{\boxplus \tau_2}$ for some τ_1', τ_2' .

33According to Lemma 17.37 (3), we should write $\operatorname{ev}_+^{\boxplus \tau_0}$ but not ev_+ . However, to simplify

the notation we drop $^{\boxplus \tau_0}$ from the notation of evaluation maps if no confusion can occur.

 $^{^{34}}$ See Remark 16.2 for the sign and the order of the fiber products.

The isomorphism (19.2) is compatible with the isomorphism

$$\partial \widehat{\mathcal{U}}(\alpha_{-}, \alpha_{+})^{\boxplus \tau_{0}} = \coprod_{\alpha} (-1)^{\dim \widehat{\mathcal{U}}(\alpha, \alpha_{+})} (\widehat{\mathcal{U}}(\alpha, \alpha_{+})^{\boxplus \tau_{0}})_{\mathrm{ev}_{-}} \times_{\mathrm{ev}_{+}} (\widehat{\mathcal{U}}(\alpha_{-}, \alpha)^{\boxplus \tau_{0}})$$

which is induced by Condition 16.1 (X) via the embedding in (1).

(5) The pull-back of $\mathfrak{S}^+(\alpha_-, \alpha_+)$ by the isomorphism (19.2) is equivalent to the fiber product

$$\widehat{\mathfrak{S}^+}(\alpha, \alpha_+)_{\mathrm{ev}_-} \times_{\mathrm{ev}_+} \widehat{\mathfrak{S}^+}(\alpha_-, \alpha).$$

This fiber product is well-defined by (2).

(6) There exists an isomorphism of τ -collared K-spaces

$$\widehat{S}_{k}(\mathcal{M}(\alpha_{-}, \alpha_{+})^{\boxplus \tau_{0}}, \widehat{\mathcal{U}^{+}}(\alpha_{-}, \alpha_{+}))$$

$$\cong \coprod_{\alpha_{1}, \dots, \alpha_{k} \in \mathfrak{A}} \left((\mathcal{M}(\alpha_{-}, \alpha_{1})^{\boxplus \tau_{0}}, \widehat{\mathcal{U}^{+}}(\alpha_{-}, \alpha_{1})) \,_{\operatorname{ev}_{+}} \times_{R_{\alpha_{1}}} \cdots$$

$$\cdots \,_{R_{\alpha_{k}}} \times_{\operatorname{ev}_{-}} \left(\mathcal{M}(\alpha_{k}, \alpha_{+})^{\boxplus \tau_{0}}, \widehat{\mathcal{U}^{+}}(\alpha_{k}, \alpha_{+})) \right). \tag{19.3}$$

The isomorphism (19.3) is compatible with the isomorphism

$$\widehat{S}_{k}(\widehat{\mathcal{U}}(\alpha_{-},\alpha_{+})^{\boxplus \tau_{0}})$$

$$\cong \coprod_{\alpha_{1},\dots,\alpha_{k} \in \mathfrak{A}} (\widehat{\mathcal{U}}(\alpha_{-},\alpha_{1})^{\boxplus \tau_{0}}_{\text{ev}_{+}} \times_{R_{\alpha_{1}}} \dots_{R_{\alpha_{k}}} \times_{\text{ev}_{-}} \widehat{\mathcal{U}}(\alpha_{k},\alpha_{+})^{\boxplus \tau_{0}})$$

which is induced by Condition 16.1 (XI) via the embedding in (1).

(7) The isomorphism (19.3) implies that $\widehat{S}_{\ell}(\widehat{S}_k(\mathcal{M}(\alpha_-, \alpha_+)^{\boxplus \tau_0}, \widehat{\mathcal{U}^+}(\alpha_-, \alpha_+)))$ is decomposed into similar fiber products as (19.3) where k is replaced by $\ell + k$. On the other hand, $\widehat{S}_{\ell+k}(\mathcal{M}(\alpha_-, \alpha_+)^{\boxplus \tau_0}, \widehat{\mathcal{U}^+}(\alpha_-, \alpha_+))$ is decomposed into similar fiber products as (19.3) where k is replaced by $\ell + k$. The map

$$\pi_{\ell,k}: \widehat{S}_{\ell}(\widehat{S}_{k}(\mathcal{M}(\alpha_{-},\alpha_{+})^{\boxplus \tau_{0}},\widehat{\mathcal{U}^{+}}(\alpha_{-},\alpha_{+}))) \to \widehat{S}_{\ell+k}(\mathcal{M}(\alpha_{-},\alpha_{+})^{\boxplus \tau_{0}},\widehat{\mathcal{U}^{+}}(\alpha_{-},\alpha_{+}))$$
in Proposition 24.16 becomes an identity map on each component under this identification.

(8) The pull-back of $\widehat{\mathfrak{S}}^+(\alpha_-, \alpha_+)$ by the isomorphism (19.3) is the fiber product:

$$\widehat{\mathfrak{S}^{+}}(\alpha_{-}, \alpha_{1}) \underset{\mathrm{ev}_{+}}{\times} \times_{R_{\alpha_{1}}} \dots_{R_{\alpha_{k}}} \times_{\mathrm{ev}_{-}} \widehat{\mathfrak{S}^{+}}(\alpha_{k}, \alpha_{+}). \tag{19.4}$$

This fiber product is well-defined by (2). This isomorphism is compatible with the covering map $\pi_{\ell,k}$.

Proof. Using the results of Subsection 17.9, we can prove the proposition in a straightforward way as follows. We take τ^+ with

$$\tau < \tau^+ < 1 = \tau_0.$$

Suppose we constructed $\widehat{\mathcal{U}^+}(\alpha_-, \alpha_+)$ and $\widehat{\mathfrak{S}^+}(\alpha_-, \alpha_+)$ satisfying (1)-(8) above for any α_- , α_+ with $E(\alpha_+) - E(\alpha_-) < E_{\mathfrak{E}}^k$ and for τ replaced by τ^+ . We consider α_- , α_+ with $E(\alpha_+) - E(\alpha_-) = E_{\mathfrak{E}}^k$ and $\mathcal{M}(\alpha_-, \alpha_+)^{\boxplus \tau_0}$.

To apply Propositions 17.46 and 17.62, we check that we are in Situation 17.43.

To apply Propositions 17.46 and 17.62, we check that we are in Situation 17.43. The space X in Situation 17.43 is $\mathcal{M}(\alpha_-, \alpha_+)^{\boxplus \tau_0}$ and τ in Situation 17.43 is τ^+ here. The τ -collared Kuranishi structure $\widehat{\mathcal{U}}$ in Situation 17.43 is the Kuranishi structure $\widehat{\mathcal{U}}(\alpha_-, \alpha_+)^{\boxplus \tau_0}$ here. The Kuranishi structure $\widehat{\mathcal{U}}_{S_k}^+$ in Situation 17.43 is the Kuranishi structure in the right hand side of (19.3), which is given by the

induction hypothesis. We write it as $\widehat{\mathcal{U}_{S_k}^+}$. Then it is easy to see from definition that $\widehat{S}_k(\widehat{\mathcal{U}_{S_\ell}^+})$ is $(k+\ell)!/k!\ell!$ disjoint union of $\widehat{\mathcal{U}_{S_{k+\ell}}^+}$. Therefore Situation 17.43 (2) holds. The commutativity of Diagram (17.22) holds because all the maps in (17.22) is the identity map via the isomorphism (19.3). We next construct the embedding $\widehat{S}_k(\mathcal{M}(\alpha_-,\alpha_+)^{\boxplus \tau_0},\widehat{\mathcal{U}}(\alpha_-,\alpha_+)^{\boxplus \tau_0}) \to \widehat{S}_k(\widehat{\mathcal{U}_{S_\ell}^+})$. By induction hypothesis the embedding $\widehat{\mathcal{U}}(\alpha,\alpha')^{\boxplus \tau_0} \to \widehat{\mathcal{U}}^+(\alpha,\alpha')$ for $E(\alpha') - E(\alpha) < E_{\mathfrak{E}}^k$ is given. By using Condition 16.1 (XI) we have

$$\widehat{S}_{k}(\widehat{\mathcal{U}}(\alpha_{-},\alpha_{1})^{\boxplus \tau_{0}})$$

$$\cong \coprod_{\alpha_{1},...,\alpha_{k} \in \mathfrak{A}} \left(\widehat{\mathcal{U}}(\alpha_{-},\alpha_{1})^{\boxplus \tau_{0}}_{\operatorname{ev}_{+}} \times_{R_{\alpha_{1}}} \cdots_{R_{\alpha_{k}}} \times_{\operatorname{ev}_{-}} \widehat{\mathcal{U}}(\alpha_{k},\alpha_{+})^{\boxplus \tau_{0}}\right).$$

Therefore, the right hand side is embedded in

$$\widehat{\mathcal{U}_{S_k}^+} \cong \coprod_{\alpha_1, \dots, \alpha_k \in \mathfrak{A}} \left((\mathcal{M}(\alpha_-, \alpha_1)^{\boxplus \tau_0}, \widehat{\mathcal{U}^+}(\alpha_-, \alpha_1)) \right|_{\text{ev}_+} \times_{R_{\alpha_1}} \dots$$

$$\dots R_{\alpha_k} \times_{\text{ev}_-} \left(\mathcal{M}(\alpha_k, \alpha_+)^{\boxplus \tau_0}, \widehat{\mathcal{U}^+}(\alpha_k, \alpha_+) \right) \right).$$

So Situation 17.43 (4) is satisfied. The commutativity of Diagram (17.22) and Diagram (17.23) follows from the fact that all the maps in of Diagrams (17.22), (17.23) become the identity maps via the isomorphism (19.3), which is a part of induction hypothesis (Proposition 19.1 (7)). We have thus checked the assumption of Propositions 17.46 and 17.62. The Kuranishi structure we obtain in Proposition 17.62 is our Kuranishi structure $\widehat{U}^+(\alpha_-, \alpha_+)$.

We next consider CF-perturbations. We check that we are in Situation 17.57. We define $\widehat{\mathfrak{S}_{S_k}^+}$ in Situation 17.57 by the right hand side of (19.4). Situation 17.57 (2) can be checked easily in our case by using inductive hypothesis. We can thus apply Propositions 17.58 and 17.65. The CF-perturbations we obtain by Proposition 17.65 is $\widehat{\mathfrak{S}^+}(\alpha_-,\alpha_+)$.

Now we will check that $\widehat{\mathcal{U}}^+(\alpha_-, \alpha_+)$ and $\widehat{\mathfrak{S}}^+(\alpha_-, \alpha_+)$ satisfy Proposition 19.1 (1)-(8).

- (1) Since $\tau < \tau^+$, the τ -collared-ness of $\widehat{\mathcal{U}^+}(\alpha_-, \alpha_+)$ is a consequence of Proposition 17.46 and $\widehat{\mathcal{U}}(\alpha_-, \alpha_+)^{\boxplus \tau} < \widehat{\mathcal{U}^+}(\alpha_-, \alpha_+)$ also follows from Proposition 17.46.
 - (2) is a consequence of Proposition 17.65.
- (3) We apply Proposition 17.46 to each \mathfrak{G} equivalence class of the pair (α_-, α_+) . Then for other $(\beta \alpha_-, \beta \alpha_+)$ we define $\widehat{\mathcal{U}^+}(\alpha_-, \alpha_+)$ so that it is identified with (α_-, α_+) by using existence of the periodicity isomorphism on the boundary. Then existence of the periodicity isomorphism for α_-, α_+ with $E(\alpha_+) E(\alpha_-) = E_{\mathfrak{E}}^k$ is immediate from the definition. The compatibility of the periodicity isomorphism with CF-perturbations can be proved in the same way.
- (4),(6),(7) This is a consequence of Proposition 17.46. Namely it is a consequence of Proposition 17.46 (1) and the induction hypothesis.
- (5),(8) This is a consequence of Proposition 17.65 (1) and the induction hypothesis. Hence the proof of Proposition 19.1 is now complete.

Remark 19.2. In the case of partial linear K-system, the induction to prove Proposition 19.1 stops in finite steps. In the case of linear K-system, the number of inductive steps is countably infinitely many.

We next rewrite the geometric conclusion of Proposition 19.1 into algebraic structures.

Definition 19.3. In the situation of Proposition 19.1, we define

$$\mathfrak{m}_{1:\alpha_{+},\alpha_{-}}^{\epsilon}:\Omega(R_{\alpha_{-}};o_{R_{\alpha_{-}}})\longrightarrow\Omega(R_{\alpha_{+}};o_{R_{\alpha_{+}}})$$
 (19.5)

by

$$\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{\epsilon}(h) = \operatorname{ev}_{+}!(\operatorname{ev}_{-}^{*}h;\widehat{\mathfrak{S}^{+\epsilon}}(\alpha_{-},\alpha_{+})).$$
 (19.6)

Here the right hand side is defined by Definition 17.67 on the K-space

$$(\mathcal{M}(\alpha_{-},\alpha_{+})^{\boxplus \tau_{0}},\widehat{\mathcal{U}^{+}}(\alpha_{-},\alpha_{+})).$$

See Theorem 27.1 for the correspondence coupled with local systems. Note that Condition 16.1 (VII) is compatible with the conventions (27.1), (27.2).

The degree of the map $\mathfrak{m}_{1;\alpha_+,\alpha_-}^{\epsilon}$ is

$$\dim R_{\alpha_+} - \dim \mathcal{M}(\alpha_-, \alpha_+) = 1 - \mu(\alpha_+) + \mu(\alpha_-)$$

by Condition 16.1 (VI) and [Part I, Definition 7.78]. Therefore after the degree shift in Definition 16.8 (2) its degree becomes +1.

Remark 19.4. In the situation of linear K-system where infinitely many different K-spaces are involved, we need to make a careful choice of $\epsilon > 0$. Note that the well-definedness of push out ([Part I, Theorem 9.14]) says that for each K-space and its CF-perturbation the push out is well-defined for sufficiently small $\epsilon > 0$. In the situation of linear K-system infimum of such ϵ over all α_{\pm} of $\widehat{\mathfrak{S}}^{+\epsilon}(\alpha_{-},\alpha_{+})$ may be 0. 35 (It might be possible to prove that we can take the same ϵ for all $\widehat{\mathfrak{S}}^{+\epsilon}(\alpha_{-},\alpha_{+})$ at the same time. However, it is cumbersome to formulate the condition without referring the construction itself. By this reason, we do not try to prove or use such a uniformity in this book.) More precisely speaking, we have the following:

(b) For any energy cut level E_0 there exists $\epsilon_0(E_0)$ such that the operator $\mathfrak{m}_{1:\alpha_+,\alpha_-}^{\epsilon}$ is defined when $0 < E(\alpha_+) - E(\alpha_-) \le E_0$ and $\epsilon < \epsilon_0(E_0)$.

Hereafter when the relationship between energy cut level and ϵ (which is the parameter of the approximation) appears in this way, we write in the sense of (\flat) in place of repeating the above sentence over again.

Lemma 19.5. The operators $\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{\epsilon}$ satisfy the following equality in the sense of (\flat) .

$$d_0 \circ \mathfrak{m}_{1;\alpha_+,\alpha_-}^{\epsilon} + \mathfrak{m}_{1;\alpha_+,\alpha_-}^{\epsilon} \circ d_0 + \sum_{\alpha;E(\alpha_-) < E(\alpha) < E(\alpha_+)} \mathfrak{m}_{1;\alpha_+,\alpha}^{\epsilon} \circ \mathfrak{m}_{1;\alpha,\alpha_-}^{\epsilon} = 0. \quad (19.7)$$

Here and hereafter we denote

$$d_0(h) = (-1)^{\dim R_\alpha + \mu(\alpha) + 1 + \deg h} d_{dR}(h)$$
(19.8)

for a differential form $h \in \Omega(R_{\alpha})$, where d_{dR} denotes the de Rham differential on R_{α} . This sign arises from d_0 being the classical part of \mathfrak{m}_1 in a filtered A_{∞} structure. See (16.17) and Remark 3.5.8 [FOOO3] for the sign.

³⁵This is a version of the 'running out' problem discussed in [FOOO4, Subsection 7.2.8]. The way we resolve it in Subsection 19.2 of this book is the same as one in [FOOO4].

Proof. By Stokes' formula and the definition (19.6) we have

$$(d_0 \circ \mathfrak{m}_{1;\alpha_+,\alpha_-}^{\epsilon} + \mathfrak{m}_{1;\alpha_+,\alpha_-}^{\epsilon} \circ d_0)(h) = \operatorname{ev}_+!(\operatorname{ev}_-^*h; \partial \widehat{\mathfrak{S}^{+\epsilon}}(\alpha_-,\alpha_+)). \tag{19.9}$$

By Proposition 19.1 (5) and the composition formula, the right hand side of (19.9) is

$$\sum_{\alpha; E(\alpha_-) < E(\alpha) < E(\alpha_+)} (\mathfrak{m}_{1;\alpha_+,\alpha}^{\epsilon} \circ \mathfrak{m}_{1;\alpha,\alpha_-}^{\epsilon}) (h).$$

The lemma follows.

Lemma 19.5 implies that

$$d_0 + \sum \mathfrak{m}_{1;\alpha_+,\alpha_-}^{\epsilon}$$

is a 'coboundary oprator modulo higher order term'. Since we need to take care of the point mentioned in Remark 19.4, we have to stop at some energy cut level. So we still need to do some more work to prove Theorem 16.9 (1).

19.2. Construction of cochain maps. We next consider morphism of linear K-systems or of partial linear K-systems.

Situation 19.6. Suppose we are in Situation 16.15. We assume that for each i=1,2 we have a CF-perturbation $\widehat{\mathfrak{S}^+}(i;\alpha_{i-},\alpha_{i+})$ of a τ -collared Kuranishi structure $\widehat{\mathcal{U}^+}(i;\alpha_{i-},\alpha_{i+})$ on $\mathcal{M}^i(\alpha_{i-},\alpha_{i+})^{\boxplus \tau_0}$ such that they satisfy the conclusion of Proposition 19.1. Here $\mathcal{M}^i(\alpha_{i-},\alpha_{i+})$ is as in Condition 16.16. (From now on, we write α_{\pm} in place of $\alpha_{i\pm}$ if no confusion can occur.)

Let $0 < \tau' < \tau < \tau_0 = 1$ where τ_0, τ are as in Proposition 19.1.

Proposition 19.7. Suppose we are in Situation 19.6 and are given a morphism of partial linear K-systems of energy cut level E_0 ($\in \mathbb{R}_{\geq 0} \cup \{\infty\}$) and energy loss c from \mathcal{F}_1 to \mathcal{F}_2 . Let $\mathcal{N}(\alpha_1, \alpha_2)$ be its interpolation space where $E(\alpha_2) - E(\alpha_1) \leq E_0$. Then for each τ' with $0 < \tau' < \tau < \tau_0 = 1$ as above, there exists a τ' -collared Kuranishi structure $\widehat{\mathcal{U}}^+(\text{mor}; \alpha_1, \alpha_2)$ on $\mathcal{N}(\alpha_1, \alpha_2)^{\boxplus \tau_0}$ and a τ' -collared CF-perturbation $\widehat{\mathfrak{S}}^+(\text{mor}; \alpha_1, \alpha_2)$ of $\widehat{\mathcal{U}}^+(\text{mor}; \alpha_1, \alpha_2)$ such that they have the following properties:

- (1) Let $\widehat{\mathcal{U}}(\mathrm{mor}; \alpha_1, \alpha_2)$ be the Kuranishi structure of $\mathcal{N}(\alpha_1, \alpha_2)$ in Situation 16.16 (IV). Then $\widehat{\mathcal{U}}(\mathrm{mor}; \alpha_1, \alpha_2)^{\boxplus \tau_0} < \widehat{\mathcal{U}}^+(\mathrm{mor}; \alpha_1, \alpha_2)$ as collared Kuranishi structures.
- (2) The CF-perturbation $\widehat{\mathfrak{S}^+}(\text{mor}; \alpha_1, \alpha_2)$ is transversal to 0. Moreover³⁶ ev₊: $\mathcal{N}(\alpha_1, \alpha_2)^{\boxplus \tau_0} \to R_{\alpha_2}$ is strongly submersive with respect to $\widehat{\mathfrak{S}^+}(\text{mor}; \alpha_1, \alpha_2)$.
- (3) We have the following periodicity isomorphism

$$\widehat{\mathcal{U}}^+(\text{mor}; \alpha_1, \alpha_2) \longrightarrow \widehat{\mathcal{U}}^+(\text{mor}; \beta \alpha_1, \beta \alpha_2),$$

which is compatible with the isomorphism in Condition 16.16 (VIII) via the embedding given in (1). The pull-back of $\widehat{\mathfrak{S}^+}(\operatorname{mor}; \beta\alpha_1, \beta\alpha_2)$ by this isomorphism is $\widehat{\mathfrak{S}^+}(\operatorname{mor}; \alpha_1, \alpha_2)$.

 $^{^{36}}$ As we note in the footnote of Proposition 19.1 (2), we simply write ev₊ in place of ev₊ $^{\boxplus \tau_0}$.

(4) There is an isomorphism of τ' -collared K-spaces ³⁷

$$_{\operatorname{ev}_{-}}\times_{\operatorname{ev}_{+}}\left(\mathcal{N}(\alpha_{1},\alpha_{2}^{\prime})^{\boxplus\tau_{0}},\widehat{\mathcal{U}^{+}}(\operatorname{mor};\alpha_{1},\alpha_{2}^{\prime}))\right).$$

Here the first union is taken over $\alpha'_1 \in \mathfrak{A}_1$ with $E(\alpha_1) < E(\alpha'_1) \le E(\alpha_2) + c$ and the second union is taken over $\alpha'_2 \in \mathfrak{A}_2$ with $E(\alpha_1) - c \le E(\alpha'_2) < E(\alpha_2)$. The number c is the energy loss of the given morphism.

The isomorphism (19.10) is compatible with the isomorphism induced by (16.25) in Condition 16.16 (X) via the embedding in (1).

- (5) The pull-back of $\widehat{\mathfrak{S}^+}(\text{mor}; \alpha_1, \alpha_2)$ by the isomorphism (19.10) is equivalent to the fiber product of $\widehat{\mathfrak{S}^+}(1; \alpha_1, \alpha_1')$, $\widehat{\mathfrak{S}^+}(\text{mor}; \alpha_1', \alpha_2)$ and of $\widehat{\mathfrak{S}^+}(\text{mor}; \alpha_1, \alpha_2')$, $\widehat{\mathfrak{S}^+}(1; \alpha_2', \alpha_2)$.
- (6) The normalized corner $\widehat{S}_k(\mathcal{N}(\alpha_1, \alpha_2)^{\boxplus \tau_0}, \widehat{\mathcal{U}^+}(\text{mor}; \alpha_1, \alpha_2))$ is the disjoint union of

$$\widehat{\mathcal{U}^{+}}(1; \alpha_{1}, \alpha_{1,1}) \times_{R_{\alpha_{1,1}}^{1}} \cdots \times_{R_{\alpha_{1,k_{1}-1}}^{1}} \widehat{\mathcal{U}^{+}}(1; \alpha_{1,k_{1}-1}, \alpha_{1,k_{1}}) \\
\times_{R_{\alpha_{1,k_{1}}}^{1}} \widehat{\mathcal{U}^{+}}(\text{mor}; \alpha_{1,k_{1}}, \alpha_{2,1}) \\
\times_{R_{\alpha_{2,1}}^{1}} \widehat{\mathcal{U}^{+}}(2; \alpha_{2}, \alpha_{2,1}) \times_{R_{\alpha_{2,1}}^{2}} \cdots \times_{R_{\alpha_{2,k_{2}-1}}^{2}} \widehat{\mathcal{U}^{+}}(2; \alpha_{2,k_{2}-1}, \alpha_{2,k_{2}}) \\
\text{where } k_{1} + k_{2} = k, \ \alpha_{1,i} \in \mathfrak{A}_{1}, \ \alpha_{2,i} \in \mathfrak{A}_{2}.$$
(19.11)

This isomorphism is compatible with the isomorphism in Condition 16.16 (XI) via the the embedding in (1).

(7) (19.3) and (19.11) imply that $\widehat{S}_{\ell}(\widehat{S}_k(\mathcal{N}(\alpha_1,\alpha_2)^{\boxplus \tau_0},\widehat{\mathcal{U}^+}(\text{mor};\alpha_1,\alpha_2)))$ is a fiber product similar to (19.11) with k replaced by $k + \ell$.

Moreover $\widehat{S}_{k+\ell}(\mathcal{N}(\alpha_1, \alpha_2)^{\boxplus \tau_0}, \widehat{\mathcal{U}^+}(\mathrm{mor}; \alpha_1, \alpha_2))$ is also a fiber product similar to (19.11) with k replaced by $k + \ell$. The map

 $\pi_{\ell,k}: \widehat{S}_{\ell}(\widehat{S}_{k}(\mathcal{N}(\alpha_{1},\alpha_{2})^{\boxplus \tau_{0}},\widehat{\mathcal{U}^{+}}(\text{mor};\alpha_{1},\alpha_{2}))) \to \widehat{S}_{k+\ell}(\mathcal{N}(\alpha_{1},\alpha_{2})^{\boxplus \tau_{0}},\widehat{\mathcal{U}^{+}}(\text{mor};\alpha_{1},\alpha_{2})))$ in Proposition 24.16 becomes the identity map via those identifications.

- (8) The pull-back of the restriction of $\widehat{\mathfrak{S}}^+(\text{mor}; \alpha_1, \alpha_2)$ to $\widehat{S}_k(\mathcal{U}^+(\text{mor}; \alpha_1, \alpha_2))$ by the isomorphism in (6) is equivalent to the fiber product of $\widehat{\mathfrak{S}}^+(\text{mor}; *, *)$, $\widehat{\mathfrak{S}}^+(1; *, *)$, $\widehat{\mathfrak{S}}^+(2; *, *)$ along (19.11).
- (9) Suppose we are given a uniform family $\widehat{\mathfrak{S}^+}(i;\alpha_-,\alpha_+)^{\sigma}$ of CF-perturbations as in Condition 19.6. Then the σ -parametrized family of CF-perturbations $\widehat{\mathfrak{S}^+}(\operatorname{mor};\alpha_1,\alpha_2)^{\sigma}$ satisfying (5)(6)(7) for each σ is also uniform.

Remark 19.8. Note in Formula (19.11) we simplify the notation and omit the symbol of the underlying topological space. Namely we write $\widehat{\mathcal{U}^+}(1;\alpha_1,\alpha_{1,1})$ in

³⁷See Remark 16.2 for the sign and the order of the fiber products.

place of $(\mathcal{M}^1(\alpha_1, \alpha_{1,1})^{\boxplus \tau_0}, \widehat{\mathcal{U}^+}(1; \alpha_1, \alpha_{1,1}))$. From now on, we use this kinds of simplified notation when no confusion can occur.

Proof. The proof is by induction on k. It is entirely similar to the proof of Proposition 19.1. So we omit it.

We next rewrite the geometric conclusion of Proposition 19.7 to algebraic language.

Definition 19.9. In the situation of Proposition 19.7, we define

$$\psi_{\alpha_2,\alpha_1}^{\epsilon}: \Omega(R_{\alpha_1}; o_{R_{\alpha_1}}) \longrightarrow \Omega(R_{\alpha_2}; o_{R_{\alpha_2}})$$
(19.12)

by

$$\psi_{\alpha_2,\alpha_1}^{\epsilon}(h) = \operatorname{ev}_+!(\operatorname{ev}_-^*h;\widehat{\mathfrak{S}^{+\epsilon}}(\operatorname{mor};\alpha_1,\alpha_2)). \tag{19.13}$$

Here the right hand side is defined by Definition 17.67 on the K-space

$$(\mathcal{N}(\alpha_1, \alpha_2)^{\boxplus \tau_0}, \widehat{\mathcal{U}^+}(\text{mor}; \alpha_1, \alpha_2)).$$

By Condition 16.16 (VI) and [Part I, Definition 7.78], the degree of $\psi_{\alpha_1,\alpha_2}^{\epsilon}$ is $\mu(\alpha_1) - \mu(\alpha_2)$. Therefore, after the degree shift as in Definition 16.8 (2), its degree becomes 0. If the energy loss of our morphism $\mathfrak N$ is c, the family $\{\psi_{\alpha_1,\alpha_2}^{\epsilon}\}$ of maps induces

$$\mathfrak{F}^{\lambda}CF(\mathcal{F}_1) \longrightarrow \mathfrak{F}^{\lambda-c}CF(\mathcal{F}_2)$$

where the filtration \mathfrak{F}^{λ} is defined in Definition 16.8 (2)(3).

Lemma 19.10. The operators $\{\psi_{\alpha_2,\alpha_1}^{\epsilon}\}$ satisfy the following equality in the sense of (b).

$$d_0 \circ \psi_{\alpha_2,\alpha_1}^{\epsilon} - \psi_{\alpha_2,\alpha_1}^{\epsilon} \circ d_0 + \sum_{\alpha_1'} \mathfrak{m}_{1;\alpha_2,\alpha_2'}^{2,\epsilon} \circ \psi_{\alpha_2',\alpha_1}^{\epsilon} - \sum_{\alpha_1'} \psi_{\alpha_2,\alpha_1'}^{\epsilon} \circ \mathfrak{m}_{1;\alpha_1',\alpha_1}^{1,\epsilon} = 0.$$
 (19.14)

Here the first sum in the second line is taken over $\alpha'_2 \in \mathfrak{A}_2$ with $E(\alpha_1) - c \leq E(\alpha'_2) < E(\alpha_2)$ and the second sum in the second line is taken over $\alpha'_1 \in \mathfrak{A}_1$ with $E(\alpha_1) < E(\alpha'_1) \leq E(\alpha_2) + c$. Here $c \geq 0$ is the energy loss of our morphism.

Proof. By Stokes' formula the sum $d_0 \circ \psi_{\alpha_2,\alpha_1}^{\epsilon} - \psi_{\alpha_2,\alpha_1}^{\epsilon} \circ d_0$ is equal to the correspondence induced by the boundary of $\widehat{\mathfrak{S}^+}(\text{mor};\alpha_1,\alpha_2)$. By Proposition 19.7 (4) and the composition formula this is equal to the second line of (19.14).

19.3. **Proof of Theorem 16.9 (1) and Theorem 16.39 (1).** In this subsection we will prove Theorem 16.9 (1). We will also prove Theorem 16.39 (1) at the same time.

Situation 19.11. We study a partial linear K-system \mathcal{F}^i .

We define \mathfrak{E} as a set of $E \in \mathbb{R}$ such that one of the following holds.

- (1) There exist α_-, α_+ and i such that $\mathcal{N}^i(\alpha_-, \alpha_+) \neq \emptyset$ and $E(\alpha_+) E(\alpha_-) = E$
- (2) There exist α_-, α_+ and i such that $\mathcal{M}^i(\alpha_-, \alpha_+) \neq \emptyset$ and $E(\alpha_+) E(\alpha_-) = E$.

Then \mathfrak{E} is a discrete set by Definition 16.36 (2)(g). We put

$$\mathfrak{E} = \{E^1_{\mathfrak{G}}, E^2_{\mathfrak{G}}, \dots\}$$

such that $0 < E_{\mathfrak{C}}^1 < E_{\mathfrak{C}}^2 < \dots$ Note that the graded and filtered vector space $CF(\mathcal{F}^i)$ as in Definition 16.8 is independent of i. We denote it by $CF(\mathcal{F})$.

We observe that for each fixed i the set $\mathfrak{E} \cap [0, E_0]$ can be strictly bigger than the set $\mathfrak{E}_{\leq E_0}$ in (19.1). Nevertheless we can replace $\mathfrak{E}_{\leq E_0}$ by $\mathfrak{E} \cap [0, E_0]$ in Proposition 19.1 etc., by putting $\mathfrak{m}_{1;\alpha_+,\alpha_-}^{\epsilon} = 0$ when $\mathcal{M}^i(\alpha_-, \alpha_+) = \emptyset$.

In Situation 19.11, we take $i \in \mathbb{Z}_+$ and study the relationship between the operators $\mathfrak{m}_{1;\alpha_+,\alpha_-}^{i,\epsilon}$ and $\mathfrak{m}_{1;\alpha_+,\alpha_-}^{i+1,\epsilon}$ defined by applying Proposition 19.1 and Lemma 19.5 to partial linear K-systems \mathcal{F}^i and \mathcal{F}^{i+1} . The next definition will be applied with $\mathfrak{m}_{1;\alpha_+,\alpha_-}^1$ and $\mathfrak{m}_{1;\alpha_+,\alpha_-}^2$ replaced by $\mathfrak{m}_{1;\alpha_+,\alpha_-}^{i,\epsilon}$ and $\mathfrak{m}_{1;\alpha_+,\alpha_-}^{i+1,\epsilon}$ respectively.

Definition 19.12. Suppose we are in Situation 19.11.

- (1) A partial cochain complex structure on $CF(\mathcal{F})$ of energy cut level E_0 assigns $\mathfrak{m}_{1;\alpha_+,\alpha_-}$ to each α_+,α_- with $0 < E(\alpha_+) E(\alpha_-) \le E_0$ such that (19.5) is satisfied.
- (2) Let $(CF(\mathcal{F}), \{\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{j}\})$ be a partial cochain complex structure on $CF(\mathcal{F})$ of energy cut level E_0 for j=1,2. A partial cochain map of energy cut level E with energy loss 0 from $(CF(\mathcal{F}), \{\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{1}\})$ to $(CF(\mathcal{F}), \{\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{2}\})$ assigns a map $\psi_{\alpha_{2},\alpha_{1}}$ to each α_{1},α_{2} with $0 \leq E(\alpha_{2}) E(\alpha_{1}) \leq E_{0}$ such that (19.14) is satisfied. For $\alpha_{1} \neq \alpha_{2}$ with $E(\alpha_{1}) = E(\alpha_{2})$, we assign $\psi_{\alpha_{2},\alpha_{1}} = 0$. For $\alpha_{1} = \alpha_{2}$, we assign $\psi_{\alpha_{2},\alpha_{1}} = 0$ identity.
- (3) If $(CF(\mathcal{F}), \{\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}\})$ is a partial cochain complex structure on $CF(\mathcal{F})$ of energy cut level E_{0} and $E'_{0} < E_{0}$, then by forgetting a part of the operations $\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}$ we may regard $(CF(\mathcal{F}), \{\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}\})$ as a partial cochain complex structure on $CF(\mathcal{F})$ of energy cut level E'_{0} . We call it the reduction by energy cut at E'_{0} .

We can define a reduction by energy cut at E'_0 of a partial cochain map of energy cut level E_0 with energy loss 0 in the same way.

(4) Let $(CF(\mathcal{F}), \{\mathfrak{m}_{1;\alpha_+,\alpha_-}\})$ be a partial cochain complex structure of energy cut level E'. A partial cochain complex structure of energy cut level E is said to be its *promotion* if its reduction by energy cut at E_0 is $(CF(\mathcal{F}), \{\mathfrak{m}_{1;\alpha_+,\alpha_-}\})$.

A promotion of partial cochain map is defined in the same way.

The next lemma is a baby version of [FOOO4, Lemma 7.2.72].

Lemma 19.13. Let $(CF(\mathcal{F}), \{\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{j}\})$ be partial cochain complexes of energy cut level $E_{\mathfrak{E}}^{k_{j}}$ for j=1,2. Suppose $k_{1}< k_{2}$. Let $\{\psi_{\alpha_{2},\alpha_{1}}\}$ be a partial cochain map of energy cut level $E_{\mathfrak{E}}^{k_{1}}$ from $(CF(\mathcal{F}), \{\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{1}\})$ to a reduction by energy cut at $E_{\mathfrak{E}}^{k_{1}}$ of $(CF(\mathcal{F}), \{\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{1}\})$. Then there exists a promotion of $(CF(\mathcal{F}), \{\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{1}\})$ to energy cut level $E_{\mathfrak{E}}^{k_{2}}$ and a promotion of $\{\psi_{\alpha_{2},\alpha_{1}}\}$ to the energy cut level $E_{\mathfrak{E}}^{k_{2}}$ from this promoted partial cochain complex structure.

Proof. By an obvious induction argument it suffices to prove the case $k_2 = k_1 + 1$. Suppose $E(\alpha_+) - E(\alpha_-) = E_{\mathfrak{E}}^{k_2}$. We define a linear map

$$o(\alpha_+, \alpha_-) : \Omega(R_{\alpha_-}; o_{R_{\alpha_-}}) \longrightarrow \Omega(R_{\alpha_+}; o_{R_{\alpha_+}})$$

by

$$o(\alpha_+,\alpha_-) = \sum_{\alpha; E(\alpha_-) < E(\alpha) < E(\alpha_+)} (\mathfrak{m}^1_{1;\alpha_+,\alpha} \circ \mathfrak{m}^1_{1;\alpha,\alpha_-}).$$

We will prove that $o(\alpha_+, \alpha_-)$ is a d_0 -coboundary.

Notation 19.14. We use the following notation. For an \mathbb{R} linear map $F: CF(\mathcal{F}) \to CF(\mathcal{F}), F_{\alpha_+,\alpha_-}$ denotes the $\operatorname{Hom}_{\mathbb{R}}(\Omega(R_{\alpha_-},o_{R_{\alpha_-}}),\Omega(R_{\alpha_+},o_{R_{\alpha_+}}))$ component of F.

We put $\psi_{\alpha,\alpha} = \text{id.}$ We define $\hat{d}^j : CF(\mathcal{F}) \to CF(\mathcal{F}) \ (j = 1, 2) \text{ and } \widehat{\psi} : CF(\mathcal{F}) \to CF(\mathcal{F})$ by

$$\hat{d}^{j} = d_{0} \oplus \bigoplus_{\substack{\alpha_{1}, \alpha_{2} \\ E(\alpha_{2}) - E(\alpha_{1}) \le E_{k_{1}}}} \mathfrak{m}_{1;\alpha_{2},\alpha_{1}}^{j}, \quad \hat{\psi} = \bigoplus_{\substack{\alpha_{1}, \alpha_{2} \\ E(\alpha_{2}) - E(\alpha_{1}) \le E_{k_{1}}}} \psi_{\alpha_{2},\alpha_{1}}. \tag{19.15}$$

We have

$$(\hat{d}^1 \circ \hat{d}^1)_{\alpha_+,\alpha_-} = o(\alpha_+, \alpha_-),$$
 (19.16)

if $E(\alpha_+) - E(\alpha_-) = E_{\mathfrak{E}}^{k_2}$. On the other hand, we have

$$(\hat{d}^1 \circ \hat{d}^1)_{\alpha_2,\alpha_1} = 0 \tag{19.17}$$

if $E(\alpha_2) - E(\alpha_1) \leq E_{\mathfrak{G}}^{k_1}$. We note

$$(\hat{d}^1 \circ \hat{d}^1) \circ \hat{d}^1 - \hat{d}^1 \circ (\hat{d}^1 \circ \hat{d}^1) = 0. \tag{19.18}$$

Then (19.17) implies

$$((\hat{d}^{1} \circ \hat{d}^{1}) \circ \hat{d}^{1})_{\alpha_{+},\alpha_{-}} = (\hat{d}^{1} \circ \hat{d}^{1})_{\alpha_{+},\alpha_{-}} \circ d_{0},$$

$$(\hat{d}^{1} \circ (\hat{d}^{1} \circ \hat{d}^{1}))_{\alpha_{+},\alpha_{-}} = d_{0} \circ (\hat{d}^{1} \circ \hat{d}^{1})_{\alpha_{+},\alpha_{-}}.$$

Therefore (19.16) and (19.18) imply

$$d_0 \circ o(\alpha_+, \alpha_-) - o(\alpha_+, \alpha_-) \circ d_0 = 0.$$

Namely $o(\alpha_+, \alpha_-)$ is a cocycle.

We continue to show that $o(\alpha_+, \alpha_-)$ is a coboundary. When $E(\alpha_+) - E(\alpha_-) = E_{\mathfrak{G}}^{k_2}$, we put

$$b(\alpha_+, \alpha_-) = (\hat{d}^2 \circ \widehat{\psi} - \widehat{\psi} \circ \hat{d}^1)_{\alpha_+, \alpha_-}.$$

Since $\widehat{\psi}$ is assumed to be a cochain map of energy cut level $E_{\mathfrak{E}}^{k_1}$, we have

$$(\hat{d}^2 \circ \widehat{\psi} - \widehat{\psi} \circ \hat{d}^1)_{\alpha_2, \alpha_1} = 0$$

if $E(\alpha_2) - E(\alpha_1) \leq E_{\mathfrak{E}}^{k_1}$. Therefore we find

$$(\hat{d}^2 \circ (\hat{d}^2 \circ \hat{\psi} - \hat{\psi} \circ \hat{d}^1) + (\hat{d}^2 \circ \hat{\psi} - \hat{\psi} \circ \hat{d}^1) \circ \hat{d}^1)_{\alpha_+,\alpha_-}$$

$$= d_0 \circ b(\alpha_+, \alpha_-) + b(\alpha_+, \alpha_-) \circ d_0.$$
(19.19)

On the other hand, an obvious calculation leads to

$$(\hat{d}^{2} \circ (\hat{d}^{2} \circ \widehat{\psi} - \widehat{\psi} \circ \hat{d}^{1}) + (\hat{d}^{2} \circ \widehat{\psi} - \widehat{\psi} \circ \hat{d}^{1}) \circ \hat{d}^{1})_{\alpha_{+},\alpha_{-}}$$

$$= ((\hat{d}^{2} \circ \hat{d}^{2}) \circ \widehat{\psi} - \widehat{\psi} \circ (\hat{d}^{1} \circ \hat{d}^{1}))_{\alpha_{+},\alpha_{-}}$$

$$= (\hat{d}^{2} \circ \hat{d}^{2})_{\alpha_{+},\alpha_{-}} \circ \widehat{\psi}_{\alpha_{-},\alpha_{-}} - \widehat{\psi}_{\alpha_{+},\alpha_{+}} \circ (\hat{d}^{1} \circ \hat{d}^{1})_{\alpha_{+},\alpha_{-}}$$

$$= (\hat{d}^{2} \circ \hat{d}^{2})_{\alpha_{+},\alpha_{-}} - o(\alpha_{+},\alpha_{-}).$$

$$(19.20)$$

We observe that the equation $\hat{d}^2 \circ \hat{d}^2 = 0$ gives rise to

$$d_0 \circ \mathfrak{m}^2_{1:\alpha_+,\alpha_-} + \mathfrak{m}^2_{1:\alpha_+,\alpha_-} \circ d_0 + (\hat{d}^2 \circ \hat{d}^2)_{\alpha_+,\alpha_-} = 0, \tag{19.21}$$

since the energy cut level of $(CF(\mathcal{F}), \{\mathfrak{m}_{1;\alpha_+,\alpha_-}^2\})$ is $E_{\mathfrak{E}}^{k_2}$. Combination of (19.19), (19.20), (19.21) implies

$$o(\alpha_+, \alpha_-) = d_0 \circ (\mathfrak{m}^2_{1;\alpha_+,\alpha_-} - b(\alpha_+, \alpha_-)) - (\mathfrak{m}^2_{1;\alpha_+,\alpha_-} - b(\alpha_+, \alpha_-)) \circ d_0. \quad (19.22)$$

Namely $o(\alpha_+, \alpha_-)$ is a d_0 -coboundary. Therefore there exists $\mathfrak{m}^1_{1;\alpha_+,\alpha_-}$ such that

$$d_0 \circ \mathfrak{m}^1_{1;\alpha_+,\alpha_-} + \mathfrak{m}^1_{1;\alpha_+,\alpha_-} \circ d_0 + o(\alpha_+,\alpha_-) = 0. \tag{19.23}$$

Hence we have a promotion of $(CF(\mathcal{F}), \{\mathfrak{m}^1_{1:\alpha_+,\alpha_-}\})$ to the energy level $E^{k_2}_{\mathfrak{E}}$.

We note that the choice of $\mathfrak{m}_{1;\alpha_+,\alpha_-}^1$ satisfying (19.23) is not unique and can be changed by adding a d_0 -cocycle. We will use this freedom in the next part of the proof.

We next promote ψ . For this purpose it suffices to find ψ_{α_+,α_-} for $E(\alpha_+) - E(\alpha_-) = E_{\mathfrak{G}}^{k_2} - c$ such that

$$d_{0} \circ \psi_{\alpha_{+},\alpha_{-}} - \psi_{\alpha_{+},\alpha_{-}} \circ d_{0} + \mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{2} \circ \psi_{\alpha_{-},\alpha_{-}} \\ - \psi_{\alpha_{+},\alpha_{+}} \circ \mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{1} + b(\alpha_{+},\alpha_{-}) = 0.$$
(19.24)

We will prove it below. We put

$$\begin{split} o'(\alpha_{+},\alpha_{-}) &= \mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{2} \circ \psi_{\alpha_{-},\alpha_{-}} - \psi_{\alpha_{+},\alpha_{+}} \circ \mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{1} + b(\alpha_{+},\alpha_{-}) \\ &= \mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{2} - \mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{1} + b(\alpha_{+},\alpha_{-}). \end{split}$$

The identities (19.22) and (19.23) imply

$$d_0 \circ o'(\alpha_+, \alpha_-) + o'(\alpha_+, \alpha_-) \circ d_0 = 0.$$

Thus $o'(\alpha_+, \alpha_-)$ is a d_0 -cocycle. Using the freedom of the choice of $\mathfrak{m}^1_{1;\alpha_+,\alpha_-}$ we mentioned above, we may assume that $o'(\alpha_+, \alpha_-)$ is a d_0 -coboundary. Hence we can find ψ_{α_+,α_-} satisfying (19.24). The proof of Lemma 19.13 is complete. \square

Proof of Theorem 16.9 (1) and Theorem 16.39 (1). Below we will prove Theorem 16.39 (1) and indicate modifications needed to prove Theorem 16.9 (1).

Suppose we are in Situation 19.11. (We note that for the proof of Theorem 16.9 (1) we consider the situation where $\mathcal{F}^i = \mathcal{F}$ and the morphisms \mathfrak{N}^i appearing in Definition 16.36 (2)(d) are the identity morphisms for all i.)

We assume that the energy cut level of \mathcal{F}^i is $E^{k_i}_{\mathfrak{E}}$ such that $k_i < k_{i+1}$. (It implies $\lim_{i \to \infty} E^{k_i}_{\mathfrak{E}} = \infty$ by the discreteness of \mathfrak{E} .) We apply Proposition 19.1 and Lemma 19.5 to \mathcal{F}^i for each i. Then we obtain $\epsilon_{0,i}$ such that for each $\epsilon < \epsilon_{0,i}$ and i we obtain a partial cochain complex of energy cut level $E^{k_i}_{\mathfrak{E}}$, which we denote by $(CF(\mathcal{F}), \{\mathfrak{m}^{i,\epsilon}_{1;\alpha_+,\alpha_-}\})$. Note that the operator $\mathfrak{m}^{i,\epsilon}_{1;\alpha_+,\alpha_-}$ is defined as the correspondence of a Kuranishi structure and a CF-perturbation. We write them as $\widehat{\mathcal{U}^+}(i;\alpha_-,\alpha_+), \widehat{\mathfrak{S}^+}(i;\alpha_-,\alpha_+)$) respectively. We denote

$$CF(\mathcal{F}^i;\epsilon) = (CF(\mathcal{F}), \{\mathfrak{m}_{1;\alpha_+,\alpha_-}^{i,\epsilon}\}).$$
 (19.25)

This is well-defined if $\epsilon < \epsilon_{0,i}$, which is a partial cochain complex of energy cut level $E_{\mathfrak{E}}^{k_i}$ by Lemma 19.5.

We next consider the morphisms $\mathfrak{N}^i: \mathcal{F}^i \to \mathcal{F}^{i+1}$. Their interpolation spaces are denoted by $\mathcal{N}(i; \alpha_-, \alpha_+)$. We apply Proposition 19.7 to obtain a τ_{i+1} -collared

Kuranishi structure on it, which we write $\widehat{\mathcal{U}^+}(\text{mor}, i; \alpha_-, \alpha_+)$. We note that here we regard $\widehat{\mathcal{U}^+}(i; \alpha_-, \alpha_+)$ as a τ_{i+1} -collared Kuranishi structure. This is possible since $\tau_{i+1} < \tau_i$ and $\widehat{\mathcal{U}^+}(i; \alpha_-, \alpha_+)$ is τ_i -collared.

We next take $\rho_i > 0$ such that

$$\rho_i \le \min\{\epsilon_{k,i+1}/\epsilon_{k,i} \mid k = 0, 1, 2, \dots\}$$
(19.26)

where $\epsilon_{k,i+1}$, $\epsilon_{k,i}$, $k = 1, 2, \ldots$ will be defined later.

Remark 19.15. There appear only a finite number of k's. Note that $\epsilon_{0,i}$ is already defined above and $\epsilon_{1,i}$, $\epsilon_{2,i}$ and $\epsilon_{3,i}$ will be chosen after Lemma 19.16, Lemma 19.22 and Lemma 19.30 below, respectively. Other $\epsilon_{4,i}$ etc will be taken during the proof of Theorem 16.39 (3) in Subsection 19.6 for studying homotopy of homotopies.

We consider a CF-perturbation $\epsilon \mapsto \widehat{\mathfrak{S}^{+\rho_i\epsilon}}(i+1;\alpha_-,\alpha_+)$ of $\widehat{\mathcal{U}^+}(i+1;\alpha_-,\alpha_+)$, which is defined as follows. Note $\widehat{\mathfrak{S}^+}(i;\alpha_-,\alpha_+)$ is a CF-perturbation of $\widehat{\mathcal{U}^+}(i+1;\alpha_-,\alpha_+)$. Its local representative on the Kuranishi charts is $(W_{\mathfrak{r}},\omega_{\mathfrak{r}},\mathfrak{s}^{\epsilon}_{\mathfrak{r}})$. (See [Part I, Definition 7.3].) We replace $\mathfrak{s}^{\epsilon}_{\mathfrak{r}}$ by $\mathfrak{s}^{\epsilon\rho_i}_{\mathfrak{r}}$ but do not change anything else. It is easy to find that it is compatible with the coordinate change etc. and defines a CF-perturbation, which we denoted by $\epsilon \mapsto \widehat{\mathfrak{S}^{+\rho_i\epsilon}}(i+1;\alpha_-,\alpha_+)$ above. Hereafter we write it as $\widehat{\mathfrak{S}^{+\rho_i}}(i+1;\alpha_-,\alpha_+)$. By definition we have

$$f!(h; (\widehat{\mathfrak{S}^{+\rho_i}} \cdot (i+1; \alpha_-, \alpha_+)^{\epsilon}) = f!(h; (\widehat{\mathfrak{S}^{+\rho_i \epsilon}} (i+1; \alpha_-, \alpha_+)))$$
(19.27)

when the integrations along the fibers of the both hand sides are defined.

Now we apply Proposition 19.7 to obtain \mathfrak{S}^{+i} on $\widehat{\mathcal{U}}^+(\text{mor}; \alpha_-, \alpha_+)$. Here we use the CF-perturbations $\widehat{\mathfrak{S}}^+(i; \alpha_-, \alpha_+)$ on the space of connecting orbits of \mathcal{F}^i and $\widehat{\mathfrak{S}}^{+\rho_i}(i+1; \alpha_-, \alpha_+)$ on the space of connecting orbits of \mathcal{F}^{i+1} . (We put ρ_i for the second one.) Then we obtain from Proposition 19.7 a CF-perturbation on $\widehat{\mathcal{U}}^+(\text{mor}; \alpha_-, \alpha_+)$. We denote it by $\widehat{\mathfrak{S}}^{+i}(\text{mor}, i, \sigma_i; \alpha_-, \alpha_+)$).

Lemma 19.16. The family $\widehat{\mathfrak{S}^{+i}}(\text{mor}, i, \rho_i; \alpha_-, \alpha_+))$ for $\rho_i \in (0, \epsilon_{1,i+1}/\epsilon_{1,i}]$ is a uniform family in the sense of [Part I, Definition 9.28].

By [Part I, Proposition 7.88] and Lemma 19.16 we can find $\epsilon_{1,i}$ such that if $\epsilon < \epsilon_{1,i}$ the integration along the fiber is defined by using the CF-perturbation $\widehat{\mathfrak{S}^{+i}}(\text{mor},i,\rho_i;\alpha_-,\alpha_+))^{\epsilon}$ for $\epsilon < \epsilon_{1,i+1}$.

Remark 19.17. The fact that we can take ϵ in a way independent of ρ_i is crucial here. Otherwise, the process of defining those numbers would become circular. Namely while working in this step of induction, we do not know in advance how small $\epsilon_{0,i+1}$, $\epsilon_{1,i+1}$ must be. So we cannot estimate ρ_i from below at this stage.

Thus for $\epsilon < \min(\epsilon_{0,i}, \epsilon_{1,i})$ we obtain $\psi^i_{\alpha_+,\alpha_-}$ by (19.13). We use (19.27) together with Lemma 19.10 to prove that $\{\psi^i_{\alpha_+,\alpha_-}\}: CF(\mathcal{F}^i;\epsilon) \to CF(\mathcal{F}^{i+1};\epsilon)$ is a partial cochain map of energy cut level $E^{k_i}_{\mathfrak{E}}$. (Here we reduce the energy cut level of $CF(\mathcal{F}^i;\epsilon)$ to $E^{k_i}_{\mathfrak{E}}$.)

Now we use Lemma 19.13 inductively to promote $CF(\mathcal{F}^{i+1};\epsilon)$ to a partial cochain complex of energy cut level $E_{\mathfrak{E}}^{k_n}$ for each n and $\{\psi_{\alpha_+,\alpha_-}^i\}$ to a partial cochain map of energy cut level $E_{\mathfrak{E}}^{k_n}$ for each n.

To prove Theorem 16.9 (1), we regard $\mathcal{F} = \mathcal{F}^i$ and use the identity morphism $\mathcal{F} \to \mathcal{F}$ as $\mathfrak{N}^i : \mathcal{F}^i \to \mathcal{F}^{i+1}$ for all i. Then Theorem 16.39 (1) implies Theorem 16.9 (1).

The proofs of Theorem 16.9 (1) and Theorem 16.39 (1) are now complete. \Box

19.4. Composition of morphisms and of induced cochain maps. In this subsection we show that the composition of morphisms (defined in Lemma-Definition 16.35 and in Subsection 18.6) induces the composition of the partial cochain maps given by Definition 19.9. Since the partial cochain map in Definition 19.9 depends on the choice of the perturbation, we need to state it a bit carefully.

We consider a situation similar but slightly different from Situation 16.15. Namely;

Situation 19.18. (1) For j = 1, 2, 3, let

$$\mathcal{C}_j = \left(\mathfrak{A}_j, \mathfrak{G}_j, \{R_{\alpha_j}^j\}_{\alpha \in \mathfrak{A}_j}, \{o_{R_{\alpha_j}^j}\}_{\alpha \in \mathfrak{A}_j}, E, \mu, \{\operatorname{PI}_{\beta_j, \alpha_j}^j\}_{\beta_j \in \mathfrak{G}_j, \alpha_j \in \mathfrak{A}_j}\right)$$

be critical submanifold data and

$$\mathcal{F}_{j} = \left(\mathcal{C}_{j}, \{\mathcal{M}^{j}(\alpha_{j-}, \alpha_{j+})\}_{\alpha_{j\pm} \in \mathfrak{A}_{j}}, (ev_{-}, ev_{+}), \{OI_{\alpha_{j-}, \alpha_{j+}}^{j}\}_{\alpha_{j\pm} \in \mathfrak{A}_{j}}, \{PI_{\beta_{j}; \alpha_{j-}, \alpha_{j+}}^{j}\}_{\beta_{j} \in \mathfrak{G}_{j}, \alpha_{j\pm} \in \mathfrak{A}_{j}}\right)$$

linear K-systems. We assume $\mathfrak{G}_1 = \mathfrak{G}_2 = \mathfrak{G}_3$ (together with energy E and the Maslov index μ on it) and denote it by \mathfrak{G} .

(2) The same as (1) except the assumption that they consist of partial linear K-systems of energy cut level E_0 .

Situation 19.19. Suppose we are in Situation 19.18 (2)

We assume that for each j=1,2,3 we have $\widehat{\mathfrak{S}^+}(j;\alpha_{j-},\alpha_{j+})$ of $\widehat{\mathcal{U}^+}(j;\alpha_{j-},\alpha_{j+})$ on $\mathcal{M}^j(\alpha_{j-},\alpha_{j+})^{\boxplus \tau_0}$ satisfying the conclusions of Proposition 19.1. Here $\mathcal{M}^j(\alpha_{j-},\alpha_{j+})$ is as in Situation 19.18. (From now on, we write α_{\pm} in place of $\alpha_{j\pm}$ if no confusion can occur.) We assume that we have a partial morphism $\mathfrak{N}_{j+1j}: \mathcal{F}_j \to \mathcal{F}_{j+1}$ for j=1,2, whose interpolation space is $\mathcal{N}_{jj+1}(\alpha_j,\alpha_{j+1})$.

Furthermore, we have $\widehat{\mathfrak{S}^+}(\text{mor}; j+1, j; \alpha_j, \alpha_{j+1})$ and $\widehat{\mathcal{U}^+}(\text{mor}; j+1, j; \alpha_j, \alpha_{j+1})$ on $\mathcal{N}_{jj+1}(\alpha_j, \alpha_{j+1})$ satisfying the conclusions of Proposition 19.7. \blacksquare

We obtain a composition $\mathfrak{N}_{31}=\mathfrak{N}_{32}\circ\mathfrak{N}_{21}$ by Lemma-Definition 16.35 and Lemma-Definition 18.40. By (19.21) we have

$$\widehat{\mathcal{U}}(\text{mor}; 3, 1; \alpha_1, \alpha_3) = \bigcup_{\alpha_2 \in \mathfrak{A}_2} \widehat{\mathcal{U}}(\text{mor}; 2, 1; \alpha_1, \alpha_2) \times_{R_{\alpha_2}}^{\boxplus \tau} \widehat{\mathcal{U}}(\text{mor}; 3, 2; \alpha_2, \alpha_3).$$
(19.28)

Here the summand of the right hand side is defined by Definition 18.37. Therefore

$$\widehat{\mathcal{U}}(\text{mor}; 3, 1; \alpha_1, \alpha_3)^{\boxplus \tau} = \bigcup_{\alpha_2 \in \mathfrak{A}_2} \widehat{\mathcal{U}}(\text{mor}; 2, 1; \alpha_1, \alpha_2)^{\boxplus \tau} \times_{R_{\alpha_2}} \widehat{\mathcal{U}}(\text{mor}; 3, 2; \alpha_2, \alpha_3)^{\boxplus \tau}.$$
(19.29)

(Note we take an appropriate smoothing of corners in the right hand side.) On its underlying topological space

$$\bigcup_{\alpha_2 \in \mathfrak{A}_2} \mathcal{N}_{12}(\alpha_1, \alpha_2)^{\boxplus \tau} \times_{R_{\alpha_2}} \mathcal{N}_{23}(\alpha_2, \alpha_3)^{\boxplus \tau}, \tag{19.30}$$

we will define a Kuranishi structure

$$\widehat{\mathcal{U}^{+}}(\text{mor}; 3, 1; \alpha_{1}, \alpha_{3}) = \bigcup_{\alpha_{2} \in \mathfrak{A}_{2}} \widehat{\mathcal{U}^{+}}(\text{mor}; 2, 1; \alpha_{1}, \alpha_{2}) \times_{R_{\alpha_{2}}} \widehat{\mathcal{U}^{+}}(\text{mor}; 3, 2; \alpha_{2}, \alpha_{3})$$

$$(19.31)$$

as follows.

Remark 19.20. The fiber product appearing in each summand of (19.31) obviously gives a Kuranishi structure on each summand of (19.30). Below we explain how we smooth the corner to obtain a Kuranishi structure of the union.

Note that the Kuranishi structure $\widehat{\mathcal{U}^+}(\text{mor}; 2, 1; \alpha_1, \alpha_2)$ is τ' -collared. We take a Kuranishi structure $\widehat{\mathcal{U}^+}(\text{mor}; 2, 1; \alpha_1, \alpha_2)^{\boxminus (\tau - \tau')}$ such that

$$\left(\widehat{\mathcal{U}^+}(\mathrm{mor};2,1;\alpha_1,\alpha_2)^{\boxminus(\tau-\tau')}\right)^{\boxminus\tau'} = \widehat{\mathcal{U}^+}(\mathrm{mor};2,1;\alpha_1,\alpha_2).$$

We define $\widehat{\mathcal{U}^+}(\text{mor}; 3, 2; \alpha_2, \alpha_3)^{\boxminus(\tau - \tau')}$ in the same way. We consider a Kuranishi structure

$$\widehat{\mathcal{U}^{+}}(\text{mor}; 2, 1; \alpha_{1}, \alpha_{2})^{\boxminus(\tau - \tau')} \times_{R_{\alpha_{2}}} \widehat{\mathcal{U}^{+}}(\text{mor}; 3, 2; \alpha_{2}, \alpha_{3})^{\boxminus(\tau - \tau')}$$
(19.32)

on

$$\mathcal{N}(\mathrm{mor}; 2, 1; \alpha_1, \alpha_2)^{\boxplus \tau'} \times_{R_{\alpha_2}} \mathcal{N}(\mathrm{mor}; 3, 2; \alpha_2, \alpha_3)^{\boxplus \tau'}.$$

See Figure 23.

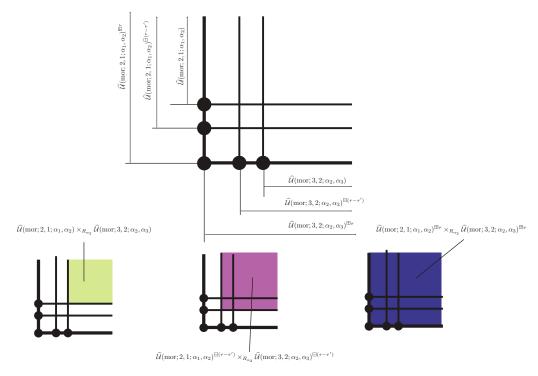


FIGURE 23. $\mathcal{N}(\text{mor}; 2, 1; \alpha_1, \alpha_2)^{\boxplus \tau} \times_{R_{\alpha_2}} \mathcal{N}(\text{mor}; 3, 2; \alpha_2, \alpha_3)^{\boxplus \tau}$.

On the other hand, let $\mathcal{N}_{123}(\alpha_1, \alpha_3)$ be as in Proposition 18.38. That is, after smoothing corners, its boundary contains $(\mathcal{N}(\alpha_1, \alpha_3), \widehat{\mathcal{U}}(\text{mor}; 3, 1; \alpha_1, \alpha_3))$. We denote the Kuranishi structure of $\mathcal{N}_{123}(\alpha_1, \alpha_3)$ by $\widehat{\mathcal{U}}_{123}(\alpha_1, \alpha_3)$.

We consider the complement \mathfrak{C}^c of the boundary components \mathfrak{C} of $\mathcal{N}_{123}(\alpha_1, \alpha_3)$ appearing in Proposition 18.38. Namely \mathfrak{C} consists of

$$\bigcup \mathcal{N}_{12}(\alpha_1, \alpha_2)^{\boxplus \tau} \times_{R_{\alpha_2}} \mathcal{N}_{23}(\alpha_2, \alpha_3)^{\boxplus \tau}.$$

The K-space $\mathcal{N}_{123}(\alpha_1, \alpha_3)$ is τ' - \mathfrak{C} -collared. We take $\mathcal{N}_{123}(\alpha_1, \alpha_3)^{\mathfrak{C} \boxminus (\tau - \tau')}$ and consider the topological space (see Figure 24)

$$\left(\mathcal{N}_{123}(\alpha_1, \alpha_3) \setminus \mathcal{N}_{123}(\alpha_1, \alpha_3)^{\mathfrak{C} \boxminus (\tau - \tau')}\right)^{\mathfrak{C}^c \boxminus \tau'}.$$
 (19.33)

Then $\widehat{\mathcal{U}}_{123}(\alpha_1, \alpha_3)$ induces a Kuranishi structure on it. We write $\widehat{\mathcal{U}}_{123}(\alpha_1, \alpha_3)^{\mathfrak{C} \boxminus (\tau - \tau')}$ by an abuse of notation.

Starting from (19.32) we apply Proposition 18.38 to obtain a τ' - \mathfrak{C} -collared Kuranishi structure on the topological space (19.33), which we denote by $\widehat{\mathcal{U}_{123}^+}(\alpha_1, \alpha_3)$. The Kuranishi structure $\widehat{\mathcal{U}}_{123}(\alpha_1, \alpha_3)^{\mathfrak{C} \boxminus (\tau - \tau')}$ is embedded in $\widehat{\mathcal{U}_{123}^+}(\alpha_1, \alpha_3)$.

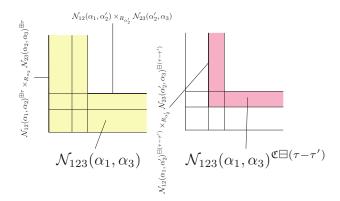


FIGURE 24. $\mathcal{N}_{123}(\alpha_1, \alpha_3)^{\mathfrak{C} \boxminus (\tau - \tau')}$

The \mathfrak{C} -partial smoothing of corners of $\widehat{\mathcal{U}_{123}^+}(\alpha_1, \alpha_3)$ defines our Kuranishi structure $\widehat{\mathcal{U}^+}(\text{mor}; 3, 1; \alpha_1, \alpha_3)$ of $\mathcal{N}_{13}(\alpha_1, \alpha_3)^{\boxplus \tau}$ appeared in (19.31), where $\mathcal{N}_{13}(\alpha_1, \alpha_3)$ is the interpolation space of the morphism \mathfrak{N}_{31} . We put

$$\widehat{\mathfrak{S}^{+}}(\operatorname{mor}; 3, 1; \alpha_{1}, \alpha_{3}) = \bigcup_{\alpha_{2} \in \mathfrak{A}_{2}} \widehat{\mathfrak{S}^{+}}(\operatorname{mor}; 2, 1; \alpha_{1}, \alpha_{2}) \times_{R_{\alpha_{2}}} \widehat{\mathfrak{S}^{+}}(\operatorname{mor}; 3, 2; \alpha_{2}, \alpha_{3}).$$

$$(19.34)$$

Here each summand of the right hand side of (19.34) gives a CF-perturbation of each summand in (19.31). Using the collared-ness, we can glue them to obtain a CF-perturbation on $\widehat{\mathcal{U}}^+$ (mor; 3, 1; α_1 , α_3). (In other words, they are automatically smooth on the part where they are glued.)

Lemma 19.21. The Kuranishi structure $\widehat{\mathcal{U}^+}(\text{mor}; 3, 1; \alpha_1, \alpha_3)$ and the CF-perturbation $\widehat{\mathfrak{S}^+}(\text{mor}; 3, 1; \alpha_1, \alpha_3)$ satisfy the conclusion of Proposition 19.7.

Proof. By construction, the corner contained in

$$\widehat{\mathcal{U}^+}(\mathrm{mor};2,1;\alpha_1,\alpha_2)\times_{R_{\alpha_2}}\widehat{\mathcal{U}^+}(2;\alpha_2,\alpha_2')\times_{R_{\alpha_2'}}\widehat{\mathcal{U}^+}(\mathrm{mor};3,2;\alpha_2',\alpha_3)$$

and

$$\widehat{\mathcal{U}}(\mathrm{mor}; 2, 1; \alpha_1, \alpha_2)^{\boxplus \tau} \times_{R_{\alpha_2}} \widehat{\mathcal{U}}(2; \alpha_2, \alpha_2')^{\boxplus \tau} \times_{R_{\alpha_2'}} \widehat{\mathcal{U}}(\mathrm{mor}; 3, 2; \alpha_2', \alpha_3)^{\boxplus \tau}$$

are smooth. Thereofore the boundary of $\widehat{\mathcal{U}^+}(\mathrm{mor};3,1;\alpha_1,\alpha_3)$ consists of

$$\widehat{\mathcal{U}^+}(1;\alpha_1,\alpha_1')\times_{R_{\alpha_1'}}\widehat{\mathcal{U}^+}(\mathrm{mor};2,1;\alpha_1',\alpha_2)\times_{R_{\alpha_2}}\widehat{\mathcal{U}^+}(\mathrm{mor};3,2;\alpha_2,\alpha_3)$$

and

$$\widehat{\mathcal{U}^+}(\mathrm{mor};2,1;\alpha_1,\alpha_2)\times_{R_{\alpha_2}}\widehat{\mathcal{U}^+}(\mathrm{mor};3,2;\alpha_2,\alpha_3')\times_{R_{\alpha_2'}}\widehat{\mathcal{U}^+}(3;\alpha_3',\alpha_3).$$

This is Proposition 19.7 (4). Since $\widehat{\mathcal{U}}_{123}(\alpha_1, \alpha_3)^{\mathfrak{C} \boxminus (\tau - \tau')}$ is embedded in $\widehat{\mathcal{U}}_{123}^+(\alpha_1, \alpha_3)$, the K-space $\widehat{\mathcal{U}}(\text{mor}; 3, 1; \alpha_1, \alpha_3)^{\boxplus \tau}$ is embedded in $\widehat{\mathcal{U}}^+(\text{mor}; 3, 1; \alpha_1, \alpha_3)$. This is Proposition 19.7 (1). The rest of the proof is obvious.

Let (i, i') be one of (3, 2), (2, 1), (3, 1). We use the pair $\widehat{\mathcal{U}^+}(\text{mor}; i', i; \alpha_i, \alpha_{i'})$ and $\widehat{\mathfrak{S}^+}(\text{mor}; i', i; \alpha_i, \alpha_{i'})$ to apply Definition 19.9. We then obtain:

$$\psi_{\alpha_{i'},\alpha_{i}}^{i'i;\epsilon}:\Omega(R_{\alpha_{i}};o_{R_{\alpha_{i}}})\longrightarrow\Omega(R_{\alpha_{i'}};o_{R_{\alpha_{i'}}}).$$

Lemma 19.22.

$$\psi_{\alpha_3,\alpha_1}^{31;\epsilon} = \sum_{\alpha_2 \in \mathfrak{A}_2} \psi_{\alpha_3,\alpha_2}^{32;\epsilon} \circ \psi_{\alpha_2,\alpha_1}^{21;\epsilon}$$
 (19.35)

in the sense of (b). (See Remark 19.4 for the meaning of (b).)

Proof. This is immediate from the composition formula and Lemma 18.34. \Box

We will choose $\epsilon_{2,i}$ so that (19.35) holds for $\epsilon < \epsilon_{2,i}$.

Corollary 19.23. Suppose we are in the Situation of Theorem 16.31 (3). Let E be an arbitrary positive number. Then we can make the choices to define \mathfrak{N}_{13} so that $\mathfrak{N}_{13} \equiv \mathfrak{N}_{23} \circ \mathfrak{N}_{12} \mod T^E$ holds.

19.5. Construction of homotopy. In this subsection we start from a homotopy of (partial) morphisms of linear K-systems and construct a (partial) cochain homotopy. We can study higher homotopy in the same way. Since the definition of parametrized morphism is a bit heavy, we discuss the case of homotopy first in this subsection. The general case of higher homotopy will be discussed in Subsection 19.7.

Situation 19.24. Suppose we are in Situation 16.20 (1). Suppose also we are given partial linear K-systems \mathcal{F}_i of energy cut level E_0 for i=1,2 and a τ -collared Kuranishi structure $\widehat{\mathcal{U}^+}(i;\alpha_{i-},\alpha_{i+})$ on $\mathcal{M}^i(\alpha_{i-},\alpha_{i+})^{\boxplus \tau_0}$ equipped with CF-perturbations $\widehat{\mathfrak{S}^+}(i;\alpha_{i-},\alpha_{i+})$ for i=1,2 which satisfy the conclusion of Proposition 19.1. Here $0 < \tau < \tau_0 = 1$ as in Proposition 19.1 and $\mathcal{M}^i(\alpha_{i-},\alpha_{i+})$ are as in Condition 16.16. (From now on, we write α_{\pm} in place of $\alpha_{i\pm}$ if no confusion can occur.)

Let
$$0 < \tau' < \tau < \tau_0 = 1$$
.

Situation 19.25. Suppose we are in Situation 19.24.

- (1) For j = 1, 2, we are given partial morphisms $\mathfrak{N}_j : \mathcal{F}_1 \to \mathcal{F}_2$ of energy cut level E_0 and energy loss c. We denote by $\mathcal{N}(j; \alpha_1, \alpha_2)$ its interpolation space.
- (2) Suppose we are given a homotopy \mathfrak{H} between partial morphisms \mathfrak{N}_1 and \mathfrak{N}_2 . By its definition, it is a [1,2]-parametrized family of partial morphisms from \mathcal{F}_1 to \mathcal{F}_2 . Suppose its energy cut level is E_0 and energy loss is c. We denote its interpolation space by $\mathcal{N}(\alpha_1, \alpha_2; [1,2])$.

Situation 19.26. Suppose we are in Situation 19.25. Suppose also that, for j=1,2, we are given a τ' -collared Kuranishi structure and a τ' -collared CF-perturbation on $\mathcal{N}(j;\alpha_1,\alpha_2)$ which satisfy the conclusions of Proposition 19.7. We denote them by $\widehat{\mathcal{U}}^+(\text{mor},j;\alpha_1,\alpha_2), \widehat{\mathfrak{S}}^+(\text{mor},j;\alpha_1,\alpha_2)$.

Before we state the main result, we explicitly write the boundary and corner compatibility conditions for the case of [1,2]-parametrized family below. The compatibility condition at the boundary (Condition 16.23) is as follows. ³⁸

$$\partial \mathcal{N}(\alpha_{1}, \alpha_{2}; [1, 2])$$

$$= \coprod_{\alpha'_{1} \in \mathfrak{A}_{1}} (-1)^{\dim \mathcal{N}(\alpha'_{1}, \alpha_{2}; [1, 2])} \mathcal{N}(\alpha'_{1}, \alpha_{2}; [1, 2]) \times_{R_{\alpha'_{1}}} \mathcal{M}^{1}(\alpha_{1}, \alpha'_{1})$$

$$\sqcup \coprod_{\alpha'_{2} \in \mathfrak{A}_{2}} (-1)^{\dim \mathcal{M}^{2}(\alpha'_{2}, \alpha_{2})} \mathcal{M}^{2}(\alpha'_{2}, \alpha_{2}) \times_{R_{\alpha'_{2}}} \mathcal{N}(\alpha_{1}, \alpha'_{2}; [1, 2])$$

$$\sqcup \mathcal{N}(2; \alpha_{1}, \alpha_{2}) \sqcup -\mathcal{N}(1; \alpha_{1}, \alpha_{2}). \tag{19.36}$$

The first of the corner compatibility conditions (Condition 16.26) says that the normalized corner $\hat{S}_k(\mathcal{N}(\alpha_1, \alpha_2; [1, 2]))$ is the disjoint union of the following two types of fiber products:

$$\mathcal{M}^{1}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \cdots \times_{R_{\alpha_{k_{1}-1}}} \mathcal{M}^{1}(\alpha_{k_{1}-1}, \alpha_{k_{1}})$$

$$\times_{R_{\alpha_{k_{1}}}} \mathcal{N}(\alpha_{k_{1}}, \alpha_{k_{1}+1}; [1, 2])$$

$$\times_{R_{\alpha_{k_{1}+1}}} \mathcal{M}^{2}(\alpha_{k_{1}+1}, \alpha_{k_{1}+2}) \times_{R_{\alpha_{k_{1}+2}}} \cdots \times_{R_{\alpha_{k_{1}+k_{2}}}} \mathcal{M}^{2}(\alpha_{k_{1}+k_{2}}, \alpha_{+}),$$
(19.37)

with $k_1 + k_2 = k$, and

$$\mathcal{M}^{1}(\alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \cdots \times_{R_{\alpha_{k_{1}-1}}} \mathcal{M}^{1}(\alpha_{k_{1}-1}, \alpha_{k_{1}})
\times_{R_{\alpha_{k_{1}}}} \mathcal{N}(j; \alpha_{k_{1}}, \alpha_{k_{1}+1})
\times_{R_{\alpha_{k_{1}+1}}} \mathcal{M}^{2}(\alpha_{k_{1}+1}, \alpha_{k_{1}+2}) \times_{R_{\alpha_{k_{1}+2}}} \cdots \times_{R_{\alpha_{k_{1}+k_{2}}}} \mathcal{M}^{2}(\alpha_{k_{1}+k_{2}}, \alpha_{+}),$$
(19.38)

with $k_1 + k_2 = k - 1$, j = 1, 2.

The second of the corner compatibility conditions (Condition 16.28) says the following: Consider the ℓ -th normalized corner of the K-spaces (19.37). According to the descriptions of $\widehat{S}_n(\mathcal{N}(\alpha_1, \alpha_2; [1, 2]))$, $\mathcal{N}(j; \alpha_1, \alpha_2)$ and $\mathcal{M}^i(\alpha_1, \alpha_2)$ we gave above, we see that the ℓ -th normalized corner of (19.37) or (19.38) is given by the disjoint union of the same type of fiber products as (19.37) or (19.38). Condition 16.28 requires that this description coincides with the description of $\widehat{S}_{k+\ell}(\mathcal{N}(\alpha_1, \alpha_2; [1, 2]))$.

³⁸See Remark 16.2 for the sign and the order of the fiber products.

Definition 19.27. We consider the evaluation map

$$\operatorname{ev}_{[1,2]}: \mathcal{N}(\alpha_1, \alpha_2; [1,2]) \longrightarrow [1,2].$$

The inverse image $\operatorname{ev}_{[1,2]}^{-1}(\partial[1,2])$ is a part of the boundary $\partial \mathcal{N}(\alpha_1,\alpha_2;[1,2])$. We denote it by $\partial_{\mathfrak{C}^v}$ and call it the *vertical boundary*. We put

$$\partial_{\mathfrak{C}^h} \mathcal{N}(\alpha_1, \alpha_2; [1, 2]) := \partial \mathcal{N}(\alpha_1, \alpha_2; [1, 2]) \setminus \partial_{\mathfrak{C}^v} \mathcal{N}(\alpha_1, \alpha_2; [1, 2])$$

and call it the horizontal boundary.

Note that we have the map

$$\operatorname{ev}_{[1,2]}: \mathcal{N}(\alpha_1, \alpha_2; [1,2])^{\mathfrak{C}^h \boxplus \tau} \longrightarrow [1,2].$$

Proposition 19.28. Suppose we are in Situations 19.24, 19.25, 19.26 and $\tau'' < \tau'$. Then for any α_1, α_2 with $E(\alpha_2) - E(\alpha_1) \leq E_0 - c$, there exist $\widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; [1, 2])$ and $\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; [1, 2])$ such that they enjoy the following properties.

- (1) $\widehat{\mathcal{U}}^+(\alpha_1, \alpha_2; [1, 2])$ is a τ'' - \mathfrak{C}^h -collared Kuranishi structure of $\mathcal{N}(\alpha_1, \alpha_2; [1, 2])^{\mathfrak{C}^h \boxplus \tau}$ and $\widehat{\mathfrak{S}}^+(\alpha_1, \alpha_2; [1, 2])$ is its τ'' - \mathfrak{C}^h -collared CF-perturbation.
- (2) $\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; [1,2])$ is transversal to 0. Moreover the map

$$(\operatorname{ev}_+, \operatorname{ev}_{[1,2]}) : \mathcal{N}(\alpha_1, \alpha_2; [1,2])^{\mathfrak{C}^h \boxplus \tau} \longrightarrow R_{\alpha_2} \times [1,2]$$

is strongly stratumwise submersive with respect to $\widehat{\mathfrak{S}}^+(\alpha_1, \alpha_2; [1, 2])$.

- (3) We have periodicity isomorphisms among $\widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; [1, 2])$'s that are compatible with $\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; [1, 2])$.
- (4) There exists an embedding of τ'' -collared Kuranishi structures from

$$\mathcal{N}(\alpha_1, \alpha_2; [1, 2])^{\mathfrak{C}^h \boxplus \tau}$$

to the τ'' - \mathfrak{C}^h -collared Kuranishi structure $\widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; [1, 2])$. Here for the source $\mathcal{N}(\alpha_1, \alpha_2; [1, 2])^{\mathfrak{C}^h \boxplus \tau}$ we use the τ'' - \mathfrak{C}^h -collared Kuranishi structure induced by that of $\mathcal{N}(\alpha_1, \alpha_2; [1, 2])$ which is given by the definition of [1, 2]-parametrized interpolation space.

(5) There is an isomorphism of τ'' - \mathfrak{C}^h -collared K-spaces ³⁹

$$\partial \widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; [1, 2])$$

$$= \coprod_{\alpha'_{1} \in \mathfrak{A}_{1}} (-1)^{\dim \widehat{\mathcal{U}^{+}}(\alpha'_{1}, \alpha_{2}; [1, 2])} \widehat{\mathcal{U}^{+}}(\alpha'_{1}, \alpha_{2}; [1, 2]) \times_{R_{\alpha'_{1}}} \widehat{\mathcal{U}^{+}}(1; \alpha_{1}, \alpha'_{1})$$

$$\sqcup \coprod_{\alpha'_{2} \in \mathfrak{A}_{2}} (-1)^{\dim \widehat{\mathcal{U}^{+}}(2; \alpha'_{2}, \alpha_{2})} \widehat{\mathcal{U}^{+}}(2; \alpha'_{2}, \alpha_{2}) \times_{R_{\alpha'_{2}}} \widehat{\mathcal{U}^{+}}(\alpha_{1}, \alpha'_{2}; [1, 2])$$

$$(19.39)$$

$$\sqcup \widehat{\mathcal{U}^+}(\text{mor}, 2; \alpha_1, \alpha_2) \sqcup -\widehat{\mathcal{U}^+}(\text{mor}; 1; \alpha_1, \alpha_2).$$

The isomorphism (19.39) is compatible with the isomorphism (19.36) via the embedding (4). It is also compatible with the periodicity isomorphism and the evaluation maps.

(6) The pull-back of $\widehat{\mathcal{U}}^+(\alpha_1, \alpha_2; [1, 2])$ by the isomorphism (19.39) is equivalent to the fiber product of $\widehat{\mathfrak{S}}^+(j; \alpha, \alpha')$, $\widehat{\mathfrak{S}}^+(\operatorname{mor}, j; \alpha, \alpha')$ and $\widehat{\mathfrak{S}}^+(\alpha, \alpha'; [1, 2])$.

 $^{^{39}}$ See Remark 16.2 for the sign and the order of the fiber products.

(7) The normalized corner $\widehat{S}_k(\widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; [1, 2]))$ is a disjoint union of the following two types of fiber products.

$$\widehat{\mathcal{U}^{+}}(1; \alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \cdots \times_{R_{\alpha_{k_{1}}-1}} \widehat{\mathcal{U}^{+}}(1; \alpha_{k_{1}-1}, \alpha_{k_{1}}) \\
\times_{R_{\alpha_{k_{1}}}} \widehat{\mathcal{U}^{+}}(\alpha_{k_{1}}, \alpha_{k_{1}+1}; [1, 2]) \\
\times_{R_{\alpha_{k_{1}+1}}} \widehat{\mathcal{U}^{+}}(2; \alpha_{k_{1}+1}, \alpha_{k_{1}+2}) \times_{R_{\alpha_{k_{1}+2}}} \cdots \times_{R_{\alpha_{k_{1}+k_{2}}}} \widehat{\mathcal{U}^{+}}(2; \alpha_{k_{1}+k_{2}}, \alpha_{+})$$
and
$$(19.40)$$

$$\widehat{\mathcal{U}^{+}}(1; \alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \cdots \times_{R_{\alpha_{k_{1}-1}}} \widehat{\mathcal{U}^{+}}(1; \alpha_{k_{1}-1}, \alpha_{k_{1}})$$

$$\times_{R_{\alpha_{k_{1}}}} \widehat{\mathcal{U}^{+}}(\operatorname{mor}; j; \alpha_{k_{1}}, \alpha_{k_{1}+1})$$
(19.41)

$$\times_{R_{\alpha_{k_1+1}}} \widehat{\mathcal{U}^+}(2; \alpha_{k_1+1}, \alpha_{k_1+2}) \times_{R_{\alpha_{k_1+2}}} \dots \times_{R_{\alpha_{k_1+k_2}}} \widehat{\mathcal{U}^+}(2; \alpha_{k_1+k_2}, \alpha_+).$$

This isomorphism is compatible with the isomorphism (19.37), (19.38) via the embedding (4). It is also compatible with the periodicity isomorphism and the evaluation maps.

- (8) The pull-back of the restriction of $\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; [1, 2])$ to $\widehat{S}_k(\widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; [1, 2]))$ by the isomorphism in (7) is equivalent to the fiber product of $\widehat{\mathfrak{S}^+}(\text{mor}; j; *, *)$, $\widehat{\mathfrak{S}^+}(1; *, *)$, $\widehat{\mathfrak{S}^+}(2; *, *)$.
- (9) The isomorphism of (7) is compatible with the covering map

$$\widehat{S}_{\ell}(\widehat{S}_{k}(\widehat{\mathcal{U}^{+}}(\alpha_{1},\alpha_{2};[1,2]))) \longrightarrow \widehat{S}_{k+\ell}(\widehat{\mathcal{U}^{+}}(\alpha_{1},\alpha_{2};[1,2])).$$

(The precise meaning of this compatibility is the same as the case of

$$\mathcal{N}(\alpha_1,\alpha_2;[1,2]),$$

which we explained right before this proposition.)

(10) If we start from a uniform family of $\widehat{\mathfrak{S}^+}(i; \alpha_-, \alpha_+)$ and $\widehat{\mathfrak{S}^+}(\operatorname{mor}, j; \alpha_1, \alpha_2)$, then we can take the family of $\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; [1, 2])$ to be uniform.

Proof. We prove the Proposition 19.28 with E_0 replaced by $E_{\mathfrak{E}}^n$, by induction on n. The corner compatibility conditions Conditions 16.26 and Conditions 16.28 are written in such a way that they immediately imply the assumptions of Proposition 17.46, which is Situation 17.43 (especially its (1)(2)).

We rewrite the geometric conclusion of Proposition 19.28 into algebraic language.

Definition 19.29. In the situation of Proposition 19.28, we define

$$\mathfrak{h}_{\alpha_2,\alpha_1}^{\epsilon}: \Omega(R_{\alpha_1};o_{R_{\alpha_1}}) \longrightarrow \Omega(R_{\alpha_2};o_{R_{\alpha_2}})$$
 (19.42)

by

$$\mathfrak{h}_{\alpha_2,\alpha_1}^{\epsilon}(h) = \operatorname{ev}_{+}!(\operatorname{ev}_{-}^{*}h;\widehat{\mathfrak{S}^{+\epsilon}}(\alpha_1,\alpha_2;[1,2])). \tag{19.43}$$

Here the right hand side is defined by Definition 17.67 on the K-space

$$(\mathcal{N}(\alpha_1, \alpha_2; [1, 2])^{\mathfrak{C}^h \boxplus \tau_0}, \widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; [1, 2])).$$

By Condition 16.21 (VI) and [Part I, Definition 7.78], the degree of $\mathfrak{h}_{\alpha_2,\alpha_1}^{\epsilon}$ is $\eta(\alpha_2) - \eta(\alpha_1) - 1$. Therefore after degree shift as in Definition 16.8 (2) its degree becomes -1.

If the energy loss of our homotopy is c, the family $\{\mathfrak{h}_{\alpha_2,\alpha_1}^{\epsilon}\}$ of maps induces

$$\mathfrak{F}^{\lambda}CF(\mathcal{F}_1) \to \mathfrak{F}^{\lambda-c}CF(\mathcal{F}_2)$$

where the filtration \mathfrak{F}^{λ} is defined in Definition 16.8 (2)(3).

Suppose we are in the situation of Proposition 19.28. We have two morphisms of partial linear K-systems equipped with CF-perturbations. Namely we have $\mathcal{N}(j;\alpha_1,\alpha_2), \widehat{\mathcal{U}^+}(\text{mor},j;\alpha_1,\alpha_2), \widehat{\mathfrak{S}^+}(\text{mor},j;\alpha_1,\alpha_2)$ for j=1,2. We use Definition 19.9 for j=1,2 to obtain maps

$$\psi_{\alpha_2,\alpha_1}^{j,\epsilon}:\Omega(R_{\alpha_1};o_{R_{\alpha_1}})\to\Omega(R_{\alpha_2};o_{R_{\alpha_2}}).$$

Lemma 19.30. The linear maps $\{\mathfrak{h}_{\alpha_1,\alpha_2}^{\epsilon}\}$ satisfy the following equality in the sense of (\flat) in Remark 19.4:

$$d_{0} \circ \mathfrak{h}_{\alpha_{2},\alpha_{1}}^{\epsilon} + \mathfrak{h}_{\alpha_{2},\alpha_{1}} \circ d_{0}$$

$$= -\sum_{\alpha_{1}'} \mathfrak{h}_{\alpha_{2},\alpha_{1}'}^{\epsilon} \circ \mathfrak{m}_{1;\alpha_{1}',\alpha_{1}}^{1,\epsilon} - \sum_{\alpha_{2}'} \mathfrak{m}_{1;\alpha_{2},\alpha_{2}'}^{2,\epsilon} \circ \mathfrak{h}_{\alpha_{2}',\alpha_{1}}^{2,\epsilon} + \psi_{\alpha_{2},\alpha_{1}}^{2,\epsilon} - \psi_{\alpha_{2},\alpha_{1}}^{1,\epsilon}.$$
(19.44)

Here the first sum in the second line is taken over $\alpha'_1 \in \mathfrak{A}_1$ with $E(\alpha_1) < E(\alpha'_1) \le E(\alpha_2) + c$ and the second sum in the second line is taken over $\alpha'_2 \in \mathfrak{A}_2$ with $E(\alpha_1) - c \le E(\alpha'_2) < E(\alpha_2)$. The number c is the energy loss of our morphism.

We will define $\epsilon_{3,i}$ so that (19.44) holds for $\epsilon < \epsilon_{3,i}$.

Proof. The proof is similar to the proof of Lemmas 19.5, 19.10. By Stokes' formula the left hand side is obtained from $\partial \widehat{\mathfrak{S}}^{+\epsilon}(\alpha_1, \alpha_2; [1, 2])$ in the same way as (19.43). We can decompose the boundary $\partial \widehat{\mathfrak{S}}^{+\epsilon}(\alpha_1, \alpha_2; [1, 2])$ into a disjoint union by Proposition 19.28. Then we use the composition formula to obtain the right hand side of (19.44). Namely the 1,2,3,4-th union of (19.39) correspond to the 1,2,3,4-th term of (19.44), respectively.

19.6. Proof of Theorem 16.9 (2)(except (f)), Theorem 16.31 (1) and Theorem 16.39 (2)(except (e)), (3). In this subsection we use the result of Subsection 19.5 to prove Theorem 16.9 (2) (a)-(e). We also prove Theorem 16.31 (1) and Theorem 16.39 (2)(3) (except (2)(e)) at the same time.

To prove Theorem 16.31 (1) we need to take the 'projective limit' in a similar way as the argument of Subsection 19.3. Proposition 19.33 below is its algebraic part. It is similar to Lemma 19.13 and is a baby version of [FOOO4, Lemma 7.2.129].

Definition 19.31. Suppose we are given two partial linear K-systems \mathcal{F}_i (i = 1, 2). Also suppose we are given $(CF(\mathcal{F}_i), \{\mathfrak{m}_{1;\alpha_+,\alpha_-}^i\})$, a partial cochain complex structure on $CF(\mathcal{F}_i)$ of energy cut level $E_{(i)}$ for i = 1, 2. We assume $0 \le c < E_0 \le E_{(1)}$ and $E_0 \le E_{(2)} - c$.

(1) A partial cochain map $\mathcal{F}_1 \to \mathcal{F}_2$ of energy cut level E_0 and energy loss c is a family $\widehat{\psi} = \{\psi_{\alpha_2,\alpha_1}\}$ consisting of the following objects ψ_{α_2,α_1} : If $\alpha_i \in \mathfrak{A}_i$ and $E(\alpha_2) \leq E(\alpha_1) + E_0 - c$, we have an \mathbb{R} linear map

$$\psi_{\alpha_2,\alpha_1}: \Omega(R_{\alpha_1};o_{R_{\alpha_1}}) \longrightarrow \Omega(R_{\alpha_2};o_{R_{\alpha_2}}).$$

We require that it satisfies (19.14) for $E(\alpha_2) \leq E(\alpha_1) + E_0 - c$.

(2) For j=1,2, let $\{\psi_{j;\alpha_2,\alpha_1}\}$ be partial cochain maps of energy cut level E_0 with energy loss c. A partial cochain homotopy of energy cut level E_0 with energy loss c from $\widehat{\psi}_1 = \{\psi_{1;\alpha_2,\alpha_1}\}$ to $\widehat{\psi}_2 = \{\psi_{2;\alpha_2,\alpha_1}\}$ is a family $\widehat{\mathfrak{h}} = \{\mathfrak{h}_{\alpha_2,\alpha_1}\}$ consisting of the following objects $\mathfrak{h}_{\alpha_2,\alpha_1}$:

If $\alpha_i \in \mathfrak{A}_i$ and $E(\alpha_2) \leq E(\alpha_1) + E_0 - c$, we have an \mathbb{R} linear map

$$\mathfrak{h}_{\alpha_2,\alpha_1}: \Omega(R_{\alpha_1};o_{R_{\alpha_1}}) \longrightarrow \Omega(R_{\alpha_2};o_{R_{\alpha_2}}).$$

We require that it satisfies (19.44) for $E(\alpha_2) \leq E(\alpha_1) + E_0 - c$.

(3) For i=1,2, let $\widehat{\psi}_{i+1i}=\{\psi_{\alpha_{i+1},\alpha_i}\}$ be partial cochain maps of energy loss c_i . Its energy cut level is E_0 for i=1 and E_0-c_1 for i=2. We define the composition $\widehat{\psi}_{31}=\widehat{\psi}_{32}\circ\widehat{\psi}_{21}$ by

$$(\psi_{31})_{\alpha_3\alpha_1} = \sum_{\alpha_2 \in \mathfrak{A}_2} (\psi_{32})_{\alpha_3\alpha_2} \circ (\psi_{21})_{\alpha_2\alpha_1}.$$

Then $\widehat{\psi}_{31}$ is a partial cochain map of energy cut level E_0 and energy loss $c_1 + c_2$.

(4) Consider E'_0 such that $c < E'_0 < E_0$. In the situation of (1) we forget all of ψ_{α_2,α_1} for $E(\alpha_2) \leq E(\alpha_1) + E'_0 - c$. We then obtain a partial cochain map $\mathcal{F}_1 \to \mathcal{F}_2$ of energy cut level E'_0 . We call it the energy cut of $\widehat{\psi}$ at energy cut level E'_0 .

The energy cut of $\hat{\mathfrak{h}}$ at energy cut level E_0' is defined in the same way.

(5) Let $\widehat{\psi}: \mathcal{F}_1 \to \mathcal{F}_2$ be a partial cochain map of energy cut level E_0 and energy loss c and let $c < E'_0 < E_0$. If $\widehat{\psi}'$ is an energy cut of $\widehat{\psi}$ at energy cut level E'_0 , we call $\widehat{\psi}$ a promotion of $\widehat{\psi}'$ to the energy cut level E_0 . A promotion of a partial cochain homotopy is defined in the same way.

Lemma-Definition 19.32. Two partial cochain maps are said to be cochain homotopic if there exists a partial cochain homotopy between them. This is an equivalence relation.

Proof. If $\widehat{\mathfrak{h}}^j = \{\mathfrak{h}^j_{\alpha_2\alpha_1}\}$ is a partial cochain homotopy from $\widehat{\psi}_j$ to $\widehat{\psi}_{j+1}$ for j=1,2, then $\{\mathfrak{h}^1_{\alpha_2\alpha_1} + \mathfrak{h}^2_{\alpha_2\alpha_1}\}$ is a partial cochain homotopy from $\widehat{\psi}_1$ to $\widehat{\psi}_3$. The other part of the proof is obvious.

Proposition 19.33. Let $(CF(\mathcal{F}_i), \{\mathfrak{m}_{1;\alpha_+\alpha_-}^i\})$ and $(CF(\mathcal{F}'_i), \{\mathfrak{m}_{1;\alpha_+\alpha_-}^{i'}\})$ be partial cochain complexes of energy cut level E_2 for i=1,2. We take $E_1 < E_2$. Suppose we have a diagram

$$(CF(\mathcal{F}'_{1}), \{\mathfrak{m}^{1\prime}_{1;\alpha_{+},\alpha_{-}}\}) \xrightarrow{\widehat{\psi}'_{21}} (CF(\mathcal{F}'_{2}), \{\mathfrak{m}^{2\prime}_{1;\alpha_{+},\alpha_{-}}\})$$

$$\widehat{\psi}_{1} \uparrow \qquad \qquad \uparrow \widehat{\psi}_{2} \qquad (19.45)$$

$$(CF(\mathcal{F}_{1}), \{\mathfrak{m}^{1}_{1;\alpha_{+},\alpha_{-}}\}) \xrightarrow{\widehat{\psi}_{21}} (CF(\mathcal{F}_{2}), \{\mathfrak{m}^{2}_{1;\alpha_{+},\alpha_{-}}\})$$

such that:

- (i) $\widehat{\psi}_{21}$, $\widehat{\psi}'_{21}$ are partial cochain maps of energy cut level E_2 and energy loss 0. We assume that $\widehat{\psi}_{21}$, $\widehat{\psi}'_{21}$ induces isomorphisms modulo T^{ϵ} for a sufficiently small $\epsilon > 0$.
- (ii) $\widehat{\psi}_2$ is a partial cochain map of energy cut level E_2 and energy loss c.
- (iii) ψ_1 is a partial cochain map of energy cut level E_1 and energy loss c.
- (iv) The Diagram 19.45 is homotopy commutative as partial cochain maps of energy cut level E_1 and energy loss c. Namely there exists a partial cochain

homotopy $\widehat{\mathfrak{h}} = \{\mathfrak{h}_{\alpha_2,\alpha_1}\}$ where $\mathfrak{h}_{\alpha_2,\alpha_1}$ is defined when $E(\alpha_2) \leq E(\alpha_1) + E_2 - c$, and satisfies

$$d_0 \circ \mathfrak{h}_{\alpha'_2,\alpha_1} + \mathfrak{h}_{\alpha'_2,\alpha_1} \circ d_0$$

$$\begin{split} &= -\sum_{\hat{\alpha}_{1} \in \mathfrak{A}_{1}} \mathfrak{h}_{\alpha'_{2}, \hat{\alpha}_{1}} \circ \mathfrak{m}_{1; \hat{\alpha}_{1}, \alpha_{1}}^{1} - \sum_{\hat{\alpha}'_{2} \in \mathfrak{A}'_{2}} \mathfrak{m}_{1; \alpha'_{2}, \hat{\alpha}'_{2}}^{2} \circ \mathfrak{h}_{\hat{\alpha}'_{2}, \alpha_{1}}^{2} \\ &+ \sum_{\hat{\alpha}_{2} \in \mathfrak{A}_{2}} (\hat{\psi}_{2})_{\alpha'_{2} \hat{\alpha}_{2}} \circ (\hat{\psi}_{21})_{\hat{\alpha}_{2} \alpha_{1}} - \sum_{\hat{\alpha}'_{1} \in \mathfrak{A}'_{1}} (\hat{\psi}'_{21})_{\alpha'_{2} \hat{\alpha}'_{1}} \circ (\hat{\psi}_{1})_{\hat{\alpha}'_{1}, \alpha_{1}}, \end{split}$$
(19.46)

if
$$E(\alpha_2') \leq E(\alpha_1) + E_1 - c$$
.

Then we can promote $\widehat{\psi}_1$ to a partial cochain map of energy cut level E_2 and energy loss c and $\widehat{\mathfrak{h}}$ to a partial cochain homotopy of energy cut level E_2 and energy loss c. That is, (19.46) holds if $E(\alpha_2') \leq E(\alpha_1) + E_2 - c$.

Proof. By using induction and the discreteness of the set of energies (which follows from uniform Gromov compactness Definition 16.36~(2)~(g)), it suffices to prove the statement for the case when

$$E(\alpha'_{2}) - E(\alpha_{1}) \notin (E_{1} - c, E_{2} - c) \qquad \text{for all } (\alpha_{1}, \alpha'_{2}) \in \mathfrak{A}_{1} \times \mathfrak{A}'_{2}.$$

$$E(\alpha'_{i}) - E(\alpha_{i}) \notin (E_{1} - c, E_{2} - c) \qquad \text{for all } (\alpha_{i}, \alpha'_{i}) \in \mathfrak{A}_{i} \times \mathfrak{A}'_{i}, i = 1, 2.$$

$$E(\alpha'_{1}) - E(\alpha_{1}) \notin (E_{1}, E_{2}) \qquad \text{for all } (\alpha_{1}, \alpha'_{1}) \in \mathfrak{A}_{1} \times \mathfrak{A}'_{1}.$$

$$E(\alpha'_{2}) - E(\alpha_{2}) \notin (E_{1} - c, E_{2} - c) \qquad \text{for all } (\alpha_{2}, \alpha'_{2}) \in \mathfrak{A}_{2} \times \mathfrak{A}'_{2}.$$

$$(19.47)$$

We will prove the proposition for this case below.

We first promote $\hat{\psi}_1$ to the energy cut level E_2 . We regard both $\hat{\psi}_j$ (j = 1, 2) as partial cochain maps of energy cut level E_1 and energy loss c. We use them to define $\hat{\psi}_j : CF(\mathcal{F}_j) \to CF(\mathcal{F}'_j)$ as

$$\widehat{\psi}_j = \bigoplus_{\alpha_j, \alpha_j'} (\psi_j)_{\alpha_j' \alpha_j}$$

where we consider α_j , α'_j with $E(\alpha'_j) - E(\alpha_j) \leq E_1$. Note that $(\psi_2)_{\alpha'_2\alpha_2}$ is defined for α'_2, α_2 satisfying $E(\alpha'_2) - E(\alpha_2) \leq E_2$. To clarify the energy cut we did for $\widehat{\psi}_2$ we write $\widehat{\psi}_2|_{E_1}$. We define $\widehat{\mathfrak{h}}: CF(\mathcal{F}_1) \to CF(\mathcal{F}'_2)$ in the same way. We also drop the energy cut level E_2 of $\widehat{\psi}'_{21}$, $\widehat{\psi}_{21}$ to E_1 , which we write $\widehat{\psi}'_{21}|_{E_1}$, $\widehat{\psi}_{21}|_{E_1}$ respectively. We regard them as

$$\widehat{\psi}_{21}'|_{E_1}: CF(\mathcal{F}_1') \to CF(\mathcal{F}_2'), \quad \widehat{\psi}_{21}|_{E_1}: CF(\mathcal{F}_1) \to CF(\mathcal{F}_2)$$

in that sense. We define

$$\widehat{d}_{j}|_{E_{1}}: CF(\mathcal{F}_{j}) \to CF(\mathcal{F}_{j}), \qquad \widehat{d}'_{j}|_{E_{1}}: CF(\mathcal{F}'_{j}) \to CF(\mathcal{F}'_{j})$$

in the same way.

Let
$$E(\alpha'_1) - E(\alpha_1) = E_2 - c$$
. We define $o(\alpha_1, \alpha'_1) \in \operatorname{Hom}_{\mathbb{R}}(o_{R_{\alpha_1}}, o_{R_{\alpha'_1}})$ by

$$o(\alpha_1, \alpha_1') = (\widehat{d}_1' \circ \widehat{\psi}_1 - \widehat{\psi}_1 \circ \widehat{d}_1)_{\alpha_1' \alpha_1}. \tag{19.48}$$

As in the proof of Lemma 19.13, we can show

$$d_0 \circ o(\alpha_1, \alpha_1') + o(\alpha_1, \alpha_1') \circ d_0 = 0.$$

We will prove that $o(\alpha_1, \alpha_1')$ is a d_0 -coboundary. We consider

$$(\widehat{\psi}'_{21}|_{E_1} \circ (\widehat{d}'_1 \circ \widehat{\psi}_1 - \widehat{\psi}_1 \circ \widehat{d}_1))_{\alpha'_1 \alpha_1}.$$

Using the fact that (19.48) is zero for $E(\alpha'_1) - E(\alpha_1) < E_0 - c$ and $\widehat{\psi}_{**}$ has energy loss 0 (Definition 19.31 (6)), we find that

$$(\widehat{\psi}_{21}' \circ (\widehat{d}_1' \circ \widehat{\psi}_1 - \widehat{\psi}_1 \circ \widehat{d}_1))_{\alpha_1 \alpha_1'} = o(\alpha_1, \alpha_1').$$

On the other hand, we have

$$(\widehat{\psi}'_{21}|_{E_1} \circ (\widehat{d}'_1|_{E_1} \circ \widehat{\psi}_1 - \widehat{\psi}_1 \circ \widehat{d}_1|_{E_1}))_{\alpha_1 \alpha'_1} \equiv (\widehat{d}'_2|_{E_1} \circ \widehat{\psi}'_{21}|_{E_1} \circ \widehat{\psi}_1 - \widehat{\psi}'_{21}|_{E_1} \circ \widehat{\psi}_1 \circ \widehat{d}_1|_{E_1})_{\alpha'_1 \alpha_1}.$$
(19.49)

Here \equiv means modulo d_0 -coboundary. Such d_0 -coboundary appears because

$$(\widehat{d}_{2}'|_{E_{1}} \circ \widehat{\psi}_{21}'|_{E_{1}} - \widehat{\psi}_{21}'|_{E_{1}} \circ \widehat{d}_{2}'|_{E_{1}})_{\alpha_{1}'\alpha_{1}} = d_{0} \circ (\widehat{\psi}_{21}')_{\alpha_{1}'\alpha_{1}} - (\widehat{\psi}_{21}')_{\alpha_{1}'\alpha_{1}} \circ d_{0}.$$

Then we use

$$(\widehat{\psi}_{21}'|_{E_1}\circ\widehat{\psi}_1-\widehat{\psi}_2|_{E_1}\circ\widehat{\psi}_{21}|_{E_1})_{\alpha_1'\alpha_1}=d_0\circ\mathfrak{h}_{\alpha_1,\alpha_1'}+\mathfrak{h}_{\alpha_1'\alpha_1}\circ d_0$$

to show that

$$(19.49) \equiv (\widehat{d}_2'|_{E_1} \circ \widehat{\psi}_2|_{E_1} \circ \widehat{\psi}_{21}|_{E_1} - \widehat{\psi}_2|_{E_1} \circ \widehat{\psi}_{21}|_{E_1} \circ \widehat{d}_2|_{E_1})_{\alpha_1',\alpha_1}. \tag{19.50}$$

The right hand side of (19.50) is a d_0 -coboundary since the partial cochain maps $\widehat{\psi}_{21}|_{E_1}$, $\widehat{\psi}_2|_{E_1}$ appearing here can be promoted to the energy cut level E_2 by assumption. Thus $o(\alpha_1, \alpha_1')$ is a d_0 -coboundary. Therefore we can define $(\widehat{\psi}_1)_{\alpha_1'\alpha_1}$ which bounds $-o(\alpha_1, \alpha_1')$ to promote $\widehat{\psi}_1$ to the energy cut level E_2 . We note that we can still change $(\widehat{\psi}_1)_{\alpha_1'\alpha_1}$ by d_0 -cocycle. We will use this freedom in the next step.

Next we promote the homotopy $\hat{\mathfrak{h}}$. Hereafter we denote by $\hat{\psi}_1$ its promotion to the energy cut level E_2 , which we have just done above. We now change our notation and regard $\hat{\psi}_j$, $\hat{\psi}_{21}$ etc. as partial cochain maps of energy cut level E_2 . So when we regard them as $: CF(*) \to CF(*)$, we include the $\alpha\alpha'$ -component where $E(\alpha') - E(\alpha) = E_2 - c$.

We consider $(\alpha_1, \alpha_2') \in \mathfrak{A}_1 \times \mathfrak{A}_2'$ with $E(\alpha_2') - E(\alpha_1) = E_2 - c$. We define

$$o(\alpha_1, \alpha_2') = \left((\widehat{\psi}_{21}' \circ \widehat{\psi}_1 - \widehat{\psi}_2 \circ \widehat{\psi}_{21}') - (\widehat{d}_2' \circ \widehat{\mathfrak{h}} + \widehat{\mathfrak{h}} \circ \widehat{d}_1) \right)_{\alpha_2' \alpha_1}. \tag{19.51}$$

We note that the right hand side is 0 if $E(\alpha'_2) - E(\alpha_1) < E_2 - c$ by (19.47) and the assumption. Moreover we have

$$0 = \left(\widehat{d}'_{2} \circ (\widehat{\psi}'_{21} \circ \widehat{\psi}_{1} - \widehat{\psi}_{2} \circ \widehat{\psi}'_{21}) - (\widehat{\psi}'_{21} \circ \widehat{\psi}_{1} - \widehat{\psi}_{2} \circ \widehat{\psi}'_{21}) \circ \widehat{d}_{1} - \widehat{d}'_{2} \circ (\widehat{d}'_{2} \circ \widehat{\mathfrak{h}} + \widehat{\mathfrak{h}} \circ \widehat{d}_{1}) + (\widehat{d}'_{2} \circ \widehat{\mathfrak{h}} + \widehat{\mathfrak{h}} \circ \widehat{d}_{1}) \circ \widehat{d}_{1}\right)_{\alpha'_{2}\alpha_{1}}.$$

$$(19.52)$$

In fact, we already promoted \widehat{d}'_i , \widehat{d}_i , $\widehat{\psi}'_{21}$, $\widehat{\psi}'_i$ to the energy cut level E_2 and in our notation (which we changed at the beginning of the construction of $\widehat{\mathfrak{h}}$) $\widehat{\psi}'_{21}$, $\widehat{\psi}'_i$ contain the components up to the energy cut level E_2 . Therefore the equalities $\widehat{d}'_2 \circ \widehat{\psi}'_{21} = \widehat{\psi}'_{21} \circ \widehat{d}'_1$ etc. hold up to the energy cut level E_2 . It implies that the first line of (19.52) vanishes. The second line vanishes obviously. We use them to obtain

$$d_0 \circ o(\alpha_1, \alpha_2') + o(\alpha_1, \alpha_2') \circ d_0 = 0. \tag{19.53}$$

We now use the freedom to change $(\widehat{\psi}_1)_{\alpha_1\alpha'_1}$ by d_0 -cocycle. Then we may assume (19.51) is a d_0 -coboundary. Thus we can promote the cochain homotopy $\widehat{\mathfrak{h}}$.

Proof of Theorem 16.39 (3). Let $\mathcal{F}\mathcal{F}_j$ (j=a,b) be inductive systems of partial linear K-systems which consist of partial linear K-systems \mathcal{F}^i_j , $i=1,2,\ldots$ and partial morphisms $\mathfrak{N}^{i+1i}_j: \mathcal{F}^i_j \to \mathcal{F}^{i+1}_j$. The morphism $\mathcal{F}\mathcal{F}_a \to \mathcal{F}\mathcal{F}_b$ consists of morphisms $\mathfrak{N}\mathfrak{N}^i: \mathcal{F}^i_a \to \mathcal{F}^i_b$ and homotopy. Namely, we have a homotopy commutative diagram of partial morphisms:

$$\mathcal{F}_{b}^{i} \xrightarrow{\mathfrak{N}_{b}^{i+1i}} \mathcal{F}_{b}^{i+1}$$

$$\mathfrak{N}\mathfrak{N}^{i} \qquad \qquad \uparrow \mathfrak{N}\mathfrak{N}^{i+1}$$

$$\mathcal{F}_{a}^{i} \xrightarrow{\mathfrak{N}_{a}^{i+1i}} \mathcal{F}_{a}^{i+1}$$
(19.54)

In the proof of Theorem 16.39 (1) given in Subsection 19.3, we made a choice of

$$\widehat{\mathcal{U}}^+(j,i;\alpha_-,\alpha_+), \quad \widehat{\mathfrak{S}}^+(j,i;\alpha_-,\alpha_+)$$

that are Kuranishi structures of the spaces of connecting orbits and their CF-perturbations to define partial cochain complex $(CF(\mathcal{F}_j^i), \{\mathfrak{m}_{1;\alpha_+\alpha_-}^{ji;\epsilon}\})$ and this partial cochain complex is defined from \mathcal{F}_i^i . We also made a choice of

$$\widehat{\mathcal{U}^+}(\mathrm{mor};j,i,i+1;\alpha_-,\alpha_+),\quad \widehat{\mathfrak{S}^+}(\mathrm{mor};j,i,i+1;\alpha_-,\alpha_+;\rho_j^{ii+1})$$

that are Kuranishi structures of the spaces of interpolation spaces of \mathfrak{N}_{j}^{i+1i} and their CF-perturbations. Here the parameter $\rho=\rho_{j}^{ii+1}\in(0,1]$ is taken as follows. We need to fix CF-perturbations of $\widehat{\mathcal{U}^{+}}(j,i;\alpha_{-},\alpha_{+})$ and of $\widehat{\mathcal{U}^{+}}(j,i+1;\alpha_{-},\alpha_{+})$ with which our CF-perturbations on the interpolation spaces are compatible. On one of those Kuranishi structures, we take $\widehat{\mathfrak{S}^{+}}(j,i;\alpha_{-},\alpha_{+})$. On the other Kuranishi structure, we take $\epsilon\mapsto\widehat{\mathfrak{S}^{+}}(j,i+1;\alpha_{-},\alpha_{+})^{\rho\epsilon}$. The parameter ρ appears here. Using these choices, we obtain partial cochain maps at the horizontal arrows in Diagram (19.54).

Next we apply Proposition 19.7 to the vertical arrows of Diagram (19.54). Then we can choose

$$\widehat{\mathcal{U}}^+(\text{mor}; ab, i; \alpha_-, \alpha_+), \quad \widehat{\mathfrak{S}}^+(\text{mor}; ab, i; \alpha_-, \alpha_+; \rho_{ab}^i),$$

and

$$\widehat{\mathcal{U}^+}(\mathrm{mor};ab,i+1;\alpha_-,\alpha_+),\quad \widehat{\mathfrak{S}^+}(\mathrm{mor};ab,i+1;\alpha_-,\alpha_+;\rho_{ab}^{i+1}),$$

that are Kuranishi structures and CF-perturbations of the interpolation spaces of \mathfrak{NN}^i , \mathfrak{NN}^{i+1} , respectively.

For each i we take $\epsilon_{bi} \leq \epsilon_{ai} \leq \epsilon_{4,i}$ such that $\epsilon_{ai+1} \leq \epsilon_{ai}$, $\epsilon_{bi+1} \leq \epsilon_{bi}$. (We specify our choice of $\epsilon_{4,i}$ later.) We take $\rho_j^{ii+1} = \epsilon_{ji+1}/\epsilon_{ji}$. We put

$$CF_j^i := \left(CF(\mathcal{F}_j^i), \{\mathfrak{m}_{1;\alpha_+\alpha_-}^{ji;\epsilon} \} \right).$$
 (19.55)

Then we have a diagram of partial cochain complexes.

$$CF_b^i \longrightarrow CF_b^{i+1}$$

$$\uparrow \qquad \qquad \uparrow$$

$$CF_a^i \longrightarrow CF_a^{i+1}$$
(19.56)

By construction the vertical arrows are partial cochain maps of energy cut level E^i and E^{i+1} , respectively.

We now use Proposition 19.28 to obtain Kuranishi structures and CF-perturbations on the interpolation spaces of the homotopy of Diagram (19.54) which are compatible with the choices we had made for the interpolation spaces of the arrows of Diagram (19.54) and the space of connecting orbits. (We had made the choice of them already as explained above.) We can take $\epsilon_{4,i}$ so small that this choice of Kuranishi structures and CF-perturbations determines a cochain homotopy which makes Diagram (19.56) commutative up to cochain homotopy. (This is a consequence of Lemma 19.30.) Recall from Remark 19.15 the numbers $\epsilon_{0,i}, \epsilon_{1,i}, \epsilon_{2,i}, \epsilon_{3,i}$ in (19.26) are already chosen. We will need $\epsilon_{4,i}, \epsilon_{5,i}$ etc. for homotopy of homotopies etc. (However we need only finitely many of them.)

Therefore by Proposition 19.33 we can promote Diagram (19.56) to homotopy commutative diagram of the partial cochain maps of energy cut level E_{i+1} .

The rest of the proof is purely algebraic. We now consider the following diagram:

$$CF_b^1 \longrightarrow \cdots \longrightarrow CF_b^i \longrightarrow CF_b^{i+1} \longrightarrow \cdots$$

$$\uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow$$

$$CF_a^1 \longrightarrow \cdots \longrightarrow CF_a^i \longrightarrow CF_a^{i+1} \longrightarrow \cdots$$

$$(19.57)$$

Note that our construction of cochain complex $CF(\mathcal{F}_a)$ is done by promoting the horizontal lines inductively to one of energy cut level E_k and taking limit $k \to \infty$. We do so for both of the horizontal lines. Then we continue to promote the vertical lines so that the whole diagram becomes homotopy commutative. Thus we obtain a cochain map $CF(\mathcal{F}_a) \to CF(\mathcal{F}_b)$. This finishes the proof of Theorem 16.39 (3). \square

Proof of Theorem 16.39 (2) (a)-(d). When we proved Theorem 16.39 (1) in Subsection 19.3, we made the following choices.

(1) For each i the pairs $(\widehat{\mathcal{U}}^+(i; \alpha_-, \alpha_+), \widehat{\mathfrak{S}}^+(i; \alpha_-, \alpha_+))$ for various α_-, α_+ that are a system of Kuranishi structures of the spaces of connecting orbits and their CF-perturbations to define a partial cochain complex

$$(CF(\mathcal{F}^i), \{\mathfrak{m}_{1:\alpha_+\alpha_-}^{i;\epsilon}\})$$

and this partial cochain complex is defined from the partial linear K-system \mathcal{F}^i with energy cut level $E^{k_i}_{\sigma}$.

(2) For each i the pairs

$$\left(\widehat{\mathcal{U}^+}(\mathrm{mor};i,i+1;\alpha_-,\alpha_+),\widehat{\mathfrak{S}^+}(\mathrm{mor};i,i+1;\alpha_-,\alpha_+;\rho_i^{ii+1})\right)$$

for various α_-, α_+ are a system of Kuranishi structures of the interpolation spaces of \mathfrak{N}^{i+1i} and their CF-perturbations. Here the parameter $\rho = \rho_i^{ii+1} \in (0,1]$ is as explained during the proof of Theorem 16.39 (3).

(3) The small numbers ϵ_i .

We then defined $(CF(\mathcal{F}^i), \{\mathfrak{m}_{1;\alpha_+\alpha_-}^{i;\epsilon_i}\})$ by using the choices (1),(3). Moreover using the choices (2), (3), we defined partial cochain maps

$$\widehat{\psi}^{i} : \left(CF(\mathcal{F}^{i}), \{\mathfrak{m}_{1;\alpha_{+}\alpha_{-}}^{i;\epsilon_{i}}\} \right) \longrightarrow \left(CF(\mathcal{F}^{i+1}), \{\mathfrak{m}_{1;\alpha_{+}\alpha_{-}}^{i+1;\epsilon_{i+1}}\} \right). \tag{19.58}$$

Finally using (19.58) and an algebraic result (Lemma 19.13), we promoted partial cochain complexes $\left(CF(\mathcal{F}^i), \{\mathfrak{m}_{1;\alpha_+\alpha_-}^{i;\epsilon_i}\}\right)$ and partial cochain maps $\widehat{\psi}^i$ to cochain complexes and cochain maps. This algebraic process also involves choices.

We will prove that the resulting cochain complex is independent of those choices up to cochain homotopy equivalence. Let

$$\left(\widehat{\mathcal{U}^+}(ji;\alpha_-,\alpha_+),\widehat{\mathfrak{S}^+}(ji;\alpha_-,\alpha_+)\right)$$

and

$$\left(\widehat{\mathcal{U}^+}(\mathrm{mor},j;i,i+1;\alpha_-,\alpha_+),\widehat{\mathfrak{S}^+}(\mathrm{mor},j;i,i+1;\alpha_-,\alpha_+;\rho_j^{ii+1})\right),$$

and ϵ_{ji} be two choices, where j = a, b.

We consider the next diagram, which defines a morphism $\mathcal{FF} \to \mathcal{FF}$.

$$\mathcal{F}^{1} \xrightarrow{\mathfrak{N}^{21}} \dots \xrightarrow{\mathfrak{N}^{ii-1}} \mathcal{F}^{i} \xrightarrow{\mathfrak{N}^{i+1i}} \mathcal{F}^{i+1} \xrightarrow{\mathfrak{N}^{i+2i+1}} \dots$$

$$\uparrow \mathfrak{ID} \qquad \qquad \uparrow \mathfrak{ID} \qquad \qquad \uparrow \mathfrak{ID} \qquad (19.59)$$

$$\mathcal{F}^{1} \xrightarrow{\mathfrak{N}^{21}} \dots \xrightarrow{\mathfrak{N}^{ii-1}} \mathcal{F}^{i} \xrightarrow{\mathfrak{N}^{i+1i}} \mathcal{F}^{i+1} \xrightarrow{\mathfrak{N}^{i+2i+1}} \dots$$

The homotopy commutativity of Diagram (19.59) follows from Proposition 18.63.

Now we apply Theorem 16.39 (3). Namely we apply the choice we made for j = b for the first line and the choice we made for j = a for the second line. We also use the particular way to promote the inductive system of partial cochain complexes which we used for choices j = b and j = a. (We obtain those cochain complexes from the first and second lines of Diagram (19.59).)

Now we apply Proposition 19.33 and obtain a cochain map $CF_a \to CF_b$. Here CF_a (resp. CF_b) is the cochain complex we obtain by this promotion of the second line (resp. first line). The proof of Theorem 16.39 (2) (except (e)) is complete. \square

Proof of Theorem 16.9 (2) (a)-(e). This is nothing but a special case of Theorem 16.39 (2) where \mathfrak{N}^{i+1i} is the identity morphism.

Proof of Theorem 16.31 (1). We can use Proposition 18.63 to obtain the following homotopy commutative diagram:

$$\mathcal{F}_{2} \xrightarrow{\mathfrak{ID}} \dots \xrightarrow{\mathfrak{ID}} \mathcal{F}_{2} \xrightarrow{\mathfrak{ID}} \mathcal{F}_{2} \xrightarrow{\mathfrak{ID}} \dots$$

$$\uparrow \mathfrak{M} \qquad \qquad \uparrow \mathfrak{M} \qquad \qquad \uparrow \mathfrak{M} \qquad (19.60)$$

$$\mathcal{F}_{1} \xrightarrow{\mathfrak{ID}} \dots \xrightarrow{\mathfrak{ID}} \mathcal{F}_{1} \xrightarrow{\mathfrak{ID}} \mathcal{F}_{1} \xrightarrow{\mathfrak{ID}} \dots$$

In fact, we have $\mathfrak{ID} \circ \mathfrak{N} \sim \mathfrak{N} \sim \mathfrak{N} \circ \mathfrak{ID}$, by Proposition 18.63.

Following the proof of Theorem 16.9 (1), we make the following choices.

- (1) For each j = 1, 2 we regard \mathcal{F}_j as a partial linear K-system of energy cut level E^i . We write it as \mathcal{F}_i^i .
- (2) We take $\widehat{\mathcal{U}^+}(j, i; \alpha_-, \alpha_+)$ and $\widehat{\mathfrak{S}^+}(j, i; \alpha_-, \alpha_+)$ that are a Kuranishi structure of the space of connection orbits of \mathcal{F}^i_j and its CF-perturbation, respectively. They satisfy the conclusion of Proposition 19.1.
- (3) We take $\widehat{\mathcal{U}^+}(\text{mor}, j, ii + 1; \alpha_-, \alpha_+)$ and $\widehat{\mathfrak{S}^+}(\text{mor}, j, ii + 1; \alpha_-, \alpha_+)$ that are a Kuranishi structure of the interpolation space of $\mathfrak{ID}: \mathcal{F}^i_j \to \mathcal{F}^{i+1}_j$ and its CF-perturbation, respectively. They satisfy the conclusion of Proposition 19.7.
- (4) We also take $\widehat{\mathcal{U}^+}(\text{mor}, 12, i; \alpha_1, \alpha_2)$ and $\widehat{\mathfrak{S}^+}(\text{mor}, 12, i; \alpha_1, \alpha_2)$ that are a Kuranishi structure of the interpolation space of $\mathfrak{N}: \mathcal{F}_1^i \to \mathcal{F}_2^i$ and its CF-perturbation, respectively. They satisfy the conclusion of Proposition 19.7.

(5) We take $\widehat{\mathcal{U}}^+$ (mor, i; α_- , α_+ ; [0, 1]) and $\widehat{\mathfrak{S}}^+$ (mor, i; α_- , α_+ ; [0, 1]) that are a Kuranishi structure of the interpolation space of the homotopy $\mathfrak{ID} \circ \mathfrak{N} \sim$ $\mathfrak{N}\circ\mathfrak{ID}:\mathcal{F}_1^i o\mathcal{F}_2^{i+1}$ and its CF-perturbation, respectively. The Kuranishi structure is compatible with ones in Item (1)(2)(3)(4) at the boundary.

By these choices the geometric Diagram (19.60) is converted to the algebraic diagram below:

$$\begin{pmatrix}
CF(\mathcal{F}_{2}), \{\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{2i\epsilon_{2i}}\} \rangle \xrightarrow{\widehat{\psi}_{2}^{i+1i}} \begin{pmatrix}
CF(\mathcal{F}_{2}), \{\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{2i\epsilon_{2i+1}}\} \rangle \\
\mathfrak{n}^{i} \uparrow & \uparrow \mathfrak{n}^{i+1} \end{pmatrix}$$

$$\begin{pmatrix}
CF(\mathcal{F}_{1}), \{\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{1i,\epsilon_{1i}}\} \rangle \xrightarrow{\widehat{\psi}_{1}^{i+1i}} \begin{pmatrix}
CF(\mathcal{F}_{1}), \{\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{1i+1,\epsilon_{1i+1}}\} \end{pmatrix}$$

$$(19.61)$$

Here

- $\left(CF(\mathcal{F}_j), \{\mathfrak{m}_{1;\alpha_+,\alpha_-}^{ji,\epsilon_{ji}}\}\right)$ is obtained by the choice (2) above. $\widehat{\psi}_j^{i+1i}, \ j=1,2$ is obtained by the choice (3) above.
- \mathfrak{n}^{i} is obtained by the choice (4) above.
- By the choice (5) above we obtain a cochain homotopy between $\widehat{\psi}_2^{i+1i} \circ \mathfrak{n}^i$ and $\mathfrak{n}^{i+1} \circ \widehat{\psi}_1^{i+1i}$. We denote it by \mathfrak{h}_i .

Here we use Lemmas 19.21 and 19.22 to prove that the composition $\widehat{\psi}_2^{i+1} \circ \mathfrak{n}^i$ (resp. $\mathfrak{n}^{i+1} \circ \widehat{\psi}_1^{i+1i}$) is the cochain map associated to the morphism that is the composition of the identity morphism and \mathfrak{N}^i (resp. \mathfrak{N}^i and the identity morphism).

We note that at this stage the energy cut level of the objects (partial cochain maps and partial cochain complexes) in the right vertical line is E^{i+1} and the energy cut level of all the other objects in (19.61) (and the partial cochain homotopy \mathfrak{h}_i)

Now we start promoting the objects in Diagram (19.61). First according to the proof of Theorem 16.9 (1), we promote all the objects in the upper and lower horizontal lines in Diagram (19.61) to a cochain complex and cochain map. (Namely we promote them to the energy cut level ∞ .)

Thus all the objects other than \mathfrak{n}^i , \mathfrak{n}^{i+1} and the cochain homotopy in Diagram (19.61) are promoted to the energy cut level ∞ . We now use the fact that the energy cut level of \mathfrak{n}^{i+1} is E^{i+1} to promote \mathfrak{n}^i and \mathfrak{h}^i to the energy cut level E^{i+1} . We use Proposition 19.33 to do so.

Then by induction using Diagrams (19.61) for all i, we can promote \mathfrak{n}^i , \mathfrak{n}^{i+1} to the energy cut level ∞ so that Diagram (19.61) commutes up to cochain homotopy. The proof of Theorem 16.31 (1) is complete.

19.7. Construction of higher homotopy. In this subsection, we generalize Proposition 19.28 to the case of P-parametrized morphisms.

Situation 19.34. Suppose we are given a P-parametrized family of morphisms of (partial) linear K-system. Let E_0 be its energy cut level and c be its energy loss. Let $\mathcal{N}(\alpha_1, \alpha_2; P)$ be its interpolation spaces.

Let $\alpha_i \in \mathfrak{A}_i$ with $\alpha_2 - \alpha_1 \leq E_0 - c$.

(1) We are given a τ - \mathfrak{C}^h -collared Kuranishi structure $\widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; \widehat{S}_k(P))$ on the underlying topological space of $\mathcal{N}(\alpha_1, \alpha_2; S_k(P))^{\mathfrak{C}^h \boxplus \tau_0}$. (Here \mathfrak{C}^h is the horizontal boundary component, that is the complement of the inverse image $\operatorname{ev}_{S_k(P)}^{-1}(\partial S_k(P))$ of the evaluation map

$$\operatorname{ev}_{S_k(P)}: \mathcal{N}(\alpha_1, \alpha_2; S_k(P)) \longrightarrow \widehat{S}_k(P)$$

in $\partial \mathcal{N}(\alpha_1, \alpha_2; S_k(P))$. See Definition 19.27 for the case P = [1, 2].)

We assume that there exists an embedding to $\widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; \widehat{S}_k(P))$ from the corner trivialization of the Kuranishi structure of $\mathcal{N}(\alpha_1, \alpha_2; S_k(P))$ in Lemma 16.24, by which $\widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; \widehat{S}_k(P))$ becomes a thickening of

$$\mathcal{N}(\alpha_1, \alpha_2; S_k(P))^{\mathfrak{C}^h \boxplus \tau_0}$$
.

- (2) Evaluation maps ev_{\pm} and $\operatorname{ev}_{S_k(P)}$ extend to $\widehat{\mathcal{U}}^+(\alpha_1, \alpha_2; \widehat{S}_k(P))$ and the above mentioned embedding is compatible with the evaluation maps. There exists a periodicity isomorphism which commutes with the embedding.
- (3) We are given a τ -collared CF-perturbation $\mathfrak{S}^+(\alpha_1, \alpha_2; \widehat{S}_k(P))$ of the Kuranishi structure in (1). It is compatible with the periodicity isomorphism.
- (4) The τ -collared CF-perturbation $\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; \widehat{S}_k(P))$ is transversal to zero. Moreover the evaluation map

$$(\mathrm{ev}_+,\mathrm{ev}_{S_k(P)}):\widehat{\mathcal{U}^+}(\alpha_1,\alpha_2;\widehat{S}_k(P))\longrightarrow R_{\alpha_2}\times\widehat{S}_k(P)$$

is stratified strongly submersive with respect to $\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; \widehat{S}_k(P))$.

(5) The Kuranishi structure $\widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; \widehat{S}_k(P))$ and the CF-perturbation

$$\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; \widehat{S}_k(P))$$

are compatible with the isomorphisms in Conditions 16.23, 16.26, 16.28.

Proposition 19.35. Suppose we are in Situation 19.34. Then there exists a τ' - \mathfrak{C}^h -collared Kuranishi structure $\widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; P)$ on the underlying topological space of $\mathcal{N}(\alpha_1, \alpha_2; P)^{\mathfrak{C}^h \boxplus \tau_0}$ and a τ' - \mathfrak{C}^h -collared CF-perturbation $\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; P)$ of $\widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; P)$ such that they have the following properties.

- (1) There exists an embedding from $\mathcal{N}(\alpha_1, \alpha_2; P)^{\mathfrak{C}^h \boxplus \tau_0}$ to $\widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; P)$ by which the later becomes a thickening of the former.
- (2) Evaluation maps ev_{\pm} and ev_{P} extend to $\widehat{\mathcal{U}}^{+}(\alpha_{1}, \alpha_{2}; P)$ and the above mentioned embedding is compatible with them.
- (3) There exists a periodicity isomorphism

$$\widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; P) \longrightarrow \widehat{\mathcal{U}^+}(\beta \alpha_1, \beta \alpha_2; P)$$

which commutes with the embedding. The pull-back of $\widehat{\mathfrak{S}^+}(\beta\alpha_1, \beta\alpha_2; P)$ by the periodicity isomorphism is isomorphic to $\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; P)$.

(4) The CF-perturbation $\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; P)$ is transversal to 0. Moreover the evaluation map

$$(\mathrm{ev}_+,\mathrm{ev}_P): \mathcal{N}(\alpha_1,\alpha_2;P)^{\mathfrak{C}^h \boxplus \tau_0} \longrightarrow R_{\alpha_2} \times P$$

is strongly submersive with respect to $\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; P)$.

(5) The fiber product

$$\widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; P) \times_P \widehat{S}_k(P)$$

is isomorphic to $\widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; \widehat{S}_k(P))$. This isomorphism is compatible with the periodicity isomorphism and evaluation maps. It is also compatible with the embedding in (1) and one in Situation 19.34 (1).

(6) The fiber product

$$\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; P) \times_P \widehat{S}_k(P)$$

is equivalent to $\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; \widehat{S}_k(P))$.

(7) There exists an isomorphism: 40

$$\partial \left(\widehat{\mathcal{U}^+}(\alpha_1, \alpha_2; P)\right)$$

$$\cong \coprod_{\alpha \in \mathfrak{A}_{1}} (-1)^{\dim \widehat{\mathcal{U}^{+}}(\alpha, \alpha_{2}; P)} \left(\widehat{\mathcal{U}^{+}}(\alpha, \alpha_{2}; P) \right|_{\operatorname{ev}_{-}} \times_{\operatorname{ev}_{+}} \widehat{\mathcal{U}^{+}}(1; \alpha_{1}, \alpha)$$

$$= \coprod_{\alpha \in \mathfrak{A}_{2}} (-1)^{\dim \widehat{\mathcal{U}^{+}}(2; \alpha, \alpha_{2})} \left(\widehat{\mathcal{U}^{+}}(2; \alpha, \alpha_{2}) \right|_{\operatorname{ev}_{-}} \times_{\operatorname{ev}_{+}} \widehat{\mathcal{U}^{+}}(\alpha_{1}, \alpha; P)$$

$$(19.62)$$

$$\sqcup \widehat{\mathcal{U}^+}(\alpha_1, \alpha; \partial P).$$

(Recall that $\widehat{\mathcal{U}^+}(1; \alpha_1, \alpha)$, $\widehat{\mathcal{U}^+}(2; \alpha, \alpha_2)$ are Kuranishi structures on $\mathcal{M}^1(\alpha_1, \alpha)$, $\mathcal{M}^2(\alpha, \alpha_2)$, respectively, given in Proposition 19.1.)

This isomorphism is compatible with the isomorphism in Condition 16.23 via the embedding in (1). It is also compatible with the orientation isomorphism, the periodicity isomorphism and the evaluation maps.

(8) The normalized corner $\widehat{S}_k(\widehat{\mathcal{U}}^+(\alpha_1, \alpha_2; P))$ is decomposed to the disjoint union of the following fiber products.

$$\widehat{\mathcal{U}^{+}}(1; \alpha_{-}, \alpha_{1}) \times_{R_{\alpha_{1}}} \cdots \times_{R_{\alpha_{k_{1}-1}}} \widehat{\mathcal{U}^{+}}(1; \alpha_{k_{1}-1}, \alpha_{k_{1}})$$

$$\times_{R_{\alpha_{k_{1}}}} \widehat{\mathcal{U}^{+}}(\alpha_{k_{1}}, \alpha_{k_{1}+1}; \widehat{S}_{k_{3}}(P))$$

$$(19.63)$$

$$\times_{R_{\alpha_{k_1+1}}} \widehat{\mathcal{U}^+}(2; \alpha_{k_1+1}, \alpha_{k_1+2}) \times_{R_{\alpha_{k_1+2}}} \cdots \times_{R_{\alpha_{k_1+k_2}}} \widehat{\mathcal{U}^+}(2; \alpha_{k_1+k_2}, \alpha_+).$$

This isomorphism is compatible with the isomorphism in Condition 16.26 via the embedding in (1). It is also compatible with the periodicity isomorphism and the evaluation maps.

- (9) The isomorphism (19.63) satisfies the same compatibility conditions claimed for $\mathcal{N}(\alpha_1, \alpha_2; P)$ in Condition 16.28.
- (10) The CF-perturbation $\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; P)$ is compatible with the isomorphisms (19.62) and (19.63).
- (11) When the τ -collared CF-perturbation $\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; \widehat{S}_k(P))$ given in Situation 19.34 varies in a uniform family, we can take $\widehat{\mathfrak{S}^+}(\alpha_1, \alpha_2; P)$ to be uniform.

Proof. The proof is entirely the same as the proof of Proposition 19.1. \Box

The translation to algebra is fairly immediate.

Definition 19.36. In the situation of Proposition 19.35, we define

$$\psi_{\alpha_2,\alpha_1}^{P,\epsilon}: \Omega(R_{\alpha_1}; o_{R_{\alpha_1}}) \longrightarrow \Omega(R_{\alpha_2}; o_{R_{\alpha_2}})$$
 (19.64)

by

$$\psi_{\alpha_2,\alpha_1}^{P,\epsilon}(h) = \text{ev}_+!(\text{ev}_-^*h;\widehat{\mathfrak{S}^{+\epsilon}}(\alpha_1,\alpha_2;P)). \tag{19.65}$$

⁴⁰See Remark 16.2).

Here the right hand side is defined by Definition 17.67 on the K-spaces

$$(\mathcal{N}(\alpha_1, \alpha_2; P)^{\boxplus \tau_0}, \widehat{\mathcal{U}^+}(\text{mor}; \alpha_1, \alpha_2; P)).$$

The degree of $\psi_{\alpha_2,\alpha_1}^{P,\epsilon}$ after shifted is $-\dim P$. If the energy loss of our parametrized family of morphisms is c, the map $\psi_{\alpha_2,\alpha_1}^{P,\epsilon}$ induces

$$\mathfrak{F}^{\lambda}CF(\mathcal{F}_1) \longrightarrow \mathfrak{F}^{\lambda-c}CF(\mathcal{F}_2)$$

where the filtration \mathfrak{F}^{λ} is defined in Definition 16.8 (2)(3).

Lemma 19.37. The operators $\psi_{\alpha_2,\alpha_1}^{P,\epsilon}$ satisfy the following equality in the sense of (\flat) .

$$d_{0} \circ \psi_{\alpha_{2},\alpha_{1}}^{P,\epsilon} - (-1)^{\deg P} \psi_{\alpha_{2},\alpha_{1}}^{P,\epsilon} \circ d_{0} + \psi_{\alpha_{2},\alpha_{1}}^{\partial P,\epsilon} + (-1)^{\deg P} \sum_{\alpha'_{1}} \psi_{\alpha_{2},\alpha'_{1}}^{P,\epsilon} \circ \mathfrak{m}_{1;\alpha'_{1},\alpha_{1}}^{1,\epsilon} - \sum_{\alpha'_{2}} \mathfrak{m}_{1;\alpha_{2},\alpha'_{2}}^{2,\epsilon} \circ \psi_{\alpha'_{2},\alpha_{1}}^{P,\epsilon} = 0.$$

$$(19.66)$$

Here the first sum in the second line is taken over $\alpha'_1 \in \mathfrak{A}_1$ with $E(\alpha_1) < E(\alpha'_1) \le E(\alpha_2) + c$ and the second sum in the second line is taken over $\alpha'_2 \in \mathfrak{A}_2$ with $E(\alpha_1) - c \le E(\alpha'_2) < E(\alpha_2)$. The number c is the energy loss of our morphism.

Proof. The proof is by Stokes' formula and the composition formula and is entirely similar to the proof of Lemma 19.10. \Box

19.8. Proof of Theorem 16.39 (2)(e), (4)-(6) and Theorem 16.9 (2)(f). We begin with an algebraic result that is similar to Proposition 19.33 and is a baby version of [FOOO4, Theorem 7.2.212].

Situation 19.38. For j=1,2, let $(CF(\mathcal{F}_j^i),\hat{d}_j^i),$ $(CF(\mathcal{F}_j^{i+1}),\hat{d}_j^{i+1})$ be partial cochain complexes of energy cut level E^{i+1} .

- (1) For $j=1,2,\ \psi_j^{i+1i}: CF(\mathcal{F}_j^i)\to CF(\mathcal{F}_j^{i+1})$ is a partial cochain map of energy cut level E^{i+1} and energy loss 0. Moreover, we assume ψ_j^{i+1i} induces an isomorphism modulo T^ϵ for small $\epsilon>0$.
- (2) For k=a,b and $\ell=i,i+1,\,\mathfrak{n}_k^\ell:CF(\mathcal{F}_1^\ell)\to CF(\mathcal{F}_2^\ell)$ is a partial cochain map of energy cut level E^{i+1} and energy loss c.
- (3) $\mathfrak{h}_{ab}^{i}: CF(\mathcal{F}_{1}^{i}) \to CF(\mathcal{F}_{2}^{i})$ is a cochain homotopy between \mathfrak{n}_{a}^{i} and \mathfrak{n}_{b}^{i} of energy cut level E^{i} and energy loss c. $\mathfrak{h}_{ab}^{i+1}: CF(\mathcal{F}_{1}^{i+1}) \to CF(\mathcal{F}_{2}^{i+1})$ is a cochain homotopy between \mathfrak{n}_{a}^{i+1} and \mathfrak{n}_{b}^{i+1} of energy cut level E^{i+1} and energy loss c. See Diagram (19.67).
- energy loss c. See Diagram (19.67). (4) For $k=a,b, \ \mathfrak{h}_k^{i+1i}: CF(\mathcal{F}_1^i) \to CF(\mathcal{F}_2^{i+1})$ is a partial cochain homotopy between $\mathfrak{n}_k^{i+1} \circ \psi_1^{i+1i}$ and $\psi_2^{i+1i} \circ \mathfrak{n}_k^i$ of energy cut level E^{i+1} and energy loss c.

$$CF(\mathcal{F}_{1}^{i}) \xrightarrow{\psi_{1}^{i+1i}} CF(\mathcal{F}_{1}^{i+1})$$

$$\mathfrak{n}_{a}^{i} \xrightarrow{\mathfrak{h}_{ab}^{i}} \mathfrak{n}_{b}^{i} \qquad \mathfrak{n}_{a}^{i+1} \xrightarrow{\mathfrak{h}_{ab}^{i+1}} \mathfrak{n}_{b}^{i+1}$$

$$CF(\mathcal{F}_{2}^{i}) \xrightarrow{\psi_{2}^{i+1i}} CF(\mathcal{F}_{2}^{i+1})$$

$$(19.67)$$

Proposition 19.39. Suppose in Situation 19.38 there exists (homotopy of homotopies)

$$\mathfrak{H}_{ab}^{i+1i}: CF(\mathcal{F}_1^i) \longrightarrow CF(\mathcal{F}_2^{i+1})$$

which satisfies

$$\begin{split} \hat{d}_{2}^{i+1} &\circ \mathfrak{H}_{ab}^{i+1i} - \mathfrak{H}_{ab}^{i+1i} \circ \hat{d}_{1}^{i} \\ &= \mathfrak{h}_{b}^{i+1i} - \mathfrak{h}_{a}^{i+1i} + \mathfrak{h}_{ab}^{i+1} \circ \psi_{1}^{i+1i} - \psi_{2}^{i+1i} \circ \mathfrak{h}_{ab}^{i} \end{split} \tag{19.68}$$

as equality of maps of energy cut level E^i and energy loss c. Then we can promote \mathfrak{h}^i_{ab} and \mathfrak{H}^{i+1i}_{ab} to the energy cut level E^{i+1} so that Formula (19.68) holds as an equality of maps of energy cut level E^{i+1} and energy loss c.

Proof. We first promote \mathfrak{h}_{ab}^i . Let us consider α_1, α_2 with $E(\alpha_2) - E(\alpha_1) = E^{i+1} - c$. We will find

$$(\mathfrak{h}_{ab}^i)_{\alpha_2\alpha_1}: CF(\mathcal{F}_1^i) \longrightarrow CF(\mathcal{F}_2^i)$$

such that

$$d_{0} \circ (\mathfrak{h}_{ab}^{i})_{\alpha_{2}\alpha_{1}} + (\mathfrak{h}_{ab}^{i})_{\alpha_{2}\alpha_{1}} \circ d_{0}$$

$$= -\sum_{\alpha'_{2}} \mathfrak{m}_{1,\alpha_{2}\alpha'_{2}}^{2} \circ (\mathfrak{h}_{ab}^{i})_{\alpha'_{2}\alpha_{1}} - \sum_{\alpha'_{1}} (\mathfrak{h}_{ab}^{i})_{\alpha_{2}\alpha'_{1}} \circ \mathfrak{m}_{1,\alpha'_{1}\alpha_{1}}^{1}$$

$$+ (\mathfrak{n}_{b}^{i})_{\alpha_{2}\alpha_{1}} - (\mathfrak{n}_{a}^{i})_{\alpha_{2}\alpha_{1}}.$$

$$(19.69)$$

Let $o(\alpha_1, \alpha_2)$ be the right hand side of (19.69). Then we have

$$o(\alpha_1, \alpha_2) = \left(-\hat{d}_2^i \circ \mathfrak{h}_{ab}^i - \mathfrak{h}_{ab}^i \circ \hat{d}_1^i + \mathfrak{n}_b^i - \mathfrak{n}_a^i\right)_{\alpha_2 \alpha_1}.$$

Lemma 19.40.

$$d_0 \circ o(\alpha_1, \alpha_2) - o(\alpha_1, \alpha_2) \circ d_0 = 0.$$

Proof. We observe

$$\begin{split} &\left(\hat{d}_2^i \circ \left(-\hat{d}_2^i \circ \mathfrak{h}_{ab}^i - \mathfrak{h}_{ab}^i \circ \hat{d}_1^i + \mathfrak{n}_b^i - \mathfrak{n}_a^i\right) \\ &- \left(-\hat{d}_2^i \circ \mathfrak{h}_{ab}^i - \mathfrak{h}_{ab}^i \circ \hat{d}_1^i + \mathfrak{n}_b^i - \mathfrak{n}_a^i\right) \circ \hat{d}_1^i\right)_{\Omega \supseteq \Omega \downarrow} = 0. \end{split}$$

This is a consequence of $\hat{d}_2^i \circ \hat{d}_2^i = \hat{d}_1^i \circ \hat{d}_1^i = 0$ and Situation 19.38 (2). On the other hand we have

$$\left(-\hat{d}_2^i \circ \mathfrak{h}_{ab}^i - \mathfrak{h}_{ab}^i \circ \hat{d}_1^i + \mathfrak{n}_b^i - \mathfrak{n}_a^i\right)_{\alpha_2' \alpha_1'} = 0$$

if
$$E(\alpha_2') - E(\alpha_1') \leq E^i - c$$
 by Situation 19.38 (4). The lemma follows.

We will show that $o(\alpha_1, \alpha_2)$ is a coboundary. Below \equiv stands for modulo d_0 coboundary. Let $o: CF(\mathcal{F}_1^i) \to CF(\mathcal{F}_2^i)$ be a homomorphism such that $o_{\alpha_2\alpha_1} = o(\alpha_1, \alpha_2)$ for $E(\alpha_2) - E(\alpha_1) = E^{i+1} - c$ and $o_{\alpha_2\alpha_1} = 0$ otherwise.

First by definition we find

$$\begin{split} &(\psi_{2}^{i+1i} \circ o)_{\alpha'_{2}\alpha_{1}} \\ &= \left(\psi_{2}^{i+1i} \circ (-\hat{d}_{2}^{i} \circ \mathfrak{h}_{ab}^{i} - \mathfrak{h}_{ab}^{i} \circ \hat{d}_{1}^{i} + \mathfrak{n}_{b}^{i} - \mathfrak{n}_{a}^{i})\right)_{\alpha'_{2}\alpha_{1}} \\ &= \left(-\hat{d}_{2}^{i+1i} \circ \psi_{2}^{i+1i} \circ \mathfrak{h}_{ab}^{i} - \psi_{2}^{i+1i} \circ \mathfrak{n}_{a}^{i} - \psi_{2}^{i+1i} \circ \mathfrak{n}_{a}^{i}\right)_{\alpha'_{2}\alpha_{1}}. \end{split} \tag{19.70}$$

Here and hereafter $E(\alpha'_2) - E(\alpha_1) = E^{i+1} - c$. We used Situation 19.38 (1) in (19.70). We observe

$$\begin{aligned} & (-\hat{d}_{2}^{i+1} \circ \psi_{2}^{i+1i} \circ \mathfrak{h}_{ab}^{i} - \psi_{2}^{i+1i} \circ \mathfrak{h}_{ab}^{i} \circ \hat{d}_{1}^{i})_{\alpha'_{2}\alpha_{1}} \\ & \equiv (-\hat{d}_{2}^{i+1} \circ (\psi_{2}^{i+1i} \circ \mathfrak{h}_{ab}^{i})|_{E^{i}} - (\psi_{2}^{i+1i} \circ \mathfrak{h}_{ab}^{i})|_{E^{i}} \circ \hat{d}_{1}^{i})_{\alpha'_{2}\alpha_{1}}. \end{aligned}$$
(19.71)

Here we used the following equality

$$\begin{split} &(\hat{d}_{2}^{i+1} \circ A - (-1)^{\deg A} A \circ \hat{d}_{1}^{i})_{\alpha'_{2}\alpha_{1}} \\ &- (\hat{d}_{2}^{i+1} \circ A|_{E^{i}} - (-1)^{\deg A} A|_{E^{i}} \circ \hat{d}_{1}^{i})_{\alpha'_{2}\alpha_{1}} \\ &= d_{0} \circ A_{\alpha'_{2}\alpha_{1}} - (-1)^{\deg A} A_{\alpha'_{2}\alpha_{1}} \circ d_{0}. \end{split}$$
(19.72)

By induction hypothesis we have

$$-(\psi_{2}^{i+1i} \circ \mathfrak{h}_{ab}^{i})|_{E^{i}} + (\mathfrak{h}_{ab}^{i+1} \circ \psi_{1}^{i+1i})|_{E^{i}} - (\mathfrak{h}_{a}^{i+1i})|_{E^{i}} + (\mathfrak{h}_{b}^{i+1i})|_{E^{i}}$$

$$= (\hat{d}_{2}^{i+1} \circ \mathfrak{H}_{ab}^{i+1i} - \mathfrak{H}_{ab}^{i+1i} \circ \hat{d}_{1}^{i})|_{E^{i}}.$$

$$(19.73)$$

Therefore we obtain

$$(-\hat{d}_{2}^{i+1} \circ (\psi_{2}^{i+1i} \circ \mathfrak{h}_{ab}^{i})|_{E^{i}})_{\alpha'_{2}\alpha_{1}}$$

$$= \left(\hat{d}_{2}^{i+1} \circ \left(-(\mathfrak{h}_{ab}^{i+1} \circ \psi_{1}^{i+1i})|_{E^{i}} + (\mathfrak{h}_{a}^{i+1i})|_{E^{i}} - (\mathfrak{h}_{b}^{i+1i})|_{E^{i}}\right)\right)_{\alpha'_{2}\alpha_{1}}$$

$$+ \left(\hat{d}_{2}^{i+1} \circ (\hat{d}_{2}^{i+1} \circ \mathfrak{H}_{ab}^{i+1i} - \mathfrak{H}_{ab}^{i+1i} \circ \hat{d}_{1}^{i})|_{E^{i}}\right)_{\alpha'_{2}\alpha_{1}}.$$

$$(19.74)$$

By (19.73) we have

$$(-(\psi_{2}^{i+1i} \circ \mathfrak{h}_{ab}^{i})|_{E^{i}} \circ \hat{d}_{1}^{i})_{\alpha'_{2}\alpha_{1}}$$

$$= ((-(\mathfrak{h}_{b}^{i+1} \circ \psi_{1}^{ii+1})|_{E^{i}} + (\mathfrak{h}_{a}^{i+1i})|_{E^{i}} - (\mathfrak{h}_{b}^{i+1i})|_{E^{i}}) \circ \hat{d}_{1}^{i})_{\alpha'_{2}\alpha_{1}}$$

$$+ ((\hat{d}_{2}^{i+1} \circ \mathfrak{H}_{ab}^{i+1i} - \mathfrak{H}_{ab}^{i+1i} \circ \hat{d}_{1}^{i})|_{E^{i}} \circ \hat{d}_{1}^{i})_{\alpha'_{2}\alpha_{1}}.$$
(19.75)

It follows that

$$(19.74) + (19.75)$$

$$= \left(\hat{d}_{2}^{i+1} \circ \left((-\mathfrak{h}_{ab}^{i+1} \circ \psi_{1}^{ii+1})|_{E^{i}} + (\mathfrak{h}_{a}^{i+1i})|_{E^{i}} - (\mathfrak{h}_{b}^{i+1i})|_{E^{i}} \right) \right)_{\alpha'_{2}\alpha_{1}}$$

$$+ \left(\left(-(\mathfrak{h}_{ab}^{i+1} \circ \psi_{1}^{ii+1})|_{E^{i}} + (\mathfrak{h}_{a}^{i+1i})|_{E^{i}} - (\mathfrak{h}_{b}^{i+1i})|_{E^{i}} \right) \circ \hat{d}_{1}^{i} \right)_{\alpha'_{2}\alpha_{1}}.$$

$$(19.76)$$

Here we used $\hat{d}_{1}^{i} \circ \hat{d}_{1}^{i} = \hat{d}_{2}^{i} \circ \hat{d}_{2}^{i} = 0$ and (19.72) with $A = \hat{d}_{2}^{i+1} \circ \mathfrak{H}_{ab}^{i+1i} - \mathfrak{H}_{ab}^{i+1i} \circ \hat{d}_{1}^{i}$. Using (19.72) again we have

$$\begin{aligned} (19.76) &= \left(\hat{d}_{2}^{i+1} \circ \left(-\mathfrak{h}_{ab}^{i+1} \circ \psi_{1}^{i+1i} + \mathfrak{h}_{a}^{i+1i} - \mathfrak{h}_{b}^{i+1i} \right) \right)_{\alpha'_{2}\alpha_{1}} \\ &+ \left(\left(-\mathfrak{h}_{ab}^{i+1} \circ \psi_{1}^{i+1i} + \mathfrak{h}_{a}^{i+1i} - \mathfrak{h}_{b}^{i+1i} \right) \circ \hat{d}_{1}^{i} \right)_{\alpha'_{1}\alpha_{1}}. \end{aligned}$$
 (19.77)

By Situation 19.38 (4) we have

$$(\psi_2^{i+1i} \circ \mathfrak{n}_k^i - \mathfrak{n}_k^{i+1} \circ \psi_1^{i+1i})_{\alpha_2'\alpha_1} = (\hat{d}_2^{i+1} \circ \mathfrak{h}_k^{i+1i} + \mathfrak{h}_k^{i+1i} \circ \hat{d}_1^{i+1})_{\alpha_2'\alpha_1} \qquad (19.78)$$

for k = a, b. Thus (19.70), (19.71), (19.76), (19.77), (19.78) imply

$$\begin{split} &(\psi_{2}^{i+1i} \circ o)_{\alpha'_{2}\alpha_{1}} \\ &\equiv \left(\hat{d}_{2}^{i+1} \circ (-\mathfrak{h}_{b}^{i+1i} + \mathfrak{h}_{a}^{i+1i} - \mathfrak{h}_{ab}^{i+1} \circ \psi_{1}^{i+1i}) \right. \\ &\quad + \left. (-\mathfrak{h}_{b}^{i+1i} + \mathfrak{h}_{a}^{i+1i} - \mathfrak{h}_{ab}^{i+1} \circ \psi_{1}^{i+1i}) \circ \hat{d}_{1}^{i} \right. \\ &\quad + \left. (\mathfrak{n}_{b}^{i+1i} + \hat{d}_{a}^{i+1i} - \hat{d}_{ab}^{i+1i} + \mathfrak{h}_{b}^{i+1i} \circ \hat{d}_{1}^{i} \right) \\ &\quad + \left. (-\mathfrak{n}_{a}^{i+1} \circ \psi_{1}^{i+1i} - \hat{d}_{2}^{i+1} \circ \mathfrak{h}_{a}^{i+1i} - \mathfrak{h}_{a}^{i+1i} \circ \hat{d}_{1}^{i} \right) \\ &\quad + \left. (-\mathfrak{n}_{a}^{i+1} \circ \psi_{1}^{i+1i} - \hat{d}_{2}^{i+1} \circ \mathfrak{h}_{a}^{i+1i} - \mathfrak{h}_{a}^{i+1i} \circ \hat{d}_{1}^{i} \right) \\ &\quad = \left. \left(-\hat{d}_{2}^{i+1} \circ \mathfrak{h}_{ab}^{i+1} \circ \psi_{1}^{i+1i} - \mathfrak{h}_{ab}^{i+1} \circ \psi_{1}^{i+1i} \circ \hat{d}_{1}^{i} \right. \\ &\quad + \mathfrak{n}_{b}^{i+1} \circ \psi_{1}^{i+1i} - \mathfrak{n}_{a}^{i+1} \circ \psi_{1}^{i+1i} \right)_{\alpha'_{1}\alpha_{1}}. \end{split} \tag{19.79}$$

By Situation 19.38(3) we have

$$(\hat{d}_2^{i+1} \circ \mathfrak{h}_{ab}^{i+1} + \mathfrak{h}_{ab}^{i+1} \circ \hat{d}_1^{i+1})_{\alpha_2'\alpha_1} = (\mathfrak{n}_b^{i+1} - \mathfrak{n}_a^{i+1})_{\alpha_2'\alpha_1}. \tag{19.80}$$

Moreover, by Situation 19.38 (1) we find

$$(\hat{d}_2^{i+1} \circ \mathfrak{h}_{ab}^{i+1} \circ \psi_1^{i+1i})_{\alpha_2'\alpha_1} = (\hat{d}_2^{i+1} \circ \psi_1^{i+1i} \circ \mathfrak{h}_{ab}^{i+1})_{\alpha_2'\alpha_1}.$$

Therefore the right hand side of (19.79) vanishes. Namely $(\psi_2^{i+1i} \circ o)_{\alpha_2'\alpha_1}$ is a d_0 -coboundary. Since ψ_2^{i+1i} induces an isomorphism on d_0 -cohomology in energy level 0 (this follows from Definition 19.31 (6)), $o(\alpha_1, \alpha_2)$ is also a d_0 -coboundary.

Thus we can find $(\mathfrak{h}_{ab}^i)_{\alpha_2\alpha_1}$ and promote \mathfrak{h}_{ab}^i to the energy level E^{i+1} . Note that we have a freedom to change $(\mathfrak{h}_{ab}^i)_{\alpha_2\alpha_1}$ by d_0 -cocycle.

Lemma 19.41.

$$\begin{pmatrix}
\hat{d}_{2}^{i+1} \circ \mathfrak{H}_{ab}^{i+1i} - \mathfrak{H}_{ab}^{i+1i} \circ \hat{d}_{1}^{i} + \mathfrak{h}_{b}^{i+1i} - \mathfrak{h}_{a}^{i+1i} \\
- \mathfrak{h}_{ab}^{i+1i} \circ \psi_{1}^{i+1i} + \psi_{2}^{i+1i} \circ \mathfrak{h}_{ab}^{i} \rangle_{\alpha',\alpha},
\end{pmatrix} (19.81)$$

is a d_0 -cocycle for $E(\alpha_2') - E(\alpha_1) = E^{i+1} - c$.

Proof. We first note that

$$\begin{split} &(\hat{d}_{2}^{i+1}\circ(\hat{d}_{2}^{i+1}\circ\mathfrak{H}_{ab}^{i+1i}-\mathfrak{H}_{ab}^{i+1i}\circ\hat{d}_{1}^{i})\\ &+(\hat{d}_{2}^{i+1}\circ\mathfrak{H}_{ab}^{ii+1}-\mathfrak{H}_{ab}^{i+1i}\circ\hat{d}_{1}^{i})\circ\hat{d}_{1}^{i})_{\alpha'_{2}\alpha_{1}}=0. \end{split} \tag{19.82}$$

Next we observe that Situation 19.38 (4) implies

$$\begin{aligned} &(\hat{d}_{2}^{i+1} \circ (\mathfrak{h}_{b}^{ii+1} - \mathfrak{h}_{a}^{i+1i}) + (\mathfrak{h}_{b}^{i+1i} - \mathfrak{h}_{a}^{ii+1}) \circ \hat{d}_{1}^{i})_{\alpha'_{2}\alpha_{1}} \\ &= (\mathfrak{n}_{b}^{i+1} \circ \psi_{1}^{i+1i} - \psi_{2}^{ii+1} \circ \mathfrak{n}_{b}^{i} - \mathfrak{n}_{a}^{i} - \mathfrak{n}_{a}^{i+1} \circ \psi_{1}^{i+1i} + \psi_{2}^{ii+1} \circ \mathfrak{n}_{a}^{i})_{\alpha'_{2}\alpha_{1}}. \end{aligned}$$
 (19.83)

Moreover Situation 19.38 (1)(3) imply

$$\begin{split} &(\hat{d}_{2}^{i+1} \circ \mathfrak{h}_{ab}^{i+1} \circ \psi_{1}^{i+1i} + \mathfrak{h}_{ab}^{i+1} \circ \psi_{1}^{ii+1} \circ \hat{d}_{1}^{i})_{\alpha'_{2}\alpha_{1}} \\ &= (\hat{d}_{2}^{i+1} \circ \mathfrak{h}_{ab}^{i+1} \circ \psi_{1}^{i+1i} + \mathfrak{h}_{ab}^{i+1} \circ \hat{d}_{1}^{i+1} \circ \psi_{1}^{i+1i})_{\alpha'_{2}\alpha_{1}} \\ &= ((\mathfrak{n}_{b}^{i+1} - \mathfrak{n}_{a}^{i+1}) \circ \psi_{1}^{i+1i})_{\alpha'_{2}\alpha_{1}}. \end{split} \tag{19.84}$$

In a similar way we have

$$\begin{split} &(\hat{d}_{2}^{i+1} \circ \psi_{2}^{i+1i} \circ \mathfrak{h}_{ab}^{i} + \psi_{2}^{i+1i} \circ \mathfrak{h}_{ab}^{i} \circ \hat{d}_{1}^{i})_{\alpha'_{2}\alpha_{1}} \\ &= (\psi_{2}^{i+1i} \circ \hat{d}_{2}^{i} \circ \mathfrak{h}_{ab}^{i} + \psi_{2}^{i+1i} \circ \mathfrak{h}_{ab}^{i} \circ \hat{d}_{1}^{i})_{\alpha'_{2}\alpha_{1}} \\ &= (\psi_{2}^{i+1i} \circ (\mathfrak{n}_{b}^{i} - \mathfrak{n}_{b}^{i}))_{\alpha'_{2}\alpha_{1}}. \end{split} \tag{19.85}$$

The sum of (19.82)-(19.85) is 0.

On the other hand,

$$\hat{d}_{2}^{i+1} \circ \mathfrak{H}_{ab}^{i+1i} - \mathfrak{H}_{ab}^{i+1i} \circ \hat{d}_{1}^{i} + \mathfrak{h}_{b}^{i+1i} - \mathfrak{h}_{a}^{ii+1} - \mathfrak{h}_{ab}^{i+1} \circ \psi_{1}^{i+1i} + \psi_{2}^{i+1i} \circ \mathfrak{h}_{ab}^{i}$$

is zero up to energy level E^i by assumption. Therefore the sum of (19.82)-(19.85) is d_0 differential of (19.81).

Therefore we can choose $(\mathfrak{h}_{ab}^i)_{\alpha_2\alpha_1}$ so that (19.81) is a d_0 -coboundary for α_1,α_2 with $E(\alpha_2) - E(\alpha_1) = E^{i+1} - c$. We now choose $(\mathfrak{H}_{ab}^{i+1i})_{\alpha_2'\alpha_1}$ which bounds (19.81). The proof of Proposition 19.39 is now complete.

Remark 19.42. In Situation 19.38 (1) we assumed that the energy zero part of the partial cochain map of energy loss 0 is the identity map. Actually we only need a milder assumption that the energy 0 part of the partial cochain map of energy loss 0 induces an injection on d_0 cohomology.

Proof of Theorem 16.39 (4). We are given \mathfrak{MN}_a , \mathfrak{MN}_b , that are morphisms of inductive systems $\mathcal{FF}_1 \to \mathcal{FF}_2$. Each of them consists of partial morphisms

$$\mathfrak{N}_{k}^{i}:\mathcal{F}_{1}^{i}\to\mathcal{F}_{2}^{i},\quad k=a,b,$$

respectively. They induce partial cochain maps

$$\mathfrak{n}_k^i: CF(\mathcal{F}_1^i) \to CF(\mathcal{F}_2^i), \quad k=a,b$$

for each i. We are also given a homotopy $\mathfrak{H}\mathfrak{H}$ from $\mathfrak{N}\mathfrak{N}_a$ to $\mathfrak{N}\mathfrak{N}_b$. It consists of partial homotopies from \mathfrak{N}_a^i to \mathfrak{N}_b^i . By Proposition 19.28 and Lemma 19.37, it induces a partial homotopy

$$\mathfrak{h}_{ab}^i: CF(\mathcal{F}_1^i) \to CF(\mathcal{F}_2^i)$$

from \mathfrak{n}_a^i to \mathfrak{n}_b^i .

Since \mathfrak{NN}_a , \mathfrak{NN}_b are morphisms of partial linear K-system, we are given a partial homotopy from $\mathfrak{N}_2^{i+1i} \circ \mathfrak{N}_k^i$ to $\mathfrak{N}_k^{i+1} \circ \mathfrak{N}_1^{i+1i}$. Again by Proposition 19.28 and Lemma 19.37, it induces a partial homotopy

$$\mathfrak{h}_k^{i+1i}: CF(\mathcal{F}_1^i) \to CF(\mathcal{F}_2^{i+1})$$

from $\psi_2^{i+1i} \circ \mathfrak{n}_k^i$ to $\mathfrak{n}_k^{i+1} \circ \psi_1^{i+1i}$. (Note $\psi_j^{i+1i} : CF(\mathcal{F}_j^i) \to CF(\mathcal{F}_j^{i+1})$ is a cochain map induced by \mathfrak{N}_i^{i+1i} .) Thus we are in Situation 19.38.

Now the existence of homotopy of homotopy \mathcal{H}^i which is Definition 16.36 (4) (b), (c), together with Proposition 19.35 and Lemma 19.37 implies that there exists

$$\mathfrak{H}_{ab}^{i+1i}: CF(\mathcal{F}_1^i) \to CF(\mathcal{F}_2^{i+1})$$

that satisfies (19.68). Thus we can apply Proposition 19.39 to promote our partial homotopy \mathfrak{h}_{ab}^i to one of energy cut level ∞ . Therefore \mathfrak{n}_a^i is cochain homotopic to \mathfrak{n}_b^i .

Proof of Theorem 16.31 (2). This is a special case of Theorem 16.39 (4) where \mathfrak{N}^{i+1i} is the identity morphism.

Proof of Theorem 16.39 (5) and Theorem 16.31 (3). We will prove Theorem 16.39 (5). Theorem 16.31 (3) is its special case.

We recall that $\mathfrak{N}_{ba}: \mathcal{F}\mathcal{F}_a \to \mathcal{F}\mathcal{F}_b$ consists of partial morphisms

$$\mathfrak{N}^i_{ba}:\mathcal{F}^i_a o\mathcal{F}^i_b$$

and partial homotopies \mathfrak{H}^i_{ba} between $\mathfrak{N}^{i+1i}_b \circ \mathfrak{N}^i_{ba}$ and $\mathfrak{N}^{i+1}_{ba} \circ \mathfrak{N}^{i+1i}_a$. (See Diagram 19.92 below.)

$$\mathcal{F}_{b}^{i} \xrightarrow{\mathfrak{N}_{b}^{i+1i}} \mathcal{F}_{b}^{i+1} \\
\mathfrak{N}_{ba}^{i} \uparrow \qquad \qquad \uparrow \mathfrak{N}_{ba}^{i+1} \\
\mathcal{F}_{a}^{i} \xrightarrow{\mathfrak{N}_{a}^{i+1i}} \mathcal{F}_{a}^{i+1}$$
(19.86)

Also $\mathfrak{N}_{cb}: \mathcal{F}\mathcal{F}_b \to \mathcal{F}\mathcal{F}_c$ consists of partial morphisms $\mathfrak{N}^i_{cb}: \mathcal{F}^i_b \to \mathcal{F}^i_c$ and partial homotopies \mathfrak{N}^i_{cb} in a similar way. The definition of the composition $\mathfrak{N}_{ca} = \mathfrak{N}_{cb} \circ \mathfrak{N}_{ba}$ is given in Lemma-Definition 16.38.

Let $\mathcal{N}_{ab}^{i}(\alpha_{a}, \alpha_{b})$ and $\mathcal{N}_{bc}^{i}(\alpha_{b}, \alpha_{c})$ be interpolation spaces of \mathfrak{N}_{ba}^{i} and \mathfrak{N}_{cb}^{i} , respectively. For k=a,b,c, let $\mathcal{N}_{k}^{i+1i}(\alpha_{k},\alpha_{k}')$ be an interpolation space of \mathfrak{N}_{k}^{i+1i} . We denote by $\mathcal{N}_{ab}^{i}(\alpha_{a},\alpha_{b};[0,1])$ and $\mathcal{N}_{bc}^{i}(\alpha_{b},\alpha_{c};[0,1])$ interpolation spaces of \mathfrak{H}_{ba}^{i} and \mathfrak{H}_{cb}^{i} , respectively. By definition we have

$$\partial \mathcal{N}_{ab}^{i}(\alpha_{a}, \alpha_{b}; [0, 1])$$

$$= \bigcup_{\alpha'_{a}} \mathcal{N}_{a}^{i+1i}(\alpha_{a}, \alpha'_{a}) \times_{R_{\alpha'_{a}}}^{\boxplus \tau} \mathcal{N}_{ab}^{i}(\alpha'_{a}, \alpha_{b})$$

$$\cup \bigcup_{\alpha'_{a}} \mathcal{N}_{ab}^{i}(\alpha_{a}, \alpha'_{b}) \times_{R_{\alpha'_{b}}}^{\boxplus \tau} \mathcal{N}_{b}^{i+1i}(\alpha'_{b}, \alpha_{b}).$$

$$(19.87)$$

Here $\times_{R_{\alpha'_a}}^{\boxplus \tau}$ and $\times_{R_{\alpha'_b}}^{\boxplus \tau}$ are as in Definition 18.37. Similarly we have

$$\partial \mathcal{N}_{bc}^{i}(\alpha_{b}, \alpha_{c}; [0, 1])$$

$$= \bigcup_{\alpha'_{b}} \mathcal{N}_{b}^{i+1i}(\alpha_{b}, \alpha'_{b}) \times_{R_{\alpha'_{b}}}^{\boxplus \tau} \mathcal{N}_{bc}^{i}(\alpha'_{b}, \alpha_{c})$$

$$\cup \bigcup_{\alpha'_{c}} \mathcal{N}_{bc}^{i}(\alpha_{a}, \alpha'_{c}) \times_{R_{\alpha'_{c}}}^{\boxplus \tau} \mathcal{N}_{c}^{i+1i}(\alpha'_{c}, \alpha_{c}).$$

$$(19.88)$$

The composition $\mathfrak{N}_{ca} = \mathfrak{N}_{cb} \circ \mathfrak{N}_{ba}$ consists of \mathfrak{N}_{ca}^i and \mathfrak{H}_{ca}^i . Here the interpolation space $\mathcal{N}_{ac}^i(\alpha_a, \alpha_c)$ of \mathfrak{N}_{ca} is given by

$$\mathcal{N}_{ac}^{i}(\alpha_{a},\alpha_{c}) = \bigcup_{\alpha_{b}} \mathcal{N}_{ab}^{i}(\alpha_{a},\alpha_{b}) \times_{R_{\alpha_{b}}}^{\boxplus \tau} \mathcal{N}_{bc}^{i}(\alpha_{b},\alpha_{c}).$$

Note that the union in the above formula is different from the disjoint union and is defined as in Lemma-Definition 18.40.

The homotopy \mathfrak{H}^i_{ca} is obtained by gluing $\mathfrak{H}^i_{cb} \circ \mathfrak{N}^i_{ba}$ and $\mathfrak{N}^i_{cb} \circ \mathfrak{H}^i_{ba}$ as follows. The interpolation space of $\mathfrak{H}_{cb}^i \circ \mathfrak{N}_{ba}^i$ is

$$\bigcup_{\alpha_b} \mathcal{N}_{ab}^i(\alpha_a, \alpha_b; [0, 1]) \times_{R_{\alpha_b}}^{\boxplus \tau} \mathcal{N}_{bc}^i(\alpha_b, \alpha_c). \tag{19.89}$$

The interpolation space of $\mathfrak{N}_{cb}^i \circ \mathfrak{H}_{ba}^i$ is

$$\bigcup_{\alpha_b} \mathcal{N}_{ab}^i(\alpha_a, \alpha_b) \times_{R_{\alpha_b}}^{\boxplus \tau} \mathcal{N}_{bc}^i(\alpha_b, \alpha_c; [0, 1]). \tag{19.90}$$

We observe that both (19.89) and (19.90) contain

$$\bigcup_{\alpha_b} \bigcup_{\alpha'_b} \mathcal{N}_{ab}^i(\alpha_a, \alpha_b) \times_{R_{\alpha_b}}^{\boxplus \tau} \mathcal{N}_b^{i+1i}(\alpha_b, \alpha'_b) \times_{R_{\alpha'_b}}^{\boxplus \tau} \mathcal{N}_{bc}^i(\alpha_b, \alpha_c)$$
(19.91)

in its boundary. We smooth the corners contained in (19.91). See Section 18. Next we recall that while we constructed a partial cochain map

$$\psi_{ba}^i: CF_a^i \to CF_b^i$$

(see (19.55) for the notation CF_*^i), we took a thickening $\widehat{\mathcal{U}_{ab}^{i,+}}(\alpha_a,\alpha_b)$ of $\widehat{\mathcal{U}_{ab}^i}(\alpha_a,\alpha_b)$ (note they are Kuranishi structures of $\mathcal{N}_{ab}^i(\alpha_a,\alpha_b)^{\mathfrak{C}^h \boxplus \tau}$) and a CF-perturbation $\widehat{\mathfrak{S}_{ab}^{i}}(\alpha_a, \alpha_b)$ of $\widehat{\mathcal{U}_{ab}^{i,+}}(\alpha_a, \alpha_b)$. (Here \mathfrak{C}^h stands for the horizontal boundary as in Definition 19.27.) During the construction of a partial cochain map

$$\psi_{ch}^i: CF_h^i \to CF_c^i$$

we took a thickening $\widehat{\mathcal{U}_{bc}^{i,+}}(\alpha_b,\alpha_c)$ (of $\widehat{\mathcal{U}_{bc}^i}(\alpha_b,\alpha_c)$) and its CF-perturbation $\widehat{\mathfrak{S}_{bc}^i}(\alpha_b,\alpha_c)$. During the construction of partial cochain maps

$$\psi_k^{i+1i}: CF_k^i \to CF_k^{i+1}, \quad k = a, b, c,$$

we took thickenings $\widehat{\mathcal{U}_k^{i+1i,+}}(\alpha_k,\alpha_k')$ of $\widehat{\mathcal{U}_k^{i+1i}}(\alpha_k,\alpha_k')$ which are Kuranishi structures on $\mathcal{N}_k^{i+1i}(\alpha_k, \alpha_k')^{\boxplus \tau}$), and their CF-perturbations $\widehat{\mathfrak{S}_k^{i+1i}}(\alpha_k, \alpha_k')$. Furthermore, in the course of our construction of a partial cochain homotopy

$$\mathfrak{h}_{ba}^{i}$$
 between $\mathfrak{n}_{b}^{i+1i} \circ \psi_{ba}^{i}$ and $\psi_{ba}^{i+1} \circ \mathfrak{n}_{a}^{i+1i}$

(where $\mathfrak{n}_b^{i+1i} \circ \psi_{ba}^i$ and $\psi_{ba}^{i+1} \circ \mathfrak{n}_a^{i+1i}$ are cochain maps : $CF_a^i \to CF_b^{i+1}$), we took a thickening $\widehat{\mathcal{U}}_{ab}^{i,+}(\alpha_a,\alpha_b;[0,1])$ of $\widehat{\mathcal{U}}_{ab}^{i}(\alpha_a,\alpha_b;[0,1])$ which is a Kuranishi structure on $\mathcal{N}_{ab}^{i}(\alpha_a,\alpha_b;[0,1])^{\mathfrak{C}^h \boxplus \tau}$, and its CF-perturbation $\widehat{\mathfrak{S}_{ab}^{i}}(\alpha_a,\alpha_b;[0,1])$.

$$CF_{a}^{i} \xrightarrow{\psi_{ba}^{i}} CF_{b}^{i} \xrightarrow{\psi_{cb}^{i}} CF_{c}^{i}$$

$$\mathfrak{n}_{a}^{i+1i} \downarrow \qquad \mathfrak{n}_{b}^{i+1i} \downarrow \qquad \mathfrak{n}_{c}^{i+1i} \downarrow$$

$$CF_{a}^{i+1} \xrightarrow{\psi_{ba}^{i+1}} CF_{b}^{i+1} \xrightarrow{\psi_{cb}^{i+1}} CF_{c}^{i+1}$$

$$(19.92)$$

During our construction of a partial cochain homotopy

$$\mathfrak{h}_{cb}^{i} \quad \text{between } \mathfrak{n}_{c}^{i+1i} \circ \psi_{cb}^{i} \text{ and } \psi_{cb}^{i+1} \circ \mathfrak{n}_{b}^{i+1i},$$

we took a thickening $\widehat{\mathcal{U}_{bc}^{i,+}}(\alpha_b, \alpha_c; [0,1])$ of $\widehat{\mathcal{U}_{bc}^i}(\alpha_b, \alpha_c; [0,1])$ which is a Kuranishi structure on $\mathcal{N}_{bc}^{i}(\alpha_{b}, \alpha_{c}; [0, 1])^{\mathfrak{C}^{h} \boxplus \tau}$ and its CF-perturbation $\widehat{\mathfrak{S}_{bc}^{i}}(\alpha_{b}, \alpha_{c}; [0, 1])$.

Note that by Lemma 19.21 we may use

$$\bigcup_{\alpha_b} \widehat{\mathcal{U}_{ab}^{i,+}}(\alpha_a, \alpha_b) \times_{R_{\alpha_b}} \widehat{\mathcal{U}_{bc}^{i,+}}(\alpha_b, \alpha_c)$$
 (19.93)

and

$$\bigcup_{\alpha_i} \widehat{\mathfrak{S}_{ab}^i}(\alpha_a, \alpha_b) \times_{R_{\alpha_b}} \widehat{\mathfrak{S}_{bc}^i}(\alpha_b, \alpha_c)$$
 (19.94)

to define a partial cochain map

$$\psi_{ca}^i: CF_a^i \to CF_c^i.$$

Here (19.93) is a thickening of $\bigcup_{\alpha_b} \widehat{\mathcal{U}_{ab}^i}(\alpha_a, \alpha_b) \times_{R_{\alpha_b}} \widehat{\mathcal{U}_{bc}^i}(\alpha_b, \alpha_c)$ which is a Kuranishi structure on $\mathcal{N}_{ac}^i(\alpha_a, \alpha_c)^{\mathfrak{C}^h \boxplus \tau}$, and (19.94) is its CF-perturbation. Therefore composition formula and Lemma 18.34 yield

$$\psi_{ca}^i = \psi_{cb}^i \circ \psi_{ba}^i, \tag{19.95}$$

if we define ψ_{ca}^{i} by this particular choice. (Lemma 19.22.)

Next we consider \mathfrak{h}_{ca}^i . We take

$$\left(\bigcup_{\alpha_{b}}\widehat{\mathcal{U}_{ab}^{i,+}}(\alpha_{a},\alpha_{b};[0,1]) \times_{R_{\alpha_{b}}}\widehat{\mathcal{U}_{bc}^{i,+}}(\alpha_{b},\alpha_{c})\right) \\
\cup \left(\bigcup_{\alpha_{b}}\widehat{\mathcal{U}_{ab}^{i,+}}(\alpha_{a},\alpha_{b}) \times_{R_{\alpha_{b}}}\widehat{\mathcal{U}_{bc}^{i,+}}(\alpha_{b},\alpha_{c};[0,1])\right).$$
(19.96)

Here we take a partial smoothing of corner of the right hand side and glue them. Then (19.96) is a thickening of $\mathcal{N}_{ac}^i(\alpha_a,\alpha_c)^{\mathfrak{C}^h\boxplus \tau}$ and

$$\left(\bigcup_{\alpha_{b}}\widehat{\mathfrak{S}_{ab}^{i}}(\alpha_{a}, \alpha_{b}; [0, 1]) \times_{R_{\alpha_{b}}} \widehat{\mathfrak{S}_{bc}^{i}}(\alpha_{b}, \alpha_{c})\right) \\
\cup \left(\bigcup_{\alpha_{b}}\widehat{\mathfrak{S}_{ab}^{i}}(\alpha_{a}, \alpha_{b}) \times_{R_{\alpha_{b}}} \widehat{\mathfrak{S}_{bc}^{i}}(\alpha_{b}, \alpha_{c}; [0, 1])\right)$$
(19.97)

is a CF-perturbation of (19.96). We use (19.96) and (19.97) to define \mathfrak{h}_{ca}^i . Then by composition formula and Lemma 18.34 again we find

$$\mathfrak{h}_{ca}^{i} = \psi_{cb}^{i+1} \circ \mathfrak{h}_{ba}^{i} + \mathfrak{h}_{cb}^{i} \circ \psi_{ba}^{i}. \tag{19.98}$$

Lemma 19.43.

$$\hat{d}\circ\mathfrak{h}_{ca}^{i}+\mathfrak{h}_{ca}^{i}\circ\hat{d}=\psi_{ca}^{i+1}\circ\mathfrak{n}_{a}^{i+1i}-\mathfrak{n}_{c}^{i+1i}\circ\psi_{ca}^{i}. \tag{19.99}$$

Proof.

$$\begin{split} \hat{d} &\circ \mathfrak{h}^{i}_{ca} + \mathfrak{h}^{i}_{ca} \circ \hat{d} \\ = & \psi^{i+1}_{cb} \circ \hat{d} \circ \mathfrak{h}^{i}_{ba} + \psi^{i+1}_{cb} \circ \mathfrak{h}^{i}_{ba} \circ \hat{d} \\ &+ \mathfrak{h}^{i}_{cb} \circ \hat{d} \circ \psi^{i}_{ba} + \hat{d} \circ \mathfrak{h}^{i}_{cb} \circ \psi^{i}_{ba} \\ = & \psi^{i+1}_{cb} \circ \psi^{i+1}_{ba} \circ \mathfrak{n}^{i+1i}_{a} - \psi^{i+1}_{cb} \circ \mathfrak{n}^{i+1i}_{b} \circ \psi^{i}_{ba} \\ &+ \psi^{i+1}_{cb} \circ \mathfrak{n}^{i+1i}_{b} \circ \psi^{i}_{ba} - \mathfrak{n}^{i+1i}_{c} \circ \psi^{i+1}_{cb} \circ \psi^{i}_{ba} \\ = & \psi^{i+1}_{ca} \circ \mathfrak{n}^{i+1i}_{a} - \mathfrak{n}^{i+1i}_{c} \circ \psi^{i}_{ca}. \end{split}$$

We note that the equalities (19.95), (19.98), (19.99) are ones of energy cut level E^i . We recall that while we constructed ψ_{ba} and ψ_{cb} we promoted \mathfrak{n}_k^{i+1i} (k=a,b,c), ψ_{ba}^i , ψ_{cb}^i , \mathfrak{h}_{ba}^i and \mathfrak{h}_{cb}^i to energy cut level ∞ . We now promote ψ_{ca}^i and \mathfrak{h}_{ca}^i to energy cut level ∞ so that (19.95), (19.98) hold as the equalities at the energy cut level ∞ . Then (19.99) holds as an equality at the energy cut level ∞ .

Thus for this particular choice of promotion, the equality $\psi^i_{ca} = \psi^i_{cb} \circ \psi^i_{ba}$ holds not only up to cochain homotopy but also as a strict identity. Since ψ^i_{ca} is independent of various choices up to cochain homotopy (Theorem 16.39, which we proved in Subsection 19.6), $\psi^i_{ca} = \psi^i_{cb} \circ \psi^i_{ba}$ holds for any choice of ψ^i_{ca} up to cochain homotopy. The proof of Theorem 16.39 (5) is complete.

We next prove a partial cochain map version of Theorem 16.31 (4).

Lemma 19.44. In the situation of Theorem 16.31 (4) we have $\mathcal{ID}_* \sim \operatorname{id} \mod T^E$ for any E. Here \sim means cochain homotopic.

Proof. This is a consequence of Theorem 16.31 (2) and Proposition 18.63. In fact, Theorem 16.31 (2) implies that $\mathcal{ID}_* \circ \mathcal{ID}_* \sim \mathcal{ID}_*$. (Here \mathfrak{ID}_* is the cochain map induced by the identity morphism and \sim is a chan homotopy). On the other hand, Lemma 19.45 below implies that \mathfrak{ID}_* is an isomorphism. Therefore $\mathfrak{ID}_* \sim \mathrm{id}$. \square

Lemma 19.45. Let $\widehat{\psi}_{21} = \{\psi_{\alpha_2\alpha_1}\}: \mathcal{F}_1 \to \mathcal{F}_2$ be a partial cochain map of energy cut level E_0 and energy loss 0. We assume it is congruent to the isomorphism in the sense of Definition 16.19. Then there exists a partial cochain map $\widehat{\psi}_{12} = \{(\psi_{12})_{\alpha_1\alpha_2}\}: \mathcal{F}_2 \to \mathcal{F}_1$ of energy cut level E_0 and energy loss 0 such that $\widehat{\psi}_{12} \circ \widehat{\psi}_{21}$ and $\widehat{\psi}_{21} \circ \widehat{\psi}_{12}$ are identity maps.

Proof. We construct $(\psi_{12})_{\alpha_1\alpha_2}$ by induction on $E(\alpha_1) - E(\alpha_2)$. This induction is possible because the set of values of $E(\alpha_1) - E(\alpha_2)$ is a discrete set by uniform Gromov compactness Definition 16.36 (2)(g).

We start with the case when $E(\alpha_1) - E(\alpha_2) = 0$. By definition of partial cochain map of energy loss 0 congruent to the isomorphism (Definition 16.19), we have

$$(\psi_{12})_{\alpha_1\alpha_2} = \begin{cases} 0 & \text{if } \alpha_1 \neq \alpha_2\\ \text{id} & \text{if } \alpha_1 = \alpha_2. \end{cases}$$

In fact, the interpolation space $\mathcal{N}(\alpha_2, \alpha_1)$ is an empty set if $\alpha_1 \neq \alpha_2$ and $E(\alpha_1) - E(\alpha_2) = 0$ by Condition 16.16 (V). This implies the first equality. If $\alpha_1 = \alpha_2 = \alpha$, we have $\mathcal{N}(\alpha, \alpha) = R_{\alpha}$ by Definition 16.19. Moreover, $\operatorname{ev}_{\pm} : \mathcal{N}(\alpha, \alpha) \to R_{\alpha}$ is the identity map. This implies the second equality.

Suppose $E(\alpha_1) - E(\alpha_2) = E_0$ and we have defined $(\psi_{12})_{\alpha'_1\alpha'_2}$ for $E(\alpha_1) - E(\alpha_2) < E_0$. Then the condition that $\widehat{\psi}_{21}$ is a right inverse of $\widehat{\psi}_{12}$ at the energy cut level E_0 is written as

$$(\psi_{12})_{\alpha_1\alpha_2} + \sum_{\alpha'_2} (\psi_{12})_{\alpha_1\alpha'_2} \circ (\psi_{21})_{\alpha'_2\alpha_1} = 0.$$

Since the second term is already defined we can find $(\psi_{12})_{\alpha_1\alpha_2}$ uniquely so that this condition holds. Thus we have found the left inverse by induction. We can find the right inverse in the same way. A standard fact in group theory yields that the right and left inverse coincide. It is also easy to see that partial inverse of a partial cochain map is a partial cochain map.

Proof of Theorem 16.9 (2)(f) and Theorem 16.39 (2)(e). Theorem 16.39 (2)(e) follows from Theorem 16.39 (4) applied to the identity morphism. Theorem 16.9 (2)(f) is a special case of Theorem 16.39 (2)(e) when \mathfrak{N}^{i+1i} are the identity morphisms. \square

Proof of Theorem 16.39 (6) and Theorem 16.31 (4). We will prove Theorem 16.39 (6). Theorem 16.31 (4) is its special case.

We first define the identity morphism of an inductive system of linear K-systems. Let $\mathcal{FF} = (\{\mathcal{F}^i\}, \{\mathfrak{N}^i\})$ be an inductive system of linear K-systems. (Definition 16.36 (2).) We put $\mathcal{FF}_k = \mathcal{FF}$ for k = a, b and $\mathfrak{N}_{ba}^i = \mathcal{ID}_{\mathcal{F}^i}$ the identity morphism of \mathcal{F}^i . By Proposition 18.63 we have

$$\mathfrak{N}^i \circ \mathcal{ID}_{\mathcal{F}^i} \sim \mathfrak{N}^i \sim \mathcal{ID}_{\mathcal{F}^i} \circ \mathfrak{N}^i.$$

Let \mathfrak{H}_{ba}^{i} be this homotopy. (We take the particular choice of the homotopy which we gave during the proof of Proposition 18.63.)

Definition 19.46. We define $\mathcal{ID}_{\mathcal{F}\mathcal{F}} = (\{\mathcal{ID}_{\mathcal{F}^i}\}, \{\mathfrak{H}_{ba}^i\})$ the identity morphism from $\mathcal{F}\mathcal{F}$ to itself.

Theorem 16.39 (6) claims that the cochain map induced by $\mathcal{ID}_{\mathcal{F}\mathcal{F}}$ is cochain homotopic to the identity. To prove this it suffices to show the following lemma.

Lemma 19.47. If $\mathfrak{N}_{cb}: \mathcal{FF}_b \to \mathcal{FF}_c$ be a morphism of inductive systems of linear K-systems, then the composition $\mathfrak{N}_{cb} \circ \mathcal{ID}_{\mathcal{F}\mathcal{F}}$ is homotopic to \mathfrak{N}_{cb} . The same holds for $\mathcal{ID}_{\mathcal{F}\mathcal{F}} \circ \mathfrak{N}_{cb}$.

Proof. Write $\mathcal{FF}_c = (\{\mathcal{F}_c^i\}, \{\mathfrak{N}_c^{i+1i}\})$ and $\mathfrak{N}_{cb} = (\{\mathfrak{N}_{cb}^i\}, \{\mathfrak{H}_{cb}^i\}) : \mathcal{FF}_b \to \mathcal{FF}_c$. Here $\mathfrak{N}_k^{i+1i} : \mathcal{F}_k^i \to \mathcal{F}_k^{i+1} \ (k=b,c), \, \mathfrak{N}_{cb}^i : \mathcal{F}_b^i \to \mathcal{F}_c^i$ are partial morphisms of partial linear K-systems, and \mathfrak{H}_{cb}^i is a homotopy between $\mathfrak{N}_{cb}^{i+1} \circ \mathfrak{N}_b^{i+1i}$ and $\mathfrak{N}_c^{i+1i} \circ \mathfrak{N}_{cb}^i$. (They are partial morphisms : $\mathcal{F}_b^i \to \mathcal{F}_c^{i+1}$.)

We denote by $\mathcal{M}(ki; \alpha_-, \alpha_+)$ the moduli space of connecting orbits for \mathcal{F}_k^i . (k =a, b, c). Note that

$$\mathcal{M}(ai; \alpha_-, \alpha_+) = \mathcal{M}(bi; \alpha_-, \alpha_+).$$

Let $\mathcal{N}(k, ii + 1; \alpha_-, \alpha_+)$ and $\mathcal{N}(bc, i; \alpha_-, \alpha'_+)$ be interpolation spaces of \mathfrak{N}_k^{i+1i} and \mathfrak{R}_{cb}^i , respectively. Let $\mathcal{N}(bc, ii+1; \alpha_-, \alpha'_+; [1,2])$ be the interpolation space of \mathfrak{H}_{cb}^i .

$$\mathcal{F}_{c}^{i} \xrightarrow{\mathfrak{N}_{c}^{i+1i}} \mathcal{F}_{c}^{i+1}$$

$$\mathfrak{N}_{cb}^{i} \uparrow \qquad \uparrow \mathfrak{N}_{cb}^{i+1}$$

$$\mathcal{F}_{b}^{i} \xrightarrow{\mathfrak{N}_{b}^{i+1i}} \mathcal{F}_{b}^{i+1}$$

$$\mathcal{ID} \uparrow \qquad \uparrow \mathcal{ID}$$

$$\mathcal{F}_{a}^{i} \xrightarrow{\mathfrak{N}_{a}^{i+1i}} \mathcal{F}_{c}^{i+1}$$
(19.100)

Here we also note that $\mathcal{F}_a^i = \mathcal{F}_b^i$ and $\mathfrak{N}_a^{i+1i} = \mathfrak{N}_b^{i+1i}$. We put $\mathfrak{N}^i \circ \mathcal{ID}_{\mathcal{F}^i} = (\{\mathfrak{N}_{cb}^i \circ \mathcal{ID}\}, \{\mathfrak{H}_{ca}^i\})$. By definition the homotopy \mathfrak{H}_{ca}^i is obtained as

$$\mathfrak{H}_{ca}^{i} = (\mathfrak{N}_{cb}^{i+1} \circ \mathfrak{H}_{ba}^{i}) \cup (\mathfrak{H}_{cb}^{i} \circ \mathcal{ID}). \tag{19.101}$$

Note that $\mathfrak{N}_{cb}^{i+1} \circ \mathfrak{H}_{ba}^{i}$ is a homotopy from $\mathfrak{N}_{cb}^{i+1} \circ \mathcal{ID} \circ \mathfrak{N}_{a}^{i+1i}$ to $\mathfrak{N}_{cb}^{i+1i} \circ \mathfrak{ID}$ and $\mathfrak{H}_{cb}^{i} \circ \mathcal{ID}$ is a homotopy from $\mathfrak{N}_{cb}^{i+1} \circ \mathfrak{N}_{b}^{i+1i} \circ \mathcal{ID}$ to $\mathfrak{N}_{c}^{i+1i} \circ \mathfrak{N}_{cb}^{i} \circ \mathcal{ID}$. (See Diagram (19.100).)

Now we start the construction of the homotopy between $\mathfrak{N}_{cb} \circ \mathcal{ID}_{\mathcal{F}\mathcal{F}}$ and \mathfrak{N}_{cb} . By Proposition 18.63 we have $\mathfrak{N}_{cb}^i \circ \mathcal{ID} \sim \mathfrak{N}_{cb}^i$. Let \mathfrak{H}^i be this homotopy. Note that as \mathfrak{H}^i we take the specific homotopy we constructed during the proof of Proposition 18.63. To prove Lemma 19.47 it suffices to construct a homotopy of homotopies \mathcal{H}^i appearing in Definition 16.36 (4). Recall that \mathcal{H}^i is a $[0,1]^2$ -parametrized partial morphism from \mathcal{F}_a^i to \mathcal{F}_c^{i+1} such that its normalized boundary $\partial \mathcal{H}^i$ is a disjoint union of the following 4 homotopies.

- $\begin{array}{ll} \text{(i)} & \mathfrak{H}_{cb}^{i}.\\ \text{(ii)} & \mathfrak{H}^{i+1} \circ \mathfrak{N}_{a}^{i+1i}.\\ \text{(iii)} & \mathfrak{H}_{ca}^{i} = (\mathfrak{N}_{cb}^{i+1} \circ \mathfrak{H}_{ba}^{i}) \cup (\mathfrak{H}_{cb}^{i} \circ \mathcal{ID}).\\ \text{(iv)} & \mathfrak{N}_{c}^{i+1i} \circ \mathfrak{H}^{i}. \end{array}$

We will construct an interpolation space

$$\mathcal{N}(ac, ii + 1, \alpha_{-}, \alpha_{+}; [1, 2]^{2})$$

of the homotopy of homotopies \mathcal{H}^i by modifying the interpolation space

$$\mathcal{N}(bc, ii + 1, \alpha_{-}, \alpha_{+}; [1, 2])$$

of \mathfrak{H}_{cb}^i in a way similar to the proof of Proposition 18.63 as follows. We note that the restriction of $\mathcal{N}(bc, ii+1; \alpha_-, \alpha_+; [1,2])$ to $1 \in \partial[1,2]$ and to $2 \in \partial[1,2]$ is the union of the following fiber products, respectively.

- (I) $\mathcal{N}(b, ii+1; \alpha_{-}, \alpha) \times_{R_{\alpha}}^{\boxplus \tau} \mathcal{N}(bc, i+1; \alpha, \alpha'_{+}).$ (II) $\mathcal{N}(bc, i; \alpha_{-}, \alpha') \times_{R_{\alpha'}}^{\boxplus \tau} \mathcal{N}(c, ii+1; \alpha', \alpha'_{+}).$

There are two other kinds of boundary of $\mathcal{N}(bc, ii+1; \alpha_-, \alpha_+; [1, 2])$ as follows.

- (III) $\mathcal{M}(b, i; \alpha_{-}, \alpha) \times_{R_{\alpha}} \mathcal{N}(bc, ii + 1; \alpha, \alpha_{+}; [1, 2]).$
- (IV) $\mathcal{N}(bc, ii + 1; \alpha, \alpha'; [1, 2]) \times_{R_{\alpha'}} \mathcal{M}(b, i; \alpha', \alpha'_{+}).$

Let C be a sufficiently large positive number. We assume that it is large enough compared to the energy loss of \mathfrak{N}_{i}^{cb} . The top dimensional stratum of our interpolation space $\mathcal{N}(ac, ii + 1; \alpha_-, \alpha_+; [1, 2])$ is

(1)
$$\overset{\circ}{\mathcal{N}}(bc, ii+1; \alpha_{-}, \alpha_{+}; [1,2]) \times (E(\alpha_{-}), E(\alpha'_{+}) + C).$$

This is the only stratum of top dimension. Below we list up the codimension one strata:

- (2) $R_{\alpha_{-}} \times \{E(\alpha_{-})\}$ $\times_{R_{\alpha}} \mathring{\mathcal{N}}(bc, ii+1; \alpha_{-}, \alpha_{+}; [1, 2])$
- (3) $\mathring{\mathcal{M}}(b, i; \alpha_{-}, \alpha) \times (E(\alpha_{-}), E(\alpha))$ $\times_{R_{\alpha}}^{\boxplus \tau} \mathring{\mathcal{N}}(bc, ii+1; \alpha, \alpha_{+}; [1, 2])$
- (4) $\overset{\circ}{\mathcal{M}}(b,i;\alpha_{-},\alpha)$ $\times_{R_{\alpha}} \overset{\circ}{\mathcal{N}} (bc, ii + 1; \alpha, \alpha'_{+}; [1, 2]) \times (E(\alpha), E(\alpha'_{+}) + C)$
- (5) $\mathring{\mathcal{N}}(b, ii + 1; \alpha_{-}, \alpha) \times (E(\alpha_{-}), E(\alpha))$ $\times_{R_{\alpha}}^{\boxplus \tau} \mathring{\mathcal{N}}(bc, i+1; \alpha, \alpha'_{+})$
- (6) $\mathring{\mathcal{N}}(b, ii+1; \alpha_-, \alpha)$ $\times_{R}^{\boxplus \tau} \mathring{\mathcal{N}}(bc, i+1; \alpha, \alpha'_{+}) \times (E(\alpha), E(\alpha'_{+}))$

(7)
$$R_{\alpha_{-}} \times \{E(\alpha'_{+}) + C\}$$

 $\times_{R_{\alpha}} \overset{\circ}{\mathcal{N}}(bc, ii + 1; \alpha_{-}, \alpha_{+}; [1, 2])$

(8)
$$\mathring{\mathcal{N}}(bc, i; \alpha_{-}, \alpha') \times (E(\alpha_{-}), E(\alpha'_{+}) + C) \times_{R_{\alpha'}}^{\boxplus \tau} \mathring{\mathcal{N}}(c, ii + 1; \alpha', \alpha'_{+})$$

(9)
$$\overset{\circ}{\mathcal{N}}(bc, ii+1; \alpha_{-}, \alpha'; [1,2]) \times (E(\alpha_{-}), E(\alpha'_{+}) + C) \times_{R_{\alpha'}} \overset{\circ}{\mathcal{M}}(c, i; \alpha', \alpha'_{+})$$

Observe that (3) \cup (4), (5) \cup (6), (8), (9) are product of (III), (I), (II), (IV) and the interval $(E(\alpha_{-}), E(\alpha'_{+}) + C)$, respectively.

We also find that $(2) \cup (3) = \mathfrak{H}_i^{cb} \circ \mathcal{ID}$ and $(5) = \mathfrak{N}_{i+1}^{cb} \circ \mathfrak{H}_i^{ba}$. Therefore $(2) \cup (3) \cup (5) = (iii)$. We also have (6) = (ii), (7) = (i).

We can regard (8) = (iv). Here the interval $(E(\alpha_{-}), E(\alpha'_{+}) + C)$ appearing in (8) is not $(E(\alpha_{-}), E(\alpha') + C)$, but they are diffeomorphic.

Moreover, (4) is nothing but

$$\overset{\circ}{\mathcal{M}}(a,i;\alpha_{-},\alpha)\times_{R_{\alpha}}\overset{\circ}{\mathcal{N}}(ac,ii+1,\alpha,\alpha_{+};[1,2]^{2})$$

and we can regard (9) as

$$\mathring{\mathcal{N}}(ac, ii+1, \alpha_-, \alpha'; [1, 2]^2) \times_{R_{\alpha'}} \mathring{\mathcal{M}}(c, i+1; \alpha', \alpha).$$

Thus the boundary of $\mathcal{N}(ac, ii + 1, \alpha_-, \alpha_+; [1, 2]^2)$ has the required properties.

We note that we need to smooth corners and bend a part of the boundary so that corner structure stratification of $\mathcal{N}(ac, ii+1, \alpha_-, \alpha_+; [1, 2]^2)$ becomes a correct one. In fact, since the union of (2)(3)(5) is (iv), we need to smooth the corner where they intersect. For example, the intersection of (3) and (5) is

$$\mathring{\mathcal{M}}(b,i;\alpha_{-},\alpha_{1}) \times_{R_{\alpha_{1}}}^{\boxplus \tau} \mathring{\mathcal{N}}(b,ii+1;\alpha_{1},\alpha_{2}) \times_{R_{\alpha}}^{\boxplus \tau} \mathring{\mathcal{N}}(bc,ii+1;\alpha_{2},\alpha_{+};[1,2]) \times (E(\alpha_{1}),E(\alpha_{2})).$$

So we smooth the corner here. On the other hand

$$\mathring{\mathcal{M}}(b,i;\alpha_{-},\alpha_{1}) \times_{R_{\alpha_{1}}}^{\boxplus \tau} \mathring{\mathcal{N}}(b,ii+1;\alpha_{1},\alpha_{2}) \times_{R_{\alpha}}^{\boxplus \tau} \mathring{\mathcal{N}}(bc,ii+1;\alpha_{2},\alpha_{+};[1,2]) \times (E(\alpha_{-}),E(\alpha_{1}))$$

is a part of the boundary of (3) which is not contained in the boundary of (5). We bend the boundary component

$$\mathring{\mathcal{M}}(b,i;\alpha_{-},\alpha_{1}) \times_{R_{\alpha_{1}}}^{\boxplus \tau} \mathring{\mathcal{N}}(b,ii+1;\alpha_{1},\alpha_{2}) \times_{R_{\alpha}}^{\boxplus \tau} \mathring{\mathcal{N}}(bc,ii+1;\alpha_{2},\alpha_{+};[1,2]) \times (E(\alpha_{-}),E(\alpha_{2}))$$

at

$$\mathring{\mathcal{M}}(b,i;\alpha_{-},\alpha_{1}) \times_{R_{\alpha_{1}}}^{\boxplus \tau} \mathring{\mathcal{N}}(b,ii+1;\alpha_{1},\alpha_{2}) \times_{R_{\alpha}}^{\boxplus \tau} \mathring{\mathcal{N}}(bc,ii+1;\alpha_{2},\alpha_{+};[1,2]) \times \{E(\alpha_{1})\}.$$

This bending is included in the process we constructed the homotopy \mathfrak{H}^i between \mathfrak{N}^i_{cb} and $\mathcal{ID} \sim \mathfrak{N}^i_{cb}$. (See the proof of Proposition 18.63.)

We also note that in (4)(9) we take $\times_{R_{\alpha}}$ etc. but in other places we take $\times_{R_{\alpha}}^{\boxplus}$. This does *not* cause inconsistency at the intersection of (3) and (4) by the following reason. We first take the fiber product

$$\mathring{\mathcal{M}}(b,i;\alpha_{-},\alpha) \times_{R_{\alpha}} \mathring{\mathcal{N}}(bc,ii+1;\alpha,\alpha_{+};[1,2]) \times (E(\alpha_{-}),E(\alpha'_{+}))$$

and bend it at

$$\mathring{\mathcal{M}}(b, i; \alpha_{-}, \alpha) \times_{R_{\alpha}} \mathring{\mathcal{N}}(bc, ii + 1; \alpha, \alpha_{+}; [1, 2]) \times \{E(\alpha)\}.$$

We then partially trivialize by choosing \mathfrak{C} = union of components

$$\mathring{\mathcal{M}}(b,i;\alpha_{-},\alpha) \times_{R_{\alpha}} \mathring{\mathcal{N}}(bc,ii+1;\alpha,\alpha_{+};[1,2]) \times (E(\alpha_{-}),E(\alpha)).$$

Then when we consider the corner

$$\overset{\circ}{\mathcal{M}}(b,i;\alpha_{-},\alpha_{1})\times_{R_{\alpha_{1}}}\mathcal{M}(b,i;\alpha_{1},\alpha_{2})\times_{R_{\alpha_{2}}}\overset{\circ}{\mathcal{N}}(bc,ii+1;\alpha_{2},\alpha_{+};[1,2])\times(E(\alpha),E(\alpha'_{+})),$$

the part where the \mathfrak{C} -trivialization is performed is nothing but

$$\mathring{\mathcal{M}}(b,i;\alpha_{-},\alpha_{1})\times_{R_{\alpha_{1}}}\mathcal{M}(b,i;\alpha_{1},\alpha_{2})\times_{R_{\alpha_{2}}}\mathring{\mathcal{N}}(bc,ii+1;\alpha_{2},\alpha_{+};[1,2])\times(E(\alpha_{2}),E(\alpha'_{+})).$$

This is because the \mathfrak{C} -trivialization is performed at the intersection of the two boundary components which belongs to \mathfrak{C} . We smooth those corners to obtain the union of (3) and (4).

In the same way as in the proof of Proposition 18.63, we can describe the higher codimension boundary of our K-space $\mathcal{N}(ac, ii+1, \alpha_-, \alpha_+; [1, 2]^2)$, and can check the consistency in a straight forward way. This finishes the proof of Lemma 19.47. \square

The proof of Theorem 16.39 (6) is now complete.

Thus we have completed the proof of all the results claimed in Section 16.

Remark 19.48. In this section we use the homotopy of homotopies version of algebraic lemmas (Proposition 19.39) for the promotion of homotopies. We can avoid it if we use the next lemma instead.

Lemma 19.49. Let $\psi_1, \psi_2 : CF(\mathcal{F}) \to CF(\mathcal{F}')$ be two gapped cochain maps. (See Definition Definition 16.11 for gapped-ness.) Suppose $\psi_1|_E$ is partially cochain homotopic to $\psi_2|_E$ for any E, up to energy cut level E. Then ψ_1 is cochain homotopic to ψ_2 .

We can prove Lemma 19.49 in the same way as in the proof of [FOOO4, Lemma 7.2.177]. Note that Lemma 19.49 itself is valid for any ground ring R, while [FOOO4, Lemma 7.2.177] is proved only for the case when the ground ring is $\mathbb C$ or a finite field. (In fact, [FOOO4, Remark 7.2.181] gives a counter example to [FOOO4, Lemma 7.2.177] for the case when the ground ring is $\mathbb Q$.) The reason why Lemma 19.49 is valid for any ground ring R is that the equation for a map to be a cochain map or a cochain homotopy is linear. On the other hand, the equation for a map to be an A_{∞} map is nonlinear.

We choose to prove Proposition 19.39 rather than using Lemma 19.49, it is more direct to apply Proposition 19.39 to other situations.

20. Linear K-system: Floer cohomology III: Morse case by Multisection

In this section we provide the technical detail of the way how we associate Floer cohomology with $ground\ ring\ \mathbb{Q}$ to a linear K-system when the critical submanifolds are 0 dimensional.

Definition 20.1. We say a (partial) linear K-system as in Condition 16.1 and Definition 16.6 is of Morse type if all the critical submanifolds R_{α} consist of finite set.

Theorem 20.2. For (partial) linear K-system of Morse type, Theorems 16.9, 16.31, 16.39 hold with \mathbb{R} replaced by \mathbb{Q} .

The proof is done by replacing *CF-perturbations* used in Section 19 by *multivalued* perturbations (or multisections) in this section. We recall that we defined the notion of fiber product of CF-perturbations in [Part I, Section 10]. There is certain trouble to define the notion of fiber product of multivalued perturbations. (In fact, it is not a correct way to assume that the evaluation map restricted to the zero set of multisection is submersive, because this assumption is not satisfied even in the case of generic perturbations by an obvious dimensional reason.) On the other hand, in the case of direct product or fiber product over 0 dimensional spaces, we can define the notion of (fiber) product of multivalued perturbations in an obvious way. This is the main idea of the proof. To work it out in detail, we need to check carefully that the whole proof in Section 19 (and ones in Section 17 which are used in Section 19) can be carried out using multivalued perturbation in place of CF-perturbations. Indeed, once we correctly state a series of lemmas, their proofs are either automatic or straightforward analogue of the corresponding results in Sections 17 or 19.

The contents of this section are not used in the other part of this article. So many of the readers may skip this section and directly go to Section 21.

20.1. Bundle extension data revisited. In [Part I, Subsections 13.1, 13.2, 13.4] we proved existence of multisection or multivalued perturbation on good coordinate system. To resolve a technical problem mentioned in [Part I, Subsection 13.5], we used the notion of bundle extension data ([Part I, Definition 12.24]). In our situation where we have a system of Kuranishi structures (K-system), we first define a similar notion for a Kuranishi structure and then discuss the compatibility of them with the fiber product description of boundaries. We will study this point in detail in this subsection.

Definition 20.3. Let $\mathcal{U}_i = (U_i, \mathcal{E}_i, \psi_i, s_i)$ (i = 1, 2) be Kuranishi charts of X and $\Phi_{21} = (U_{21}, \varphi_{21}, \widehat{\varphi}_{21}) : \mathcal{U}_1 \to \mathcal{U}_2$ a coordinate change. A bundle extension data associated with Φ_{21} is $\mathfrak{D}_{12} = (\pi_{12}, \widetilde{\varphi}_{21}, \Omega_{12})$ satisfying the following:

- (1) $\pi_{12}: \Omega_{12} \to U_1$ is a continuous map, where Ω_{12} is a neighborhood of $\varphi_{21}(U_{21})$ in U_2 .
- (2) π_{12} is diffeomorphic to the projection of the normal bundle. (See [Part I, Definition 12.23].)
- (3) $\tilde{\varphi}_{21}: \pi_{12}^* \mathcal{E}_1 \to \mathcal{E}_2$ is an embedding of vector bundle. (See Definition 23.18.)
- (4) The map $\varphi_{21}^*\pi_{12}^*\mathcal{E}_1 \to \varphi_{21}^*\mathcal{E}_2$ that is induced from $\tilde{\varphi}_{21}$ and φ_{21} coincides with the bundle map $\hat{\varphi}_{21}: \mathcal{E}_1 \to \mathcal{E}_2$ which covers φ_{21} .

Definition 20.3 is mostly the same as [Part I, Definition 12.24]. In [Part I, Definition 12.24] we take a compact subset Z in U_1 and Ω_{12} as a neighborhood of $\varphi(Z)$, while in Definition 20.3 we do not take a compact subset $Z \subset U_1$ but take Ω_{12} as a neighborhood of $\varphi(U_1)$ itself. This difference is not essential since when we use such structures we may restrict them to compact sets.

Definition 20.4. Let $U_i = (U_i, \mathcal{E}_i, \psi_i, s_i)$ (i = 1, 2, 3) be Kuranishi charts of X, $\Phi_{ji} = (U_{ji}, \varphi_{ji}, \widehat{\varphi}_{ji}) : \mathcal{U}_i \to \mathcal{U}_j \ (i, j \in \{1, 2, 3\}, i < j)$ coordinate changes and let $\mathfrak{O}_{ji} = (\pi_{ij}, \tilde{\varphi}_{ji}, \Omega_{ij})$ be bundle extension data associated to Φ_{ji} .

(1) We say Φ_{21} , Φ_{32} are compatible with Φ_{31} if

$$\Phi_{31}|_{U_{321}} = \Phi_{32} \circ \Phi_{21}|_{U_{321}}. \tag{20.1}$$

holds on $U_{321} = \varphi_{21}^{-1}(U_{32}) \cap U_{31}$. (Note this is the same as [Part I, Definition

- (2) In the situation of (1), we say \mathfrak{O}_{21} , \mathfrak{O}_{32} are compatible with \mathfrak{O}_{31} if the following holds.

(a) $\pi_{23} \circ \pi_{12} = \pi_{13}$ on $\pi_{12}^{-1}(\Omega_{23}) \cap \Omega_{12} \cap \Omega_{13}$. (b) $\tilde{\varphi}_{32} \circ \tilde{\varphi}_{21} = \tilde{\varphi}_{31}$ on $\pi_{12}^{-1}(\Omega_{23}) \cap \Omega_{12} \cap \Omega_{13}$. This is mostly the same as the compatibility in [Part I, Definition 13.7].

- (1) Let $\widehat{\mathcal{U}}$ be a Kuranishi structure of X. A bundle extension Definition 20.5. data of $\widehat{\mathcal{U}}$ associates \mathfrak{O}_{pq} to each coordinate change Φ_{pq} of $\widehat{\mathcal{U}}$ so that they are compatible in the sense of Definition 20.4 (2).
 - (2) Let $\widehat{\mathcal{U}}$ be a good coordinate system of X. A bundle extension data of $\overline{\mathcal{U}}$ associates $\mathfrak{O}_{\mathfrak{pq}}$ to each coordinate change $\Phi_{\mathfrak{pq}}$ of $\overline{\mathcal{U}}$ so that they are compatible in the sense of Definition 20.4 (2).
 - (3) We can define a bundle extension data of various kinds of embeddings between Kuranishi structures and/or good coordinate systems, in the same

Lemma 20.6. Let $\widehat{\mathcal{U}}$ be a good coordinate system of X. Then there exists a bundle extension data for any proper open substructure of $\widehat{\mathcal{U}}$.

Proof. This is an immediate consequence of [Part I, Proposition 13.9].

In general, for a given Kuranishi structure, it seems hard to prove existence of bundle extension data. In fact, the proof of [Part I, Proposition 13.9] is done by induction on the charts. The definition of good coordinate system is designed so that such an induction does work. On the other hand, since there may be infinitely many charts in Kuranishi structure, the definition of Kuranishi structure is not suitable to work out such an induction. This point is the same as for the construction of multisection. The way to resolve this trouble is also the same. Namely we first take a good coordinate system and construct a bundle extension data associated to the good coordinate system. Then we restrict it to obtain one on (slightly different) Kuranishi structure. Namely we have the next lemma.

Lemma 20.7. Suppose we are in the situation of [Part I, Proposition 6.44]. We assume that there exists a bundle extension data $\overline{\mathfrak{D}_0}$ on good coordinate system $\overline{\mathcal{U}_0}$. Then we can take a Kuranishi structure $\widehat{\mathcal{U}}$ so that there exists a bundle extension data $\widehat{\mathfrak{O}}$ on it such that $\widehat{\mathfrak{O}}_0$ and $\widehat{\mathfrak{O}}$ are compatible with the embedding $\widehat{\mathcal{U}}_0 \to \widehat{\mathcal{U}}$.

Proof. By the construction of $\widehat{\mathcal{U}}$ given in the proof of [Part I, Proposition 6.44], all the coordinate changes appearing in $\widehat{\mathcal{U}}$ are restrictions of coordinate changes of $\widehat{\mathcal{U}}_0$. Therefore we can obtain the bundle extension data $\widehat{\mathfrak{D}}$ by restricting $\widehat{\mathfrak{D}}_0$. The compatibility among members of $\widehat{\mathfrak{D}}_0$ follows from the compatibility among members of $\widehat{\mathfrak{D}}_0$. The embedding $\widehat{\mathcal{U}}_0 \to \widehat{\mathcal{U}}$ also consists of restriction of coordinate changes of $\widehat{\mathfrak{D}}_0$. Therefore the compatibility of $\widehat{\mathfrak{D}}_0$ and $\widehat{\mathfrak{D}}$ with the embedding also follows from the compatibility among members of $\widehat{\mathfrak{D}}_0$.

We can prove a similar result in the situation of [Part I, Theorem 3.30].

Lemma 20.8. Suppose we are in the situation of [Part I, Theorem 3.30]. We assume that we have a bundle extension data $\widehat{\mathfrak{D}}$ of $\widehat{\mathcal{U}}$. Then there exists a bundle extension data $\widehat{\mathfrak{D}}$ of the good coordinate system $\widehat{\mathcal{U}}$ such that $\widehat{\mathfrak{D}}$ and $\widehat{\mathfrak{D}}$ are compatible with respect to the embedding $\widehat{\mathcal{U}} \to \widehat{\mathcal{U}}$.

Proof. The proof of [Part I, Theorem 3.30] is done by induction via the statement formulated as [Part I, Proposition 11.3]. We can amplify [Part I, Proposition 11.3] by adding the statement that all the embeddings and coordinate changes are associated with bundle extension data and they are all compatible with respect to compositions appearing there.

To prove this amplified statement, we proceed in the same way as in the proof of [Part I, Proposition 11.3]. All we need to note are the following two points:

- (1) We glued two coordinate changes sometimes on open sets where they coincide each other.
- (2) We invert coordinate changes which are isomorphisms.

As for point (1) we note that whenever two coordinate changes coincide during our construction, the bundle extension data coincide as well. Therefore we can also glue the bundle extension data in an obvious way.

As for point (2), we note that in the case when coordinate changes or embeddings of Kuranishi charts are local isomorphisms, bundle extension data exists and is unique up to the process of restricting Ω_i etc. to its open subsets. So we can obviously invert them. Thus we obtain the required bundle obstruction data $\widehat{\mathfrak{D}}$ of $\widehat{\mathcal{U}}$.

There are various relative versions etc. in Part I which we can prove together with bundle extension data. The way to generalize them is straightforward so we omit them and mention only when we use them.

Next we discuss compatibility of multisection and multivalued perturbation with bundle extension data.

Definition 20.9. For each i = 1, 2 let \mathcal{U}_i , Φ_{21} , \mathfrak{D}_{21} be as in Definition 20.3 and \mathfrak{s}_i (resp. $\widehat{\mathfrak{s}}_i = \{\mathfrak{s}_i^{\epsilon}\}$) a multisection (resp. multivalued perturbation) of \mathcal{U}_i . We say \mathfrak{s}_1 is *compatible* with \mathfrak{s}_2 with respect to Φ_{21} , \mathfrak{D}_{21} if they satisfy

$$(y, \mathfrak{s}_2(y)) = \tilde{\varphi}_{21}(y, \mathfrak{s}_1(\pi_{12}(y))),$$
 (20.2)

(resp.

$$(y, \mathfrak{s}_2^{\epsilon}(y)) = \tilde{\varphi}_{21}(y, \mathfrak{s}_1^{\epsilon}(\pi_{12}(y)))), \tag{20.3}$$

holds for all $y \in \Omega_{21}$.

This definition is mostly the same as [Part I, Definition 13.15].

Remark 20.10. The equalities (20.2), (20.3) are point-wise equalities which are supposed to hold branch-wise. So the ambiguity of the notion of branches explained in [Part I, Subsection 13.5] does not cause any trouble here.

Definition 20.11. Let $\widehat{\mathcal{U}}$ be a Kuranishi structure and $\widehat{\mathfrak{D}}$ its bundle extension data. A multivalued perturbation $\widehat{\mathfrak{s}} = \{\widehat{\mathfrak{s}_p^{\epsilon}}\}\$ of $\widehat{\mathcal{U}}$ is said to be *compatible* with $\widehat{\mathfrak{D}}$ if it is compatible with the pair $(\Phi_{pq}, \mathfrak{D}_{pq})$ of the coordinate change and a member of $\widehat{\mathfrak{D}}$ in the sense of Definition 20.9.

For the case of good coordinate system with bundle extension data, the compatibility of multivalued perturbation with extension data is defined in the same way.

Lemma 20.12. Let $\widehat{\mathcal{U}}$ be a good coordinate system of X and $\widehat{\mathfrak{D}}$ its bundle extension data. Let $\widehat{\mathcal{U}}_0$ be a proper open substructure of $\widehat{\mathcal{U}}$. We restrict $\widehat{\mathfrak{D}}$ to it. Then there exists a multivalued perturbation $\widehat{\mathfrak{s}} = \{\widehat{\mathfrak{s}}_{\widehat{\mathfrak{p}}}^{\mathsf{n}}\}$ thereof with the following properties:

- (1) $\hat{\mathfrak{s}}$ is transversal to 0.
- (2) It is compatible with $\widehat{\mathfrak{D}}|_{\widehat{\mathcal{U}}_0}$.

Proof. In the proof of the [Part I, Theorem 6.37] given in [Part I, Seciton 13] we used induction to prove [Part I, Proposition 13.23]. The support system \mathcal{K} which appears in the statement of [Part I, Proposition 13.23] can be chosen so that $\mathcal{U}_{0,\mathfrak{p}} \subset \mathcal{K}_{\mathfrak{p}}$. Then Lemma 20.12 is an immediate consequence of [Part I, Proposition 13.23].

We can also include multivalued perturbations in Lemmas 20.7 and 20.8.

Lemma 20.13. Suppose we are in the situation of Lemma 20.7. We assume in addition that there exists a multivalued perturbation $\widehat{\mathfrak{s}_0}$ of $\widehat{\mathcal{U}_0}$ compatible with the bundle extension data $\widehat{\mathfrak{D}_0}$. Then there exists a multivalued perturbation $\widehat{\mathfrak{s}}$ of $\widehat{\mathcal{U}}$ compatible with the bundle extension data $\widehat{\mathfrak{D}}$. Moreover, $\widehat{\mathfrak{s}_0}$ and $\widehat{\mathfrak{s}}$ are compatible with the embedding $\widehat{\mathcal{U}_0} \to \widehat{\mathcal{U}}$ with respect to $\widehat{\mathfrak{D}_0}$, $\widehat{\mathfrak{D}}$. If $\widehat{\mathfrak{s}_0}$ is transversal to 0, so is $\widehat{\mathfrak{s}}$.

Proof. The proof is the same as the proof of Lemma 20.7. \Box

Lemma 20.14. Suppose we are in the situation of Lemma 20.8. We assume in addition that we have a multivalued perturbation $\hat{\mathfrak{s}}$ of $\widehat{\mathcal{U}}$ compatible with the bundle extension data $\widehat{\mathfrak{D}}$. Then there exists a multivalued perturbation $\widehat{\mathfrak{s}_0}$ of $\widehat{\mathcal{U}}$ compatible with the bundle extension data $\widehat{\mathfrak{D}}$. Moreover, $\widehat{\mathfrak{s}}$ and $\widehat{\mathfrak{s}_0}$ are compatible with the embedding $\widehat{\mathcal{U}} \to \widehat{\mathcal{U}}$ with respect to $\widehat{\mathfrak{D}}$, $\widehat{\widehat{\mathfrak{D}}}$. If $\widehat{\mathfrak{s}}$ is transversal to 0, so is $\widehat{\mathfrak{s}_0}$.

Proof. The proof is the same as the proof of Lemma 20.8. \Box

20.2. Virtual fundamental chain of 0 dimensional K-space.

Definition 20.15. Suppose X_1 , X_2 have Kuranishi structures $\widehat{\mathcal{U}}_1$, $\widehat{\mathcal{U}}_2$ respectively, and $\widehat{f}_i:(X_i,\widehat{\mathcal{U}}_i)\to R$ are strongly smooth. We assume that R is a 0-dimensional compact manifold, that is nothing but a finite set. Then they are automatically transversal and we have a fiber product $(X_1\times_R X_2,\widehat{\mathcal{U}}_1\times_R \widehat{\mathcal{U}}_2)$. We call this fiber product the *direct-like product*.

The next lemma is trivial to prove.

Lemma-Definition 20.16. Suppose we are in the situation of Definition 20.15.

- (1) If $\widehat{\mathfrak{D}}_i$ are bundle extension data of $\widehat{\mathcal{U}}_i$ for i=1,2, then they induce bundle extension data of their direct-like product in a canonical way. We call it the direct-like product $\widehat{\mathfrak{D}}_1 \times_R \widehat{\mathfrak{D}}_2$.
- (2) If $\widehat{\mathfrak{s}}_i$ (resp. \mathfrak{s}_i) are multivalued perturbations (resp. multisections) of $\widehat{\mathcal{U}}_i$ for i=1,2, then they induce a multivalued perturbation (resp. multisection) of their fiber product in a canonical way. ⁴¹ We call it fiber product of the multivalued perturbation (resp. multisection) and write $\widehat{\mathfrak{s}}_1 \times_R \widehat{\mathfrak{s}}_2$ (resp. $\mathfrak{s}_1 \times_R \mathfrak{s}_2$).
- (3) If $\widehat{\mathfrak{s}}_i$ (resp. \mathfrak{s}_i) are transversal to 0, then its direct-like product is transversal to 0.
- (4) Let $\widehat{\mathfrak{O}}_i$ be bundle extension data of $\widehat{\mathcal{U}}_i$ and $\widehat{\mathfrak{s}}_i$ (resp. \mathfrak{s}_i) multivalued perturbations (resp. multisections) of $\widehat{\mathcal{U}}_i$ for i=1,2. We assume $\widehat{\mathfrak{s}}_i$ (resp. \mathfrak{s}_i) are compatible with $\widehat{\mathfrak{O}}_i$. Then the direct-like product $\widehat{\mathfrak{s}}_1 \times_R \widehat{\mathfrak{s}}_2$ (resp. $\mathfrak{s}_1 \times_R \mathfrak{s}_2$) is compatible with $\widehat{\mathfrak{O}}_1 \times_R \widehat{\mathfrak{O}}_2$.

Next we define the virtual fundamental chain, which is identified with a rational number, of a 0 dimensional K-space by using multivalued perturbation. We recall that in [Part I, Definition 14.6] we defined a virtual fundamental chain $(\in \mathbb{Q})$ of $(X, \widehat{\mathcal{U}}, \widehat{\mathfrak{s}})$ where $\widehat{\mathcal{U}}$ is a good coordinate system of X and $\widehat{\mathfrak{s}}$ is its multivalued perturbation transversal to 0. We adopt this story and proceed in the same way as in [Part I, Subsection 9.2] to define a virtual fundamental chain $(\in \mathbb{Q})$ for a 0 dimensional K-space.

Situation 20.17. (1) We consider a quadruple $(X, \widehat{\mathcal{U}}, \widehat{\mathfrak{D}}, \widehat{\mathfrak{s}})$ such that:

- (a) $(X, \widehat{\mathcal{U}})$ is a K-space.
- (b) $\widehat{\mathfrak{O}}$ is a bundle extension data of $\widehat{\mathcal{U}}$.
- (c) $\widehat{\mathfrak{s}}$ is a multivalued perturbation of $\widehat{\mathcal{U}}$ compatible with $\widehat{\mathfrak{O}}$.
- (2) We consider a quadruple $(X, \widehat{\mathcal{U}}, \widehat{\mathfrak{D}}, \widehat{\mathfrak{s}})$ such that:
 - (a) $(X, \widehat{\mathcal{U}})$ is a good coordinate system.
 - (b) $\widehat{\mathfrak{D}}$ is a bundle extension data of $\widehat{\mathcal{U}}$.
 - (c) $\widehat{\mathfrak{s}}$ is a multivalued perturbation of $\widehat{\mathcal{U}}$ compatible with $\widehat{\mathfrak{D}}$.
- **Definition 20.18.** (1) Let $(X, \mathfrak{X}_i) = (X, \widehat{\mathcal{U}}_i, \widehat{\mathfrak{D}}_i, \widehat{\mathfrak{s}}_i)$ (i = 1, 2) be as in Situation 20.17 (1). We say \mathfrak{X}_2 is a *thickening* of \mathfrak{X}_1 and write $(X, \widehat{\mathcal{U}}_1, \widehat{\mathfrak{D}}_1, \widehat{\mathfrak{s}}_1) < (X, \widehat{\mathcal{U}}_2, \widehat{\mathfrak{D}}_2, \widehat{\mathfrak{s}}_2)$ if the following holds.
 - (a) $(X, \widehat{\mathcal{U}}_1) \to (X, \widehat{\mathcal{U}}_2)$ by which $(X, \widehat{\mathcal{U}}_2)$ is a thickening of $(X, \widehat{\mathcal{U}}_1)$. (See [Part I, Definition 5.3].)
 - (b) $\widehat{\mathfrak{O}}_2$, $\widehat{\mathfrak{O}}_1$ are compatible with respect to this embedding.
 - (c) $\widehat{\mathfrak{s}_2}$, $\widehat{\mathfrak{s}_1}$ are compatible with this embedding with respect to $\widehat{\mathfrak{O}_2}$, $\widehat{\mathfrak{O}_1}$.
 - (2) In the case one or both of the objects are good coordinate systems, we can define the notion of thickening such as

$$(X,\widehat{\mathcal{U}}_{1},\widehat{\mathfrak{D}}_{1},\widehat{\mathfrak{s}}_{1}) < (X,\widehat{\mathcal{U}}_{2},\widehat{\mathfrak{D}}_{2},\widehat{\mathfrak{s}}_{2}),$$

$$(X,\widehat{\mathcal{U}}_{1},\widehat{\mathfrak{D}}_{1},\widehat{\mathfrak{s}}_{1}) < (X,\widehat{\mathcal{U}}_{2},\widehat{\mathfrak{D}}_{2},\widehat{\mathfrak{s}}_{2}),$$

$$(X,\widehat{\mathcal{U}}_{1},\widehat{\mathfrak{D}}_{1},\widehat{\mathfrak{s}}_{1}) < (X,\widehat{\mathcal{U}}_{2},\widehat{\mathfrak{D}}_{2},\widehat{\mathfrak{s}}_{2}),$$

⁴¹We do not need to assume that our fiber product is direct-like in Item (2).

in the same way.

Below we will define virtual fundamental chain of a quadruple $(X, \widehat{\mathcal{U}}, \widehat{\mathfrak{D}}, \widehat{\mathfrak{s}})$ as in Situation 20.17 (1) provided that $\hat{\mathfrak{s}}$ is transversal to 0. We begin with the following

Lemma 20.19. Let $(X,\widehat{\mathcal{Q}}_i,\widehat{\mathfrak{S}}_i)$ i=1,2 be as in Situation 20.17 (2) such that $\dim(X,\widehat{\mathcal{U}}_i) = 0$. We assume

$$(X,\widehat{\mathcal{U}_1},\widehat{\mathfrak{O}_1},\widehat{\mathfrak{s}_1}) < (X,\widehat{\mathcal{U}_2},\widehat{\mathfrak{O}_2},\widehat{\mathfrak{s}_2}).$$

Then there exists $\epsilon_0 > 0$ such that the following holds for $0 < \epsilon < \epsilon_0$: If $\widehat{\mathfrak{s}}_1$ and $\widehat{\mathfrak{s}}_2$ are transversal to 0, then

$$[(X,\widehat{\mathcal{U}}_1,\widehat{\mathfrak{s}}_1^{\widehat{\epsilon}})] = [(X,\widehat{\mathcal{U}}_2,\widehat{\mathfrak{s}}_2^{\widehat{\epsilon}})].$$

Proof. The proof is entirely the same as the proof of [Part I, Proposition 9.16]. \Box

Definition 20.20. Let $(X, \widehat{\mathcal{U}}, \widehat{\mathfrak{D}}, \widehat{\mathfrak{s}})$ be as in Situation 20.17 (1) and dim $(X, \widehat{\mathcal{U}}) = 0$. We assume that $\hat{\mathfrak{s}}$ is transversal to 0 as a family in the sense of [Part I, Remark 14.11 (2)]. Using Lemma 20.14, we take $(X, \widehat{\mathcal{U}}, \widehat{\mathcal{D}'}, \widehat{\mathfrak{s}'})$ such that

$$(X,\widehat{\mathcal{U}},\widehat{\mathfrak{O}},\widehat{\mathfrak{s}}) < (X,\widehat{\mathcal{U}},\widehat{\overline{\mathfrak{O}}'},\widehat{\mathfrak{s}'}).$$

Then we define the virtual fundamental chain of $(X, \widehat{\mathcal{U}}, \widehat{\mathfrak{D}}, \widehat{\mathfrak{s}})$ (at ϵ) by

$$[(X,\widehat{\mathcal{U}},\widehat{\mathcal{D}},\widehat{\mathfrak{s}^{\epsilon}})] = [(X,\widehat{\mathcal{U}},\widehat{\mathcal{D}'},\widehat{\mathfrak{s}'^{\epsilon}})] \in \mathbb{Q}, \tag{20.4}$$

for sufficiently small $\epsilon > 0$ such that $\widehat{\mathfrak{s}^{\epsilon}}$ is transversal to 0 at ϵ .

Lemma 20.21. The right hand side of (20.4) is independent of $(X, \widehat{\mathcal{U}}, \widehat{\mathcal{D}}', \widehat{\mathfrak{s}}')$ but depends only on $(X, \widehat{\mathcal{U}}, \widehat{\mathfrak{D}}, \widehat{\mathfrak{s}^{\epsilon}})$ and ϵ , if $\epsilon > 0$ is sufficiently small.

Proof. The proof is entirely the same as the proof of [Part I, Theorem 9.14].

Now we state the multivalued perturbation versions of Stokes' formula and composition formula.

Proposition 20.22. Let $(X, \widehat{\mathcal{U}}, \widehat{\mathfrak{D}}, \widehat{\mathfrak{s}^{\epsilon}})$ be as in Situation 20.17 (1) and $\dim(X, \widehat{\mathcal{U}}) =$ 1. We assume that $\widehat{\mathfrak{s}^{\epsilon}}$ is transversal to 0 as a family in the sense of [Part I, Remark [14.1 (2)]. Then we have

$$[\partial(X,\widehat{\mathcal{U}},\widehat{\mathfrak{D}},\widehat{\mathfrak{s}^{\epsilon}})] = 0.$$

Here $\partial(X,\widehat{\mathcal{U}},\widehat{\mathfrak{D}},\widehat{\mathfrak{s}^{\epsilon}})$ is the normalized boundary $\partial(X,\widehat{\mathcal{U}})$ together with the restrictions of $\widehat{\mathfrak{O}}$ and $\widehat{\mathfrak{s}^{\epsilon}}$ to the boundary.

Proof. This immediately follows from [Part I, Proposition 14.10] and the definition.

To state the analogue of the composition formula we begin with explaining the situation and introducing the notation.

Situation 20.23. Suppose we are in Situation 20.15. We assume that we are given the quadruples $(X_i, \widehat{\mathcal{U}}_i, \widehat{\mathfrak{I}}_i, \widehat{\mathfrak{s}}_i)$ for i = 1, 2 of bundle extension data $\widehat{\mathfrak{I}}_i$ and multivalued perturbations $\widehat{\mathfrak{s}_i}$.

For $r \in R$ we put $f_i^{-1}(r) = \{x \in X_i \mid f_i(x) = r\}$. They carry various structures such as Kuranishi structures, which are induced by $\widehat{\mathcal{U}}_i, \widehat{\mathfrak{I}}_i, \widehat{\mathfrak{s}}_i$. Thus we obtain quadruples $f_i^{-1}(r) \cap (X_i, \widehat{\mathcal{U}}_i, \widehat{\mathfrak{D}}_i, \widehat{\mathfrak{s}}_i)$ as in Situation 20.17 (1).

Proposition 20.24. In Situation 20.23 we assume in addition that for i = 1, 2

- (1) $\dim(X_i, \widehat{\mathcal{U}}_i) = 0$,
- (2) $\widehat{\mathfrak{s}}_i$ are transversal to 0 as a family in the sense of [Part I, Remark 14.11 (2)].

Then the direct-like product

$$(X_1,\widehat{\mathcal{U}_1},\widehat{\mathfrak{O}_1},\widehat{\mathfrak{s}_1}) \times_R (X_2,\widehat{\mathcal{U}_2},\widehat{\mathfrak{O}_2},\widehat{\mathfrak{s}_2})$$

is 0 dimensional and transversal to 0 as a family. Moreover we have

$$[(X_{1},\widehat{\mathcal{U}_{1}},\widehat{\mathfrak{I}_{1}},\widehat{\mathfrak{s}_{1}^{\epsilon}}) \times_{R} (X_{2},\widehat{\mathcal{U}_{2}},\widehat{\mathfrak{I}_{2}},\widehat{\mathfrak{s}_{2}^{\epsilon}})]$$

$$= \sum_{r \in R} [f_{1}^{-1}(r) \cap (X_{1},\widehat{\mathcal{U}_{1}},\widehat{\mathfrak{I}_{1}},\widehat{\mathfrak{s}_{1}^{\epsilon}})][f_{2}^{-1}(r) \cap (X_{2},\widehat{\mathcal{U}_{2}},\widehat{\mathfrak{I}_{2}},\widehat{\mathfrak{s}_{2}^{\epsilon}})].$$
(20.5)

Proof. The proof is the same as the proof of [Part I, Proposition 10.23]. 42

We also note the following.

Lemma 20.25. In the situation of Proposition 20.24 we replace (1) by the following assumption.

(1),
$$\dim(X_1,\widehat{\mathcal{U}_1}) = -\dim(X_2,\widehat{\mathcal{U}_2}) \neq 0.$$

Except this point, we assume that the same condition as in Proposition 20.24. Then we have

$$[(X_1,\widehat{\mathcal{U}_1},\widehat{\mathfrak{O}_1},\widehat{\mathfrak{s}_1^{\epsilon}})\times_R (X_2,\widehat{\mathcal{U}_2},\widehat{\mathfrak{O}_2},\widehat{\mathfrak{s}_2^{\epsilon}})]=0.$$

Proof. We may assume $\dim(X_1,\widehat{\mathcal{U}}_1) < 0$. Then by [Part I, Lemma 14.1] $(\mathfrak{s}_1^{\epsilon})^{-1}(0)$ is an empty set. Therefore $(\mathfrak{s}_1^{\epsilon} \times_R \mathfrak{s}_2^{\epsilon})^{-1}(0)$ is an empty set. The lemma follows. \square

20.3. Extension of multisection from boundary to its neighborhood. In this subsection we discuss an analogue of the story in Section 17 for multivalued perturbations.

Definition 20.26. Let $(X', \widehat{\mathcal{U}'})$ be a τ -collared Kuranishi structure as in Definition 17.34 (1). A τ -collared bundle extension data of this τ -collared Kuranishi structure assigns a bundle extension data $\mathfrak{O}_{p'q'}$ to each embedding $\Phi_{p'q'}$ given in Definition 17.34 (1) (b) such that in the situation of Definition 17.34 (1) (c), the bundle extension data $\mathfrak{O}_{p'q'}$, $\mathfrak{O}_{q'r'}$ are compatible with $\mathfrak{O}_{p'r'}$ in the sense of Definition 20.4.

The compatibility of τ -collared multivalued perturbations with τ -collared bundle extension data is defined in the same way as in Definition 20.9.

The next lemma is obvious from the definition.

Lemma 20.27. We consider the situation of Lemma-Definition 17.35.

- (1) Let $\widehat{\mathfrak{D}}$ be a bundle extension data of $\widehat{\mathcal{U}}$. Then it induces a τ -collared bundle extension data of $\widehat{\mathcal{U}}^{\boxplus \tau}$. We denote it by $\widehat{\mathfrak{D}}^{\boxplus \tau}$.
- (2) Suppose we are in the situation of (1). Let $\widehat{\mathfrak{s}} = \{\mathfrak{s}^{\epsilon}\}$ be a multivalued perturbation of $(X,\widehat{\mathcal{U}})$ compatible with $\widehat{\mathfrak{D}}$. Then $\widehat{\mathfrak{s}^{\boxplus \tau}}$ obtained in Lemma 17.37 (6) is compatible with $\widehat{\mathfrak{D}}^{\boxplus \tau}$.

⁴²It is actually easier than that.

Next we study the situation in Subsection 17.6. We note that bundle extension data can be restricted to normalized corner in an obvious way. We can also pull it back by a covering map. Moreover, the compatibility of multivalued perturbation with bundle extension data is preserved by restriction to normalized corner and pull-back by a covering map.

Situation 20.28. Suppose we are in Situation 17.43. We assume the following in addition.

- (1) We are given a τ -collared bundle extension data $\widehat{\mathfrak{O}_{S_k}^+}$ of a τ -collared Ku-
- ranishi structure $\widehat{\mathcal{U}_{S_k}^+}$.

 (2) The restriction of $\widehat{\mathfrak{O}_{S_\ell}^+}$ to $\widehat{S}_k(\widehat{S}_\ell(X), \widehat{\mathcal{U}_{S_\ell}^+})$ coincides with the pull-back of $\mathfrak{O}_{S_{\ell+k}}^+$ by the isomorphism in Situation 17.43 (2).

Remark 20.29. Here and in several similar places, we can say two bundle extension data or multisections are 'being the same' or 'coincide'. In the case of CF-perturbations, for example, we say two such objects 'being equivalent' or 'isomorphic' etc., instead. Recall that bundle extension data consists of maps and subsets. The maps are ones between the spaces common in the two bundle extension data in question and the subsets are ones of the sets which are common in the two bundle extension data in question. On the other hand in the case of CF-perturbations, for example, we also have a parameter space W of perturbations as a part of the object. So we can say two of them are equivalent but it does not make so much sense to say that they are the same. The fact that we can say two bundle extension data are the same slightly simplifies discussion here.

Lemma 20.30. Suppose we are in Situation 20.28. Then the τ' -collared Kuranishi structure $\widehat{\mathcal{U}^+}$ obtained in Proposition 17.46 carries a τ' -collared bundle extension data $\widehat{\mathfrak{O}^+}$ such that the restriction of $\widehat{\mathfrak{O}^+}$ to $\widehat{S}_k(X)$ coincides with $\widehat{\mathfrak{O}^+_{S_k}}$ under the isomorphism of Proposition 17.46 (1).

Proof. The proof is immediate from the proof of Proposition 17.46.

Remark 20.31. We do not assume that we have bundle extension data of $(X, \widehat{\mathcal{U}})$. In our application, the Kuranishi structure $(X, \widehat{\mathcal{U}})$ is one we obtain from *geometry*. As we explained in Subsection 20.1, it seems hard to find bundle extension data, in general. We take a good coordinate system compatible with the Kuranishi structure $\widehat{\mathcal{U}}$ and use it to find a multivalued perturbation and bundle extension data. Then $\widehat{\mathcal{U}^+}$ is a Kuranishi structure obtained from this good coordinate system. We will construct bundle extension data and multisections compatible with the direct-like product description of the corners inductively. The results we are explaining here will be used for this purpose.

Nevertheless, we can also prove the following type of statements. If we are given bundle extension data of $\widehat{\mathcal{U}}$ such that this bundle extension data and $\widehat{\mathfrak{O}}_{S_k}^+$ etc. are compatible with the embeddings, covering maps etc. appearing in Situation 17.43 (3)(4)(5), then the τ' -collared bundle extension data $\widehat{\mathfrak{D}^+}$ in Lemma 20.30 can be taken so that it is compatible with the embeddings and covering maps appearing in Proposition 17.46. We do not state or prove it here since we do not use it.

Lemma 20.32. Suppose we are in the situation of Situation 20.28. We assume in addition that we are given a τ -collared multivalued perturbation $\widehat{\mathfrak{s}_{S_k}^+}$ of $\widehat{\mathcal{U}_{S_k}^+}$ such that

- (i) $\widehat{\mathfrak{s}_{S_k}^+}$ is compatible with $\widehat{\mathfrak{O}_{S_k}^+}$.
- (ii) The pull-back of $\widehat{\mathfrak{s}_{S_{k+\ell}}^+}$ by the map $\pi_{k,\ell}:\widehat{S}_k(\widehat{S}_\ell(X),\widehat{\mathcal{U}_{S_\ell}^+})\to (\widehat{S}_{k+\ell}(X),\widehat{\mathcal{U}_{S_{k+\ell}}^+})$ coincides with the restriction of $\widehat{\mathfrak{s}_{S_\ell}^+}$.

Then for any $0 < \tau' < \tau$ there exists a τ' -collared multivalued perturbation $\widehat{\mathfrak{s}^+}$ on the Kuranishi structure $\widehat{\mathcal{U}^+}$ obtained in Proposition 17.46 with the following properties.

(1) $\widehat{\mathfrak{s}^+}$ is compatible with the τ' -collared bundle extension data $\widehat{\mathfrak{O}^+}$ obtained in Lemma 20.30.

(2) Its restriction to $(\widehat{S}_k(X), \widehat{\mathcal{U}_{S_k}^+})$ coincides with $\widehat{\mathfrak{s}_{S_k}^+}$.

Proof. The proof is the same as the proof of Proposition 17.58.

Now we discuss results corresponding to those in Subsection 17.9.

Lemma 20.33. Suppose we are in the situation of Lemma 20.32. We assume that $\widehat{\mathfrak{s}_k^+}$ are transversal to 0 as a family. Then there exists a τ' -collared multivalued perturbations $\widehat{\mathfrak{s}^{++}}$ on the Kuranishi structure $\widehat{\mathcal{U}^{++}}$ obtained in Proposition 17.62 such that

- (1) Its restriction to $(\widehat{S}_k(X), \widehat{\mathcal{U}_{S_k}^+})$ coincides with $\widehat{\mathfrak{s}_k^+}$.
- (2) $\widehat{\mathfrak{s}^{++}}$ is transversal to 0 as a family.
- (3) The bundle extension data $\widehat{\mathfrak{O}^+}$ obtained in Lemma 20.30 can be extended to a collared bundle extension data $\widehat{\mathfrak{O}^{++}}$ of $\widehat{\mathcal{U}^{++}}$ which coincides with $\widehat{\mathfrak{O}^+}$ in a neighborhood of the boundary.
- (4) $\widehat{\mathfrak{s}^{++}}$ is compatible with $\widehat{\mathfrak{O}^{++}}$.

Proof. The proof is the same as the proof of Proposition 17.65.

20.4. Completion of the proof of Theorem 20.2.

Proof of Theorem 20.2. We first show a version of Proposition 19.1.

Proposition 20.34. Suppose we are in the situation of Proposition 19.1. Moreover we assume that the linear K-system is of Morse type.

Then for any $0 < \tau < 1$ there exists a τ -collared Kuranishi structure $\widehat{\mathcal{U}^+}(\alpha_-, \alpha_+)$ of $\mathcal{M}(\alpha_-, \alpha_+)^{\boxplus \tau_0}$, its bundle extension data $\widehat{\mathfrak{D}}(\alpha_-, \alpha_+)$, multivalued perturbations $\widehat{\mathfrak{s}^+}(\alpha_-, \alpha_+)$ of $\widehat{\mathcal{U}^+}(\alpha_-, \alpha_+)$ for every α_- , α_+ with $E(\alpha_+) - E(\alpha_-) \leq E_{\mathfrak{E}}^k$, such that $\widehat{\mathfrak{s}^+}(\alpha_-, \alpha_+)$ is compatible with $\widehat{\mathfrak{D}}(\alpha_-, \alpha_+)$ and enjoy the following properties.

- (1) The same as Proposition 19.1 (1).
- (2) $\widehat{\mathfrak{s}^+}(\alpha_-, \alpha_+)$ is transversal to 0.
- (3) The pull-back of $\widehat{\mathfrak{s}^+}(\beta\alpha_-,\beta\alpha_+)$ by the periodicity isomorphism coincides with $\widehat{\mathfrak{s}^+}(\alpha_-,\alpha_+)$.
- (4) Proposition 19.1 (4) holds. Moreover $\widehat{\mathfrak{D}}(\alpha_-, \alpha_+)$ is preserved by the periodicity isomorphism.

(5) The pull-back of $\widehat{\mathfrak{s}^+}(\alpha_-, \alpha_+)$ by the isomorphism (19.2) coincides with the fiber product

$$\widehat{\mathfrak{s}^+}(\alpha_-,\alpha)_{\mathrm{ev}_+} \times_{\mathrm{ev}_-} \widehat{\mathfrak{s}^+}(\alpha,\alpha_+).$$

The pull-back of $\widehat{\mathfrak{D}}(\alpha_-, \alpha_+)$ by the isomorphism (19.2) coincides with the fiber product

$$\widehat{\mathfrak{D}}(\alpha_{-}, \alpha)_{\mathrm{ev}_{+}} \times_{\mathrm{ev}_{-}} \widehat{\mathfrak{D}}(\alpha, \alpha_{+}).$$

These fiber products are well-defined since they are direct-like products.

- (6) Proposition 19.1 (6) holds. Moreover (19.3) preserves bundle extension data.
- (7) Proposition 19.1 (7) holds. Moreover $\pi_{\ell,k}$ preserves bundle extension data.
- (8) The pull-back of $\widehat{\mathfrak{s}^+}(\alpha_-, \alpha_+)$ by the isomorphism (19.3) is the fiber product

$$\widehat{\mathfrak{s}^+}(\alpha_-, \alpha_1)_{\text{ev}_+} \times_{R_{\alpha_1}} \dots_{R_{\alpha_k}} \times_{\text{ev}_-} \widehat{\mathfrak{s}^+}(\alpha_k, \alpha_+).$$
 (20.6)

This fiber product is well-defined because it is direct-like.

Proof. Using the results of previous subsections, the proof is the same as the proof of Proposition 19.1. \Box

We next rewrite Proposition 20.34 in the algebraic language. In our situation (linear K-system of Morse type) we define

$$\Omega(R_{\alpha}) = \bigoplus_{r \in \Omega(R_{\alpha})} \mathbb{Q}[r]. \tag{20.7}$$

We next define

$$\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{\epsilon}:\Omega(R_{\alpha_{-}})\to\Omega(R_{\alpha_{+}})$$
 (20.8)

in the case when dim $\mathcal{M}(\alpha_-, \alpha_+) = 0$ by the formula

$$\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{\epsilon}([r_{-}])$$

$$= \sum_{r_{+} \in R_{\alpha_{+}}} [(ev_{-}, ev_{+})^{-1}((r_{-}, r_{+}))$$
(20.9)

$$\cap \left(\mathcal{M}(\alpha_{-}, \alpha_{+})^{\boxplus \tau_{0}}, \widehat{\mathcal{U}^{+}}(\alpha_{-}, \alpha_{+}), \widehat{\mathfrak{D}}(\alpha_{-}, \alpha_{+}), \widehat{\mathfrak{s}^{+\epsilon}}(\alpha_{-}, \alpha_{+}) \right)] [r_{+}].$$

Here the coefficient of $[r_+]$ in the right hand side is the virtual fundamental chain as in Definition 20.20, which is a rational number, and $r_- \in R_{\alpha_-}$.

We modify (b) appearing in Remark 19.4 as follows.

(b') For any energy cut level E_0 there exists $\epsilon_0(E_0) > 0$ such that the operator $\mathfrak{m}_{1;\alpha_+,\alpha_-}^{\epsilon}$ is defined when $0 < E(\alpha_+) - E(\alpha_-) \le E_0$ and ϵ is in a dense open subset of $\{\epsilon \mid 0 < \epsilon < \epsilon_0(E_0)\}$.

Lemma 20.35. The operators $\mathfrak{m}_{1;\alpha_{+},\alpha_{-}}^{\epsilon}$ in (20.9) satisfy the following equality in the sense of (b'):

$$\sum_{\alpha; E(\alpha_{-}) < E(\alpha) < E(\alpha_{+})} \mathfrak{m}_{1;\alpha_{+},\alpha}^{\epsilon} \circ \mathfrak{m}_{1;\alpha,\alpha_{-}}^{\epsilon} = 0.$$
 (20.10)

Proof. Using Propositions 20.22, 20.24 and 20.34, the proof is the same as the proof of Lemma 19.5. \Box

We have thus rewritten Subsection 19.1 by using multivalued perturbations in the case of linear K-system of Morse type. It is now obvious that we can rewrite Subsections 19.2-19.8 in the same way and complete the proof of Theorem 20.2. \Box

21. Tree-like K-system: A_{∞} structure I: statement

In Sections 21-22 we discuss construction of a filtered A_{∞} structure associated to a relatively spin Lagrangian submanifold L of a symplectic manifold. This construction had been written in great detail in the article [FOOO6], [FOOO7] based on singular homology. Furthermore its de Rham version was given in [FOOO7, Section 12], [FOOO9], [Fu2].

In this article, based on de Rham cohomology, we give its detail again. We also provide a package so that the construction part of various Kuranishi structures and their usage part to obtain a filtered A_{∞} structure are clearly separated from each other. For this purpose, we take an axiomatic approach as in the case of linear Ksystem developed up to the previous sections. A similar axiomatic treatment was written in [Fu3]. The axiom we give here is slightly different from that of [Fu3]. In [Fu3] we used a geometric operad (the Stasheff operad) to formulate Kuranishi A_{∞} correspondence. In this article we use the K-system organized by a tree, sometimes called tree-like K-system in short, (plus certain additional data) rather than the Stasheff operad. In fact, in [Fu3] the Stasheff operad was used only to describe the combinatorial data on the way how various strata (which are fiber products of moduli spaces of pseudo-holomorphic disks) are glued. Since the members of the Stasheff operad are cells, they do not carry a nontrivial homology class. So in [Fu3] we actually did not use the geometric data of the Stasheff operad but used only its combinatorial structure, (that is, the way how various strata intersect).

21.1. Axiom of tree-like K-system: A_{∞} correspondence. In the rest of Part 2, we assume that L is a smooth oriented closed manifold.

Situation 21.1. Let \mathfrak{G} be an additive group and $\mu:\mathfrak{G}\to\mathbb{Z}$, $E:\mathfrak{G}\to\mathbb{R}$ group homomorphisms. We call $\mu(\beta)$ the Maslov index of β and $E(\beta)$ the energy of β .

Definition 21.2. A decorated rooted metric ribbon tree is $(\mathcal{T}, \beta(\cdot))$ such that:

- (1) \mathcal{T} is a connected tree. Let $C_0(\mathcal{T})$, $C_1(\mathcal{T})$ be the set of all vertices and edges of \mathcal{T} , respectively.
- (2) For each $v \in C_0(\mathcal{T})$ we fix a cyclic order of the set of edges containing v. This is equivalent to fixing an isotopy type of an embedding of \mathcal{T} to the plane \mathbb{R}^2 . (Namely, the cyclic order of the edges is given by the orientation of the plane so that the edges are enumerated according to the counter clockwise orientation. We call it a ribbon structure at the vertex v.)
- (3) $C_0(\mathcal{T})$ is divided into the set of exterior vertices $C_{0,\text{ext}}(\mathcal{T})$ and the set of interior vertices $C_{0,int}(\mathcal{T})$.
- (4) We fix one element of $C_{0,\text{ext}}(\mathcal{T})$, which we call the *root*.
- (5) The valency of all the exterior vertices are 1.
- (6) $\beta(): C_{0,int}(\mathcal{T}) \to \mathfrak{G}$ is a map. We require $E(\beta(\mathbf{v})) \geq 0$. Moreover if $E(\beta(\mathbf{v})) = 0$ then $\beta(\mathbf{v})$ is required to be the unit.
- (7) (Stability) For each $v \in C_{0,int}(\mathcal{T})$ we assume that one of the following holds.
 - (a) $E(\beta(v)) > 0$.
 - (b) The valency of v is not smaller than 3.

We denote by $\mathcal{G}(k+1,\beta)$ the set of all decorated ribbon trees $(\mathcal{T},\beta(\cdot))$ such that:

- (I) $\#C_{0,\text{ext}}(\mathcal{T}) = k+1$. (II) $\sum_{\mathbf{v} \in C_{0,\text{int}}(\mathcal{T})} (\beta(\mathbf{v})) = \beta$.

We decompose the set of edges $C_1(\mathcal{T})$ as follows. If an edge e contains an exterior vertex, we call e an *exterior edge*. Otherwise we call e an *interior edge*. We denote by $C_{1,\text{int}}(\mathcal{T})$, (resp. $C_{1,\text{ext}}(\mathcal{T})$) the set of all interior (resp. exterior) edges.

Next we define fiber product of K-spaces along an element $(\mathcal{T}, \beta(\cdot))$ of $\mathcal{G}(k+1, \beta)$. Suppose we are given K-spaces $\mathcal{M}_{k+1}(\beta)$ and maps

$$\operatorname{ev} = (\operatorname{ev}_0, \dots, \operatorname{ev}_k) : \overset{\circ}{\mathcal{M}}_{k+1}(\beta) \to L^{k+1}$$

for each β and $k \in \mathbb{Z}_{\geq 0}$. Let $(\mathcal{T}, \beta(\cdot)) \in \mathcal{G}(k+1, \beta)$. We define the K-space

$$\prod_{(\mathcal{T},\beta(\cdot))} \mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v})) \tag{21.1}$$

as follows. We consider the direct product $\prod_{v \in C_{0,int}(\mathcal{T})} \mathcal{M}_{k_v+1}(\beta(v))$, where $k_v + 1$ is the valency of the vertex v. We take two copies of L for each interior edge $e \in C_{1,int}(\mathcal{T})$. We define a map

$$\operatorname{ev}: \prod_{\mathbf{v} \in C_{0,\operatorname{int}}(\mathcal{T})} \mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v})) \to \prod_{\mathbf{e} \in C_{1,\operatorname{int}}(\mathcal{T})} L^{2}$$
(21.2)

as follows. For each $v \in C_{0,int}(\mathcal{T})$ we enumerate the edges containing v as

$$e_{v,0},\ldots,e_{v,k_v}$$

such that the following conditions are satisfied.

Condition 21.3. (1) The edge $e_{v,0}$ is contained in the connected component of $\mathcal{T} \setminus \{v\}$ which contains the root.

(2) $(e_{v,0}, \ldots, e_{v,k_v})$ respects the cyclic ordering of the edges given by the ribbon structure at v.

Such an enumeration is unique. Each edge e contains two vertices. For one of them \mathbf{v}_{-} we have $\mathbf{e} = \mathbf{e}_{\mathbf{v}_{-},0}$. For the other vertex \mathbf{v}_{+} contained in \mathbf{e} , we have $\mathbf{e} = \mathbf{e}_{\mathbf{v}_{+},i}$ for some $i \in \{1,\ldots,k_{\mathbf{v}_{+}}\}$. We define the e component of $\mathbf{ev}((\mathbf{x}_{\mathbf{v}})_{\mathbf{v} \in C_{0,\mathrm{int}}(\mathcal{T})})$ as $(\mathbf{ev}_{0}(\mathbf{x}_{\mathbf{v}_{-}}),\mathbf{ev}_{i}(\mathbf{x}_{\mathbf{v}_{+}}))$, where $\mathbf{e} = \mathbf{e}_{\mathbf{v}_{+},i}$ and $\mathbf{x}_{\mathbf{v}} \in \mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v}))$. (Here \mathbf{ev} is the map in (21.2).)

Definition 21.4. The fiber product (21.1) is defined by

$$\left(\prod_{\mathbf{v}\in C_{0,\mathrm{int}}(\mathcal{T})} \mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v}))\right) \stackrel{\text{ev}}{=} \times \prod_{\mathbf{e}\in C_{1,\mathrm{int}}(\mathcal{T})} L^{2} \left(\prod_{\mathbf{e}\in C_{1,\mathrm{int}}(\mathcal{T})} L\right). \tag{21.3}$$

Here ev is as in (21.2) and $\prod_{e \in C_{1,int}(\mathcal{T})} L$ is the product of the diagonal $L \subset L^2$ and is contained in $\prod_{e \in C_{1,int}(\mathcal{T})} L^2$. We call (21.3) the fiber product of $\mathcal{M}_{k+1}(\beta)$ along $(\mathcal{T}, \beta(\cdot))$.

Remark 21.5. The fiber product (21.3) in the sense of K-spaces may not be defined because of the transversality problem. It is defined if the following Condition 21.6 is satisfied.

Condition 21.6. The map $ev_0 : \mathcal{M}_{k+1}(\beta) \to L$ is weakly submersive.

Condition 21.7. We consider the following objects.

(I) \mathfrak{G} is an additive group. (We denote the unit $0 \in \mathfrak{G}$ by β_0 .) $E : \mathfrak{G} \to \mathbb{R}$ and $\mu : \mathfrak{G} \to \mathbb{Z}$ are group homomorphisms. We call $E(\beta)$ the energy of β and $\mu(\beta)$ the Maslov index of β .

- (II) L is a smooth oriented manifold without boundary.
- (III) (Moduli space) For each $\beta \in \mathfrak{G}$ and $k \in \mathbb{Z}_{\geq 0}$ we have a K-space with corners $\mathcal{M}_{k+1}(\beta)$ and a strongly smooth map

$$\operatorname{ev} = (\operatorname{ev}_0, \dots, \operatorname{ev}_k) : \mathcal{M}_{k+1}(\beta) \to L^{k+1}.$$

We assume that ev_0 is weakly submersive. We call $\mathcal{M}_{k+1}(\beta)$ the moduli space of A_{∞} operations.

- (IV) (Positivity of energy) We assume $\mathcal{M}_{k+1}(\beta) = \emptyset$ if $E(\beta) < 0$.
- (V) (Energy zero part) In case $E(\beta) = 0$, we have $\mathcal{M}_{k+1}(\beta) = \emptyset$ unless $\beta = 0$ and $k \geq 2$. If $\beta = \beta_0 = 0$ then $\mathcal{M}_{k+1}(\beta_0) = L \times D^{k-2}$ and $\operatorname{ev}_i : \mathcal{M}_{k+1}(\beta_0) \to L$ is the projection. Here we regard D^{k-2} as a Stasheff cell which is a manifold with corners. (See [FOh, Section 10], for example.)
- (VI) (Dimension) The dimension of the moduli space of A_{∞} operations is given by

$$\dim \mathcal{M}_{k+1}(\beta) = \mu(\beta) + \dim L + k - 2. \tag{21.4}$$

(VII) (Orientation) $\mathcal{M}_{k+1}(\beta)$ is oriented.

(VIII) (Gromov compactness) For any E_0 the set

$$\{\beta \in \mathfrak{G} \mid \exists k \ \mathcal{M}_{k+1}(\beta) \neq \emptyset, \ E(\beta) \leq E_0\}$$
 (21.5)

is a finite set.

(IX) (Compatibility at the boundary) The normalized boundary of the moduli space of A_{∞} operations is decomposed into the disjoint union of fiber products as follows. ⁴³

$$\partial \mathcal{M}_{k+1}(\beta) \cong \coprod_{\beta_1, \beta_2, k_1, k_2, i} (-1)^{\epsilon} \mathcal{M}_{k_1+1}(\beta_1) \underset{\text{ev}_i}{\text{ev}_i} \times_{\text{ev}_0} \mathcal{M}_{k_2+1}(\beta_2)$$
 (21.6)

where

$$\epsilon = (k_1 - 1)(k_2 - 1) + \dim L + k_1 + (i - 1)\left(1 + (\mu(\beta_2) + k_2)\dim L\right)$$
 (21.7)

and the union is taken over $\beta_1, \beta_2, k_1, k_2, i$ such that $\beta_1 + \beta_2 = \beta$, $k_1 + k_2 = k+1$, $i=1,\ldots,k_2$ and i. This isomorphism is compatible with orientation and is compatible with evaluation maps in the following sense. Let $\mathbf{x}_1 \in \mathcal{M}_{k_1+1}(\beta_1)$ and $\mathbf{x}_2 \in \mathcal{M}_{k_2+1}(\beta_2)$. We denote by \mathbf{x} the element of $\partial \mathcal{M}_{k+1}(\beta)$ by the isomorphism (21.6). Then

$$\operatorname{ev}_{j}(\mathbf{x}) = \begin{cases} \operatorname{ev}_{j}(\mathbf{x}_{1}) & \text{if } j = 0, \dots, i - 1, \\ \operatorname{ev}_{j-i+1}(\mathbf{x}_{2}) & \text{if } j = i, \dots, i + k_{2} - 1, \\ \operatorname{ev}_{j-k_{2}+1}(\mathbf{x}_{1}) & \text{if } j = i + k_{2}, \dots, k. \end{cases}$$
(21.8)

See Remark 21.8 below for the sign (21.7) and (21.8).

In case $\beta = 0$, we require that (21.15) coincides with the standard decomposition appearing at the boundary of Stasheff cell. (See [FOh, Section 10].)

(X) (Compatibility at the corner I) Let $\widehat{S}_m(\mathcal{M}_{k+1}(\beta))$ be the normalized corner of the K-space $\mathcal{M}_{k+1}(\beta)$ in the sense of Definition 24.17. Then it is isomorphic to the disjoint union of

$$\prod_{(\mathcal{T},\beta(\cdot))} \mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v})). \tag{21.9}$$

⁴³See Remark 16.2 for the sign and the order of the fiber products.

Here the union is taken over all $(\mathcal{T}, \beta(\cdot)) \in \mathcal{G}(k+1, \beta)$ such that $\#C_{1,\text{int}}(\mathcal{T}) = m$. This isomorphism is compatible with the evaluation map in the following sense.

Let v_i be the *i*-th exterior vertex of \mathcal{T} . The (unique) edge e containing v_i contains one interior vertex denoted by v. Suppose e is the *j*-th edge of v. Let an element $(\mathbf{x}_v)_{v \in C_{0,int}(\mathcal{T})}$ of (21.9) correspond to an element \mathbf{x} in $\widehat{S}_m(\mathcal{M}_{k+1}(\beta))$. Then we require

$$ev_i(\mathbf{x}) = ev_i(\mathbf{x}_v). \tag{21.10}$$

When $\beta = 0$, we require that (21.9) coincides with the standard decomposition appearing at the corner of the Stasheff cell.

(XI) (Compatibility at the corner II) Condition (X) implies that

$$\widehat{S}_{\ell}(\widehat{S}_m(\mathcal{M}_{k+1}(\beta)))$$

is a disjoint union of $(m + \ell)!/m!\ell!$ copies of (21.9), where the union is taken over all $(\mathcal{T}, \beta(\cdot)) \in \mathcal{G}(k+1, \beta)$ such that $\#C_{1,\text{int}}(\mathcal{T}) = m + \ell$.

The map $\widehat{S}_{\ell}(\widehat{S}_m(\mathcal{M}_{k+1}(\beta))) \to \widehat{S}_{m+\ell}(\mathcal{M}_{k+1}(\beta))$ is identified with the identity map on each of the component (21.9).

Remark 21.8. The sign in (21.6) is consistent with our conventions adopted in [FOOO4]. [FOOO4, Proposition 8.3.3] is the same as (21.6) for the case i = 1. Also we can derive the sign in (21.6) by using [FOOO4, Proposition 8.3.3] and [FOOO4, (8.4.5)]. Indeed, the formula [FOOO4, (8.4.5)] yields

$$\mathcal{M}_{k_1+1}(\beta_1) \,_{\text{ev}_i} \times_{\text{ev}_0} \mathcal{M}_{k_2+1}(\beta_2) = (-1)^{\delta} \mathcal{M}_{k_1+1}(\beta_1) \,_{\text{ev}_1} \times_{\text{ev}_0} \mathcal{M}_{k_2+1}(\beta_2)$$
 (21.11)

where

$$\delta = (i - 1)(1 + \dim L \dim \mathcal{M}_{k_2 + 1}(\beta_2) + \dim L)$$

$$\equiv (i - 1)(1 + (\mu(\beta_2) + k_2) \dim L) \mod 2,$$
(21.12)

by taking the dimension formula (21.4) into account. Moreover, the convention of the order of boundary marked points after gluing described in [FOOO4, Remark 8.3.4] is nothing but the convention (21.8) for the case i = 1.

Definition 21.9. A tree-like K-system, or sometimes called an A_{∞} correspondence over L, is a system of $(\mathcal{M}_{k+1}(\beta), \text{ev}, \mu, E)$ satisfying Condition 21.7.

We next define a notion of partial A_{∞} correspondence. The moduli space $\mathcal{M}_{k+1}(\beta)$ depends on k and β . In the version of partial A_{∞} correspondence which we use in this article, we include the Kuranishi structure on $\mathcal{M}_{k+1}(\beta)$ for only a finite number of the pairs (β, k) . This coincides with the way taken in [FOOO4, Section 7], where we used the notion of $A_{n,K}$ structure. In [Fu2] the notion of A_{∞} structure modulo E_0 was used. It includes only a finite number of β 's but infinitely many k's are included. In [Fu2] the Kuranishi structure of $\mathcal{M}_{k+1}(\beta)$ such that it is compatible with the forgetful map $\mathcal{M}_{k+1}(\beta) \to \mathcal{M}_1(\beta)$ was used. This is the reason why infinitely many of k's were included in [Fu2].

Since we postpone technical detail concerning the forgetful map to [FOOO18], we use the formulation where only a finitely many k's are included in the partial structure. Since we are working in de Rham theory, it is certainly possible to include infinitely many k's at this stage. However, it seems easier to use only a finite number of moduli spaces at each step of the construction.

Definition 21.10. A partial A_{∞} correspondence of energy cut level E_0 and minimal energy e_0 over L is defined in the same way as A_{∞} correspondence except the following:

- (1) The moduli space $\mathcal{M}_{k+1}(\beta)$ of A_{∞} operations is defined only when $E(\beta) + ke_0 \leq E_0$.
- (2) The compatibility conditions that are Conditions 21.7 (IX)(X)(XI) are assumed only when $E(\beta) + ke_0 \leq E_0$.
- (3) We assume that $\mathcal{M}_{k+1}(\beta) = \emptyset$ if $0 < E(\beta) < e_0$.

Hereafter we say $\mathcal{M}_{k+1}(\beta)$ is a (partial) A_{∞} correspondence, (and omit ev, μ , E) for simplicity.

We next describe a parametrized version of Condition 21.7.

Condition 21.11. We consider the following objects.

- (I) \mathfrak{G} is an additive group. (We denote the unit 0 by β_0 .) $E:\mathfrak{G}\to\mathbb{R}$ and $\mu:\mathfrak{G}\to\mathbb{Z}$ are group homomorphisms. We call $E(\beta)$ the energy of β and $\mu(\beta)$ the Maslov index of β .
- (II) L is a smooth oriented manifold without boundary. P is a smooth oriented manifold with corners.
- (III) (Moduli space) For each $\beta \in \mathfrak{G}$ and $k \in \mathbb{Z}_{\geq 0}$ we have a K-space with corners $\mathcal{M}_{k+1}(\beta; P)$ and a strongly smooth map

$$\operatorname{ev} = (\operatorname{ev}_P, \operatorname{ev}_0, \dots, \operatorname{ev}_k) : \mathcal{M}_{k+1}(\beta; P) \to P \times L^{k+1}.$$

We assume that (ev_P, ev_0) is weakly submersive stratumwisely. We call $\mathcal{M}_{k+1}(\beta; P)$ the moduli space of P-parametrized A_{∞} operations.

- (IV) (Positivity of energy) We assume $\mathcal{M}_{k+1}(\beta; P) = \emptyset$ if $E(\beta) < 0$.
- (V) (Energy zero part) In case $E(\beta) = 0$, we have $\mathcal{M}_{k+1}(\beta; P) = \emptyset$ unless $\beta = 0$ and $k \geq 2$. If $\beta = \beta_0 = 0$, then $\mathcal{M}_{k+1}(\beta_0; P) = P \times L \times D^{k-2}$ and $\text{ev}_i : \mathcal{M}_{k+1}(\beta_0; P) \to L$ is the projection. Also $\text{ev}_P : \mathcal{M}_{k+1}(\beta_0; P) \to P$ is the projection. Here we again identify D^{k-2} with the Stasheff cell.
- (VI) (Dimension) The dimension of the moduli space of P-parametrized A_{∞} operations is given by

$$\dim \mathcal{M}_{k+1}(\beta; P) = \mu(\beta) + \dim L + k - 2 + \dim P.$$
 (21.13)

- (VII) (Orientation) $\mathcal{M}_{k+1}(\beta; P)$ is oriented.
- (VIII) (Gromov compactness) For any E_0 the set

$$\{\beta \in \mathfrak{G} \mid \exists k \ \mathcal{M}_{k+1}(\beta; P) \neq \emptyset, \ E(\beta) \leq E_0\}$$
 (21.14)

is a finite set.

(IX) (Compatibility at the boundary) The normalized boundary of the moduli space of A_{∞} operations is decomposed into the fiber products as follows. ⁴⁴

$$\partial \mathcal{M}_{k+1}(\beta;P)$$

$$\cong \coprod_{\beta_1,\beta_2,k_1,k_2,i} (-1)^{\epsilon} \mathcal{M}_{k_1+1}(\beta_1;P) \underset{(\text{ev}_P,\text{ev}_i)}{\text{(ev}_P,\text{ev}_i)} \times_{(\text{ev}_P,\text{ev}_0)} \mathcal{M}_{k_2+1}(\beta_2;P)$$

$$\sqcup (\partial P_P \times_{\text{ev}_P} \mathcal{M}_{k+1}(\beta;P)).$$
(21.15)

 $^{^{44}}$ Following our convention in [FOOO4, (8.9.1)], that the parameter space P is put on the first factor in the fiber product. So there is no extra sign contribution from the parameter space in the second line on the right hand side of (21.15).

where

$$\epsilon = (k_1 - 1)(k_2 - 1) + \dim L + k_1 + (i - 1)\left(1 + (\mu(\beta_2) + k_2 + \dim P)\dim L\right)$$
(21.16)

and the union in the second line is taken over $\beta_1, \beta_2, k_1, k_2, i$ such that $\beta_1 + \beta_2 = \beta$, $k_1 + k_2 = k + 1$, $i = 1, ..., k_2$. The fiber product in the second line is taken over $P \times L$. This isomorphism is compatible with orientation. It is compatible with evaluation maps, in the same sense as (21.8). In case $\beta = 0$, we require that (21.15) coincides with the decomposition induced from the standard decomposition appearing at the boundary of Stasheff cell.

(X) (Compatibility at the corner I) Let $\widehat{S}_m(\mathcal{M}_{k+1}(\beta; P))$ be the normalized corner of the K-space $\mathcal{M}_{k+1}(\beta; P)$ in the sense of Definition 24.17. Then it is isomorphic to the disjoint union of

$$\prod_{(\mathcal{T},\beta(\cdot))} \mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v}); \widehat{S}_{m'}(P)). \tag{21.17}$$

(We will explain the fiber product (21.17) right after Condition 21.11.) Here the union is taken over all $(\mathcal{T}, \beta(\cdot)) \in \mathcal{G}(k+1, \beta)$ and $m' \in \mathbb{Z}_{\geq 0}$ such that $\#C_{1,int}(\mathcal{T}) + m' = m$ and we put

$$\mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v}); \widehat{S}_{m'}(P)) = \widehat{S}_{m'}(P) P \times_{\mathrm{ev}_P} \mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v}); P).$$

This isomorphism is compatible with the evaluation map in the same sense as (21.10). In case $\beta = 0$, we require that (21.17) coincides with the decomposition induced from the standard decomposition appearing at the corner of the Stasheff cell.

(XI) (Compatibility at the corner II) Condition (X) implies that $\widehat{S}_{\ell}(\widehat{S}_m(\mathcal{M}_{k+1}(\beta;P)))$ is a disjoint union of copies

$$\prod_{(\mathcal{T},\beta(\cdot))} \mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v}); \widehat{S}_{\ell'}(\widehat{S}_{m'}(P)))$$
(21.18)

where the union is taken over all $(\mathcal{T}, \beta(\cdot)) \in \mathcal{G}(k+1, \beta)$, and m', ℓ' such that $\#C_{1,\text{int}}(\mathcal{T}) + m' + \ell' = m + \ell$ and we put

$$\mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v}); \widehat{S}_{\ell'}(\widehat{S}_{m'}(P))) = \widehat{S}_{\ell'}(\widehat{S}_{m'}(P)) \,\,_{P} \times_{\mathrm{ev}_{P}} \mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v}); P).$$

The covering map $\widehat{S}_{\ell}(\widehat{S}_m(\mathcal{M}_{k+1}(\beta;P))) \to \widehat{S}_{m+\ell}(\mathcal{M}_{k+1}(\beta;P))$ is identified with the map induced from the covering map $\widehat{S}_{\ell'}(\widehat{S}_{m'}(P)) \to \widehat{S}_{\ell'+m'}(P)$ on each of the component (21.18).

Now we define the fiber product (21.17) as follows. We consider the direct product $\prod_{v \in C_{0,int}(\mathcal{T})} \mathcal{M}_{k_v+1}(\beta(v); P)$ as in (21.2). We have

$$\operatorname{ev}: \prod_{\mathbf{v} \in C_{0,\operatorname{int}}(\mathcal{T})} \mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v}); P) \to \prod_{\mathbf{e} \in C_{1,\operatorname{int}}(\mathcal{T})} L^2.$$

Using $\operatorname{ev}_P: \mathcal{M}_{k_v+1}(\beta(\mathbf{v}); P) \to P$ we have

$$\operatorname{ev}_P: \prod_{v \in C_{0,\operatorname{int}}(\mathcal{T})} \mathcal{M}_{k_v+1}(\beta(v); P) \to \prod_{e \in C_{1,\operatorname{int}}(\mathcal{T})} P^2.$$

The fiber product (21.17) is by definition

$$\left(\prod_{\mathbf{e} \in C_{1, \text{int}}(\mathcal{T})} P \times L\right) \\
\prod_{\mathbf{e} \in C_{1, \text{int}}(\mathcal{T})} (P^{2} \times L^{2}) \times_{(\text{ev}_{P}, \text{ev})} \left(\prod_{\mathbf{v} \in C_{0, \text{int}}(\mathcal{T})} \mathcal{M}_{k_{\mathbf{v}} + 1}(\beta(\mathbf{v}); P)\right).$$
(21.19)

Lemma 21.12. The fiber product (21.19) is well-defined.

Proof. We assumed that the map

$$(ev_P, ev_0) : \mathcal{M}_{k_v+1}(\beta(v); P) \to P \times L$$

is weakly submersive stratumwisely. We also note that for each $e \in C_{1,int}(\mathcal{T})$ there exists a unique vertex v such that e is its 0-th vertex. These two facts imply the lemma immediately.

Definition 21.13. A *P*-parametrized A_{∞} correspondence over *L* is a system of $(\mathcal{M}_{k+1}(\beta; P), \text{ev}, \mu, E)$ satisfying Condition 21.11.

A partial P-parametrized A_{∞} correspondence of energy cut level E_0 and minimal energy e_0 over L is defined in the same way as P-parametrized A_{∞} correspondence except the following:

- (1) The moduli space of P-parametrized A_{∞} operations $\mathcal{M}_{k+1}(\beta; P)$ is defined only when $E(\beta) + ke_0 \leq E_0$.
- (2) The compatibility conditions that is Condition 21.11 (IX)(X)(XI) are assumed only when $E(\beta) + ke_0 \leq E_0$.
- (3) We assume $\mathcal{M}_{k+1}(\beta, P) = \emptyset$ if $0 < E(\beta) < e_0$.

Lemma 21.14. Let $\mathcal{M}_{k+1}(\beta; P)$ be a P-parametrized A_{∞} correspondence over L and $\partial_i P$ a connected component of the normalized boundary of P. Then

$$\mathcal{M}_{k+1}(\beta; \partial_i P) = \partial_i P_P \times_{\text{ev}_P} \mathcal{M}_{k+1}(\beta; P)$$

defines a $\partial_i P$ -parametrized A_{∞} correspondence over L. The same holds for partial P-parametrized A_{∞} correspondence.

The proof is obvious.

Definition 21.15. Suppose we are given two A_{∞} correspondences over L denoted by $\mathcal{M}_{k+1}^{j}(\beta)$ with j=1,2. Then a pseudo-isotopy between them is a P=[1,2] parametrized A_{∞} correspondence $\mathcal{M}_{k+1}(\beta;[1,2])$ such that for $\{1\} \subset \partial[1,2]$ (resp. $\{2\} \subset \partial[1,2]$) the A_{∞} correspondence $\mathcal{M}_{k+1}(\beta;\{1\})$ (resp. $\mathcal{M}_{k+1}(\beta;\{2\})$) is isomorphic to $\mathcal{M}_{k+1}^{1}(\beta)$ (resp $\mathcal{M}_{k+1}^{2}(\beta)$). We define the notion of pseudo-isotopy of partial A_{∞} correspondences in the same way.

Remark 21.16. In this article we use the notion of pseudo-isotopy of A_{∞} correspondences to prove well-definedness of the filtered A_{∞} algebra induced by the A_{∞} correspondence. (See Theorem 21.35 (2).) On the other hand, we can define the notion of morphism of A_{∞} correspondences and use it instead to prove well-definedness. In other words, we are using the bifurcation method here but not the cobordism method. (See [FOOO7, Subsection 7.2.14] for these two methods. In a slightly different formulation, a morphism of A_{∞} correspondences is defined in [Fu3, Definition 8].) In [FOOO7] and [Fu3] we used the cobordism method. In [AJ] and

[Fu2] the bifurcation method was used. They will give the same morphism at the end of the day as explained in [Fu2, Remark 12.3].

Definition 21.17. An inductive system of A_{∞} correspondences over L consists of the following objects.

- (1) We are given a sequence $\{E^i\}_{i=1}^{\infty}$ of positive real numbers such that $E^i < E^{i+1}$ and $\lim_{i\to\infty} E^i = \infty$. We are also given $e_0 > 0$ independent of i.
- (2) For each i, we are given a partial A_{∞} correspondence $\mathcal{M}_{k+1}^{i}(\beta)$ over L of energy cut level E^{i} and minimal energy e_{0} .
- (3) For each i we are given a pseudo-isotopy $\mathcal{M}_{k+1}(\beta; [i, i+1])$ between $\mathcal{M}_{k+1}^i(\beta)$ and $\mathcal{M}_{k+1}^{i+1}(\beta)$. Here the energy cut level of $\mathcal{M}_{k+1}(\beta; [i, i+1])$ is E^i and its minimal energy is e_0 .
- (4) We assume the following uniform Gromov compactness. For each $E^\prime>0$ the next set is of finite order.

$$\{\beta \in \mathfrak{G} \mid \exists k \,\exists i \, \mathcal{M}_{k+1}^i(\beta) \neq \emptyset, \, E(\beta) \leq E'\}.$$
 (21.20)

The next set is also of finite order for each E' > 0:

$$\{\beta \in \mathfrak{G} \mid \exists k \,\exists i \, \mathcal{M}_{k+1}^i(\beta; [i, i+1]) \neq \emptyset, \, E(\beta) \leq E'\}.$$
 (21.21)

Remark 21.18. In the situation of Definition 21.17, suppose E'^i is another sequence so that $E'^i < E'^{i+1}$, $\lim_{i \to \infty} E'^i = \infty$ and $E'^i < E^i$. We forget the K-spaces $\mathcal{M}_{k+1}(\beta; [i, i+1])$ and $\mathcal{M}_{k+1}^i(\beta + ke_0)$ for $E(\beta) > E'^i$. Then we obtain another inductive system of A_{∞} correspondences over L.

Let $e'_0 < e_0$. We replace E^i by $E'^i = E^i e'_0/e_0$. Then $ke'_0 + E \leq E'^i$ implies $ke_0 + E \leq E^i$. Therefore any partial A_{∞} correspondence of energy cut level E^i and of minimal energy e_0 induces one of energy cut level E'^i and of minimal energy e'_0 by forgetting certain moduli spaces. In this way when we compare two inductive systems of A_{∞} correspondences over L we may always assume that the numbers E^i , e_0 are common without loss of generality. We will assume it in the next definition.

Definition 21.19. Suppose for j=0,1 we are given inductive systems of A_{∞} correspondences over L, denoted by $\mathcal{M}_{k+1}^{ji}(\beta)$, $\mathcal{M}_{k+1}^{j}(\beta;[i,i+1])$. (We take the same E^{i} and e_{0} for j=0,1 as we explained in Remark 21.18.) A pseudo-isotopy between these two inductive systems consists of the following objects.

- (1) For each i we are given a pseudo-isotopy $\mathcal{M}_{k+1}^{i}(\beta; [0, 1])$ between $\mathcal{M}_{k+1}^{0i}(\beta)$ and $\mathcal{M}_{k+1}^{1i}(\beta)$. The energy cut level and minimal energy of this pseudo-isotopy are E^{i} and e_{0} , respectively.
- (2) For each i we are given a $P = [0,1] \times [i,i+1]$ parametrized A_{∞} correspondence $\mathcal{M}_{k+1}(\beta;[0,1] \times [i,i+1])$ satisfying the following properties.
 - (a) Its restriction to the boundary component $\{j\} \times [i, i+1]$ is isomorphic to $\mathcal{M}_{k+1}^{j}(\beta; [i, i+1])$. Here j=0,1.
 - (b) Its restriction to the boundary component $[0,1] \times \{i\}$ is isomorphic to $\mathcal{M}_{k+1}^{1i}(\beta)$.
 - (c) Its restriction to the boundary component $[0,1] \times \{i+1\}$ is isomorphic to $\mathcal{M}_{k+1}^{1i+1}(\beta)$.
 - (d) The isomorphisms in (a)(b)(c) are consistent at $\mathcal{M}_{k+1}(\beta; \widehat{S}_2([0,1] \times [i,i+1]))$.
 - (e) The energy cut level of $\mathcal{M}_{k+1}(\beta; [0,1] \times [i,i+1])$ is E^i and its minimal energy is e_0 .

- (f) The isomorphisms in Item (a)(b)(c)(d) satisfy appropriate corner compatibility conditions.
- (3) We assume the following uniform Gromov compactness. For each E' > 0 the next set is of finite order.

$$\{\beta \in \mathfrak{G} \mid \exists k \; \exists i \; \mathcal{M}_{k+1}(\beta; [0,1] \times [i,i+1]) \neq \emptyset, \; E(\beta) \leq E'\}.$$
 (21.22)

21.2. Filtered A_{∞} algebra and its pseudo-isotopy. In this subsection we review certain algebraic material in [FOOO3], [FOOO4], [Fu2].

Definition 21.20. We call a subset $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ a discrete submonoid⁴⁵ if the following holds. We denote by $E: G \to \mathbb{R}_{\geq 0}$ and $\mu: G \to 2\mathbb{Z}$ the natural projections.

- (1) If $\beta_1, \beta_2 \in G$, then $\beta_1 + \beta_2 \in G$. $(0,0) \in G$.
- (2) The image $E(G) \subset \mathbb{R}_{>0}$ is discrete.
- (3) For each $E_0 \in \mathbb{R}_{\geq 0}$ the inverse image $G \cap E^{-1}([0, E_0])$ is a finite set.

Let $\Omega(L)$ be the de Rham complex of L. We put $\Omega(L)[1]^d = \Omega^{d+1}(L)$, where $\Omega^d(L)$ is the space of degree d smooth forms. We put

$$B_k(\Omega(L)[1]) = \underbrace{\Omega(L)[1] \otimes \cdots \otimes \Omega(L)[1]}_{k \text{ times}}.$$
 (21.23)

Let G be a discrete submonoid as in Definition 21.20.

Definition 21.21. ([FOOO3, Definition 3.2.26, Definition 3.5.6, Remark 3.5.8]) A G-gapped filtered A_{∞} algebra structure on $\Omega(L)$ is a sequence of multilinear maps

$$\mathfrak{m}_{k,\beta}: B_k(\Omega(L)[1]) \to \Omega(L)[1] \tag{21.24}$$

for each $\beta \in G$ and $k \in \mathbb{Z}_{\geq 0}$ of degree $1 - \mu(\beta)$ with the following properties:

- (1) $\mathfrak{m}_{k,\beta_0} = 0$ for $\beta_0 = (0,0), k \neq 1,2$.
- (2) $\mathfrak{m}_{2,\beta_0}(h_1,h_2) = (-1)^*h_1 \wedge h_2$, where $* = \deg h_1(\deg h_2 + 1)$.
- (3) $\mathfrak{m}_{1,\beta_0}(h) = (-1)^* dh$, where $* = n + 1 + \deg h$.
- (4)

$$\sum_{k_1+k_2=k+1} \sum_{\beta_1+\beta_2=\beta} \sum_{i=1}^{k-k_2+1}$$
 (21.25)

$$(-1)^*\mathfrak{m}_{k_1,\beta_1}(h_1,\ldots,\mathfrak{m}_{k_2,\beta_2}(h_i,\ldots,h_{i+k_2-1}),\ldots,h_k)=0$$

holds for any $\beta \in G$ and k. The sign is given by $* = \deg' h_1 + \cdots + \deg' h_{i-1}$. Here \deg' is the shifted degree by +1.

Definition 21.22. A partial G-gapped filtered A_{∞} algebra structure of energy cut level E and minimal energy e_0 is a sequence of operators (21.24) for $E(\beta) + ke_0 \leq E$ satisfying the same properties, except (21.25) is assumed only for $E(\beta) + ke_0 \leq E$. We require $\mathfrak{m}_{k,\beta} = 0$ if $0 < E(\beta) < e_0$.

 $^{^{45}}$ In the situation of oriented Lagrangian submanifolds, the Maslov index is even. Thus we assume $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$, while we consider $G \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}$ in Definition 16.11.

 $^{^{46}}$ See [FOOO3, (3.5.9)] for the sign.

 $^{^{47}}$ The sign here is different from the one given in [FOOO3, (3.2.5)]. The latter is a convention when we regard a DGA as an A_{∞} algebra with trivial higher multiplications. (Actually it does not matter if we change the sign of differential in DGA.) But the signs here and in [FOOO3, Remark 3.5.8] provide the signs in the filtered A_{∞} algebra which are compatible with the signs in its unfiltered DGA.

Definition 21.23. A multilinear map $F: B_k(\Omega(L)[1]) \to \Omega(L)[1]$ is said to be continuous in C^{∞} topology if the following holds. Suppose $h_{i,a} \in \Omega(L)$ converges to $h_i \in \Omega(L)$ in C^{∞} topology as $a \to \infty$, then $F(h_{1,a}, \ldots, h_{k,a})$ converges to $F(h_1, \ldots, h_k)$ in C^{∞} topology as $a \to \infty$.

Definition 21.24. A (partial) G gapped filtered A_{∞} algebra structure on $\Omega(L)$ is said to be *continuous* if the operations $\mathfrak{m}_{k,\beta}$ is continuous in C^{∞} topology. Hereafter we assume that all the operations of (partial) A_{∞} algebra structure on $\Omega(L)$ are continuous in C^{∞} topology.

Definition 21.25. ([Fu2, Definition 8.5]) For each $t \in [0, 1]$, $\beta \in G$ and $k \in \mathbb{Z}_{\geq 0}$, let $\mathfrak{m}_{k,\beta}^t$ be as in Definition 21.21 and $\mathfrak{c}_{k,\beta}^t$ a sequence of multilinear maps

$$\mathfrak{c}_{k,\beta}^t: B_k(\Omega(L)[1]) \to \Omega(L)[1] \tag{21.26}$$

of degree $-\mu(\beta)$. We say $(\{\mathfrak{m}_{k,\beta}^t\}, \{\mathfrak{c}_{k,\beta}^t\})$ is a pseudo-isotopy of G-gapped filtered A_{∞} algebra structures, or in short, gapped pseudo-isotopy on $\Omega(L)$ if the following holds:

- (1) $\mathfrak{m}_{k,\beta}^t$ and $\mathfrak{c}_{k,\beta}^t$ are continuous in C^∞ topology. The map sending t to $\mathfrak{m}_{k,\beta}^t$ or $\mathfrak{c}_{k,\beta}^t$ is smooth. Here we use the operator topology with respect to the C^∞ topology for $\mathfrak{m}_{k,\beta}^t$ and $\mathfrak{c}_{k,\beta}^t$ to define their smoothness.
- (2) For each (but fixed) t, the set of operators $\{\mathfrak{m}_{k,\beta}^t\}$ defines a G-gapped filtered A_{∞} algebra structure on $\Omega(L)$.
- (3) For each $h_i \in \Omega(L)[1]$ the following equality holds:

$$\frac{d}{dt}\mathfrak{m}_{k,\beta}^{t}(h_{1},\ldots,h_{k}) + \sum_{k_{1}+k_{2}=k+1}\sum_{\beta_{1}+\beta_{2}=\beta}\sum_{i=1}^{k-k_{2}+1}(-1)^{*}\mathfrak{c}_{k_{1},\beta_{1}}^{t}(h_{1},\ldots,\mathfrak{m}_{k_{2},\beta_{2}}^{t}(h_{i},\ldots),\ldots,h_{k}) - \sum_{k_{1}+k_{2}=k+1}\sum_{\beta_{1}+\beta_{2}=\beta}\sum_{i=1}^{k-k_{2}+1}\mathfrak{m}_{k_{1},\beta_{1}}^{t}(h_{1},\ldots,\mathfrak{c}_{k_{2},\beta_{2}}^{t}(h_{i},\ldots),\ldots,h_{k})$$
(21.27)

Here
$$* = \deg' h_1 + \dots + \deg' h_{i-1}$$
.
(4) $\mathfrak{c}_{k,\beta}^t = 0$ if $E(\beta) \le 0$.

Sometimes we say that $(\{\mathfrak{m}_{k,\beta}^t\}, \{\mathfrak{c}_{k,\beta}^t\})$ is a pseudo-isotopy or *gapped pseudo-isotopy* between $\{\mathfrak{m}_{k,\beta}^0\}$ and $\{\mathfrak{m}_{k,\beta}^1\}$.

Definition 21.26. We define the notion of pseudo-isotopy of partial G-gapped filtered A_{∞} algebra structures on $\Omega(L)$ of energy cut level E and minimal energy e_0 in the same way, except we replace (2) and (3) by the following (2)' and (3)' and we further require (5) below.

- (2)' We require that the set of operators $\{\mathfrak{m}_{k,\beta}^t\}$ defines a partial G-gapped filtered A_{∞} algebra structures on $\Omega(L)$ of energy cut level E and of minimal energy e_0 .
- (3)' We require (21.27) only for β, k with $E(\beta) + ke_0 \leq E$.
- (5) $\mathfrak{m}_{k,\beta}^t = \mathfrak{c}_{k,\beta}^t = 0 \text{ if } 0 < E(\beta) < e_0.$

We also use the notion of pesudo-isotopy of pseudo-isotopies. It seems simpler to define a more general notion of the P-parametrized family of G-gapped filtered A_{∞} algebra structures on $\Omega(L)$ in general. So we define this notion below.

We consider the totality of smooth differential forms on $P \times L$ which we write $\Omega(P \times L)$. Here $\Omega(P \times L)[1]$ is its degree shift as before. Let t_1, \ldots, t_d be local coordinates of P. (In this article we only consider the case $P \subset \mathbb{R}^d$ for $d = \dim P$. So we actually have canonical global coordinates.) For $h \in \Omega(P \times L)$ we put

$$h = \sum_{I \subset \{1, \dots, d\}} dt_I \wedge h_I. \tag{21.28}$$

Here $I = \{i_1, \ldots, i_{|I|}\}$ $(i_1 < \cdots < i_{|I|})$, $dt_I = dt_{i_1} \wedge \cdots \wedge dt_{i_{|I|}}$ and h_I does not contain dt_i .

Definition 21.27. A multilinear map $F: B_k(\Omega(P \times L)[1]) \to \Omega(P \times L)[1]$ is said to be *pointwise in P direction* if the following holds:

For each $I \subset \{1, ..., d\}$ and $\mathbf{t} \in P$ there exists a continuous map

$$F_{I;j_1,\ldots,j_k}^{\mathbf{t}}: B_k(\Omega(L)[1]) \to \Omega(L)[1]$$

such that

$$F(dt_{j_1} \wedge h_1, \dots, dt_{j_k} \wedge h_k)|_{\{\mathbf{t}\} \times L}$$

$$= \sum_{I} dt_{I} \wedge dt_{j_1} \wedge \dots \wedge dt_{j_k} \wedge F_{I;j_1,\dots,j_k}^{\mathbf{t}}(h_1^{\mathbf{t}}, \dots, h_k^{\mathbf{t}}), \qquad (21.29)$$

where $|_{\{\mathbf{t}\}\times L}$ means the restriction to $\{\mathbf{t}\}\times L$. Moreover $F_{I;j_1,...,j_k}^{\mathbf{t}}$ depends smoothly on \mathbf{t} with respect to the operator topology. Here $h_i^{\mathbf{t}}$ is the restriction of h_i to $\{\mathbf{t}\}\times L$.

Remark 21.28. This condition is equivalent to the following one.

(*) For smooth differential forms σ_i on P, we have

$$F(\sigma_1 h_1, \dots, \sigma_k h_k) = \pm \sigma_1 \wedge \dots \wedge \sigma_k \wedge F(h_1, \dots, h_k).$$

Definition 21.29. A *P*-parametrized family of *G*-gapped filtered A_{∞} algebra structures on $\Omega(L)$ is $\{\mathfrak{m}_{k,\beta}^P\}$ satisfying the following properties:

- (1) $\mathfrak{m}_{k,\beta}^P: B_k(\Omega(P \times L)[1]) \to \Omega(P \times L)[1]$ is a multilinear map of degree 1.
- (2) $\mathfrak{m}_{k,\beta}^{P}$ is pointwise in P direction if $\beta \neq \beta_0$.
- (3) $\mathfrak{m}_{k,\beta_0}^P = 0 \text{ for } k \neq 1,2.$
- (4) $\mathfrak{m}_{1,\beta_0}^{p^n}(h) = (-1)^*dh$. Here d is the de Rham differential and $*=n+1+\deg h$.
- (5) $\mathfrak{m}_{2,\beta_0}^P(h_1 \wedge h_2) = (-1)^*h_1 \wedge h_2$. Here \wedge is the wedge product and $* = \deg h_1(\deg h_2 + 1)$.
- (6) $\mathfrak{m}_{k,\beta}^P$ satisfies the following A_{∞} relation.

$$\sum_{k_1+k_2=k+1} \sum_{\beta_1+\beta_2=\beta} \sum_{i=1}^{k-k_2+1} (21.30)$$

$$(-1)^* \mathfrak{m}_{k_1,\beta_1}^P(h_1,\ldots,\mathfrak{m}_{k_2,\beta_2}^P(h_i,\ldots,h_{i+k_2-1}),\ldots,h_k) = 0,$$
where $*= \operatorname{deg}' h_1 + \ldots + \operatorname{deg}' h_{i-1}.$

Definition 21.30. A partial P-parametrized family of G-gapped filtered A_{∞} algebra structures on $\Omega(L)$ of energy cut level E and of minimal energy e_0 is $\{\mathfrak{m}_{k,\beta}^P\}$ satisfying the same properties as above except the following points:

- (a) $\mathfrak{m}_{k,\beta}^P$ is defined only for β, k with $E(\beta) + ke_0 \leq E$.
- (b) We require the A_{∞} relation (21.30) only for β, k with $E(\beta) + ke_0 \leq E$.
- (c) $\mathfrak{m}_{k,\beta}^P = 0 \text{ if } 0 < E(\beta) < e_0.$

Lemma 21.31. The notion of pseudo-isotopy of G-gapped filtered A_{∞} algebra structures on $\Omega(L)$ is the same as the notion of the P = [0,1] parametrized family of G-gapped filtered A_{∞} algebra structures on $\Omega(L)$. The same holds for the partial G-gapped filtered A_{∞} algebra structure on $\Omega(L)$.

Proof. Let $(\{\mathfrak{m}_{k,\beta}^t\}, \{\mathfrak{c}_{k,\beta}^t\})$ be the objects as in Definition 21.25. We define $\mathfrak{m}_{k,\beta}^P$ as follows. It suffices to consider the case $\beta \neq \beta_0$.

Suppose h_i does not contain dt. Then we put

$$\mathfrak{m}_{k,\beta}^P(h_1,\ldots,h_k) = \mathfrak{m}_{k,\beta}^t(h_1,\ldots,h_k) + dt \wedge \mathfrak{c}_{k,\beta}^t(h_1,\ldots,h_k).$$

We also put

$$\mathfrak{m}_{k,\beta}^{P}(h_1,\ldots,dt\wedge h_i,\ldots,h_k)=(-1)^*dt\wedge\mathfrak{m}_{k,\beta}^{t}(h_1,\ldots,h_k)$$

where $* = \deg' h_1 + \cdots + \deg' h_{i-1}$. If at least two of $\hat{h}_1, \ldots, \hat{h}_k$ contain dt, then

$$\mathfrak{m}_{k,\beta}^P(\hat{h}_1,\ldots\hat{h}_k)=0.$$

It is straightforward to check that (21.27) is equivalent to (21.30).

Lemma-Definition 21.32. Let Q be a connected component of the normalized corner $\widehat{S}_k(P)$. Then a P-parametrized family of G-gapped filtered A_{∞} algebra structures on $\Omega(L)$ induces a Q-parametrized family of G-gapped filtered A_{∞} algebra structures on $\Omega(L)$.

Proof. This is a consequence of the following fact. If $F: B_k(\Omega(P \times L)[1]) \to \Omega(P \times L)[1]$ is pointwise in P direction, it induces : $B_k(\Omega(Q \times L)[1]) \to \Omega(Q \times L)[1]$ which is pointwise in Q direction. This fact is a consequence of the definition of pointwise-ness.

Remark 21.33. If two G-gapped filtered A_{∞} algebra structures on $\Omega(L)$ are pseudo-isotopic, then the two filtered A_{∞} algebras induced from those two structures are homotopy equivalent in the sense of [FOOO3, Definition 4.2.42]. This fact is proved in [Fu2, Theorem 8.2].

21.3. Statement of the results. In this subsection and hereafter we say a filtered A_{∞} structure, pseudo-isotopy or P-parametrised A_{∞} structure is gapped when it is G-gapped for some discrete submonoid $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ in Definition 21.20.

Situation 21.34. Let μ , E be as in Situation 21.1. Let L be a compact oriented smooth manifold without boundary. In addition to those, we consider one of the following situations.

- (1) We are given $\mathcal{AF} = \{(\mathcal{M}_{k+1}(\beta), \text{ev}) \mid \beta, k\}$, which defines an A_{∞} correspondence over L. Hereafter we will not include ev in the notation for simplicity.
- (2) We are given $\mathcal{AF} = \{\mathcal{M}_{k+1}(\beta) \mid \beta, k\}$, which defines a partial A_{∞} correspondence over L of energy cut level E_0 and minimal energy e_0 .
- (3) We are given

$$\mathcal{AF}^{j} = \{ \mathcal{M}_{k+1}^{j}(\beta) \mid \beta, k \} \ (j = 0, 1), \quad \mathcal{AF}^{[0,1]} = \{ \mathcal{M}_{k+1}(\beta; [0, 1]) \mid \beta, k \},$$

which are two A_{∞} correspondences over L and a pseudo-isotopy among them, respectively.

(4) We are given

$$\mathcal{AF}^{j} = \{ \mathcal{M}_{k+1}^{j}(\beta) \mid \beta, k \} \ (j = 0, 1), \quad \mathcal{AF}^{[0,1]} = \{ \mathcal{M}_{k+1}(\beta; [0, 1]) \mid \beta, k \},$$

which are two partial A_{∞} correspondences over L and a pseudo-isotopy among them, respectively. Their energy cut level are E_0 and minimal energy are e_0 .

(5) We are given

$$\mathcal{AF}^i = \{ \mathcal{M}_{k+1}^i(\beta) \mid \beta, k \}, \quad \mathcal{AF}^{[i,i+1]} = \{ \mathcal{M}_{k+1}(\beta; [i,i+1]) \mid \beta, k \}$$

for $i = 1, 2, \ldots$ such that

$$\mathcal{IAF} = \left(\{ \mathcal{AF}^i \mid i = 1, 2, \dots \}, \{ \mathcal{AF}^{[i, i+1]} \mid i = 1, 2, \dots \} \right)$$

consists of an inductive system of A_{∞} correspondences over L.

(6) For j = 0, 1, we are given

$$\mathcal{AF}^{ji} = \{\mathcal{M}^{ji}_{k+1}(\beta) \mid \beta, k\}, \quad \mathcal{AF}^{j,[i,i+1]} = \{\mathcal{M}^{j}_{k+1}(\beta;[i,i+1]) \mid \beta, k\}$$

for $i = 1, 2, \ldots$ such that

$$\mathcal{IAF}^{j} = \left(\{ \mathcal{AF}^{ji} \mid i = 1, 2, \dots \}, \{ \mathcal{AF}^{j,[i,i+1]} \mid i = 1, 2, \dots \} \right)$$

consist of two inductive systems of A_{∞} correspondences over L. Moreover we are given $\{\mathcal{M}_{k+1}^i(\beta;[0,1])\mid \beta,k\}$ and $\{\mathcal{M}_{k+1}(\beta;[0,1]\times[i,i+1])\mid \beta,k\}$ which consist of a pseudo-isotopy of the inductive systems \mathcal{IAF}^0 , \mathcal{IAF}^1 of A_{∞} correspondences.

(7) Let $\mathcal{AF}^j = \{\mathcal{M}_{k+1}^j(\beta) \mid \beta, k\}$ define A_{∞} correspondences over L for j = 0, 1. Let $\mathcal{AF}^{[0,1],\ell} = \{\mathcal{M}_{k+1}(\beta; [0,1]) \mid \beta, k\}, \ \ell = a, b$ be two pseudo-isotopies from \mathcal{AF}^1 to \mathcal{AF}^2 . Let

$$AF^{[0,1]\times[1,2]}$$

be a $[0,1] \times [1,2]$ -parametrized family of A_{∞} correspondences over L with the following properties.

- (a) On $[0,1] \times \{1\}$ it is isomorphic to $\mathcal{AF}^{[0,1],a}$.
- (b) On $[0,1] \times \{2\}$ it is isomorphic to $\mathcal{AF}^{[0,1],b}$.
- (c) Let j = 1 or j = 2 then on $\{j\} \times [1, 2]$ it is isomorphic to the direct product $\mathcal{AF}^j \times [1, 2]$.
- (d) At the corner $\{0,1\} \times \{1,2\}$ the isomorphisms (a)(b)(c)(d) and the various isomorphisms included in the definitions of A_{∞} correspondence and its pseudo-isotopies are compatible in the same sense as Condition 21.11 (X)(XI).
- (8) For j = 0, 1, we are given

$$\mathcal{AF}^{ji} = \{\mathcal{M}^{ji}_{k+1}(\beta) \mid \beta, k\}, \quad \mathcal{AF}^{j,[i,i+1]} = \{\mathcal{M}^{j}_{k+1}(\beta;[i,i+1]) \mid \beta, k\}$$

for $i = 1, 2, \ldots$ such that

$$\mathcal{IAF}^{j} = (\{\mathcal{AF}^{ji} \mid i = 1, 2, \dots\}, \{\mathcal{AF}^{j,[i,i+1]} \mid i = 1, 2, \dots\})$$

consist of inductive systems of A_{∞} correspondences over L. Moreover for $\ell = a, b$, we are given $\{\mathcal{M}_{k+1}^{i,\ell}(\beta; [0,1]) \mid \beta, k\}$ and $\{\mathcal{M}_{k+1}^{\ell}(\beta; [0,1] \times [i,i+1]) \mid \beta, k\}$

 β, k which consist of two pseudo-isotopies of the inductive systems \mathcal{IAF}^0 , \mathcal{IAF}^1 of A_{∞} correspondences.

Furthermore we assume that we have a pseudo-isotopy of pseudo-isotopies among these two pseudo-isotopies in the following sense: For each i we have a $[0,1]\times[i,i+1]\times[1,2]$ -parametrized family of partial A_{∞} correspondence $\mathcal{AF}^{[0,1]\times[i,i+1]\times[1,2]}$

over L of energy cut level E^i and minimal energy e_0 with the following properties.

- (i) On $[0,1] \times \{i\} \times [1,2]$, it satisfies the same condition as (7) (a)-(d) up to energy level E^i .
- (ii) On $[0,1] \times [i,i+1] \times \{c\}$ with c=1 (resp. c=2) it is isomorphic to $\{\mathcal{M}_{k+1}^a(\beta;[0,1] \times [i,i+1]) \mid \beta,k\}$ (resp. $\{\mathcal{M}_{k+1}^b(\beta;[0,1] \times [i,i+1]) \mid \beta,k\}$.)
- (iii) On $\{j\} \times [i, i+1] \times [1, 2]$ with j=0 or j=1, it is isomorphic to the direct product $\mathcal{AF}^{j,[i,i+1]} \times [1, 2]$.
- (iv) Various isomorphisms in (i)(ii)(iii) above and those appearing in the definitions of A_{∞} correspondences or its pseudo-isotopies are compatible at $S_m([0,1] \times [i,i+1] \times [1,2])$ in the same sense as Condition 21.11 (X)(XI).
- (v) A similar uniform Gromov compactness as in Definition 21.19 (3) is satisfied.

Theorem 21.35. Suppose we are given μ , E as in Situation 21.1. Let L be a compact oriented smooth manifold without boundary.

- (1) Suppose we are in Situation 21.34 (1). We can associate a gapped filtered A_{∞} structure on $\Omega(L)$. This filtered A_{∞} structure is independent of the choices made for its construction up to pseudo-isotopy.
- (2) Suppose we are in Situation 21.34 (2). We can associate a gapped partial filtered A_{∞} structure on $\Omega(L)$ of energy cut level E_0 and of minimal energy e_0 . This partial filtered A_{∞} structure is independent of the choices made for its construction up to gapped pseudo-isotopy.
- (3) Suppose we are in Situation 21.34 (3). We can associate a gapped pseudo-isotopy of filtered A_{∞} structures on $\Omega(L)$ among the two gapped filtered A_{∞} algebras which associate by (1) to $\mathcal{M}_{k+1}^j(\beta)$ for j=0,1.

In particular, it induces a gapped homotopy equivalence between those two gapped filtered A_{∞} algebras.

- (4) Suppose we are in Situation 21.34 (4). We can associate a gapped partial pseudo-isotopy of filtered A_{∞} structures on $\Omega(L)$ of energy cut level E_0 and of minimal energy e_0 among the two gapped partial filtered A_{∞} algebras which we associate by (2) to $\mathcal{M}_{k+1}^{j}(\beta)$ for j=0,1.
- (5) Suppose we are in Situation 21.34 (5). We can associate a gapped filtered A_{∞} structure on $\Omega(L)$. This gapped filtered A_{∞} structure is independent of the choices made for its construction up to gapped pseudo-isotopy.
- (6) Suppose we are in Situation 21.34 (6). We can associate a gapped pseudo-isotopy of gapped filtered A_{∞} structures on $\Omega(L)$ among the two gapped filtered A_{∞} algebras which we associate in (5) to $\mathcal{M}_{k+1}^{ji}(\beta)$ and $\mathcal{M}_{k+1}^{j}(\beta;[i,i+1])$ for j=0,1.

In particular, it induces a gapped homotopy equivalence between those two gapped filtered A_{∞} algebras.

(7) Suppose we are in Situation 21.34 (7). By (1) we obtain two gapped filtered A_{∞} algebras $(\Omega(L), \mathfrak{m}_k^j)$ for j = 1, 2. By (3) we obtain two gapped homotopy equivalences φ_a and φ_b from $(\Omega(L), \mathfrak{m}_k^1)$ to $(\Omega(L), \mathfrak{m}_k^2)$, corresponding to $\ell = a$ and $\ell = b$, respectively.

Then the claim of (7) is that φ_a is gapped homotopic to φ_b .

(8) We obtain the same conclusion as (7) in the Situation 21.34 (8).

22. Tree-like K-system: A_{∞} structure II: proof

In this section we prove Theorem 21.35. The proof is mostly parallel to the proofs given in Section 19, and also similar to those given in [FOOO4, Subsection 7.2], [Fu2].

22.1. Existence of CF-perturbations.

Definition 22.1. Let $\mathcal{M}_{k+1}(\beta)$ be the moduli spaces of the A_{∞} operations of a (partial) A_{∞} correspondence \mathcal{AC} . We use the notation of Condition 21.11. Consider the submonoid of $\mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ generated by the subset

$$\{(E(\beta), \mu(\beta)) \mid \beta \in \mathfrak{G}, \mathcal{M}_{k+1}(\beta) \neq \emptyset\}$$

of $\mathbb{R}_{\geq 0} \times 2\mathbb{Z}$. This submonoid is discrete by Condition 21.11 (VIII). We call it the discrete submonoid associated to \mathcal{AC} , and denote it by $G(\mathcal{AC})$.

When we have a partial P-parametrized family \mathcal{AC}_P of A_{∞} correspondences whose moduli spaces of P-parametrized A_{∞} operations are $\mathcal{M}_{k+1}(\beta; P)$, we define the discrete submonoid $G(\mathcal{AC}_P)$ associated to \mathcal{AC}_P as the submonoid generated by the subset

$$\{(E(\beta), \mu(\beta)) \mid \beta \in \mathfrak{G}, \mathcal{M}_{k+1}(\beta; P) \neq \emptyset\}.$$

Definition 22.2. Consider a discrete submonoid $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ in the sense of Definition 21.20.

(1) We put

$$e_{\min}(G) = \inf\{E(\beta) \mid \beta \in G, E(\beta) > 0\},\$$

if the image of $E: G \to \mathbb{R}$ is not 0. Otherwise we put $e_{\min}(G) = 1$.

(2) For $E_0, e_0 > 0$ with $e_0 \le e_{\min}(G)$ we define

$$\mathcal{GK}(G; E_0, e_0) = \{ (\beta, k) \mid \beta \in G, k \in \mathbb{Z}_{\geq 0}, \ E(\beta) = 0 \Rightarrow k > 0$$
$$E(\beta) + ke_0 \leq E_0 \}.$$

Note that if e_0 is a minimal energy of a G-gapped (partial) A_{∞} correspondence \mathcal{AC} and $G \supseteq G(\mathcal{AC})$, then $e_0 \le e_{\min}(G)$. In case $G \ne G(\mathcal{AC})$ we put $\mathcal{M}_{k+1}(\beta) = \emptyset$ for $\beta \notin G(\mathcal{AC})$ as convention.

Proposition 22.3. Let \mathcal{AC} be a partial A_{∞} correspondence of energy cut level E_0 and minimal energy e_0 , and G a discrete submonoid containing $G(\mathcal{AC})$. Suppose $e_0 \leq e_{\min}(G)$ and $0 < \tau < \tau_0 = 1$. We can find a system of τ -collared Kuranishi structures and CF-perturbations, $\{(\widehat{\mathcal{U}}_{k+1}^+(\beta), \widehat{\mathfrak{S}}_{k+1}(\beta)) \mid (\beta, k) \in \mathcal{GK}(G; E_0, e_0)\}$, with the following properties:

- (1) $\mathcal{U}_{k+1}^+(\beta)$ is a τ -collared Kuranishi structure of $\mathcal{M}_{k+1}(\beta)^{\boxplus \tau_0}$ and is a thickening of the Kuranishi structure obtained from one in Condition 21.7 (III) by trivialization of the corner. (Lemma-Definition 17.35). Evaluation maps are extended to τ -collared strongly smooth maps on this τ -collared Kuranishi structure and the associated evaluation map ev_0 is weakly submersive.
- (2) $\widehat{\mathfrak{S}}_{k+1}(\beta)$ is a τ -collared CF-perturbation of the τ -collared Kuranishi structure $\widehat{\mathcal{U}}_{k+1}^+(\beta)$. It is transversal to 0 and ev_0 is strongly submersive with respect to $\widehat{\mathfrak{S}}_{k+1}(\beta)$.
- (3) There exists an isomorphism of τ -collared K-spaces ⁴⁸

$$\partial(\mathcal{M}_{k+1}(\beta)^{\boxplus \tau_0}, \widehat{\mathcal{U}_{k+1}^+}(\beta)) \cong \coprod_{\beta_1, \beta_2, k_1, k_2, i} (-1)^{\epsilon} (\mathcal{M}_{k_1+1}(\beta_1)^{\boxplus \tau_0}, \widehat{\mathcal{U}_{k_1+1}^+}(\beta_1))$$

$${}_{\text{ev}_i} \times_{\text{ev}_0} (\mathcal{M}_{k_2+1}(\beta_2)^{\boxplus \tau_0}, \widehat{\mathcal{U}_{k_2+1}^+}(\beta_2)),$$
(22.1)

where

$$\epsilon = (k_1 - 1)(k_2 - 1) + \dim L + k_1 + (i - 1)(1 + (\mu(\beta_2) + k_2) \dim L).$$
 (22.2)

- (4) The restriction of $\widehat{\mathfrak{S}}_{k+1}(\beta)$ to the boundary is equivalent (see [Part I, Definition 7.5] for the definition of equivalence of CF-perturbations) to the fiber product of $\widehat{\mathfrak{S}}_{k_1+1}(\beta_1)$ and $\widehat{\mathfrak{S}}_{k_2+1}(\beta_2)$ under the isomorphism (22.1).
- (5) On the normalized corner $\widehat{S}_m(\mathcal{M}_{k+1}(\beta)^{\boxplus \tau_0})$, we put the Kuranishi structure that is the restriction of $\widehat{\mathcal{U}_{k+1}^+}(\beta)$. Then it is isomorphic to the disjoint union of the fiber products

$$\prod_{(\mathcal{T},\beta(\cdot))} (\mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v}))^{\boxplus \tau_0}, \widehat{\mathcal{U}_{k_{\mathbf{v}}+1}^+}(\beta(\mathbf{v}))). \tag{22.3}$$

Here the union is taken over all $(\mathcal{T}, \beta(\cdot)) \in \mathcal{G}(k+1, \beta)$ with $\#C_{1,int}(\mathcal{T}) = m$. The fiber product (22.3) is defined as in Definition 21.4. This isomorphism is compatible with the evaluation maps. It is also compatible with the embedding of the Kuranishi structures. (Here we mean the embedding of the Kuranishi structure obtained from one in Condition 21.7 (III) by trivialization of the corner to $\widehat{\mathcal{U}_{k+1}^+}(\beta)$.)

- (6) The restriction of $\widehat{\mathfrak{S}}_{k+1}(\beta)$ to $\widehat{S}_m(\mathcal{M}_{k+1}(\beta)^{\boxplus \tau_0})$ is equivalent to the fiber product of $\widehat{\mathfrak{S}}_{k_v+1}(\beta(v))$ under the isomorphism (22.3).
- (7) (5) implies that the restriction of $\mathcal{U}_{k+1}^+(\beta)$ to $\widehat{S}_{\ell}(\widehat{S}_m(\mathcal{M}_{k+1}(\beta)^{\boxplus \tau_0}))$ is a disjoint union of copies of fiber products

$$\prod_{(\mathcal{T},\beta(\cdot))} (\mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v}))^{\boxplus \tau_0}, \widehat{\mathcal{U}_{k_{\mathbf{v}}+1}^+}(\beta(\mathbf{v})))$$
 (22.4)

where the union is taken over all $(\mathcal{T}, \beta(\cdot)) \in \mathcal{G}(k+1, \beta)$ with $\#C_{1,int}(\mathcal{T}) = m + \ell$. The fiber product (22.4) is defined as in Definition 21.4.

The covering maps $\widehat{S}_{\ell}(\widehat{S}_m(\mathcal{M}_{k+1}(\beta)^{\boxplus \tau_0})) \to \widehat{S}_{m+\ell}(\mathcal{M}_{k+1}(\beta)^{\boxplus \tau_0})$ are the underlying map of the covering maps of the K-spaces (where the Kuranishi

⁴⁸See Remark 16.2 for the sign and the order of the fiber products.

structure is $\mathcal{U}_{k+1}^+(\beta)$ etc.) that is the identity map on each of the compo-

To prove Proposition 22.3 we use the following:

Lemma 22.4. If $(\mathcal{T}, \beta(\cdot)) \in \mathcal{G}(k+1, \beta)$, $(k, \beta) \in \mathcal{GK}(G; E_0, e_0)$ and $v \in C_{0,int}(\mathcal{T})$, then $(\beta(\mathbf{v}), k_{\mathbf{v}}) \in \mathcal{GK}(G; E_1, e_0)$ with $E_1 < E_0$.

Proof. We consider $\mathcal{T}\setminus\{v\}$. It has k_v+1 connected components \mathcal{T}_i , $i=0,\ldots,k$. Let ℓ be the number of its connected components that do not contain exterior vertices. If \mathcal{T}_i does not contain exterior vertices, then by Definition 21.2 (7) we have

$$\sum_{\mathbf{v}' \in C_{0,\mathrm{int}}(\mathcal{T}) \cap \mathcal{T}_i} E(\beta_{\mathbf{v}'}) > 0.$$

Therefore $\ell e_0 + E(\beta_{\rm v}) \leq E_0$. The lemma follows immediately from this fact.

Proof of Proposition 22.3. The proof is given by induction on E_0 . Lemma 22.4 implies that we already obtained $\widehat{\mathcal{U}_{k_{\mathrm{v}}+1}^+}(\beta(\mathrm{v}))$ and $\widehat{\mathfrak{S}}_{k_{\mathrm{v}}+1}(\beta_{\mathrm{v}})$ while we construct $\mathcal{U}_{k+1}^+(\beta)$ and $\widehat{\mathfrak{S}}_{k+1}(\beta)$. Here $k_{\rm v}$ and $\beta({\rm v})$ are as in (22.4). Therefore using Propositions 17.62, 17.65, we can prove Proposition 22.3 in the same way as the proof of Proposition 19.1.

Next we consider the case of *P*-parametrized family.

Situation 22.5. Suppose we have a partial P-parametrized family \mathcal{AC}_P of A_∞ correspondences whose moduli spaces of P-parametrized A_{∞} operations are $\mathcal{M}_{k+1}(\beta; P)$. Let E_0 be its energy cut level and e_0 its minimal energy. We assume that we are given the family of the pairs $\widehat{\mathcal{U}_{k+1}^+}(\beta;\widehat{S}_m(P))$, $\widehat{\mathfrak{S}}_{k+1}(\beta;\widehat{S}_m(P))$ for $m\geq 1$ and $0<\tau<\tau_0=1$ with the following properties:

- (1) $\mathcal{U}_{k+1}^+(\beta; \widehat{S}_m(P))$ is a τ -collared Kuranishi structure of $\mathcal{M}_{k+1}(\beta; \widehat{S}_m(P))^{\boxplus \tau_0}$ and is a thickening of the τ -collared Kuranishi structure of $\mathcal{M}_{k+1}(\beta; \widehat{S}_m(P))^{\boxplus \tau_0}$ that is a trivialization of the corner of the Kuranishi structure given by Condition 21.11 (III).
- (2) $\widehat{\mathfrak{S}}_{k+1}(\beta; \widehat{S}_m(P))$ is a CF-perturbation of $\widehat{\mathcal{U}_{k+1}^+}(\beta; \widehat{S}_m(P))$. (3) $\widehat{\mathfrak{S}}_{k+1}(\beta; \widehat{S}_m(P))$ is transversal to 0. The evaluation map $(\mathrm{ev}_0, \mathrm{ev}_{\widehat{S}_m(P)})$ is strongly submersive with respect to $\widehat{\mathfrak{S}}_{k+1}(\beta;\widehat{S}_m(P))$. Here $\operatorname{ev}_{\widehat{S}_{-1}(P)}$ is the restriction of ev_P to $\widehat{S}_m(P)$.
- (4) By Condition 21.11 (XI), $\widehat{S}_{\ell}(\mathcal{M}_{k+1}(\beta;\widehat{S}_m(P))^{\boxplus \tau_0})$ is isomorphic to the disjoint union of copies of

$$\prod_{(\mathcal{T},\beta(\cdot))} \mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v}); \widehat{S}_{\ell'+m}(P))^{\boxplus \tau_0}.$$
(22.5)

This fiber product is defined in the same way as (21.18), (21.19). Here the union is taken over all $(\mathcal{T}, \beta(\cdot)) \in \mathcal{G}(k+1, \beta)$ and $\ell' \in \mathbb{Z}_{>0}$ with $\#C_{1,\text{int}}(\mathcal{T})$ +

(a) The restriction of $\widehat{\mathcal{U}_{k+1}^+}(\beta; \widehat{S}_m(P))$ to $\widehat{S}_{\ell}(\mathcal{M}_{k+1}(\beta; \widehat{S}_m(P)))^{\boxplus \tau_0}$ is isomorphic to the fiber product of $\widehat{\mathcal{U}_{\ell'+k_v+1}^+}(\beta(\mathbf{v}); \widehat{S}_{\ell'+m}(P))$ on (22.5).

- (b) The isomorphism in (a) is compatible with the covering map from $\widehat{S}_{\ell}(\mathcal{M}_{k+1}(\beta;\widehat{S}_m(P)))^{\boxplus \tau_0}$ to $\mathcal{M}_{k+1}(\beta;\widehat{S}_{\ell+m}(P))^{\boxplus \tau_0}$. Namely it is induced by the covering map of the Kuranishi structures.
- (c) The restriction of the CF-perturbation $\widehat{\mathfrak{S}}_{k+1}(\beta;\widehat{S}_m(P))$ to

$$\widehat{S}_{\ell}(\mathcal{M}_{k+1}(\beta;\widehat{S}_m(P)))^{\boxplus \tau_0}$$

is equivalent to the fiber product of the CF-perturbations

$$\widehat{\mathfrak{S}}_{k_{\mathrm{v}}+1}(\beta(\mathrm{v});\widehat{S}_{\ell'+m}(P))$$

under the isomorphism in (a).

Proposition 22.6. In Situation 22.5 let G be a discrete monoid containing $G(\mathcal{AC}_P)$. Then we can find a system of Kuranishi structures and CF-perturbations

$$\{(\widehat{\mathcal{U}_{k+1}^+}(\beta;P),\widehat{\mathfrak{S}}_{k+1}(\beta;P)) \mid (\beta,k) \in \mathcal{GK}(G;E_0,e_0)\}$$

with the following properties:

- (1) $\mathcal{U}_{k+1}^+(\beta;P)$ is a τ -collared Kuranishi structure of $\mathcal{M}_{k+1}(\beta;P)^{\boxplus \tau_0}$ and is a thickening of the Kuranishi structure obtained from one in Condition 21.11 (III) by trivialization of the corner. (Lemma-Definition 17.35). The evaluation maps are extended to strongly smooth maps on this Kuranishi structure and (ev_0, ev_P) is stratumwise weakly submersive.
- (2) $\widehat{\mathfrak{S}}_{k+1}(\beta;P)$ is a CF-perturbation of the Kuranishi structure $\widehat{\mathcal{U}}_{k+1}^+(\beta;P)$. It is transversal to 0 and $(\mathrm{ev}_0,\mathrm{ev}_P)$ is stratumwise strongly submersive with respect to $\widehat{\mathfrak{S}}_{k+1}(\beta;P)$.
- (3) There exists an isomorphism of K-spaces⁴⁹

$$\partial(\mathcal{M}_{k+1}(\beta; P)^{\boxplus \tau_0}, \widehat{\mathcal{U}_{k+1}^+}(\beta; P))$$

$$\cong \coprod_{\beta_1, \beta_2, k_1, k_2, i} (-1)^{\epsilon} (\mathcal{M}_{k_1+1}(\beta_1; P)^{\boxplus \tau_0}, \widehat{\mathcal{U}_{k_1+1}^+}(\beta_1; P))$$

$$\underset{(\text{ev}_P, \text{ev}_i)}{(\text{ev}_P, \text{ev}_i)} \times_{(\text{ev}_P, \text{ev}_0)} (\mathcal{M}_{k_2+1}(\beta_2; P)^{\boxplus \tau_0}, \widehat{\mathcal{U}_{k_2+1}^+}(\beta_2; P))$$

$$\sqcup (\mathcal{M}_{k+1}(\beta; \partial P)^{\boxplus \tau_0}, \widehat{\mathcal{U}_{k+1}^+}(\beta; \partial P)), \tag{22.6}$$

where

$$\epsilon = (k_1 - 1)(k_2 - 1) + \dim L + k_1 + (i - 1)(1 + (\mu(\beta_2) + k_2 + \dim P) \dim L).$$

(4) The restriction of $\widehat{\mathfrak{S}}_{k+1}(\beta;P)$ to the boundary is equivalent to the fiber product of $\widehat{\mathfrak{S}}_{k+1}(\beta_1;P)$ and $\widehat{\mathfrak{S}}_{k+1}(\beta_2;P)$ on the first summand of the right hand side of the isomorphism (22.6). It is equivalent to $\widehat{\mathfrak{S}}_{k+1}(\beta;\widehat{S}_1(P)) = \widehat{\mathfrak{S}}_{k+1}(\beta;\partial P)$ on the second summand of the right hand side of (22.6).

⁴⁹ As we note in the footnote at Proposition 19.1 (2), we simply write ev_i in place of $\operatorname{ev}_i^{\boxplus \tau_0}$. Similarly we write ev_P in place of $\operatorname{ev}_P^{\ni \tau_0}$.

(5) On the normalized corner $\widehat{S}_m(\mathcal{M}_{k+1}(\beta; P)^{\boxplus \tau_0})$, we put the Kuranishi structure that is the restriction of $\widehat{\mathcal{U}_{k+1}^+}(\beta; P)$. Then it is isomorphic to the disjoint union of the fiber products

$$\prod_{(\mathcal{T},\beta(\cdot))} (\mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v}); \widehat{S}_{m'}(P))^{\boxplus \tau_0}, \widehat{\mathcal{U}_{k_{\mathbf{v}}+1}^+}(\beta(\mathbf{v}); \widehat{S}_{m'}(P))). \tag{22.7}$$

This fiber product is defined in the same way as (21.18), (21.19). Here the union is taken over all $(\mathcal{T}, \beta(\cdot)) \in \mathcal{G}(k+1, \beta)$ with $\#C_{1,\mathrm{int}}(\mathcal{T}) + m' = m$. This isomorphism is compatible with the evaluation maps. It is also compatible with the embedding of the Kuranishi structures. (Here we mean the embedding of the Kuranishi structures obtained from one in Condition 21.11 (III) by trivialization of the corner to $\widehat{U_{k+1}^+}(\beta; P)$.)

- (6) The restriction of $\widehat{\mathfrak{S}}_{k+1}(\beta; P)$ to $\widehat{S}_m(\mathcal{M}_{k+1}(\beta; P)^{\boxplus \tau_0})$ is equivalent to the fiber product of $\widehat{\mathfrak{S}}_{k_v+1}(\beta(v), \widehat{S}_{m'}(P))$ under the isomorphism (22.7).
- (7) (5) and Situation 22.5 imply that the restriction of $\mathcal{U}_{k+1}^+(\beta; P)$ to the iterated normalized corner $\widehat{S}_{\ell}(\widehat{S}_m(\mathcal{M}_{k+1}(\beta; P)^{\boxplus \tau_0}))$ is a disjoint union of copies of the fiber products

$$\prod_{(\mathcal{T},\beta(\cdot))} (\mathcal{M}_{k_{\mathbf{v}}+1}(\beta(\mathbf{v}); \widehat{S}_{m'}(P))^{\boxplus \tau_0}, \widehat{\mathcal{U}_{k_{\mathbf{v}}+1}^+}(\beta(\mathbf{v}); \widehat{S}_{m'}(P)))$$
(22.8)

where the union is taken over all $(\mathcal{T}, \beta(\cdot)) \in \mathcal{G}(k+1,\beta)$ with $\#C_{1,int}(\mathcal{T}) + m' = m+\ell$. The fiber product (22.8) is defined in the same way as (21.18), (21.19).

The covering map $\widehat{S}_{\ell}(\widehat{S}_m(\mathcal{M}_{k+1}(\beta;P)^{\boxplus \tau_0})) \to \widehat{S}_{m+\ell}(\mathcal{M}_{k+1}(\beta;P)^{\boxplus \tau_0})$ is the underlying map of the covering map of the K-spaces (where the Kuranishi structure is $\widehat{\mathcal{U}_{k+1}^+}(\beta;P)$ etc.) that is the identity map on each component of (22.8).

(8) If we have a uniform family of $\widehat{\mathfrak{S}}_{k+1}(\beta; \widehat{S}_m(P))$ as in Situation 22.5, then we obtain a uniform family of $\widehat{\mathfrak{S}}_{k+1}(\beta; P)$.

Proof. Using Lemma 22.4, Propositions 17.62, 17.65, we can prove Proposition 22.6 in the same way as the proof of Proposition 19.1. \Box

Remark 22.7. In Section 19 we use the \mathfrak{C}^h -partial trivialization of the corner, so the P-parametrized family after trivialization of the corner remains to be P-parametrized. Here we use the trivialization of the corner. Therefore after the trivialization of the corner we will get a $P^{\boxplus \tau_0}$ -parametrized family.

Later in the proof of Proposition 22.14, we use the collared-ness of the parametrized family in an algebraic model. To obtain the collared family, we also need to trivialize the corner in P direction.

22.2. Algebraic lemmas: promotion lemmas via pseudo-isotopy.

Definition 22.8. Let $E'_0 < E_0$.

(1) Suppose $\{\mathfrak{m}_{k,\beta}\}$ is a partial G-gapped filtered A_{∞} structure on $\Omega(L)$ of energy cut level E_0 and of minimal energy e_0 . We forget all the $\mathfrak{m}_{k,\beta}$'s with $E(\beta) > E'_0$ and obtain a partial G-gapped filtered A_{∞} structure on $\Omega(L)$ of energy cut level E'_0 . We call it the partial filtered A_{∞} structure on $\Omega(L)$ obtained by the energy cut at E'_0 .

- (2) Suppose $\{\mathfrak{m}_{k,\beta}\}$ is obtained from $\{\mathfrak{m}'_{k,\beta}\}$ by the energy cut at E'_0 and $\{\mathfrak{m}'_{k,\beta}\}$ is a partial G-gapped filtered A_{∞} structure on $\Omega(L)$ of energy cut level E_0 , we call $\{\mathfrak{m}'_{k,\beta}\}$ a promotion of $\{\mathfrak{m}_{k,\beta}\}$ to the energy cut level E_0 .
- (3) We define an energy cut or a promotion of pseudo-isotopy or a P-parametrized family of partial A_{∞} structures in the same way.

We will use the next proposition which says that we can extend the promotion of partial A_{∞} structures using pseudo-isotopy.

Proposition 22.9. Fix a discrete submonoid G and $e_0 leq e_{\min}(G)$. Let $E_0 < E_1$. For each j = 0, 1 let $\{\mathfrak{m}_{k,\beta}^j\}$ be a G-gapped partial filtered A_{∞} structure of energy cut level E_j and minimal energy e_0 on $\Omega(L)$. Suppose that we are given a G-gapped partial filtered A_{∞} pseudo-isotopy $(\{\mathfrak{m}_{k,\beta}^t\}, \{\mathfrak{c}_{k,\beta}^t\})$ of energy cut level E_0 and minimal energy e_0 , from $\{\mathfrak{m}_{k,\beta}^0\}$ to the energy cut of $\{\mathfrak{m}_{k,\beta}^1\}$ at E_0 . Then we can promote $\{\mathfrak{m}_{k,\beta}^0\}$ to energy cut level E_1 and $(\{\mathfrak{m}_{k,\beta}^t\}, \{\mathfrak{c}_{k,\beta}^t\})$ to energy cut level E_1 .

Proof. The proof is the same as the proof of [Fu2, Theorem 8.1]. (The only difference is the following point: In [Fu2, Theorem 8.1] partial structures are ones where we take only finitely many β 's: Here we take finitely many (β, k) 's.) We repeat the proof for completeness.

We consider the set

$$\mathfrak{E} = \{ E(\beta) + ke_0 \mid (\beta, k) \in G \times \mathbb{Z}_{\geq 0} \}.$$

This is a discrete set. So by applying an induction we may and will assume that

$$\#(\mathfrak{E} \cap (E_0, E_1]) = 1.$$
 (22.9)

Let $(\beta, k) \in G \times \mathbb{Z}_{\geq 0}$ such that $E(\beta) + ke_0 \in (E_0, E_1]$. We will define $\mathfrak{m}_{k,\beta}^t$ and $\mathfrak{c}_{k,\beta}^t$ for each such (β, k) .

We put $\mathfrak{c}_{k,\beta}^t = 0$. Then there exists a unique $\mathfrak{m}_{k,\beta}^t$ such that it satisfies (21.27) and $\mathfrak{m}_{k,\beta}^t = \mathfrak{m}_{k,\beta}^1$ for t = 1. Note that (21.27) can be regarded as an ordinary differential equation for each fixed (h_1, \ldots, h_k) . Therefore $\mathfrak{m}_{k,\beta}^t$ depends smoothly on t and is local in the [0,1]-direction.

Next we check the A_{∞} relation for each fixed t in the case of (β, k) . We calculate

$$\frac{d}{dt} \sum_{k_1+k_2=k+1} \sum_{\beta_1+\beta_2=\beta} \sum_{i=1}^{k-k_2+1} \mathfrak{m}_{k_1,\beta_1}^t(h_1,\ldots,\mathfrak{m}_{k_2,\beta_2}^t(h_i,\ldots),\ldots,h_k)$$

$$= \sum_{k_1+k_2=k+1} \sum_{\beta_1+\beta_2=\beta} \sum_{i=1}^{k-k_2+1} \frac{d\mathfrak{m}_{k_1,\beta_1}^t}{dt}(h_1,\ldots,\mathfrak{m}_{k_2,\beta_2}^t(h_i,\ldots),\ldots,h_k)$$

$$+ \sum_{k_1+k_2=k+1} \sum_{\beta_1+\beta_2=\beta} \sum_{i=1}^{k-k_2+1} \mathfrak{m}_{k_1,\beta_1}^t(h_1,\ldots,\frac{d\mathfrak{m}_{k_2,\beta_2}^t}{dt}(h_i,\ldots),\ldots,h_k).$$
(22.10)

Using (21.27) and the A_{∞} relation, (that is, the induction hypothesis), it is easy to see that (22.10) is zero. We define $\mathfrak{m}_{k,\beta}^0$ as the case t=0 of $\mathfrak{m}_{k,\beta}^t$. The proof of Proposition 22.9 is complete.

We next study the promotion of pseudo-isotopy using pseudo-isotopy of pseudo-isotopies. The proof is similar to that of [Fu2, Theorem 14.1]. We repeat the detail of the proof for completeness.

We first define the notion of a $P^{\boxplus \tau}$ -parametrized (partial) A_{∞} structure to be collared. Here $P^{\boxplus \tau}$ is the trivialization of the corner of our manifold with corner P. (A similar assumption appeared in [Fu2, Assumption 14.1].)

Definition 22.10. Let $\{\mathfrak{m}_{k,\beta}^{P^{\boxplus \tau}}\}$ be a $P^{\boxplus \tau}$ -parametrized partial A_{∞} -structure of energy cut level E_0 and minimal energy e_0 . We say it is τ -collared if the following conditions (1)(2) are satisfied.

Let $\mathbf{t} \in \overset{\circ}{S}_k(P^{\boxplus \tau})$. Its τ -collared neighborhood is identified with $V \times [-\tau, 0)^k$. Let (t'_1, \dots, t'_m) be a coordinate of V and let (t''_1, \dots, t''_k) be the standard coordinate of $[-\tau,0)^k$. A differential form on P in a neighborhood is written as $\sum f_{I'I''}dt'_{I'} \wedge dt''_{I''}$ where $dt'_{I'}$ are wedge products of $dt'_{i'}$'s, and $dt''_{I''}$ are wedges products of $dt''_{i'}$'s.

By definition, $\mathfrak{m}_{k,\beta}^{P^{\boxplus \tau}}$ is written on this neighborhood as the form

$$\mathfrak{m}_{k,\beta}^{P^{\boxplus \tau}}(h_1,\ldots,h_k) = \sum_{I,I'} dt'_{I'} \wedge dt''_{I''} \wedge \mathfrak{m}_{k,\beta;I',I''}^{t',t''}(h_1,\ldots,h_k).$$

Now we require:

- (1) $\mathfrak{m}_{k,\beta;I',I''}^{t',t''}(h_1,\ldots,h_k) = 0 \text{ unless } I'' = \emptyset.$ (2) If $I'' = \emptyset$, $\mathfrak{m}_{k,\beta;I',\emptyset}^{t',t''}(h_1,\ldots,h_k)$ is independent of $t'' \in [-\tau,0)^k$.

We say a P-parametrized partial A_{∞} structure is collared if there exist $\tau > 0$ and P' such that $P = P'^{\boxplus \tau}$

Example 22.11. The case when P = [0, 1] in Definition 22.10 is nothing but the case of pseudo-isotopy in Definition 21.25. In this case, $P^{\boxplus \tau} = [-\tau, 1+\tau]$ and the τ -collared-ness property (1), (2) in Definition 22.10 implies the following property of pseudo-isotopy ($\{\mathfrak{m}_{k,\beta}^t\}, \{\mathfrak{c}_{k,\beta}^t\}$), respectively:

- $\begin{array}{l} (1) \ \, \mathfrak{c}_{k,\beta}^t = 0 \ \, \text{for} \, \, t \in [-\tau,0] \cup [1,1+\tau]. \\ (2) \ \, \frac{d}{dt} \mathfrak{m}_{k,\beta}^t = 0 \ \, \text{for} \, \, t \in [-\tau,0] \cup [1,1+\tau]. \end{array}$

Situation 22.12. Let P be a manifold with corner and $E_1 > E_0 \ge 0$, $e_0 > 0$. We assume that we are given the following objects.

- (1) A $P \times [0, 1]$ -parametrized collared partial A_{∞} structure $\{\mathfrak{m}_{k,\beta}^{P \times [0,1]}\}$ of energy
- cut level E_0 and of minimal energy e_0 on $\Omega(L)$. (2) A collared promotion of the restriction of $\{\mathfrak{m}_{k,\beta}^{P\times[0,1]}\}$ to $P\times\{1\}$ to energy cut level E_1 .
- (3) Let $\partial P = \prod \partial_i P$ be the decomposition of the normal boundary of P into the connected components. Then we also assume that a collared promotion of the restriction of $\{\mathfrak{m}_{k,\beta}^{P\times[0,1]}\}$ to $\partial_i P\times[0,1]$ to energy cut level E_1 is given for each i.
- (4) We assume that the restriction of the promotion in (2) coincides with the promotion in (3) on $\partial_i P \times \{1\}$.
- (5) Suppose that the images of $\partial_i P$ and $\partial_j P$ intersect each other in P at the component $\partial_{ij}P$ of the codimension 2 corner of P. (Note that the case i=j is included. In this case, $\partial_{ii}P$ is the 'self intersection' of $\partial_{i}P$.) Then we assume that the promotions of the restrictions on $\partial_i P \times [0,1]$ and on $\partial_j P \times [0,1]$ in (3) coincide with each other on $\partial_{ij} P \times [0,1]$.

Remark 22.13. In Situation 22.12 we assumed the compatibility of the promotion only at the codimension 2 corners. In this situation it automatically implies that they coincide at higher codimensional corners. This is because our assumptions are their exact coincidence and not coincidence up to certain equivalence relation.

Proposition 22.14. In Situation 22.12 there exists a promotion of $\{\mathfrak{m}_{k,\beta}^{P\times[0,1]}\}$ to energy cut level E_1 such that the promotion coincides with those given in Situation 22.12 (2) (resp. (3)) on $P \times \{1\}$ (resp. $\partial_i P \times [0,1]$).

Proof. We first prove Proposition 22.14 for the case $P = [-\tau, 1 + \tau]$. We regard $P \times [0,1]$ as $P \times [-\tau,1+\tau] = ([0,1]^2)^{\boxplus \tau}$ and assume that our structures are τ -

We change the corner structure of $([0,1]^2)^{\boxplus \tau}$ so that we smooth the corners at two points $(-\tau, 1+\tau)$, $(1+\tau, 1+\tau)$ and make two points $(-\tau, -\tau/4)$, $(1+\tau, -\tau/4)$ into new corners instead. We leave two other corners $(-\tau, -\tau)$, $(1+\tau, -\tau)$ as corners. We then get a new cornered 2 manifold Q diffeomorphic to $[-\tau, 1+\tau]^2$. We denote this diffeomorphism by $F: [-\tau, 1+\tau]^2 \to Q$. The diffeomorphism F is different from the set theoretical identity map id: $[-\tau, 1+\tau]^2 \to Q$. In fact, we can take F satisfying the following properties. See Figure 25.

- (1) $F(-\tau, -\tau/4) = (-\tau, 1+\tau)$ and $F(1+\tau, -\tau/4) = (1+\tau, 1+\tau)$.
- (2) F is identity on the edge $[-\tau, 1+\tau] \times \{-\tau\}$.

Using the τ -collared-ness, our structures give a Q-parametrized family of partial A_{∞} structures. We regard it as a $[-\tau, 1+\tau]^2$ -parametrized family of partial A_{∞} structures under the diffeomorphism F. By assumption, its energy cut level is E_0 and the energy cut level of its restriction to $[-\tau, 1+\tau] \times \{1+\tau\}$ is E_1 . Therefore we can apply Proposition 22.9 to promote this $[-\tau, 1+\tau]^2$ -parametrized family to energy cut level E_1 . Using collared-ness again, we find that on $\partial [-\tau, 1+\tau] \times$ $[-\tau, 1+\tau] \subset [-\tau, 1+\tau]^2 \stackrel{F}{\cong} Q$ this promotion coincides with the structure of energy cut level E_1 given at the beginning.

Now we identify $Q \stackrel{\text{id}}{\cong} [-\tau, 1+\tau]^2$. At the place where we smooth corners or make new corners, we can use the τ -collared-ness to show that the promotion coincides with the one originally given at the beginning. Thus we obtain the required promotion. Note the structure obtained is τ' -collared for some $0 < \tau' < \tau$ by construction.

Thus we have proved Proposition 22.14 for the case P = [0, 1]. (We use only this case in this book.)

The general case can be proved in a similar way. Namely we smooth some of corners of $(P \times [0,1])^{\boxplus \tau}$ and make certain points into new corners to obtain a cornered manifold Q so that the following holds.

- (1) There exists a diffeomorphism $F: (P \times [0,1])^{\boxplus \tau} \to Q$ which is identity on $P^{\boxplus \tau} \times \{-\tau\}.$ (2) $\{\mathfrak{m}_{k,\beta}^{P \times [0,1]}\}$ induces a Q parameter family of partial A_{∞} structures.

By the diffeomorphism in (2), the set $P^{\boxplus \tau} \times \{1+\tau\} \subset (P \times [0,1])^{\boxplus \tau} = Q$, where the last equality is the set-theoretical one, is mapped from a subset of $(\partial P^{\boxplus \tau} \times [-\tau, 1 +$ $\tau]) \cup (P^{\boxplus \tau} \times \{1 + \tau\})$. Therefore the partial structure in (2) is one of energy cut level E_1 on $P^{\boxplus \tau} \times \{1+\tau\}$. On $(P \times [0,1])^{\boxplus \tau}$ it is of energy cut level E_0 . We apply Proposition 22.9 to promote it to the energy cut level E_1 . Using the diffeomorphism

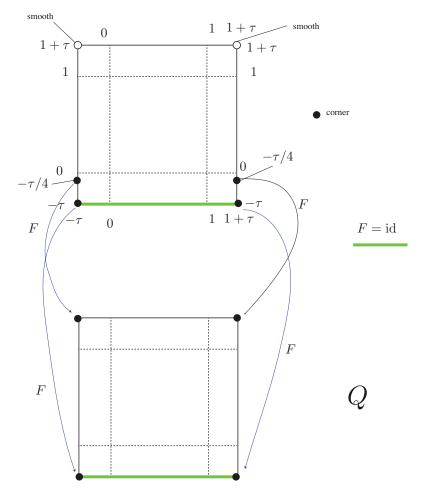


Figure 25. Q and F

F, we regard it as a $(P \times [0,1])^{\boxplus \tau}$ -parametrized structure. Using collared-ness, we find that it induces a $(P \times [0,1])$ -parametrized structure via the identity map. (Note that identity map $(P \times [0,1])^{\boxplus \tau} \cong Q$ is not a diffeomorphism. However the structures are constant at the place where differentiability breaks down.)

Thus we have obtained the required promotion.

22.3. Pointwise-ness of parametrized family of smooth correspondences. In this subsection we prove Proposition 22.18 which reads that the operation defined as a smooth correspondence associated to a P-parametrized family of A_{∞} correspondences is pointwise in P direction in the sense of Definition 21.27. To state the result in the way we can utilize in similar but different situations, we slightly generalize Definition 21.27. We use the notation t_J etc. of Definition 21.27 in the next definition.

Definition 22.15. Let M_s , M_t be smooth manifolds (without boundary) and P a smooth manifold with corner. A linear map $F: \Omega(P \times M_s) \to \Omega(P \times M_t)$ is said to be *pointwise in* P *direction* if the following holds:

For each $I \subset \{1, ..., d\}$ and $\mathbf{t} \in P$ there exists a linear and continuous map $F_{I:J}^{\mathbf{t}}: \Omega(M_s) \to \Omega(M_t)$ such that

$$F(dt_J \wedge h)|_{\{\mathbf{t}\} \times M_t} = \sum_I dt_I \wedge dt_J \wedge F_{I;J}^{\mathbf{t}}(h|_{\{\mathbf{t}\} \times M_t}). \tag{22.11}$$

Moreover $F_{I;J}^{\mathbf{t}}$ depends smoothly on \mathbf{t} (with respect to the operator topology) and is independent of J up to sign.

In case $M_s = L^k$ and $M_t = L$, Definition 22.15 is nothing but Definition 21.27.

Situation 22.16. Let $(X,\widehat{\mathcal{U}})$ be a K-space, and let M_s , M_t be smooth manifolds without boundary and P a smooth manifold with corners. Let $f_s:(X,\widehat{\mathcal{U}})\to M_s$, $f_t:(X,\widehat{\mathcal{U}})\to M_t$ and $f_P:(X,\widehat{\mathcal{U}})\to P$ be strongly smooth maps. We assume that $(f_t,f_P):(X,\widehat{\mathcal{U}})\to P\times M$ is stratumwise weakly submersive.

Let $\widehat{\mathfrak{S}}$ be a CF-perturbation of $(X,\widehat{\mathcal{U}})$. We assume that (f_t, f_P) is stratumwise strongly submersive with respect to $\widehat{\mathfrak{S}}$.

Definition 22.17. We call $\mathfrak{X}_P = ((X, \widehat{\mathcal{U}}), f_s, f_t, f_P)$ as in Situation 22.16 a P parametrized family of smooth correspondences.

Let $\widehat{\mathfrak{S}}$ be a CF-perturbation such that (f_t, f_P) is strongly submersive with respect to $\widehat{\mathfrak{S}}$. Then for any $\epsilon > 0$ we associate a linear map

$$\operatorname{Corr}_{\mathfrak{X}_P}(\cdot;\widehat{\mathfrak{S}}^{\epsilon}) : \Omega(P \times M_s) \longrightarrow \Omega(P \times M_t)$$

by

$$\operatorname{Corr}_{\mathfrak{X}_P}(h;\widehat{\mathfrak{S}}^{\epsilon}) = (f_P, f_t)!((f_P, f_s)^*h; \widehat{\mathfrak{S}}^{\epsilon}). \tag{22.12}$$

Then we have

Proposition 22.18. The map $Corr_{\mathfrak{X}_P}(\cdot;\widehat{\mathfrak{S}}^{\epsilon})$ is pointwise in P direction.

Proof. Let $h \in \Omega(M_s)$. We put

$$(f_P, f_t)!((f_s)^*h; \widehat{\mathfrak{S}}^{\epsilon}) = \sum_I dt_I \wedge F_I(h).$$

Let $F_I^{\mathbf{t}}(h)$ be the restriction of $F_I(h)$ to $\{\mathbf{t}\} \times M_t$. Then it is easy to see that this $F_I^{\mathbf{t}}$ satisfies (22.11) up to sign.

22.4. **Proof of Theorem 21.35.** In this subsection, we complete the proof of Theorem 21.35.

Proof of Theorem 21.35 (2). Suppose $\mathcal{AF} = \{\mathcal{M}_{k+1}(\beta) \mid \beta, k\}$ defines a partial A_{∞} correspondence over L of energy cut level E_0 and minimal energy e_0 . Let G a discrete submonoid containing the discrete submonoid $G(\mathcal{AC})$ in Definition 22.1.

Remark 22.19. To prove Theorem 21.35 (2) itself, it suffices to take $G = G(\mathcal{AC})$. However we may also take G which is strictly bigger than $G(\mathcal{AC})$. We may replace e_0 by a smaller one.

We apply Proposition 22.3 and find a system of τ -collared Kuranishi structures and CF-perturbations, $\{(\widehat{\mathcal{U}_{k+1}^+}(\beta), \widehat{\mathfrak{S}}_{k+1}(\beta)) \mid (\beta, k) \in \mathcal{GK}(G; E_0, e_0)\}$. We regard

$$\left(\left(\mathcal{M}_{k+1}(\beta)^{\boxplus \tau_0},\widehat{\mathcal{U}_{k+1}^+}(\beta)\right);(\mathrm{ev}_1,\ldots,\mathrm{ev}_k),\mathrm{ev}_0\right)$$

as a smooth correspondence from L^k to L and write it as $\mathfrak{M}_{k+1}(\beta)$. We now define:

$$\mathfrak{m}_{k,\beta}^{\epsilon}(h_1,\ldots,h_k) := \operatorname{Corr}_{\mathfrak{M}_{k+1}(\beta)} \left(h_1 \times \cdots \times h_k; \widehat{\mathfrak{S}}_{k+1}^{\epsilon}(\beta) \right).$$
 (22.13)

Here and hereafter we denote

$$h_1 \times \cdots \times h_k := \pi_1^* h_1 \wedge \cdots \wedge \pi_k^* h_k$$

where $\pi_i: L^k \to L$ is the *i*-th projection. By Stokes' formula ([Part I, Theorem 9.26]) we have

$$(d \circ \mathfrak{m}_{k,\beta}^{\epsilon})(h_1, \dots, h_k) \pm (\mathfrak{m}_{k,\beta}^{\epsilon} \circ d)(h_1, \dots, h_k)$$

= $\operatorname{Corr}_{\partial \mathfrak{M}_{k+1}(\beta)} (h_1 \times \dots \times h_k; \widehat{\mathfrak{S}}_{k+1}^{\epsilon}(\beta))$.

We recall (22.1), that is,

$$\partial(\mathcal{M}_{k+1}(\beta)^{\boxplus \tau_0}, \widehat{\mathcal{U}_{k+1}^+}(\beta)) \cong \coprod_{\beta_1, \beta_2, k_1, k_2, i} (-1)^{\epsilon} (\mathcal{M}_{k_1+1}(\beta_1)^{\boxplus \tau_0}, \widehat{\mathcal{U}_{k_1+1}^+}(\beta_1))$$

$$\underset{ev_i}{\underset{}_{}} \times_{ev_0} (\mathcal{M}_{k_2+1}(\beta_2)^{\boxplus \tau_0}, \widehat{\mathcal{U}_{k_2+1}^+}(\beta_2)),$$
(22.14)

where

$$\epsilon = (k_1 - 1)(k_2 - 1) + \dim L + k_1 + (i - 1)(1 + (\mu(\beta_2) + k_2) \dim L).$$
 (22.15)

We denote by $\mathfrak{M}_{k_1,k_2,i}(\beta_1,\beta_2)$ the component corresponding to $\beta_1,\beta_2,k_1,k_2,i$ in the right hand side together with evaluation maps. Note that the evaluation maps to the source of the left hand side restrict to the evaluation maps to the source of either the first or the second fiber product factor of the right hand side. So our situation is (very slightly) different from one of the composition formula [Part I, Theorem 10.20]. However we can apply [Part I, Proposition 10.23] instead by putting

$$(X_{1}, \widehat{\mathcal{U}}_{1}, \widehat{\mathfrak{S}}_{1}, \widehat{f}_{1}) = \left(\mathcal{M}_{k_{1}+1}(\beta_{1})^{\boxplus \tau_{0}}, \widehat{\mathcal{U}}_{k_{1}+1}^{+}(\beta_{1}), \widehat{\mathfrak{S}}_{k_{1}+1}^{\epsilon}(\beta_{1}), \operatorname{ev}_{0}\right),$$

$$(X_{2}, \widehat{\mathcal{U}}_{2}, \widehat{\mathfrak{S}}_{2}, \widehat{f}_{2}) = \left(\mathcal{M}_{k_{2}+1}(\beta_{2})^{\boxplus \tau_{0}}, \widehat{\mathcal{U}}_{k_{2}+1}^{+}(\beta_{2}), \widehat{\mathfrak{S}}_{k_{2}+1}^{\epsilon}(\beta_{2}), \operatorname{ev}_{i}\right),$$

$$\widehat{h}_{1} = (\operatorname{ev}_{1}, \dots, \operatorname{ev}_{k_{1}})^{*}(h_{i} \times \dots \times h_{i+k_{1}-1}),$$

$$\widehat{h}_{2} = (\operatorname{ev}_{1}, \dots, \operatorname{ev}_{i-1}, \operatorname{ev}_{i+1}, \dots, \operatorname{ev}_{k_{1}}, \operatorname{ev}_{0})^{*}$$

$$(h_{1} \times \dots \times h_{i-1} \times h_{i+k_{1}} \times \dots \times h_{k_{2}} \times h_{0}).$$

Then [Part I, (10.14)] and (22.1)=(22.14) imply

$$\int_{L} \operatorname{Corr}_{\partial \mathfrak{M}_{k+1}(\beta)} \left(h_{1} \times \cdots \times h_{k}; \widehat{\mathfrak{S}}_{k+1}^{\epsilon}(\beta) \right) \wedge h_{0}$$

$$= \sum_{\beta_{1}, \beta_{2}, k_{1}, k_{2}, i} \operatorname{Corr}_{\mathfrak{M}_{k_{2}+1}(\beta_{2})} \left(\diamond; \widehat{\mathfrak{S}}_{k_{2}+1}^{\epsilon}(\beta_{2}) \right) \wedge h_{0}, \tag{22.16}$$

where

$$\diamond = h_1 \times \cdots \times h_{i-1} \times \times \operatorname{Corr}_{\mathfrak{M}_{k_1+1}(\beta_1)} \left(h_i \times \cdots \times h_{i+k_1-1}; \widehat{\mathfrak{S}}_{k_1+1}^{\epsilon}(\beta_1) \right) \times h_{i+k_1} \times \cdots \times h_{k_2}.$$

(22.16) implies that $\{\mathfrak{m}_{k,\beta}^{\epsilon}\}$ defines a G-gapped partial A_{∞} -structure of energy loss E_0 and minimal energy e_0 .

Thus we have constructed the required partial A_{∞} -correspondence. Its well-definedness up to pseudo-isotopy will have been proved if Theorem 21.35 (4) is proved.

Remark 22.20. In the formulation of this article, we do not perturb \mathfrak{m}_{2,β_0} . (Recall $\beta_0=0$.) Namely \mathfrak{m}_{2,β_0} coincides with the wedge product up to sign and $\mathfrak{m}_{k,\beta_0}=0$ for $k\geq 3$. We take $\widehat{\mathfrak{S}}_{2+1}^{\epsilon}(\beta_0)$ as the trivial perturbation. Note that $\mathcal{M}_{2+1}(\beta_0)=L$ and he evaluation map $\mathrm{ev}_0:\mathcal{M}_{2+1}(\beta_0)\to L$ is the identity map (that is a submersion). So we do not need to perturb it. We also take $\mathcal{M}_{k+1}(\beta_0)=L\times D^{k-2}$, where we identify D^{k-2} with the Stasheff cell. Note that $\mathrm{ev}_0:\mathcal{M}_{k+1}(\beta_0)\to L$ factors through the projection $L\times D^{k-2}\to L$ whose fiber is of positive dimension. Therefore this smooth correspondence induces the zero map.

We can proceed in a different way and perturb $\mathcal{M}_{2+1}(\beta_0)$ so that \mathfrak{m}_{2,β_0} has a smooth Schwartz kernel. Then we necessarily include nonzero $\mathfrak{m}_{k+1,\beta_0}$ for k>2. We need to take such a choice of perturbation to generalize our story to the case of bordered Riemann surfaces of higher genus and/or those which have more than one boundary components, because the corresponding moduli space of constant maps is not transversal.

Proof of Theorem 21.35 (4). We are given two partial A_{∞} correspondences over L $\mathcal{AF}^j = \{\mathcal{M}_{k+1}^j(\beta) \mid \beta, k\}$ (j=0,1), and a pseudo-isotopy $\mathcal{AF}^{[0,1]} = \{\mathcal{M}_{k+1}(\beta; [0,1]) \mid \beta, k\}$ between them. We assume that both of their energy cut levels are E_0 and minimal energies are e_0 . Let G be a discrete submonoid containing both $G(\mathcal{AC}^j)$ for j=0,1. We can make such a choice by Remark 21.18. We also assume that G contains $G(\mathcal{AF}^{[0,1]})$ and $e_0 \leq e_{\min}(G)$.

We assume that we have obtained partial filtered A_{∞} structures

$$\{\mathfrak{m}_{k,\beta}^{j,\epsilon_j} \mid (k,\beta) \in \mathcal{GK}(G; E_0, e_0)\}$$

associated with the partial A_{∞} correspondences \mathcal{AF}^{j} given in the proof of Theorem 21.35 (2) above. It means that we have taken a system of

$$\{(\widehat{\mathcal{U}_{k+1}^{j+}}(\beta),\widehat{\mathfrak{S}}_{k+1}^{j}(\beta))\mid (\beta,k)\in\mathcal{GK}(G;E_0,e_0)\},$$

where $\widehat{\mathcal{U}_{k+1}^{j+}}(\beta)$ is a τ -collared Kuranishi structure on $\mathcal{M}_{k+1}^{j}(\beta)^{\boxplus \tau_0}$ and $\widehat{\mathfrak{S}}_{k+1}^{j}(\beta)$ is a CF-perturbation of $\widehat{\mathcal{U}_{k+1}^{j+}}(\beta)$ such that they satisfy (22.14). Here recall that τ and τ_0 satisfies the following inequality:

$$0 < \tau < \tau_0 = 1.$$

Now we apply Proposition 22.6 to $P = [0, 1]^{\boxplus (\tau_0 - \tau)}$ (then $P^{\boxplus \tau} = [0, 1]^{\boxplus \tau_0}$) to obtain objects

$$\widehat{\mathcal{U}_{k+1}^+}(\beta;[0,1]), \quad \widehat{\mathfrak{S}}_{\rho,k+1}(\beta;[0,1])$$

with $\rho \in (0,1]$ described below. Firstly, $\widehat{\mathcal{U}_{k+1}^+}(\beta;[0,1])$ is a τ -collared Kuranishi structure on $\mathcal{M}_{k+1}(\beta;[0,1])^{\boxplus \tau_0}$ with the following properties.

Property 22.21. (1) Its restriction to
$$\mathcal{M}_{k+1}^{j}(\beta)^{\boxplus \tau_0} \subset \partial(\mathcal{M}_{k+1}(\beta; [0,1])^{\boxplus \tau_0})$$
 is $\widehat{\mathcal{U}_{k+1}^{j+}}(\beta)$.

(2) Its restriction to $\mathcal{M}_{k_1+1}(\beta_1; [0,1])^{\boxplus \tau_0} {}_{(ev_0, ev_{[0,1]})} \times_{(ev_i, ev_{[0,1]})} \mathcal{M}_{k_2+1}(\beta_2; [0,1])^{\boxplus \tau_0}$ is $\widehat{\mathcal{U}_{k_1+1}^+}(\beta_1;[0,1])_{(\mathrm{ev}_0,\mathrm{ev}_{[0,1]})} \times_{(\mathrm{ev}_i,\mathrm{ev}_{[0,1]})} \widehat{\mathcal{U}_{k_2+1}^+}(\beta_2;[0,1]).$

Note that $\mathcal{U}_{k+1}^{j+}(\beta)$ is a $[0,1]^{\boxplus \tau_0} = [-\tau_0, 1+\tau_0]$ -parametrized family.

The τ -collared Kuranishi structure $\widehat{\mathcal{U}_{k+1}^+}(\beta;[0,1])$ also satisfies the compatibility conditions at the corner. However, we do not describe them here since they are special cases of the statement of Proposition 22.6 and we do not use them below directly.

Secondly, $\widehat{\mathfrak{S}}_{\rho,k+1}(\beta;[0,1])$ is a family of CF-perturbations of $\widehat{\mathcal{U}_{k+1}^+}(\beta;[0,1])$ parametrized by $\rho \in (0,1]$ with the following properties.

(1) (a) Its restriction to $\mathcal{M}_{k+1}^{j}(\beta)^{\boxplus \tau_0} \subset \partial \mathcal{M}_{k+1}(\beta; [0,1])^{\boxplus \tau_0}$ Property 22.22. with j = 0 is $\widehat{\mathfrak{S}}_{k+1}^0(\beta)$.

- (b) Its restriction to $\mathcal{M}_{k+1}^{j}(\beta)^{\boxplus \tau_0} \subset \partial \mathcal{M}_{k+1}(\beta;[0,1])^{\boxplus \tau_0}$ with j=1 is $\epsilon \mapsto \widehat{\mathfrak{S}}_{k+1}^{1\rho\epsilon}(\beta).$
- (2) Its restriction to $\mathcal{M}_{k_1+1}(\beta_1; [0,1])^{\boxplus \tau_0} {}_{(\mathrm{ev}_0,\mathrm{ev}_{[0,1]})} \times_{(\mathrm{ev}_i,\mathrm{ev}_{[0,1]})} \mathcal{M}_{k_2+1}(\beta_2; [0,1])^{\boxplus \tau_0}$ is $\widehat{\mathfrak{S}}_{\rho,k_1+1}(\beta_1;[0,1])_{(\mathrm{ev}_0,\mathrm{ev}_{[0,1]})} \times_{(\mathrm{ev}_i,\mathrm{ev}_{[0,1]})} \widehat{\mathfrak{S}}_{\rho,k_2+1}(\beta_2;[0,1]).$ (3) It is transversal to 0. The map $(\mathrm{ev}_0,\mathrm{ev}_{[0,1]})$ is strongly submersive with
- respect to $\widehat{\mathfrak{S}}_{\rho,k+1}(\beta;[0,1])$.
- (4) $\{\widehat{\mathfrak{S}}_{\rho,k+1}(\beta;[0,1]) \mid \rho \in (0,1]\}$ is a uniform family.

We regard

$$\left(\left(\mathcal{M}_{k+1}(\beta)^{\boxplus \tau_0}, \widehat{\mathcal{U}_{k+1}^+}(\beta; [0,1])\right); \left((\mathrm{ev}_1, \mathrm{ev}_{[0,1]}), \ldots, (\mathrm{ev}_k, \mathrm{ev}_{[0,1]})\right), \left(\mathrm{ev}_0, \mathrm{ev}_{[0,1]}\right)\right)$$

together with $\widehat{\mathfrak{S}}_{\rho,k+1}(\beta;[0,1])$ as a smooth correspondence from $(L\times[0,1]^{\boxplus\tau_0})^k$ to $L \times [0,1]^{\boxplus \tau_0}$ and write it as

$$\mathfrak{M}_{\rho,k+1}(\beta;[0,1]^{\boxplus \tau_0}).$$

Now for differential forms h_1, \ldots, h_k on $L \times [0,1]^{\boxplus \tau_0}$ we put

$$\mathfrak{m}_{k,\beta}^{\rho,[0,1]^{\boxplus \tau_0}}(h_1,\ldots,h_k) = \operatorname{Corr}_{\mathfrak{M}_{\rho,k+1}(\beta;[0,1]^{\boxplus \tau_0})}\left(h_1,\ldots,h_k;\widehat{\mathfrak{S}}_{\rho,k+1}^{\epsilon}(\beta;[0,1])\right).$$

Using Property 22.22 (2) and Stokes' formula (Theorem 26.12) in the same way as in the proof of Theorem 21.35 (2), we can prove

$$\sum_{k_1+k_2=k+1} \sum_{\beta_1+\beta_2=\beta} \sum_{i=1}^{k-k_2+1}$$

$$(-1)^* \mathfrak{m}_{k_1,\beta_1}^{\epsilon,\rho,[0,1]^{\boxplus \tau_0}} (h_1, \dots, \mathfrak{m}_{k_2,\beta_2}^{\epsilon,\rho,[0,1]^{\boxplus \tau_0}} (h_i, \dots, h_{i+k_2-1}), \dots, h_k) = 0,$$

where $* = \deg' h_1 + \ldots + \deg' h_{i-1}$. By Proposition 22.18, $\mathfrak{m}_{k,\beta}^{\epsilon,\rho,[0,1]^{\boxplus \tau_0}}$ is pointwise in $[0,1]^{\boxplus \tau_0}$ -direction. Moreover, Property 22.22 (1) (a) implies that the restriction of $\mathfrak{m}_{k,\beta}^{\epsilon,\rho,[0,1]^{\boxplus \tau_0}}(h_1,\ldots,h_k)$ to $L \times \{-\tau\}$ is $\mathfrak{m}_{k,\beta}^{0,\epsilon}(h_1,\ldots,h_k)$ and Property 22.22 (1) (b) implies that the restriction of $\mathfrak{m}_{k,\beta}^{\epsilon,\rho,[0,1]}(h_1,\ldots,h_k)$ to $L\times\{1+\tau\}$ is $\mathfrak{m}_{k,\beta}^{1,\epsilon\rho}(h_1,\ldots,h_k)$. Therefore $\mathfrak{m}_{k,\beta}^{\epsilon,\rho,[0,1]^{\boxplus\tau_0}}$ gives a partial A_{∞} pseudo-isotopy between $\mathfrak{m}_{k,\beta}^{0,\epsilon}$ and $\mathfrak{m}_{k,\beta}^{1,\epsilon\rho}$.

Remark 22.23. We introduced the parameter ρ and constructed a pseudo-isotopy between $\mathfrak{m}_{k,\beta}^{\epsilon,\rho,[0,1]^{\boxplus \tau_0}}$ and $\mathfrak{m}_{k,\beta}^{1,\epsilon\rho}$ since we need it in the inductive construction of a filtered A_{∞} structure associated to an inductive system of A_{∞} correspondences. See Remark 19.17.

Proof of Theorem 21.35 (5). Recall from Definition 21.17 that we have a divergent sequence $\{E^i\}_i$ with

$$0 < \dots < E^i < E^{i+1} < \dots \to +\infty$$

of energy cut levels in the definition of the inductive system. By (2) there exists a filtered A_{∞} structure of energy cut level E^i and minimal energy e_0 on $\Omega(L)$ induced

$$\mathcal{AF}^{i} = \{ \mathcal{M}_{k+1}^{i}(\beta) \mid \beta, k \}.$$

We denote it by $\{\mathfrak{m}_k^i\}$. By (4) there exists a partial pseudo-isotopy from $\{\mathfrak{m}_k^i\}$ to $\{\mathfrak{m}_k^{i+1}\}$. Its energy cut level is E^i and minimal energy is e_0 . It is induced by

$$\mathcal{AF}^{[i,i+1]} = \{ \mathcal{M}_{k+1}(\beta; [i,i+1]) \mid \beta, k \}.$$

We denote it by $\{\mathfrak{m}_k^{[i,i+1]}\}$. Then we can prove the following lemma by induction on N.

Lemma 22.24. For each $n \leq N$ we have the following.

- (1) For $i \leq n$, there exists a promotion of $\{\mathfrak{m}_k^i\}$ to the energy cut level E^n .
- (2) For $i \leq n-1$, there exists a promotion of $\{\mathfrak{m}_k^{[i,i+1]}\}$ to the energy cut level E^n . It is a pseudo-isotopy between the promotions in (1).

Moreover if n' < n the structures in (1)(2) for n' is the energy cut at $E^{n'}$ of the structures in (1)(2) for n.

Proof. This is immediate from Proposition 22.9.

Now by mathematical induction we obtain the same conclusion as in Lemma 22.24 in the case $N=\infty$. This implies Theorem 21.35 (5). Namely the filtered A_{∞} structure associated to our inductive system of linear K-systems is one $\{\mathfrak{m}_{k}^{[0,1],1}\mid$ $i=1,2,\ldots$ obtained by promotion to energy cut level ∞ .

Proof of Theorem 21.35 (1). Using trivial pseudo-isotopy (the direct product), this is a special case of Theorem 21.35 (5).

Proof of Theorem 21.35 (6). Suppose we are in Situation 21.34 (6). We apply Lemma 22.24 to each of the two inductive systems \mathcal{IAF}^0 , \mathcal{IAF}^1 . Namely we start with $\{\mathfrak{m}_k^{ji} \mid i=1,2,\ldots\}$ which are partial A_{∞} structures of energy cut level E^i and minimal energy e_0 on $\Omega(L)$ (where j=0,1) and with $\{\mathfrak{m}_k^{j,[i,i+1]}\mid i=1,2,\dots\}$ which are partial pseudo-isotopies of energy cut level E^i and minimal energy e_0 among them. Then by Lemma 22.24 and induction, we promote them to the energy cut level ∞ .

Next we consider $\{\mathcal{M}_{k+1}^{i}(\beta; [0,1]) \mid \beta, k\}$ and $\{\mathcal{M}_{k+1}(\beta; [0,1] \times [i,i+1]) \mid \beta, k\}$ in Situation 21.34 (6).

The former defines $\{\mathfrak{m}_k^{[0,1],i}\mid i=1,2,\dots\}$ that is a pseudo-isotopy of energy cut level E^i and minimal energy e_0 from $\{\mathfrak{m}_k^{0i}\mid i=1,2,\dots\}$ to $\{\mathfrak{m}_k^{1i}\mid i=1,2,\dots\}$. The latter defines $\{\mathfrak{m}_k^{[0,1],[i,i+1]}\mid i=1,2,\dots\}$ that is a pseudo-isotopy of pseudo

isotopies. In other words, it is a $([0,1] \times [i,i+1])$ -parametrized family of partial

 A_{∞} algebras of energy cut level E^i and minimal energy e_0 and their restrictions to the normalized boundary are disjoint union of $\{\mathfrak{m}_k^{[0,1],i}\mid i=1,2,\ldots\}$, $\{\mathfrak{m}_k^{[0,1],i+1}\mid i=1,2,\ldots\}$ and $\{\mathfrak{m}_k^{1,[i,i+1]}\mid i=1,2,\ldots\}$.

Now we apply Proposition 22.9 and the same induction argument as in the proof of Lemma 22.24 to promote $\{\mathfrak{m}_k^{[0,1],i}\mid i=1,2,\ldots\}$ and $\{\mathfrak{m}_k^{[0,1],[i,i+1]}\mid i=1,2,\ldots\}$ to the energy cut level ∞ . Thus after promotion, $\{\mathfrak{m}_k^{[0,1],1}\mid i=1,2,\ldots\}$ gives a pseudo-isotopy from the promotion of $\{\mathfrak{m}_k^{01}\mid i=1,2\ldots\}$ to the promotion of $\{\mathfrak{m}_k^{01}\mid i=1,2\ldots\}$ of energy cut level ∞ . This is what we want to construct.

We note that there exist A_{∞} homomorphisms

$$\{\mathfrak{m}_{k}^{[0,1],1} \mid i=1,2,\dots\} \longrightarrow \{\mathfrak{m}_{k}^{j1} \mid i=1,2\dots\}, \quad j=0,1,$$
 (22.18)

which are linear homotopy equivalences induced by the inclusion $L = L \times \{j\} \to L \times [0,1]$. Therefore inverting one of them and using the Whitehead theorem for A_{∞} algebra [FOOO6, Theorem 4.2.45], we obtain the homotopy equivalence $\{\mathfrak{m}_k^{01} \mid i=1,2\ldots\} \to \{\mathfrak{m}_k^{11} \mid i=1,2\ldots\}$.

Proof of Theorem 21.35 (3). This is a special case of Theorem 21.35 (6). \Box

Proof of Theorem 21.35 (8). For each i we use $\mathcal{AF}^{[0,1]\times[i,i+1]\times[1,2]}$ to obtain a $[0,1]\times[i,i+1]\times[1,2]$ parametrized family of partial A_{∞} structures of energy cut level E^i and minimal energy e_0 . On $[0,1]\times[i,i+1]\times\{1\}$ and $[0,1]\times[i,i+1]\times\{2\}$ this family restricts to the family we obtain in the above proof of Theorem 21.35 (6) applied to $\{\mathcal{M}_{k+1}^{i,\ell}(\beta;[0,1])\mid \beta,k\}, \{\mathcal{M}_{k+1}^{\ell}(\beta;[0,1]\times[i,i+1])\mid \beta,k\}$ with $\ell=a$ and $\ell=b$, respectively. Moreover it restricts to the direct product on $\{j\}\times[i,i+1]\times[1,2]$ with j=0 or j=1.

Now applying Proposition 22.9, we use the same induction argument as in the proof of Lemma 22.24 to obtain at the part i=1 the $[0,1]\times\{1\}\times[1,2]$ parametrized family of A_{∞} structures of energy cut level ∞ . We denote it by $\{\mathfrak{m}_k^{[0,1]\times\{1\}\times[1,2]}\mid i=1,2,\ldots\}$. We have a commutative diagram:

$$\{\mathfrak{m}_{k}^{10}\} \qquad \longleftarrow \qquad \{\mathfrak{m}_{k}^{[0,1],1,b}\} \qquad \longrightarrow \qquad \{\mathfrak{m}_{k}^{11}\} \\ \qquad \qquad \qquad \uparrow \qquad \qquad \uparrow \\ \{\mathfrak{m}_{k}^{10}\} \times [1,2] \qquad \longleftarrow \qquad \{\mathfrak{m}_{k}^{[0,1]\times\{1\}\times[1,2]}\} \qquad \longrightarrow \qquad \{\mathfrak{m}_{k}^{11}\} \times [1,2] \qquad \qquad (22.19) \\ \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \\ \{\mathfrak{m}_{k}^{10}\} \qquad \longleftarrow \qquad \{\mathfrak{m}_{k}^{[0,1],1,a}\} \qquad \longrightarrow \qquad \{\mathfrak{m}_{k}^{11}\}$$

The first and the third horizontal lines define homotopy equivalences we obtain in Theorem 21.35 (6) applied for $\ell=a$ and $\ell=b$ respectively. (We invert the first arrow.)

By the symbol $\{\mathfrak{m}_k^{10}\} \times [1,2]$, we denote the direct product pseudo-isotopy, that is, the isotopy $\{\mathfrak{m}_k^t,\mathfrak{c}_k^t\}$ such that \mathfrak{m}_k^t is independent of t and \mathfrak{c}_k^t are all zero. Then by inverting one of the arrows in the first or third vertical lines we obtain identity maps. Thus the commutativity of (22.19) implies that the two homotopy equivalences obtained for $\ell=a$ and $\ell=b$ are homotopic. This is the conclusion of Theorem 21.35 (8).

Proof of Theorem 21.35 (7). This is a special case of proof of Theorem 21.35 (8).

Remark 22.25. The proof we gave here using Diagram (22.19) is similar to those in [AJ]. It is based on a way to define homotopy equivalence we took here, that is, to invert homotopy equivalence (22.18).

There is an alternative way to define homotopy equivalence, given in [Fu2, Proof of Theorem 8.2], where we take an appropriate integration and sum over trees to construct homotopy equivalence from pseudo-isotopy directly. This method has an advantage that in case the pseudo-isotopy has cyclic symmetry the resulting homotopy equivalence is also cyclic. (We do not know the version of [FOOO6, Theorem 4.2.45] with cyclic symmetry.)

We can use the same method to prove Theorem 21.35 (7). Namely we regard $\{\mathfrak{m}_k^{[0,1]\times\{1\}\times[1,2]}\}$ as the pseudo-isotopy from $\{\mathfrak{m}_k^{10}\}\times[1,2]$ to $\{\mathfrak{m}_k^{11}\}\times[1,2]$ and apply the same formula [Fu2, Definition 9.4]. Using the fact that this is a pseudo-isotopy between direct product A_{∞} algebras, we can easily check that the resulting filtered A_{∞} homomorphism becomes the required homotopy.

Remark 22.26. We stop here at the stage where we prove consistency up to homotopy of homotopies. It is fairly obvious from the proof that we can prove as many higher consistency of the homotopies as we want.

Remark 22.27. We heard from some people that using iterated homotopies and homological algebra such as those we developed in this section is cumbersome and had better be avoided. Actually we do not think so at all.

One of the origin of 'homotopy everything structure' in algebraic topology is to study 'homotopy limit'. So the language of A_{∞} structures which we are using here is very much suitable for such a discussion. Moreover, taking an inductive limit is necessary anyway to study, for example, symplectic homology.

Furthermore, the general strategy taken here does not use any of the special feature of the problem and uses only the facts which are 'intuitively obvious'. By this reason it should work in almost all the situations we meet and will meet in the future. (Once we get used to it, applying this strategy becomes a routine.) The general strategy is summarized as follows.

- (1) The problem is that it is hard and sometimes impossible to perturb infinitely many moduli spaces simultaneously.
- (2) Those moduli spaces are filtered by certain quantities, typically by energy.
- (3) We fix certain 'energy cut level' and perturb the (finitely many) moduli spaces only up to that level.
- (4) Then we obtain a partial structure, such as partial A_{∞} structure of energy cut level E.
- (5) We may take E as large as we want though E should be finite.
- (6) Let E < E'. we obtain the structures of energy cut level E and of E'. The later can be regarded as the structure of energy cut level E. Those two structures are not the same but are 'homotopy equivalent' in a sense of 'homotopy everything structure'.
- (7) Then by a general method of homological algebra we can take homotopy limit to obtain the desired structure.

We note that a similar technique also appears in the renormalization theory. Especially Costello [Co] uses a similar technique.

Part 3. Appendices

23. Orbifold and orbibundle by local coordinate

In this section we describe the story of orbifold as much as we need in this article. We restrict ourselves to effective orbifolds and regard only embeddings as morphisms. The category $\mathcal{OB}_{\text{ef},\text{em}}$ where objects are effective orbifolds and morphisms are embeddings among them is naturally a 1 category. Moreover it has the following property. We consider the forgetful map

$$\mathfrak{forget}: \mathcal{OB}_{\mathrm{ef,em}}
ightarrow \mathcal{TOP}$$

where \mathcal{TOP} is the category of topological spaces. Then

$$forget : \mathcal{OB}_{ef.em}(c, c') \to \mathcal{TOP}(forget(c), forget(c'))$$

is injective. In other words, we can check the equality between morphisms settheoretically. This is a nice property, which we use extensively in the main body of this article. If we go beyond this category, then we need to distinguish carefully the two notions, two morphisms are equal, two morphisms are isomorphic. It will make the argument much more complicated. ⁵⁰

We emphasize that there is nothing new in this section. The story of orbifold is classical and is well-established. It has been used in various branches of mathematics since its invention by Satake [Sa] in more than 50 years ago. Especially, if we restrict ourselves to effective orbifolds, the story of orbifolds is nothing more than a straightforward generalization of the theory of smooth manifolds. The only important issue is the observation that for effective orbifolds almost everything works in the same way as manifolds.

23.1. Orbifolds and embeddings between them.

Definition 23.1. Let X be a paracompact Hausdorff space.

- (1) An orbifold chart of X (as a topological space) is a triple (V, Γ, ϕ) such that V is a manifold, Γ is a finite group acting smoothly and effectively on V and $\phi: V \to X$ is a Γ equivariant continuous map⁵¹ which induces a homeomorphism $\overline{\phi}: V/\Gamma \to X$ onto an open subset of X. We assume that there exists $o \in V$ such that $\gamma o = o$ for all $\gamma \in \Gamma$. We call o the base point. We say (V, Γ, ϕ) is an orbifold chart at x if $x = \phi(o)$. We call Γ the isotropy group, ϕ the local uniformization map and $\overline{\phi}$ the parametrization.
- (2) Let (V, Γ, ϕ) be an orbifold chart and $p \in V$. We put $\Gamma_p = \{ \gamma \in \Gamma \mid \gamma p = p \}$. Let V_p be a Γ_p invariant open neighborhood of p in V. We assume the map $\overline{\phi}: V_p/\Gamma_p \to X$ is injective. (In other words, we assume that $\gamma V_p \cap V_p \neq \emptyset$ implies $\gamma \in \Gamma_p$.) We call such a triple $(V_p, \Gamma_p, \phi|_{V_p})$ a subchart of (V, Γ, ϕ) .
- (3) Let (V_i, Γ_i, ϕ_i) (i = 1, 2) be orbifold charts of X. We say that they are compatible if the following holds for each $p_1 \in V_1$ and $p_2 \in V_2$ with $\phi_1(p_1) = \phi_2(p_2)$.
 - (a) There exists a group isomorphism $h: (\Gamma_1)_{p_1} \to (\Gamma_2)_{p_2}$.

⁵⁰We need to use several maps between underlying topological spaces of orbifolds, such as projection of bundles or covering maps. In case we include those maps, we need to carefully examine whether set-theoretical equality is enough to show their various properties are preserved.

⁵¹The Γ action on X is trivial.

- (b) There exists an h equivariant diffeomorphism $\tilde{\varphi}: V_{1,p_1} \to V_{2,p_2}$. Here V_{i,p_i} is a $(\Gamma_i)_{p_i}$ equivariant subset of V_i such that $(V_{i,p_i},(\Gamma_i)_{p_i},\phi|_{V_{i,p_i}})$ is a subchart.
- (c) $\phi_2 \circ \tilde{\varphi} = \phi_1 \text{ on } V_{1,p_1}$.
- (4) A representative of an orbifold structure on X is a set of orbifold charts $\{(V_i, \Gamma_i, \phi_i) \mid i \in I\}$ such that each two of the charts are compatible in the sense of (3) above and $\bigcup_{i\in I} \phi_i(V_i) = X$, is a locally finite open cover of X.

Definition 23.2. Suppose that X, Y are equipped with representatives of orbifold structures $\{(V_i^X, \Gamma_i^X, \phi_i^X) \mid i \in I\}$ and $\{(V_i^Y, \Gamma_i^Y, \phi_i^Y) \mid j \in J\}$, respectively. A continuous map $f: X \to Y$ is said to be an *embedding* if the following holds.

- (1) f is an embedding of topological spaces.
- (2) Let $p \in V_i^X$, $q \in V_i^Y$ with $f(\phi_i(p)) = \phi_j(q)$. Then we have the following.

 - (a) There exists an isomorphism of groups $h_{p;ji}:(\Gamma_i^X)_p\to (\Gamma_j^Y)_q$. (b) There exist $V_{i,p}^X$ and $V_{j,q}^Y$ such that $(V_{i,p}^X,(\Gamma_i^X)_p,\phi_i|_{V_{i,p}^X})$ is a subchart for i = 1, 2. There exists an $h_{p;ji}$ equivariant embedding of manifolds
 $$\begin{split} \tilde{f}_{p;ji}:V_{i,p}^X \to V_{j,q}^Y.\\ \text{(c) The diagram below commutes.} \end{split}$$

$$V_{i,p}^{X} \xrightarrow{\tilde{f}_{p;ji}} V_{j,q}^{Y}$$

$$\phi_{i} \downarrow \qquad \qquad \downarrow \phi_{j}$$

$$X \xrightarrow{f} Y$$

$$(23.1)$$

Two orbifold embeddings are said to be equal if they coincide set-theoretically.

Remark 23.3. Note that an embedding of effective orbifolds is a continuous map $f: X \to Y$ of underlying topological spaces, which has the properties (2) above.

When we study morphisms among ineffective orbifolds or morphisms between effective orbifolds which is not an embedding, such a morphism is a continuous map $f: X \to Y$ of underlying topological spaces plus certain additional data. For example, if we consider an ineffective orbifold that is a point with an action of a nontrivial finite group Γ , then the morphism from this ineffective orbifold to itself contains the datum of an automorphism of the group Γ . (Two such morphisms h_1, h_2 are equivalent if there exists an inner automorphism h such that $h_1 = h \circ h_2$.)

(1) The composition of embeddings is an embedding. Lemma 23.4.

- (2) The identity map is an embedding.
- (3) If an embedding is a homeomorphism, then its inverse is also an embedding.

The proof is easy and is left to the reader.

Definition 23.5. (1) We call an embedding of orbifolds a diffeomorphism if it is a homeomorphism in addition.

- (2) We say that two representatives of orbifold structures on X are equivalent if the identity map regarded as a map between X equipped with those two representatives of orbifold structures is a diffeomorphism. This is an equivalence relation by Lemma 23.4.
- (3) An equivalence class of the equivalence relation (2) is called an orbifold structure on X. An orbifold is a pair of topological space and its orbifold structure.

- (4) The condition for a map $X \to Y$ to be an embedding does not change if we replace representatives of orbifold structures to equivalent ones. So we can define the notion of an *embedding of orbifolds*.
- (5) If U is an open subset of an orbifold X, then there exists a unique orbifold structure on U such that the inclusion $U \to X$ is an embedding. We call U with this orbifold structure an *open suborbifold*.
- **Definition 23.6.** (1) Let X be an orbifold. An orbifold chart (V, Γ, ϕ) of underlying topological space X in the sense of Definition 23.1 (1) is called an *orbifold chart of an orbifold* X if the map $\overline{\phi}: V/\Gamma \to X$ induced by ϕ is an embedding of orbifolds.
 - (2) Hereafter when X is an orbifold, an 'orbifold chart' always means an orbifold chart of an orbifold in the sense of (1).
 - (3) In case when an orbifold structure on X is given, a representative of its orbifold structure is called an *orbifold atlas*.
 - (4) Two orbifold charts (V_i, Γ_i, ϕ_i) are said to be isomorphic if there exist a group isomorphism $h: \Gamma_1 \to \Gamma_2$ and an h-equivariant diffeomorphism $\tilde{\varphi}: V_1 \to V_2$ such that $\phi_2 \circ \tilde{\varphi} = \phi_1$. The pair $(h, \tilde{\varphi})$ is called an isomorphism or a coordinate change between two orbifold charts.

Proposition 23.7. In the situation of Definition 23.6 (4), suppose $(h, \tilde{\varphi})$ and $(h', \tilde{\varphi}')$ are both isomorphisms between two orbifold charts (V_1, Γ_1, ϕ_1) and (V_2, Γ_2, ϕ_2) . Then there exists $\mu \in \Gamma_2$ such that

$$h'(\gamma) = \mu h(\gamma)\mu^{-1}, \qquad \tilde{\varphi}'(x) = \mu \tilde{\varphi}(x).$$
 (23.2)

Conversely, if $(h, \tilde{\varphi})$ is an isomorphism between orbifold charts, then $(h', \tilde{\varphi}')$ defined by (23.2) is also an isomorphism between orbifold charts. In particular, any automorphism of an orbifold chart $(h, \tilde{\varphi})$ is given by $h(\gamma) = \mu \gamma \mu^{-1}$, $\tilde{\varphi}(x) = \mu x$ for some element $\mu \in \Gamma$.

Proof. The proposition immediately follows from the next lemma.

Lemma 23.8. Let V_1 , V_2 be manifolds on which finite groups Γ_1 , Γ_2 act effectively and smoothly. Assume that V_1 is connected. Let $(h_i, \tilde{\varphi}_i)$ (i = 1, 2) be pairs such that $h_i : \Gamma_1 \to \Gamma_2$ are injective group homomorphisms and $\tilde{\varphi}_i : V_1 \to V_2$ are h_i -equivariant embeddings of manifolds. Moreover, we assume that the induced maps $\varphi_i : V_1/\Gamma_1 \to V_2/\Gamma_2$ are embeddings of orbifolds and φ_1 coincides with φ_2 settheoretically. Then there exists $\mu \in \Gamma_2$ such that

$$\tilde{\varphi}_2(x) = \mu \tilde{\varphi}_1(x), \qquad h_2(\gamma) = \mu h_1(\gamma) \mu^{-1}.$$

Proof. For the sake of simplicity we prove only the case when Condition 23.9 below is satisfied. Let X be an orbifold. For a point $x \in X$ we take its orbifold chart (V_x, Γ_x, ψ_x) . We say $x \in \text{Reg}(X)$ if $\Gamma_x = \{1\}$, and put $\text{Sing}(X) = X \setminus \text{Reg}(X)$.

Condition 23.9. We assume that $\dim \operatorname{Sing}(X) \leq \dim X - 2$.

This condition is satisfied if X is oriented. (In fact, Condition 23.9 fails only when there exists an element of Γ_x (an isotropy group of some orbifold chart) whose action is given by $(x_1, x_2, \ldots, x_n) \mapsto (-x_1, x_2, \ldots, x_n)$ for some coordinate (x_1, \ldots, x_n) . Therefore we can always assume Condition 23.9 in the study of Kuranishi structure, by adding a trivial factor which is acted by the induced representation of $t \mapsto -t$ to both the obstruction bundle and to the Kuranishi neighborhood.)

Let $x_0 \in V_1^0$. By assumption there exists a unique $\mu \in \Gamma_2$ such that $\tilde{\varphi}_2(x_0) = \mu \tilde{\varphi}_1(x_0)$. By Condition 23.9 the subset V_1^0 is connected. Therefore the above element μ is independent of $x_0 \in V_1^0$ by uniqueness. Since V_1^0 is dense, we conclude $\tilde{\varphi}_2(x) = \mu \tilde{\varphi}_1(x)$ for any $x \in V_1$. Now, for $\gamma \in \Gamma_1$, we calculate

$$h_1(\gamma)\tilde{\varphi}_1(x_0) = \tilde{\varphi}_1(\gamma x_0) = \mu^{-1}\tilde{\varphi}_2(\gamma x_0) = \mu^{-1}h_2(\gamma)\tilde{\varphi}_2(x_0) = \mu^{-1}h_1(\gamma)\mu\tilde{\varphi}_1(x_0).$$

Since the induced map is an embedding of orbifold, it follows that the isotropy group of $\tilde{\varphi}_1(x_0)$ is trivial. Therefore $h_1(\gamma) = \mu^{-1}h_2(\gamma)\mu$ as required.

The proof of Proposition 23.7 is complete.

Definition 23.10. Let X be an orbifold.

- (1) A function $f: X \to \mathbb{R}$ is said to be a *smooth function* if for any orbifold chart (V, Γ, ϕ) the composition $f \circ \phi: V \to \mathbb{R}$ is smooth.
- (2) A differential form on an orbifold X assigns a Γ invariant differential form $h_{\mathfrak{V}}$ on V to each orbifold chart $\mathfrak{V} = (V, \Gamma, \phi)$ such that the following holds.
 - (a) If (V_1, Γ_1, ϕ_1) is isomorphic to (V_2, Γ_2, ϕ_2) and $(h, \tilde{\varphi})$ is an isomorphism, then $\tilde{\varphi}^* h_{\mathfrak{V}_2} = h_{\mathfrak{V}_1}$.
 - (b) If $\mathfrak{V}_p = (V_p, \Gamma_p, \phi_p)$ is a subchart of $\mathfrak{V} = (V, \Gamma, \phi)$, then $h_{\mathfrak{V}}|_{V_p} = h_{\mathfrak{V}_p}$.
- (3) An n dimensional orbifold X is said to be *orientable* if there exists a differential n-form ω such that $\omega_{\mathfrak{V}}$ never vanishes.
- (4) Let ω be an *n*-form as in (3) and $\mathfrak{V} = (V, \Gamma, \phi)$ an orbifold chart. Then we give V an orientation so that it is compatible with $\omega_{\mathfrak{V}}$. The Γ action preserves the orientation. We call such (V, Γ, ϕ) equipped with an orientation of V, an *oriented orbifold chart*.
- (5) Let $\bigcup_{i \in I} U_i = X$ be an open covering of an orbifold X. A smooth partition of unity subordinate to the covering $\{U_i\}$ is a set of functions $\{\chi_i \mid i \in I\}$ such that:
 - (a) χ_i are smooth functions.
 - (b) The support of χ_i is contained in U_i .
 - (c) $\sum_{i \in I} \chi_i = 1$.

Lemma 23.11. For any locally finite open covering of an orbifold X there exists a smooth partition of unity subordinate thereto.

We omit the proof, which is an obvious analogue of the standard proof for the case of manifolds.

Definition 23.12. An *orbifold with corner* is defined in the same way. We require the following.

- (1) In Definition 23.1 (1) we assume that V is a manifold with corners.
- (2) Let $S_k(V)$ be the set of points which lie on the codimension k corner and $\overset{\circ}{S}_k(V) = S_k(V) \setminus \bigcup_{k'>k} S_{k'}(V)$. We require that Γ action on each connected component of $\overset{\circ}{S}_k(V)$ is effective. (Compare [Part I, Condition 4.14].)
- (3) For an embedding of orbifolds with corners we require that the map \tilde{f} in Definition 23.2 (c) satisfies $\tilde{f}(\mathring{S}_k(V_1)) \subset \mathring{S}_k(V_2)$.

Lemma 23.13. Let X_i (i = 1, 2) be orbifolds and $\varphi_{21} : X_1 \to X_2$ an embedding. Then we can find an orbifold atlas $\{\mathfrak{V}^i_{\mathfrak{r}} = \{(V^i_{\mathfrak{r}}, \Gamma^i_{\mathfrak{r}}, \phi^i_{\mathfrak{r}})\} \mid \mathfrak{r} \in \mathfrak{R}_i\}$ with the following properties.

- (1) $\mathfrak{R}_1 \subseteq \mathfrak{R}_2$.
- (2) $V_{\mathfrak{r}}^2 \cap \varphi_{21}(X_1) \neq \emptyset$ if and only if $\mathfrak{r} \in \mathfrak{R}_1$. (3) If $\mathfrak{r} \in \mathfrak{R}_1$ then $\varphi_{21}^{-1}(\phi_{\mathfrak{r}}^2(V_{\mathfrak{r}}^2)) = \phi_{\mathfrak{r}}^1(V_{\mathfrak{r}}^1)$ and there exists $(h_{\mathfrak{r},21}, \tilde{\varphi}_{\mathfrak{r},21})$ such
 - (a) $h_{\mathfrak{r},21}:\Gamma^1_{\mathfrak{r}}\to\Gamma^2_{\mathfrak{r}}$ is a group isomorphism.
 - (b) $\tilde{\varphi}_{\mathfrak{r},21}:V^1_{\mathfrak{r}}\to V^2_{\mathfrak{r}}$ is an $h_{\mathfrak{r},21}$ -equivariant embedding of smooth mani-
 - (c) The next diagram commutes.

$$V_{\mathfrak{r}}^{1} \xrightarrow{\tilde{\varphi}_{\mathfrak{r},21}} V_{\mathfrak{r}}^{2}$$

$$\phi_{1}^{\mathfrak{r}} \downarrow \qquad \qquad \downarrow \phi_{2}^{\mathfrak{r}} \qquad (23.3)$$

$$X_{1} \xrightarrow{\varphi_{21}} X_{2}$$

- (4) In case X_i has boundary or corners we may choose our charts so that the following is satisfied.
 - (a) $V_{\mathfrak{r}}^i$ is an open subset of $\overline{V}_{\mathfrak{r}}^i \times [0,1)^{d(\mathfrak{r})}$, where $d(\mathfrak{r})$ is independent of i and $\overline{V}_{\mathbf{r}}^{i}$ is a manifold without boundary.
 - (b) There exists a point $o^i(\mathfrak{r})$ which is fixed by all $\gamma \in \Gamma^i_{\mathfrak{r}}$ such that $[0,1)^{d(\mathfrak{r})}$ components of $o^i(\mathfrak{r})$ are all 0.
 - (c) If we write

$$\varphi_{\mathfrak{r},21}(\overline{y}',(t_1',\ldots,t_{d(\mathfrak{r})}'))=(\overline{y},(t_1,\ldots,t_{d(\mathfrak{r})})),$$

then $t_i = 0$ if and only if $t'_i = 0$.

We may take our atlas that are refinements of the given coverings of X_1 and X_2 .

Proof. For each $x \in X_1$ we can find orbifold charts \mathfrak{V}_x^i for i = 1, 2, such that $\varphi_{21}^{-1}(U_x^2) = U_x^1$, $x \in U_x^1$ and that there exists a representative $(h_{x,21}, \tilde{\varphi}_{x,21})$ of embedding $U_x^1 \to U_x^2$ that is a restriction of φ_{21} . In case X_i has boundary or corners, we choose them so that (4) is also satisfied.

We cover X_1 by finitely many such $U_{x_i}^1$. This is our choice of atlas $\{\mathfrak{V}_{\mathfrak{r}}^1 \mid \mathfrak{r} \in \mathfrak{R}_1\}$. Then the associated $\{\mathfrak{V}^2_{\mathfrak{r}} \mid \mathfrak{r} \in \mathfrak{R}_1\}$ satisfies (3)(4) and covers $\varphi_{21}(X_1)$. We can extend it to $\{\mathfrak{V}^2_{\mathfrak{r}} \mid \mathfrak{r} \in \mathfrak{R}_2\}$ so that (1)(2) are also satisfied.

Definition 23.14. We call $(h_{r,21}, \tilde{\varphi}_{r,21})$ a local representative of embedding $\varphi_{r,21}$ on the charts $\mathfrak{V}^1_{\mathbf{r}}$, $\mathfrak{V}^2_{\mathbf{r}}$

Lemma 23.15. If $(h_{\mathfrak{r},21}, \tilde{\varphi}_{\mathfrak{r},21})$, $(h'_{\mathfrak{r},21}, \tilde{\varphi}'_{\mathfrak{r},21})$ are local representatives of an embedding of the same charts $\mathfrak{V}^1_{\mathbf{r}}$, $\mathfrak{V}^2_{\mathbf{r}}$, then there exists $\mu \in \Gamma_2$ such that

$$\tilde{\varphi}'_{\mathfrak{r},21}(x) = \mu \tilde{\varphi}_{\mathfrak{r},21}(x), \qquad h'_{\mathfrak{r},21}(\gamma) = \mu h_{\mathfrak{r},21}(\gamma) \mu^{-1}.$$

This is a consequence of Lemma 23.8.

Lemma 23.16. Let X be a topological space, Y an orbifold, and $f: X \to Y$ an embedding of topological spaces. Then the orbifold structure on X by which f becomes an embedding of orbifolds is unique if there exists one.

Proof. Let X_1, X_2 be orbifolds whose underlying topological spaces are both X and satisfy that $f_i: X_i \to Y$ are embeddings of orbifolds for i=1,2. We will prove that the identity map id: $X_1 \to X_2$ is a diffeomorphism of orbifods. Since the condition for a homeomorphism to be a diffeomorphism of orbifolds is a local condition, it suffices to check it on a neighborhood of each point. Let $p \in X$ and q = f(p). We take a representative $(h_i, \tilde{\varphi}_i)$ of the orbifold embeddings $f_i : X_i \to Y$ using the orbifold charts $\mathfrak{V}_p^i = (V_p^i, \Gamma_p^i, \phi_p^i)$ of X and $\mathfrak{V}_q = (V_q, \Gamma_q, \phi_q)$ of Y. The maps $h_i : \Gamma_p^i \to \Gamma_q^i$ are group isomorphisms. So we have a group isomorphism $h = h_2^{-1} \circ h_1 : \Gamma_p^1 \to \Gamma_p^2$. Since $\tilde{\varphi}_1(V^1)/\Gamma_p = \tilde{\varphi}_2(V^2)/\Gamma_p$ set-theoretically, we have $\tilde{\varphi}_1(V_p^1) = \tilde{\varphi}_2(V_p^2) \subset V_q$. They are smooth submanifolds since f_i are embeddings of orbifolds. Therefore $\varphi = \tilde{\varphi}_2^{-1} \circ \tilde{\varphi}_1$ is defined in a neighborhood of the base point o_p^i and is a diffeomorphism. Then $(h, \tilde{\varphi})$ is a local representative of id.

23.2. Vector bundle on orbifold.

Definition 23.17. Let (X, \mathcal{E}, π) be a pair of orbifolds X and \mathcal{E} with a continuous map $\pi : \mathcal{E} \to X$ between their underlying topological spaces. Hereafter we write (X, \mathcal{E}) in place of (X, \mathcal{E}, π) .

- (1) An orbifold chart of (X, \mathcal{E}) is a quintuple $(V, E, \Gamma, \phi, \widehat{\phi})$ with the following properties:
 - (a) $\mathfrak{V} = (V, \Gamma, \phi)$ is an orbifold chart of the orbifold X.
 - (b) E is a finite dimensional vector space equipped with a linear Γ action.
 - (c) $(V \times E, \Gamma, \widehat{\phi})$ is an orbifold chart of the orbifold \mathcal{E} .
 - (d) The diagram below commutes set-theoretically.

$$V \times E \xrightarrow{\widehat{\phi}} \mathcal{E}$$

$$\downarrow \qquad \qquad \downarrow^{\pi}$$

$$V \xrightarrow{\phi} X$$

$$(23.4)$$

Here the left vertical arrow is the projection to the first factor.

- (2) In the situation of (1), let $p \in V$ and $(V_p, \Gamma_p, \phi|_{V_p})$ be a subchart of (V, Γ, ϕ) in the sense of Definition 23.1 (2). Then $(V_p, E, \Gamma_p, \phi|_{V_p}, \widehat{\phi}|_{V_p \times E})$ is an orbifold chart of (X, \mathcal{E}) . We call it a *subchart* of $(V, E, \Gamma, \phi, \widehat{\phi})$.
- (3) Let $(V^i, E^i, \Gamma^i, \phi^i, \widehat{\phi^i})$ (i = 1, 2) be orbifold charts of (X, \mathcal{E}) . We say that they are *compatible* if the following holds for each $p_1 \in V^1$ and $p_2 \in V^2$ with $\phi^1(p_1) = \phi^2(p_2)$: There exist open neighborhoods $V^i_{p_i}$ of $p_i \in V^i$ such that:
 - (a) There exists an isomorphism $(h, \tilde{\varphi}): (V^1, \Gamma^1, \phi^1)|_{V^1_{p_1}} \to (V^2, \Gamma^2, \phi^2)|_{V^2_{p_2}}$ between orbifold charts of X, which are subcharts.
 - (b) There exists an isomorphism $(h, \tilde{\hat{\varphi}}): (V^1 \times E^1, \Gamma^1, \phi^1)|_{V^1_{p_1} \times E^1} \to (V^2 \times E^2, \Gamma^2, \phi^2)|_{V^2_{p_2} \times E^2}$ between orbifold charts of \mathcal{E} , which are subcharts.
 - (c) For each $y \in V_{p_1}^1$ the map $E^1 \to E^2$ given by $\xi \to \pi_{E^2} \tilde{\hat{\varphi}}(y,\xi)$ is a linear isomorphism. Here $\pi_{E^2} : V^2 \times E^2 \to E^2$ is the projection.
- (4) A representative of a vector bundle structure on (X, \mathcal{E}) is a set of orbifold charts $\{(V_i, E_i, \Gamma_i, \phi_i, \widehat{\phi}_i) \mid i \in I\}$ such that any two of the charts are compatible in the sense of (3) above and

$$\bigcup_{i \in I} \phi_i(V_i) = X, \quad \bigcup_{i \in I} \widehat{\phi}_i(V_i \times E_i) = \mathcal{E},$$

are locally finite open covers.

Definition 23.18. Suppose (X^*, \mathcal{E}^*) (* = a, b) have representatives of vector bundle structures $\{(V_i^*, E_i^*, \Gamma_i^*, \phi_i^*, \widehat{\phi}_i^*) \mid i \in I^*\}$, respectively. A pair of orbifold embeddings $(f, \widehat{f}), f: X^a \to X^b, \widehat{f}: \mathcal{E}^a \to \mathcal{E}^b$ is said to be an *embedding of vector bundles* if the following holds.

- (1) Let $p \in V_i^a$, $q \in V_j^b$ with $f(\phi_i^a(p)) = \phi_j^b(q)$. Then there exist open subcharts $(V_{i,p}^a \times E_{i,p}^a, \Gamma_{i,p}^a, \widehat{\phi}_{i,p}^a)$ and $(V_{j,q}^b \times E_{j,q}^b, \Gamma_{j,q}^b \widehat{\phi}_{j,q}^b)$ and a local representative $(h_{p;i,j}, f_{p;i,j}, \widehat{f}_{p;i,j})$ of the embeddings f and \widehat{f} such that for each $y \in V_i^a$ the map $\xi \mapsto \pi_{E^b}(\widehat{f}_{p;i,j}(y,\xi))$, $E_{i,p}^a \to E_{j,q}^b$ is a linear embedding. Here $\pi_{E^b}: V^b \times E^b \to E^b$ is the projection.
- (2) The diagram below commutes set-theoretically.

$$\begin{array}{ccc}
\mathcal{E}^{a} & \xrightarrow{\widehat{f}} & \mathcal{E}^{b} \\
\pi_{E^{a}} \downarrow & & \downarrow \pi_{E^{b}} \\
X^{a} & \xrightarrow{f} & X^{b}
\end{array} (23.5)$$

Two orbifold embeddings of vector bundles are said to be *equal* if they coincide set-theoretically as pairs of maps.

Lemma 23.19. (1) A composition of embeddings of vector bundles is an embedding.

- (2) The pair of identity maps (id, \widehat{id}) is an embedding.
- (3) If an embedding of vector bundles is a pair of homeomorphisms, then the pair of their inverses is also an embedding.

The proof is easy and is omitted.

Definition 23.20. Let (X, \mathcal{E}) be as in Definition 23.17.

- (1) An embedding of vector bundles is said to be an *isomorphism* if it is a pair of diffeomorphisms of orbifolds.
- (2) We say that two representatives of a vector bundle structure on (X, \mathcal{E}) are *equivalent* if the pair of identity maps regarded as a self-map of vector bundle (X, \mathcal{E}) equipped with those two representatives of vector bundle structure is an isomorphism. This is an equivalence relation by Lemma 23.19.
- (3) An equivalence class of the equivalence relation (2) is called a *vector bundle* structure on (X, \mathcal{E}) .
- (4) A pair (X, \mathcal{E}) together with its vector bundle structure is called a vector bundle on X. We call \mathcal{E} the total space, X the base space, and $\pi : \mathcal{E} \to X$ the projection.
- (5) The condition for the pair $(f, \hat{f}): (X^a, \mathcal{E}^a) \to (X^b, \mathcal{E}^b)$ to be an embedding depends only on the equivalence class of vector bundle structures independent of its representatives. This enable us to define the notion of an embedding of vector bundles.
- (6) We say (f, \hat{f}) is an embedding over the orbifold embedding f.

Remark 23.21. (1) We may use the terminology 'orbibundle' in place of vector bundle. We use this terminology in case we emphasize that it is different from the vector bundle over the underlying topological space.

(2) We sometimes simply say \mathcal{E} is a vector bundle on an orbifold X.

Definition 23.22. (1) Let (X, \mathcal{E}) be a vector bundle. We call an orbifold chart $(V, E, \Gamma, \phi, \widehat{\phi})$ in the sense of Definition 23.17 (1) of underlying pair of topological spaces (X, \mathcal{E}) an orbifold chart of a vector bundle (X, \mathcal{E}) if the pair of maps $(\overline{\phi}, \overline{\widehat{\phi}}) : (V/\Gamma, (V \times E)/\Gamma) \to (X, \mathcal{E})$ induced from $(\phi, \widehat{\phi})$ is an embedding of vector bundles.

- (2) If $(V, E, \Gamma, \phi, \widehat{\phi})$ is an orbifold chart of a vector bundle, we call a pair $(E, \widehat{\phi})$ a trivialization of our vector bundle on V/Γ .
- (3) Hereafter when (X, \mathcal{E}) is a vector bundle, its 'orbifold chart' always means an orbifold chart of a vector bundles in the sense of (1).
- (4) In case when a vector bundle structure on (X, \mathcal{E}) is given, a representative of this vector bundle structure is called an *orbifold atlas* of (X, \mathcal{E}) .
- (5) Two orbifold charts $(V_i, E_i, \Gamma_i, \phi_i, \widehat{\phi}_i)$ of a vector bundle are said to be *isomorphic* if there exist an isomorphism $(h, \widetilde{\varphi})$ of orbifold charts $(V_1, \Gamma_1, \phi_1) \to (V_2, \Gamma_2, \phi_2)$ and an isomorphism $(h, \widetilde{\varphi})$ of orbifold charts $(V_1 \times E_1, \Gamma_1, \widehat{\phi}_1) \to (V_2 \times E_2, \Gamma_2, \widehat{\phi}_2)$ such that they induce an embedding of vector bundles $(\varphi, \widehat{\varphi}) : (V_1/\Gamma_1, (V_1 \times E_1)/\Gamma_1) \to (V_2/\Gamma_2, (V_2 \times E_2)/\Gamma_2)$. The triple $(h, \widetilde{\varphi}, \widetilde{\varphi})$ is called an *isomorphism* or a *coordinate change* between orbifold charts of the vector bundle.

Lemma 23.23. Let (X^b, \mathcal{E}^b) be a vector bundle over an orbifold X^b and $f: X^a \to X^b$ an embedding of orbifolds. Let $\mathcal{E}^a = X^a \times_{X^b} \mathcal{E}^b$ be the fiber product in the category of topological space. By definition of the fiber product, we have maps $\pi: \mathcal{E}^a \to X^a$ and $\widehat{f}: \mathcal{E}^a \to \mathcal{E}^b$. Then the exists a unique structure of vector bundle on (X^a, \mathcal{E}^a) such that the projection is given the above map π and (f, \widehat{f}) is an embedding of vector bundles.

Proof. Let $\{\mathfrak{V}^*_{\mathfrak{r}} \mid \mathfrak{r} \in \mathfrak{R}_*\}$, *=a,b be orbifold at lases where $\mathfrak{V}^*_{\mathfrak{r}} = (V^*_{\mathfrak{r}}, \Gamma^*_{\mathfrak{r}}, \phi^*_{\mathfrak{r}})$. Let $(V^b_{\mathfrak{r}}, E^b_{\mathfrak{r}}, \Gamma^b_{\mathfrak{r}}, \phi^b_{\mathfrak{r}}, \widehat{\phi}^b_{\mathfrak{r}})$ be orbifold at las of the vector bundle (X^b, \mathcal{E}^b) . Let $(h_{\mathfrak{r},ba}, \widetilde{\varphi}_{\mathfrak{r},ba})$ be a local representative of the embedding f on the charts $\mathfrak{V}^a_{\mathfrak{r}}, \mathfrak{V}^b_{\mathfrak{r}}$. We put $E^a_{\mathfrak{r}} = E^b_{\mathfrak{r}}$, on which $\Gamma^a_{\mathfrak{r}}$ acts by the isomorphism $h_{\mathfrak{r},ba}$. By definition of fiber product, there exists uniquely a map $\widehat{\phi}^a_{\mathfrak{r}}: V^b_{\mathfrak{r}} \times E^b_{\mathfrak{r}} \to \mathcal{E}^a$ such that the next diagram commutes.

$$V_{\mathfrak{r}}^{a} \leftarrow {}^{\pi} V_{\mathfrak{r}}^{a} \times E_{\mathfrak{r}}^{b} \xrightarrow{\widetilde{\varphi}_{\mathfrak{r},ba} \times id} V_{\mathfrak{r}}^{b} \times E_{\mathfrak{r}}^{b}$$

$$\phi_{\mathfrak{r}}^{a} \downarrow \qquad \qquad \downarrow \widehat{\phi}_{\mathfrak{r}}^{a} \qquad \qquad \downarrow \widehat{\phi}_{\mathfrak{r}}^{b}$$

$$X^{a} \leftarrow {}^{\pi} \mathcal{E}^{a} \xrightarrow{\widehat{f}} \mathcal{E}^{b}$$

$$(23.6)$$

In fact,

$$f \circ \phi_{\mathfrak{r}}^{a} \circ \pi = \phi_{\mathfrak{r}}^{b} \circ \varphi_{\mathfrak{r},ba} \circ \pi = \phi_{\mathfrak{r}}^{b} \circ \pi \circ (\tilde{\varphi}_{\mathfrak{r},ba} \times id) = \pi \circ \widehat{\phi_{\mathfrak{r}}}^{b} \circ (\tilde{\varphi}_{\mathfrak{r},ba} \times id).$$

Thus
$$\{(V_{\mathfrak{r}}^a, E_{\mathfrak{r}}^a, \Gamma_{\mathfrak{r}}^a, \phi_{\mathfrak{r}}^a, \widehat{\phi}_{\mathfrak{r}}^a) \mid \mathfrak{r} \in \mathfrak{R}\}$$
 is an atlas of the vector bundle (X^a, \mathcal{E}^a) .

Definition 23.24. We call the vector bundle in Lemma 23.23 the *pull-back* and write $f^*(X^b, \mathcal{E}^b)$. (Sometimes we write $f^*\mathcal{E}^b$ by an abuse of notation.)

In case X^a is an open subset of X^b equipped with open substructure we call restriction in place of pull-back of \mathcal{E}^b and write $\mathcal{E}^b|_{X^a}$ in place of $f^*\mathcal{E}^b$.

Lemma 23.25. In the situation of Lemma 23.13 suppose in addition that \mathcal{E}^i is a vector bundle over X^i and $\widehat{\varphi}_{21}: \mathcal{E}^1 \to \mathcal{E}^2$ is an embedding of vector bundles over φ_{21} . Then in addition to the conclusion of Lemma 23.13, there exists $\widetilde{\widehat{\varphi}}_{\mathfrak{r};21}:$

 $V_{\mathfrak{r}}^1 \times E_{\mathfrak{r}}^1 \to V_{\mathfrak{r}}^2 \times E_{\mathfrak{r}}^2$ that is an $h_{\mathfrak{r};21}$ equivariant embedding of manifolds with the following properties:

(1) The next diagram commutes.

$$V_{\mathfrak{r}}^{1} \times E_{\mathfrak{r}}^{1} \xrightarrow{\widetilde{\varphi}_{\mathfrak{r},21}} V_{\mathfrak{r}}^{2} \times E_{\mathfrak{r}}^{2}$$

$$\widehat{\varphi}_{\mathfrak{r}}^{1} \downarrow \qquad \qquad \downarrow \widehat{\varphi}_{\mathfrak{r}}^{2} \qquad (23.7)$$

$$\mathcal{E}^{1} \xrightarrow{\widehat{\varphi}_{21}} \mathcal{E}^{2}$$

(2) For each $y \in V_{\mathfrak{r}}^1$ the map $\xi \mapsto \pi_2(\tilde{\varphi}_{\mathfrak{r},21}(y,\xi)) : E_{\mathfrak{r}}^1 \to E_{\mathfrak{r}}^2$ is a linear embedding.

The proof is similar to the proof of Lemma 23.13 and is omitted.

Definition 23.26. We call $(h_{\tau,21}, \tilde{\varphi}_{\tau,21}, \tilde{\hat{\varphi}}_{\tau,21})$ a local representative of embedding $(\varphi_{21}, \hat{\varphi}_{21})$ on the charts $(V^1 \times E^1, \Gamma^1, \hat{\phi}^1), (V^2 \times E^2, \Gamma^2, \hat{\phi}^2)$.

Lemma 23.27. If $(h_{\mathfrak{r},21}, \tilde{\varphi}_{\mathfrak{r},21}, \tilde{\varphi}_{\mathfrak{r},21})$, $(h'_{\mathfrak{r},21}, \tilde{\varphi}'_{\mathfrak{r},21}, \tilde{\varphi}'_{\mathfrak{r},21})$ are local representatives of an embedding of vector bundles of the same charts $(V^1 \times E^1, \Gamma^1, \hat{\phi}^1)$, $(V^2 \times E^2, \Gamma^2, \hat{\phi}^2)$, then there exists $\mu \in \Gamma^2$ such that

$$\tilde{\varphi}_{\mathfrak{r},21}'(x) = \mu \tilde{\varphi}_{\mathfrak{r},21}(x), \quad \tilde{\hat{\varphi}}_{\mathfrak{r},21}'(x,\xi) = \mu \tilde{\hat{\varphi}}_{\mathfrak{r},21}(x,\xi) \quad h_{\mathfrak{r},21}'(\gamma) = \mu h_{\mathfrak{r},21}(\gamma) \mu^{-1}.$$

Proof. This is a consequence of Lemma 23.8.

Remark 23.28. In [Part I, Situation 6.3] we introduced the notation $(h_{\mathfrak{r},21}, \tilde{\varphi}_{\mathfrak{r},21}, \check{\varphi}_{\mathfrak{r},21})$ where $\check{\varphi}_{\mathfrak{r},21}$ is related to $\tilde{\hat{\varphi}}_{\mathfrak{r},21}$ by the formula

$$\tilde{\hat{\varphi}}_{\mathfrak{r},21}(y,\xi) = (\tilde{\varphi}_{\mathfrak{r},21}(y), \check{\varphi}_{\mathfrak{r},21}(y,\xi)).$$

We use the pull-back of vector bundles in a different situation. Let \mathcal{E}^i , i=1,2, be vector bundles over an orbifold X. We take the Whitney sum bundle $\mathcal{E}^1 \oplus \mathcal{E}^2$ and denote by $|\mathcal{E}^1 \oplus \mathcal{E}^2|$ its total space. There exists a projection

$$|\mathcal{E}^1 \oplus \mathcal{E}^2| \to |\mathcal{E}^2|.$$
 (23.8)

Definition-Lemma 23.29. The total space $|\mathcal{E}^1 \oplus \mathcal{E}^2|$ has a structure of vector bundle over $|\mathcal{E}^2|$ such that (23.8) is the projection. We write it as $\pi_{\mathcal{E}^2}^*\mathcal{E}^1$ and call the *pull-back* of \mathcal{E}^1 by the projection $\pi_{\mathcal{E}^2}: |\mathcal{E}^2| \to X$.

When U is an open subset of $|\mathcal{E}^2|$ and $\pi: U \to X$ is the restriction of $\pi_{\mathcal{E}^2}$ to U, the pull-back $\pi_{\mathcal{E}^2}^*\mathcal{E}^1$ is by definition the restriction of $\pi_{\mathcal{E}^2}^*\mathcal{E}^1$ to U.

The proof is immediate from definition.

Remark 23.30. We note that the total space $|\mathcal{E}^1 \oplus \mathcal{E}^2|$ is *not* a fiber product $|\mathcal{E}^1| \times_X |\mathcal{E}^2|$. In fact, if X is a point and $\mathcal{E}^1 = \mathcal{E}^2 = \mathbb{R}^n/\Gamma$ with linear Γ action, then the fiber of $|\pi_{\mathcal{E}^2}^*\mathcal{E}^1| \to |\mathcal{E}^2| \to X$ at [0] is $(E^1 \times E^2)/\Gamma$. The fiber of the map $|\mathcal{E}^1| \times_X |\mathcal{E}^2| \to X$ at [0] is $(E^1/\Gamma) \times (E^2/\Gamma)$.

Definition 23.31. Let (X, \mathcal{E}) be a vector bundle. A *section* of (X, \mathcal{E}) is an embedding of orbifolds $s: X \to \mathcal{E}$ such that the composition of s and the projection is the identity map set-theoretically.

Lemma 23.32. Let $\{(V_{\mathfrak{r}}, E_{\mathfrak{r}}, \Gamma_{\mathfrak{r}}, \psi_{\mathfrak{r}}, \widehat{\psi}_{\mathfrak{r}}) \mid \mathfrak{r} \in \mathfrak{R}\}$ be an atlas of (X, \mathcal{E}) . Then a section of (X, \mathcal{E}) corresponds one to one to the following object.

- (1) For each \mathfrak{r} we have a Γ_r equivariant map $s_{\mathfrak{r}}: V_{\mathfrak{r}} \to E_{\mathfrak{r}}$, which is compatible in the sense of (2) below.
- (2) Suppose $\phi_{\mathfrak{r}_1}(x_1) = \phi_{\mathfrak{r}_2}(x_2)$. Then the definition of orbifold atlas implies that there exist subcharts $(V_{\mathfrak{r}_i,x_i},E_{\mathfrak{r}_i,x_i},\Gamma_{\mathfrak{r}_i,x_i},\phi_{\mathfrak{r}_i,x_i},\widehat{\phi}_{\mathfrak{r}_i})$ of the orbifold charts $(V_{\mathfrak{r}_i},E_{\mathfrak{r}_i},\Gamma_{\mathfrak{r}_i},\phi_{\mathfrak{r}_i},\widehat{\phi}_{\mathfrak{r}_i})$ at $x_i \in V_{\mathfrak{r}_i}$ for i=1,2 and an isomorphism of charts

$$(h_{12}^{\mathfrak{r},p}, \tilde{\varphi}_{12}^{\mathfrak{r},p}, \tilde{\hat{\varphi}}_{12}^{\mathfrak{r},p}) : (V_{\mathfrak{r}_{2},x_{2}}, E_{\mathfrak{r}_{2}}, \Gamma_{\mathfrak{r}_{2},x_{2}}, \phi_{\mathfrak{r}_{2},x_{2}}, \widehat{\phi}_{\mathfrak{r}_{2}}) \\ \to (V_{\mathfrak{r}_{1},x_{1}}, E_{\mathfrak{r}_{1},x_{1}}, \Gamma_{\mathfrak{r}_{1},x_{1}}, \phi_{\mathfrak{r}_{1},x_{1}}, \widehat{\phi}_{\mathfrak{r}_{1}}).$$

Now we require the following equality:

$$\tilde{\hat{\varphi}}_{12}^{\mathfrak{r},p}(s_{\mathfrak{r}_1}(y,\xi)) = s_{\mathfrak{r}_2}(\tilde{\varphi}_{12}^{\mathfrak{r},p}(y),\xi). \tag{23.9}$$

Proof. The proof is mostly the same as the corresponding standard result for the case of vector bundle on a manifold or on a topological space. Let $s:X\to \mathcal{E}$ be a section, which is an orbifold embedding. Let $p\in\phi_{\mathfrak{r}}(V_{\mathfrak{r}})$. Then there exist a subchart $(V_{\mathfrak{r},p},\Gamma_{\mathfrak{r},p},\phi_{\mathfrak{r},p})$ of $\mathfrak{V}_{\mathfrak{r}}$ and a subchart $(\widehat{V}_{\mathfrak{r},\tilde{p}},\Gamma_{\mathfrak{r},\tilde{p}},\phi_{\mathfrak{r},\tilde{p}})$ of $(V_{\mathfrak{r}}\times E_{\mathfrak{r}},\Gamma_{\mathfrak{r}},\phi_{\mathfrak{r},\tilde{p}})$ such that a representative $(h',\tilde{\varphi}')$ of s exists on the subcharts. Since $\pi\circ s$ =identity set-theoretically, it follows that $\pi_1(\tilde{\varphi}(y))\equiv y\mod\Gamma_p$ for any $y\in V_{\mathfrak{r},p}$. We take y such that $\Gamma_y=\{1\}$. Then, there exists a unique $\mu\in\Gamma_p$ such that $\pi_1(\tilde{\varphi}'(y))\equiv\mu y$. By continuity this μ is independent of y. (We use Condition 23.9 here.)

We replace \tilde{p} by $\mu^{-1}\tilde{p}$ and $(\widehat{V}_{\mathfrak{r},\tilde{p}},\Gamma_{\mathfrak{r},\tilde{p}},\phi_{\mathfrak{r},\tilde{p}})$ by $(\mu^{-1}\widehat{V}_{\mathfrak{r},\tilde{p}},\mu^{-1}\Gamma_{\tau,\tilde{p}}\mu,\phi_{\mathfrak{r},\tilde{p}}\circ\mu)$ and $(h',\tilde{\varphi}')$ by $(h'\circ\operatorname{conj}_{\mu},\tilde{\varphi}'\circ\mu^{-1})$. (Here $\operatorname{conj}_{\mu}(\gamma)=\mu\gamma\mu^{-1}$.) Therefore we may assume $\pi_1(\tilde{\varphi}'(y))=y$. Note that $\tilde{\varphi}'$ is h'-equivariant and π_1 is id-equivariant. Here id is the identity map $\Gamma_{\mathfrak{r},y}\to\Gamma_{\mathfrak{r},y}$. Therefore the identity map $V_{\mathfrak{r},p}\to V_{\mathfrak{r},p}$ is h' equivariant. Hence $h'=\operatorname{id}$.

In sum, we have the following. (We put $s_{\mathfrak{r},p} = \tilde{\varphi}'$.) For a sufficiently small $\mathfrak{V}_{\mathfrak{r},p}$ there exists uniquely a map $s_{\mathfrak{r},p}: V_{\mathfrak{r},p} \to V_{\mathfrak{r},p} \times E_{\mathfrak{r}}$ such that

- (a) $\pi_1(s_{\mathfrak{r},p}(x)) = x$
- (b) $s_{\mathfrak{r},p}$ is equivariant with respect to the embedding $\Gamma_{\mathfrak{r},p} \to \Gamma_{\mathfrak{r}}$. (Recall $\Gamma_{\mathfrak{r},p} = \{\gamma \in \Gamma_{\mathfrak{r}} \mid \gamma p = p\}$.)
- (c) $(id, s_{\mathfrak{r},p})$ is a local representative of s.

We can use uniqueness of such $s_{\mathfrak{r},p}$ to glue them to obtain a map $V_{\mathfrak{r}} \to V_{\mathfrak{r}} \times E_{\mathfrak{r}}$. By (a) this map is of the form $x \mapsto (x, \mathfrak{s}_{\mathfrak{r}}(x))$ for some map $\mathfrak{s}_{\mathfrak{r}} : V_{\mathfrak{r}} \to E_{\mathfrak{r}}$. This is the map $\mathfrak{s}_{\mathfrak{r}}$ required in (1). Since $x \mapsto \gamma^{-1}\mathfrak{s}_{\mathfrak{r}}(\gamma x)$ also has the same property, the uniqueness implies that $\mathfrak{s}_{\mathfrak{r}}$ is $\Gamma_{\mathfrak{r}}$ equivariant. (23.9) is also a consequence of the uniqueness.

Thus we find a map from the set of sections to the set of $(s_{\mathfrak{r}})_{\mathfrak{r} \in \mathfrak{R}}$ satisfying (1)(2). The construction of the converse map is obvious.

The next lemma is proved during the proof of Lemma 23.32.

Lemma 23.33. Let $(V_{\mathfrak{r}}, E_{\mathfrak{r}}, \Gamma_{\mathfrak{r}}, \phi_{\mathfrak{r}}, \widehat{\phi}_{\mathfrak{r}})$ be an orbifold chart of (X, \mathcal{E}) and s a section of (X, \mathcal{E}) . Then there exists uniquely a Γ equivariant map $s_{\mathfrak{r}}: V_{\mathfrak{r}} \to E_{\mathfrak{r}}$ such that the following diagram commutes.

$$V_{\mathfrak{r}} \times E_{\mathfrak{r}} \xrightarrow{\widehat{\phi}_{\mathfrak{r}}} \mathcal{E}_{\mathfrak{r}}$$

$$\downarrow id \times s_{\mathfrak{r}} \uparrow \qquad \qquad \uparrow s \qquad (23.10)$$

$$V_{\mathfrak{r}} \xrightarrow{\phi_{\mathfrak{r}}} X$$

Definition 23.34. We call the system of maps $s_{\mathfrak{r}}$ the *local expression* of s in the orbifold chart $(V_{\mathfrak{r}}, E_{\mathfrak{r}}, \Gamma_{\mathfrak{r}}, \phi_{\mathfrak{r}}, \widehat{\phi}_{\mathfrak{r}})$.

Next we review the proofs of a few well-known facts on pull-back bundle etc.. Those proofs are straightforward generalization of the corresponding proofs for the pull-back in the manifold theory. We include them only for completeness' sake.

Proposition 23.35. Let \mathcal{E} be a vector bundle on $X \times [0,1]$, where X is an orbifold. We identify $X \times \{0\}$, $X \times \{1\}$ with X in an obvious way. Then there exists an isomorphism of vector bundles

$$I: \mathcal{E}|_{X \times \{0\}} \cong \mathcal{E}|_{X \times \{1\}}.$$

Suppose in addition that we are given a compact set $K \subset X$, its neighborhood V and an isomorphism

$$I_0: \mathcal{E}|_{V \times [0,1]} \cong \mathcal{E}|_{V \times \{0\}} \times [0,1].$$

Then we may choose I so that it coincides with the isomorphism induced by I_0 on K. If K is a submanifold, we may take K = V.

For the proof of the proposition we use the notion of connection on vector bundle on orbifolds, which we now recall here. Note that a vector field on an orbifold is a section of the tangent bundle.

Definition 23.36. A connection on a vector bundle (X, \mathcal{E}) is an \mathbb{R} linear map

$$\nabla: C^{\infty}(TX) \otimes_{\mathbb{R}} C^{\infty}(\mathcal{E}) \to C^{\infty}(\mathcal{E})$$

such that $\nabla_X(V) = \nabla(X, V)$ satisfies

$$\nabla_{fX}(V) = f\nabla_X(V), \qquad \nabla_X(fV) = f\nabla_X(V) + X(f)V.$$

Here $C^{\infty}(\mathcal{E})$ is the vector space consisting of all smooth sections of \mathcal{E} .

For any connection ∇ and piecewise smooth map $\ell:[a,b]\to X$ we obtain parallel transport

$$\operatorname{Pal}^{\nabla}: \mathcal{E}_{\ell(a)} \to \mathcal{E}_{\ell(b)}$$

in the same way as in the case of manifolds.

Remark 23.37. Here $\mathcal{E}_{\ell(a)}$ is the fiber of \mathcal{E} at $\ell(a) \in X$ and is defined as follows. We take a chart $(V_{\mathbf{r}}, E_{\mathbf{r}}, \Gamma_{\mathbf{r}}, \psi_{\mathbf{r}}, \widehat{\psi}_{\mathbf{r}})$ of (\mathcal{E}, X) at $\ell(a)$. Then $\mathcal{E}_{\ell(a)} = E_{\mathbf{r}}$. If $(V_{\mathbf{r}'}, E_{\mathbf{r}'}, \Gamma_{\mathbf{r}'}, \psi_{\mathbf{r}'}, \widehat{\psi}_{\mathbf{r}'})$ is another chart, we can identify $E_{\mathbf{r}}$ with $E_{\mathbf{r}'}$ by $\xi \mapsto \widecheck{\varphi}_{\mathbf{r}'\mathbf{r}}(\xi, y)$ where $\psi_{\mathbf{r}}(y) = \ell(a)$ and $\widecheck{\varphi}_{\mathbf{r}'\mathbf{r}} : V_{\mathbf{r}} \times E_{\mathbf{r}} \to E_{\mathbf{r}'}$ is a part of the coordinate change. ([Part I, Situaion 6.3].) Note that the identification $\xi \mapsto \widecheck{\varphi}_{\mathbf{r}'\mathbf{r}}(\xi, y)$ is well-defined up to the $\Gamma_{\ell(a)} = \{\gamma \in \Gamma_{\mathbf{r}} \mid \gamma(y) = y\}$ action. Thus the parallel transport $\mathrm{Pal}^{\nabla} : \mathcal{E}_{\ell(a)} \to \mathcal{E}_{\ell(b)}$ is well-defined up to the $\Gamma_{\ell(a)} \times \Gamma_{\ell(b)}$ action.

Lemma 23.38. Any vector bundle \mathcal{E} over orbifold X has a connection. Moreover if a connection is given for $\mathcal{E}|_V$ where V is an open neighborhood of a compact subset K of X, then we can extend it without changing it on K. If K is a submanifold, we may take K = V.

The proof is an obvious analogue of the proof of the existence of connection on a vector bundle over a manifold, which uses a partition of unity.

We are now ready to give the proof of Proposition 23.35.

Proof of Proposition 23.35. We take a connection on $\mathcal{E}|_V$. We then take direct product connection on $\mathcal{E}|_{V\times\{0\}}\times[0,1]$, and use I_0 to obtain a connection on $\mathcal{E}|_{V\times[0,1]}$. We extend it to a connection on \mathcal{E} without changing it on $K\times[0,1]$. For each fixed $x\in X$, we can use parallel transportation along the path $t\mapsto(x,t)$ to get an isomorphism $\mathcal{E}_{(x,0)}\cong\mathcal{E}_{(x,1)}$. We have thus obtained a set theoretical map

$$|\mathcal{E}|_{X\times\{0\}}|\cong |\mathcal{E}|_{X\times\{1\}}|.$$

It is easy to see that it induces an isomorphism of vector bundles. Using the fact that our connection is direct product on $K \times [0,1]$, we can check the second half of the statement.

Definition 23.39. We say two embeddings of orbifold $f_i: X \to Y$ (i = 1, 2) are *isotopic* to each other if there exists an embedding of orbifolds $H: X \times [0, 1] \to Y \times [0, 1]$ such that the second factor of H(x, t) is t and that

$$H(x,0) = (f_1(x),0)$$
 $H(x,1) = (f_2(x),1).$

Suppose $V \subset X$ and $f_1 = f_2$ on a neighborhood V of K. We say f_1 is isotopic to f_2 relative to K if we may take H such that

$$H(x,t) = (f_1(x),t) = (f_2(x),t)$$
(23.11)

for x in a neighborhood of K. In case K is a submanifold, we may take K = V and then (23.11) holds for $x \in K$.

Corollary 23.40. Let $f_i: X \to Y$ be two isotopic embeddings and \mathcal{E} a vector bundle on Y. Then the pull-back bundle $f_1^*\mathcal{E}$ is isomorphic to $f_2^*\mathcal{E}$. If $f_1 = f_2$ on a neighborhood of $K \subset X$ and f_1 is isotopic to f_2 relative to K, then we may choose the isomorphism $f_1^*\mathcal{E} \cong f_2^*\mathcal{E}$ so that its restriction to K is the identity map.

Proof. This is immediate from Proposition 23.35 and the definition. \Box

We next recall [Part I, Definition 12.23] which we re-state here.

Definition 23.41. Let $f: X \to Y$ be an embedding of orbifolds and $K \subset X$ a compact subset and U an open neighborhood of K in Y.⁵² We say that a continuous map $\pi: U \to X$ is diffeomorphic to the projection of the normal bundle if the following holds.

Let pr: $N_XY \to X$ be the normal bundle. Then there exists a neighborhood U' of K in N_XY , (Note $K \subset X \subset N_XY$.) and a diffeomorphism of orbifolds $h: U' \to U$ such that $\pi \circ h = \operatorname{pr}$. We also require that h(x) = x for x in a neighborhood of K in X.

Definition-Lemma 23.42. Let $\pi: U \to X$ be diffeomorphic to the projection of the normal bundle as in Definition 23.41 and \mathcal{E} a vector bundle on X. We define $\pi^*\mathcal{E}$, the pull-back bundle as follows.

Let h, U' be as in Definition 23.41. We defined a pull-back bundle $\operatorname{pr}^*\mathcal{E}$ on N_XY in Definition 23.29. We put

$$\pi^* \mathcal{E} = (h^{-1})^* \operatorname{pr}^* \mathcal{E}|_{U'}.$$

 $^{^{52}\}mathrm{Here}$ we regard the image f(K) as a subset of Y via the embedding f, and write f(K) as K to simply the notation.

This is independent of the choice of (U', h) in the following sense. Let U'_i , h_i (i = 1, 2) be two choices. Then we can shrink U and U'_i so that for each i the restriction of h_i becomes an isomorphism between them. Then

$$(h_1^{-1})^* \operatorname{pr}^* \mathcal{E}|_{U_1'} \cong (h_2^{-1})^* \operatorname{pr}^* \mathcal{E}|_{U_2'}.$$
 (23.12)

Moreover the isomorphism (23.12) can be taken so that the following holds in addition. We regard $K \subset U$. Then by definition it is easy to see that the restriction of both sides of (23.12) is canonically identified with the restriction of \mathcal{E} to $K \subset X$. The isomorphism (23.12) becomes the identity map on K under this isomorphism.

Proof. We can replace U by a smaller open neighborhood so that $h_1^{-1}: U \to N_X Y$ is isotopic to $h_2^{-1}: U \to N_X Y$. (See the proof of Proposition 23.43 below.) Then (23.12) follows from Corollary 23.40. The second half of the claim also follows from the second half of Corollary 23.40.

The pull-back bundle is independent of the projection π but depends only on the neighborhood U in the situation of Definition 23.41. In fact, we have

Proposition 23.43. Let $\pi_i: U \to X$ be as in Definition 23.41 for i = 1, 2. Then there exist a neighborhood U_0 of X in Y and a map $f: U_0 \to U$ such that

- (1) $\pi_2 \circ f = \pi_1$
- (2) $f: U_0 \to U$ is isotopic to the inclusion map $U_0 \hookrightarrow U$ relative to X.

Proof. Let $h_i: U_i' \to U_i$ be as in Definition 23.41. We put $f = h_2 \circ h_1^{-1}$ which is defined for sufficiently small U_0 . If suffices to show that f is isotopic to the inclusion map. We first prove the proposition in the case when the following additional assumption is satisfied. (We will remove this assumption later.)

Assumption 23.44. For any $x \in K \subset N_X Y$ the first derivative at x, $D_x f : T_x(N_X Y) \to T_x(N_X Y)$ is the identity map.

For the case of manifolds, this assumption enables us to prove Proposition 23.43 by observing that f is C^1 -close to the inclusion map. Then, for example, using minimal geodesic, we can show that f is isotopic to the inclusion map.

For the case of orbifolds, we need to work out this last step a bit more carefully since the number

 $\inf\{r \mid \text{if } d(x,y) < r, \text{ the minimal geodesic joining } x \text{ and } y \text{ is unique}\}$

can be 0 in general unlike the case of manifolds.

To clarify this point, we need to prepare certain lemmas whose statements require some digression. We can define the notion of Riemannian metric of orbifold X in a straightforward way similarly as in the manifold case. For $p \in X$ we have a geodesic coordinate $(TB_p(c_p), \Gamma_p, \psi_p)$ where

$$TB_p(c_p) = \{ \xi \in T_p X \mid ||\xi|| < c_p \}$$

and the group Γ_p is the isotropy group of the orbifold chart of X at p. The uniformization map $\psi: TB_p(c_p) \to X$ is defined by using minimal geodesic in the same way as the construction of the exponential map in the standard Riemannian geometry. We note that this map is well defined up to the action of Γ_p . We need to take the number c_p small so that the exponential map ψ induces a homeomorphism $TB_p(c_p)/\Gamma_p \to X$. We may not be able to choose c_p uniformly away from 0 even on the compact subset of the given orbifold in general. (This is because $d(p,q) < c_p$

must imply $\#\Gamma_q \leq \#\Gamma_p$, when c_p is sufficiently small.) However we can prove the following. Let X be an orbifold and Z a compact set. Suppose $B_{c_0}(Z) = \{x \mid$ $d(x,Z) \leq c_0$ is complete with respect to the metric induced by the Riemannian metric.

Lemma 23.45. Let $Z \subset X$ be a compact subset. Then there exists a finite set $\{p_i \mid j \in J\} \subset Z \text{ and } 0 < c_j < c_0 \text{ such that }$

- (1) The geodesic coordinate $(TB_{p_i}(c_j), \Gamma_{p_i}, \psi_{p_i})$ exists.
- (2)

$$Z \subset \bigcup_{j} \psi_{p_j}(TB_{p_j}(c_j/2)).$$

The proof is immediate from the compactness of Z. We call such $\{(TB_{p_j}(c_j), \Gamma_{p_j}, \psi_{p_j}) \mid$ j} a geodesic coordinate system of (X, Z). We put $P = \{p_j \mid j = 1, \dots, \tilde{J}\}$.

Definition 23.46. We fix a geodesic coordinate system of (X, Z). Let $Z_0 \subset Z$ be a compact subset containing P and $F:U\to X$ be an embedding of orbifolds where $U \supset Z$ is an open neighborhood of Z. We say F is C^1 ϵ -close to the identity on Z_0 if the following holds.

- (1) $F(B_{p_i}(c_j/2)) \subset B_{p_i}(c_j)$.
- (2) There exists $\tilde{F}_j: B_{p_j}(c_j/2) \to B_{p_j}(c_j)$ such that:
 - (a) $\psi_{p_j} \circ \tilde{F}_j = F \circ \psi_{p_j}$.

 - (b) $d(x, \tilde{F}_{j}(x)) < \epsilon$ for $x \in TB_{p_{j}}(c_{j}/2) \cap \psi_{p_{j}}^{-1}(Z_{0})$. (c) $d(D_{x}\tilde{F}_{j}, id) < \epsilon$ for $x \in TB_{p_{j}}(c_{j}/2) \cap \psi_{p_{j}}^{-1}(Z_{0})$.

Here d in Item (b) is the standard metric on Euclidean space $T_{p_i}X$ (together with the metric induced by our Riemannian metric), d in Item (c) is a distance in the space of $n \times n$ matrices. (Here $n = \dim X$. We use our Riemannian metric to define a metric on this space of matrices, which is a vector space of dimension n^2 with metric.)

Lemma 23.47. For each Z and a geodesic coordinate system of (X, Z), there exists $\epsilon > 0$ such that the following holds for any given $Z_0 \subset Z$ and $F: U \to X$:

If F is C^1 ϵ -close to the identity on Z_0 , then F is isotopic to the identity on Z_0 . Moreover for any $\delta > 0$ there exists $\epsilon(\delta) > 0$ such that if F is C^1 $\epsilon(\delta)$ -close to the identity on Z_0 , then the isotopy from F to the identity map is taken to be C^1 δ -close to the identity on Z_0 .

Proof. We first observe that if $\epsilon > 0$ is sufficiently small, then for each j the map F_j satisfying Definition 23.46 (2) (a),(b) and (c) is unique. Indeed, it easily follows that such F_j is unique up to the action of Γ_{p_j} . Since the Γ_{p_j} action is effective, Γ_{p_j} is a finite group and $p_j \in Z_0$, we find that at most one such F_j can satisfy (c). (Note that the map $\Gamma_{p_j} \to O(n)$ taking the linear part at p_j of the action is injective since the Γ_{p_j} action has a fixed point and is effective.)

Next for each $t \in [0,1]$, we define a map

$$\tilde{F}_{t,j}: V_j \to TB_{p_j}(c_j/2)$$

as follows. Here V_j is a sufficiently small neighborhood of $TB_{p_j}(c_j/2) \cap \psi_j^{-1}(Z_0)$. We take a Riemannian metric on V_i that is the pull-back of the given Riemannian metric on X by the map ψ_{p_i} . By choosing ϵ sufficiently small and using (b), we find a unique minimal geodesic $\ell_{x,j}:[0,1]\to TB_{p_j}(c_j/2+2\epsilon)$ joining x to $\tilde{F}_j(x)$. We put

$$\tilde{F}_{t,j}(x) = \ell_{x,j}(t).$$

In the same way as in the proof of the uniqueness of \tilde{F}_j we can show that there exists a map F_t such that $\psi_{p_j} \circ \tilde{F}_{t,j} = F_t \circ \psi_{p_j}$. Using Definition 23.46 (2)(b) and (c), we can show that F_t is C^1 ϵ -close to the identity map. By choosing $\epsilon > 0$ sufficiently small, we derive that F_t is a diffeomorphism to its image. Thus F_t is the required isotopy from F to the identity map.

Now we use Lemma 23.47 to complete the proof of Proposition 23.43 under Assumption 23.44. Recall from the beginning of the proof of Proposition 23.43 that we put $f = h_2 \circ h_1^{-1}$ which we want to prove is isotopic to the identity map in a neighborhood of $K \subset X \subset Y$.

We take a finite set of points $p_j \in K$ so that $K \subset \bigcup_j \psi_{p_j}(TB_{p_j}(c_j/2))$. Then we take a compact neighborhood $Z \supset K$ such that $Z \subset \bigcup_j \psi_{p_j}(TB_{p_j}(c_j/2))$. We apply Lemma 23.47 to obtain ϵ . Note that f is the identity map on K and its first derivative at any point in K is also the identity by Assumption 23.44. Therefore we can find a sufficiently small compact neighborhood Z_0 of K so that f is C^1 ϵ -close to the identity on Z_0 . Thus Lemma 23.47 implies that f is isotopic to the identity map. The proof of Proposition 23.43 under the additional Assumption 23.44 is now complete.

To remove Assumption 23.44, we use the following lemma.

Lemma 23.48. Let U be an open neighborhood of K in N_XY and $F: U \to N_XY$ be an open embedding of orbifolds. Assume $F = \operatorname{id}$ on a neighborhood of K in X and $D_xF(V) \equiv V \mod T_xX$ at any x contained in a neighborhood of K.

Then there exists a smaller neighborhood U' of K such that the restriction of F to U' is isotopic to the embedding satisfying Assumption 23.44.

Proof. We take the first derivative of F

$$D_xF:T_xN_XY\to N_XY$$

at x contained in a neighborhood of K in X. Note that $T_xX \subset N_XY$ is preserved under this map and we have a decomposition $T_xN_XY = T_xX \oplus (N_XY)_x$. Therefore there exists a linear bundle map

$$H: N_X Y \to TX$$

on a neighborhood of K such that

$$D_x F(\xi, \eta) = (\xi + H_x(\eta), \eta),$$

where $\xi \in T_x X$ and $\eta \in (N_X Y)_x$. Now we define $G_t : U' \to N_X Y$ as follows. (Here U' is a small neighborhood of K in $N_X Y$.) Let $(x, \eta) \in U'$ where x is in a neighborhood of K in X and $\eta \in (N_X Y)_x$. We take a geodesic $\ell : [0, 1] \to X$ of constant speed with $\ell(0) = x$ and $D\ell/dt(0) = H_x(\eta)$. Let $\ell_{\leq t}$ be its restriction to [0, t]. Then $G_t(x, \eta) = (\ell(t), \operatorname{Pal}_{\ell \leq t}(\eta))$, where $\operatorname{Pal}_{\ell \leq t}(\eta) \in (N_X Y)_{\ell(t)}$ is the parallel transport along $\ell_{\leq t}$. By construction the first derivative of G_t at a point in K is given by

$$(\xi, \eta) \mapsto (\xi + tH_x(\eta), \eta),$$

which is invertible. Therefore, if V' is a sufficiently small neighborhood of K, then the restriction of G_t is an embedding $V' \to N_X Y$. Note that $F \circ G_1^{-1}$ satisfies Assumption 23.44. Thus the proof of Lemma 23.48 is complete.

Using Lemma 23.48, we can reduce the general case of Proposition 23.43 to the case when Assumption 23.44 is satisfied. The proof of Proposition 23.43 is complete. \Box

We used the next result in [Part I, Subsection 13.2].

Proposition 23.49. Let $f: X \to Y$ be an embedding of orbifolds and K_i compact subsets of X for i = 1, 2 such that $K_1 \subset \operatorname{Int} K_2$. ⁵³ Suppose U_1 is an open neighborhood K_1 in Y and $\pi_1: U_1 \to X$ is diffeomorphic to the projection of the normal bundle in the sense of Definition 23.41

Then there exists an open neighborhood U_2 of K_2 in Y and $\pi_2: U_2 \to X$ such that it is diffeomorphic to the projection of the normal bundle and $\pi_1 = \pi_2$ on an open neighborhood of K_1 .

Proof. For the case of manifolds, the corresponding result is standard and can be proved by applying the isotopy extension lemma. We can substitute the isotopy extension lemma by Lemma 23.47 in the same way to handle the orbifold case. For completeness' sake we give detail of the proof below.

We first apply [FOOO13, Lemma 6.5] to obtain an open neighborhood U_2' of K_2 in Y and a submersion $\pi_2': U_2' \to X$ such that (U_2', π_2') is diffeomorphic to the normal bundle N_XY . We may adjust the map π_2' to π_2 so that $\pi_1 = \pi_2$ on an open neighborhood of K_1 whose detail is in order.

Let $W_1^{(i)}$ be a neighborhood of K_1 in X such that

$$\overline{W_1^{(1)}} \subset W_1^{(2)} \subset \overline{W_1^{(2)}} \subset U_1 \cap X.$$

Let Ω be an open subset of U_1 with

$$\overline{W_1^{(2)}} \subset \Omega \subset \overline{\Omega} \subset U_1.$$

Later on we will choose Ω so that it is contained in a sufficiently small neighborhood of X in Y. We put

$$V_1^{(i)} = \pi_1^{-1}(W_1^{(i)}) \cap \Omega.$$

Let $\tilde{\chi}: \operatorname{Int}(K_2) \cup \Omega \to [0,1]$ be a smooth function such that

$$\tilde{\chi} = \begin{cases} 1 & \text{on } W_1^{(1)} \\ 0 & \text{on the complement of } W_1^{(2)}, \end{cases}$$

and put $\chi = \tilde{\chi} \circ \pi'_2$. We may choose Ω so small that the following holds.

$$\chi = \begin{cases} 1 & \text{on } V_1^{(1)} \\ 0 & \text{on the complement of } V_1^{(2)}. \end{cases}$$

Let $Z = \overline{W_1^{(2)}} \setminus W_1^{(1)}$. We take a neighborhood U' of Z and restrict π_1 and π_2' there. Then we can apply Proposition 23.43 to prove that there exists an isotopy

⁵³Hereafter we simply write f(X), $f(K_i)$ as X, K_i and regard them as subsets of Y via the embedding f, when no confusion can occur.

 $F_t: U' \to X$ such that $\pi'_2 \circ F_1 = \pi_1$ and F_0 is the identity map. Now we put

$$\pi_2(x) = \begin{cases} (\pi_2 \circ F_{\chi(x)})(x) & \text{on } U' \\ \pi_1 & \text{on } V_1^{(1)} \\ \pi_2'(x) & \text{elsewhere on } U_2'. \end{cases}$$

It is easy to see that they are glued to define a map. To complete the proof it suffices to show that $x \mapsto F_{\chi(x)}(x)$ is an embedding $: U' \to X$. We will prove this below.

We first consider the case that Assumption 23.44 is satisfied for the map $f: U' \to X$ with $\pi'_2 \circ f = \pi_1$. In this case we may choose the isotopy F_t arbitrarily close to the identity map in C^1 sense by taking Ω sufficiently small. (This is the consequence of the second half of Lemma 23.47.) Therefore the first derivative of $x \mapsto F_{\chi(x)}(x)$ is close to the identity. It follows that this map is an embedding.

We finally show that we can choose (U_2', π_2') so that Assumption 23.44 is satisfied for the map $f: U' \to X$ with $\pi_2' \circ f = \pi_1$. We consider the fiber $\pi_1^{-1}(x)$ of π_1 . We may choose a Riemannian metric of X in a neighborhood of K_1 so that this fiber is perpendicular to X at any x in a neighborhood of K_1 . We now extend this Riemannian metric to the whole X. We use this Riemannian metric and the associated exponential map in the normal direction to identify a neighborhood of K_2 with its normal bundle and obtain U_2' and π_2' . Then Assumption 23.44 is satisfied. The proof of Proposition 23.49 is now complete.

24. Covering space of effective orbifold and K-space

24.1. Covering space of orbifold. We first define the notion of a covering space of an orbifold. Let U_1, U_2 be orbifolds and let $\pi: U_1 \to U_2$ be a continuous map between their underlying topological spaces.

Definition 24.1. For i = 1, 2 let $x_i \in U_i$ with $\pi(x_1) = x_2$ and $\mathfrak{V}_i = (V_i, \Gamma_i, \phi_i)$ be orbifold charts of U_i at x_i . We say that $(\mathfrak{V}_1, \mathfrak{V}_2)$ is a *covering chart* if the following holds:

- (1) There exists an injective group homomorphism $h_{21}: \Gamma_1 \to \Gamma_2$.
- (2) There exists an h_{21} -equivariant diffeomorphism $\varphi_{21}: V_1 \to V_2$.
- (3) $\phi_2 \circ \varphi_{21} = \phi_1$.

The index $[\Gamma_2: h_{21}(\Gamma_1)]$ is called the *covering index* of the covering chart $(\mathfrak{V}_1, \mathfrak{V}_2)$.

Definition 24.2. The map $\pi: U_1 \to U_2$ is called a *covering map* if the following holds at each $x \in U_2$.

- (1) The set $\pi^{-1}(x)$ is a finite set, which we write $\{\tilde{x}_1, \dots, \tilde{x}_{m_x}\}$.
- (2) There exist an orbifold chart \mathfrak{V}_x of U_2 at x and orbifold charts $\mathfrak{V}_{\tilde{x}_j}$ of U_1 at \tilde{x}_j respectively for $j=1,\ldots,m_x$ such that $(\mathfrak{V}_{\tilde{x}_j},\mathfrak{V}_x)$ is a covering chart. (Here \mathfrak{V}_x is independent of j.) We write its covering index $n_j(x)$.
- (3) $\sum_{j=1}^{m_x} n_j(x)$ is independent of x.

We call $\sum_{j=1}^{m_x} n_j(x)$ the covering index of π .

Remark 24.3. (1) We only define a finite covering here since we do not use an infinite covering in the present article.

(2) If Definition 24.2 (1)-(2) is satisfied and U_2 is connected, then (3) is equivalent to the following condition:

(3)' $\pi^{-1}(U_x) = \bigcup_{j=1}^{m_x} U_{\tilde{x}_j}$ and the right hand side is a disjoint union.

In fact, (3)' implies that $\sum_{j=1}^{m_x} n_j(x)$ is a locally constant function. We can replace (3) by (3)' and define an infinite covering in the same way.

(3) The composition of covering maps is a covering map.

Lemma 24.4. Let $\varphi_{21}: U_1 \to U_2$ be an embedding of orbifolds and $\pi_2: \widetilde{U}_2 \to U_2$ a covering map of orbifolds. We consider the fiber product $U_1 \times_{U_2} \widetilde{U}_2$ in the category of topological spaces. It comes with continuous maps $\pi_1: U_1 \times_{U_2} \widetilde{U}_2 \to U_1$ and $\widetilde{\varphi}_{21}: U_1 \times_{U_2} \widetilde{U}_2 \to \widetilde{U}_2$. Then $U_1 \times_{U_2} \widetilde{U}_2$ has a structure of orbifolds such that:

- (1) π_1 is a covering map.
- (2) $\widetilde{\varphi}_{21}$ is an embedding of orbifolds.

The conditions (1) (2) uniquely determine the orbifold structure on $U_1 \times_{U_2} \widetilde{U}_2$.

Proof. We take the atlas $\{\mathfrak{V}_{\mathfrak{r}}^i \mid \mathfrak{r} \in \mathfrak{R}_i\}$ as in Lemma 23.13. We may choose it sufficiently fine so that for each $\mathfrak{r} \in \mathfrak{R}_2$ there exist $\mathfrak{V}_j^{2,\mathfrak{r}}$, $j=1,\ldots,m_{\mathfrak{r}}$ such that $(\mathfrak{V}_j^{2,\mathfrak{r}},\mathfrak{V}_{\mathfrak{r}}^2)$ is a covering atlas and satisfies $\pi^{-1}(U_{\mathfrak{r}}^2) = \bigcup_{j=1}^{m_{\mathfrak{r}}} U_{\mathfrak{r},j}^2$. For each $j=1,\ldots,m_{\mathfrak{r}}$, the map π determines a finite index subgroup $\Gamma_{j,\mathfrak{r}}^2$ of $\Gamma_{\mathfrak{r}}^2$ for $\mathfrak{r} \in \mathfrak{R}_2$. Note that in case $\mathfrak{r} \in \mathfrak{R}_1$ the group $\Gamma_{\mathfrak{r}}^2$ is isomorphic to $\Gamma_{\mathfrak{r}}^1$. Then a finite subgroup $\Gamma_{j,\mathfrak{r}}^1$ of $\Gamma_{\mathfrak{r}}^1$ determines $\Gamma_{j,\mathfrak{r}}^2 \subset \Gamma_{\mathfrak{r}}^2$. Therefore the collection $(V_{\mathfrak{r}}^1, \Gamma_{\mathfrak{r},j}^1, \phi_{\mathfrak{r},j}^1)$ determines an orbifold chart. Here the map $\phi_{\mathfrak{r},j}^1$ is defined by

$$\phi_{\mathfrak{r},j}^{1}(y) = (\phi_{\mathfrak{r}}^{1}(y), \phi_{\mathfrak{r},j}^{2}(\varphi_{21}^{\mathfrak{r}}(y))),$$

where $\varphi_{21}^{\mathfrak{r}}: V_{1}^{\mathfrak{r}} \to V_{2}^{\mathfrak{r}}$ is determined by the orbifold embedding φ_{21} and $\phi_{\mathfrak{r},j}^{2}: V_{2}^{\mathfrak{r}} \to \widetilde{U}_{2}$ is a part of the orbifold chart $\mathfrak{V}_{j}^{2,\mathfrak{r}}$. It is easy to check that the charts $(V_{\mathfrak{r}}^{1}, \Gamma_{\mathfrak{r},j}^{1}, \phi_{\mathfrak{r},j}^{1})$ for various \mathfrak{r} and j determine an orbifold structure on the fiber product. We can easily check (1), (2). The uniqueness is also easy to check. \square

Lemma-Definition 24.5. We can pull back a vector bundle and its section by a covering map of an orbifold.

Proof. Let $\widetilde{U} \to U$ be a covering and $\mathcal{E} \to U$ a vector bundle. We take a local coordinate $(V \times E, \Gamma, \widehat{\phi})$ of $\mathcal{E} \to U$ at $p \in U$ where (V, Γ, ϕ) is the corresponding chart of U. We may shrink V and may assume that there is a covering chart (V, Γ_i, ϕ_i) for each $p_i \in \widetilde{U}$ with $\pi(p_i) = p$. The two maps $\phi_i \circ \operatorname{pr}_V : V \times E \to \widetilde{U}$ and $\widehat{\phi} : V \times E \to |\mathcal{E}|$ give rise to a map

$$\widehat{\phi}_i: V \times E \to \widetilde{U} \times_U |\mathcal{E}|.$$

It is easy to see that $(V \times E, \Gamma_i, \widehat{\phi}_i)$ gives a structure of vector bundle on $\widetilde{U} \times_U |\mathcal{E}| \to \widetilde{U}$.

Remark 24.6. Our covering map in the sense of Definition 24.2 defines a *good map* in the sense of [ALR]. Then the above lemma is a special case of [ALR, Theorem 2.43].

Lemma 24.7. Let X be a topological space, Y an orbifold, and let $f: X \to Y$ be a continuous map. Then the orbifold structure on X, under which f becomes a covering map of orbifolds, is unique if it exists.

Proof. Suppose we have two such structures. It suffices to show that the identity map is a diffeomorphism of orbifold. Let $p \in Y$ and (V, Γ, ϕ) be an orbifold chart of X at p. We may shrink V if necessary and assume that there are covering charts (V_i, Γ_i, ϕ_i) (i = 1, 2) for two orbifold structures on X such that $f: X \to Y$ is a covering map. We put

$$S(V) = \{x \in V \mid \exists \gamma \in \Gamma \setminus \{1\}, \ \gamma x = x\}, \quad U^{\text{reg}} = (V \setminus S(V)) / \Gamma.$$

Note that $f^{-1}(U^{\text{reg}}) \to U^{\text{reg}}$ is a covering space. An element of $\gamma \in \Gamma$ is contained in the image of Γ_i if and only if the corresponding loop in U^{reg} lifts to a loop in $f^{-1}(U^{\text{reg}})$. Therefore $\Gamma_1 = \Gamma_2$ as subgroups of Γ . It is now easy to see that the identity map is a diffeomorphism at p_i .

24.2. Covering space of K-space. Using Lemma 24.4 and Lemma-Definition 24.5, it is fairly straightforward to define the notion of a covering space of a K-space and show that various objects are pulled back by a covering map. We will describe them below for completeness.

Situation 24.8. Let $\mathcal{U} = (U, \mathcal{E}, s, \psi)$ be a Kuranishi chart of X. Suppose we are given a topological space \widetilde{X} and a continuous map $\pi : \widetilde{X} \to X$.

Definition 24.9. We call $\widetilde{\mathcal{U}} = (\widetilde{U}, \widetilde{\mathcal{E}}, \widetilde{s}, \widetilde{\psi})$ a covering chart if \widetilde{U} is a covering space of the orbifold U, $\widetilde{\mathcal{E}}$ is the pull-back of \mathcal{E} by $\widetilde{U} \to U$, \widetilde{s} is obtained from s by the pull-back and $\widetilde{\psi}$ is a homeomorphism from $\widetilde{s}^{-1}(0)$ to \widetilde{X} . We require $\pi \circ \widetilde{\psi} = \psi$.

Lemma-Definition 24.10. Suppose we are in Situation 24.8. Let $\mathcal{U}' = (U', \mathcal{E}', s', \psi')$ be another Kuranishi chart and $\Phi : \mathcal{U}' \to \mathcal{U}$ an embedding of Kuranishi chart. Let $\widetilde{\mathcal{U}}$ be a covering chart of \mathcal{U} . Then we can define a covering chart $\widetilde{\mathcal{U}}' = (\widetilde{U}', \widetilde{\mathcal{E}}', \widetilde{s}', \widetilde{\psi}')$ of \mathcal{U}' and an embedding of Kuranishi chart $\widetilde{\Phi} : \widetilde{\mathcal{U}}' \to \widetilde{\mathcal{U}}$ such that

$$\widetilde{U}' = U'_{\Phi} \times_{U} \widetilde{U} \tag{24.1}$$

with the orbifold structure given by Lemma 24.4. We call $\widetilde{\mathcal{U}}'$ the pull-back covering chart of $\widetilde{\mathcal{U}}$ by Φ . In particular, if U_0 is an open subset of U, then we can define a restriction $\widetilde{\mathcal{U}}'|_{U_0}$ of $\widetilde{\mathcal{U}}'$.

Proof. We define \widetilde{U}' by (24.1). Then $\widetilde{U}' \to U'$ is a covering space. Therefore by Lemma 24.5 we can pull back E to \widetilde{U}' . We define E' by the pull-back. We can then define \widetilde{s}' , $\widetilde{\psi}'$ from definition. The existence of $\widetilde{\Phi}$ is immediate from construction. \square

Definition 24.11. Let $\widehat{\mathcal{U}}$ be a Kuranishi structure on X, $\pi:\widetilde{X}\to X$ a finite to one continuous map. Let $\widehat{\widetilde{\mathcal{U}}}$ be a Kuranishi structure on \widetilde{X} . We say that $(\widetilde{X},\widehat{\widetilde{\mathcal{U}}})$ is a *covering space* of $(X,\widehat{\mathcal{U}})$ if the following holds.

- (1) For each $p \in X$ we are given a covering chart $\widetilde{\mathcal{U}}_p$ of the Kuranishi chart \mathcal{U}_p , which is a part of the data consisting $\widehat{\mathcal{U}}$.
- (2) If $q \in \psi_p(s_p^{-1}(0))$, then the restriction $\widetilde{\mathcal{U}}_q|_{U_{pq}}$ of the covering chart $\widetilde{\mathcal{U}}_q$ given in Item (1) is an open subchart of the pull-back covering chart of $\widetilde{\mathcal{U}}_p$ by the coordinate change Φ_{pq} .
- (3) Let $p_j \in \widetilde{X}$ and $p \in X$ with $\pi(p_j) = p$ and $\mathcal{U}_{p_j} = (U_{p_j}, \mathcal{E}_{p_j}, s_{p_j}, \psi_{p_j})$ a Kuranishi chart of \widetilde{X} at p_j which is a part of the data of $(\widetilde{X}, \widehat{\mathcal{U}})$. Then there

exists an open embedding Φ_{p_j} of the Kuranishi chart \mathcal{U}_{p_j} to the covering chart $\widetilde{\mathcal{U}}_p$ given in Item (1).

(4) For any $q \in \psi_p(s_p^{-1}(0))$, $q_j \in \psi_{p_j}(s_{p_j}^{-1}(0))$, the following diagram commutes.

$$\mathcal{U}_{q_{j}}|_{U_{p_{j}q_{j}}} \xrightarrow{\Phi_{p_{j}q_{j}}} \mathcal{U}_{p_{j}}$$

$$\Phi_{q_{j}} \downarrow \qquad \qquad \downarrow \Phi_{p_{j}}$$

$$\widetilde{\mathcal{U}}_{q}|_{U_{p_{q}}} \xrightarrow{\widetilde{\Phi}_{p_{q}}} \widetilde{\mathcal{U}}_{p}$$

$$(24.2)$$

Here $\Phi_{p_jq_j}$ is the coordinate change of the Kuranishi structure $(\widetilde{X},\widehat{\widetilde{\mathcal{U}}})$ and $\widetilde{\Phi}_{pq}$ is obtained by Lemma-Definition 24.10.

(5) Let $n_{p,j}$ be the covering index of the covering $\widetilde{U}_{p,j} \to U_p$. Then the number

$$\sum_{p_j \in \widetilde{X}: \pi(p_j) = p} n_{p,j} \tag{24.3}$$

is independent of p. We call it the *covering index* of the covering $(\widetilde{X},\widehat{\widetilde{\mathcal{U}}})$ of $(X,\widehat{\mathcal{U}})$.

- Remark 24.12. (1) If X is connected, Condition (5) follows from Conditions (1) (2). In fact, Conditions (1) (2) imply that the covering index of $\tilde{U}_p \to U_p$ is locally constant. However it does not seem to be a good idea to assume connectivity of X in our situation, since the topology of X can be pathological.
 - (2) The commutativity of Diagram 24.2 means the set theoretical equalities of the underlying topological spaces of orbifolds and of the total spaces of obstruction bundles. We can safely do so since all the maps involved are embeddings.
 - (3) We can define the notion of a covering space of a space equipped with a good coordinate system and can prove that for a given covering space of K-space we can construct a covering space equipped with a good coordinate system in the same way.

Lemma 24.13. Let $(\widetilde{X}, \widehat{\widetilde{\mathcal{U}}})$ be a covering space of $(X, \widehat{\mathcal{U}})$ in the sense of Definition 24.11.

- (1) If $\widehat{\mathfrak{S}}$ is a CF-perturbation of $\widehat{\mathcal{U}}$, it induces a CF-perturbation $\widehat{\widehat{\mathfrak{S}}}$ of $\widehat{\widetilde{\mathcal{U}}}$.
- (2) A strongly continuous (strongly smooth) map \widehat{f} from $(X,\widehat{\mathcal{U}})$ can be pulled back to a strongly continuous (strongly smooth) map $\widehat{\widehat{f}}$ from $(\widetilde{X},\widehat{\widetilde{\mathcal{U}}})$.
- (3) A differential form \widehat{h} on $(X,\widehat{\mathcal{U}})$ can be pulled back to $\widehat{\widetilde{h}}$ on $(\widetilde{X},\widehat{\widetilde{\mathcal{U}}})$.
- (4) A strongly smooth map \widehat{f} is strongly submersive with respect to $\widehat{\mathfrak{S}}$ if and only if $\widehat{\widehat{f}}$ in (2) is strongly submersive with respect to $\widehat{\widetilde{\mathfrak{S}}}$.
- (5) The statements such that 'CF-perturbations' in (2) (4) are replaced by 'multivalued perturbations' also hold.
- (6) In the situation of (4) we have

$$n\widehat{f}!(\widehat{h};\widehat{\mathfrak{S}}) = \widehat{\widetilde{f}}(\widehat{\widetilde{h}};\widehat{\widetilde{\mathfrak{S}}}).$$

Here n is the covering index.

The proof is straightforward and so omitted.

24.3. Covering spaces associated to the corner structure stratification. One of the main reasons we introduced the notion of covering space of a K-space is that we use the next lemma to clarify the discussion of normalized boundary.

Proposition 24.14. If $(X,\widehat{\mathcal{U}})$ is a K-space with corners, then $\overset{\circ}{S}_{k-1}(\partial(X,\widehat{\mathcal{U}}))$ is a covering space of $\overset{\circ}{S}_k(X,\widehat{\mathcal{U}})$ with covering index k.

Proof. We first prove the proposition for the case of orbifolds.

Lemma 24.15. Let U be an orbifold with corner. The map $\overset{\circ}{S}_{k-1}(\partial U) \to \overset{\circ}{S}_k U$ which is the restriction of the map π in [Part I, Lemma 8.8 (2)] is a k-fold covering of orbifolds. If \mathcal{E} is a vector bundle on U, then the bundle induced on $\overset{\circ}{S}_{k-1}(\partial U)$ is canonically isomorphic to the pull-back of the restriction of \mathcal{E} to $\overset{\circ}{S}_k U$.

Proof. Let $x \in \overset{\circ}{S}_k U$ and $\mathfrak{V}_x = (V_x, \Gamma_x, \phi_x)$ be an orbifold chart of U at x. We may assume $V_x \subset \overline{V_x} \times [0,1)^k$ and $o_x = (\overline{o_x}, (0,\ldots,0))$, where $\overline{V_x}$ is a manifold without boundary. There exists a group homomorphism $\sigma: \Gamma_x \to \operatorname{Perm}(k)$ such that if $\gamma(\overline{y}, (t_1,\ldots,t_k)) = (\overline{y}', (t_1',\ldots,t_k'))$ then $t_k' = 0$ if and only if $t_{\sigma(\gamma)(k)} = 0$.

We consider $I \subset \{1, ..., k\}$ a complete set of representatives of $\{1, ..., k\}/\Gamma_x$. For each $i \in I$, we put

$$\Gamma_{x,i} = \{ \gamma \in \Gamma_x \mid \sigma(\gamma)i = i \},$$

$$\partial_i V_x = \{ (\overline{y}, (t_1, \dots, t_k)) \in V_x \mid t_i = 0 \}.$$

The given Γ_x action on V_x induces a $\Gamma_{x,i}$ action on $\partial_i V_x$. By definition of normalized boundary, there exists a map $\phi_{x,i}:\partial_i V_x\to \partial U$ such that $(\partial_i V_x,\Gamma_{x,i},\phi_{x,i})$ is an orbifold chart of ∂U at \tilde{x}_i . Here $\{\tilde{x}_i\mid i\in I\}=\pi^{-1}(x)\subset \mathring{S}_{k-1}(\partial U)$. An orbifold chart of $\mathring{S}_{k-1}(\partial U)$ at \tilde{x}_i is

$$(\overline{V_x}, \Gamma_{x,i}, \phi_{x,i}|_{\overline{V_x}}),$$

where $\overline{V_x}$ is identified with the subset $\overline{V_x} \times \{0\}$ of $\partial_i V_x$.

On the other hand, an orbifold chart of $\overset{\circ}{S}_k U$ at x is $(\overline{V_x}, \Gamma_x, \phi_x|_{\overline{V_x}})$. Since $\#(\Gamma_x(i)) = \#(\Gamma_x/\Gamma_{x,i})$, we have

$$\sum_{i \in I} \#(\Gamma_x/\Gamma_{x,i}) = k.$$

We have thus proved that $\pi: S_{k-1}(\partial U) \to S_k(U)$ is a k-fold covering of orbifolds. We can prove the second half of the lemma using the above description of the orbifold charts.

Now we consider the case of Kuranishi structure. Let $\mathcal{U}_p = (U_p, \mathcal{E}_p, \psi_p, s_p)$ be a Kuranishi chart of $\widehat{\mathcal{U}}$ at $p \in \overset{\circ}{S}_k(X, \widehat{\mathcal{U}})$. We put

$$\overset{\circ}{S}_k(\mathcal{U}_p) := (\overset{\circ}{S}_k(U_p), \mathcal{E}_p|_{\overset{\circ}{S}_k}, \psi_p|_{\overset{\circ}{S}_k}, s_p|_{\overset{\circ}{S}_k}),$$

which is a Kuranishi chart at p of the K-space $\overset{\circ}{S}_k(X,\widehat{\mathcal{U}})$. Let π be the map from the underlying topological space of $\overset{\circ}{S}_{k-1}(\partial(X,\widehat{\mathcal{U}}))$ to that of $\overset{\circ}{S}_k(X,\widehat{\mathcal{U}})$. We use the notation in the proof of Lemma 24.15 by putting $x=o_p\in U_p,\,U=U_p$ and $\mathcal{E}=\mathcal{E}_p$. Then $\pi^{-1}(p)$ consists of #I points which we write $\widetilde{p}_i,\,i\in I$. Then $(\overline{V_x},\Gamma_{x,i},\psi_{x,i}|_{\overline{V_x}})$ is an orbifold chart of $\overset{\circ}{S}_{k-1}(\partial(X,\widehat{\mathcal{U}}))$ at \widetilde{p}_i . We restrict $\overset{\circ}{S}_k(\mathcal{U}_p)$ to $\overline{V_x}/\Gamma_x\subset U_p$ to get a Kuranishi chart of $\overset{\circ}{S}_{k-1}(\partial(X,\widehat{\mathcal{U}}))$.

The second half of Lemma 24.15 implies that the pull-back of the restriction of the obstruction bundle \mathcal{E}_p to $\overline{V_x}/\Gamma_x \subset U_p$ defines a vector bundle on $\overline{V_x}/\Gamma_{x,i}$ which is isomorphic to the obstruction bundle of the Kuranishi structure $\overset{\circ}{S}_{k-1}(\partial(X,\widehat{\mathcal{U}}))$. It is easy to see from construction that Kuranishi maps are preserved by this isomorphism.

We have thus constructed open substructures of $\overset{\circ}{S}_{k-1}(\partial(X,\widehat{\mathcal{U}}))$ and $\overset{\circ}{S}_k(X,\widehat{\mathcal{U}})$ so that the Kuranishi chart of the former is a k fold covering of the Kuranishi chart of the latter. It is easy to see that this isomorphism is compatible with the coordinate change. Hence the proof of Proposition 24.14 is complete.

We next generalize Proposition 24.14 to the corners of arbitrary codimension.

Proposition 24.16. Let $(X,\widehat{\mathcal{U}})$ be an n-dimensional K-space. Then for each k there exists an (n-k)-dimensional K-space $\widehat{S}_k(X,\widehat{\mathcal{U}})$ with corners and maps $\pi_k: \widehat{S}_k(X,\widehat{\mathcal{U}}) \to S_k(X,\widehat{\mathcal{U}}), \ \pi_{\ell,k}: \widehat{S}_\ell(\widehat{S}_k(X,\widehat{\mathcal{U}})) \to \widehat{S}_{k+\ell}(X,\widehat{\mathcal{U}})$ with the following properties:

- (1) π_k is a continuous map between underlying topological spaces.
- (2) $\widehat{S}_1(X,\widehat{\mathcal{U}})$ is the normalized boundary $\partial(X,\widehat{\mathcal{U}})$.
- (3) The interior of $\widehat{S}_k(X,\widehat{\mathcal{U}})$ is isomorphic to $\widecheck{S}_k(X,\widehat{\mathcal{U}})$. The underlying homeomorphism of this isomorphism is the restriction of π_k .
- (4) $\pi_{\ell,k}$ is an $(\ell+k)!/\ell!k!$ fold covering map of K-spaces.
- (5) The following objects on $(X, \widehat{\mathcal{U}})$ induce the corresponding ones on $\widehat{S}_k(X, \widehat{\mathcal{U}})$. The induced objects are compatible with the covering maps $\pi_{\ell,k}$.
 - (a) CF-perturbation.
 - (b) Multivalued perturbation.
 - (c) Differential form.
 - (d) Strongly continuous map. Strongly smooth map.
 - (e) Covering map.
- (6) The following diagram commutes.

$$\widehat{S}_{k_1}(\widehat{S}_{k_2}(\widehat{S}_{k_3}(X,\widehat{\mathcal{U}}))) \xrightarrow{\pi_{k_1,k_2}} \widehat{S}_{k_1+k_2}(\widehat{S}_{k_3}(X,\widehat{\mathcal{U}}))$$

$$\widehat{S}_{k_1}(\pi_{k_2,k_3}) \downarrow \qquad \qquad \downarrow \pi_{k_1+k_2,k_3}$$

$$\widehat{S}_{k_1}(\widehat{S}_{k_2+k_3}(X,\widehat{\mathcal{U}})) \xrightarrow{\pi_{k_1,k_2+k_3}} \widehat{S}_{k_1+k_2+k_3}(X,\widehat{\mathcal{U}})$$

$$(24.4)$$

Here $\hat{S}_{k_1}(\pi_{k_2,k_3})$ is the covering map induced from π_{k_2,k_3} .

(7) Let $f_i:(X_i,\widehat{\mathcal{U}}_i)\to M$ be a strongly smooth map and assume that f_1 is transversal to f_2 . Then

$$\widehat{S}_k\left((X_1,\widehat{\mathcal{U}}_1)\times_M(X_2,\widehat{\mathcal{U}}_2)\right)\cong\coprod_{k_1+k_2=k}\widehat{S}_{k_1}(X_1,\widehat{\mathcal{U}}_1)\times_M\widehat{S}_{k_2}(X_2,\widehat{\mathcal{U}}_2).$$

Here the right hand side is the disjoint union.

- (8) (1)-(6) also hold when we replace 'Kuranishi structure' by 'good coordinate system'.
- (9) Various kinds of embeddings of Kuranishi structures and/or good coordinate systems induce the corresponding ones of $\widehat{S}_k(X,\widehat{\mathcal{U}})$.

We note that Proposition 24.16 (7) implies

$$\widehat{S}_{\ell}(X_1 \times_{M_1} \dots \times_{M_{n-1}} X_n) = \coprod_{\ell_1 + \dots + \ell_n = \ell} (\widehat{S}_{\ell_1}(X_1) \times_{M_1} \dots \times_{M_{n-1}} \widehat{S}_{\ell_n}(X_n)).$$
 (24.5)

Definition 24.17. We call $\widehat{S}_k(X,\widehat{\mathcal{U}})$ the normalized (codimension k) corner of $(X,\widehat{\mathcal{U}})$.

Proof of Proposition 24.16. Let M be a manifold with corners. We first define $\widehat{S}_k(M)$. Let $x \in \overset{\circ}{S}_m(M)$. We take a chart $\mathfrak{V}_x = (V_x, \psi_x)$ so that $V_x \cong \overline{V}_x \times [0, 1)^m$. Let $A \subset \{1, \ldots, m\}$ with #A = k. A pair (x, A) becomes an element of $\widehat{S}_k(M)$.

We next define a topology on $\widehat{S}_k(M)$. Let $y = \psi(\widetilde{y})$ with $\widetilde{y} \in V_x$. We write $\widetilde{y} = (\widetilde{y}_0, (t_1, \ldots, t_m))$. If $t_i = 0$ for all $i \in A$, we consider elements $y_A = (y, A) \in \widehat{S}_k(M)$ as follows. Suppose $B = \{i \mid t_i = 0\} \supset A$. Let W be a neighborhood of $(t_i)_{i \notin B}$ in $(0, 1)^{\{1, \ldots, m\} \setminus B}$. Then $\overline{V} \times W \times [0, 1)^B$ together with the restriction of ψ_x is a chart of y. Thus we have $(y, A) \in \widehat{S}_k(U)$. We say (y^a, A) above converges to (x, A) if \widetilde{y}^a converges to o_x , where o_x is the point such that $\psi_x(o_x) = x$.

It is easy to see that $\widehat{S}_k(M)$ with this topology becomes a manifold with corners. This construction is canonical so that it induces one of orbifolds and of Kuranishi structure. (The proof of this part is entirely similar to the case of normalized boundary and so is omitted.)

We next construct the covering map $\pi_{\ell,k}$. We consider the case of manifolds. Let $x \in \overset{\circ}{S}_m(M)$ and let \mathfrak{V}_x , A be as above. For simplicity of notation, we put $A = \{1, \ldots, k\}$. Suppose $(x, A) \in S_{\ell}(\widehat{S}_k(M))$. It implies $m \geq k + \ell$.

By definition the neighborhood of (x,A) in $\widehat{S}_k(M)$ is (y,A) where $y \in \overline{V} \times \{(0,\ldots,0)\} \times [0,1)^{m-k}$. Therefore a point \widetilde{x} in $\widehat{S}_{\ell}(\widehat{S}_k(M))$ such that $\pi_{\ell}(\widetilde{x}) = (x,A)$ corresponds one to one to the set $A^+ \supset A$ with $\#A^+ = \ell + k$. We put $B = A^+ \setminus A$. We thus may regard $(x,A,B) \in \widehat{S}_{\ell}(\widehat{S}_k(M))$. (Here $\#B = \ell$.) Now we define the map $\pi_{\ell,k} : \widehat{S}_{\ell}(\widehat{S}_k(M)) \to \widehat{S}_{\ell+k}(M)$ by

$$\pi_{\ell,k}(x,A,B) = (x,A \cup B).$$

Given $(x, C) \in \widehat{S}_{\ell+k}(M)$, the element in the fiber of $\pi_{\ell,k}$ corresponds one to one to the partition of C into $A \cup B$ where #A = k and $\#B = \ell$. We can use this fact to show that $\pi_{\ell,k}$ is a covering map of covering index $(k+\ell)!/k!\ell!$.

We have thus constructed the covering map $\pi_{\ell,k}$ in the case of manifolds. To prove the case of orbifolds and of K-spaces, it suffices to observe that this construction is canonical and so is compatible with various kinds of coordinate changes.

Once the K-space $\widehat{S}_k(X,\widehat{\mathcal{U}})$ and the covering map $\pi_{\ell,k}$ are defined as above, it is very easy to check the properties (1)-(9).

Remark 24.18. As well as other parts of this article, Proposition 24.16 is *not* new. Especially we would like to mention that mostly the same construction appeared in D. Joyce's paper [Jo3]. (The article [Jo3] discusses the case of manifolds. However

it is straightforward to generalize the story to the case of K-spaces.) In [Jo3] the notion of the boundary ∂X is defined, which is the same as our definition of normalized boundary. Then the action of $\operatorname{Perm}(k)$ on $\underbrace{\partial \cdots \partial}_{} X$ is introduced. The

quotient space $\underbrace{\partial \cdots \partial}_{k \text{ times}} X/\text{Perm}(k)$ (which is denoted by $C_k X$ in [Jo3]), coincides with our $\widehat{S}_k(X)$.

24.4. Finite group action on K-space.

Definition 24.19. Let X be an orbifold and G a finite group. A G action on X as a topological space is said to be a G action on the orbifold X if the homeomorphism $X \to X$ induced by each element of G is a diffeomorphism of orbifold.

Two actions are said to be the same if they are the same as maps $G \times X \to X$, set-theoretically.

Lemma 24.20. Let X be an orbifold on which a finite group G acts (as orbifold). We assume that the action is effective on each connected component. Then there exists a unique orbifold structure on X/G such that X is a covering space of X/G and the natural map $X \to X/G$ is a covering map.

Proof. Let $x \in X$ and $G_x = \{g \in G \mid gx = x\}$. We take an orbifold chart $\mathfrak{V}_x = (V_x, \Gamma_x, \psi_x)$ such that U_x is G_x invariant (by using a G invariant Riemannian metric, for example). Using the effectivity of G action on each connected component, we can easily show that the G_x action on U_x is effective. For each $g \in G_x$ we obtain a map $\varphi_g : V_x \to V_x$ and a group homomorphism $h_g : \Gamma_x \to \Gamma_x$. Since $\varphi_{g_1}\varphi_{g_2}$ induces the same continuous map as $\varphi_{g_1g_2}$ between the underlying topological spaces, there exists a unique element $\gamma_{g_1g_2g_3} \in \Gamma_x$ such that

$$\varphi_{g_1}\varphi_{g_2}=\gamma_{g_1g_2}\varphi_{g_1g_2}.$$

Moreover we have

$$h_{g_1}h_{g_2} = \operatorname{conj}_{\gamma_{g_1g_2}}h_{g_1g_2}.$$

Note that φ_g is h_g equivariant. Then we define a group structure on the direct product set $\Gamma_x \times G_x$ by

$$(\gamma_1, g_1) \circ (\gamma_2, g_2) = (\gamma_1 h_{g_1}(\gamma_2) \gamma_{g_1, g_2}, g_1 g_2). \tag{24.6}$$

We define $\cdot : (\Gamma_x \times G_x) \times V_x \to V_x$ by

$$(\gamma, g) \cdot x = \gamma(\varphi_q(x)).$$

Then we observe

$$\begin{split} &(\gamma_1,g_1)\cdot ((\gamma_2,g_2))\cdot x) = (\gamma_1,g_1)\cdot \gamma_2(\varphi_{g_2}(x)) = \gamma_1(\varphi_{g_1}(\gamma_2(\varphi_{g_2}(x)))) \\ &= \gamma_1 h_{g_1}(\gamma_2)\varphi_{g_1}(\varphi_{g_2}(x)) = \gamma_1 h_{g_1}(\gamma_2)\gamma_{g_1g_2}(\varphi_{g_1g_2}(x)) = ((\gamma_1,g_1)\circ (\gamma_2,g_2))\cdot x. \end{split}$$

It follows from effectivity that \circ defines a group structure. We denote this group by $\Gamma_x \tilde{\times} G_x$.

Now we define $\overline{\phi}_x: V_x \to X/G$ by the composition of $\phi_x: V_x \to X$ and the projection $X \to X/G$. Then it is easy to see that $(V_x, \Gamma_x \tilde{\times} G_x, \overline{\phi}_x)$ defines an orbifold structure on X/G. The rest of the proof is obvious.

Now we define the definition of the action of finite group on K-space.

Definition 24.21. Let $(X, \widehat{\mathcal{U}})$ be a K-space. An automorphism Φ consists of a pair $(|\Phi|, \{\Phi_p\})$ of a homeomorphism $|\Phi|: X \to X$ and an assignment $X \ni p \mapsto \Phi_p =$ $(\varphi_p, \widehat{\varphi}_p)$ with the following properties:

- (1) $\varphi_p: U_p \to U_{|\Phi|(p)}$ is a diffeomorphism of orbifolds.
- (2) $\widehat{\varphi}_p: E_p \to E_{|\Phi|(p)}$ is an isomorphism of vector bundles over φ_p .
- (3) $\widehat{\varphi}_p \circ s_p = s_{|\Phi|(p)} \circ \varphi_p$ holds on U_p .
- (4) $|\Phi| \circ \varphi_p = \varphi_{|\Phi|(p)} \circ \varphi_p \text{ holds on } s_p^{-1}(0).$ (5) Let $q \in \psi(s_p^{-1}(0))$. Suppose $\Phi_{pq} = (U_{pq}, \varphi_{pq}, \widehat{\varphi}_{pq})$ and

$$\Phi_{|\Phi|(p)|\Phi|(q)} = \left(U_{|\Phi|(p)|\Phi|(q)}, \varphi_{|\Phi|(p)|\Phi|(q)}, \widehat{\varphi}_{|\Phi|(p)|\Phi|(q)}\right)$$

are the coordinate changes. Then we have the following:

- (a) $\varphi_q(U_{pq}) = U_{|\Phi|(p)|\Phi|(q)}$.
- (b) $\varphi_{|\Phi|(p)|\Phi|(q)} \circ \varphi_q = \varphi_p \circ \varphi_{pq}$.
- (c) $\widehat{\varphi}_{|\Phi|(p)|\Phi|(q)} \circ \widehat{\varphi}_q = \widehat{\varphi}_p \circ \widehat{\varphi}_{pq}$

We say $(|\Phi|, {\Phi_p})$ is the same as $(|\Phi'|, {\Phi'_p})$ if $|\Phi| = |\Phi'|$ and $\Phi_p = \Phi'_p$ for all p.

- (1) The equality $\Phi_p = \Phi_p'$ has an obvious meaning. Namely Remark 24.22. we defined the notion of two diffeomorphisms and bundle isomorphisms of orbifolds to be the same. (That is, they are the same set-theoretically.)
 - (2) It happens that two automorphisms Φ and Φ' with the same underlying homeomorphisms $|\Phi|$ and $|\Phi'|$ could be different.

Definition-Lemma 24.23. (1) We can compose two automorphisms of Kspaces. The composition is again an automorphism.

- (2) The set of automorphisms of a given K-space $(X, \hat{\mathcal{U}})$ is a group whose product is the composition of automorphisms. We denote this group by $\operatorname{Aut}(X,\mathcal{U}).$
- (3) An action of a finite group G on $(X, \widehat{\mathcal{U}})$ is, by definition, a group homomorphism $G \to \operatorname{Aut}(X, \mathcal{U})$.
- (4) A G action on $(X, \widehat{\mathcal{U}})$ induces a G action on the underlying topological space X.

Definition 24.24. An action of a finite group G on a K-space $(X, \widehat{\mathcal{U}})$ is said to be effective if the following holds for each $p \in X$.

We put $G_p = \{g \in G \mid gp = p\}$. Let U_p be the Kuranishi neighborhood of p. By definition G_p acts on U_p . We require that this action is effective on each connected component of U_p .

Lemma 24.25. Suppose a finite group G acts effectively on a K-space $(X, \widehat{\mathcal{U}})$. Then there exists a unique Kuranishi structure on X/G such that the projection $X \to X/G$ is an underlying map of the covering map and each \mathcal{U}_p gives a covering chart of this covering.

Proof. This follows from Lemma 24.20.

25. Admissible Kuranishi structure

In this section we introduce the notion of admissible Kuranishi structure. For this purpose we introduce the notion of admissible orbifold, admissible vector bundle, and various admissible objects associated to them, like admissible section, admissible Riemannian metric, etc., and provide their fundamental properties. Roughly speaking, 'admissibility' in this section is some condition that objects in question obey certain exponential decay estimates at asymptotic ends. Here we regard boundary or corner points as the end points, so the 'exponential decay estimates at asymptotic ends' means the exponential decay estimates in the direction normal to boundary or corners.

25.1. **Admissible orbifold.** Firstly, we discuss admissibility for the case of manifold with corner before going to the case of orbifold with corner, because the key idea can be seen even for the case of manifold.

Situation 25.1. Let $V \subset \overline{V} \times [0,1)^k$ be an open subset where \overline{V} is a manifold without corner. We denote $\mathbf{t} = (t_1, \dots, t_k) \in [0,1)^k$.

Convention 25.2. (1) We put

$$T_i = e^{1/t_i},$$
 (i.e. $t_i = \frac{1}{\log T_i}$). (25.1)

We consider the corner structure stratification of V. Then each connected component of open stratum $\overset{\circ}{S}_{\ell}V$ has coordinates that are union of the coordinate of \overline{V} and $k-\ell$ of T_i 's. Here $T_i \in [0,\infty)$. We consider the C^m norm of a function $f:V\to\mathbb{R}$ stratumwisely (i.e., the norm of the differential in the stratum direction) using the above coordinate T_i . (Namely we use T_i and not t_i to define the C^m norm.)

- (2) For a function $f:V\to\mathbb{R}$, we denote by $|f|_{C^m}$ the pointwise C^m norm in the above sense. Thus $|f|_{C^m}$ can be regarded as a non negative real valued function on V. To define it we use a certain Riemannian metric on \overline{V} . (We use the standard metric for $T_i\in[0,\infty)$.) Since we only consider its value on a compact subset, the difference of the metric affects only by a bounded ratio. So we do not need to care about the difference of the metric.
- **Definition 25.3.** (1) We say a function $f: V \to \mathbb{R}$ is admissible if for each compact subset K and m > 0, there exist $\sigma(m, K) > 0$ and C(m, K) > 0 such that the following holds for each i.

$$\left| \frac{\partial f}{\partial T_i} \right|_{C^m} \le C(m, K) e^{-\sigma(m, K)T_i}. \tag{25.2}$$

(2) We say a function $f: V \to \mathbb{R}$ is exponentially small near the boundary if for each each compact subset K and m > 0, there exist $\sigma(m, K) > 0$ and C(m, K) > 0 such that the following holds for each i.

$$|f|_{C^m} \le C(m, K)e^{-\sigma(m, K)T_i}.$$
 (25.3)

Example 25.4. If k=1, an admissible function $f(\overline{y},t)$ is written as the form

$$f(\overline{y},t) = f_0(\overline{y}) + f_1(\overline{y},t)$$

such that $f_1(\overline{y}, 1/\log T)$ decays in an exponential order in T. (Here $t = 1/\log T$.)

In a way similar to the above example, we can prove the following.

Lemma 25.5. On a subset $K \times [0,c)^k$, any admissible function f is written in the following form:

$$f(\overline{y},(t_1,\ldots,t_k)) = \sum_{I\subseteq\{1,\ldots,k\}} f_I(\overline{y},t_I).$$
 (25.4)

Here $t_I = (t_i)_{i \in I}$ and f_I is a function on $K \times [0, c)^I$ which is exponentially small near the boundary.

Proof. Firstly we observe that the set of functions of the form (25.4) forms an \mathbb{R} vector space. We also note that if $\overline{V} \times [0,c)^k \to \overline{V} \times [0,c)^I$ is a projection then the pull-back of admissible functions are admissible. The same holds for 'exponentially small near the boundary' and 'of the form (25.4)'.

Sublemma 25.6. If an admissible function is zero on the boundary, then it is exponentially small near the boundary.

Proof. We have

$$f(\overline{y};(t_1,\ldots,t_k)) = -\int_{T_i=e^{1/t_i}}^{\infty} \frac{\partial f}{\partial T_i} dT_i.$$

This implies the sublemma.

We are going to construct f_I in (25.4) by an upward induction on #I such that

$$f(\overline{y},(t_1,\ldots,t_k)) - \sum_{I\subseteq\{1,\ldots,k\},\#I\leq m} f_I(\overline{y},t_I) = 0$$

on $S_{k-m}(K \times [0,c)^k)$.

For m=0 we put $f_{\emptyset}(\overline{y})=f(\overline{y};(0,\ldots,0))$. Suppose we have $f_{I'}$ for I' with #I' < m. Let $I \subset \{1,\ldots,k\}$ with #I=m. By induction hypothesis, we may assume that f is zero on $S_{k-m+1}(K\times[0,c)^k)$. We embed $K\times[0,c)^I$ into $K\times[0,c)^k$ by putting $t_i=0$ for $i\notin I$. Restricting f to its image we obtain an admissible function on $K\times[0,c)^I$. Since we assumed f is zero on $S_{k-m+1}(K\times[0,c)^k)$, then f=0 on $\partial(K\times[0,c)^I)$. We define $f_I=f|_{\partial(K\times[0,c)^I)}$. Then f_I is exponentially small near the boundary and therefore its pull-back to $K\times[0,c)^k$ is of the form (25.4). By taking

$$S_{k-m}(K \times [0,c)^k) = \bigcup_{I \subseteq \{1,\dots,k\}, \#I = m} K \times [0,c)^I$$

into account, we can see that

$$f - \sum_{I \subseteq \{1,\dots,k\}, \#I = m} f_I$$

is zero on $S_{k-m}(K\times[0,c)^k)$. The proof of the lemma is complete by induction. \square

- **Definition 25.7.** (1) Let $V_i \subset \overline{V}_i \times [0,1)^k$ (i=1,2) be open subsets as in Situation 25.1 and let $\varphi_{21}: V_1 \to V_2$ be an embedding of manifolds. We say that φ_{21} is an admissible embedding if there exists a permutation $\sigma: \{1,\ldots,k\} \to \{1,\ldots,k\}$ such that $\varphi(\overline{y},(t_1,\ldots,t_k)) = (\overline{y}',(t_1',\ldots,t_k'))$ has the following properties.
 - (a) The coordinates of \overline{y}' are admissible in the sense of Definition 25.3.
 - (b) If we put $T'_i = e^{1/\tilde{t}'_i}$, then for each i, $T'_{\sigma(i)} T_i$ is admissible in the sense of Definition 25.3 near the boundary.
 - (2) An admissible embedding is said to be an *admissible diffeomorphism* if it is also a homeomorphism.
 - (3) An action of a finite group Γ on $V \subset \overline{V} \times [0,1)^k$ is said to be an *admissible action* if each element of Γ induces an admissible diffeomorphism.

(4) An orbifold chart in the sense of Definition 23.1 (1) is said to be an *admissible chart* if the Γ action is admissible.

Remark 25.8. (1) In the geometric context of pseudo-holomorphic curves we took T to be the 'length' of the neck region in [FOOO4] and etc.. For this choice, Definition 25.7 (1)(b) is satisfied. See Subsection 25.5.

(2) The choice of the coordinate t=1/T used in [FOOO4] is different from that of (25.1). See also Subsection 25.5 for this point.

Lemma 25.9. Let $\varphi_{21}: V_1 \to V_2$ be an admissible embedding as in Definition 25.7.

(1) An admissible embedding φ_{21} induces a smooth embedding

$$\varphi_{21}: \overline{V}_1 \times [0,1)^k \to \overline{V}_2 \times [0,1)^k.$$

(2) Denote by φ_{21}^j the j-th component of the $[0,1)^k$ factor of φ_{21} . If we put

$$S_i = \frac{1}{t_i},\tag{25.5}$$

then for each compact set $K \subset V$ and $m \geq 0$ there exist C(m,K) > 0 and $\sigma(m,K) > 0$ such that

$$\left\| \varphi_{21}^{\sigma(i)} - t_i \right\|_{C_K^{m,S}} \le C(m, K) e^{-\sigma(m, K)S_i}.$$
 (25.6)

Here $\sigma(i) \in \{1, ..., k\}$ and we identify $V \cong \overline{V} \times (1, \infty]^k$ using S_i as the coordinates of the second factor and $C_K^{m,S}$ stands for the C^m norm on K with respect to the S_i coordinates.

Proof. If $T'_{\sigma(i)} = T_i$ for all i in addition, then it is easy to see that an admissible embedding induces a smooth embedding. So it suffices to consider the case when $\overline{y}' = \overline{y}$ and $\sigma(i) = i$. We also note that (1) follows from (2). Then by Definition 25.7 (1)(b) and (25.1) we have

$$t'_i = (\log(e^{\frac{1}{t_i}} + f_i(x, (t_1, \dots, t_k))))^{-1}$$

for some admissible function f_i . The right hand side is equal to

$$\frac{t_i}{1 + t_i \log(1 + e^{-S_i} f_i(x, (t_1, \dots, t_k)))}.$$

Statement (2) easily follows from this formula.⁵⁴

Remark 25.10. The admissible function f_i above may not be zero at $t_i = 0$. However, $\log(1 + e^{-S_i}f_i(x, (t_1, \dots, t_k)))$ goes to 0 in exponential order as $S_i \to \infty$. Therefore we can smoothly extend φ_{21} to a collared neighborhood. Here is the key point that coordinate changes of Kuranishi structure can be extended smoothly to a collared neighborhood, once we establish the exponential decay estimate (25.6) of the coordinate changes with respect to the S_i coordinates. See also Remark 25.45. On the other hand, we recall from Definition 25.3 that admissible functions are required that their derivatives 55 with respect to the T_i coordinates satisfy the exponential decay estimate.

Next, we go to the case of orbifold with corner.

 $^{^{54}}$ Recall $T_i = e^{S_i}$. Therefore a function which decays in exponential order in T_i coordinates also decays in exponential order in S_i coordinates.

⁵⁵We assume m > 0 in Definition 25.3, while we assume $m \ge 0$ in (25.6).

- **Definition 25.11.** (1) Two admissible charts are said to be *compatible as admissible charts* if the diffeomorphism $\tilde{\varphi}$ in Definition 23.1 (3) is admissible.
 - (2) A representative of an orbifold structure (with boundary or corner) is said to be a representative of an admissible orbifold structure if
 - (a) Each member is an admissible chart.
 - (b) Two of them are compatible as admissible charts.
 - (3) In Definition 23.2, suppose $\{(V_i^X, \Gamma_i^X, \phi_i^X) \mid i \in I\}$ and $\{(V_j^Y, \Gamma_j^Y, \phi_j^Y) \mid j \in J\}$ are representatives of admissible orbifold structures. Then the embedding f in Definition 23.2 (2) is said to be an admissible embedding if $\tilde{f}_{p;ji}$ in Definition 23.2 (2) is an admissible embedding in the sense of Definition 25.7 (1).
- **Lemma 25.12.** (1) Composition of admissible embeddings is an admissible embedding.
 - (2) The identity map is an admissible embedding.
 - (3) If an admissible embedding is a homeomorphism, the inverse is also an admissible embedding.

The proof is obvious.

- **Definition 25.13.** (1) We say an admissible embedding of orbifolds is an *admissible diffeomorphism* if it is a homeomorphism in addition.
 - (2) We say that two representatives of admissible orbifold structures on X are equivalent if the identity map regarded as a map between X equipped with those two representatives of admissible orbifold structures is an admissible diffeomorphism. This is an equivalence relation by Lemma 25.12.
 - (3) An equivalence class of the equivalence relation (2) is called an *admissible* orbifold structure on X.
 - (4) An orbifold X with an admissible orbifold structure is called an *admissible* orbifold.
 - (5) The condition for a map $X \to Y$ to be an admissible embedding does not change if we replace representatives of admissible orbifold structures to equivalent ones. So we can define the notion of an admissible embedding of admissible orbifolds.
 - (6) If U is an open subset of an admissible orbifold X, then there exists a unique admissible orbifold structure on U such that the inclusion $U \to X$ is an admissible embedding. We call U with this admissible orbifold structure an open admissible suborbifold.
- **Definition 25.14.** (1) Let X be an admissible orbifold. An admissible orbifold chart (V, Γ, ϕ) of underlying topological space X is an admissible orbifold chart of orbifold X if the map $V/\Gamma \to X$ induced by ϕ is an admissible embedding of orbifolds.
 - (2) Hereafter when X is an admissible orbifold, an admissible orbifold chart always means an admissible orbifold chart of orbifold in the sense of (1).
 - (3) In case when an admissible orbifold structure on X is given, a representative of its admissible orbifold structure is called an *admissible orbifold atlas*.
 - (4) Two admissible orbifold charts (V_i, Γ_i, ϕ_i) are said to be *isomorphic* if there exists a group isomorphism $h: \Gamma_1 \to \Gamma_2$ and an h equivariant admissible diffeomorphism $\varphi: V_1 \to V_2$ such that $\phi_2 \circ \varphi = \phi_1$. The pair (h, φ) is

called an admissible isomorphism or admissible coordinate change between two admissible orbifold charts.

The proofs of the following lemmas are obvious from definition.

Lemma 25.15. Suppose V, V_1, V_2 are as in Situation 25.1.

- (1) A restriction of an admissible function on V to its open subset is also admissible.
- (2) Let $f: V_2 \to \mathbb{R}$ be an admissible function and $\varphi_{21}: V_{21} \to V_1$ an admissible embedding, then the composition $f \circ \varphi_{21}: V_1 \to \mathbb{R}$ is admissible.

Lemma 25.16. A subchart (in the sense of Definition 23.1 (2)) of an admissible chart is also admissible.

Definition 25.17. Let X be an admissible orbifold. A function $f: X \to \mathbb{R}$ is said to be *admissible* if for all the orbifold charts (V, Γ, ϕ) of X the composition $f \circ \phi: V \to \mathbb{R}$ is admissible in the sense of Definition 25.3.

Lemma 25.18. Let X be an admissible orbifold and $f: X \to \mathbb{R}$ a function.

- (1) If there exists a representative $\{(V_i, \Gamma_i, \phi_i) \mid i \in I\}$ of the orbifold structure on X such that $f \circ \phi_i : V_i \to \mathbb{R}$ is admissible for any i, then f is admissible.
- (2) The composition of an admissible embedding and an admissible function (resp. a function of exponential decay) is again admissible (resp. a function of exponential decay).

This is a consequence of Lemma 25.15.

25.2. Admissible tensor. Next we discuss tensor calculus etc. on an admissible orbifold.

Convention 25.19. Let X be an admissible orbifold and let V be as in Situation 25.1. The coordinates of V consist of those in \overline{V} direction and in \mathbf{t} direction. We call the coordinates in \mathbf{t} direction the *normal coordinates*, and the coordinates in \overline{V} direction the *horizontal coordinates*.

- **Definition-Lemma 25.20.** (1) Let \mathcal{T} be a tensor on an admissible orbifold and $\mathcal{T}_{j_1,\ldots,j_{m'}}^{i_1,\ldots,i_m}$ be its expression by admissible local coordinates. We call it admissible if all $\mathcal{T}_{j_1,\ldots,j_{m'}}^{i_1,\ldots,i_m}$ are admissible functions. This notion is invariant under the admissible coordinate change defined in Definition 25.14 (4).
 - (2) A tensor \mathcal{T} is called *strongly admissible* if the following holds.
 - (a) We rewrite $\mathcal{T}_{j_1,\ldots,j_{m'}}^{i_1,\ldots,i_m}$ using $T_i=e^{1/t_i}$ in place of t_i as coordinates to obtain $\widehat{\mathcal{T}}_{j_1,\ldots,j_{m'}}^{i_1,\ldots,i_m}$. Then if i_1,\ldots,i_m and j_1,\ldots,j_m contain the normal coordinates, $\widehat{\mathcal{T}}_{j_1,\ldots,j_{m'}}^{i_1,\ldots,i_m}$ is exponentially small near the boundary. ⁵⁶
 - (b) \mathcal{T} is admissible.

This notion is also invariant under the admissible coordinate change.

- (3) We define an *admissibile differential form* as a special case of an admissible tensor.
- (4) Various algebraic operations of tensors and differential forms preserve the admissibility.

⁵⁶In particular it is zero if the corresponding t_i coordinate is zero.

- (5) The exterior derivative of an admissible differential form is admissible. Moreover the exterior derivative of an admissible function is strongly admissible.
- (6) The Lie derivative by an admissible vector field preserves admissibility. The bracket of admissible vector fields (resp. strongly admissible vector fieles) is admissible (resp. strongly admissible).
- (7) An admissible Riemannian metric of an admissible orbifold is a stratumwise Riemannian metric with the following properties. (Here we consider the corner structure stratification.)

Let $\overline{V} \times [0,1)^k$ be an admissible chart. We write the $[0,1)^k$ coordinates as t_i and put $T_i = e^{1/t_i}$. Then there exists a symmetric 2-tensor g' on $\overline{V} \times [0,1)^k$ with the following properties:

- (a) g' is strongly admissible as a 2-tensor.
- (b) For $A \subseteq \{1, \ldots, k\}$ we put

$$V^A = \{(x, (t_1, \dots, t_k)) \mid x \in \overline{V}, \ t_i = 0 \text{ if } i \in A, t_i \neq 0 \text{ if } i \notin A\},\$$

which is an open subset of a stratum. We require that on V^A the stratumwise Riemannian metric is given by

$$g' + \sum_{i \notin A} (dT_i)^{2\otimes}. \tag{25.7}$$

In particular, this condition implies that the Riemannian metric is stratumwise complete. Note that we use the T_i coordinates and not the t_i coordinates in (25.7). We also note that g' C^{∞} -converges to the stratumwise metric on \overline{V} as $T_i \to \infty$.

(8) Let $\bigcup_{i \in I} U_i = X$ be an open covering of an orbifold X. A smooth partition of unity $\{\chi_i \mid i \in I\}$ subordinate to this covering is called an *admissible* partition of unity if each χ_i is an admissible function.

The proof is straightforward by definition, so is omitted.

Lemma 25.21. Let V be as in Situation 25.1.

- (1) Pull-back of an admissible differential form on V by an admissible embedding is admissible.
- (2) Pull-back of a differential form on a smooth manifold M by an admissible map from V to M is admissible.
- (3) Any locally finite covering of an admissible orbifold admits an admissible partition of unity.
- (4) An admissible Riemannian metric exists.

Proof. We can prove the lemma by modifying the standard proof of the corresponding results in manifold theory in an obvious way, so omit it. \Box

When we apply certain operations on tensors, admissibility is mostly preserved under the operations. In certain case if we take derivative on the normal direction, an admissible object changes to a strongly admissible one. Since it is easy to see when it happens, we do not provide a detailed account thereon here. We will state that kinds of facts in case we need it.

25.3. Admissible vector bundle. We next describe the admissible version of Definition 23.17.

Definition 25.22. Let (X, \mathcal{E}, π) be a pair of admissible orbifolds X and \mathcal{E} with a continuous map $\pi : \mathcal{E} \to X$ between their underlying topological spaces. Hereafter we write (X, \mathcal{E}) in place of (X, \mathcal{E}, π) .

- (1) An admissible orbifold chart of (X, \mathcal{E}) is a quintuple $(V, E, \Gamma, \phi, \widehat{\phi})$ with the following properties:
 - (a) $\mathfrak{V} = (V, \Gamma, \phi)$ is an admissible orbifold chart of X.
 - (b) E is a finite dimensional vector space equipped with a linear Γ action.
 - (c) $(V \times E, \Gamma, \widehat{\phi})$ is an admissible orbifold chart of \mathcal{E} .
 - (d) The diagram below commutes set-theoretically.

$$V \times E \xrightarrow{\widehat{\phi}} \mathcal{E}$$

$$\downarrow \qquad \qquad \downarrow_{\pi}$$

$$V \xrightarrow{\phi} X$$

$$(25.8)$$

Here the left vertical arrow is the projection to the first factor.

- (2) In the situation of (1), let $p \in V$ and $(V_p, \Gamma_p, \phi|_{V_p})$ be a subchart of (V, Γ, ϕ) in the sense of Definition 23.1 (2). Then $(V_p, E, \Gamma_p, \phi|_{V_p}, \widehat{\phi}|_{V_p \times E})$ is an admissible orbifold chart of (X, \mathcal{E}) . We call it a *subchart* of $(V, E, \Gamma, \phi, \widehat{\phi})$.
- (3) Let $(V_i, E_i, \Gamma_i, \phi_i, \widehat{\phi}_i)$ (i = 1, 2) be admissible orbifold charts of (X, \mathcal{E}) . We say that they are *compatible as admissible charts* if the following holds for each $p_1 \in V_1$ and $p_2 \in V_2$ with $\phi_1(p_1) = \phi_2(p_2)$.
 - (a) There exists an isomorphism $(h,\varphi):(V_1,\Gamma_1,\phi_1)\to (V_2,\Gamma_2,\phi_2)$ of admissible orbifold charts of X.
 - (b) There exists an isomorphism $(h, \widehat{\varphi}) : (V_1 \times E_1, \Gamma_1, \widehat{\phi}_1) \to (V_2 \times E_2, \Gamma_2, \widehat{\phi}_2)$ of admissible orbifold charts of \mathcal{E} .
 - (c) For each $y \in V_1$ the map $E_1 \to E_2$ given by $\xi \to \pi_{E_2} \widehat{\varphi}(y, \xi)$ is a linear isomorphism. Here $\pi_{E_2} : V_2 \times E_2 \to E_2$ is the projection.
 - (d) Each component of the map $V_1 \times E_1 \to E_2$ that is a composition of $\widehat{\varphi}$ and the projection $V_2 \times E_2 \to E_2$ is an admissible function.
- (4) A representative of an admissible vector bundle structure on (X, \mathcal{E}) is a set of admissible orbifold charts $\{(V_i, E_i, \Gamma_i, \phi_i, \widehat{\phi}_i) \mid i \in I\}$ such that any two of the charts are compatible in the sense of (3) above and

$$\bigcup_{i \in I} \phi_i(V_i) = X, \quad \bigcup_{i \in I} \widehat{\phi}_i(V_i \times E_i) = \mathcal{E},$$

are locally finite open coverings.

Definition 25.23. Let (X^*, \mathcal{E}^*) (* = a, b) have representatives of vector bundle structures $\{(V_i^*, E_i^*, \Gamma_i^*, \phi_i^*, \widehat{\phi}_i^*) \mid i \in I^*\}$, respectively. A pair of admissible orbifold embeddings $(f, \widehat{f}), f: X^a \to X^b, \widehat{f}: \mathcal{E}^a \to \mathcal{E}^b$ is said to be an admissible embedding of admissible vector bundles if the following holds.

(1) Let $p \in V_i^a$, $q \in V_j^b$ with $f(\phi_i^a(p)) = \phi_j^b(q)$. Then there exist admissible open subcharts $(V_{i,p}^a \times E_{i,p}^a, \Gamma_{i,p}^a, \widehat{\phi}_{i,p}^a)$ and $(V_{j,q}^b \times E_{j,q}^b, \Gamma_{j,q}^b \widehat{\phi}_{j,q}^b)$ and a local representative $(h_{p;ji}, f_{p;ji}, \widehat{f}_{p;ji})$ of the embeddings f and \widehat{f} such that the

following holds. For each $y \in V_i^a$ the map $\xi \mapsto \pi_2(\widehat{f}_{p;ji}(y,\xi)), E_1^a \to E_2^b$ is a linear embedding.

- (2) Each component of the map $\pi_2 \circ \widehat{f}_{p;ji} : V_1^a \times E_1^a \to E_2^b$ is admissible.
- (3) The diagram below commutes set-theoretically.

$$\begin{array}{ccc}
\mathcal{E}^{a} & \xrightarrow{\widehat{f}} & \mathcal{E}^{b} \\
\pi \downarrow & & \downarrow \pi \\
X^{a} & \xrightarrow{f} & X^{b}
\end{array} (25.9)$$

Two orbifold embeddings are said to be *equal* if they coincide set-theoretically as pairs of maps.

Lemma 25.24. (1) The composition of admissible embeddings of vector bundles is an admissible embedding.

- (2) The pair of identity maps (id, id) is an admissible embedding.
- (3) If an admissible embedding of vector bundles is a pair of homeomorphisms, then the pair of their inverses is also an admissible embedding.

The proof is obvious.

Definition 25.25. Let (X, \mathcal{E}) be as in Definition 23.17.

- (1) An admissible embedding of vector bundles is said to be an *isomorphism* if it is a pair of admissible diffeomorphisms of admissible orbifolds.
- (2) We say that two representatives of an admissible vector bundle structure on (X, \mathcal{E}) are *equivalent* if the pair of identity maps regarded as a map between (X, \mathcal{E}) equipped with those two representatives of vector bundle structures is an admissible embedding. This is an equivalence relation by Lemma 25.24.
- (3) An equivalence class of the equivalence relation (1) is called an *admissible* vector bundle structure on (X, \mathcal{E}) .
- (4) A pair (X, \mathcal{E}) together with its admissible vector bundle structure is called an admissible vector bundle on X. We call \mathcal{E} the total space, X the base space, and $\pi: \mathcal{E} \to X$ the projection.
- (5) The condition for $(f, \widehat{f}): (X^a, \mathcal{E}^a) \to (X^b, \mathcal{E}^b)$ to be an admissible embedding does not change if we replace representatives of admissible vector bundle structures to equivalent ones. So we can define the notion of an admissible embedding of vector bundles.
- (6) We say (f, \hat{f}) is an admissible embedding over the admissible orbifold embedding f.
- **Definition 25.26.** (1) Let (X, \mathcal{E}) be an admissible vector bundle. We call an admissible orbifold chart $(V, E, \Gamma, \phi, \widehat{\phi})$ in the sense of Definition 25.22 (1) of underlying pair of topological spaces (X, \mathcal{E}) an admissible orbifold chart of an admissible vector bundle (X, \mathcal{E}) if the pair of maps $(\overline{\phi}, \overline{\widehat{\phi}})$: $(V/\Gamma, (V \times E)/\Gamma) \to (X, \mathcal{E})$ induced from $(\phi, \widehat{\phi})$ is an admissible embedding of admissible vector bundles.
 - (2) If $(V, E, \Gamma, \phi, \widehat{\phi})$ is an admissible orbifold chart of an admissible vector bundle, we call a pair $(E, \widehat{\phi})$ a *trivialization* of our admissible vector bundle on V/Γ .

- (3) Hereafter when (X, \mathcal{E}) is an admissible vector bundle, its 'admissible orbifold chart' always means an admissible orbifold chart of an admissible vector bundle in the sense of (1).
- (4) In case when an admissible vector bundle structure on (X, \mathcal{E}) is given, a representative of this admissible vector bundle structure is called an *admissible orbifold atlas* of (X, \mathcal{E}) .
- (5) Two admissible orbifold charts $(V_i, E_i, \Gamma_i, \phi_i, \widehat{\phi_i})$ of an admissible vector bundle are said to be *isomorphic* if there exist an isomorphism $(h, \widetilde{\varphi})$ of admissible orbifold charts $(V_1, \Gamma_1, \phi_1) \to (V_2, \Gamma_2, \phi_2)$ and an admissible isomorphism $(h, \widetilde{\varphi})$ of admissible orbifold charts $(V_1 \times E_1, \Gamma_1, \widehat{\phi_1}) \to (V_2 \times E_2, \Gamma_2, \widehat{\phi_2})$ such that they induce an admissible embedding of admissible vector bundles $(\varphi, \widehat{\varphi}) : (V_1/\Gamma_1, (V_1 \times E_1)/\Gamma_1) \to (V_2/\Gamma_2, (V_2 \times E_2)/\Gamma_2)$. The triple $(h, \widetilde{\varphi}, \widetilde{\varphi})$ is called an *admissible isomorphism* or *admissible coordinate change* between admissible orbifold charts of the admissible vector bundle.

Once we have established these basic notions related to admissible vector bundles as above, the next lemma obviously holds.

Lemma 25.27. The tangent bundle of an admissible orbifold has a canonical structure of admissible vector bundle.

Taking Whitney sum, tensor product, dual, exterior power, quotient of admissible vector bundles preserve admissibility.

Lemma-Definition 25.28. If (X^b, \mathcal{E}^b) is an admissible vector bundle and $f: X^a \to X^b$ is an admissible embedding, then the pull-back $f^*\mathcal{E}^b$ has a unique structure of admissible vector bundle such that the embedding of vector bundles $(f, \hat{f}): (X^a, f^*\mathcal{E}^b) \to (X^b, \mathcal{E}^b)$ becomes an admissible embedding.

We call $f^*\mathcal{E}^b$ equipped with this admissible vector bundle structure the pull-back in the sense of admissible vector bundles.

The proof is straightforward, so is omitted. The following lemmas are also straightforward consequences from definitions.

Lemma 25.29. A covering space \tilde{X} of an admissible orbifold X has a canonical structure of an admissible orbifold such that admissible functions of X are pulled back to admissible functions.

Lemma 25.30. The normalized boundary or corner of an admissible orbifold is admissible. The covering maps in Lemma 24.15 or Proposition 24.16 are admissible.

Definition 25.31. An admissible section of an admissible vector bundle (X, \mathcal{E}) is an admissible embedding of orbifolds $s: X \to \mathcal{E}$ such that the composition of s and the projection is the identity map set-theoretically.

The next lemma obviously follows from definition.

Lemma 25.32. An admissible tensor of an admissible orbifold is regarded as an admissible section of an appropriate tensor product bundle of the tangent bundle or its dual

Definition-Lemma 25.33. A connection ∇ on an admissible vector bundle (X, \mathcal{E}) is called *strongly admissible* if for an admissible orbifold chart $(V, E, \Gamma, \phi, \widehat{\phi})$ of

 (X,\mathcal{E}) with $V \subset \overline{V} \times [0,1)^k$ as in Situation 25.1, it is locally expressed by a 1-form $\sum_{\alpha=1}^{\overline{v}} A^a_{b,\alpha} dx_\alpha + \sum_{i=1}^k A^a_{b,\overline{v}+i} dT_i$ satisfying the following: (Here $x_1,\ldots,x_{\overline{v}}$ are the local coordinates of \overline{V} and $A^a_{b,*}$ is the (a,b)-component of a matrix $(A^a_{b,*})$ defined by an endmorphism of E, and $T_i = e^{1/t_i}$ $(i=1,\ldots,k)$.)

- (1) the function $A_{b,\alpha}^a$ is admissible, and
- (2) the function $A_{b,\overline{v}+i}^{a}$ is exponentially small near the boundary.

This notion is independent of the choices of the admissible orbifold charts of (X, \mathcal{E}) up to admissible coordinate changes in the sense of Definition 25.26 (5). (Note that the exterior derivative of an admissible function is strongly admissible by Definition-Lemma 25.20 (5).)

The next lemma is used in the proof of Lemma 25.39.

Lemma 25.34. The Levi-Civita connection of an admissible Riemannian metric is strongly admissible. Moreover, the Christoffel symbol Γ_{ij}^k of the Levi-Civita connection enjoys the following property: If i, j, k contain the horizontal coordinates (see Convention 25.19), Γ_{ij}^k is admissible, and if i, j, k contain the normal coordinates, Γ_{ij}^k is exponentially small near the boundary (under the coordinates $T_* = e^{1/t_*}$).

The next is an analog of Proposition 23.35 in admissible vector bundles.

Proposition 25.35. Let \mathcal{E} be an admissible vector bundle on $X \times [0,1]$, where X is an admissible orbifold. We identify $X \times \{0\}$, $X \times \{1\}$ with X in an obvious way. Then there exists an isomorphism of admissible vector bundles

$$I: \mathcal{E}|_{X \times \{0\}} \cong \mathcal{E}|_{X \times \{1\}}.$$

Suppose in addition that we are given a compact set $K \subset X$, its neighborhood V and an isomorphism

$$I_0: \mathcal{E}|_{V \times [0,1]} \cong \mathcal{E}|_{V \times \{0\}} \times [0,1].$$

Then we may choose I so that it coincides with the isomorphism induced by I_0 on K. If K is a submanifold, we may take K = V.

Proof. Using an admissible connection defined in Definition-Lemma 25.33, we can define a parallel transform for admissible vector bundles. Then the proof goes in a way similar to that of Proposition 23.35. See also the proof of Lemma 25.41 below. \Box

In this way, we can translate various stories for manifolds to the admissible realm. Especially we can define the notion of admissibility of Kuranishi structure. In fact, the whole story works just by adding the word admissible to various constructions.

Definition 25.36. A Kuranishi structure

$$\widehat{\mathcal{U}} = \left(\left\{ \mathcal{U}_p = \left(U_p, \mathcal{E}_p, \psi_p, s_p \right) \right\}, \left\{ \Phi_{pq} = \left(U_{pq}, \varphi_{pq}, \widehat{\varphi}_{pq} \right) \right\} \right)$$

is admissible if

- U_p is an admissible orbifold in the sense of Definition 25.13.
- \mathcal{E}_p is an admissible vector bundle in the sense of Definition 25.25.
- s_p is an admissible section of \mathcal{E}_p in the sense of Definition 25.31.
- U_{pq} is an open admissible suborbifold of U_q in the sense of Definition 25.13.
- $(\varphi_{pq}, \widehat{\varphi}_{pq})$ is an admissible embedding of admissible vector bundles in the sense of Definition 25.25.

25.4. Admissibility of bundle extension data. In [Part I, Subsections 12.2 and 13.3] we used a pair of the projection of the normal bundle and an isomorphism of bundles, which we called a bundle extension data. In this subsection we introduce the corresponding notion in the admissible category and prove its existence.

Definition 25.37. In [Part I, Situation 12.21], we assume that Kuranishi charts $\mathcal{U}_i = (U_i, \mathcal{E}_i, \psi_i, s_i)$ (i = 1, 2) are admissible and the embedding $\Phi_{21} : \mathcal{U}_1 \to \mathcal{U}_2$ is admissible. Then we say the bundle extension data $(\pi_{12}, \tilde{\varphi}_{21}, \Omega_{12})$ associated with Φ_{21} defined as in Definition 20.3 is an admissible bundle extension data if both of $\pi_{12} : \Omega_{12} \to U_1$ and $\tilde{\varphi}_{21} : \pi_{12}^* \mathcal{E}_1 \to \mathcal{E}_2$ are admissible.

Proposition 25.38. An admissible bundle extension data exists.

Proof. We first construct a tubular neighborhood in the admissible category.

Let $f: X \to Y$ be an admissible embedding of admissible orbifolds. Its normal bundle N_XY is defined and is an admissible vector bundle over X. We take an admissible Riemannian metric on Y. We will use it to define an exponential map

$$\operatorname{Exp}:BN_XY\to Y$$

that is a map from the neighborhood of the zero section of the total space of the normal bundle $N_X Y$ to Y.

Lemma 25.39. The exponential map Exp is an admissible diffeomorphism between admissible orbifolds.

Proof. Let Y be an admissible suborbifold. Using local coordinates we can represent them as follows. Let $p \in X \subset Y$ and $\mathfrak{V}_{X,p} = (V_{X,p}, \Gamma_{X,p}, \phi_{X,p})$ (resp. $\mathfrak{V}_{Y,p} = (V_{Y,p}, \Gamma_{Y,p}, \phi_{Y,p})$) be an orbifold chart of X (resp. Y) at p. We may take $V_{X,p} = \overline{V}_{X,p} \times [0,1)^k$ and $V_{Y,p} = \overline{V}_{Y,p} \times [0,1)^k$. Here $\overline{V}_{X,p}$, $\overline{V}_{Y,p}$ are open subsets of Euclidean space.

We change the coordinates from $(t_1, \ldots, t_k) \in [0, 1)^k$ to $(T_1, \ldots, T_k) \in (1, \infty]$ by $T_i = e^{1/t_i}$.

Then the embedding $X \to Y$ in the coordinates is given by

$$\varphi: \overline{V}_{X,p} \times (0,\infty]^k \to \overline{V}_{Y,p} \times (0,\infty]^k$$

such that

$$\varphi(\overline{y},(T_1,\ldots,T_k)) = (\varphi_0(\overline{y},(T_1,\ldots,T_k)), (T_i + \varphi_i(\overline{y},(T_1,\ldots,T_k))_{i=1}^k)),$$

where φ_0 , φ_i are admissible. Therefore the normal bundle has an orthonormal frame of the form

$$e^a = (e_0^a, (e_1^a, \dots, e_k^a)),$$
 (25.10)

where e_0^a is admissible and e_i^a are exponentially small at the boundary. (We also note that we use an admissible Riemannian metric of the form (25.7).)

Now let $\ell(s)$ be a geodesic of Y such that $\ell(0) \in \varphi(X)$ and

$$\dot{\ell}(0) = \sum_{a=1}^{\dim X - \dim Y} c_a e^a.$$

Let us write down the equation that ℓ is a geodesic by local coordinates. We use Greek letters for the indices of the horizontal coordinates and h, i, j for the indices

of the normal coordinates in the sense of Convention 25.19. (Recall that we use $T_i = e^{1/t_i}$ for the normal coordinates.) Then the equation of the geodesic is:

$$\frac{d^{2}\ell_{\alpha}}{ds^{2}} + \sum_{i,j} \Gamma_{ij}^{\alpha} \frac{d\ell_{i}}{ds} \frac{d\ell_{j}}{ds} + \sum_{i,\beta} \Gamma_{i\beta}^{\alpha} \frac{d\ell_{i}}{ds} \frac{d\ell_{\beta}}{ds} + \sum_{\beta,\gamma} \Gamma_{\beta\gamma}^{\alpha} \frac{d\ell_{\beta}}{ds} \frac{d\ell_{\gamma}}{ds} = 0,$$

$$\frac{d^{2}\ell_{h}}{ds^{2}} + \sum_{i,j} \Gamma_{ij}^{h} \frac{d\ell_{i}}{ds} \frac{d\ell_{j}}{ds} + \sum_{i,\beta} \Gamma_{i\beta}^{h} \frac{d\ell_{i}}{ds} \frac{d\ell_{\beta}}{ds} + \sum_{\beta,\gamma} \Gamma_{\beta\gamma}^{h} \frac{d\ell_{\beta}}{ds} \frac{d\ell_{\gamma}}{ds} = 0.$$
(25.11)

Since $\ell(0) \perp T\varphi(X)$, the functions $\frac{d\ell_{\alpha}}{ds}(0)$ are admissible and $\frac{d\ell_{h}}{ds}(0)$ are exponentially small. We also note that the Christoffel symbols Γ_{**}^* are admissible and Γ_{i*}^* , Γ_{**}^h are exponentially small. (See Lemma 25.34.) Therefore the equation (25.11) implies that the map which associates $\ell(s)$ to the initial value $(\ell(0), (c_1, \ldots, c_a))$ is admissible. The proof of Lemma 25.39 is complete.

Remark 25.40. The above proof shows that the geodesic $s \mapsto \ell(s)$ such that $\ell(0) \in \varphi(X)$ and $\ell(0) \perp T\varphi(X)$, $\|\ell(0)\| = 1$ is defined on $s \in [-S, S]$, where S is independent of the initial condition $\ell(0)$ and $\ell(0)$.

In the situation of Lemma 25.39 let \mathcal{E} be an admissible vector bundle on X. Its pull-back $f^*\mathcal{E}$ is an admissible vector bundle on Y. We identity a neighborhood of f(X) in Y with an open subset BN_XY of N_XY that is a neighborhood of the origin. The pull-back $\pi^*f^*\mathcal{E}$ is an admissible vector bundle. We define a map

$$\tilde{f}:\pi^*f^*\mathcal{E}\to\mathcal{E}$$

as follows. We describe the construction of \tilde{f} for the case when X,Y are manifolds. For the case of orbifolds we construct the map locally in a way equivariant under the action of the isotropy group. Then we can globalize the construction using the invariance under the coordinate change.

Let $p \in BN_XY$. The fiber $\pi^*f^*\mathcal{E}_p$ is identified with $\mathcal{E}_{f(p)}$. We assume that our neighborhood BN_XY of X is sufficiently small. Then there exists a unique geodesic joining p = f(q) and q. Taking an admissible connection on \mathcal{E} , we use the parallel transport along this geodesic to send elements of $\mathcal{E}_{f(p)}$ to \mathcal{E}_p . Thus we have obtained the map $\tilde{f}: \pi^*f^*\mathcal{E} \to \mathcal{E}$.

Lemma 25.41. The map \tilde{f} is an admissible isomorphism of admissible vector bundles.

Proof. Let $\ell(s)$ be the geodesic joining p to q. Note $\ell(0) \in \varphi(X)$ and $\ell(0) \perp T\varphi(X)$. Let $\xi(s) \in \mathcal{E}_{\ell(s)}$ is a parallel section along ℓ . We take an admissible orbifold chart of (X, \mathcal{E}) and write ξ_a be the expression of ξ . We use the letter a, b for the indices of ξ . The equation that $\xi(s)$ is parallel is written as

$$\frac{d\xi_a}{ds} + \sum_{b,i} A_{b,i}^a \frac{d\ell_i}{ds} \xi_b + \sum_{b,\alpha} A_{b,\alpha}^a \frac{d\ell_\alpha}{ds} \xi_b = 0.$$

Here A_{**}^* is an admissible connection on our admissible vector bundle. Note that $A_{b,i}^a$ and $\frac{d\ell_i}{ds}$ are exponentially small near the boundary and $A_{b,\alpha}^a$, $\frac{d\ell_\alpha}{ds}$ are admissible. Therefore the map which associates $\xi_a(s)$ to $\xi_a(0)$ is admissible.

We put $\pi = \text{pr} \circ \text{Exp}^{-1}: Y \to X$, where Exp is as in Lemma 25.39 and pr: $BN_XY \to X$ is the projection of the normal bundle. The pair (π, \tilde{f}) where \tilde{f} is as in Lemma 25.41 is an admissible bundle extension data.

Proposition 25.42. In the situation of [Part I, Proposition 13.9] we assume that $(\widehat{\mathcal{U}}, \mathcal{K})$ is admissible. Then there exists a system of bundle extension data of $(\widehat{\mathcal{U}}, \mathcal{K})$ consisting of admissible bundle extension data.

Proof. The proof is the same as that of [Part I, Proposition 13.9], using Proposition 25.38. In the course of the proof we need to extend the given admissible extension data without changing it at the compact set where the bundle extension data is already defined. (See the last part of the proof of [Part I, Lemma 13.10].) We can do the procedure in the admissible category using an admissible partition of unity.

25.5. Admissibility of the moduli spaces of pseudo-holomorphic curves. In [FOOO17, Theorem 6.4, Proposition 8.27, Proposition 8.32] (see also [FOOO15, Proposition 16.11]), we proved the exponential decay of the coordinate change. It implies

$$\left\| \frac{\partial}{\partial T_{\rm e}'} (T_{\rm e}' - T_{\rm e}) \right\| \le C e^{-\delta T_{\rm e}'}. \tag{25.12}$$

Here $T_{\rm e}$ is the coordinate corresponding to a singular point which is resolved and $T'_{\rm e}$ is the coordinate corresponding to the same singular point after coordinate change.

In this subsection, we axiomatize the properties of the coordinate change proved in [FOOO15] [FOOO17] under an abstract setting described in Situation 25.43. Then we can directly see that the results proved in [FOOO15] [FOOO17] actually imply that the Kuranishi structure of the moduli space of pseudo-holomorphic curves is admissible in the sense of Definition 25.36. This is tautological, but such an axiomatization will be useful when we prove admissibility of coordinate change of a Kuranishi structure constructed in other situation.

Situation 25.43. We consider open subsets $V_1 \subset \overline{V}_1 \times (0, \infty]^k$ and $V_2 \subset \overline{V}_2 \times (0, \infty]^k$, where \overline{V}_i are open subsets of \mathbb{R}^{n_i} . Let $\varphi : V_1 \to V_2$ be an embedding of topological spaces such that the following holds: We consider the stratification of V_i such that $S_m V_i$ consists of the points where at least m of the coordinates of the $(0, \infty)^k$ factor are ∞ . We put $\overset{\circ}{S}_m V_i = S_m V_i \setminus S_{m+1} V_i$. We assume the following:

- (1) $\varphi(p) \in S_m V_2$ if and only if $p \in S_m V_1$.
- (2) The restriction of φ to $\overset{\circ}{S}_m V_1$ is a smooth embedding $\overset{\circ}{S}_m V_1 \to \overset{\circ}{S}_m V_2$.
- (3) We write

$$\varphi(x;T_1,\ldots,T_k)=(\overline{\varphi}(x;T_1,\ldots,T_k);T_1'(x;T_1,\ldots,T_k),\ldots,T_k'(x;T_1,\ldots,T_k)).$$

Then the following holds: (a)

$$\left\| \frac{\partial \overline{\varphi}}{\partial T_i} \right\|_{C^k} \le C_k e^{-c_k T_i}.$$

Here C_k, c_k are positive numbers depending on k and φ , and $\|\cdot\|_{C^k}$ is the C^k norm with respect to all of x, T_i .

(b)
$$\left\| \frac{\partial T_j'}{\partial T_i} - \delta_{ij} \right\|_{C_k} \le C_k e^{-c_k T_i}. \tag{25.13}$$

Here $C_k, c_k, \|\cdot\|_{C^k}$ are the same as above and $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$.

We assume the above inequality (a)(b) stratumwise. Namely in case when certain coordinate T_i is ∞ , we require the inequality only for T_j derivatives which are not ∞ .

Lemma 25.44. In Situation 25.43 the coordinate change φ is admissible in the sense of Definition 25.7.

Proof. This is immediate from the definition.

We can use this lemma to obtain an admissible coordinate in the geometric situation appearing in the moduli space of pseudo-holomorphic curves.

Remark 25.45. As we explained in [FOOO4, Remark A1.63], the coordinate appearing in algebraic geometry is $e^{-cT_{\rm e}}$ which decays faster than $1/T_{\rm e}$. On the other hand, $1/T_{\rm e}$ is the coordinate used in [FOOO7] [FOOO15] etc. Here the coordinate change is smooth with respect to this coordinate $1/T_{\rm e}$. In this article we take an even more slower coordinate $1/\log T_{\rm e}$ than $1/T_{\rm e}$ (see (25.1)) so that the coordinate change is admissible.

When we use the coordinate $s_{\rm e}=1/T_{\rm e}$ in place of $1/\log T_{\rm e}$, Lemma 25.9 (1) still holds while Lemma 25.9 (2) does not. In fact, f(x,T)=1 is an admissible coordinate and so

$$T' = T + 1 (25.14)$$

is an admissible coordinate change. Then putting s'=1/T' and s=1/T, we have $s'=\frac{1}{1+1/s}=\frac{s}{s+1}$. Hence $\frac{d^2s'}{ds^2}(0)=-2\neq 0$. As we mentioned in Remark 25.10, Lemma 25.9 (2) is necessary to extend the coordinate change to the collared neighborhood. Indeed, in Section 17 of this article we extended the coordinate change from $\overline{V}\times[0,1)$ to $\overline{V}\times(-1,1)$ by taking the (-1,1) component to be the identity map on (-1,0). Lemma 25.9 (2) implies that this extended coordinate change is smooth at $t=1/\log T=0$.

On the other hand, in [FOOO4] we used the fact that the coordinate change is smooth in the coordinate s=1/T. However, we note that we did not need Lemma 25.9 (2) in [FOOO4] because we did not use the *collared* Kuranishi neighborhood and did not need to extend the coordinate change.

The fact that the extended coordinate change is smooth will be used when we make Kuranishi structures on the moduli spaces of pseudo-holomorphic curves compatible with the *forgetful map of marked points*, by adopting the framework of *collared* Kuranishi structure developed in Section 17. We will discuss this point in detail in the forthcoming paper [FOOO18].

Remark 25.46. In the geometric situation, the coordinate change where T'-T is positive at $T=\infty$ actually occurs. So the coordinate change of the form (25.14) should be considered. In fact, the parameter T corresponds to the 'length of the neck' region. Namely if the neck of the source curve is $[0,1] \times [-5T,5T]$, the corresponding element in the (thickened) moduli space has the coordinate T.

However the value of T depends on the choice of the coordinate at the nodes. Actually when nodes have coordinates z and w, we identify zw = -r and

$$r = e^{10\pi T}.$$

See [FOOO17, Section 8] right above Figure 12. If we take a different choice of z, say z' = ez, then r becomes r' = er. So T' = T + 1.

Remark 25.47. (1) In [FOOO17, Section 8] it is also proved that the coordinate change of the obstruction bundle is admissible and the Kuranishi map is also admissible. (The former is a consequence of [FOOO17, Proposition 8.19] and the latter is proved in the course of the proof of [FOOO17, Proposition 8.31].) Therefore it turns out to be that the Kuranishi structure we constructed on the moduli space of pseudo-holomorphic curves is admissible in the sense of Definition 25.36.

(2) The argument of [FOOO17, Section 8] quoted above is the cases of the Kuranishi chart of a stable map $((\Sigma, \vec{z}), u)$ when the marked source curve (Σ, \vec{z}) is stable. There are cases when the pair of a marked source curve (Σ, \vec{z}) and a map $u: \Sigma \to X$ is stable but (Σ, \vec{z}) is not stable. We can also prove admissibility of the Kuranishi chart in such cases. See [FOOO15, Part 4 especially Section 21].

Remark 25.48. There are different kind of boundaries or corners appearing in applications. For example, to prove independence of the filtered A_{∞} structure associated to a Lagrangian submanifold under the change of compatible almost complex structures chosen in the course of the construction, we take a one parameter family of almost complex structures $\{J_s\}$ joining two almost complex structures J_0 and J_1 which we choose for the construction. Then we consider the union of moduli spaces of pseudo-holomorphic discs bounding our Lagrangian submanifold with respect to the almost complex structures J_s for $s \in [0,1]$. In this case the part s=0,1 becomes a boundary. To prove that our Kuranishi structure is admissible at this boundary, we choose our family $\{J_s\}$ so that $J_s=J_0$ (resp. $J_s=J_1$) for $s \in [0,\epsilon]$ (resp. for $s \in [1-\epsilon,1]$). Then the boundary corresponding to s=0,1 has a canonical collar. Therefore admissibility is obviously satisfied for this collar.

We can study the case when we consider a homotopy of Hamiltonians or homotopy of homotopies of almost complex structures (or Hamiltonians) in the same way. The study of such boundaries is much easier than that of the boundary corresponding to the boundary node.

26. Stratified submersion to a manifold with corners.

In [Part I, Definition 3.39] we defined the notions of strongly smooth map and weekly submersive map from a K-space to a manifold without boundary or corners. In this section we give the corresponding definitions for the case when the target manifold P has boundary or corner. In Sections 16 and 19, we used them to define and study homotopy and/or higher homotopy of morphisms of linear K-systems. In Sections 21 and 22, we also used them to define and study pseudo-isotopy of filtered A_{∞} structure associated to a Lagrangian submanifold.

Let P be a manifold with corners (cornered manifold). For $p \in P$ we take a coordinate (V_p, ϕ_p) such that $V_p = \overline{V}_p \times [0, 1)^k$ where \overline{V}_p is an open set of $\mathbb{R}^{\dim P - k}$ and $\phi_p : V_p \to P$ is a parametrization. Here $p \in \overset{\circ}{S}_k(P)$ and $p = \phi(y_p, (0, \dots, 0))$. In this section we take the coordinate of P in this form.

Let X be an orbifold with corners (cornered orbifold), and $f: X \to P$ a continuous map.

Definition 26.1. Under the situation above, we say $f: X \to P$ is a corner stratified smooth map if for each $q \in X$ and p = f(q) we can choose coordinates $\mathfrak{V}_q=(V_q,\Gamma_q,\phi_q)$ and $\mathfrak{V}_p=(V_p,\phi_p)$ respectively with the following properties. (Note that since P is a smooth manifold, $\Gamma_p=\{\mathrm{id}\}$.)

- (1) $V_p = \overline{V}_p \times [0,1)^k$ is as above and $V_q = \overline{V}_q \times [0,1)^{\ell+k}$. Here $p \in \overset{\circ}{S}_k(P)$, $q \in \overset{\circ}{S}_{k+\ell}(X)$ and $q = \phi(y_q, (0, \dots, 0))$. (2) There exists a map $f_q: V_q \to V_p$ of the form

$$f_q(y;(s_1,\ldots,s_{\ell},t_1,\ldots,t_k)) = (\overline{f}_q(y;(s_1,\ldots,s_{\ell},t_1,\ldots,t_k)),(t_1,\ldots,t_k))$$

such that $\overline{f}_q:V_q\to \overline{V}_p$ is admissible.

(3) The following diagram commutes.

$$V_{q} \xrightarrow{\phi_{q}} X$$

$$f_{p} \downarrow \qquad \qquad \downarrow f$$

$$V_{p} \xrightarrow{\phi_{p}} P$$

$$(26.1)$$

See Figure 26. In the case X, P are admissible, we require \mathfrak{V}_p , \mathfrak{V}_q are admissible charts.

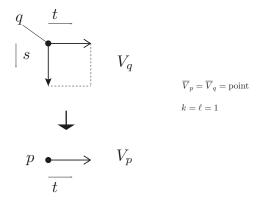


Figure 26. Figure of Definition 26.1

Remark 26.2. Throughout this section, we can work either in the category of smooth manifold (or orbifold) with corners, or in the admissible category. We do not mention about admissibility from now on in this section.

Definition 26.3. In the situation of Definition 26.1, we say $f: X \to P$ is a corner stratified submersion if $\overline{f}_q: V_q \to \overline{V}_p$ is a submersion for any $q \in X$.

Lemma 26.4. Let X_1, X_2 be cornered orbifolds and let P_1, P_2, P be cornered manifolds, and R a smooth manifold without boundary.

- (1) Let $f_i: X_i \to P \times R$ be smooth maps. Suppose f_1 is a corner stratified submersion. Then the fiber product $X_1 \times_{P \times R} X_2$ carries a structure of cornered orbifold. In addition, if $\pi_P \circ f_2: X_2 \to P$ is also a corner stratified submersion, then the map $X_1 \times_P X_2 \to P$ induced from f_1 and f_2 in an obvious way is a corner stratified submersion.
- (2) Let $f_i: X_i \to P_i \times R$ be smooth maps. Suppose f_1 is a corner stratified submersion. Then the fiber product $X_1 \times_R X_2$ carries a structure of cornered orbifold. In addition, if $\pi_{P_2} \circ f_2: X_2 \to P_2$ is also a corner stratified submersion, then the map $X_1 \times_R X_2 \to P_1 \times P_2$ induced from f_1 and f_2 in an obvious way is a corner stratified submersion.

The proof is obvious.

Lemma 26.5. Let $f: X \to P$ be a corner stratified smooth map from a cornered orbifold to a cornered manifold. Let $\widehat{S}_k(P)$ be the normalized corner of P and $\pi: \widehat{S}_k(P) \to S_k(P) \subset P$ the projection.

- (1) The fiber product $\widehat{S}_k(P) \times_P X$ as topological space carries a structure of cornered orbifold. The projection $\widehat{S}_k(P) \times_P X \to \widehat{S}_k(P)$ is a corner stratified smooth map.
- (2) The projection $\widehat{S}_k(P) \times_P X \to \widehat{S}_k(P)$ is a corner stratified submersion if $f: X \to P$ is a corner stratified submersion.
- (3) The map $\widehat{S}_{\ell}(\widehat{S}_k(P)) \times_P X \to \widehat{S}_{\ell+k}(P) \times_P X$ is a $(k+\ell)!/k!\ell!$ fold covering map.

The proof is again obvious.

Now it is straightforward to generalize the story for an orbifold X to the case when X is a K-space.

Definition 26.6. Let $(X, \widehat{\mathcal{U}})$ be a K-space and P a manifold with corner.

- (1) A strongly continuous map $\widehat{f}:(X,\widehat{\mathcal{U}})\to P$ is said to be a *corner stratified* smooth map if $f_p:U_p\to P$ is a corner stratified smooth map for any $p\in X$.
- (2) A corner stratified smooth map $\widehat{f}:(X,\widehat{\mathcal{U}})\to P$ is said to be a *corner stratified weak submersion* if $f_p:U_p\to P$ is a corner stratified submersion for any $p\in X$.
- (3) Let $\widehat{\mathfrak{S}}$ be a CF-perturbation of X. We say that a corner stratified smooth map $\widehat{f}:(X,\widehat{\mathcal{U}})\to P$ is a corner stratified strong submersion with respect to $\widehat{\mathfrak{S}}$ if the following holds. Let $p\in X$ and (U_p,E_p,ψ_p,s_p) be a Kuranishi chart at $p\in X$. Let $(V_{\mathfrak{r}},\Gamma_{\mathfrak{r}},\phi_{\mathfrak{r}})$ be an orbifold chart of U_p at some point and $(W_{\mathfrak{r}},\omega_{\mathfrak{r}},\mathfrak{s}_{\mathfrak{r}}^{\epsilon})$ be a representative of $\widehat{\mathfrak{S}}$ in this orbifold chart. Then

$$f \circ \psi_p \circ \phi_{\mathfrak{r}} \circ \operatorname{pr} : (\mathfrak{s}^{\epsilon}_{\mathfrak{r}})^{-1}(0) \to P$$

is a corner stratified submersion. Here $\operatorname{pr}: V_{\mathfrak{r}} \times W_{\mathfrak{r}} \to V_{\mathfrak{r}}$ is the projection. We can define a corner stratified smooth map and a corner stratified weak submersion from a space equipped with a good coordinate system in the same way.

Lemma-Definition 26.7. Let P be a cornered manifold and R a manifold with boundary. Let $\widehat{f}:(X,\widehat{\mathcal{U}})\to P\times R$ be a corner stratified strong submersion with respect to $\widehat{\mathfrak{S}}$. Then for any differential form h on $(X,\widehat{\mathcal{U}})$ and for each sufficiently small $\epsilon>0$, we can define the push out

$$\widehat{f}!(h;\widehat{\mathfrak{S}}^{\epsilon})$$

which is a smooth differential form on $P \times R$, in the same way as in [Part I, Theorem 9.14] using a good coordinate system compatible with the given Kuranishi structure. It is independent of the choice of the compatible good coordinate system if $\epsilon > 0$ is sufficiently small.

The proof is the same as that of [Part I, Theorem 9.14] and so is omitted.

Lemma 26.8. For each i=1,2 let $(X_i,\widehat{\mathcal{U}}_i)$ be a K-space and $\widehat{f}_i:(X_i,\widehat{\mathcal{U}}_i)\to P\times R$ a corner stratified smooth map, where P is a cornered manifold and R is a manifold without boundary. We assume that \widehat{f}_1 is a corner stratified weak submersion and $\pi\circ\widehat{f}_2:(X_2,\widehat{\mathcal{U}}_2)\to P$ is a corner stratified weak submersion.

- (1) The fiber product $X_1 \times_{P \times R} X_2$ carries a Kuranishi structure and the map $X_1 \times_{P \times R} X_2 \to P$ induced from \widehat{f}_1 and \widehat{f}_2 in an obvious way is a corner stratified weak submersion.
- (2) Let $\widehat{\mathfrak{S}}_i$ be a CF-perturbation of $(X_i,\widehat{\mathcal{U}}_i)$. We assume that \widehat{f}_1 is a corner stratified strong submersion with respect to $\widehat{\mathfrak{S}}_1$ and $\pi_P \circ \widehat{f}_2 : (X_2,\widehat{\mathcal{U}}_2) \to P$ is a corner stratified strong submersion with respect to $\widehat{\mathfrak{S}}_2$. Then we can define the fiber product $\widehat{\mathfrak{S}}_1 \times_{P \times R} \widehat{\mathfrak{S}}_2$ of CF-perturbations. The map $X_1 \times_{P \times R} X_2 \to P$ is a corner stratified strong submersion with respect to $\widehat{\mathfrak{S}}_1 \times_{P \times R} \widehat{\mathfrak{S}}_2$.

Proof. This follows from Lemma 26.4 (1). \Box

There is a slightly different situation we take the fiber product as follows:

Lemma 26.9. For each i=1,2 let $(X_i,\widehat{\mathcal{U}}_i)$ be a K-space and $\widehat{f}_i:(X_i,\widehat{\mathcal{U}}_i)\to R\times P_i$ a corner stratified smooth map, where P_i is a cornered manifold and R is a manifold without boundary. We assume that \widehat{f}_1 is a corner stratified weak submersion and $\pi_{P_2}\circ\widehat{f}_2$ is a corner stratified weak submersion.

- (1) The fiber product $X_1 \times_R X_2$ carries a Kuranishi structure and the map $X_1 \times_R X_2 \to P_1 \times P_2$ induced from \widehat{f}_1 and \widehat{f}_2 in an obvious way is a corner stratified weak submersion.
- (2) Let $\widehat{\mathfrak{S}}_i$ be a CF-perturbation of $(X_i,\widehat{\mathcal{U}}_i)$. We assume that \widehat{f}_i is a corner stratified strong submersion with respect to $\widehat{\mathfrak{S}}_i$. Then we can define the fiber product $\widehat{\mathfrak{S}}_1 \times_R \widehat{\mathfrak{S}}_2$ of CF-perturbations. The map $X_1 \times_R X_2 \to P_1 \times P_2$ is a corner stratified strong submersion with respect to $\widehat{\mathfrak{S}}_1 \times_R \widehat{\mathfrak{S}}_2$.

Proof. This follows from Lemma 26.4 (2). \Box

Next we discuss Stokes' formula and the composition formula under the correspondence.

Definition 26.10. Let $\widehat{f}:(X,\widehat{\mathcal{U}})\to P$ be a corner stratified weak submersion. We divide the boundary of X into two components:

$$\partial_{\mathfrak{C}^v}(X,\widehat{\mathcal{U}}) = \widehat{f}^{-1}(\partial P) \tag{26.2}$$

and

$$\partial_{\mathfrak{C}^h}(X,\widehat{\mathcal{U}}) = \partial(X,\widehat{\mathcal{U}}) \setminus \partial_{\mathfrak{C}^v}(X,\widehat{\mathcal{U}}). \tag{26.3}$$

We call (26.2) the *vertical boundary* and (26.3) the *horizontal boundary*. See Figure 27. They are induced by the decomposition of the boundary satisfying (18.9) in Situation 18.4.

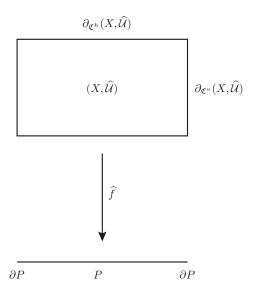


FIGURE 27. vertical/horizontal boundary

Lemma 26.11. In the situation of Definition 26.10, the restriction of \widehat{f} to the horizontal boundary induces a corner stratified weak submersion $\widehat{f}|_{\partial_{\mathfrak{C}^h}(X,\widehat{\mathcal{U}})}:\partial_{\mathfrak{C}^h}(X,\widehat{\mathcal{U}})\to P$. If \widehat{f} is a corner stratified strong submersion with respect to a CF-perturbation $\widehat{\mathfrak{S}}$, then the restriction $\widehat{f}|_{\partial_{\mathfrak{C}^h}(X,\widehat{\mathcal{U}})}$ is a corner stratified strong submersion with respect to $\widehat{\mathfrak{S}}|_{\partial_{\sigma^h}(X,\widehat{\mathcal{U}})}$.

The proof is obvious.

Theorem 26.12. In the situation of Lemma 26.11, let h be a differential form on $(X, \widehat{\mathcal{U}})$. Then for each sufficiently small $\epsilon > 0$ we have

$$d\widehat{f}!(h;\widehat{\mathfrak{S}}^{\epsilon}) = \widehat{f}!(dh;\widehat{\mathfrak{S}}^{\epsilon}) + \widehat{f}!(h|_{\partial_{\sigma h}(X,\widehat{\mathcal{U}})};\widehat{\mathfrak{S}}^{\epsilon}|_{\partial_{\sigma h}(X,\widehat{\mathcal{U}})}). \tag{26.4}$$

Proof. Since both hand sides are smooth forms, it suffices to prove the formula pointwise on Int P. Let $p \in \text{Int } P$ and take a compact set $K \subset \text{Int } P$ containing an open neighborhood of p. Using a partition of unity, we may assume without loss of generality that h is supported in $\widehat{f}^{-1}(K)$. Then we can apply [Part I, Theorem 9.26] to $X \setminus \partial_{\mathfrak{C}^v}(X,\widehat{\mathcal{U}})$ to prove the equality (26.4) at p.

Definition 26.13. Let P be a manifold with corner and let M_s, M_t be manifolds without boundary.

(1) A P-parametrized smooth correspondence is a system

$$\mathfrak{X} = \left((X, \widehat{\mathcal{U}}), \widehat{f}_s, \widehat{f}_t, \pi_{s,P}, \pi_{t,P} \right)$$

where $(X, \widehat{\mathcal{U}})$ is a K-space and $\widehat{f}_s: (X, \widehat{\mathcal{U}}) \to P \times M_s$ and $\widehat{f}_t: (X, \widehat{\mathcal{U}}) \to P \times M_t$ are strongly smooth maps. We assume that \widehat{f}_t is a corner stratified weak submersion and satisfies

$$\pi_{s,P} \circ \widehat{f}_s = \pi_{t,P} \circ \widehat{f}_t. \tag{26.5}$$

Here $\pi_{s,P}$ and $\pi_{t,P}$ are the projections to the P factor.

- (2) A P-parametrized perturbed smooth correspondence is $(\mathfrak{X}, \widehat{\mathfrak{S}})$ where $\mathfrak{X} = ((X, \widehat{\mathcal{U}}), \widehat{f_s}, \widehat{f_t})$ is a P-parametrized smooth correspondence and $\widehat{\mathfrak{S}}^{\epsilon}$ is a CF-perturbation of $(X, \widehat{\mathcal{U}})$ such that $\widehat{f_t}$ is a corner stratified strong submersion with respect to $\widehat{\mathfrak{S}}$.
- (3) Let $(\mathfrak{X},\widehat{\mathfrak{S}})$ $(\mathfrak{X} = ((X,\widehat{\mathcal{U}}),\widehat{f_s},\widehat{f_t},\pi_{s,P},\pi_{t,P}))$ be a P-parametrized perturbed smooth correspondence. The restrictions of $\widehat{f_s},\widehat{f_t},\widehat{\mathfrak{S}}$ to the horizontal boundary $\partial_{\mathfrak{C}^h}(X,\widehat{\mathcal{U}})$ define a P-parametrized perturbed smooth correspondence. We call it the *boundary* of $(\mathfrak{X},\widehat{\mathfrak{S}})$ and denote it by $\partial(\mathfrak{X},\widehat{\mathfrak{S}})$.

Definition 26.14. A P parametrized perturbed smooth correspondence $(\mathfrak{X}, \widehat{\mathfrak{S}})$ from M_s to M_t induces a map

$$\operatorname{Corr}_{(\mathfrak{X},\widehat{\mathfrak{S}}^{\epsilon})}: \Omega^{k}(P \times M_{s}) \to \Omega^{k+\ell}(P \times M_{t})$$

by

$$\operatorname{Corr}_{(\mathfrak{X},\widehat{\mathfrak{S}}^{\epsilon})}(h) = \widehat{f}_t!(\widehat{f}_s^*h;\widehat{\mathfrak{S}}^{\epsilon})$$
 (26.6)

for each sufficiently small $\epsilon > 0$.

Lemma 26.15. Suppose we are in the situation of Definition 26.14. If ρ is a differential form on P, then for each sufficiently small $\epsilon > 0$ we have

$$\operatorname{Corr}_{(\mathfrak{X},\widehat{\mathfrak{S}}^{\epsilon})}(\rho \wedge h) = \pm \rho \wedge \operatorname{Corr}_{(\mathfrak{X},\widehat{\mathfrak{S}}^{\epsilon})}(h).$$

Proof. This is a consequence of (26.5).

Theorem 26.12 immediately implies the following.

Proposition 26.16. For each sufficiently small $\epsilon > 0$ we have

$$d \circ \operatorname{Corr}_{(\mathfrak{X},\widehat{\mathfrak{S}}^{\epsilon})} = \operatorname{Corr}_{(\mathfrak{X},\widehat{\mathfrak{S}}^{\epsilon})} \circ d = \operatorname{Corr}_{\partial(\mathfrak{X},\widehat{\mathfrak{S}}^{\epsilon})}.$$

Next we discuss the composition formula.

Definition-Lemma 26.17. Let M_1, M_2, M_3 be smooth manifolds and P, P_1, P_2 manifolds with corner.

- (1) Let $\mathfrak{X}_{ii+1} = (X_{ii+1}, \widehat{\mathcal{U}}_{ii+1}, \widehat{f}_{ii+1;i}, \widehat{f}_{ii+1;i+1})$ be P-parametrized smooth correspondences from M_i to M_{i+1} for i = 1, 2.
 - (a) The composition $\mathfrak{X}_{13} = \mathfrak{X}_{23} \circ \mathfrak{X}_{12}$ is

$$((X_{12},\widehat{\mathcal{U}}_{12}) \times_{P \times M_2} (X_{23},\widehat{\mathcal{U}}_{23}), \widehat{f}_{12;1} \circ \pi, \widehat{f}_{12;2} \circ \pi),$$

which is a P-parametrized smooth correspondence from M_1 to M_3 .

- (b) In addition, if $(\mathfrak{X}_{ii+1}, \widehat{\mathfrak{S}}_{ii+1})$ is a P-parametrized perturbed smooth correspondence, then together with $\widehat{\mathfrak{S}}_{23} = \widehat{\mathfrak{S}}_{12} \times_{P \times M_2} \widehat{\mathfrak{S}}_{23}$, the composition $\mathfrak{X}_{13} = \mathfrak{X}_{23} \circ \mathfrak{X}_{12}$ defines a P-parametrized perturbed smooth correspondence from M_1 to M_3 . We say $(\mathfrak{X}_{13}, \widehat{\mathfrak{S}}_{13})$ is the *composition* of $(\mathfrak{X}_{12}, \widehat{\mathfrak{S}}_{12})$ and $(\mathfrak{X}_{23}, \widehat{\mathfrak{S}}_{23})$.
- (2) Let $\Xi_{ii+1} = (X_{ii+1}, \widehat{\mathcal{U}}_{ii+1})$ be P_i -parametrized smooth correspondences from M_i to M_{i+1} for i = 1, 2.
 - (a) The composition $\mathfrak{X}_{13} = \mathfrak{X}_{23} \circ \mathfrak{X}_{12}$ is defined by $(X_{12}, \widehat{\mathcal{U}}_{12}) \times_{M_2} (X_{23}, \widehat{\mathcal{U}}_{23})$ which is a $(P_1 \times P_2)$ -parametrized smooth correspondence from M_1 to M_3 .

(b) In addition, if $(\mathfrak{X}_{ii+1}, \widehat{\mathfrak{S}}_{ii+1})$ is a P_i -parametrized perturbed smooth correspondence, then together with $\widehat{\mathfrak{S}}_{23} = \widehat{\mathfrak{S}}_{12} \times_{M_2} \widehat{\mathfrak{S}}_{23}$, the composition \mathfrak{X}_{13} defines a $(P_1 \times P_2)$ -parametrized perturbed smooth correspondence from M_1 to M_3 . We say $(\Xi_{13}, \widehat{\mathfrak{S}}_{13})$ is the *composition* of $(\mathfrak{X}_{12}, \widehat{\mathfrak{S}}_{12})$ and $(\mathfrak{X}_{23}, \widehat{\mathfrak{S}}_{23})$.

Proof. This is a consequence of Lemma 26.9.

Proposition 26.18. In the situation of Definition-Lemma 26.17 (1) we have

$$\operatorname{Corr}_{(\mathfrak{X}_{23},\widehat{\mathfrak{S}}_{23}^{\epsilon})} \circ \operatorname{Corr}_{(\mathfrak{X}_{12},\widehat{\mathfrak{S}}_{12}^{\epsilon})} = \operatorname{Corr}_{(\mathfrak{X}_{13},\widehat{\mathfrak{S}}_{13}^{\epsilon})}$$
(26.7)

for each sufficiently small $\epsilon > 0$.

We will discuss the situation of Definition-Lemma 26.17 (2) later.

Proof. Let us consider the following situation.

Situation 26.19. For i = 1, 2, let $(X_i, \widehat{\mathcal{U}}_i)$ be K-spaces, P, P_i manifolds with corner, and R a manifold without boundary. Let $\widehat{\mathfrak{S}}_i$ be CF-perturbations of $(X_i, \widehat{\mathcal{U}}_i)$.

- (1) $\widehat{f}_i: (X_i, \widehat{\mathcal{U}}_i) \to P \times R$ are corner stratified strongly smooth maps for i = 1, 2. We assume that $\widehat{f}_1: (X_1, \widehat{\mathcal{U}}_1) \to P \times R$ and $\pi_P \circ \widehat{f}_2: (X_2, \widehat{\mathcal{U}}_2) \to P$ are corner stratified weakly submersive and corner stratified strongly submersive with respect to $\widehat{\mathfrak{S}}_1, \widehat{\mathfrak{S}}_2$, respectively.
- (2) $\widehat{f}_i: (X_i, \widehat{\mathcal{U}}_i) \to P_i \times R$ are corner stratified strongly smooth maps. We assume \widehat{f}_1 and $\pi_{P_2} \circ \widehat{f}_2$ are corner stratified weakly submersive and corner stratified strongly submersive with respect to $\widehat{\mathfrak{S}}_1, \widehat{\mathfrak{S}}_2$, respectively.

Lemma 26.20. In Situation 26.19 (1) we consider differential forms h_i on $(X_i, \widehat{\mathcal{U}}_i)$. They define a differential form $h_1 \wedge h_2$ on the fiber product $(X_1, \widehat{\mathcal{U}}_1) \times_{P \times R} (X_2, \widehat{\mathcal{U}}_2)$. Then for each sufficiently small $\epsilon > 0$ we have

$$\int_{((X_{1},\widehat{\mathcal{U}_{1}})\times_{R\times P}(X_{2},\widehat{\mathcal{U}_{2}}),(\widehat{\mathfrak{S}_{1}}\times_{R\times P}\widehat{\mathfrak{S}_{2}})^{\epsilon})}^{h_{1}\wedge h_{2}} h_{1} \wedge h_{2}$$

$$= \int_{((X_{2},\widehat{\mathcal{U}_{2}}),\widehat{\mathfrak{S}_{2}})}^{f^{*}} \widehat{f}_{1}!(h_{1};(\widehat{\mathfrak{S}_{1}})^{\epsilon}) \wedge h_{2}.$$
(26.8)

Proof. We can use Sublemma 26.21 below in place of [Part I, Lemma 10.27]. Then the proof is the same as that of [Part I, Proposition 10.23]. \Box

Sublemma 26.21. For i = 1, 2 let N_i , P be smooth manifolds with corner, and $f_i : N_i \to P \times M$ smooth maps, and h_i smooth differential forms on N_i of compact support. Suppose that f_1 is a corner stratified submersion. Then we have

$$\int_{N_1} \int_{f_1 \times f_2} h_1 \wedge h_2 = \pm \int_{N_2} f_2^*(f_1!(h_1)) \wedge h_2.$$
 (26.9)

Proof. Using a partition of unity, it suffices to prove (26.9) when $P = \overline{P} \times [0,1)^b$, $N_1 = P \times M \times \mathbb{R}^{a_1} \times [0,1)^{a_2}$ and $f_i : N_i \to P \times M$ is the obvious projection. We can prove this by Fubini's theorem in the same way as in [Part I, Lemma 10.27]

Lemma 26.22. In Situation 26.19 (2) we consider differential forms h_i on $(X_i, \widehat{\mathcal{U}}_i)$ (i = 1, 2) and the wedge product $h_1 \wedge h_2$ on the fiber product $(X_1, \widehat{\mathcal{U}}_1) \times_R (X_2, \widehat{\mathcal{U}}_2)$. Then for each sufficiently small $\epsilon > 0$ we have

$$\int_{((X_1,\widehat{\mathcal{U}}_1)\times_{R\times P}(X_2,\widehat{\mathcal{U}}_2),(\widehat{\mathfrak{S}}_1\times_{R\times P}\widehat{\mathfrak{S}}_2)^{\epsilon})} h_1 \wedge h_2$$

$$= \int_{((X_2,\widehat{\mathcal{U}}_2),\widehat{\mathfrak{S}}_2)} (\pi_R \circ \widehat{f}_2)^* (\pi_R \circ \widehat{f}_1)! (h_1;(\widehat{\mathfrak{S}}_1)^{\epsilon}) \wedge h_2. \tag{26.10}$$

Proof. This is a consequence of [Part I, Proposition 10.23].

The proof of Proposition 26.18 is now complete.

To discuss the situation of Situation 26.19 (2) we slightly generalize the notion of correspondence.

Definition 26.23. Let $(\mathfrak{X},\widehat{\mathfrak{S}}) = ((X,\widehat{\mathcal{U}},\widehat{f}_s,\widehat{f}_t,\pi_{s,P_1},\pi_{t,P_1}),\widehat{\mathfrak{S}})$ be a P_1 -parametrized perturbed smooth correspondence from M_s to M_t . Let P_2 be a manifold with corner. We regard

$$(P_2 \times (X, \widehat{\mathcal{U}}), \mathrm{id}_{P_2} \times \widehat{f}_s, \mathrm{id}_{P_2} \times \widehat{f}_t, \pi_{P_2} \circ (\mathrm{id}_{P_2} \times \pi_{s, P_1}), \pi_{P_2} \circ (\mathrm{id}_{P_2} \times \pi_{t, P_1}))$$

as a $(P_2 \times P_1)$ -parametrized smooth correspondence from M_s to M_t . Here π_{P_2} : $P_2 \times P_1 \to P_2$ is the projection. Then this defines a map which we denote by

$$\operatorname{Corr}_{(\mathfrak{X},\widehat{\mathfrak{S}}^{\epsilon}),P_2}: \Omega^k(P_2 \times P_1 \times M_s) \to \Omega^{k+\ell}(P_2 \times P_1 \times M_t)$$
 (26.11)

for each sufficiently small $\epsilon > 0$. Here $\ell = \dim M_t - \dim(X, \widehat{\mathcal{U}})$. Similarly we define

$$\operatorname{Corr}_{(\mathfrak{X},\widehat{\mathfrak{S}}^{\epsilon}),P_1}: \Omega^k(P_1 \times P_2 \times M_s) \to \Omega^{k+\ell}(P_1 \times P_2 \times M_t).$$
 (26.12)

We note that when we define these maps, we do not use the orientations on P_1, P_2 . So the order of factors in the direct product does not cause the sign problem. Thus we may write as

$$\operatorname{Corr}_{(\mathfrak{X},\widehat{\mathfrak{S}}^{\epsilon}),P_2}: \Omega^k(P_1 \times P_2 \times M_s) \to \Omega^{k+\ell}(P_1 \times P_2 \times M_t).$$

Proposition 26.24. In the situation of Definition-Lemma 26.17 (2) we have

$$\operatorname{Corr}_{(\mathfrak{X}_{23},\widehat{\mathfrak{S}}_{23}^{\epsilon}),P_{1}} \circ \operatorname{Corr}_{(\mathfrak{X}_{12},\widehat{\mathfrak{S}}_{12}^{\epsilon}),P_{2}} = \operatorname{Corr}_{(\mathfrak{X}_{13},\widehat{\mathfrak{S}}_{13}^{\epsilon})}$$
(26.13)

for each sufficiently small $\epsilon > 0$.

Proof.

Lemma 26.25. Suppose we are in Situation 26.19 (2). Let h_i be differential forms on $(X_i, \widehat{\mathcal{U}}_i)$ and ρ_i differential forms on P_i for i = 1, 2. Then we obtain a differential form

$$h_1 \wedge (\pi_{P_1} \circ f_1)^* \rho_1 \wedge h_2 \wedge (\pi_{P_2} \circ f_2)^* \rho_2$$

on $(X_1, \widehat{\mathcal{U}}_1) \times_R (X_2, \widehat{\mathcal{U}}_2)$. Moreover we have the following equality:

$$\int_{(X_1,\widehat{\mathcal{U}}_1)\times_R(X_2,\widehat{\mathcal{U}}_2)} h_1 \wedge (\pi_{P_1} \circ f_1)^* \rho_1 \wedge h_2 \wedge (\pi_{P_2} \circ f_2)^* \rho_2$$

$$= \int_{((X_2,\widehat{\mathcal{U}}_2),\widehat{\mathfrak{S}}_2)} (\pi_R \circ f_2)^* (\pi_R \circ f_1)! (h_1 \wedge (\pi_{P_1} \circ f_1)^* \rho_1; (\widehat{\mathfrak{S}}_1)^{\epsilon})$$

$$\wedge h_2 \wedge (\pi_{P_2} \circ f_2)^* \rho_2.$$
(26.14)

Proof. Applying [Part I, Proposition 10.23] to $h_1 \wedge (\pi_{P_1} \circ f_1)^* \rho_1$ and $h_2 \wedge (\pi_{P_2} \circ f_2)^* \rho_2$, we obtain Lemma 26.25.

Proposition 26.24 is a consequence of Lemma 26.25.

Next we rewrite Lemma 26.5 to the case when X is a K-space.

Lemma 26.26. Let $\hat{f}:(X,\hat{\mathcal{U}})\to P$ be a corner stratified smooth map from a K-space to a cornered manifold. Let $\hat{S}_k(P)$ be the normalized corner of P and $\pi:\hat{S}_k(P)\to S_k(P)\subset P$ the projection.

- (1) The fiber product $\widehat{S}_k(P) \times_P X$ as topological space carries a Kuranishi structure. The projection $\widehat{S}_k(P) \times_P X \to \widehat{S}_k(P)$ is a corner stratified smooth map.
- (2) The projection $\widehat{S}_k(P) \times_P X \to \widehat{S}_k(P)$ is a corner stratified submersion if $\widehat{f}: X \to P$ is a corner stratified submersion.
- (3) The map $\widehat{S}_{\ell}(\widehat{S}_k(P)) \times_P X \to \widehat{S}_{\ell+k}(P) \times_P X$ is induced by a $(k+\ell)!/k!\ell!$ fold covering map of K-spaces.

The proof is again obvious.

Next we mention the relation to (partial) trivialization of the corners. (See Sections 17 and 18.) The proof of the next lemma is straightforward so omitted.

Lemma 26.27. Suppose $\widehat{f}:(X,\widehat{\mathcal{U}})\to P$ is a corner stratified submersion from a K-space to an (admissible) manifold with corner. Then \widehat{f} induces a map

$$\widehat{f}^{\boxplus \tau_0} : (X, \widehat{\mathcal{U}})^{\boxplus \tau_0} \to P^{\boxplus \tau_0}.$$

Let $\mathfrak{C}^{\mathrm{vert}}$ be a component of the corner of $(X,\widehat{\mathcal{U}})$. Then we obtain a map

$$\widehat{f}^{\mathfrak{C}^{\operatorname{vert}} \coprod \tau_0} : (X, \widehat{\mathcal{U}})^{\mathfrak{C}^{\operatorname{vert}} \coprod \tau_0} \to P.$$

In both cases, if \widehat{f} is corner stratified weakly submersive (resp. corner stratified strongly submersive with respect to $\widehat{\mathfrak{S}}$), then $\widehat{f}^{\boxplus \tau_0}$ and $\widehat{f}^{\mathfrak{C}^{\mathrm{vert} \boxplus \tau_0}}$ are corner stratified weakly submersive (resp. corner stratified strongly submersive with respect to $\widehat{\mathfrak{S}}$).

Most of the stories of Kuranishi structure, CF-perturbation, push out etc. can be generalized to the case when the target space has a corner, in a straightforward way. We will describe them when we need them.

Remark 26.28. In Section 19 etc. we are using corner stratified submersions (Definition 26.6) to define and study homotopy and higher homotopy of morphisms of linear K-systems. On the other hand, we like to mention that there is another way to define homotopy and/or higher homotopy of morphisms etc. without using corner stratified submersions to manifolds with corner. Indeed, while we were writing [FOOO2] we sometimes took this way. For example, in [FOOO2, Subsection 19.2] we take a small number $\epsilon > 0$ and consider $P = (-\epsilon, 1 + \epsilon)$ instead of P = [0, 1]. When we construct $\mathcal{N}(\alpha_1, \alpha_2; P)$ in [FOOO2], we consider the P-parametrized version of the moduli space so that it is constant on $(-\epsilon, 0)$ and $(1, 1+\epsilon)$. This method also works rigorously. However choosing P = [0, 1] and using corner stratified submersions seem more natural.

27. Local system and smooth correspondence in de Rham theory with twisted coefficients

Let \mathcal{L} be a local system, i.e., a flat vector bundle, on a manifold M. We denote by $(\Omega^{\bullet}(M; \mathcal{L}), d = d_{\mathcal{L}})$ the de Rham complex with coefficients in \mathcal{L} . We recall some basic operations on the de Rham complex with twisted coefficients.

1. (Pull-back.) Let $f: N \to M$ be a smooth map. Clearly, the pull-back $f^*\mathcal{L}$ is a flat vector bundle and we have a cochain homomorphisms

$$f^*: \Omega^{\bullet}(M; \mathcal{L}) \to \Omega^{\bullet}(N; f^*\mathcal{L}).$$

As in the usual de Rham theory, we have $d \circ f^* = f^* \circ d$.

2. (Wedge product.) Let $\mathcal{L}_1, \mathcal{L}_2$ be flat vector bundles. Then $\mathcal{L}_1 \otimes \mathcal{L}_2$ is a flat vector bundle and we have the product

$$\wedge: \Omega^{\bullet}(M; \mathcal{L}_1) \otimes \Omega^{\bullet}(M; \mathcal{L}_2) \to \Omega^{\bullet}(M; \mathcal{L}_1 \otimes \mathcal{L}_2).$$

The wedge product and the differential enjoy the Leibniz' rule.

3. (Integration along fibers.) Let $\pi: N \to M$ be a proper submersion and let O_M (resp. O_N) be the flat real line bundle associated with the orientation O(1)-bundle of M (resp. N). We denote by $O_{\pi} = O_N \otimes \pi^* O_M$ be the relative orientation bundle of the submersion π . Then we have the integration along fibers

$$\pi!: \Omega^{\bullet}(N; O_{\pi}) \to \Omega^{\bullet}(M).$$

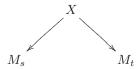
For a flat vector bundle \mathcal{L} on M, we have the integration along fibers with twisted coefficients

$$\pi!: \Omega^{\bullet}(N; \pi^*\mathcal{L}) \to \Omega^{\bullet}(M; \mathcal{L}).$$

Suppose that the boundary ∂N of N is not empty and the restriction $\pi|_{\partial N}:\partial N\to M$ is also submersion, then we have

$$d \circ \pi! = \pi! \circ d + (\pi|_{\partial N})!.$$

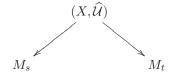
Now consider the following situation. Let $f_s: X \to M_s$ is a smooth map and $f_t: X \to M_t$ is a proper submersion.



Let \mathcal{L}_s (resp. \mathcal{L}_t) be a flat vector bundle such that $(f_s)^*\mathcal{L}_s \cong (f_t)^*\mathcal{L}_t \otimes O_{f_s}$. By composing the pull-back operation by f_s and the integration along fibers of f_t , we obtain the correspondence

$$f_t! \circ f_s^* : \Omega^{\bullet}(M_s; \mathcal{L}_s) \to \Omega^{\bullet}(M_t; \mathcal{L}_t).$$

Taking these arguments into account, we can obtain the following. Now we consider [Part I, Situation 7.1]. Let $\mathfrak{X} = ((X; \widehat{\mathcal{U}}); \widehat{f}_s, \widehat{f}_t)$ be a smooth correspondence from M_s to M_t . Namely, $(X, \widehat{\mathcal{U}})$ is a K-space, $f_s : (X, \widehat{\mathcal{U}}) \to M_s$ is a weakly smooth map and $f_t : (X, \widehat{\mathcal{U}}) \to M_t$ is a weak submersion.



Theorem 27.1. Let $\widehat{\mathfrak{S}} = \{\widehat{\mathfrak{S}}^{\epsilon}\}\$ be a CF-perturbation with respect to which f_t is a strong submersion. Let \mathcal{L}_s (resp. \mathcal{L}_t) be a flat vector bundle on M_s (resp. M_t) such that

$$(f_s)^* \mathcal{L}_s \cong (f_t)^* \mathcal{L}_t \otimes O_{f_t}. \tag{27.1}$$

Here O_{f_t} is the flat real line bundle associated with the relative orientation O(1)-bundle, i.e.,

$$O_{f_t} = (f_t)^* O_{M_t}^* \otimes O_X.$$
 (27.2)

Then for $\widehat{\mathfrak{X}} = (\mathfrak{X}, \widehat{\mathfrak{S}})$ we have the map

$$\operatorname{Corr}_{\widehat{\mathfrak{X}}}^{\epsilon}: \Omega^{\bullet}(M_s; \mathcal{L}_s) \to \Omega^{\bullet}(M_t; \mathcal{L}_t).$$

The following properties are fundamental.

Theorem 27.2. If the restriction of f_t to $(\partial X, \partial \widehat{U})$ is strongly submersive with respect to $\widehat{\mathfrak{S}}|_{\partial X}$, we have

$$d \operatorname{Corr}_{\widehat{\mathfrak{T}}}^{\epsilon}(h) = \operatorname{Corr}_{\widehat{\mathfrak{T}}}^{\epsilon}(dh) + \operatorname{Corr}_{\partial \widehat{\mathfrak{T}}}^{\epsilon}(h)$$

for $h \in \Omega^{\bullet}(M_s; \mathcal{L}_s)$.

In addition to [Part I, Situation 10.15], let \mathcal{L}_i be a flat vector bundle on M_i , i = 1, 2, 3 such that

$$f_{1,21}^* \mathcal{L}_1 \cong f_{2,21}^* \mathcal{L}_2 \otimes O_{f_{2,21}}, \quad f_{2,32}^* \mathcal{L}_2 \cong f_{3,32}^* \mathcal{L}_3 \otimes O_{f_{3,32}}.$$

Note that $O_{f_{3,31}}\cong g_{32,31}^*O_{f_{3,32}}\otimes g_{21,31}^*O_{f_{2,21}}$. Here $g_{32,321}:\mathfrak{X}_{31}\to\mathfrak{X}_{32}$ and $g_{21,31}:\mathfrak{X}_{31}\to\mathfrak{X}_{21}$ are natural projections from the fiber product. Hence we have

$$f_{1,31}^* \mathcal{L}_1 \cong f_{3,31}^* \mathcal{L}_3 \otimes O_{f_{3,31}}.$$

Then we have the following composition formula.

Theorem 27.3.

$$\operatorname{Corr}_{\widehat{\mathfrak{X}}_{32}\circ\widehat{\mathfrak{X}}_{21}}^{\epsilon}=\operatorname{Corr}_{\widehat{\mathfrak{X}}_{32}}^{\epsilon}\circ\operatorname{Corr}_{\widehat{\mathfrak{X}}_{21}}^{\epsilon}.$$

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