D-manifolds, d-orbifolds and derived differential geometry: a detailed summary

Dominic Joyce

Abstract

This is a detailed summary of the author's (rather longer) book [35]. We introduce a 2-category **dMan** of *d-manifolds*, new geometric objects which are 'derived' smooth manifolds, in the sense of the 'derived algebraic geometry' of Toën and Lurie. Manifolds **Man** embed in **dMan** as a full (2-)subcategory. There are also 2-categories **dMan**[b], **dMan**[c] of *d-manifolds with boundary* and *with corners*, and orbifold versions **dOrb**, **dOrb**[b], **dOrb**[c] of all of these, *d-orbifolds*.

Much of differential geometry extends very nicely to d-manifolds and d-orbifolds — immersions, submersions, submanifolds, transverse fibre products, orientations, bordism groups, etc. Compact oriented d-manifolds and d-orbifolds have virtual classes. Boundaries of d-manifolds and d-orbifolds with corners behave in a functorial way.

Many important areas of geometry involve forming moduli spaces $\mathcal{M}$ of geometric objects, and 'counting' them to get an enumerative invariant, or a more general structure in homological algebra, such as a Floer homology theory. These areas include Donaldson invariants and Seiberg–Witten invariants of 4-manifolds, Donaldson–Thomas invariants of Calabi–Yau 3-folds, Gromov–Witten invariants in both algebraic and symplectic geometry, and Lagrangian Floer cohomology, Fukaya categories, contact homology, and Symplectic Field Theory in symplectic geometry.

In all these areas, one first defines an appropriate geometric structure on $\mathcal{M}$, and then applies a 'virtual class' or 'virtual chain' construction to do the 'counting' and define the invariants. The geometric structures used for this purpose include $\mathbb{C}$-schemes and Deligne–Mumford $\mathbb{C}$-stacks with perfect obstruction theories in complex algebraic geometry, and polyfolds and Kuranishi spaces in symplectic geometry.

There are truncation functors from all of these classes of geometric structures on $\mathcal{M}$ to d-manifolds or d-orbifolds, with or without corners. There are also truncation functors from quasi-smooth derived $\mathbb{C}$-schemes and Spivak's derived manifolds to d-manifolds. As a result, all the areas of geometry above involving 'counting' moduli spaces can be rewritten in terms of d-manifolds and d-orbifolds. This will lead to new results and simplifications of existing proofs, particularly in areas involving moduli spaces with boundary and corners.

A (rather shorter) survey paper on the book, focussing on d-manifolds without boundary, is [36].

Contents

1 Introduction

This is a summary of the author's (rather longer) book [35] on 'D-manifolds and d-orbifolds: a theory of derived differential geometry'. A (rather shorter) survey paper on the book, focussing on d-manifolds without boundary, is [36], and readers just wanting a general overview of the theory are advised to read [36].

In this paper we aim to provide a fairly complete coverage of the main definitions and results of [35], omitting almost all proofs and some more abstruse technical details, which is suitable to be the primary reference for those wanting to use d-manifolds and d-orbifolds in their own research.

We develop a new theory of 'derived differential geometry'. The objects in this theory are *d-manifolds*, 'derived' versions of smooth manifolds, which form a (strict) 2-category **dMan**. There are also 2-categories of *d-manifolds with boundary* **dMan$^{\mathbf{b}}$** and *d-manifolds with corners* **dMan$^{\mathbf{c}}$**, and orbifold versions of all these, *d-orbifolds* **dOrb**, **dOrb$^{\mathbf{b}}$**, **dOrb$^{\mathbf{c}}$**.

Here 'derived' is intended in the sense of *derived algebraic geometry*. The original motivating idea for derived algebraic geometry, as in Kontsevich [38] for instance, was that certain moduli schemes $\mathcal{M}$ appearing in enumerative invariant problems may be very singular as schemes. However, it may be natural to realize $\mathcal{M}$ as a truncation of some 'derived' moduli space $\boldsymbol{\mathcal{M}}$, a new kind of geometric object living in a higher category. The geometric structure on $\boldsymbol{\mathcal{M}}$ should encode the full deformation theory of the moduli problem, the obstructions as well as the deformations. It was hoped that $\boldsymbol{\mathcal{M}}$ would be 'smooth', and so in some sense simpler than its truncation $\mathcal{M}$.

Early work in derived algebraic geometry focussed on *dg-schemes*, as in Ciocan-Fontanine and Kapranov [14]. These have largely been replaced by the *derived stacks* of Toën and Vezzosi [56–58], and the *structured spaces* of Lurie [40–42]. *Derived differential geometry* aims to generalize these ideas to differential geometry and smooth manifolds. A brief note about it can be found in Lurie [42, §4.5]; the ideas are worked out in detail by Lurie's student David Spivak [53], who defines an ∞-category of *derived manifolds*.

The author came to these questions from a different direction, symplectic geometry. Many important areas in symplectic geometry involve forming moduli spaces $\overline{\mathcal{M}}_{g,m}(X, J, \beta)$ of J-holomorphic curves in some symplectic manifold (X, ω), possibly with boundary in a Lagrangian Y, and then 'counting' these moduli spaces to get 'invariants' with interesting properties. Such areas include Gromov–Witten invariants (open and closed), Lagrangian Floer cohomology, Symplectic Field Theory, contact homology, and Fukaya categories.

To do this 'counting', one needs to put a suitable geometric structure on $\overline{\mathcal{M}}_{g,m}(X, J, \beta)$ — something like the 'derived' moduli spaces $\boldsymbol{\mathcal{M}}$ above — and use this to define a 'virtual class' or 'virtual chain' in $\mathbb{Z}, \mathbb{Q}$ or some homology theory. Two alternative theories for geometric structures to put on moduli spaces $\overline{\mathcal{M}}_{g,m}(X, J, \beta)$ are the *Kuranishi spaces* of Fukaya, Oh, Ohta and Ono [19, 20] and the *polyfolds* of Hofer, Wysocki and Zehnder [22–27].

The philosophies of Kuranishi spaces and of polyfolds are in a sense opposite: Kuranishi spaces remember only the minimal information needed to form virtual

chains, but polyfolds remember a huge amount more information, essentially a complete description of the functional-analytic problem which gives rise to the moduli space. There is a truncation functor from polyfolds to Kuranishi spaces.

The theory of Kuranishi spaces in [19, 20] does not go far — they define Kuranishi spaces, and construct virtual cycles upon them, but they do not define morphisms between Kuranishi spaces, for instance. The author tried to study and work with Kuranishi spaces as geometric spaces in their own right, but ran into problems, and became convinced that a new definition was needed. Upon reading Spivak's theory of derived manifolds [53], it became clear that some form of 'derived differential geometry' was required: *Kuranishi spaces in the sense of* [19, §A] *ought to be defined to be 'derived orbifolds with corners'*.

The purpose of [35], summarized here, is to build a comprehensive, rigorous theory of derived differential geometry designed for applications in symplectic geometry, and other areas of mathematics such as String Topology.

As the moduli spaces of interest in the symplectic geometry of Lagrangian submanifolds should be 'derived orbifolds with corners', it was necessary that this theory should cover not just derived manifolds without boundary, but also derived manifolds and derived orbifolds with boundary and with corners. It turns out that doing 'things with corners' properly is a complex, fascinating, and hitherto almost unexplored area. This has added considerably to the length of the project: the parts (sections 2–4 and Appendix A) dealing with d-manifolds without boundary are only a quarter of the whole.

The author wants the theory to be easily usable by symplectic geometers, and others who are not specialists in derived algebraic geometry. In applications, much of the theory can be treated as a 'black box', as they do not require a detailed understanding of what a d-manifold or d-orbifold really is, but only a general idea, plus a list of useful properties of the 2-categories $\mathbf{dMan}, \mathbf{dOrb}$.

Our theory of derived differential geometry has a major simplification compared to the derived algebraic geometry of Toën and Vezzosi [56–58] and Lurie [40–42], and the derived manifolds of Spivak [53]. All of the 'derived' spaces in [40–42, 53, 56–58] form some kind of ∞-category (simplicial category, model category, Segal category, quasicategory, . . .). In contrast, our d-manifolds and d-orbifolds form (strict) 2-categories $\mathbf{dMan}, \ldots, \mathbf{dOrb^c}$, which are the simplest and most friendly kind of higher category.

Furthermore, the ∞-categories in [40–42, 53, 56–58] are usually formed by localization (inversion of some class of morphisms), so the (higher) morphisms in the resulting ∞-category are difficult to describe and work with. But the 1- and 2-morphisms in $\mathbf{dMan}, \ldots, \mathbf{dOrb^c}$ are defined explicitly, without localization.

The essence of our simplification is this. Consider a 'derived' moduli space $\mathcal{M}$ of some objects E, e.g. vector bundles on some $\mathbb{C}$-scheme X. One expects $\mathcal{M}$ to have a 'cotangent complex' $\mathbb{L}_{\mathcal{M}}$, a complex in some derived category with cohomology $h^i(\mathbb{L}_{\mathcal{M}})|_E \cong \mathrm{Ext}^{1-i}(E, E)^*$ for $i \in \mathbb{Z}$. In general, $\mathbb{L}_{\mathcal{M}}$ can have nontrivial cohomology in many negative degrees, and because of this such objects $\mathcal{M}$ must form an ∞-category to properly describe their geometry.

However, the moduli spaces relevant to enumerative invariant problems are of a restricted kind: one considers only $\mathcal{M}$ such that $\mathbb{L}_{\mathcal{M}}$ has nontrivial coho-

mology only in degrees $-1, 0$, where $h^0(\mathbb{L}_{\boldsymbol{\mathcal{M}}})$ encodes the (dual of the) deformations $\mathrm{Ext}^1(E, E)^*$, and $h^{-1}(\mathbb{L}_{\boldsymbol{\mathcal{M}}})$ the (dual of the) obstructions $\mathrm{Ext}^2(E, E)^*$. As in Toën [56, §4.4.3], such derived spaces are called *quasi-smooth*, and this is a necessary condition on $\boldsymbol{\mathcal{M}}$ for the construction of a virtual fundamental class.

Our construction of d-manifolds replaces complexes in a derived category $D^b \mathrm{coh}(\boldsymbol{\mathcal{M}})$ with a 2-category of complexes in degrees $-1, 0$ only. For general $\boldsymbol{\mathcal{M}}$ this loses a lot of information, but for quasi-smooth $\boldsymbol{\mathcal{M}}$, since $\mathbb{L}_{\boldsymbol{\mathcal{M}}}$ is concentrated in degrees $-1, 0$, the important information is retained. In the language of dg-schemes, this corresponds to working with a subclass of derived schemes whose dg-algebras are of a special kind: they are 2-step supercommutative dg-algebras $A^{-1} \xrightarrow{\mathrm{d}} A^0$ such that $\mathrm{d}(A^{-1}) \cdot A^{-1} = 0$. Then $\mathrm{d}(A^{-1})$ is a square zero ideal in A^0, and A^{-1} is a module over $H^0\big(A^{-1} \xrightarrow{\mathrm{d}} A^0\big)$.

An important reason why this 2-category style derived geometry works successfully in our differential-geometric context is the existence of *partitions of unity* on smooth manifolds, and on nice C^∞-schemes. This means that (derived) structure sheaves are 'fine' or 'soft', which simplifies their behaviour. Partitions of unity are also essential for constructions such as gluing d-manifolds by equivalences on open d-subspaces in **dMan**. In conventional derived algebraic geometry, where partitions of unity do not exist, one needs the extra freedom of an ∞-category to glue by equivalences.

Throughout the paper, following [35], we will consistently use different typefaces to indicate different classes of geometrical objects. In particular:

- $W, X, Y, \ldots$ will denote manifolds (of any kind), or topological spaces.
- $\underline{W}, \underline{X}, \underline{Y}, \ldots$ will denote C^∞-schemes.
- $\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{Y}, \ldots$ will denote d-spaces, including d-manifolds.
- $\mathcal{W}, \mathcal{X}, \mathcal{Y}, \ldots$ will denote Deligne–Mumford C^∞-stacks, including orbifolds.
- $\boldsymbol{\mathcal{W}}, \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \ldots$ will denote d-stacks, including d-orbifolds.
- $\mathbf{W}, \mathbf{X}, \mathbf{Y}, \ldots$ will denote d-spaces with corners, including d-manifolds with corners.
- $\mathcal{W}, \mathcal{X}, \mathcal{Y}, \ldots$ will denote orbifolds with corners.
- $\boldsymbol{\mathcal{W}}, \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \ldots$ will denote d-stacks with corners, including d-orbifolds with corners.

2 C^∞-rings and C^∞-schemes

If X is a manifold then the $\mathbb{R}$-algebra $C^\infty(X)$ of smooth functions $c : X \to \mathbb{R}$ is a C^∞-*ring*. That is, for each smooth function $f : \mathbb{R}^n \to \mathbb{R}$ there is an n-fold

operation $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ acting by $\Phi_f : c_1, \ldots, c_n \mapsto f(c_1, \ldots, c_n)$, and these operations Φ_f satisfy many natural identities. Thus, $C^\infty(X)$ actually has a far richer algebraic structure than the obvious $\mathbb{R}$-algebra structure.

C^∞-*algebraic geometry* is a version of algebraic geometry in which rings or algebras are replaced by C^∞-rings. The basic objects are C^∞-*schemes*, a category of differential-geometric spaces including smooth manifolds, and also many singular spaces. They were introduced in synthetic differential geometry (see for instance Dubuc [18] and Moerdijk and Reyes [49]), and developed further by the author in [33] (surveyed in [34] and [35, App. B]).

This section briefly discusses C^∞-rings, C^∞-schemes, and quasicoherent sheaves on C^∞-schemes, following the author's treatment [33, §2–§6].

2.1 C^∞-rings

Definition 2.1. A C^∞-*ring* is a set $\mathfrak{C}$ together with operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all $n \geqslant 0$ and smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, where by convention when $n = 0$ we define $\mathfrak{C}^0$ to be the single point $\{\emptyset\}$. These operations must satisfy the following relations: suppose $m, n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ are smooth functions. Define a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g\big(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots, x_n)\big),$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $(c_1, \ldots, c_n) \in \mathfrak{C}^n$ we have

$$\Phi_h(c_1, \ldots, c_n) = \Phi_g\big(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)\big).$$

We also require that for all $1 \leqslant j \leqslant n$, defining $\pi_j : \mathbb{R}^n \to \mathbb{R}$ by $\pi_j : (x_1, \ldots, x_n) \mapsto x_j$, we have $\Phi_{\pi_j}(c_1, \ldots, c_n) = c_j$ for all $(c_1, \ldots, c_n) \in \mathfrak{C}^n$.

Usually we refer to $\mathfrak{C}$ as the C^∞-ring, leaving the operations Φ_f implicit.

A *morphism* between C^∞-rings $\big(\mathfrak{C}, (\Phi_f)_{f:\mathbb{R}^n \to \mathbb{R}\ C^\infty}\big)$, $\big(\mathfrak{D}, (\Psi_f)_{f:\mathbb{R}^n \to \mathbb{R}\ C^\infty}\big)$ is a map $\phi : \mathfrak{C} \to \mathfrak{D}$ such that $\Psi_f\big(\phi(c_1), \ldots, \phi(c_n)\big) = \phi \circ \Phi_f(c_1, \ldots, c_n)$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$ and $c_1, \ldots, c_n \in \mathfrak{C}$. We will write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

Here is the motivating example:

Example 2.2. Let X be a manifold. Write $C^\infty(X)$ for the set of smooth functions $c : X \to \mathbb{R}$. For $n \geqslant 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ smooth, define $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by

$$\big(\Phi_f(c_1, \ldots, c_n)\big)(x) = f\big(c_1(x), \ldots, c_n(x)\big), \tag{2.1}$$

for all $c_1, \ldots, c_n \in C^\infty(X)$ and $x \in X$. It is easy to see that $C^\infty(X)$ and the operations Φ_f form a C^∞-ring.

Now let $f : X \to Y$ be a smooth map of manifolds. Then pullback $f^* : C^\infty(Y) \to C^\infty(X)$ mapping $f^* : c \mapsto c \circ f$ is a morphism of C^∞-rings. Furthermore (at least for Y without boundary), every C^∞-ring morphism $\phi : C^\infty(Y) \to C^\infty(X)$ is of the form $\phi = f^*$ for a unique smooth map $f : X \to Y$.

Write $\mathbf{C^\infty Rings}^{\mathrm{op}}$ for the opposite category of $\mathbf{C^\infty Rings}$, with directions of morphisms reversed, and $\mathbf{Man}$ for the category of manifolds without boundary. Then we have a full and faithful functor $F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}} : \mathbf{Man} \to \mathbf{C^\infty Rings}^{\mathrm{op}}$ acting by $F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}}(X) = C^\infty(X)$ on objects and $F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}}(f) = f^*$ on morphisms. This embeds $\mathbf{Man}$ as a full subcategory of $\mathbf{C^\infty Rings}^{\mathrm{op}}$.

Note that C^∞-rings are far more general than those coming from manifolds. For example, if X is any topological space we could define a C^∞-ring $C^0(X)$ to be the set of *continuous* $c : X \to \mathbb{R}$, with operations Φ_f defined as in (2.1). For X a manifold with $\dim X > 0$, the C^∞-rings $C^\infty(X)$ and $C^0(X)$ are different.

Definition 2.3. Let $\mathfrak{C}$ be a C^∞-ring. Then we may give $\mathfrak{C}$ the structure of a *commutative* $\mathbb{R}$-*algebra*. Define addition '+' on $\mathfrak{C}$ by $c + c' = \Phi_f(c, c')$ for $c, c' \in \mathfrak{C}$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is $f(x, y) = x + y$. Define multiplication '$\cdot$' on $\mathfrak{C}$ by $c \cdot c' = \Phi_g(c, c')$, where $g : \mathbb{R}^2 \to \mathbb{R}$ is $g(x, y) = xy$. Define scalar multiplication by $\lambda \in \mathbb{R}$ by $\lambda c = \Phi_{\lambda'}(c)$, where $\lambda' : \mathbb{R} \to \mathbb{R}$ is $\lambda'(x) = \lambda x$. Define elements $0, 1 \in \mathfrak{C}$ by $0 = \Phi_{0'}(\emptyset)$ and $1 = \Phi_{1'}(\emptyset)$, where $0' : \mathbb{R}^0 \to \mathbb{R}$ and $1' : \mathbb{R}^0 \to \mathbb{R}$ are the maps $0' : \emptyset \mapsto 0$ and $1' : \emptyset \mapsto 1$. One can show using the relations on the Φ_f that the axioms of a commutative $\mathbb{R}$-algebra are satisfied. In Example 2.2, this yields the obvious $\mathbb{R}$-algebra structure on the smooth functions $c : X \to \mathbb{R}$.

An *ideal I* in $\mathfrak{C}$ is an ideal $I \subset \mathfrak{C}$ in $\mathfrak{C}$ regarded as a commutative $\mathbb{R}$-algebra. Then we make the quotient $\mathfrak{C}/I$ into a C^∞-ring as follows. If $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, define $\Phi_f^I : (\mathfrak{C}/I)^n \to \mathfrak{C}/I$ by

$$\left(\Phi_f^I(c_1 + I, \ldots, c_n + I)\right)(x) = f\left(c_1(x), \ldots, c_n(x)\right) + I.$$

Using Hadamard's Lemma, one can show that this is independent of the choice of representatives $c_1, \ldots, c_n$. Then $\left(\mathfrak{C}/I, (\Phi_f^I)_{f : \mathbb{R}^n \to \mathbb{R} \ C^\infty}\right)$ is a C^∞-ring.

A C^∞-ring $\mathfrak{C}$ is called *finitely generated* if there exist $c_1, \ldots, c_n$ in $\mathfrak{C}$ which generate $\mathfrak{C}$ over all C^∞-operations. That is, for each $c \in \mathfrak{C}$ there exists smooth $f : \mathbb{R}^n \to \mathbb{R}$ with $c = \Phi_f(c_1, \ldots, c_n)$. Given such $\mathfrak{C}, c_1, \ldots, c_n$, define $\phi : C^\infty(\mathbb{R}^n) \to \mathfrak{C}$ by $\phi(f) = \Phi_f(c_1, \ldots, c_n)$ for smooth $f : \mathbb{R}^n \to \mathbb{R}$, where $C^\infty(\mathbb{R}^n)$ is as in Example 2.2 with $X = \mathbb{R}^n$. Then ϕ is a surjective morphism of C^∞-rings, so $I = \mathrm{Ker}\,\phi$ is an ideal in $C^\infty(\mathbb{R}^n)$, and $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ as a C^∞-ring. Thus, $\mathfrak{C}$ is finitely generated if and only if $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ for some $n \geqslant 0$ and some ideal I in $C^\infty(\mathbb{R}^n)$.

2.2 C^∞-schemes

Next we summarize material in [33, §4] on C^∞-schemes.

Definition 2.4. A C^∞-*ringed space* $\underline{X} = (X, \mathcal{O}_X)$ is a topological space X with a sheaf $\mathcal{O}_X$ of C^∞-rings on X.

A *morphism* $\underline{f} = (f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of C^∞ ringed spaces is a continuous map $f : X \to Y$ and a morphism $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ of sheaves of C^∞-rings on X, where $f^{-1}(\mathcal{O}_Y)$ is the inverse image sheaf. There is another

way to write the data $f^\sharp$: since direct image of sheaves f_* is right adjoint to inverse image f^{-1}, there is a natural bijection

$$\mathrm{Hom}_X\big(f^{-1}(\mathcal{O}_Y),\mathcal{O}_X\big) \cong \mathrm{Hom}_Y\big(\mathcal{O}_Y, f_*(\mathcal{O}_X)\big). \qquad (2.2)$$

Write $f_\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ for the morphism of sheaves of C^∞-rings on Y corresponding to $f^\sharp$ under (2.2), so that

$$f^\sharp : f^{-1}(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X \quad \rightsquigarrow \quad f_\sharp : \mathcal{O}_Y \longrightarrow f_*(\mathcal{O}_X). \qquad (2.3)$$

Depending on the application, either $f^\sharp$ or $f_\sharp$ may be more useful. We choose to regard $f^\sharp$ as primary and write morphisms as $\underline{f} = (f, f^\sharp)$ rather than $(f, f_\sharp)$, because we find it convenient to work uniformly using pullbacks, rather than mixing pullbacks and pushforwards.

Write $\mathbf{C^\infty RS}$ for the category of C^∞-ringed spaces. As in [18, Th. 8] there is a *spectrum functor* $\mathrm{Spec} : \mathbf{C^\infty Rings}^{\mathrm{op}} \to \mathbf{C^\infty RS}$, defined explicitly in [33, Def. 4.12]. A C^∞-ringed space $\underline{X}$ is called an *affine C^∞-scheme* if it is isomorphic in $\mathbf{C^\infty RS}$ to $\mathrm{Spec}\,\mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$. A C^∞-ringed space $\underline{X} = (X, \mathcal{O}_X)$ is called a C^∞-*scheme* if X can be covered by open sets $U \subseteq X$ such that $(U, \mathcal{O}_X|_U)$ is an affine C^∞-scheme. Write $\mathbf{C^\infty Sch}$ for the full subcategory of C^∞-schemes in $\mathbf{C^\infty RS}$.

A C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$ is called *locally fair* if X can be covered by open $U \subseteq X$ with $(U, \mathcal{O}_X|_U) \cong \mathrm{Spec}\,\mathfrak{C}$ for some finitely generated C^∞-ring $\mathfrak{C}$. Roughly speaking this means that $\underline{X}$ is locally finite-dimensional. Write $\mathbf{C^\infty Sch^{lf}}$ for the full subcategory of locally fair C^∞-schemes in $\mathbf{C^\infty Sch}$.

We call a C^∞-scheme $\underline{X}$ *separated, second countable, compact, locally compact*, or *paracompact*, if the underlying topological space X is Hausdorff, second countable, compact, locally compact, or paracompact, respectively.

We define a C^∞-scheme $\underline{X}$ for each manifold X.

Example 2.5. Let X be a manifold. Define a C^∞-ringed space $\underline{X} = (X, \mathcal{O}_X)$ to have topological space X and $\mathcal{O}_X(U) = C^\infty(U)$ for each open $U \subseteq X$, where $C^\infty(U)$ is the C^∞-ring of smooth maps $c : U \to \mathbb{R}$, and if $V \subseteq U \subseteq X$ are open define $\rho_{UV} : C^\infty(U) \to C^\infty(V)$ by $\rho_{UV} : c \mapsto c|_V$. Then $\underline{X} = (X, \mathcal{O}_X)$ is a local C^∞-ringed space. It is canonically isomorphic to $\mathrm{Spec}\,C^\infty(X)$, and so is an affine C^∞-scheme. It is locally fair.

Define a functor $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}} : \mathbf{Man} \to \mathbf{C^\infty Sch^{lf}} \subset \mathbf{C^\infty Sch}$ by $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}} = \mathrm{Spec} \circ F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}}$. Then $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}$ is full and faithful, and embeds $\mathbf{Man}$ as a full subcategory of $\mathbf{C^\infty Sch}$.

By [33, Cor. 4.21 & Th. 4.33] we have:

Theorem 2.6. *Fibre products and all finite limits exist in $\mathbf{C^\infty Sch}$. The subcategory $\mathbf{C^\infty Sch^{lf}}$ is closed under fibre products and finite limits. The functor $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}$ takes transverse fibre products in $\mathbf{Man}$ to fibre products in $\mathbf{C^\infty Sch}$.*

The proof of the existence of fibre products in $\mathbf{C^\infty Sch}$ follows that for fibre products of schemes in Hartshorne [21, Th. II.3.3], together with the existence

of C^∞-scheme products $\underline{X} \times \underline{Y}$ of affine C^∞-schemes $\underline{X}, \underline{Y}$. The latter follows from the existence of coproducts $\mathfrak{C} \hat{\otimes} \mathfrak{D}$ in $\mathbf{C^\infty Rings}$ of C^∞-rings $\mathfrak{C}, \mathfrak{D}$. Here $\mathfrak{C} \hat{\otimes} \mathfrak{D}$ may be thought of as a 'completed tensor product' of $\mathfrak{C}, \mathfrak{D}$. The actual tensor product $\mathfrak{C} \otimes_\mathbb{R} \mathfrak{D}$ is naturally an $\mathbb{R}$-algebra but not a C^∞-ring, with an inclusion of $\mathbb{R}$-algebras $\mathfrak{C} \otimes_\mathbb{R} \mathfrak{D} \hookrightarrow \mathfrak{C} \hat{\otimes} \mathfrak{D}$, but $\mathfrak{C} \hat{\otimes} \mathfrak{D}$ is often much larger than $\mathfrak{C} \otimes_\mathbb{R} \mathfrak{D}$. For free C^∞-rings we have $C^\infty(\mathbb{R}^m) \hat{\otimes} C^\infty(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^{m+n})$.

In [33, Def. 4.34 & Prop. 4.35] we discuss *partitions of unity* on C^∞-schemes.

Definition 2.7. Let $\underline{X} = (X, \mathcal{O}_X)$ be a C^∞-scheme. Consider a formal sum $\sum_{a \in A} c_a$, where A is an indexing set and $c_a \in \mathcal{O}_X(X)$ for $a \in A$. We say $\sum_{a \in A} c_a$ is a *locally finite sum on* $\underline{X}$ if X can be covered by open $U \subseteq X$ such that for all but finitely many $a \in A$ we have $\rho_{XU}(c_a) = 0$ in $\mathcal{O}_X(U)$.

By the sheaf axioms for $\mathcal{O}_X$, if $\sum_{a \in A} c_a$ is a locally finite sum there exists a unique $c \in \mathcal{O}_X(X)$ such that for all open $U \subseteq X$ with $\rho_{XU}(c_a) = 0$ in $\mathcal{O}_X(U)$ for all but finitely many $a \in A$, we have $\rho_{XU}(c) = \sum_{a \in A} \rho_{XU}(c_a)$ in $\mathcal{O}_X(U)$, where the sum makes sense as there are only finitely many nonzero terms. We call c the *limit* of $\sum_{a \in A} c_a$, written $\sum_{a \in A} c_a = c$.

Let $c \in \mathcal{O}_X(X)$. Then there is a unique maximal open set $V \subseteq X$ with $\rho_{XV}(c) = 0$ in $\mathcal{O}_X(V)$. Define the *support* $\operatorname{supp} c$ to be $X \setminus V$, so that $\operatorname{supp} c$ is closed in X. If $U \subseteq X$ is open, we say that *c is supported in* U if $\operatorname{supp} c \subseteq U$.

Let $\{U_a : a \in A\}$ be an open cover of X. A *partition of unity on* $\underline{X}$ *subordinate to* $\{U_a : a \in A\}$ is $\{\eta_a : a \in A\}$ with $\eta_a \in \mathcal{O}_X(X)$ supported on U_a for $a \in A$, such that $\sum_{a \in A} \eta_a$ is a locally finite sum on $\underline{X}$ with $\sum_{a \in A} \eta_a = 1$.

Proposition 2.8. *Suppose $\underline{X}$ is a separated, paracompact, locally fair C^∞-scheme, and $\{\underline{U}_a : a \in A\}$ an open cover of $\underline{X}$. Then there exists a partition of unity $\{\eta_a : a \in A\}$ on $\underline{X}$ subordinate to $\{\underline{U}_a : a \in A\}$.*

Here are some differences between ordinary schemes and C^∞-schemes:

Remark 2.9. (i) If A is a ring or algebra, then points of the corresponding scheme $\operatorname{Spec} A$ are prime ideals in A. However, if $\mathfrak{C}$ is a C^∞-ring then (by definition) points of $\operatorname{Spec} \mathfrak{C}$ are maximal ideals in $\mathfrak{C}$ with residue field $\mathbb{R}$, or equivalently, $\mathbb{R}$-algebra morphisms $x : \mathfrak{C} \to \mathbb{R}$. This has the effect that if X is a manifold then points of $\operatorname{Spec} C^\infty(X)$ are just points of X.

(ii) In conventional algebraic geometry, affine schemes are a restrictive class. Central examples such as $\mathbb{CP}^n$ are not affine, and affine schemes are not closed under open subsets, so that $\mathbb{C}^2$ is affine but $\mathbb{C}^2 \setminus \{0\}$ is not. In contrast, affine C^∞-schemes are already general enough for many purposes. For example:

- All manifolds are fair affine C^∞-schemes.
- Open C^∞-subschemes of fair affine C^∞-schemes are fair and affine.
- Separated, second countable, locally fair C^∞-schemes are affine.

Affine C^∞-schemes are always separated (Hausdorff), so we need general C^∞-schemes to include non-Hausdorff behaviour.

(iii) In conventional algebraic geometry the Zariski topology is too coarse for many purposes, so one has to introduce the étale topology. In C^∞-algebraic geometry there is no need for this, as affine C^∞-schemes are Hausdorff.

(iv) Even very basic C^∞-rings such as $C^\infty(\mathbb{R}^n)$ for $n > 0$ are not noetherian as $\mathbb{R}$-algebras. So C^∞-schemes should be compared to non-noetherian schemes in conventional algebraic geometry.

(v) The existence of partitions of unity, as in Proposition 2.8, makes some things easier in C^∞-algebraic geometry than in conventional algebraic geometry. For example, geometric objects can often be 'glued together' over the subsets of an open cover using partitions of unity, and if $\mathcal{E}$ is a quasicoherent sheaf on a separated, paracompact, locally fair C^∞-scheme $\underline{X}$ then $H^i(\mathcal{E}) = 0$ for $i > 0$.

2.3 Modules over C^∞-rings, and cotangent modules

In [33, §5] we discuss modules over C^∞-rings.

Definition 2.10. Let $\mathfrak{C}$ be a C^∞-ring. A $\mathfrak{C}$-*module* M is a module over $\mathfrak{C}$ regarded as a commutative $\mathbb{R}$-algebra as in Definition 2.3. $\mathfrak{C}$-modules form an abelian category, which we write as $\mathfrak{C}$-mod. For example, $\mathfrak{C}$ is a $\mathfrak{C}$-module, and more generally $\mathfrak{C} \otimes_\mathbb{R} V$ is a $\mathfrak{C}$-module for any real vector space V. Let $\phi : \mathfrak{C} \to \mathfrak{D}$ be a morphism of C^∞-rings. If M is a $\mathfrak{C}$-module then $\phi_*(M) = M \otimes_\mathfrak{C} \mathfrak{D}$ is a $\mathfrak{D}$-module. This induces a functor $\phi_* : \mathfrak{C}$-mod $\to \mathfrak{D}$-mod.

Example 2.11. Let X be a manifold, and $E \to X$ a vector bundle. Write $C^\infty(E)$ for the vector space of smooth sections e of E. Then $C^\infty(X)$ acts on $C^\infty(E)$ by multiplication, so $C^\infty(E)$ is a $C^\infty(X)$-module.

In [33, §5.3] we define the *cotangent module* $\Omega_\mathfrak{C}$ of a C^∞-ring $\mathfrak{C}$.

Definition 2.12. Let $\mathfrak{C}$ be a C^∞-ring, and M a $\mathfrak{C}$-module. A C^∞-*derivation* is an $\mathbb{R}$-linear map $\mathrm{d} : \mathfrak{C} \to M$ such that whenever $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth map and $c_1, \ldots, c_n \in \mathfrak{C}$, we have

$$\mathrm{d}\Phi_f(c_1, \ldots, c_n) = \sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot \mathrm{d}c_i.$$

We call such a pair M, d a *cotangent module* for $\mathfrak{C}$ if it has the universal property that for any $\mathfrak{C}$-module M' and C^∞-derivation $\mathrm{d}' : \mathfrak{C} \to M'$, there exists a unique morphism of $\mathfrak{C}$-modules $\phi : M \to M'$ with $\mathrm{d}' = \phi \circ \mathrm{d}$.

Define $\Omega_\mathfrak{C}$ to be the quotient of the free $\mathfrak{C}$-module with basis of symbols $\mathrm{d}c$ for $c \in \mathfrak{C}$ by the $\mathfrak{C}$-submodule spanned by all expressions of the form $\mathrm{d}\big(\Phi_f(c_1, \ldots, c_n)\big) - \sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot \mathrm{d}c_i$ for $f : \mathbb{R}^n \to \mathbb{R}$ smooth and $c_1, \ldots, c_n \in \mathfrak{C}$, and define $\mathrm{d}_\mathfrak{C} : \mathfrak{C} \to \Omega_\mathfrak{C}$ by $\mathrm{d}_\mathfrak{C} : c \mapsto \mathrm{d}c$. Then $\Omega_\mathfrak{C}, \mathrm{d}_\mathfrak{C}$ is a cotangent module for $\mathfrak{C}$. Thus cotangent modules always exist, and are unique up to unique isomorphism.

Let $\mathfrak{C}, \mathfrak{D}$ be C^∞-rings with cotangent modules $\Omega_\mathfrak{C}, \mathrm{d}_\mathfrak{C}, \Omega_\mathfrak{D}, \mathrm{d}_\mathfrak{D}$, and $\phi : \mathfrak{C} \to \mathfrak{D}$ be a morphism of C^∞-rings. Then ϕ makes $\Omega_\mathfrak{D}$ into a $\mathfrak{C}$-module, and there is a unique morphism $\Omega_\phi : \Omega_\mathfrak{C} \to \Omega_\mathfrak{D}$ in $\mathfrak{C}$-mod with $\mathrm{d}_\mathfrak{D} \circ \phi = \Omega_\phi \circ \mathrm{d}_\mathfrak{C}$. This induces a morphism $(\Omega_\phi)_* : \Omega_\mathfrak{C} \otimes_\mathfrak{C} \mathfrak{D} \to \Omega_\mathfrak{D}$ in $\mathfrak{D}$-mod with $(\Omega_\phi)_* \circ (\mathrm{d}_\mathfrak{C} \otimes \mathrm{id}_\mathfrak{D}) = \mathrm{d}_\mathfrak{D}$.

Example 2.13. Let X be a manifold. Then the cotangent bundle T^*X is a vector bundle over X, so as in Example 2.11 it yields a $C^\infty(X)$-module $C^\infty(T^*X)$. The exterior derivative $\mathrm{d} : C^\infty(X) \to C^\infty(T^*X)$ is a C^∞-derivation. These $C^\infty(T^*X), \mathrm{d}$ have the universal property in Definition 2.12, and so form a *cotangent module* for $C^\infty(X)$.

Now let X, Y be manifolds, and $f : X \to Y$ be smooth. Then $f^*(TY), TX$ are vector bundles over X, and the derivative of f is a vector bundle morphism $\mathrm{d}f : TX \to f^*(TY)$. The dual of this morphism is $\mathrm{d}f^* : f^*(T^*Y) \to T^*X$. This induces a morphism of $C^\infty(X)$-modules $(\mathrm{d}f^*)_* : C^\infty\big(f^*(T^*Y)\big) \to C^\infty(T^*X)$. This $(\mathrm{d}f^*)_*$ is identified with $(\Omega_{f^*})_*$ in Definition 2.12 under the natural isomorphism $C^\infty\big(f^*(T^*Y)\big) \cong C^\infty(T^*Y) \otimes_{C^\infty(Y)} C^\infty(X)$.

Definition 2.12 abstracts the notion of cotangent bundle of a manifold in a way that makes sense for any C^∞-ring.

2.4 Quasicoherent sheaves on C^∞-schemes

In [33, §6] we discuss sheaves of modules on C^∞-schemes.

Definition 2.14. Let $\underline{X} = (X, \mathcal{O}_X)$ be a C^∞-scheme. An $\mathcal{O}_X$-*module* $\mathcal{E}$ on $\underline{X}$ assigns a module $\mathcal{E}(U)$ over $\mathcal{O}_X(U)$ for each open set $U \subseteq X$, with $\mathcal{O}_X(U)$-action $\mu_U : \mathcal{O}_X(U) \times \mathcal{E}(U) \to \mathcal{E}(U)$, and a linear map $\mathcal{E}_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$ for each inclusion of open sets $V \subseteq U \subseteq X$, such that the following commutes:

$$
\begin{array}{ccc}
\mathcal{O}_X(U) \times \mathcal{E}(U) & \xrightarrow{\ \mu_U\ } & \mathcal{E}(U) \\
{\scriptstyle \rho_{UV} \times \mathcal{E}_{UV}}\big\downarrow & & \big\downarrow {\scriptstyle \mathcal{E}_{UV}} \\
\mathcal{O}_X(V) \times \mathcal{E}(V) & \xrightarrow{\ \mu_V\ } & \mathcal{E}(V),
\end{array}
$$

and all this data $\mathcal{E}(U), \mathcal{E}_{UV}$ satisfies the usual sheaf axioms [21, §II.1] .

A *morphism of $\mathcal{O}_X$-modules* $\phi : \mathcal{E} \to \mathcal{F}$ assigns a morphism of $\mathcal{O}_X(U)$-modules $\phi(U) : \mathcal{E}(U) \to \mathcal{F}(U)$ for each open set $U \subseteq X$, such that $\phi(V) \circ \mathcal{E}_{UV} = \mathcal{F}_{UV} \circ \phi(U)$ for each inclusion of open sets $V \subseteq U \subseteq X$. Then $\mathcal{O}_X$-modules form an abelian category, which we write as $\mathcal{O}_X$-mod.

As in [33, §6.2], the spectrum functor $\mathrm{Spec} : \mathbf{C^\infty Rings}^{\mathrm{op}} \to \mathbf{C^\infty Sch}$ has a counterpart for modules: if $\mathfrak{C}$ is a C^∞-ring and $(X, \mathcal{O}_X) = \mathrm{Spec}\,\mathfrak{C}$ we can define a functor $\mathrm{MSpec} : \mathfrak{C}\text{-mod} \to \mathcal{O}_X\text{-mod}$. If $\mathfrak{C}$ is a *fair* C^∞-ring, there is a full abelian subcategory $\mathfrak{C}\text{-mod}^{\mathrm{co}}$ of *complete* $\mathfrak{C}$-modules in $\mathfrak{C}$-mod, such that $\mathrm{MSpec}\,|_{\mathfrak{C}\text{-mod}^{\mathrm{co}}} : \mathfrak{C}\text{-mod}^{\mathrm{co}} \to \mathcal{O}_X\text{-mod}$ is an equivalence of categories, with quasi-inverse the global sections functor $\Gamma : \mathcal{O}_X\text{-mod} \to \mathfrak{C}\text{-mod}^{\mathrm{co}}$. Let $\underline{X} = (X, \mathcal{O}_X)$ be a C^∞-scheme, and $\mathcal{E}$ an $\mathcal{O}_X$-module. We call $\mathcal{E}$ *quasicoherent* if $\underline{X}$ can be covered by open $\underline{U}$ with $\underline{U} \cong \mathrm{Spec}\,\mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$, and under this identification $\mathcal{E}|_U \cong \mathrm{MSpec}\,M$ for some $\mathfrak{C}$-module M. We call $\mathcal{E}$ a *vector bundle of rank $n \geqslant 0$* if $\underline{X}$ may be covered by open $\underline{U}$ such that $\mathcal{E}|_{\underline{U}} \cong \mathcal{O}_U \otimes_{\mathbb{R}} \mathbb{R}^n$.

Write $\mathrm{qcoh}(\underline{X}), \mathrm{vect}(\underline{X})$ for the full subcategories of quasicoherent sheaves and vector bundles in $\mathcal{O}_X$-mod. Then $\mathrm{qcoh}(\underline{X})$ is an abelian category. Since $\mathrm{MSpec} : \mathfrak{C}\text{-mod}^{\mathrm{co}} \to \mathcal{O}_X\text{-mod}$ is an equivalence for $\mathfrak{C}$ fair and $(X, \mathcal{O}_X) = \mathrm{Spec}\,\mathfrak{C}$,

as in [33, Cor. 6.11] we see that if $\underline{X}$ is a locally fair C^∞-scheme then every $\mathcal{O}_X$-module $\mathcal{E}$ on $\underline{X}$ is quasicoherent, that is, $\mathrm{qcoh}(\underline{X}) = \mathcal{O}_X\text{-mod}$.

Remark 2.15. (a) If $\underline{X}$ is a separated, paracompact, locally fair C^∞-scheme then vector bundles on $\underline{X}$ are projective objects in the abelian category $\mathrm{qcoh}(\underline{X})$.

(b) In [33, §6.3] we also define a subcategory $\mathrm{coh}(\underline{X})$ of *coherent sheaves* in $\mathrm{qcoh}(\underline{X})$. But we will not use them in this paper, as they do not have all the good properties we want. In conventional algebraic geometry, one usually restricts to noetherian schemes, where coherent sheaves are well behaved, and form an abelian category. However, as in Remark 2.9(iv), even very basic C^∞-schemes $\underline{X}$ such as $\mathbb{R}^n$ for $n > 0$ are non-noetherian. Because of this, $\mathrm{coh}(\underline{X})$ is not closed under kernels in $\mathrm{qcoh}(\underline{X})$, and is not an abelian category.

Definition 2.16. Let $\underline{f} : \underline{X} \to \underline{Y}$ be a morphism of C^∞-schemes, and let $\mathcal{E}$ be an $\mathcal{O}_Y$-module. Define the *pullback* $\underline{f}^*(\mathcal{E})$, an $\mathcal{O}_X$-module, by $\underline{f}^*(\mathcal{E}) = f^{-1}(\mathcal{E}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$, where $f^{-1}(\mathcal{E}), f^{-1}(\mathcal{O}_Y)$ are inverse image sheaves, and the tensor product uses the morphism $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$. If $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism in $\mathcal{O}_Y$-mod we have an induced morphism $\underline{f}^*(\phi) = f^{-1}(\phi) \otimes \mathrm{id}_{\mathcal{O}_X} : \underline{f}^*(\mathcal{E}) \to \underline{f}^*(\mathcal{F})$ in $\mathcal{O}_X$-mod. Then $\underline{f}^* : \mathcal{O}_Y\text{-mod} \to \mathcal{O}_X\text{-mod}$ is a right exact functor, which restricts to a right exact functor $\underline{f}^* : \mathrm{qcoh}(\underline{Y}) \to \mathrm{qcoh}(\underline{X})$.

Remark 2.17. Pullbacks $\underline{f}^*(\mathcal{E})$ are characterized by a universal property, and so are *unique up to canonical isomorphism*, rather than unique. Our definition of $\underline{f}^*(\mathcal{E})$ is not functorial in $\underline{f}$. That is, if $\underline{f} : \underline{X} \to \underline{Y}$, $\underline{g} : \underline{Y} \to \underline{Z}$ are morphisms and $\mathcal{E} \in \mathcal{O}_Z$-mod then $(\underline{g} \circ \underline{f})^*(\mathcal{E})$ and $\underline{f}^*(\underline{g}^*(\mathcal{E}))$ are canonically isomorphic in $\mathcal{O}_X$-mod, but may not be equal. We will write $I_{\underline{f},\underline{g}}(\mathcal{E}) : (\underline{g} \circ \underline{f})^*(\mathcal{E}) \to \underline{f}^*(\underline{g}^*(\mathcal{E}))$ for these canonical isomorphisms. Then $I_{\underline{f},\underline{g}} : (\underline{g} \circ \underline{f})^* \Rightarrow \underline{f}^* \circ \underline{g}^*$ is a natural isomorphism of functors.

Similarly, when $\underline{f}$ is the identity $\underline{\mathrm{id}}_X : \underline{X} \to \underline{X}$ and $\mathcal{E} \in \mathcal{O}_X$-mod we may not have $\underline{\mathrm{id}}_X^*(\mathcal{E}) = \mathcal{E}$, but there is a canonical isomorphism $\delta_X(\mathcal{E}) : \underline{\mathrm{id}}_X^*(\mathcal{E}) \to \mathcal{E}$, and $\delta_X : \underline{\mathrm{id}}_X^* \Rightarrow \mathrm{id}_{\mathcal{O}_X\text{-mod}}$ is a natural isomorphism of functors.

In fact it is a common abuse of notation in algebraic geometry to omit these isomorphisms $I_{\underline{f},\underline{g}}(\mathcal{E}), \underline{\mathrm{id}}_X^*(\mathcal{E})$, and just assume that $(\underline{g} \circ \underline{f})^*(\mathcal{E}) = \underline{f}^*(\underline{g}^*(\mathcal{E}))$ and $\underline{\mathrm{id}}_X^*(\mathcal{E}) = \mathcal{E}$. An author who treats them rigorously is Vistoli [59], see in particular [59, Introduction & §3.2.1]. One reason we decided to include them is to be sure that $\mathbf{dSpa}, \mathbf{dMan}, \ldots$ defined below are strict 2-categories, rather than weak 2-categories or some other structure.

Example 2.18. Let X be a manifold, and $\underline{X}$ the associated C^∞-scheme from Example 2.5, so that $\mathcal{O}_X(U) = C^\infty(U)$ for all open $U \subseteq X$. Let $E \to X$ be a vector bundle. Define an $\mathcal{O}_X$-module $\mathcal{E}$ on $\underline{X}$ by $\mathcal{E}(U) = C^\infty(E|_U)$, the smooth sections of the vector bundle $E|_U \to U$, and for open $V \subseteq U \subseteq X$ define $\mathcal{E}_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$ by $\mathcal{E}_{UV} : e_U \mapsto e_U|_V$. Then $\mathcal{E} \in \mathrm{vect}(\underline{X})$ is a vector bundle on $\underline{X}$, which we think of as a lift of E from manifolds to C^∞-schemes.

Let $f : X \to Y$ be a smooth map of manifolds, and $\underline{f} : \underline{X} \to \underline{Y}$ the corresponding morphism of C^∞-schemes. Let $F \to Y$ be a vector bundle over Y, so that $f^*(F) \to X$ is a vector bundle over X. Let $\mathcal{F} \in \mathrm{vect}(\underline{Y})$ be the

vector bundle over $\underline{Y}$ lifting F. Then $\underline{f}^*(\mathcal{F})$ is canonically isomorphic to the vector bundle over $\underline{X}$ lifting $f^*(F)$.

We define *cotangent sheaves*, the sheaf version of cotangent modules in §2.3.

Definition 2.19. Let $\underline{X}$ be a C^∞-scheme. Define $\mathcal{P}T^*\underline{X}$ to associate to each open $U \subseteq X$ the cotangent module $\Omega_{\mathcal{O}_X(U)}$, and to each inclusion of open sets $V \subseteq U \subseteq X$ the morphism of $\mathcal{O}_X(U)$-modules $\Omega_{\rho_{UV}} : \Omega_{\mathcal{O}_X(U)} \to \Omega_{\mathcal{O}_X(V)}$ associated to the morphism of C^∞-rings $\rho_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$. Then $\mathcal{P}T^*\underline{X}$ is a *presheaf of $\mathcal{O}_X$-modules on $\underline{X}$*. Define the *cotangent sheaf $T^*\underline{X}$ of $\underline{X}$* to be the sheafification of $\mathcal{P}T^*\underline{X}$, as an $\mathcal{O}_X$-module.

Let $\underline{f} : \underline{X} \to \underline{Y}$ be a morphism of C^∞-schemes. Then by Definition 2.16, $\underline{f}^*\bigl(T^*\underline{Y}\bigr) = f^{-1}(T^*\underline{Y}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$, where $T^*\underline{Y}$ is the sheafification of the presheaf $V \mapsto \Omega_{\mathcal{O}_Y(V)}$, and $f^{-1}(T^*\underline{Y})$ the sheafification of the presheaf $U \mapsto \lim_{V \supseteq f(U)}(T^*\underline{Y})(V)$, and $f^{-1}(\mathcal{O}_Y)$ the sheafification of the presheaf $U \mapsto \lim_{V \supseteq f(U)} \mathcal{O}_Y(V)$. The three sheafifications combine into one, so that $\underline{f}^*\bigl(T^*\underline{Y}\bigr)$ is the sheafification of the presheaf $\mathcal{P}(\underline{f}^*(T^*\underline{Y}))$ acting by

$$U \longmapsto \mathcal{P}(\underline{f}^*(T^*\underline{Y}))(U) = \lim_{V \supseteq f(U)} \Omega_{\mathcal{O}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U).$$

Define a morphism of presheaves $\mathcal{P}\Omega_{\underline{f}} : \mathcal{P}(\underline{f}^*(T^*\underline{Y})) \to \mathcal{P}T^*\underline{X}$ on X by

$$(\mathcal{P}\Omega_{\underline{f}})(U) = \lim_{V \supseteq f(U)}(\Omega_{\rho_{f^{-1}(V)\,U} \circ f_\sharp(V)})_*,$$

where $(\Omega_{\rho_{f^{-1}(V)\,U} \circ f_\sharp(V)})_* : \Omega_{\mathcal{O}_Y(V)} \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U) \to \Omega_{\mathcal{O}_X(U)} = (\mathcal{P}T^*\underline{X})(U)$ is constructed as in Definition 2.12 from the C^∞-ring morphisms $f_\sharp(V) : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ from $f_\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ corresponding to $f^\sharp$ in $\underline{f}$ as in (2.3), and $\rho_{f^{-1}(V)\,U} : \mathcal{O}_X(f^{-1}(V)) \to \mathcal{O}_X(U)$ in $\mathcal{O}_X$. Define $\Omega_{\underline{f}} : \underline{f}^*\bigl(T^*\underline{Y}\bigr) \to T^*\underline{X}$ to be the induced morphism of the associated sheaves.

Example 2.20. Let X be a manifold, and $\underline{X}$ the associated C^∞-scheme. Then $T^*\underline{X}$ is a vector bundle on $\underline{X}$, and is canonically isomorphic to the lift to C^∞-schemes from Example 2.18 of the cotangent vector bundle T^*X of X.

Here [33, Th. 6.17] are some properties of cotangent sheaves.

Theorem 2.21. (a) *Let $\underline{f} : \underline{X} \to \underline{Y}$ and $\underline{g} : \underline{Y} \to \underline{Z}$ be morphisms of C^∞-schemes. Then*

$$\Omega_{\underline{g} \circ \underline{f}} = \Omega_{\underline{f}} \circ \underline{f}^*(\Omega_{\underline{g}}) \circ I_{\underline{f},\underline{g}}(T^*\underline{Z})$$

as morphisms $(\underline{g} \circ \underline{f})^*(T^*\underline{Z}) \to T^*\underline{X}$. Here $\Omega_{\underline{g}} : \underline{g}^*(T^*\underline{Z}) \to T^*\underline{Y}$ is a morphism in $\mathcal{O}_Y$-mod, so applying $\underline{f}^*$ gives $\underline{f}^*(\Omega_{\underline{g}}) : \underline{f}^*(\underline{g}^*(T^*\underline{Z})) \to \underline{f}^*(T^*\underline{Y})$ in $\mathcal{O}_X$-mod, and $I_{\underline{f},\underline{g}}(T^*\underline{Z}) : (\underline{g} \circ \underline{f})^*(T^*\underline{Z}) \to \underline{f}^*(\underline{g}^*(T^*\underline{Z}))$ is as in Remark 2.17.

(b) *Suppose $\underline{W}, \underline{X}, \underline{Y}, \underline{Z}$ are locally fair C^∞-schemes with a Cartesian square*

$$\begin{array}{ccc} \underline{W} & \xrightarrow{\;\;f\;\;} & \underline{Y} \\ {\scriptstyle e}\downarrow & {\scriptstyle g} & \downarrow{\scriptstyle h} \\ \underline{X} & \xrightarrow{\;\;g\;\;} & \underline{Z} \end{array}$$

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in $\mathbf{C}^\infty\mathbf{Sch}^{\mathrm{lf}}$, *so that* $\underline{W} = \underline{X} \times_{\underline{Z}} \underline{Y}$. *Then the following is exact in* $\mathrm{qcoh}(\underline{W})$:

$$(\underline{g} \circ \underline{e})^*(T^*\underline{Z}) \xrightarrow{\substack{\underline{e}^*(\Omega_g) \circ I_{\underline{e},g}(T^*\underline{Z}) \oplus \\ -\underline{f}^*(\Omega_h) \circ I_{\underline{f},h}(T^*\underline{Z})}} \underline{e}^*(T^*\underline{X}) \oplus \underline{f}^*(T^*\underline{Y}) \xrightarrow{\Omega_{\underline{e}} \oplus \Omega_{\underline{f}}} T^*\underline{W} \longrightarrow 0.$$

3 The 2-category of d-spaces

We will now define the 2-category of *d-spaces* **dSpa**, following [35, Chap. 2]. D-spaces are 'derived' versions of C^∞-schemes. In §4 we will define the 2-category of d-manifolds **dMan** as a 2-subcategory of **dSpa**. For an introduction to 2-categories, see §A.3–§A.4.

3.1 The definition of d-spaces

Definition 3.1. A *d-space* $\boldsymbol{X}$ is a quintuple $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ such that $\underline{X} = (X, \mathcal{O}_X)$ is a separated, second countable, locally fair C^∞-scheme, and $\mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X$ fit into an exact sequence of sheaves on X

$$\mathcal{E}_X \xrightarrow{\quad \jmath_X \quad} \mathcal{O}'_X \xrightarrow{\quad \imath_X \quad} \mathcal{O}_X \longrightarrow 0,$$

satisfying the conditions:

(a) $\mathcal{O}'_X$ is a sheaf of C^∞-rings on X, with $\underline{X}' = (X, \mathcal{O}'_X)$ a C^∞-scheme.

(b) $\imath_X : \mathcal{O}'_X \to \mathcal{O}_X$ is a surjective morphism of sheaves of C^∞-rings on X. Its kernel $\kappa_X : \mathcal{I}_X \to \mathcal{O}'_X$ is a sheaf of ideals $\mathcal{I}_X$ in $\mathcal{O}'_X$, which should be a sheaf of square zero ideals. Here a *square zero ideal* in a commutative $\mathbb{R}$-algebra A is an ideal I with $i \cdot j = 0$ for all $i, j \in I$. Then $\mathcal{I}_X$ is an $\mathcal{O}'_X$-module, but as $\mathcal{I}_X$ consists of square zero ideals and $\imath_X$ is surjective, the $\mathcal{O}'_X$-action factors through an $\mathcal{O}_X$-action. Hence $\mathcal{I}_X$ is an $\mathcal{O}_X$-module, and thus a quasicoherent sheaf on $\underline{X}$, as $\underline{X}$ is locally fair.

(c) $\mathcal{E}_X$ is a quasicoherent sheaf on $\underline{X}$, and $\jmath_X : \mathcal{E}_X \to \mathcal{I}_X$ is a surjective morphism in $\mathrm{qcoh}(\underline{X})$.

As $\underline{X}$ is locally fair, the underlying topological space X is locally homeomorphic to a closed subset of $\mathbb{R}^n$, so it is *locally compact*. But Hausdorff, second countable and locally compact imply paracompact, and thus $\underline{X}$ is *paracompact*.

The sheaf of C^∞-rings $\mathcal{O}'_X$ has a sheaf of cotangent modules $\Omega_{\mathcal{O}'_X}$, which is an $\mathcal{O}'_X$-module with exterior derivative $\mathrm{d} : \mathcal{O}'_X \to \Omega_{\mathcal{O}'_X}$. Define $\mathcal{F}_X = \Omega_{\mathcal{O}'_X} \otimes_{\mathcal{O}'_X} \mathcal{O}_X$ to be the associated $\mathcal{O}_X$-module, a quasicoherent sheaf on $\underline{X}$, and set $\psi_X = \Omega_{\imath_X} \otimes \mathrm{id} : \mathcal{F}_X \to T^*\underline{X}$, a morphism in $\mathrm{qcoh}(\underline{X})$. Define $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ to be the composition of morphisms of sheaves of abelian groups on X:

$$\mathcal{E}_X \xrightarrow{\jmath_X} \mathcal{I}_X \xrightarrow{\mathrm{d}|_{\mathcal{I}_X}} \Omega_{\mathcal{O}'_X} \cong \Omega_{\mathcal{O}'_X} \otimes_{\mathcal{O}'_X} \mathcal{O}'_X \xrightarrow{\mathrm{id} \otimes \imath_X} \Omega_{\mathcal{O}'_X} \otimes_{\mathcal{O}'_X} \mathcal{O}_X = \mathcal{F}_X.$$

It turns out that ϕ_X is actually a morphism of $\mathcal{O}_X$-modules, and the following sequence is exact in $\mathrm{qcoh}(\underline{X})$:

$$\mathcal{E}_X \xrightarrow{\phi_X} \mathcal{F}_X \xrightarrow{\psi_X} T^*\underline{X} \longrightarrow 0.$$

The morphism $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ will be called the *virtual cotangent sheaf* of $\boldsymbol{X}$, for reasons we explain in §4.3.

Let $\boldsymbol{X}, \boldsymbol{Y}$ be d-spaces. A 1-*morphism* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a triple $\boldsymbol{f} = (\underline{f}, f', f'')$, where $\underline{f} = (f, f^\sharp) : \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes, $f' : f^{-1}(\mathcal{O}'_Y) \to \mathcal{O}'_X$ a morphism of sheaves of C^∞-rings on X, and $f'' : \underline{f}^*(\mathcal{E}_Y) \to \mathcal{E}_X$ a morphism in $\mathrm{qcoh}(\underline{X})$, such that the following diagram of sheaves on X commutes:

$$
\begin{array}{c}
f^{-1}(\mathcal{E}_Y) \otimes^{\mathrm{id}}_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{O}_Y) = f^{-1}(\mathcal{E}_Y) \longrightarrow f^{-1}(\mathcal{O}'_Y) \longrightarrow f^{-1}(\mathcal{O}_Y) \to 0 \\
\end{array}
$$

with vertical maps $\mathrm{id}\otimes f^\sharp$, $f^{-1}(\jmath_Y)$, $f^{-1}(\imath_Y)$, f', $f^\sharp$, $\underline{f}^*(\mathcal{E}_Y) = f^{-1}(\mathcal{E}_Y) \otimes^{f^\sharp}_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$, f'', and bottom row $\mathcal{E}_X \xrightarrow{\jmath_X} \mathcal{O}'_X \xrightarrow{\imath_X} \mathcal{O}_X \longrightarrow 0$.

Define morphisms $f^2 = \Omega_{f'} \otimes \mathrm{id} : \underline{f}^*(\mathcal{F}_Y) \to \mathcal{F}_X$ and $f^3 = \Omega_{\underline{f}} : \underline{f}^*(T^*\underline{Y}) \to T^*\underline{X}$ in $\mathrm{qcoh}(\underline{X})$. Then the following commutes in $\mathrm{qcoh}(\underline{X})$, with exact rows:

$$
\begin{array}{ccccccc}
\underline{f}^*(\mathcal{E}_Y) & \xrightarrow{\underline{f}^*(\phi_Y)} & \underline{f}^*(\mathcal{F}_Y) & \xrightarrow{\underline{f}^*(\psi_Y)} & \underline{f}^*(T^*\underline{Y}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f^2} & & \downarrow{\scriptstyle f^3} & & \\
\mathcal{E}_X & \xrightarrow{\phi_X} & \mathcal{F}_X & \xrightarrow{\psi_X} & T^*\underline{X} & \longrightarrow & 0.
\end{array}
\tag{3.1}
$$

If $\boldsymbol{X}$ is a d-space, the *identity* 1-*morphism* $\mathbf{id}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$ is $\mathbf{id}_{\boldsymbol{X}} = (\underline{\mathrm{id}}_X, \delta_X(\mathcal{O}'_X), \delta_{\underline{X}}(\mathcal{E}_X))$, where $\delta_X(*)$ are the canonical isomorphisms of Remark 2.17. Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ be d-spaces, and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms. Define the *composition of* 1-*morphisms* $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$ to be

$$\boldsymbol{g} \circ \boldsymbol{f} = \big(\underline{g} \circ \underline{f}, f' \circ f^{-1}(g') \circ I_{f,g}(\mathcal{O}'_Z), f'' \circ \underline{f}^*(g'') \circ I_{\underline{f},g}(\mathcal{E}_Z)\big), \tag{3.2}$$

where $I_{*,*}(*)$ are the canonical isomorphisms of Remark 2.17.

Let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be 1-morphisms of d-spaces, where $\boldsymbol{f} = (\underline{f}, f', f'')$ and $\boldsymbol{g} = (\underline{g}, g', g'')$. Suppose $\underline{f} = \underline{g}$. A 2-*morphism* $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a morphism $\eta : \underline{f}^*(\mathcal{F}_Y) \to \mathcal{E}_X$ in $\mathrm{qcoh}(\underline{X})$, such that

$$g' = f' + \jmath_X \circ \eta \circ \big(\mathrm{id} \otimes (f^\sharp \circ f^{-1}(\imath_Y))\big) \circ \big(f^{-1}(\mathrm{d})\big)$$
$$\text{and} \qquad g'' = f'' + \eta \circ \underline{f}^*(\phi_Y).$$

Then $g^2 = f^2 + \phi_X \circ \eta$ and $g^3 = f^3$, so (3.1) for $\boldsymbol{f}, \boldsymbol{g}$ combine to give a diagram

$$
\begin{array}{ccccccc}
\underline{f}^*(\mathcal{E}_Y) & \xrightarrow{\underline{f}^*(\phi_Y)} & \underline{f}^*(\mathcal{F}_Y) & \xrightarrow{\underline{f}^*(\psi_Y)} & \underline{f}^*(T^*\underline{Y}) & \longrightarrow & 0 \\
{\scriptstyle f''}\big\|{\scriptstyle g''=f''+\eta\circ\underline{f}^*(\phi_Y)} & {\scriptstyle \eta} & {\scriptstyle f^2}\big\|{\scriptstyle g^2=f^2+\phi_X\circ\eta} & & \big\downarrow{\scriptstyle f^3=g^3} & & \\
\mathcal{E}_X & \xleftarrow{\phi_X} & \mathcal{F}_X & \xrightarrow{\psi_X} & T^*\underline{X} & \longrightarrow & 0.
\end{array}
\tag{3.3}
$$

That is, η is a homotopy between the morphisms of complexes (3.1) from $\boldsymbol{f}, \boldsymbol{g}$.

If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism, the *identity 2-morphism* $\mathrm{id}_{\boldsymbol{f}} : \boldsymbol{f} \Rightarrow \boldsymbol{f}$ is the zero morphism $0 : \underline{f}^*(\mathcal{F}_Y) \to \mathcal{E}_X$. Suppose $\boldsymbol{X}, \boldsymbol{Y}$ are d-spaces, $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Y}$ are 1-morphisms and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$, $\zeta : \boldsymbol{g} \Rightarrow \boldsymbol{h}$ are 2-morphisms. The *vertical composition of 2-morphisms* $\zeta \odot \eta : \boldsymbol{f} \Rightarrow \boldsymbol{h}$ as in (A.1) is $\zeta \odot \eta = \zeta + \eta$.

Let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ be d-spaces, $\boldsymbol{f}, \tilde{\boldsymbol{f}} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}, \tilde{\boldsymbol{g}} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms, and $\eta : \boldsymbol{f} \Rightarrow \tilde{\boldsymbol{f}}$, $\zeta : \boldsymbol{g} \Rightarrow \tilde{\boldsymbol{g}}$ be 2-morphisms. The *horizontal composition of 2-morphisms* $\zeta * \eta : \boldsymbol{g} \circ \boldsymbol{f} \Rightarrow \tilde{\boldsymbol{g}} \circ \tilde{\boldsymbol{f}}$ as in (A.2) is

$$\zeta * \eta = \left(\eta \circ \underline{f}^*(g^2) + f'' \circ \underline{f}^*(\zeta) + \eta \circ \underline{f}^*(\phi_Y) \circ \underline{f}^*(\zeta) \right) \circ I_{\underline{f}, \underline{g}}(\mathcal{F}_Z).$$

This completes the definition of the 2-category of d-spaces $\mathbf{dSpa}$.

Regard the category $\mathbf{C^\infty Sch^{lf}_{ssc}}$ of separated, second countable, locally fair C^∞-schemes as a 2-category with only identity 2-morphisms $\mathrm{id}_{\underline{f}}$ for (1-)morphisms $\underline{f} : \underline{X} \to \underline{Y}$. Define a 2-functor $F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}} : \mathbf{C^\infty Sch^{lf}_{ssc}} \to \mathbf{dSpa}$ to map $\underline{X}$ to $\boldsymbol{X} = (\underline{X}, \mathcal{O}_X, 0, \mathrm{id}_{\mathcal{O}_X}, 0)$ on objects $\underline{X}$, to map $\underline{f}$ to $\boldsymbol{f} = (\underline{f}, f^\sharp, 0)$ on (1-)morphisms $\underline{f} : \underline{X} \to \underline{Y}$, and to map identity 2-morphisms $\mathrm{id}_{\underline{f}} : \underline{f} \Rightarrow \underline{f}$ to identity 2-morphisms $\mathrm{id}_{\boldsymbol{f}} : \boldsymbol{f} \Rightarrow \boldsymbol{f}$. Define a 2-functor $F^{\mathbf{dSpa}}_{\mathbf{Man}} : \mathbf{Man} \to \mathbf{dSpa}$ by $F^{\mathbf{dSpa}}_{\mathbf{Man}} = F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}} \circ F^{\mathbf{C^\infty Sch}}_{\mathbf{Man}}$.

Write $\hat{\mathbf{C}}^\infty \mathbf{Sch^{lf}_{ssc}}$ for the full 2-subcategory of objects $\boldsymbol{X}$ in $\mathbf{dSpa}$ equivalent to $F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}}(\underline{X})$ for some $\underline{X}$ in $\mathbf{C^\infty Sch^{lf}_{ssc}}$, and $\hat{\mathbf{Man}}$ for the full 2-subcategory of objects $\boldsymbol{X}$ in $\mathbf{dSpa}$ equivalent to $F^{\mathbf{dSpa}}_{\mathbf{Man}}(X)$ for some manifold X. When we say that a d-space $\boldsymbol{X}$ *is a C^∞-scheme*, or *is a manifold*, we mean that $\boldsymbol{X} \in \hat{\mathbf{C}}^\infty \mathbf{Sch^{lf}_{ssc}}$, or $\boldsymbol{X} \in \hat{\mathbf{Man}}$, respectively.

In [35, §2.2] we prove:

Theorem 3.2. (a) *Definition* 3.1 *defines a strict* 2-*category* $\mathbf{dSpa}$, *in which all* 2-*morphisms are* 2-*isomorphisms.*

(b) *For any* 1-*morphism* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *in* $\mathbf{dSpa}$ *the* 2-*morphisms* $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{f}$ *form an abelian group under vertical composition, and in fact a real vector space.*

(c) $F^{\mathbf{dSpa}}_{\mathbf{C^\infty Sch}}$ *and* $F^{\mathbf{dSpa}}_{\mathbf{Man}}$ *in Definition* 3.1 *are full and faithful strict* 2-*functors. Hence* $\mathbf{C^\infty Sch^{lf}_{ssc}}, \mathbf{Man}$ *and* $\hat{\mathbf{C}}^\infty \mathbf{Sch^{lf}_{ssc}}, \hat{\mathbf{Man}}$ *are equivalent* 2-*categories.*

Remark 3.3. (i) One should think of a d-space $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ as being a C^∞-scheme $\underline{X}$, which is the 'classical' part of $\boldsymbol{X}$ and lives in a category rather than a 2-category, together with some extra 'derived' information $\mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X$. 2-morphisms in $\mathbf{dSpa}$ are wholly to do with this derived part. The sheaf $\mathcal{E}_X$ may be thought of as a (dual) 'obstruction sheaf' on $\underline{X}$.

(ii) Readers familiar with derived algebraic geometry may find the following (oversimplified) explanation of d-spaces helpful; more details are given in [35, §14.4]. In conventional algebraic geometry, a $\mathbb{K}$-scheme $(X, \mathcal{O}_X)$ is a topological space X equipped with a sheaf of $\mathbb{K}$-algebras $\mathcal{O}_X$. In derived algebraic geometry, as in Toën and Vezzosi [57,58] and Lurie [40–42], a *derived $\mathbb{K}$-scheme* $(X, \mathcal{O}_X)$ is (roughly) a topological space X with a (homotopy) sheaf of (commutative)

dg-algebras over $\mathbb{K}$. Here a (*commutative*) *dg-algebra* (A_*, d) is a nonpositively graded $\mathbb{K}$-algebra $\bigoplus_{k \leqslant 0} A_k$, with differentials $\mathrm{d} : A_k \to A_{k+1}$ satisfying $\mathrm{d}^2 = 0$ and $ab = (-1)^{kl} ba$, $\mathrm{d}(ab) = (\mathrm{d}a)b + (-1)^k a(\mathrm{d}b)$ for all $a \in A_k$ and $b \in A_l$.

We call a dg-algebra (A_*, d) *square zero* if $A_k = 0$ for $k \neq 0, -1$ and $A_{-1} \cdot \mathrm{d}(A_{-1}) = 0$. This implies that $\mathrm{d}(A_{-1})$ is a square zero ideal in A_0. General dg-algebras form an ∞-category, but square zero dg-algebras form a 2-category. Ignoring C^∞-rings for the moment, we can think of the data $\mathcal{E}_X \xrightarrow{\jmath_X} \mathcal{O}'_X$ in a d-space $\boldsymbol{X}$ as a sheaf of square zero dg-algebras $A_{-1} \xrightarrow{\mathrm{d}} A_0$ on X. The remaining data $\mathcal{O}_X, \imath_X$ can be recovered from this, since $\mathcal{O}'_X \xrightarrow{\imath_X} \mathcal{O}_X$ is the cokernel of $\mathcal{E}_X \xrightarrow{\jmath_X} \mathcal{O}'_X$. Thus, a d-space $\boldsymbol{X}$ is like a special kind of derived $\mathbb{R}$-scheme, in which the dg-algebras are all square zero.

3.2 Gluing d-spaces by equivalences

Next we discuss gluing of d-spaces and 1-morphisms on open d-subspaces.

Definition 3.4. Let $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ be a d-space. Suppose $\underline{U} \subseteq \underline{X}$ is an open C^∞-subscheme. Then $\boldsymbol{U} = \big(\underline{U}, \mathcal{O}'_X|_U, \mathcal{E}_X|_U, \imath_X|_U, \jmath_X|_U\big)$ is a d-space. We call $\boldsymbol{U}$ an *open d-subspace* of $\boldsymbol{X}$. An *open cover* of a d-space $\boldsymbol{X}$ is a family $\{\boldsymbol{U}_a : a \in A\}$ of open d-subspaces $\boldsymbol{U}_a$ of $\boldsymbol{X}$ with $\underline{X} = \bigcup_{a \in A} \underline{U}_a$.

As in [35, §2.4], we can glue 1-morphisms on open d-subspaces which are 2-isomorphic on the overlap. The proof uses partitions of unity, as in §2.2.

Proposition 3.5. *Let $\boldsymbol{X}, \boldsymbol{Y}$ be d-spaces, $\boldsymbol{U}, \boldsymbol{V} \subseteq \boldsymbol{X}$ be open d-subspaces with $\boldsymbol{X} = \boldsymbol{U} \cup \boldsymbol{V}$, $\boldsymbol{f} : \boldsymbol{U} \to \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{V} \to \boldsymbol{Y}$ be 1-morphisms, and $\eta : \boldsymbol{f}|_{\boldsymbol{U} \cap \boldsymbol{V}} \Rightarrow \boldsymbol{g}|_{\boldsymbol{U} \cap \boldsymbol{V}}$ a 2-morphism. Then there exist a 1-morphism $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Y}$ and 2-morphisms $\zeta : \boldsymbol{h}|_{\boldsymbol{U}} \Rightarrow \boldsymbol{f}$, $\theta : \boldsymbol{h}|_{\boldsymbol{V}} \Rightarrow \boldsymbol{g}$ such that $\theta|_{\boldsymbol{U} \cap \boldsymbol{V}} = \eta \odot \zeta|_{\boldsymbol{U} \cap \boldsymbol{V}} : \boldsymbol{h}|_{\boldsymbol{U} \cap \boldsymbol{V}} \Rightarrow \boldsymbol{g}|_{\boldsymbol{U} \cap \boldsymbol{V}}$. This $\boldsymbol{h}$ is unique up to 2-isomorphism, and independent up to 2-isomorphism of the choice of η.*

Equivalences $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in a 2-category are defined in §A.3, and are the natural notion of when two objects $\boldsymbol{X}, \boldsymbol{Y}$ are 'the same'. In [35, §2.4] we prove theorems on gluing d-spaces by equivalences. See Spivak [53, Lem. 6.8 & Prop. 6.9] for results similar to Theorem 3.6 for his 'local C^∞-ringed spaces', an ∞-categorical analogue of our d-spaces.

Theorem 3.6. *Suppose $\boldsymbol{X}, \boldsymbol{Y}$ are d-spaces, $\boldsymbol{U} \subseteq \boldsymbol{X}$, $\boldsymbol{V} \subseteq \boldsymbol{Y}$ are open d-subspaces, and $\boldsymbol{f} : \boldsymbol{U} \to \boldsymbol{V}$ is an equivalence in $\mathbf{dSpa}$. At the level of topological spaces, we have open $U \subseteq X$, $V \subseteq Y$ with a homeomorphism $f : U \to V$, so we can form the quotient topological space $Z := X \amalg_f Y = (X \amalg Y)/\sim$, where the equivalence relation $\sim$ on $X \amalg Y$ identifies $u \in U \subseteq X$ with $f(u) \in V \subseteq Y$.*

Suppose Z is Hausdorff. Then there exist a d-space $\boldsymbol{Z}$ with topological space Z, open d-subspaces $\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}}$ in $\boldsymbol{Z}$ with $\boldsymbol{Z} = \hat{\boldsymbol{X}} \cup \hat{\boldsymbol{Y}}$, equivalences $\boldsymbol{g} : \boldsymbol{X} \to \hat{\boldsymbol{X}}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \hat{\boldsymbol{Y}}$ in $\mathbf{dSpa}$ such that $\boldsymbol{g}|_{\boldsymbol{U}}$ and $\boldsymbol{h}|_{\boldsymbol{V}}$ are both equivalences with $\hat{\boldsymbol{X}} \cap \hat{\boldsymbol{Y}}$, and a 2-morphism $\eta : \boldsymbol{g}|_{\boldsymbol{U}} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f} : \boldsymbol{U} \to \hat{\boldsymbol{X}} \cap \hat{\boldsymbol{Y}}$. Furthermore, $\boldsymbol{Z}$ is independent of choices up to equivalence.

In Theorem 3.6, $\boldsymbol{Z}$ is a *pushout* $\boldsymbol{X} \amalg_{\mathrm{id}_U, \boldsymbol{U}, \boldsymbol{f}} \boldsymbol{Y}$ in the 2-category **dSpa**.

Theorem 3.7. *Suppose I is an indexing set, and $<$ is a total order on I, and $\boldsymbol{X}_i$ for $i \in I$ are d-spaces, and for all $i < j$ in I we are given open d-subspaces $\boldsymbol{U}_{ij} \subseteq \boldsymbol{X}_i$, $\boldsymbol{U}_{ji} \subseteq \boldsymbol{X}_j$ and an equivalence $\boldsymbol{e}_{ij} : \boldsymbol{U}_{ij} \to \boldsymbol{U}_{ji}$, such that for all $i < j < k$ in I we have a 2-commutative diagram*

$$
\begin{array}{ccc}
& \boldsymbol{U}_{ji} \cap \boldsymbol{U}_{jk} & \\
\boldsymbol{e}_{ij}|_{\boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik}} \nearrow & \Downarrow \eta_{ijk} & \searrow \boldsymbol{e}_{jk}|_{\boldsymbol{U}_{ji} \cap \boldsymbol{U}_{jk}} \\
\boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik} & \xrightarrow{\boldsymbol{e}_{ik}|_{\boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik}}} & \boldsymbol{U}_{ki} \cap \boldsymbol{U}_{kj}
\end{array}
$$

for some η_{ijk}, where all three 1-morphisms are equivalences.

On the level of topological spaces, define the quotient topological space $Y = (\coprod_{i \in I} X_i)/\sim$, where $\sim$ is the equivalence relation generated by $x_i \sim x_j$ if $i < j$, $x_i \in U_{ij} \subseteq X_i$ and $x_j \in U_{ji} \subseteq X_j$ with $e_{ij}(x_i) = x_j$. Suppose Y is Hausdorff and second countable. Then there exist a d-space $\boldsymbol{Y}$ and a 1-morphism $\boldsymbol{f}_i : \boldsymbol{X}_i \to \boldsymbol{Y}$ which is an equivalence with an open d-subspace $\hat{\boldsymbol{X}}_i \subseteq \boldsymbol{Y}$ for all $i \in I$, where $\boldsymbol{Y} = \bigcup_{i \in I} \hat{\boldsymbol{X}}_i$, such that $\boldsymbol{f}_i|_{\boldsymbol{U}_{ij}}$ is an equivalence $\boldsymbol{U}_{ij} \to \hat{\boldsymbol{X}}_i \cap \hat{\boldsymbol{X}}_j$ for all $i < j$ in I, and there exists a 2-morphism $\eta_{ij} : \boldsymbol{f}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{f}_i|_{\boldsymbol{U}_{ij}}$. The d-space $\boldsymbol{Y}$ is unique up to equivalence, and is independent of choice of 2-morphisms η_{ijk}.

Suppose also that $\boldsymbol{Z}$ is a d-space, and $\boldsymbol{g}_i : \boldsymbol{X}_i \to \boldsymbol{Z}$ are 1-morphisms for all $i \in I$, and there exist 2-morphisms $\zeta_{ij} : \boldsymbol{g}_j \circ \boldsymbol{e}_{ij} \Rightarrow \boldsymbol{g}_i|_{\boldsymbol{U}_{ij}}$ for all $i < j$ in I. Then there exist a 1-morphism $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ and 2-morphisms $\zeta_i : \boldsymbol{h} \circ \boldsymbol{f}_i \Rightarrow \boldsymbol{g}_i$ for all $i \in I$. The 1-morphism $\boldsymbol{h}$ is unique up to 2-isomorphism, and is independent of the choice of 2-morphisms ζ_{ij}.

Remark 3.8. In Proposition 3.5, it is surprising that $\boldsymbol{h}$ is independent of η up to 2-isomorphism. It holds because of the existence of partitions of unity on nice C^∞-schemes, as in Proposition 2.8. Here is a sketch proof: suppose $\eta, \boldsymbol{h}, \zeta, \theta$ and $\eta', \boldsymbol{h}', \zeta', \theta'$ are alternative choices in Proposition 3.5. Then we have 2-morphisms $(\zeta')^{-1} \odot \zeta : \boldsymbol{h}|_{\boldsymbol{U}} \Rightarrow \boldsymbol{h}'|_{\boldsymbol{U}}$ and $(\theta')^{-1} \odot \theta : \boldsymbol{h}|_{\boldsymbol{V}} \Rightarrow \boldsymbol{h}'|_{\boldsymbol{V}}$. Choose a partition of unity $\{\alpha, 1 - \alpha\}$ on $\underline{X}$ subordinate to $\{\underline{U}, \underline{V}\}$, so that $\alpha : \underline{X} \to \mathbb{R}$ is smooth with α supported on $\underline{U} \subseteq \underline{X}$ and $1 - \alpha$ supported on $\underline{V} \subseteq \underline{X}$. Then $\alpha \cdot \big((\zeta')^{-1} \odot \zeta\big) + (1 - \alpha) \cdot \big((\theta')^{-1} \odot \theta\big)$ is a 2-morphism $\boldsymbol{h} \Rightarrow \boldsymbol{h}'$, where $\alpha \cdot \big((\zeta')^{-1} \odot \zeta\big)$ makes sense on all of $\underline{X}$ (rather than just on $\underline{U}$ where $(\zeta')^{-1} \odot \zeta$ is defined) as α is supported on $\underline{U}$, so we extend by zero on $\underline{X} \setminus \underline{U}$.

Similarly, in Theorem 3.7, the compatibility conditions on the gluing data $\boldsymbol{X}_i, \boldsymbol{U}_{ij}, \boldsymbol{e}_{ij}$ are significantly weaker than you might expect, because of the existence of partitions of unity. The 2-morphisms η_{ijk} on overlaps $\hat{\boldsymbol{X}}_i \cap \hat{\boldsymbol{X}}_j \cap \hat{\boldsymbol{X}}_k$ are only required to exist, not to satisfy any further conditions. In particular, one might think that on overlaps $\hat{\boldsymbol{X}}_i \cap \hat{\boldsymbol{X}}_j \cap \hat{\boldsymbol{X}}_k \cap \hat{\boldsymbol{X}}_l$ we should require

$$\eta_{ikl} \odot (\mathrm{id}_{\boldsymbol{f}_{kl}} * \eta_{ijk})|_{\boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik} \cap \boldsymbol{U}_{il}} = \eta_{ijl} \odot (\eta_{jkl} * \mathrm{id}_{\boldsymbol{f}_{ij}})|_{\boldsymbol{U}_{ij} \cap \boldsymbol{U}_{ik} \cap \boldsymbol{U}_{il}}, \qquad (3.4)$$

but we do not. Also, one might expect the ζ_{ij} should satisfy conditions on triple overlaps $\hat{\boldsymbol{X}}_i \cap \hat{\boldsymbol{X}}_j \cap \hat{\boldsymbol{X}}_k$, but they need not.

The moral is that constructing d-spaces by gluing together patches $\boldsymbol{X}_i$ is straightforward, as one only has to verify mild conditions on triple overlaps

$\boldsymbol{X}_i \cap \boldsymbol{X}_j \cap \boldsymbol{X}_k$. Again, this works because of the existence of partitions of unity on nice C^∞-schemes, which are used to construct the glued d-spaces $\boldsymbol{Z}$ and 1- and 2-morphisms in Theorems 3.6 and 3.7.

In contrast, for gluing d-stacks in §10.3, we do need compatibility conditions of the form (3.4). The problem of gluing geometric spaces in an ∞-category $\mathcal{C}$ by equivalences, such as Spivak's derived manifolds [53], is discussed by Toën and Vezzosi [57, §1.3.4] and Lurie [40, §6.1.2]. It requires nontrivial conditions on overlaps $\boldsymbol{\mathcal{X}}_{i_1} \cap \cdots \cap \boldsymbol{\mathcal{X}}_{i_n}$ for all $n = 2, 3, \dots$.

3.3 Fibre products in dSpa

Fibre products in 2-categories are explained in §A.4. In [35, §2.5–§2.6] we discuss fibre products in **dSpa**, and their relation to transverse fibre products in **Man**.

Theorem 3.9. (a) *All fibre products exist in the 2-category* **dSpa**.

(b) *Let $g : X \to Z$ and $h : Y \to Z$ be smooth maps of manifolds, and write $\boldsymbol{X} = F^{\mathbf{dSpa}}_{\mathbf{Man}}(X)$, and similarly for $\boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{g}, \boldsymbol{h}$. If g, h are transverse, so that a fibre product $X \times_{g,Z,h} Y$ exists in* **Man**, *then the fibre product $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ in* **dSpa** *is equivalent in* **dSpa** *to $F^{\mathbf{dSpa}}_{\mathbf{Man}}(X \times_{g,Z,h} Y)$. If g, h are not transverse then $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ exists in* **dSpa**, *but is not a manifold.*

To prove (a), given 1-morphisms $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$, we write down an explicit d-space $\boldsymbol{W} = (\underline{W}, \mathcal{O}'_W, \mathcal{E}_W, \imath_W, \jmath_W)$, 1-morphisms $\boldsymbol{e} = (\underline{e}, e', e'') : \boldsymbol{W} \to \boldsymbol{X}$ and $\boldsymbol{f} = (\underline{f}, f', f'') : \boldsymbol{W} \to \boldsymbol{Y}$ and a 2-morphism $\eta : \boldsymbol{g} \circ \boldsymbol{e} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f}$, and verify the universal property for

$$
\begin{array}{ccc}
\boldsymbol{W} & \xrightarrow{\quad\boldsymbol{f}\quad} & \boldsymbol{Y} \\
{\scriptstyle\boldsymbol{e}}\downarrow & {\scriptstyle\eta}\Uparrow & \downarrow{\scriptstyle\boldsymbol{h}} \\
\boldsymbol{X} & \xrightarrow[\quad\boldsymbol{g}\quad]{} & \boldsymbol{Z}
\end{array}
$$

to be a 2-Cartesian square in **dSpa**. The underlying C^∞-scheme $\underline{W}$ is the fibre product $\underline{W} = \underline{X} \times_{\underline{g},\underline{Z},\underline{h}} \underline{Y}$ in $\mathbf{C^\infty Sch}$, and $\underline{e} : \underline{W} \to \underline{X}$, $\underline{f} : \underline{W} \to \underline{Y}$ are the projections from the fibre product. The definitions of $\mathcal{O}'_W, \imath_W, \jmath_W, e', f'$ in [35, §2.5] are complex, and we will not give them here. The remaining data $\mathcal{E}_W, e'', f'', \eta$, as well as the virtual cotangent sheaf $\phi_W : \mathcal{E}_W \to \mathcal{F}_W$, is characterized by the following commutative diagram in $\mathrm{qcoh}(\underline{W})$, with exact top row:

$$
\begin{array}{ccccccc}
& \begin{pmatrix} \underline{e}^*(g'')\circ I_{\underline{e},g}(\mathcal{E}_Z) \\ -\underline{f}^*(h'')\circ I_{\underline{f},\underline{h}}(\mathcal{E}_Z) \\ (\underline{g}\circ\underline{e})^*(\phi_Z) \end{pmatrix} & \begin{array}{c} \underline{e}^*(\mathcal{E}_X)\oplus \\ \underline{f}^*(\mathcal{E}_Y)\oplus \\ (\underline{g}\circ\underline{e})^*(\mathcal{F}_Z) \end{array} & \begin{pmatrix} e'' & f'' & \eta \end{pmatrix} & & \\
(\underline{g}\circ\underline{e})^*(\mathcal{E}_Z) & \xrightarrow{\hspace{2.5cm}} & & \cdots\cdots\cdots\cdots\cdots\cdots> & \mathcal{E}_W & \longrightarrow & 0 \\
\begin{pmatrix} -\underline{e}^*(\phi_X) & 0 & \underline{e}^*(g^2)\circ I_{\underline{e},g}(\mathcal{F}_Z) \\ 0 & -\underline{f}^*(\phi_Y) & -\underline{f}^*(h^2)\circ I_{\underline{f},\underline{h}}(\mathcal{F}_Z) \end{pmatrix}\Big\downarrow & & & & \Big\vdots{\scriptstyle\phi_W} & & \\
& & \begin{array}{c} \underline{e}^*(\mathcal{F}_X)\oplus \\ \underline{f}^*(\mathcal{F}_Y) \end{array} & \cdots\cdots\xrightarrow[\cong]{\begin{pmatrix} e^2 & f^2 \end{pmatrix}}\cdots\cdots> & \mathcal{F}_W. & &
\end{array}
$$

3.4 Fixed point loci of finite groups in d-spaces

If a finite group Γ acts on a manifold X by diffeomorphisms, then the fixed point locus X^Γ is a disjoint union of closed, embedded submanifolds of X. In a similar way, if Γ acts on a d-space $\boldsymbol{X}$ by 1-isomorphisms, in [35, §2.7] we define a d-space $\boldsymbol{X}^\Gamma$ called the *fixed d-subspace of* Γ *in* $\boldsymbol{X}$, with an inclusion 1-morphism $\boldsymbol{j}_{\boldsymbol{X},\Gamma} : \boldsymbol{X}^\Gamma \hookrightarrow \boldsymbol{X}$, whose topological space X^Γ is the fixed point locus of Γ in X.

Note that by an *action* $\boldsymbol{r} : \Gamma \to \mathrm{Aut}(\boldsymbol{X})$ *of* Γ *on* $\boldsymbol{X}$ we shall always mean a *strict* action, that is, $\boldsymbol{r}(\gamma) : \boldsymbol{X} \to \boldsymbol{X}$ is a 1-isomorphism for all $\gamma \in \Gamma$ and $\boldsymbol{r}(\gamma\delta) = \boldsymbol{r}(\gamma)\boldsymbol{r}(\delta)$ for all $\gamma, \delta \in \Gamma$, rather than $\boldsymbol{r}(\gamma\delta)$ only being 2-isomorphic to $\boldsymbol{r}(\gamma)\boldsymbol{r}(\delta)$. The next theorem summarizes our results.

Theorem 3.10. *Let* $\boldsymbol{X}$ *be a d-space,* Γ *a finite group, and* $\boldsymbol{r} : \Gamma \to \mathrm{Aut}(\boldsymbol{X})$ *an action of* Γ *on* $\boldsymbol{X}$ *by* 1-*isomorphisms. Then we can define a d-space* $\boldsymbol{X}^\Gamma$ *called the* **fixed d-subspace of** Γ **in** $\boldsymbol{X}$, *with an inclusion* 1-*morphism* $\boldsymbol{j}_{\boldsymbol{X},\Gamma} : \boldsymbol{X}^\Gamma \to \boldsymbol{X}$. *It has the following properties:*

(a) *Let* $\boldsymbol{X}, \Gamma, \boldsymbol{r}$ *and* $\boldsymbol{j}_{\boldsymbol{X},\Gamma} : \boldsymbol{X}^\Gamma \to \boldsymbol{X}$ *be as above. Suppose* $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{X}$ *is a* 1-*morphism in* **dSpa**. *Then* $\boldsymbol{f}$ *factorizes as* $\boldsymbol{f} = \boldsymbol{j}_{\boldsymbol{X},\Gamma} \circ \boldsymbol{g}$ *for some* 1-*morphism* $\boldsymbol{g} : \boldsymbol{W} \to \boldsymbol{X}^\Gamma$ *in* **dSpa**, *which must be unique, if and only if* $\boldsymbol{r}(\gamma) \circ \boldsymbol{f} = \boldsymbol{f}$ *for all* $\gamma \in \Gamma$.

(b) *Suppose* $\boldsymbol{X}, \boldsymbol{Y}$ *are d-spaces,* Γ *is a finite group,* $\boldsymbol{r} : \Gamma \to \mathrm{Aut}(\boldsymbol{X})$, $\boldsymbol{s} : \Gamma \to \mathrm{Aut}(\boldsymbol{Y})$ *are actions of* Γ *on* $\boldsymbol{X}, \boldsymbol{Y}$, *and* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *is a* Γ-*equivariant* 1-*morphism in* **dSpa**, *that is,* $\boldsymbol{f} \circ \boldsymbol{r}(\gamma) = \boldsymbol{s}(\gamma) \circ \boldsymbol{f}$ *for* $\gamma \in \Gamma$. *Then there exists a unique* 1-*morphism* $\boldsymbol{f}^\Gamma : \boldsymbol{X}^\Gamma \to \boldsymbol{Y}^\Gamma$ *such that* $\boldsymbol{j}_{\boldsymbol{Y},\Gamma} \circ \boldsymbol{f}^\Gamma = \boldsymbol{f} \circ \boldsymbol{j}_{\boldsymbol{X},\Gamma}$.

(c) *Let* $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ *be* Γ-*equivariant* 1-*morphisms as in* (b), *and* $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ *be a* Γ-*equivariant* 2-*morphism, that is,* $\eta * \mathrm{id}_{\boldsymbol{r}(\gamma)} = \mathrm{id}_{\boldsymbol{s}(\gamma)} * \eta$ *for* $\gamma \in \Gamma$. *Then there exists a unique* 2-*morphism* $\eta^\Gamma : \boldsymbol{f}^\Gamma \Rightarrow \boldsymbol{g}^\Gamma$ *such that* $\mathrm{id}_{\boldsymbol{j}_{\boldsymbol{Y},\Gamma}} * \eta^\Gamma = \eta * \mathrm{id}_{\boldsymbol{j}_{\boldsymbol{X},\Gamma}}$.

Note that (a) *is a universal property that determines* $\boldsymbol{X}^\Gamma, \boldsymbol{j}_{\boldsymbol{X},\Gamma}$ *up to canonical* 1-*isomorphism.*

We will use fixed d-subspaces $\boldsymbol{X}^\Gamma$ in Theorem 10.14 below to describe orbifold strata $\boldsymbol{\mathcal{X}}^\Gamma$ of quotient d-stacks $\boldsymbol{\mathcal{X}} = [\boldsymbol{X}/G]$. If $\boldsymbol{X}$ is a d-manifold, as in §4, then in general the fixed d-subspaces $\boldsymbol{X}^\Gamma$ are disjoint unions of d-manifolds of different dimensions.

4 The 2-category of d-manifolds

We can now define and discuss *d-manifolds*, our derived version of smooth manifolds (without boundary), following [35, Chap.s 3 & 4].

4.1 The definition of d-manifolds

Definition 4.1. A d-space $\boldsymbol{U}$ is called a *principal d-manifold* if is equivalent in **dSpa** to a fibre product $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ with $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \hat{\mathbf{Man}}$. That is,

$$\boldsymbol{U} \simeq F_{\mathbf{Man}}^{\mathbf{dSpa}}(X) \times_{F_{\mathbf{Man}}^{\mathbf{dSpa}}(g), F_{\mathbf{Man}}^{\mathbf{dSpa}}(Z), F_{\mathbf{Man}}^{\mathbf{dSpa}}(h)} F_{\mathbf{Man}}^{\mathbf{dSpa}}(Y)$$

for manifolds X, Y, Z and smooth maps $g : X \to Z$ and $h : Y \to Z$. The *virtual dimension* $\operatorname{vdim} \boldsymbol{U}$ of $\boldsymbol{U}$ is defined to be $\operatorname{vdim} \boldsymbol{U} = \dim X + \dim Y - \dim Z$. Proposition 4.11(b) below shows that if $\boldsymbol{U} \neq \emptyset$ then $\operatorname{vdim} \boldsymbol{U}$ depends only on the d-space $\boldsymbol{U}$, and not on the choice of X, Y, Z, g, h, and so is well defined.

A d-space $\boldsymbol{W}$ is called a *d-manifold of virtual dimension* $n \in \mathbb{Z}$, written $\operatorname{vdim} \boldsymbol{W} = n$, if $\boldsymbol{W}$ can be covered by nonempty open d-subspaces $\boldsymbol{U}$ which are principal d-manifolds with $\operatorname{vdim} \boldsymbol{U} = n$.

Write **dMan** for the full 2-subcategory of d-manifolds in **dSpa**. If $\boldsymbol{X} \in \hat{\mathbf{Man}}$ then $\boldsymbol{X} \simeq \boldsymbol{X} \times_* *$, so $\boldsymbol{X}$ is a principal d-manifold, and thus a d-manifold. Therefore $\hat{\mathbf{Man}}$ in §3.1 is a 2-subcategory of **dMan**. We say that a d-manifold $\boldsymbol{X}$ *is a manifold* if it lies in $\hat{\mathbf{Man}}$. The 2-functor $F_{\mathbf{Man}}^{\mathbf{dSpa}} : \mathbf{Man} \to \mathbf{dSpa}$ maps into **dMan**, and we will write $F_{\mathbf{Man}}^{\mathbf{dMan}} = F_{\mathbf{Man}}^{\mathbf{dSpa}} : \mathbf{Man} \to \mathbf{dMan}$.

Here, as in [35, §3.2], are alternative descriptions of principal d-manifolds:

Proposition 4.2. *The following are equivalent characterizations of when a d-space $\boldsymbol{W}$ is a principal d-manifold:*

(a) $\boldsymbol{W} \simeq \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ *for* $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \hat{\mathbf{Man}}$.

(b) $\boldsymbol{W} \simeq \boldsymbol{X} \times_{\boldsymbol{i},\boldsymbol{Z},\boldsymbol{j}} \boldsymbol{Y}$, *where* X, Y, Z *are manifolds,* $i : X \to Z$, $j : Y \to Z$ *are embeddings,* $\boldsymbol{X} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(X)$, *and similarly for* Y, Z, i, j. *That is,* $\boldsymbol{W}$ *is an intersection of two submanifolds* X, Y *in* Z, *in the sense of d-spaces.*

(c) $\boldsymbol{W} \simeq \boldsymbol{V} \times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V}$, *where* V *is a manifold,* $E \to V$ *is a vector bundle,* $s : V \to E$ *is a smooth section,* $0 : V \to E$ *is the zero section,* $\boldsymbol{V} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(V)$, *and similarly for* $E, s, 0$. *That is,* $\boldsymbol{W}$ *is the zeroes* $s^{-1}(0)$ *of a smooth section* s *of a vector bundle* E, *in the sense of d-spaces.*

Example 4.3. Let $X \subseteq \mathbb{R}^n$ be any closed subset. By a lemma of Whitney's, we can write X as the zero set of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$. Then $\boldsymbol{X} = \mathbb{R}^{\boldsymbol{n}} \times_{\boldsymbol{f},\mathbb{R},\boldsymbol{0}} *$ is a principal d-manifold, with topological space X.

This example shows that the topological spaces X underlying d-manifolds $\boldsymbol{X}$ can be fairly wild, for example, X could be a fractal such as the Cantor set.

4.2 'Standard model' d-manifolds, 1- and 2-morphisms

The next three examples, taken from [35, §3.2 & §3.4], give explicit models for principal d-manifolds in the form $\boldsymbol{V} \times_{\boldsymbol{s},\boldsymbol{E},\boldsymbol{0}} \boldsymbol{V}$ from Proposition 4.2(c) and their 1- and 2-morphisms, which we call *standard models*.

Example 4.4. Let V be a manifold, $E \to V$ a vector bundle (which we sometimes call the *obstruction bundle*), and $s \in C^\infty(E)$. We will write down an explicit principal d-manifold $\boldsymbol{S} = (\underline{S}, \mathcal{O}'_S, \mathcal{E}_S, \imath_S, \jmath_S)$ which is equivalent to $\boldsymbol{V} \times_{\boldsymbol{s}, \boldsymbol{E}, \boldsymbol{0}} \boldsymbol{V}$ in Proposition 4.2(c). We call $\boldsymbol{S}$ the *standard model* of (V, E, s), and also write it $\boldsymbol{S}_{V,E,s}$. Proposition 4.2 shows that every principal d-manifold $\boldsymbol{W}$ is equivalent to $\boldsymbol{S}_{V,E,s}$ for some V, E, s.

Write $C^\infty(V)$ for the C^∞-ring of smooth functions $c : V \to \mathbb{R}$, and $C^\infty(E)$, $C^\infty(E^*)$ for the vector spaces of smooth sections of E, E^* over V. Then s lies in $C^\infty(E)$, and $C^\infty(E), C^\infty(E^*)$ are modules over $C^\infty(V)$, and there is a natural bilinear product $\cdot : C^\infty(E^*) \times C^\infty(E) \to C^\infty(V)$. Define $I_s \subseteq C^\infty(V)$ to be the ideal generated by s. That is,

$$I_s = \big\{ \alpha \cdot s : \alpha \in C^\infty(E^*) \big\} \subseteq C^\infty(V). \tag{4.1}$$

Let $I_s^2 = \langle fg : f, g \in I_s \rangle_\mathbb{R}$ be the square of I_s. Then I_s^2 is an ideal in $C^\infty(V)$, the ideal generated by $s \otimes s \in C^\infty(E \otimes E)$. That is,

$$I_s^2 = \big\{ \beta \cdot (s \otimes s) : \beta \in C^\infty(E^* \otimes E^*) \big\} \subseteq C^\infty(V).$$

Define C^∞-rings $\mathfrak{C} = C^\infty(V)/I_s$, $\mathfrak{C}' = C^\infty(V)/I_s^2$, and let $\pi : \mathfrak{C}' \to \mathfrak{C}$ be the natural projection from the inclusion $I_s^2 \subseteq I_s$. Define a topological space $S = \{v \in V : s(v) = 0\}$, as a subspace of V. Now $s(v) = 0$ if and only if $(s \otimes s)(v) = 0$. Thus S is the underlying topological space for both $\operatorname{Spec} \mathfrak{C}$ and $\operatorname{Spec} \mathfrak{C}'$. So $\operatorname{Spec} \mathfrak{C} = \underline{S} = (S, \mathcal{O}_S)$, $\operatorname{Spec} \mathfrak{C}' = \underline{S}' = (S, \mathcal{O}'_S)$, and $\operatorname{Spec} \pi = \imath_S = (\operatorname{id}_S, \imath_S) : \underline{S}' \to \underline{S}$, where $\underline{S}, \underline{S}'$ are fair affine C^∞-schemes, and $\mathcal{O}_S, \mathcal{O}'_S$ are sheaves of C^∞-rings on S, and $\imath_S : \mathcal{O}'_S \to \mathcal{O}_S$ is a morphism of sheaves of C^∞-rings. Since π is surjective with kernel the square zero ideal I_s/I_s^2, $\imath_S$ is surjective, with kernel $\mathcal{I}_S$ a sheaf of square zero ideals in $\mathcal{O}'_S$.

From (4.1) we have a surjective $C^\infty(V)$-module morphism $C^\infty(E^*) \to I_s$ mapping $\alpha \mapsto \alpha \cdot s$. Applying $\otimes_{C^\infty(V)} \mathfrak{C}$ gives a surjective $\mathfrak{C}$-module morphism

$$\sigma : C^\infty(E^*)/(I_s \cdot C^\infty(E^*)) \longrightarrow I_s/I_s^2, \quad \sigma : \alpha + (I_s \cdot C^\infty(E^*)) \longmapsto \alpha \cdot s + I_s^2.$$

Define $\mathcal{E}_S = \operatorname{MSpec}\big(C^\infty(E^*)/(I_s \cdot C^\infty(E^*))\big)$. Also $\operatorname{MSpec}(I_s/I_s^2) = \mathcal{I}_S$, so $\jmath_S = \operatorname{MSpec} \sigma$ is a surjective morphism $\jmath_S : \mathcal{E}_S \to \mathcal{I}_S$ in $\operatorname{qcoh}(\underline{S})$. Therefore $\boldsymbol{S}_{V,E,s} = \boldsymbol{S} = (\underline{S}, \mathcal{O}'_S, \mathcal{E}_S, \imath_S, \jmath_S)$ is a d-space.

In fact $\mathcal{E}_S$ is a vector bundle on $\underline{S}$ naturally isomorphic to $\mathcal{E}^*|_{\underline{S}}$, where $\mathcal{E}$ is the vector bundle on $\underline{V} = F^{\mathbf{C}^\infty \mathbf{Sch}}_{\mathbf{Man}}(V)$ corresponding to $E \to V$. Also $\mathcal{F}_S \cong T^*\underline{V}|_{\underline{S}}$. The morphism $\phi_S : \mathcal{E}_S \to \mathcal{F}_S$ can be interpreted as follows: choose a connection ∇ on $E \to V$. Then $\nabla s \in C^\infty(E \otimes T^*V)$, so we can regard ∇s as a morphism of vector bundles $E^* \to T^*V$ on V. This lifts to a morphism of vector bundles $\hat{\nabla} s : \mathcal{E}^* \to T^*\underline{V}$ on the C^∞-scheme $\underline{V}$, and ϕ_S is identified with $\hat{\nabla} s|_{\underline{S}} : \mathcal{E}^*|_{\underline{S}} \to T^*\underline{V}|_{\underline{S}}$ under the isomorphisms $\mathcal{E}_S \cong \mathcal{E}^*|_{\underline{S}}$, $\mathcal{F}_S \cong T^*\underline{V}|_{\underline{S}}$.

Proposition 4.2 implies that every principal d-manifold $\boldsymbol{W}$ is equivalent to $\boldsymbol{S}_{V,E,s}$ for some V, E, s. The notation $O(s)$ and $O(s^2)$ used below should be interpreted as follows. Let V be a manifold, $E \to V$ a vector bundle, and

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$s \in C^\infty(E)$. If $F \to V$ is another vector bundle and $t \in C^\infty(F)$, then we write $t = O(s)$ if $t = \alpha \cdot s$ for some $\alpha \in C^\infty(F \otimes E^*)$, and $t = O(s^2)$ if $t = \beta \cdot (s \otimes s)$ for some $\beta \in C^\infty(F \otimes E^* \otimes E^*)$. Similarly, if W is a manifold and $f, g : V \to W$ are smooth then we write $f = g + O(s)$ if $c \circ f - c \circ g = O(s)$ for all smooth $c : W \to \mathbb{R}$, and $f = g + O(s^2)$ if $c \circ f - c \circ g = O(s^2)$ for all c.

Example 4.5. Let V, W be manifolds, $E \to V$, $F \to W$ be vector bundles, and $s \in C^\infty(E)$, $t \in C^\infty(F)$. Write $\boldsymbol{X} = \boldsymbol{S}_{V,E,s}$, $\boldsymbol{Y} = \boldsymbol{S}_{W,F,t}$ for the 'standard model' principal d-manifolds from Example 4.4. Suppose $f : V \to W$ is a smooth map, and $\hat{f} : E \to f^*(F)$ is a morphism of vector bundles on V satisfying

$$\hat{f} \circ s = f^*(t) + O(s^2) \quad \text{in } C^\infty\big(f^*(F)\big). \tag{4.2}$$

We will define a 1-morphism $\boldsymbol{g} = (g, g', g'') : \boldsymbol{X} \to \boldsymbol{Y}$ in **dMan** using $f, \hat{f}$. We will also write $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ as $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$, and call it a *standard model 1-morphism*. If $x \in X$ then $x \in V$ with $s(x) = 0$, so (4.2) implies that

$$t\big(f(x)\big) = \big(f^*(t)\big)(x) = \hat{f}\big(s(x)\big) + O\big(s(x)^2\big) = 0,$$

so $f(x) \in Y \subseteq W$. Thus $g := f|_X$ maps $X \to Y$.

Define morphisms of C^∞-rings

$$\phi : C^\infty(W)/I_t \longrightarrow C^\infty(V)/I_s, \qquad \phi' : C^\infty(W)/I_t^2 \longrightarrow C^\infty(V)/I_s^2,$$
$$\text{by} \quad \phi : c + I_t \longmapsto c \circ f + I_s, \qquad\quad \phi' : c + I_t^2 \longmapsto c \circ f + I_s^2.$$

Here ϕ is well-defined since if $c \in I_t$ then $c = \gamma \cdot t$ for some $\gamma \in C^\infty(F^*)$, so

$$c \circ f = (\gamma \cdot t) \circ f = f^*(\gamma) \cdot f^*(t) = f^*(\gamma) \cdot \big(\hat{f} \circ s + O(s^2)\big) = \big(\hat{f} \circ f^*(\gamma)\big) \cdot s + O(s^2) \in I_s.$$

Similarly if $c \in I_t^2$ then $c \circ f \in I_s^2$, so ϕ' is well-defined. Thus we have C^∞-scheme morphisms $g = (g, g^\sharp) = \operatorname{Spec} \phi : \underline{X} \to \underline{Y}$, and $(g, g') = \operatorname{Spec} \phi' : (X, \mathcal{O}_X') \to (Y, \mathcal{O}_Y')$, both with underlying map g. Hence $g^\sharp : g^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ and $g' : g^{-1}(\mathcal{O}_Y') \to \mathcal{O}_X'$ are morphisms of sheaves of C^∞-rings on X.

Since $g^*(\mathcal{E}_Y) = \operatorname{MSpec}\big(C^\infty(f^*(F^*))/(I_s \cdot C^\infty(f^*(F^*)))\big)$, we may define $g'' : g^*(\mathcal{E}_Y) \to \mathcal{E}_X$ by $g'' = \operatorname{MSpec}(G'')$, where

$$G'' : C^\infty(f^*(F^*))/(I_s \cdot C^\infty(f^*(F^*)) \longrightarrow C^\infty(E^*)/(I_s \cdot C^\infty(E^*))$$

is defined by $\quad G'' : \gamma + I_s \cdot C^\infty(f^*(F^*)) \longmapsto \gamma \circ \hat{f} + I_s \cdot C^\infty(E^*).$

This defines $\boldsymbol{g} = (g, g', g'')$. One can show it is a 1-morphism $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ in **dMan**, which we also write as $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$.

Suppose $\tilde{V} \subseteq V$ is open, with inclusion $i_{\tilde{V}} : \tilde{V} \to V$. Write $\tilde{E} = E|_{\tilde{V}} = i_{\tilde{V}}^*(E)$ and $\tilde{s} = s|_{\tilde{V}}$. Define $\boldsymbol{i}_{\tilde{V},V} = \boldsymbol{S}_{i_{\tilde{V}}, \operatorname{id}_{\tilde{E}}} : \boldsymbol{S}_{\tilde{V}, \tilde{E}, \tilde{s}} \to \boldsymbol{S}_{V,E,s}$. If $s^{-1}(0) \subseteq \tilde{V}$ then $\boldsymbol{i}_{\tilde{V},V}$ is a 1-isomorphism, with inverse $\boldsymbol{i}_{\tilde{V},V}^{-1}$. That is, making V smaller without making $s^{-1}(0)$ smaller does not really change $\boldsymbol{S}_{V,E,s}$; the d-manifold $\boldsymbol{S}_{V,E,s}$ depends only on E, s in an arbitrarily small open neighbourhood of $s^{-1}(0)$ in V.

Example 4.6. Let V, W be manifolds, $E \to V$, $F \to W$ be vector bundles, and $s \in C^\infty(E)$, $t \in C^\infty(F)$. Suppose $f, g : V \to W$ are smooth and $\hat{f} : E \to f^*(F)$, $\hat{g} : E \to g^*(F)$ are vector bundle morphisms with $\hat{f} \circ s = f^*(t) + O(s^2)$ and $\hat{g} \circ s = g^*(t) + O(s^2)$, so we have 1-morphisms $\boldsymbol{S}_{f,\hat{f}}, \boldsymbol{S}_{g,\hat{g}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$. It is easy to show that $\boldsymbol{S}_{f,\hat{f}} = \boldsymbol{S}_{g,\hat{g}}$ if and only if $g = f + O(s^2)$ and $\hat{g} = \hat{f} + O(s)$.

Now suppose $\Lambda : E \to f^*(TW)$ is a morphism of vector bundles on V. Taking the dual of Λ and lifting to $\underline{V}$ gives $\Lambda^* : \underline{f}^*(T^*\underline{W}) \to \mathcal{E}^*$. Restricting to the C^∞-subscheme $\underline{X} = s^{-1}(0)$ in $\underline{V}$ gives $\lambda = \Lambda^*|_{\underline{X}} : \underline{f}^*(\mathcal{F}_Y) \cong \underline{f}^*(T^*\underline{W})|_{\underline{X}} \to \mathcal{E}^*|_{\underline{X}} = \mathcal{E}_X$. One can show that λ is a 2-morphism $\boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{g,\hat{g}}$ if and only if

$$g = f + \Lambda \circ s + O(s^2) \quad \text{and} \quad \hat{g} = \hat{f} + f^*(\mathrm{d}t) \circ \Lambda + O(s).$$

Then we write λ as $\boldsymbol{S}_\Lambda : \boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{g,\hat{g}}$, and call it a *standard model 2-morphism*. Every 2-morphism $\eta : \boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{g,\hat{g}}$ is $\boldsymbol{S}_\Lambda$ for some Λ. Two vector bundle morphisms $\Lambda, \Lambda' : E \to f^*(TW)$ have $\boldsymbol{S}_\Lambda = \boldsymbol{S}_{\Lambda'}$ if and only if $\Lambda = \Lambda' + O(s)$.

If $\boldsymbol{X}$ is a d-manifold and $x \in \boldsymbol{X}$ then x has an open neighbourhood $\boldsymbol{U}$ in $\boldsymbol{X}$ equivalent in **dSpa** to $\boldsymbol{S}_{V,E,s}$ for some manifold V, vector bundle $E \to V$ and $s \in C^\infty(E)$. In [35, §3.3] we investigate the extent to which $\boldsymbol{X}$ determines V, E, s near a point in $\boldsymbol{X}$ and V, and prove:

Theorem 4.7. *Let $\boldsymbol{X}$ be a d-manifold, and $x \in \boldsymbol{X}$. Then there exists an open neighbourhood $\boldsymbol{U}$ of x in $\boldsymbol{X}$ and an equivalence $\boldsymbol{U} \simeq \boldsymbol{S}_{V,E,s}$ in **dMan** for some manifold V, vector bundle $E \to V$ and $s \in C^\infty(E)$ which identifies $x \in \boldsymbol{U}$ with a point $v \in V$ such that $s(v) = \mathrm{d}s(v) = 0$, where $\boldsymbol{S}_{V,E,s}$ is as in Example 4.4. These V, E, s are determined up to non-canonical isomorphism near v by $\boldsymbol{X}$ near x, and in fact they depend only on the underlying C^∞-scheme $\underline{X}$ and the integer $\mathrm{vdim}\,\boldsymbol{X}$.*

Thus, if we impose the extra condition $\mathrm{d}s(v) = 0$, which is in fact equivalent to choosing V, E, s with $\dim V$ as small as possible, then V, E, s are determined uniquely near v by $\boldsymbol{X}$ near x (that is, V, E, s are determined locally up to isomorphism, but not up to canonical isomorphism). If we drop the condition $\mathrm{d}s(v) = 0$ then V, E, s are determined uniquely near v by $\boldsymbol{X}$ near x and $\dim V$.

Theorem 4.7 shows that any d-manifold $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ is determined up to equivalence in **dSpa** near any point $x \in \boldsymbol{X}$ by the 'classical' underlying C^∞-scheme $\underline{X}$ and the integer $\mathrm{vdim}\,\boldsymbol{X}$. So we can ask: what extra information about $\boldsymbol{X}$ is contained in the 'derived' data $\mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X$? One can think of this extra information as like a vector bundle $\mathcal{E}$ over $\underline{X}$. The only local information in a vector bundle $\mathcal{E}$ is $\mathrm{rank}\,\mathcal{E} \in \mathbb{Z}$, but globally it also contains nontrivial algebraic-topological information.

Suppose now that $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism in **dMan**, and $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$. Then by Theorem 4.7 we have $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$ near x and $\boldsymbol{Y} \simeq \boldsymbol{S}_{W,F,t}$ near y. So up to composition with equivalences, we can identify $\boldsymbol{f}$ near x with a 1-morphism $\boldsymbol{g} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$. Thus, to understand arbitrary 1-morphisms $\boldsymbol{f}$ in **dMan** near a point, it is enough to study 1-morphisms $\boldsymbol{g} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$. Our next theorem, proved in [35, §3.4], shows that after making V smaller, every 1-morphism $\boldsymbol{g} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ is of the form $\boldsymbol{S}_{f,\hat{f}}$.

Theorem 4.8. *Let V, W be manifolds, $E \to V$, $F \to W$ be vector bundles, and $s \in C^\infty(E)$, $t \in C^\infty(F)$. Define principal d-manifolds $\boldsymbol{X} = \boldsymbol{S}_{V,E,s}$, $\boldsymbol{Y} = \boldsymbol{S}_{W,F,t}$, with topological spaces $X = \{v \in V : s(v) = 0\}$ and $Y = \{w \in W : t(w) = 0\}$. Suppose $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism. Then there exist an open neighbourhood $\tilde{V}$ of X in V, a smooth map $f : \tilde{V} \to W$, and a morphism of vector bundles $\hat{f} : \tilde{E} \to f^*(F)$ with $\hat{f} \circ \tilde{s} = f^*(t)$, where $\tilde{E} = E|_{\tilde{V}}$, $\tilde{s} = s|_{\tilde{V}}$, such that $\boldsymbol{g} = \boldsymbol{S}_{f,\hat{f}} \circ \boldsymbol{i}_{\tilde{V},V}^{-1}$, where $\boldsymbol{i}_{\tilde{V},V} = \boldsymbol{S}_{\mathrm{id}_{\tilde{V}}, \mathrm{id}_{\tilde{E}}} : \boldsymbol{S}_{\tilde{V},\tilde{E},\tilde{s}} \to \boldsymbol{S}_{V,E,s}$ is a 1-isomorphism, and $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{\tilde{V},\tilde{E},\tilde{s}} \to \boldsymbol{S}_{W,F,t}$ is as in Example 4.5.*

These results give a good differential-geometric picture of d-manifolds and their 1- and 2-morphisms near a point. The $O(s)$ and $O(s^2)$ notation helps keep track of what information from V, E, s and $f, \hat{f}$ and Λ is remembered and what forgotten by the d-manifolds $\boldsymbol{S}_{V,E,s}$, 1-morphisms $\boldsymbol{S}_{f,\hat{f}}$ and 2-morphisms $\boldsymbol{S}_\Lambda$.

4.3 The 2-category of virtual vector bundles

In our theory of derived differential geometry, it is a general principle that categories in classical differential geometry should often be replaced by 2-categories, and classical concepts be replaced by 2-categorical analogues.

In classical differential geometry, if X is a manifold, the vector bundles $E \to X$ and their morphisms form a category $\mathrm{vect}(X)$. The cotangent bundle T^*X is an important example of a vector bundle. If $f : X \to Y$ is smooth then pullback $f^* : \mathrm{vect}(Y) \to \mathrm{vect}(X)$ is a functor. There is a natural morphism $\mathrm{d}f^* : f^*(T^*Y) \to T^*X$. We now explain 2-categorical analogues of all this for d-manifolds, following [35, §3.1–§3.2].

Definition 4.9. Let $\underline{X}$ be a C^∞-scheme, which will usually be the C^∞-scheme underlying a d-manifold $\boldsymbol{X}$. We will define a 2-category $\mathrm{vqcoh}(\underline{X})$ of *virtual quasicoherent sheaves* on $\underline{X}$. *Objects* of $\mathrm{vqcoh}(\underline{X})$ are morphisms $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ in $\mathrm{qcoh}(\underline{X})$, which we also may write as $(\mathcal{E}^1, \mathcal{E}^2, \phi)$ or $(\mathcal{E}^\bullet, \phi)$. Given objects $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ and $\psi : \mathcal{F}^1 \to \mathcal{F}^2$, a *1-morphism* $(f^1, f^2) : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ is a pair of morphisms $f^1 : \mathcal{E}^1 \to \mathcal{F}^1$, $f^2 : \mathcal{E}^2 \to \mathcal{F}^2$ in $\mathrm{qcoh}(\underline{X})$ such that $\psi \circ f^1 = f^2 \circ \phi$. We write $f^\bullet$ for (f^1, f^2).

The *identity* 1-*morphism* of $(\mathcal{E}^\bullet, \phi)$ is $(\mathrm{id}_{\mathcal{E}^1}, \mathrm{id}_{\mathcal{E}^2})$. The *composition* of 1-morphisms $f^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ and $g^\bullet : (\mathcal{F}^\bullet, \psi) \to (\mathcal{G}^\bullet, \xi)$ is $g^\bullet \circ f^\bullet = (g^1 \circ f^1, g^2 \circ f^2) : (\mathcal{E}^\bullet, \phi) \to (\mathcal{G}^\bullet, \xi)$.

Given $f^\bullet, g^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$, a 2-*morphism* $\eta : f^\bullet \Rightarrow g^\bullet$ is a morphism $\eta : \mathcal{E}^2 \to \mathcal{F}^1$ in $\mathrm{qcoh}(\underline{X})$ such that $g^1 = f^1 + \eta \circ \phi$ and $g^2 = f^2 + \psi \circ \eta$. The *identity* 2-*morphism* for $f^\bullet$ is $\mathrm{id}_{f^\bullet} = 0$. If $f^\bullet, g^\bullet, h^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ are 1-morphisms and $\eta : f^\bullet \Rightarrow g^\bullet$, $\zeta : g^\bullet \Rightarrow h^\bullet$ are 2-morphisms, the *vertical composition of 2-morphisms* $\zeta \odot \eta : f^\bullet \Rightarrow h^\bullet$ is $\zeta \odot \eta = \zeta + \eta$. If $f^\bullet, \tilde{f}^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ and $g^\bullet, \tilde{g}^\bullet : (\mathcal{F}^\bullet, \psi) \to (\mathcal{G}^\bullet, \xi)$ are 1-morphisms and $\eta : f^\bullet \Rightarrow \tilde{f}^\bullet$, $\zeta : g^\bullet \Rightarrow \tilde{g}^\bullet$ are 2-morphisms, the *horizontal composition of 2-morphisms* $\zeta * \eta : g^\bullet \circ f^\bullet \Rightarrow \tilde{g}^\bullet \circ \tilde{f}^\bullet$ is $\zeta * \eta = g^1 \circ \eta + \zeta \circ f^2 + \zeta \circ \psi \circ \eta$. This defines a strict 2-category $\mathrm{vqcoh}(\underline{X})$, the obvious 2-category of 2-term complexes in $\mathrm{qcoh}(\underline{X})$.

If $\underline{U} \subseteq \underline{X}$ is an open C^∞-subscheme then restriction from $\underline{X}$ to $\underline{U}$ defines a strict 2-functor $|_{\underline{U}} : \mathrm{vqcoh}(\underline{X}) \to \mathrm{vqcoh}(\underline{U})$. An object $(\mathcal{E}^\bullet, \phi)$ in $\mathrm{vqcoh}(\underline{X})$ is

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called a *virtual vector bundle of rank* $d \in \mathbb{Z}$ if $\underline{X}$ may be covered by open $\underline{U} \subseteq \underline{X}$ such that $(\mathcal{E}^\bullet, \phi)|_{\underline{U}}$ is equivalent in vqcoh($\underline{U}$) to some $(\mathcal{F}^\bullet, \psi)$ for $\mathcal{F}^1, \mathcal{F}^2$ vector bundles on $\underline{U}$ with rank $\mathcal{F}^2 - \text{rank} \, \mathcal{F}^1 = d$. We write rank($\mathcal{E}^\bullet, \phi) = d$. If $\underline{X} \neq \emptyset$ then rank($\mathcal{E}^\bullet, \phi$) depends only on $\mathcal{E}^1, \mathcal{E}^2, \phi$, so it is well-defined. Write vvect($\underline{X}$) for the full 2-subcategory of virtual vector bundles in vqcoh($\underline{X}$).

If $\underline{f} : \underline{X} \to \underline{Y}$ is a C^∞-scheme morphism then pullback gives a strict 2-functor $\underline{f}^* : \text{vqcoh}(\underline{Y}) \to \text{vqcoh}(\underline{X})$, which maps vvect($\underline{Y}$) $\to$ vvect($\underline{X}$).

We apply these ideas to d-spaces.

Definition 4.10. Let $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ be a d-space. Define the *virtual cotangent sheaf* $T^*\boldsymbol{X}$ of $\boldsymbol{X}$ to be the morphism $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ in qcoh($\underline{X}$) from Definition 3.1, regarded as a virtual quasicoherent sheaf on $\underline{X}$.

Let $\boldsymbol{f} = (\underline{f}, f', f'') : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism in **dSpa**. Then $T^*\boldsymbol{X} = (\mathcal{E}_X, \mathcal{F}_X, \phi_X)$ and $\underline{f}^*(T^*\boldsymbol{Y}) = \big(\underline{f}^*(\mathcal{E}_Y), \underline{f}^*(\mathcal{F}_Y), \underline{f}^*(\phi_Y)\big)$ are virtual quasicoherent sheaves on $\underline{X}$, and $\Omega_{\boldsymbol{f}} := (f'', f^2)$ is a 1-morphism $\underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ in vqcoh($\underline{X}$), as (3.1) commutes.

Let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be 1-morphisms in **dSpa**, and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a 2-morphism. Then $\eta : \underline{f}^*(\mathcal{F}_Y) \to \mathcal{E}_X$ with $g'' = f'' + \eta \circ \underline{f}^*(\phi_Y)$ and $g^2 = f^2 + \phi_X \circ \eta$, as in (3.3). It follows that η is a 2-morphism $\Omega_{\boldsymbol{f}} \Rightarrow \Omega_{\boldsymbol{g}}$ in vqcoh($\underline{X}$). Thus, objects, 1-morphisms and 2-morphisms in **dSpa** lift to objects, 1-morphisms and 2-morphisms in vqcoh($\underline{X}$).

The next proposition justifies the definition of virtual vector bundle. Because of part (b), if $\boldsymbol{X}$ is a d-manifold we call $T^*\boldsymbol{X}$ the *virtual cotangent bundle* of $\boldsymbol{X}$, rather than the virtual cotangent sheaf.

Proposition 4.11. (a) *Let V be a manifold, $E \to V$ a vector bundle, and $s \in C^\infty(E)$. Then Example 4.4 defines a d-manifold $\boldsymbol{S}_{V,E,s}$. Its cotangent bundle $T^*\boldsymbol{S}_{V,E,s}$ is a virtual vector bundle on $\underline{S}_{V,E,s}$ of rank $\dim V - \text{rank} \, E$.*

(b) *Let $\boldsymbol{X}$ be a d-manifold. Then $T^*\boldsymbol{X}$ is a virtual vector bundle on $\underline{X}$ of rank* vdim $\boldsymbol{X}$. *Hence if $\boldsymbol{X} \neq \emptyset$ then* vdim $\boldsymbol{X}$ *is well-defined.*

The virtual cotangent bundle $T^*\boldsymbol{X}$ of a d-manifold $\boldsymbol{X}$ is a d-space analogue of the *cotangent complex* in algebraic geometry, as in Illusie [29]. It contains only a fraction of the information in $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$, but many interesting properties of d-manifolds $\boldsymbol{X}$ and 1-morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ can be expressed solely in terms of virtual cotangent bundles $T^*\boldsymbol{X}, T^*\boldsymbol{Y}$ and 1-morphisms $\Omega_{\boldsymbol{f}} : \underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$. Here is an example of this.

Definition 4.12. Let $\underline{X}$ be a C^∞-scheme. We say that a virtual vector bundle $(\mathcal{E}^1, \mathcal{E}^2, \phi)$ on $\underline{X}$ *is a vector bundle* if it is equivalent in vvect($\underline{X}$) to $(0, \mathcal{E}, 0)$ for some vector bundle $\mathcal{E}$ on $\underline{X}$. One can show $(\mathcal{E}^1, \mathcal{E}^2, \phi)$ is a vector bundle if and only if ϕ has a left inverse in qcoh($\underline{X}$).

Proposition 4.13. *Let $\boldsymbol{X}$ be a d-manifold. Then $\boldsymbol{X}$ is a manifold (that is, $\boldsymbol{X} \in \hat{\mathbf{Man}}$) if and only if $T^*\boldsymbol{X}$ is a vector bundle, or equivalently, if $\phi_X : \mathcal{E}_X \to \mathcal{F}_X$ has a left inverse in* qcoh($\underline{X}$).

4.4 Equivalences in dMan, and gluing by equivalences

Equivalences in a 2-category are defined in §A.3. Equivalences in **dMan** are the best derived analogue of isomorphisms in **Man**, that is, of diffeomorphisms of manifolds. A smooth map of manifolds $f : X \to Y$ is called *étale* if it is a local diffeomorphism. Here is the derived analogue.

Definition 4.14. Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism in **dMan**. We call $\boldsymbol{f}$ *étale* if it is a *local equivalence*, that is, if for each $x \in \boldsymbol{X}$ there exist open $x \in \boldsymbol{U} \subseteq \boldsymbol{X}$ and $\boldsymbol{f}(x) \in \boldsymbol{V} \subseteq \boldsymbol{Y}$ such that $\boldsymbol{f}(\boldsymbol{U}) = \boldsymbol{V}$ and $\boldsymbol{f}|_{\boldsymbol{U}} : \boldsymbol{U} \to \boldsymbol{V}$ is an equivalence.

If $f : X \to Y$ is a smooth map of manifolds, then f is étale if and only if $\mathrm{d}f^* : f^*(T^*Y) \to T^*X$ is an isomorphism of vector bundles. (The analogue is false for schemes.) In [35, §3.5] we prove a version of this for d-manifolds:

Theorem 4.15. *Suppose $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism of d-manifolds. Then the following are equivalent:*

(i) *$\boldsymbol{f}$ is étale;*

(ii) *$\Omega_{\boldsymbol{f}} : \underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ is an equivalence in $\mathrm{vqcoh}(\underline{X})$; and*

(iii) *the following is a split short exact sequence in $\mathrm{qcoh}(\underline{X})$:*

$$ 0 \longrightarrow \underline{f}^*(\mathcal{E}_Y) \xrightarrow{f'' \oplus -\underline{f}^*(\phi_Y)} \mathcal{E}_X \oplus \underline{f}^*(\mathcal{F}_Y) \xrightarrow{\phi_X \oplus f^2} \mathcal{F}_X \longrightarrow 0. \quad (4.3) $$

*If in addition $f : X \to Y$ is a bijection, then $\boldsymbol{f}$ is an equivalence in **dMan**.*

Here a complex $0 \to E \to F \to G \to 0$ in an abelian category $\mathcal{A}$ is called a *split short exact sequence* if there exists an isomorphism $F \cong E \oplus G$ in $\mathcal{A}$ identifying the complex with $0 \to E \xrightarrow{\mathrm{id} \oplus 0} E \oplus G \xrightarrow{0 \oplus \mathrm{id}} G \to 0$.

The analogue of Theorem 4.15 for d-spaces is false. When $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 'standard model' 1-morphism $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$, as in §4.2, we can express the conditions for $\boldsymbol{S}_{f,\hat{f}}$ to be étale or an equivalence in terms of $f, \hat{f}$.

Theorem 4.16. *Let V, W be manifolds, $E \to V$, $F \to W$ be vector bundles, $s \in C^\infty(E)$, $t \in C^\infty(F)$, $f : V \to W$ be smooth, and $\hat{f} : E \to f^*(F)$ be a morphism of vector bundles on V with $\hat{f} \circ s = f^*(t) + O(s^2)$. Then Example 4.5 defines a 1-morphism $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ in **dMan**. This $\boldsymbol{S}_{f,\hat{f}}$ is étale if and only if for each $v \in V$ with $s(v) = 0$ and $w = f(v) \in W$, the following sequence of vector spaces is exact:*

$$ 0 \longrightarrow T_v V \xrightarrow{\mathrm{d}s(v) \oplus \mathrm{d}f(v)} E_v \oplus T_w W \xrightarrow{\hat{f}(v) \oplus -\mathrm{d}t(w)} F_w \longrightarrow 0. \quad (4.4) $$

Also $\boldsymbol{S}_{f,\hat{f}}$ is an equivalence if and only if in addition $f|_{s^{-1}(0)} : s^{-1}(0) \to t^{-1}(0)$ is a bijection, where $s^{-1}(0) = \{v \in V : s(v) = 0\}$, $t^{-1}(0) = \{w \in W : t(w) = 0\}$.

Section 3.2 discussed gluing d-spaces by equivalences on open d-subspaces. It generalizes immediately to d-manifolds: if in Theorem 3.7 we fix $n \in \mathbb{Z}$ and take the initial d-spaces $\boldsymbol{X}_i$ to be d-manifolds with $\mathrm{vdim}\,\boldsymbol{X}_i = n$, then the glued d-space $\boldsymbol{Y}$ is also a d-manifold with $\mathrm{vdim}\,\boldsymbol{Y} = n$.

Here is an analogue of Theorem 3.7, taken from [35, §3.6], in which we take the d-spaces $\boldsymbol{X}_i$ to be 'standard model' d-manifolds $\boldsymbol{S}_{V_i,E_i,s_i}$, and the 1-morphisms $\boldsymbol{e}_{ij}$ to be 'standard model' 1-morphisms $\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}$. We also use Theorem 4.16 in (ii) to characterize when $\boldsymbol{e}_{ij}$ is an equivalence.

Theorem 4.17. *Suppose we are given the following data:*

(a) *an integer n;*

(b) *a Hausdorff, second countable topological space X;*

(c) *an indexing set I, and a total order $<$ on I;*

(d) *for each i in I, a manifold V_i, a vector bundle $E_i \to V_i$ with $\dim V_i -$ $\operatorname{rank} E_i = n$, a smooth section $s_i : V_i \to E_i$, and a homeomorphism $\psi_i :$ $X_i \to \hat{X}_i$, where $X_i = \{v_i \in V_i : s_i(v_i) = 0\}$ and $\hat{X}_i \subseteq X$ is open; and*

(e) *for all $i < j$ in I, an open submanifold $V_{ij} \subseteq V_i$, a smooth map $e_{ij} : V_{ij} \to$ V_j, and a morphism of vector bundles $\hat{e}_{ij} : E_i|_{V_{ij}} \to e_{ij}^*(E_j)$.*

Using notation $O(s_i), O(s_i^2)$ as in §4.2, let this data satisfy the conditions:

(i) *$X = \bigcup_{i \in I} \hat{X}_i$;*

(ii) *if $i < j$ in I then $\hat{e}_{ij} \circ s_i|_{V_{ij}} = e_{ij}^*(s_j) + O(s_i^2)$, $\psi_i(X_i \cap V_{ij}) = \hat{X}_i \cap \hat{X}_j$, and $\psi_i|_{X_i \cap V_{ij}} = \psi_j \circ e_{ij}|_{X_i \cap V_{ij}}$, and if $v_i \in V_{ij}$ with $s_i(v_i) = 0$ and $v_j = e_{ij}(v_i)$ then the following is exact:*

$$0 \longrightarrow T_{v_i} V_i \xrightarrow{\mathrm{d}s_i(v_i) \oplus \mathrm{d}e_{ij}(v_i)} E_i|_{v_i} \oplus T_{v_j} V_j \xrightarrow{\hat{e}_{ij}(v_i) \oplus -\mathrm{d}s_j(v_j)} E_j|_{v_j} \longrightarrow 0;$$

(iii) *if $i < j < k$ in I then*

$$e_{ik}|_{V_{ij} \cap V_{ik}} = e_{jk} \circ e_{ij}|_{V_{ij} \cap V_{ik}} + O(s_i^2) \qquad and$$
$$\hat{e}_{ik}|_{V_{ij} \cap V_{ik}} = e_{ij}|_{V_{ij} \cap V_{ik}}^*(\hat{e}_{jk}) \circ \hat{e}_{ij}|_{V_{ij} \cap V_{ik}} + O(s_i).$$

Then there exist a d-manifold $\boldsymbol{X}$ with $\operatorname{vdim} \boldsymbol{X} = n$ and underlying topological space X, and a 1-morphism $\boldsymbol{\psi}_i : \boldsymbol{S}_{V_i,E_i,s_i} \to \boldsymbol{X}$ with underlying continuous map ψ_i, which is an equivalence with the open d-submanifold $\hat{\boldsymbol{X}}_i \subseteq \boldsymbol{X}$ corresponding to $\hat{X}_i \subseteq X$ for all $i \in I$, such that for all $i < j$ in I there exists a 2-morphism $\eta_{ij} : \boldsymbol{\psi}_j \circ \boldsymbol{S}_{e_{ij},\hat{e}_{ij}} \Rightarrow \boldsymbol{\psi}_i \circ \boldsymbol{i}_{V_{ij},V_i}$, where $\boldsymbol{S}_{e_{ij},\hat{e}_{ij}} : \boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}} \to$ $\boldsymbol{S}_{V_j,E_j,s_j}$ and $\boldsymbol{i}_{V_{ij},V_i} : \boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}} \to \boldsymbol{S}_{V_i,E_i,s_i}$ are as in Example 4.4. This d-manifold $\boldsymbol{X}$ is unique up to equivalence in $\mathbf{dMan}$.

Suppose also that Y is a manifold, and $g_i : V_i \to Y$ are smooth maps for all $i \in I$, and $g_j \circ e_{ij} = g_i|_{V_{ij}} + O(s_i)$ for all $i < j$ in I. Then there exist a 1-morphism $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Y}$ unique up to 2-isomorphism, where $\boldsymbol{Y} = F_{\mathbf{Man}}^{\mathbf{dMan}}(Y) =$ $\boldsymbol{S}_{Y,0,0}$, and 2-morphisms $\zeta_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow \boldsymbol{S}_{g_i,0}$ for all $i \in I$. Here $\boldsymbol{S}_{Y,0,0}$ is from Example 4.4 with vector bundle E and section s both zero, and $\boldsymbol{S}_{g_i,0} :$ $\boldsymbol{S}_{V_i,E_i,s_i} \to \boldsymbol{S}_{Y,0,0} = \boldsymbol{Y}$ is from Example 4.5 with $\hat{g}_i = 0$.

The hypotheses of Theorem 4.17 are similar to the notion of *good coordinate system* in the theory of Kuranishi spaces of Fukaya and Ono [20, Def. 6.1], as discussed in §11.9. The importance of Theorem 4.17 is that all the ingredients are described wholly in differential-geometric or topological terms. So we can use the theorem as a tool to prove the existence of d-manifold structures on spaces coming from other areas of geometry, for instance, on moduli spaces.

4.5   Submersions, immersions and embeddings

Let $f : X \to Y$ be a smooth map of manifolds. Then $\mathrm{d}f^* : f^*(T^*Y) \to T^*X$ is a morphism of vector bundles on X, and f is a *submersion* if $\mathrm{d}f^*$ is injective, and f is an *immersion* if $\mathrm{d}f^*$ is surjective. Here the appropriate notions of injective and surjective for morphisms of vector bundles are stronger than the corresponding notions for sheaves: $\mathrm{d}f^*$ is *injective* if it has a left inverse, and *surjective* if it has a right inverse.

In a similar way, if $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism of d-manifolds, we would like to define $\boldsymbol{f}$ to be a submersion or immersion if the 1-morphism $\Omega_{\boldsymbol{f}} : \underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ in $\mathrm{vvect}(\underline{X})$ is injective or surjective in some suitable sense. It turns out that there are two different notions of injective and surjective 1-morphisms in the 2-category $\mathrm{vvect}(\underline{X})$, a weak and a strong:

Definition 4.18. Let $\underline{X}$ be a C^∞-scheme, $(\mathcal{E}^1, \mathcal{E}^2, \phi)$ and $(\mathcal{F}^1, \mathcal{F}^2, \psi)$ be virtual vector bundles on $\underline{X}$, and $(f^1, f^2) : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ be a 1-morphism in $\mathrm{vvect}(\underline{X})$. Then we have a complex in $\mathrm{qcoh}(\underline{X})$:

$$0 \longrightarrow \mathcal{E}^1 \underset{\gamma}{\overset{f^1 \oplus -\phi}{\rightleftarrows}} \mathcal{F}^1 \oplus \mathcal{E}^2 \underset{\delta}{\overset{\psi \oplus f^2}{\rightleftarrows}} \mathcal{F}^2 \longrightarrow 0. \qquad (4.5)$$

One can show that $f^\bullet$ is an equivalence in $\mathrm{vvect}(\underline{X})$ if and only if (4.5) is a *split short exact sequence* in $\mathrm{qcoh}(\underline{X})$. That is, $f^\bullet$ is an equivalence if and only if there exist morphisms γ, δ as shown in (4.5) satisfying the conditions:

$$\begin{array}{cc} \gamma \circ \delta = 0, & \gamma \circ (f^1 \oplus -\phi) = \mathrm{id}_{\mathcal{E}^1}, \\ (f^1 \oplus -\phi) \circ \gamma + \delta \circ (\psi \oplus f^2) = \mathrm{id}_{\mathcal{F}^1 \oplus \mathcal{E}^2}, & (\psi \oplus f^2) \circ \delta = \mathrm{id}_{\mathcal{F}^2}. \end{array} \qquad (4.6)$$

Our notions of $f^\bullet$ injective or surjective impose some but not all of (4.6):

(a) We call $f^\bullet$ *weakly injective* if there exists $\gamma : \mathcal{F}^1 \oplus \mathcal{E}^2 \to \mathcal{E}^1$ in $\mathrm{qcoh}(\underline{X})$ with $\gamma \circ (f^1 \oplus -\phi) = \mathrm{id}_{\mathcal{E}^1}$.

(b) We call $f^\bullet$ *injective* if there exist $\gamma : \mathcal{F}^1 \oplus \mathcal{E}^2 \to \mathcal{E}^1$ and $\delta : \mathcal{F}^2 \to \mathcal{F}^1 \oplus \mathcal{E}^2$ with $\gamma \circ \delta = 0$, $\gamma \circ (f^1 \oplus -\phi) = \mathrm{id}_{\mathcal{E}^1}$ and $(f^1 \oplus -\phi) \circ \gamma + \delta \circ (\psi \oplus f^2) = \mathrm{id}_{\mathcal{F}^1 \oplus \mathcal{E}^2}$.

(c) We call $f^\bullet$ *weakly surjective* if there exists $\delta : \mathcal{F}^2 \to \mathcal{F}^1 \oplus \mathcal{E}^2$ in $\mathrm{qcoh}(\underline{X})$ with $(\psi \oplus f^2) \circ \delta = \mathrm{id}_{\mathcal{F}^2}$.

(d) We call $f^\bullet$ *surjective* if there exist $\gamma : \mathcal{F}^1 \oplus \mathcal{E}^2 \to \mathcal{E}^1$ and $\delta : \mathcal{F}^2 \to \mathcal{F}^1 \oplus \mathcal{E}^2$ with $\gamma \circ \delta = 0$, $\gamma \circ (f^1 \oplus -\phi) = \mathrm{id}_{\mathcal{E}^1}$ and $(\psi \oplus f^2) \circ \delta = \mathrm{id}_{\mathcal{F}^2}$.

If $\underline{X}$ is separated, paracompact, and locally fair, these are local conditions on $\underline{X}$.

Using these we define weak and strong forms of submersions, immersions, and embeddings for d-manifolds.

Definition 4.19. Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism of d-manifolds. Definition 4.10 defines a 1-morphism $\Omega_{\boldsymbol{f}} : \underline{f}^*(T^*\boldsymbol{Y}) \to T^*\boldsymbol{X}$ in vvect($\underline{X}$). Then:

- (a) We call $\boldsymbol{f}$ a *w-submersion* if $\Omega_{\boldsymbol{f}}$ is weakly injective.
- (b) We call $\boldsymbol{f}$ a *submersion* if $\Omega_{\boldsymbol{f}}$ is injective.
- (c) We call $\boldsymbol{f}$ a *w-immersion* if $\Omega_{\boldsymbol{f}}$ is weakly surjective.
- (d) We call $\boldsymbol{f}$ an *immersion* if $\Omega_{\boldsymbol{f}}$ is surjective.
- (e) We call $\boldsymbol{f}$ a *w-embedding* if it is a w-immersion and $f : X \to f(X)$ is a homeomorphism, so in particular f is injective.
- (f) We call $\boldsymbol{f}$ an *embedding* if it is an immersion and f is a homeomorphism with its image.

Here w-submersion is short for *weak submersion*, etc. Conditions (a)–(d) all concern the existence of morphisms γ, δ in the next equation satisfying identities:

$$0 \longrightarrow \underline{f}^*(\mathcal{E}_Y) \underset{\gamma}{\overset{f'' \oplus -\underline{f}^*(\phi_Y)}{\rightleftarrows}} \mathcal{E}_X \oplus \underline{f}^*(\mathcal{F}_Y) \underset{\delta}{\overset{\phi_X \oplus f^2}{\rightleftarrows}} \mathcal{F}_X \longrightarrow 0.$$

Parts (c)–(f) enable us to define *d-submanifolds* of d-manifolds. *Open d-submanifolds* are open d-subspaces of a d-manifold. More generally, we call $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{Y}$ a *w-immersed*, or *immersed*, or *w-embedded*, or *embedded d-submanifold* of $\boldsymbol{Y}$, if $\boldsymbol{X}, \boldsymbol{Y}$ are d-manifolds and $\boldsymbol{i}$ is a w-immersion, immersion, w-embedding, or embedding, respectively.

Here are some properties of these, taken from [35, §4.1–§4.2]:

Theorem 4.20. (i) *Any equivalence of d-manifolds is a w-submersion, submersion, w-immersion, immersion, w-embedding and embedding.*

(ii) *If $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ are 2-isomorphic 1-morphisms of d-manifolds then $\boldsymbol{f}$ is a w-submersion, submersion, ..., embedding, if and only if $\boldsymbol{g}$ is.*

(iii) *Compositions of w-submersions, submersions, w-immersions, immersions, w-embeddings, and embeddings are 1-morphisms of the same kind.*

(iv) *The conditions that a 1-morphism of d-manifolds $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a w-submersion, submersion, w-immersion or immersion are local in $\boldsymbol{X}$ and $\boldsymbol{Y}$. That is, for each $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$, it suffices to check the conditions for $\boldsymbol{f}|_{\boldsymbol{U}} : \boldsymbol{U} \to \boldsymbol{V}$ with $\boldsymbol{V}$ an open neighbourhood of y in $\boldsymbol{Y}$, and $\boldsymbol{U}$ an open neighbourhood of x in $\boldsymbol{f}^{-1}(\boldsymbol{V}) \subseteq \boldsymbol{X}$. The conditions that $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a w-embedding or embedding are local in $\boldsymbol{Y}$, but not in $\boldsymbol{X}$.*

(v) *Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a submersion of d-manifolds. Then $\operatorname{vdim}\boldsymbol{X} \geqslant \operatorname{vdim}\boldsymbol{Y}$, and if $\operatorname{vdim}\boldsymbol{X} = \operatorname{vdim}\boldsymbol{Y}$ then $\boldsymbol{f}$ is étale.*

(vi) *Let $f : X \to Y$ be an immersion of d-manifolds. Then $\operatorname{vdim} X \leqslant \operatorname{vdim} Y$, and if $\operatorname{vdim} X = \operatorname{vdim} Y$ then f is étale.*

(vii) *Let $f : X \to Y$ be a smooth map of manifolds, and $f = F^{\mathbf{dMan}}_{\mathbf{Man}}(f)$. Then f is a submersion, immersion, or embedding in $\mathbf{dMan}$ if and only if f is a submersion, immersion, or embedding in $\mathbf{Man}$, respectively. Also f is a w-immersion or w-embedding if and only if f is an immersion or embedding.*

(viii) *Let $f : X \to Y$ be a 1-morphism of d-manifolds, with Y a manifold. Then f is a w-submersion.*

(ix) *Let X, Y be d-manifolds, with Y a manifold. Then $\pi_X : X \times Y \to X$ is a submersion.*

(x) *Let $f : X \to Y$ be a submersion of d-manifolds, and $x \in X$ with $f(x) = y \in Y$. Then there exist open $x \in U \subseteq X$ and $y \in V \subseteq Y$ with $f(U) = V$, a manifold Z, and an equivalence $i : U \to V \times Z$, such that $f|_U : U \to V$ is 2-isomorphic to $\pi_V \circ i$, where $\pi_V : V \times Z \to V$ is the projection.*

(xi) *Let $f : X \to Y$ be a submersion of d-manifolds with Y a manifold. Then X is a manifold.*

4.6 D-transversality and fibre products

From §3.3, if $g : X \to Z$ and $h : Y \to Z$ are 1-morphisms of d-manifolds then a fibre product $W = X_{g,Z,h} Y$ exists in $\mathbf{dSpa}$, and is unique up to equivalence. We want to know whether W is a d-manifold. We will define when g, h are *d-transverse*, which is a sufficient condition for W to be a d-manifold.

Recall that if $g : X \to Z$, $h : Y \to Z$ are smooth maps of manifolds, then a fibre product $W = X \times_{g,Z,h} Y$ in $\mathbf{Man}$ exists if g, h are *transverse*, that is, if $T_z Z = \mathrm{d}g|_x(T_x X) + \mathrm{d}h|_y(T_y Y)$ for all $x \in X$ and $y \in Y$ with $g(x) = h(y) = z \in Z$. Equivalently, $\mathrm{d}g|_x^* \oplus \mathrm{d}h|_y^* : T_z Z^* \to T_x^* X \oplus T_y^* Y$ should be injective. Writing $W = X \times_Z Y$ for the topological fibre product and $e : W \to X$, $f : W \to Y$ for the projections, with $g \circ e = h \circ f$, we see that g, h are transverse if and only if

$$e^*(\mathrm{d}g^*) \oplus f^*(\mathrm{d}h^*) : (g \circ e)^*(T^* Z) \to e^*(T^* X) \oplus f^*(T^* Y) \qquad (4.7)$$

is an injective morphism of vector bundles on the topological space W, that is, it has a left inverse. The condition that (4.8) has a left inverse is an analogue of this, but on (dual) obstruction rather than cotangent bundles.

Definition 4.21. Let X, Y, Z be d-manifolds and $g : X \to Z$, $h : Y \to Z$ be 1-morphisms. Let $\underline{W} = \underline{X} \times_{g,\underline{Z},h} \underline{Y}$ be the C^∞-scheme fibre product, and write $\underline{e} : \underline{W} \to \underline{X}$, $\underline{f} : \underline{W} \to \underline{Y}$ for the projections. Consider the morphism

$$\alpha = \begin{pmatrix} \underline{e}^*(g'') \circ I_{\underline{e},g}(\mathcal{E}_Z) \\ -\underline{f}^*(h'') \circ I_{\underline{f},h}(\mathcal{E}_Z) \\ (g \circ \underline{e})^*(\phi_Z) \end{pmatrix} : (g \circ \underline{e})^*(\mathcal{E}_Z) \longrightarrow \\ \qquad\qquad \underline{e}^*(\mathcal{E}_X) \oplus \underline{f}^*(\mathcal{E}_Y) \oplus (g \circ \underline{e})^*(\mathcal{F}_Z) \qquad (4.8)$$

in $\operatorname{qcoh}(\underline{W})$. We call g, h *d-transverse* if α has a left inverse. Note that this is a local condition in $\underline{W}$, since local choices of left inverse for α can be combined using a partition of unity on $\underline{W}$ to make a global left inverse.

In the notation of §4.3 and §4.5, we have 1-morphisms $\Omega_{\boldsymbol{g}} : \boldsymbol{g}^*(T^*\boldsymbol{Z}) \to T^*\boldsymbol{X}$ in vvect($\underline{X}$) and $\Omega_{\boldsymbol{h}} : \underline{h}^*(T^*\boldsymbol{Z}) \to T^*\boldsymbol{Y}$ in vvect($\underline{Y}$). Pulling these back to vvect($\underline{W}$) using $\underline{e}^*, \underline{f}^*$ we form the 1-morphism in vvect($\underline{W}$):

$$
\left(\underline{e}^*(\Omega_{\boldsymbol{g}}) \circ I_{\underline{e},g}(T^*\boldsymbol{Z})\right) \oplus \left(\underline{f}^*(\Omega_{\boldsymbol{h}}) \circ I_{\underline{f},\underline{h}}(T^*\boldsymbol{Z})\right) : (g \circ \underline{e})^*(T^*\boldsymbol{Z})
$$
$$
\longrightarrow \underline{e}^*(T^*\boldsymbol{X}) \oplus \underline{f}^*(T^*\boldsymbol{Y}). \tag{4.9}
$$

For (4.8) to have a left inverse is equivalent to (4.9) being weakly injective, as in Definition 4.18. This is the d-manifold analogue of (4.7) being injective.

Here are the main results of [35, §4.3]:

Theorem 4.22. *Suppose* $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ *are d-manifolds and* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ *are d-transverse* 1-*morphisms, and let* $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ *be the d-space fibre product. Then* $\boldsymbol{W}$ *is a d-manifold, with*

$$
\mathrm{vdim}\, \boldsymbol{W} = \mathrm{vdim}\, \boldsymbol{X} + \mathrm{vdim}\, \boldsymbol{Y} - \mathrm{vdim}\, \boldsymbol{Z}. \tag{4.10}
$$

Theorem 4.23. *Suppose* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ *are* 1-*morphisms of d-manifolds. The following are sufficient conditions for* $\boldsymbol{g}, \boldsymbol{h}$ *to be d-transverse, so that* $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ *is a d-manifold of virtual dimension* (4.10):

(a) $\boldsymbol{Z}$ *is a manifold, that is,* $\boldsymbol{Z} \in \hat{\mathbf{Man}}$; *or*

(b) $\boldsymbol{g}$ *or* $\boldsymbol{h}$ *is a w-submersion.*

The point here is that roughly speaking, $\boldsymbol{g}, \boldsymbol{h}$ are d-transverse if they map the direct sum of the obstruction spaces of $\boldsymbol{X}, \boldsymbol{Y}$ surjectively onto the obstruction spaces of $\boldsymbol{Z}$. If $\boldsymbol{Z}$ is a manifold its obstruction spaces are zero. If $\boldsymbol{g}$ is a w-submersion it maps the obstruction spaces of $\boldsymbol{X}$ surjectively onto the obstruction spaces of $\boldsymbol{Z}$. In both cases, d-transversality follows. See [53, Th. 8.15] for the analogue of Theorem 4.23(a) for Spivak's derived manifolds.

Theorem 4.24. *Let* $\boldsymbol{X}, \boldsymbol{Z}$ *be d-manifolds,* $\boldsymbol{Y}$ *a manifold, and* $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ *be* 1-*morphisms with* $\boldsymbol{g}$ *a submersion. Then* $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ *is a manifold, with* $\dim \boldsymbol{W} = \mathrm{vdim}\, \boldsymbol{X} + \dim \boldsymbol{Y} - \mathrm{vdim}\, \boldsymbol{Z}$.

Theorem 4.24 shows that we may think of submersions as 'representable 1-morphisms' in $\mathbf{dMan}$. We can locally characterize embeddings and immersions in $\mathbf{dMan}$ in terms of fibre products with $\mathbb{R}^n$ in $\mathbf{dMan}$.

Theorem 4.25. (i) *Let* $\boldsymbol{X}$ *be a d-manifold and* $\boldsymbol{g} : \boldsymbol{X} \to \mathbb{R}^n$ *a* 1-*morphism in* $\mathbf{dMan}$. *Then the fibre product* $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\mathbb{R}^n,\boldsymbol{0}} *$ *exists in* $\mathbf{dMan}$ *by Theorem* 4.23(a)*, and the projection* $\boldsymbol{\pi}_{\boldsymbol{X}} : \boldsymbol{W} \to \boldsymbol{X}$ *is an embedding.*

(ii) *Suppose* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *is an immersion of d-manifolds, and* $x \in \boldsymbol{X}$ *with* $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$. *Then there exist open d-submanifolds* $x \in \boldsymbol{U} \subseteq \boldsymbol{X}$ *and* $y \in$

$V \subseteq Y$ with $f(U) \subseteq V$, and a 1-morphism $g : V \to \mathbb{R}^n$ with $g(y) = 0$, where $n = \text{vdim}\, Y - \text{vdim}\, X \geqslant 0$, fitting into a 2-Cartesian square in **dMan** :

$$
\begin{array}{ccc}
U & \xrightarrow{\hspace{3cm}} & * \\
{\scriptstyle f|_U}\downarrow & \Uparrow_{\;g} & \downarrow{\scriptstyle 0} \\
V & \xrightarrow{\hspace{3cm}g} & \mathbb{R}^n.
\end{array}
$$

If f is an embedding we may take $U = f^{-1}(V)$.

Remark 4.26. For the applications the author has in mind, it will be crucial that if $g : X \to Z$ and $h : Y \to Z$ are 1-morphisms with X, Y d-manifolds and Z a manifold then $W = X \times_Z Y$ is a d-manifold, with $\text{vdim}\, W = \text{vdim}\, X + \text{vdim}\, Y - \dim Z$, as in Theorem 4.23(a). We will show by example, following Spivak [53, Prop. 1.7], that if d-manifolds **dMan** were an ordinary category containing manifolds as a full subcategory, then this would be false.

Consider the fibre product $* \times_{0,\mathbb{R},0} *$ in **dMan**. If **dMan** were a category then as $*$ is a terminal object, the fibre product would be $*$. But then

$$\text{vdim}(* \times_{0,\mathbb{R},0} *) = \text{vdim}\, * = 0 \neq -1 = \text{vdim}\, * + \text{vdim}\, * - \text{vdim}\, \mathbb{R},$$

so equation (4.10) and Theorem 4.23(a) would be false. Thus, if we want fibre products of d-manifolds over manifolds to be well behaved, then **dMan** must be at least a 2-category. It could be an ∞-category, as for Spivak's derived manifolds [53], or some other kind of higher category. Making d-manifolds into a 2-category, as we have done, is the simplest of the available options.

4.7 Embedding d-manifolds into manifolds

Let V be a manifold, $E \to V$ a vector bundle, and $s \in C^\infty(E)$. Then Example 4.4 defines a 'standard model' principal d-manifold $S_{V,E,s}$. When E and s are zero, we have $S_{V,0,0} = V = F_{\mathbf{Man}}^{\mathbf{dMan}}(V)$, so that $S_{V,0,0}$ is a manifold. For general V, E, s, taking $f = \text{id}_V : V \to V$ and $\hat{f} = 0 : E \to 0$ in Example 4.5 gives a 'standard model' 1-morphism $S_{\text{id}_V,0} : S_{V,E,s} \to S_{V,0,0} = V$. One can show $S_{\text{id}_V,0}$ is an embedding, in the sense of Definition 4.19. Any principal d-manifold U is equivalent to some $S_{V,E,s}$. Thus we deduce:

Lemma 4.27. *Any principal d-manifold U admits an embedding $i : U \to V$ into a manifold V.*

Theorem 4.32 below is a converse to this: if a d-manifold X can be embedded into a manifold Y, then X is principal. So it will be useful to study embeddings of d-manifolds into manifolds. The following classical facts are due to Whitney [60].

Theorem 4.28. (a) *Let X be an m-manifold and $n \geqslant 2m$. Then a generic smooth map $f : X \to \mathbb{R}^n$ is an immersion.*
(b) *Let X be an m-manifold and $n \geqslant 2m + 1$. Then there exists an embedding $f : X \to \mathbb{R}^n$, and we can choose such f with $f(X)$ closed in $\mathbb{R}^n$. Generic smooth maps $f : X \to \mathbb{R}^n$ are embeddings.*

In [35, §4.4] we generalize Theorem 4.28 to d-manifolds.

Theorem 4.29. *Let X be a d-manifold. Then there exist immersions and/or embeddings $f : X \to \mathbb{R}^n$ for some $n \gg 0$ if and only if there is an upper bound for $\dim T_x^* \underline{X}$ for all $x \in \underline{X}$. If there is such an upper bound, then immersions $f : X \to \mathbb{R}^n$ exist provided $n \geqslant 2 \dim T_x^* \underline{X}$ for all $x \in \underline{X}$, and embeddings $f : X \to \mathbb{R}^n$ exist provided $n \geqslant 2 \dim T_x^* \underline{X} + 1$ for all $x \in \underline{X}$. For embeddings we may also choose f with $f(X)$ closed in $\mathbb{R}^n$.*

Here is an example in which the condition does not hold.

Example 4.30. $\mathbb{R}^k \times_{\mathbf{0}, \mathbb{R}^k, \mathbf{0}} *$ is a principal d-manifold of virtual dimension 0, with C^∞-scheme $\underline{\mathbb{R}}^k$, and obstruction bundle $\mathbb{R}^k$. Thus $X = \coprod_{k \geqslant 0} \mathbb{R}^k \times_{\mathbf{0}, \mathbb{R}^k, \mathbf{0}} *$ is a d-manifold of virtual dimension 0, with C^∞-scheme $\underline{X} = \coprod_{k \geqslant 0} \underline{\mathbb{R}}^k$. Since $T_x^* \underline{X} \cong \mathbb{R}^n$ for $x \in \mathbb{R}^n \subset \coprod_{k \geqslant 0} \mathbb{R}^k$, $\dim T_x^* \underline{X}$ realizes all values $n \geqslant 0$. Hence there cannot exist immersions or embeddings $f : X \to \mathbb{R}^n$ for any $n \geqslant 0$.

As $x \mapsto \dim T_x^* \underline{X}$ is an upper semicontinuous map $X \to \mathbb{N}$, if X is compact then $\dim T_x^* \underline{X}$ is bounded above, giving:

Corollary 4.31. *Let X be a compact d-manifold. Then there exists an embedding $f : X \to \mathbb{R}^n$ for some $n \gg 0$.*

If a d-manifold X can be embedded into a manifold Y, we show in [35, §4.4] that we can write X as the zeroes of a section of a vector bundle over Y near its image. See [53, Prop. 9.5] for the analogue for Spivak's derived manifolds.

Theorem 4.32. *Suppose X is a d-manifold, Y a manifold, and $f : X \to Y$ an embedding, in the sense of Definition 4.19. Then there exist an open subset V in Y with $f(X) \subseteq V$, a vector bundle $E \to V$, and $s \in C^\infty(E)$ fitting into a 2-Cartesian diagram in $\mathbf{dSpa}$:*

$$
\begin{array}{ccc}
X & \longrightarrow & V \\
{\scriptstyle f}\downarrow \quad {\scriptstyle f} \quad {\scriptstyle \eta}\Uparrow & & {\scriptstyle \mathbf{0}}\downarrow \\
V & \xrightarrow{\;\;s\;\;} & E.
\end{array}
$$

Here $Y = F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$, and similarly for $V, E, s, \mathbf{0}$, with $\mathbf{0} : V \to E$ the zero section. Hence X is equivalent to the 'standard model' d-manifold $S_{V,E,s}$ of Example 4.4, and is a principal d-manifold.

Combining Theorems 4.29 and 4.32, Lemma 4.27, and Corollary 4.31 yields:

Corollary 4.33. *Let X be a d-manifold. Then X is a principal d-manifold if and only if $\dim T_x^* \underline{X}$ is bounded above for all $x \in \underline{X}$. In particular, if X is compact, then X is principal.*

Corollary 4.33 suggests that most interesting d-manifolds are principal, in a similar way to most interesting C^∞-schemes being affine in Remark 2.9(ii). Example 4.30 gives a d-manifold which is not principal.

4.8 Orientations on d-manifolds

Let X be an n-manifold. Then T^*X is a rank n vector bundle on X, so its top exterior power $\Lambda^n T^*X$ is a line bundle (rank 1 vector bundle) on X. In algebraic geometry, $\Lambda^n T^*X$ would be called the canonical bundle of X. We define an *orientation* ω on X to be an *orientation on the fibres of* $\Lambda^n T^*X$. That is, ω is an equivalence class $[\tau]$ of isomorphisms of line bundles $\tau : O_X \to \Lambda^n T^*X$, where O_X is the trivial line bundle $\mathbb{R} \times X \to X$, and τ, τ' are equivalent if $\tau' = \tau \cdot c$ for some smooth $c : X \to (0, \infty)$.

To generalize all this to d-manifolds, we will need a notion of the 'top exterior power' $\mathcal{L}_{(\mathcal{E}^\bullet, \phi)}$ of a virtual vector bundle $(\mathcal{E}^\bullet, \phi)$ in §4.3. As the definition in [35, §4.5] is long, we will not give it, but just state its important properties:

Theorem 4.34. *Let $\underline{X}$ be a C^∞-scheme, and $(\mathcal{E}^\bullet, \phi)$ a virtual vector bundle on $\underline{X}$. Then in [35, §4.5] we define a line bundle (rank 1 vector bundle) $\mathcal{L}_{(\mathcal{E}^\bullet, \phi)}$ on $\underline{X}$, which we call the **orientation line bundle** of $(\mathcal{E}^\bullet, \phi)$. This satisfies:*

(a) *Suppose $\mathcal{E}^1, \mathcal{E}^2$ are vector bundles on $\underline{X}$ with ranks k_1, k_2, and $\phi : \mathcal{E}^1 \to \mathcal{E}^2$ is a morphism. Then $(\mathcal{E}^\bullet, \phi)$ is a virtual vector bundle of rank $k_2 - k_1$, and there is a canonical isomorphism $\mathcal{L}_{(\mathcal{E}^\bullet, \phi)} \cong \Lambda^{k_1}(\mathcal{E}^1)^* \otimes \Lambda^{k_2} \mathcal{E}^2$.*

(b) *Let $f^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ be an equivalence in $\mathrm{vvect}(\underline{X})$. Then there is a canonical isomorphism $\mathcal{L}_{f^\bullet} : \mathcal{L}_{(\mathcal{E}^\bullet, \phi)} \to \mathcal{L}_{(\mathcal{F}^\bullet, \psi)}$ in $\mathrm{qcoh}(\underline{X})$.*

(c) *If $(\mathcal{E}^\bullet, \phi) \in \mathrm{vvect}(\underline{X})$ then $\mathcal{L}_{\mathrm{id}_\phi} = \mathrm{id}_{\mathcal{L}_{(\mathcal{E}^\bullet, \phi)}} : \mathcal{L}_{(\mathcal{E}^\bullet, \phi)} \to \mathcal{L}_{(\mathcal{E}^\bullet, \phi)}$.*

(d) *If $f^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ and $g^\bullet : (\mathcal{F}^\bullet, \psi) \to (\mathcal{G}^\bullet, \xi)$ are equivalences in $\mathrm{vvect}(\underline{X})$ then $\mathcal{L}_{g^\bullet \circ f^\bullet} = \mathcal{L}_{g^\bullet} \circ \mathcal{L}_{f^\bullet} : \mathcal{L}_{(\mathcal{E}^\bullet, \phi)} \to \mathcal{L}_{(\mathcal{G}^\bullet, \xi)}$.*

(e) *If $f^\bullet, g^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ are 2-isomorphic equivalences in $\mathrm{vvect}(\underline{X})$ then $\mathcal{L}_{f^\bullet} = \mathcal{L}_{g^\bullet} : \mathcal{L}_{(\mathcal{E}^\bullet, \phi)} \to \mathcal{L}_{(\mathcal{F}^\bullet, \psi)}$.*

(f) *Let $\underline{f} : \underline{X} \to \underline{Y}$ be a morphism of C^∞-schemes, and $(\mathcal{E}^\bullet, \phi) \in \mathrm{vvect}(\underline{Y})$. Then there is a canonical isomorphism $I_{\underline{f}, (\mathcal{E}^\bullet, \phi)} : \underline{f}^*(\mathcal{L}_{(\mathcal{E}^\bullet, \phi)}) \to \mathcal{L}_{\underline{f}^*(\mathcal{E}^\bullet, \phi)}$.*

Now we can define orientations on d-manifolds.

Definition 4.35. Let $\boldsymbol{X}$ be a d-manifold. Then the virtual cotangent bundle $T^*\boldsymbol{X}$ is a virtual vector bundle on $\underline{X}$ by Proposition 4.11(b), so Theorem 4.34 gives a line bundle $\mathcal{L}_{T^*\boldsymbol{X}}$ on $\underline{X}$. We call $\mathcal{L}_{T^*\boldsymbol{X}}$ the *orientation line bundle* of $\boldsymbol{X}$.

An *orientation* ω on $\boldsymbol{X}$ is an orientation on $\mathcal{L}_{T^*\boldsymbol{X}}$. That is, ω is an equivalence class $[\tau]$ of isomorphisms $\tau : \mathcal{O}_X \to \mathcal{L}_{T^*\boldsymbol{X}}$ in $\mathrm{qcoh}(\underline{X})$, where τ, τ' are equivalent if they are proportional by a smooth positive function on $\underline{X}$.

If $\omega = [\tau]$ is an orientation on $\boldsymbol{X}$, the *opposite orientation* is $-\omega = [-\tau]$, which changes the sign of the isomorphism $\tau : \mathcal{O}_X \to \mathcal{L}_{T^*\boldsymbol{X}}$. When we refer to $\boldsymbol{X}$ as an oriented d-manifold, $-\boldsymbol{X}$ will mean $\boldsymbol{X}$ with the opposite orientation, that is, $\boldsymbol{X}$ is short for $(\boldsymbol{X}, \omega)$ and $-\boldsymbol{X}$ is short for $(\boldsymbol{X}, -\omega)$.

Example 4.36. (a) Let X be an n-manifold, and $\boldsymbol{X} = F_{\mathbf{Man}}^{\mathbf{dMan}}(X)$ the associated d-manifold. Then $\underline{X} = F_{\mathbf{Man}}^{\mathbf{C}^\infty\mathbf{Sch}}(X)$, $\mathcal{E}_X = 0$ and $\mathcal{F}_X = T^*\underline{X}$. So $\mathcal{E}_X, \mathcal{F}_X$ are vector bundles of ranks $0, n$. As $\Lambda^0 \mathcal{E}_X \cong \mathcal{O}_X$, Theorem 4.34(a) gives a

canonical isomorphism $\mathcal{L}_{T^*\boldsymbol{X}} \cong \Lambda^n T^*\underline{X}$. That is, $\mathcal{L}_{T^*\boldsymbol{X}}$ is isomorphic to the lift to C^∞-schemes of the line bundle $\Lambda^n T^*X$ on the manifold X.

As above, an orientation on X is an orientation on the line bundle $\Lambda^n T^*X$. Hence orientations on the d-manifold $\boldsymbol{X} = F_{\mathbf{Man}}^{\mathbf{dMan}}(X)$ in the sense of Definition 4.35 are equivalent to orientations on the manifold X in the usual sense.

(b) Let V be an n-manifold, $E \to V$ a vector bundle of rank k, and $s \in C^\infty(E)$. Then Example 4.4 defines a 'standard model' principal d-manifold $\boldsymbol{S} = \boldsymbol{S}_{V,E,s}$, which has $\mathcal{E}_S \cong \mathcal{E}^*|_{\underline{S}}$, $\mathcal{F}_S \cong T^*\underline{V}|_{\underline{S}}$, where $\mathcal{E}, T^*\underline{V}$ are the lifts of the vector bundles E, T^*V on V to $\underline{V}$. Hence $\mathcal{E}_S, \mathcal{F}_S$ are vector bundles on $\underline{S}_{V,E,s}$ of ranks k, n, so Theorem 4.34(a) gives an isomorphism $\mathcal{L}_{T^*\boldsymbol{S}_{V,E,s}} \cong (\Lambda^k \mathcal{E} \otimes \Lambda^n T^*\underline{V})|_{\underline{S}}$.

Thus $\mathcal{L}_{T^*\boldsymbol{S}_{V,E,s}}$ is the lift to $\underline{S}_{V,E,s}$ of the line bundle $\Lambda^k E \otimes \Lambda^n T^*V$ over the manifold V. Therefore we may induce an orientation on the d-manifold $\boldsymbol{S}_{V,E,s}$ from an orientation on the line bundle $\Lambda^k E \otimes \Lambda^n T^*V$ over V. Equivalently, we can induce an orientation on $\boldsymbol{S}_{V,E,s}$ from an orientation on the total space of the vector bundle E^* over V, or from an orientation on the total space of E.

We can construct orientations on d-transverse fibre products of oriented d-manifolds. Note that (4.11) depends on an *orientation convention*: a different choice would change (4.11) by a sign depending on $\mathrm{vdim}\,\boldsymbol{X}, \mathrm{vdim}\,\boldsymbol{Y}, \mathrm{vdim}\,\boldsymbol{Z}$. Our conventions follow those of Fukaya et al. [19, §8.2] for Kuranishi spaces.

Theorem 4.37. *Work in the situation of Theorem 4.22, so that $\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ are d-manifolds with $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ for $\boldsymbol{g}, \boldsymbol{h}$ d-transverse, where $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f} : \boldsymbol{W} \to \boldsymbol{Y}$ are the projections. Then we have orientation line bundles $\mathcal{L}_{T^*\boldsymbol{W}}, \ldots, \mathcal{L}_{T^*\boldsymbol{Z}}$ on $\underline{W}, \ldots, \underline{Z}$, so $\mathcal{L}_{T^*\boldsymbol{W}}, \underline{\boldsymbol{e}}^*(\mathcal{L}_{T^*\boldsymbol{X}}), \underline{\boldsymbol{f}}^*(\mathcal{L}_{T^*\boldsymbol{Y}}), (\boldsymbol{g} \circ \underline{\boldsymbol{e}})^*(\mathcal{L}_{T^*\boldsymbol{Z}})$ are line bundles on $\underline{W}$. With a suitable choice of orientation convention, there is a canonical isomorphism*

$$\Phi : \mathcal{L}_{T^*\boldsymbol{W}} \longrightarrow \underline{\boldsymbol{e}}^*(\mathcal{L}_{T^*\boldsymbol{X}}) \otimes_{\mathcal{O}_W} \underline{\boldsymbol{f}}^*(\mathcal{L}_{T^*\boldsymbol{Y}}) \otimes_{\mathcal{O}_W} (\underline{\boldsymbol{g}} \circ \underline{\boldsymbol{e}})^*(\mathcal{L}_{T^*\boldsymbol{Z}})^*. \qquad (4.11)$$

Hence, if $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ are oriented d-manifolds, then $\boldsymbol{W}$ also has a natural orientation, since trivializations of $\mathcal{L}_{T^*\boldsymbol{X}}, \mathcal{L}_{T^*\boldsymbol{Y}}, \mathcal{L}_{T^*\boldsymbol{Z}}$ induce a trivialization of $\mathcal{L}_{T^*\boldsymbol{W}}$ by (4.11).

Fibre products have natural commutativity and associativity properties. When we include orientations, the orientations differ by some sign. Here is an analogue of results of Fukaya et al. [19, Lem. 8.2.3] for Kuranishi spaces.

Proposition 4.38. *Suppose $\boldsymbol{V}, \ldots, \boldsymbol{Z}$ are oriented d-manifolds, $\boldsymbol{e}, \ldots, \boldsymbol{h}$ are 1-morphisms, and all fibre products below are d-transverse. Then the following hold, in oriented d-manifolds:*

(a) *For $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ we have*

$$\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y} \simeq (-1)^{(\mathrm{vdim}\,\boldsymbol{X} - \mathrm{vdim}\,\boldsymbol{Z})(\mathrm{vdim}\,\boldsymbol{Y} - \mathrm{vdim}\,\boldsymbol{Z})} \boldsymbol{Y} \times_{\boldsymbol{h},\boldsymbol{Z},\boldsymbol{g}} \boldsymbol{X}.$$

In particular, when $\boldsymbol{Z} = *$ so that $\boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y} = \boldsymbol{X} \times \boldsymbol{Y}$ we have

$$\boldsymbol{X} \times \boldsymbol{Y} \simeq (-1)^{\mathrm{vdim}\,\boldsymbol{X}\,\mathrm{vdim}\,\boldsymbol{Y}} \boldsymbol{Y} \times \boldsymbol{X}.$$

(b) *For $e : V \to Y$, $f : W \to Y$, $g : W \to Z$, and $h : X \to Z$ we have*

$$V \times_{e,Y,f \circ \pi_W} \left(W \times_{g,Z,h} X \right) \simeq \left(V \times_{e,Y,f} W \right) \times_{g \circ \pi_W, Z, h} X.$$

(c) *For $e : V \to Y$, $f : V \to Z$, $g : W \to Y$, and $h : X \to Z$ we have*

$$V \times_{(e,f), Y \times Z, g \times h} \left(W \times X \right) \simeq$$
$$(-1)^{\mathrm{vdim}\, Z (\mathrm{vdim}\, Y + \mathrm{vdim}\, W)} \left(V \times_{e,Y,g} W \right) \times_{f \circ \pi_V, Z, h} X.$$

5 Manifolds with boundary and corners

So far we have discussed only manifolds *without boundary* (locally modelled on $\mathbb{R}^n$). One can also consider *manifolds with boundary* (locally modelled on $[0, \infty) \times \mathbb{R}^{n-1}$) and *manifolds with corners* (locally modelled on $[0, \infty)^k \times \mathbb{R}^{n-k}$). The author [32] studied manifolds with corners, giving a new definition of smooth map $f : X \to Y$ between manifolds with corners X, Y, satisfying extra conditions over $\partial^k X, \partial^l Y$. This yields categories $\mathbf{Man^b}, \mathbf{Man^c}$ of manifolds with boundary and with corners with good properties *as categories*.

In [35, Chap. 5] we surveyed [32], changing some notation, and including some new material. This section summarizes [32], [35, Chap. 5], following the notation of [35, Chap. 5]. See [32] and [35, Chap. 5] for further references on manifolds with corners.

5.1 Boundaries and smooth maps

The definition of an n-manifold with corners X in [32, §2] involves an atlas of charts (U, ϕ) on X with $U \subseteq [0, \infty)^k \times \mathbb{R}^{n-k}$ open and $\phi : U \hookrightarrow X$ a homeomorphism with an open set in X. Apart from taking $U \subseteq [0, \infty)^k \times \mathbb{R}^{n-k}$ rather than $U \subseteq \mathbb{R}^n$, there is no difference with the usual definition of n-manifold. The definitions of the *boundary* ∂X of X in [32, §2], and of *smooth map* $f : X \to Y$ between manifolds with corners in [32, §3], may be surprising for readers who have not thought much about corners, so we give them here.

Definition 5.1. Let X be a manifold with corners, of dimension n. Then there is a natural stratification $X = \coprod_{k=0}^{n} S^k(X)$, where $S^k(X)$ is the *depth k stratum* of X, that is, the set of points $x \in X$ such that X near x is locally modelled on $[0, \infty)^k \times \mathbb{R}^{n-k}$ near 0. Then $S^k(X)$ is an $(n-k)$-manifold without boundary, and $\overline{S^k(X)} = \coprod_{l=k}^{n} S^l(X)$. The *interior* of X is $X^\circ = S^0(X)$.

A *local boundary component β of X at x* is a local choice of connected component of $S^1(X)$ near x. That is, for each sufficiently small open neighbourhood V of x in X, β gives a choice of connected component W of $V \cap S^1(X)$ with $x \in \overline{W}$, and any two such choices V, W and V', W' must be compatible in the sense that $x \in \overline{(W \cap W')}$. As a set, define the *boundary*

$$\partial X = \left\{ (x, \beta) : x \in X, \ \beta \text{ is a local boundary component for } X \text{ at } x \right\}.$$

Then ∂X is an $(n-1)$-manifold with corners if $n > 0$, and $\partial X = \emptyset$ if $n = 0$.

Define a smooth map $i_X : \partial X \to X$ by $i_X : (x, \beta) \mapsto x$.

Example 5.2. The manifold with corners $X = [0, \infty)^2$ has strata $S^0(X) = (0, \infty)^2$, $S^1(X) = (\{0\} \times (0, \infty)) \amalg ((0, \infty) \times \{0\})$ and $S^2(X) = \{(0, 0)\}$. A point (a, b) in X has local boundary components $\{x = 0\}$ if $a = 0$ and $\{y = 0\}$ if $b = 0$. Thus

$$\partial X = \{((x, 0), \{y = 0\}) : x \in [0, \infty)\} \amalg \{((0, y), \{x = 0\}) : y \in [0, \infty)\}$$
$$\cong [0, \infty) \amalg [0, \infty).$$

Note that $i_X : \partial X \to X$ maps two points $((0, 0), \{x = 0\}), ((0, 0), \{y = 0\})$ to $(0, 0)$. In general, if a manifold with corners X has $\partial^2 X \neq \emptyset$ then i_X *is not injective*, so *the boundary ∂X is not a subset of X*.

Definition 5.3. Let X, Y be manifolds with corners of dimensions m, n. A continuous map $f : X \to Y$ is called *weakly smooth* if whenever $(U, \phi), (V, \psi)$ are charts on X, Y then

$$\psi^{-1} \circ f \circ \phi : (f \circ \phi)^{-1}(\psi(V)) \longrightarrow V$$

is a smooth map from $(f \circ \phi)^{-1}(\psi(V)) \subset \mathbb{R}^m$ to $V \subset \mathbb{R}^n$.

Let $(x, \beta) \in \partial X$. A *boundary defining function for X at (x, β)* is a pair (V, b), where V is an open neighbourhood of x in X and $b : V \to [0, \infty)$ is a weakly smooth map, such that $db|_v : T_v V \to T_{b(v)}[0, \infty)$ is nonzero for all $v \in V$, and there exists an open neighbourhood U of (x, β) in $i_X^{-1}(V) \subseteq \partial X$, with $b \circ i_X|_U = 0$, and $i_X|_U : U \longrightarrow \{v \in V : b(v) = 0\}$ is a homeomorphism.

A weakly smooth map of manifolds with corners $f : X \to Y$ is called *smooth* if it satisfies the following additional condition over $\partial X, \partial Y$. Suppose $x \in X$ with $f(x) = y \in Y$, and β is a local boundary component of Y at y. Let (V, b) be a boundary defining function for Y at (y, β). We require that either:

(i) There exists an open $x \in \tilde{V} \subseteq f^{-1}(V) \subseteq X$ such that $(\tilde{V}, b \circ f|_{\tilde{V}})$ is a boundary defining function for X at $(x, \tilde{\beta})$, for some unique local boundary component $\tilde{\beta}$ of X at x; or

(ii) There exists an open $x \in W \subseteq f^{-1}(V) \subseteq X$ with $b \circ f|_W = 0$.

Form the fibre products of topological spaces

$$\partial X \times_{f \circ i_X, Y, i_Y} \partial Y = \{((x, \tilde{\beta}), (y, \beta)) \in \partial X \times \partial Y : f \circ i_X(x, \tilde{\beta}) = y = i_Y(y, \beta)\},$$
$$X \times_{f, Y, i_Y} \partial Y = \{(x, (y, \beta)) \in X \times \partial Y : f(x) = y = i_Y(y, \beta)\}.$$

Define subsets $S_f \subseteq \partial X \times_Y \partial Y$ and $T_f \subseteq X \times_Y \partial Y$ by $((x, \tilde{\beta}), (y, \beta)) \in S_f$ in case (i) above, and $(x, (y, \beta)) \in T_f$ in case (ii) above. Define maps $s_f : S_f \to \partial X$, $t_f : T_f \to X$, $u_f : S_f \to \partial Y$, $v_f : T_f \to \partial Y$ to be the projections from the fibre products. Then S_f, T_f are open and closed in $\partial X \times_Y \partial Y, X \times_Y \partial Y$ and have the structure of manifolds with corners, with $\dim S_f = \dim X - 1$ and $\dim T_f = \dim X$, and s_t, t_f, u_f, v_f are smooth maps with s_f, t_f étale.

We write $\mathbf{Man^c}$ for the category of manifolds with corners, with morphisms smooth maps, and $\mathbf{Man^b}$ for the full subcategory of manifolds with boundary.

5.2 (Semi)simple maps, submersions, immersions, and embeddings

In [35, §5.4 & §5.7] we define some interesting classes of smooth maps:

Definition 5.4. Let $f : X \to Y$ be a smooth map of manifolds with corners.

(a) We call f *simple* if $s_f : S_f \to \partial X$ in Definition 5.3 is bijective.

(b) We call f *semisimple* if $s_f : S_f \to \partial X$ is injective.

(c) We call f *flat* if $T_f = \emptyset$ in Definition 5.3.

(d) We call f a *diffeomorphism* if it has a smooth inverse $f^{-1} : Y \to X$.

(e) We call f a *submersion* if for all $x \in S^k(X) \subseteq Y$ with $f(x) = y \in S^l(Y) \subseteq Y$, then $df|_x : T_x X \to T_{f(x)} Y$ and $df|_x : T_x(S^k(X)) \to T_{f(x)}(S^l(Y))$ are surjective. Submersions are automatically semisimple. We call f an *s-submersion* if it is a simple submersion.

(f) We call f an *immersion* if $df|_x : T_x X \to T_{f(x)} Y$ is injective for all $x \in X$. We call f an *s-immersion* (or *sf-immersion*) if f is also simple (or simple and flat). We call f an *embedding* (or *s-embedding*, or *sf-embedding*) if f is an immersion (or s-immersion, or sf-immersion), and $f : X \to f(X)$ is a homeomorphism with its image.

For manifolds without boundary, one considers immersed or embedded submanifolds. Part (f) gives six different notions of submanifolds X of manifolds with corners Y: *immersed, s-immersed, sf-immersed, embedded, s-embedded* and *sf-embedded submanifolds*.

Example 5.5. (i) The inclusion $i : [0, \infty) \hookrightarrow \mathbb{R}$ is an embedding. It is semisimple and flat, but not simple, as $s_i : S_i \to \partial[0, \infty)$ maps $\emptyset \to \{0\}$, and is not surjective, so i is not an s- or sf-embedding. Thus $[0, \infty)$ is an embedded submanifold of $\mathbb{R}$, but not an s- or sf-embedded submanifold.

(ii) The map $f : [0, \infty) \to [0, \infty)^2$ mapping $f : x \mapsto (x, x)$ is an embedding. It is flat, but not semisimple, as $s_f : S_f \to \partial[0, \infty)$ maps two points to one point, and is not injective. Hence f is not an s- or sf-embedding, and $\{(x, x) : x \in [0, \infty)\}$ is an embedded submanifold of $[0, \infty)^2$, but not s- or sf-embedded.

(iii) The inclusion $i : \{0\} \hookrightarrow [0, \infty)$ has $di|_0$ injective, so it is an embedding. It is simple, but not flat, as $T_i = \{(0, (0, \{x = 0\}))\} \neq \emptyset$. Thus i is an s-embedding, but not an sf-embedding. Hence $\{0\}$ is an s-embedded but not sf-embedded submanifold of $[0, \infty)$.

(iv) Let X be a manifold with corners with $\partial X \neq \emptyset$. Then $i_X : \partial X \to X$ is an immersion. Also $s_{i_X} : S_{i_X} \to \partial^2 X$ is a bijection, so i_X is simple, but $T_{i_X} \cong \partial X \neq \emptyset$, so i_X is not flat. Hence i_X is an s-immersion, but not an sf-immersion. If $\partial^2 X = \emptyset$ then i_X is an s-embedding, but not an sf-embedding.

(v) Let $f : [0, \infty) \to \mathbb{R}$ be smooth. Define $g : [0, \infty) \to [0, \infty) \times \mathbb{R}$ by $g(x) = (x, f(x))$. Then g is an sf-embedding, and $\Gamma_f = \{(x, f(x) : x \in [0, \infty)\}$ is an sf-embedded submanifold of $[0, \infty) \times \mathbb{R}$.

Simple and semisimple maps have a property of lifting to boundaries:

Proposition 5.6. *Let $f : X \to Y$ be a semisimple map of manifolds with corners. Then there exists a natural decomposition $\partial X = \partial_+^f X \amalg \partial_-^f X$ with $\partial_\pm^f X$ open and closed in ∂X, and semisimple maps $f_+ = f \circ i_X|_{\partial_+^f X} : \partial_+^f X \to Y$ and $f_- : \partial_-^f X \to \partial Y$, such that the following commutes in $\mathbf{Man^c}$:*

$$
\begin{array}{ccc}
\partial_-^f X & \xrightarrow{\quad f_- \quad} & \partial Y \\
{\scriptstyle i_X|_{\partial_-^f X}}\downarrow & & \downarrow{\scriptstyle i_Y} \\
X & \xrightarrow{\quad f \quad} & Y.
\end{array}
\tag{5.1}
$$

If f is also flat, then (5.1) is a Cartesian square, so that $\partial_-^f X \cong X \times_Y \partial Y$. If f is simple then $\partial_+^f X = \emptyset$ and $\partial_-^f X = \partial X$. If f is simple, flat, a submersion, or an s-submersion, then $f_\pm$ are also simple, ..., s-submersions, respectively.

In fact we define $\partial_-^f X = s_f(S_f)$, so that $s_f : S_f \to \partial_-^f X$ is a bijection since s_f is injective as f is semisimple, and then $f_- = u_f \circ s_f^{-1}$, using the notation of Definition 5.3. If $f : X \to Y$ is simple then $f_- : \partial X \to \partial Y$ is also simple, so $f_{-k} : \partial^k X \to \partial^k Y$ is simple for $k = 1, 2, \ldots$. If f is also flat then f_{-k} is flat and $\partial^k X \cong X \times_Y \partial^k Y$. A smooth map $f : X \to Y$ is flat if and only if $f(X^\circ) \subseteq Y^\circ$, or equivalently, if $f : X \to Y$ and $i_Y : \partial Y \to Y$ are transverse.

(S-)submersions are locally modelled on projections $\pi_X : X \times Y \to X$:

Proposition 5.7. (a) *Let X, Y be manifolds with corners. Then the projection $\pi_X : X \times Y \to X$ is a submersion, and an s-submersion if $\partial Y = \emptyset$.*

(b) *Let $f : X \to Y$ be a submersion of manifolds with corners, and $x \in X$ with $f(x) = y \in Y$. Then there exist open neighbourhoods V of x in X and W of y in Y with $f(V) = W$, a manifold with corners Z, and a diffeomorphism $V \cong W \times Z$ which identifies $f|_V : V \to W$ with $\pi_W : W \times Z \to W$. If f is an s-submersion then $\partial Z = \emptyset$.*

S-immersions and sf-immersions are also locally modelled on products:

Proposition 5.8. (a) *Let X be a manifold with corners and $0 \leqslant k \leqslant n$. Then $\mathrm{id}_X \times 0 : X \to X \times \big([0,\infty)^k \times \mathbb{R}^{n-k}\big)$ mapping $x \mapsto (x,0)$ is an s-embedding, and an sf-embedding if $k = 0$.*

(b) *Let $f : X \to Y$ be an s-immersion of manifolds with corners, and $x \in X$ with $f(x) = y \in Y$. Then there exist open neighbourhoods V of x in X and W of y in Y with $f(V) \subseteq W$, an open neighbourhood Z of 0 in $[0,\infty)^k \times \mathbb{R}^{n-k}$, and a diffeomorphism $W \cong V \times Z$ which identifies $f|_V : V \to W$ with $\mathrm{id}_V \times 0 : V \to V \times Z$. If f is an sf-immersion then $k = 0$.*

Example 5.5(ii) shows general immersions are not modelled on products.

5.3 Corners and the corner functors

As in [32, §2], [35, §5.5], we define the *k-corners* $C_k(X)$ of a manifold with corners X.

Definition 5.9. Let X be an n-manifold with corners. Applying ∂ repeatedly gives manifolds with corners $\partial X, \partial^2 X, \ldots$. There is a natural identification

$$\partial^k X \cong \big\{ (x, \beta_1, \ldots, \beta_k) : x \in X, \, \beta_1, \ldots, \beta_k \text{ are distinct}$$
$$\text{local boundary components for } X \text{ at } x \big\}. \tag{5.2}$$

Using (5.2), we see that the symmetric group S_k of permutations of $\{1, \ldots, k\}$ has a natural, free action on $\partial^k X$ by diffeomorphisms, given by

$$\sigma : (x, \beta_1, \ldots, \beta_k) \longmapsto (x, \beta_{\sigma(1)}, \ldots, \beta_{\sigma(k)}).$$

Define the *k-corners* of X, as a set, to be

$$C_k(X) = \big\{ (x, \{\beta_1, \ldots, \beta_k\}) : x \in X, \, \beta_1, \ldots, \beta_k \text{ are distinct}$$
$$\text{local boundary components for } X \text{ at } x \big\}.$$

Then $C_k(X)$ is naturally a manifold with corners of dimension $n - k$, with $C_k(X) \cong \partial^k X / S_k$. The interior $C_k(X)^\circ$ is naturally diffeomorphic to $S^k(X)$. We have natural diffeomorphisms $C_0(X) \cong X$ and $C_1(X) \cong \partial X$.

A surprising fact about manifolds with corners X is that the disjoint union $C(X) := \coprod_{k=0}^{\dim X} C_k(X)$ has strong functorial properties. Since $C(X)$ is not a manifold with corners, it is helpful to enlarge our category $\mathbf{Man^c}$:

Definition 5.10. Write $\mathbf{\check{M}an^c}$ for the category whose objects are disjoint unions $\coprod_{m=0}^{\infty} X_m$, where X_m is a manifold with corners of dimension m, and whose morphisms are continuous maps $f : \coprod_{m=0}^{\infty} X_m \to \coprod_{n=0}^{\infty} Y_n$, such that $f|_{X_m \cap f^{-1}(Y_n)} : \big(X_m \cap f^{-1}(Y_n) \big) \to Y_n$ is a smooth map of manifolds with corners for all $m, n \geqslant 0$.

Definition 5.11. Define *corner functors* $C, \hat{C} : \mathbf{Man^c} \to \mathbf{\check{M}an^c}$ by $C(X) = \hat{C}(X) = \coprod_{k=0}^{\dim X} C_k(X)$ on objects, and on morphisms $f : X \to Y$ in $\mathbf{Man^c}$,

$$C(f) : \big(x, \{\tilde{\beta}_1, \ldots, \tilde{\beta}_i\} \big) \longmapsto \big(y, \{\beta_1, \ldots, \beta_j\} \big), \quad \text{where } y = f(x),$$
$$\{\beta_1, \ldots, \beta_j\} = \big\{ \beta : \big((x, \tilde{\beta}_l), (y, \beta) \big) \in S_f, \text{ some } l = 1, \ldots, i \big\},$$
$$\hat{C}(f) : \big(x, \{\tilde{\beta}_1, \ldots, \tilde{\beta}_i\} \big) \longmapsto \big(y, \{\beta_1, \ldots, \beta_j\} \big), \quad \text{where } y = f(x),$$
$$\{\beta_1, \ldots, \beta_j\} = \big\{ \beta : \big((x, \tilde{\beta}_l), (y, \beta) \big) \in S_f, \, l = 1, \ldots, i \big\} \cup \big\{ \beta : \big(x, (y, \beta) \big) \in T_f \big\}.$$

Write $C_j^{f,k}(X) = C_j(X) \cap C(f)^{-1}(C_k(Y))$ and $C_j^k(f) = C(f)|_{C_j^{f,k}(X)} : C_j^{f,k}(X) \to C_k(Y)$ for all j, k, and similarly for $\hat{C}_j^{f,k}(X), \hat{C}_j^k(f)$. Then $C_j^k(f)$ and $\hat{C}_j^k(f)$ are smooth maps of manifolds with corners. Note that $C_0^{f,0}(X) = C_0(X) \cong X$ and $C_0(Y) \cong Y$, and these isomorphisms identify $C_0^0(f) : C_0(X) \to C_0(Y)$ with $f : X \to Y$.

It turns out that $C, \hat{C}$ are both functors $\mathbf{Man^c} \to \mathbf{\check{M}an^c}$. Furthermore:

(i) For each $X \in \mathbf{Man^c}$ we have a natural diffeomorphism $C(\partial X) \cong \partial C(X)$ identifying $C(i_X) : C(\partial X) \to C(X)$ with $i_{C(X)} : \partial C(X) \to C(X)$

(ii) For all X, Y in $\mathbf{Man^c}$ we have a natural diffeomorphism $C(X \times Y) \cong C(X) \times C(Y)$. These diffeomorphisms commute with product morphisms and direct product morphisms in $\mathbf{Man^c}, \mathbf{\check{M}an^c}$.

(iii) If $g : X \to Z$ and $h : Y \to Z$ are *strongly transverse* maps in $\mathbf{Man^c}$ then C maps the fibre product $X \times_{g,Z,h} Y$ in $\mathbf{Man^c}$ to the fibre product $C(X) \times_{C(g),C(Z),C(h)} C(Y)$ in $\mathbf{\check{M}an^c}$.

(iv) If $f : X \to Y$ is semisimple, then $C(f)$ maps $C_k(X) \to \coprod_{l=0}^{k} C_l(Y)$ for all $k \geqslant 0$. The natural diffeomorphisms $C_1(X) \cong \partial X$, $C_0(Y) \cong Y$ and $C_1(Y) \cong \partial Y$ identify $C_1^{f,0}(X) \cong \partial_+^f X$, $C_1^0(f) \cong f_+$, $C_1^{f,1}(X) \cong \partial_-^f X$ and $C_1^1(f) \cong f_-$. If f is simple then $C(f)$ maps $C_k(X) \to C_k(Y)$ for all $k \geqslant 0$.

The analogues hold for $\hat{C}$, except for (iv) and the last part of (i).

5.4 (Strong) transversality and fibre products

In [32, §6], [35, §5.6] we discuss conditions for fibre products to exist in $\mathbf{Man^c}$.

Definition 5.12. Let $g : X \to Z$, $h : Y \to Z$ be smooth maps of manifolds with corners. We call g, h *transverse* if whenever $x \in S^j(X) \subseteq X$, $y \in S^k(Y) \subseteq Y$ and $z \in S^l(Z) \subseteq Z$ with $g(x) = h(y) = z$, then $T_z Z = \mathrm{d}g|_x(T_x X) + \mathrm{d}h|_y(T_y Y)$ and $T_z(S^l(Z)) = \mathrm{d}g|_x(T_x(S^j(X))) + \mathrm{d}h|_y(T_y(S^k(Y)))$.

We call g, h *strongly transverse* if they are transverse, and whenever there are points in $C_j(X), C_k(Y), C_l(Z)$ with

$$C(g)(x, \{\beta_1, \ldots, \beta_j\}) = C(h)(y, \{\tilde{\beta}_1, \ldots, \tilde{\beta}_k\}) = (z, \{\dot{\beta}_1, \ldots, \dot{\beta}_l\})$$

we have either $j + k > l$ or $j = k = l = 0$.

If one of g, h is a submersion then g, h are strongly transverse. It is well known that transverse fibre products of manifolds without boundary exist. Here is the (more difficult to prove) analogue for manifolds with corners.

Theorem 5.13. *Let $g : X \to Z$, $h : Y \to Z$ be transverse smooth maps of manifolds with corners. Then a fibre product $W = X \times_{g,Z,h} Y$ exists in $\mathbf{Man^c}$.*

As a topological space, the fibre product in Theorem 5.13 is just the topological fibre product $W = \{(x, y) \in X \times Y : g(x) = h(y)\}$. In general, the boundary ∂W is difficult to describe explicitly: it is the quotient of a subset of $(\partial X \times_Z Y) \amalg (X \times_Z \partial Y)$ by an equivalence relation. Here are some special cases in which we can give an explicit formula for ∂W.

Proposition 5.14. *Let $g : X \to Z$, $h : Y \to Z$ be transverse smooth maps in $\mathbf{Man^c}$, so that $X \times_{g,Z,h} Y$ exists by Theorem 5.13. Then:*

(a) *If $\partial Z = \emptyset$ then*

$$\partial\big(X \times_{g,Z,h} Y\big) \cong \big(\partial X \times_{g \circ i_X, Z, h} Y\big) \amalg \big(X \times_{g,Z,h \circ i_Y} \partial Y\big). \qquad (5.3)$$

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(b) *If g is semisimple then*

$$\partial\big(X \times_{g,Z,h} Y\big) \cong \big(\partial_+^g X \times_{g_+,Z,h} Y\big) \amalg \big(X \times_{g,Z,h\circ i_Y} \partial Y\big). \qquad (5.4)$$

(c) *If both g, h are semisimple then*

$$\partial\big(X \times_{g,Z,h} Y\big) \cong$$
$$\big(\partial_+^g X \times_{g_+,Z,h} Y\big) \amalg \big(X \times_{g,Z,h_+} \partial_+^h Y\big) \amalg \big(\partial_-^g X \times_{g_-,\partial Z,h_-} \partial_-^h Y\big). \qquad (5.5)$$

Here all fibre products in (5.3)–(5.5) are transverse, and so exist.

For *strongly* transverse smooth maps, fibre products commute with the corner functors $C, \hat{C} : \mathbf{Man^c} \to \check{\mathbf{Man}}^\mathbf{c}$. Since $C_1(W) \cong \partial W$, equation (5.6) with $i = 1$ gives another explicit description of ∂W in this case.

Theorem 5.15. *Let $g : X \to Z$, $h : Y \to Z$ be strongly transverse smooth maps of manifolds with corners, and write W for the fibre product $X \times_{g,Z,h} Y$ given by Theorem* 5.13*. Then there is a canonical diffeomorphism*

$$C_i(W) \cong \coprod_{j,k,l \geqslant 0 : i = j+k-l} C_j^{g,l}(X) \times_{C_j^l(g),C_l(Z),C_k^l(h)} C_k^{h,l}(Y) \qquad (5.6)$$

for all $i \geqslant 0$, where the fibre products are all transverse and so exist. Hence

$$C(W) \cong C(X) \times_{C(g),C(Z),C(h)} C(Y) \quad in \ \check{\mathbf{Man}}^\mathbf{c}.$$

5.5 Orientations on manifolds with corners

In [32, §7], [35, §5.8] we discuss orientations on manifolds with corners.

Definition 5.16. Let X be an n-manifold with corners. An *orientation* ω on X is an orientation on the fibres of the real line bundle $\Lambda^n T^*X$ over X. That is, ω is an equivalence class $[\tau]$ of isomorphisms $\tau : O_X \to \Lambda^n T^*X$, where $O_X = \mathbb{R} \times X \to X$ is the trivial line bundle on X, and τ, τ' are equivalent if $\tau' = \tau \cdot c$ for some smooth $c : X \to (0, \infty)$.

If $\omega = [\tau]$ is an orientation, we write $-\omega$ for the *opposite orientation* $[-\tau]$.

We call the pair (X, ω) an *oriented manifold*. Usually we suppress the orientation ω, and just refer to X as an oriented manifold. When X is an oriented manifold, we write $-X$ for X with the opposite orientation.

If X, Y, Z are oriented manifolds with corners, then we can define orientations on boundaries ∂X, products $X \times Y$, and transverse fibre products $X \times_Z Y$. To do this requires a choice of *orientation convention*. Our orientation conventions are given in [35, §5.8]. Having fixed an orientation convention, natural isomorphisms of manifolds with corners such as $X \times_Z Y \cong Y \times_Z X$ lift to isomorphisms of oriented manifolds of corners, modified by signs depending on the dimensions.

For example, if $g : X \to Z$ and $h : Y \to Z$ are transverse maps of oriented manifolds with corners then

$$X \times_{g,Z,h} Y \cong (-1)^{(\dim X - \dim Z)(\dim Y - \dim Z)} Y \times_{h,Z,g} X,$$

and with orientations equations (5.3)–(5.5) become

$$\partial\big(X \times_{g,Z,h} Y\big) \cong \big(\partial X \times_{g \circ i_X, Z, h} Y\big) \amalg (-1)^{\dim X + \dim Z}\big(X \times_{g,Z,h \circ i_Y} \partial Y\big),$$
$$\partial\big(X \times_{g,Z,h} Y\big) \cong \big(\partial_+^g X \times_{g_+, Z, h} Y\big) \amalg (-1)^{\dim X + \dim Z}\big(X \times_{g,Z,h \circ i_Y} \partial Y\big),$$
$$\partial\big(X \times_{g,Z,h} Y\big) \cong \big(\partial_+^g X \times_{g_+, Z, h} Y\big) \amalg (-1)^{\dim X + \dim Z}\big(X \times_{h,Z,h_+} \partial_+^h Y\big)$$
$$\amalg \big(\partial_-^g X \times_{g_-, \partial Z, h_-} \partial_-^h Y\big).$$

5.6 Fixed point loci in manifolds with corners

In [35, §5.5] we study the fixed point locus X^Γ of a group Γ acting on a manifold with corners X. These are related to orbifold strata $\mathcal{X}^\Gamma$ of orbifolds with corners $\mathcal{X}$, which we will discuss in §12.5. Here is our main result.

Proposition 5.17. *Suppose X is a manifold with corners, Γ a finite group, and $r : \Gamma \to \mathrm{Aut}(X)$ an action of Γ on X by diffeomorphisms. Applying the corner functor C of §5.3 gives an action $C(r) : \Gamma \to \mathrm{Aut}(C(X))$ of Γ on $C(X)$ by diffeomorphisms. Write $X^\Gamma, C(X)^\Gamma$ for the subsets of $X, C(X)$ fixed by Γ, and $j_{X,\Gamma} : X^\Gamma \to X$ for the inclusion. Then:*

(a) *X^Γ has the structure of an object in $\check{\mathbf{Man}}^{\mathbf{c}}$ (a disjoint union of manifolds with corners of different dimensions, as in §5.3) in a unique way, such that $j_{X,\Gamma} : X^\Gamma \to X$ is an embedding. This $j_{X,\Gamma}$ is flat, but need not be (semi)simple.*

(b) *By (a) we have a smooth map $C(j_{X,\Gamma}) : C(X^\Gamma) \to C(X)$. This $C(j_{X,\Gamma})$ is a diffeomorphism $C(X^\Gamma) \to C(X)^\Gamma$. As $j_{X,\Gamma}$ need not be simple, $C(j_{X,\Gamma})$ need not map $C_k(X^\Gamma) \to C_k(X)$ for $k > 0$.*

(c) *By (b), $C(j_{X,\Gamma})$ identifies $C_1(X^\Gamma) \cong \partial(X^\Gamma)$ with a subset of $C(X)^\Gamma \subseteq C(X)$. This gives the following description of $\partial(X^\Gamma)$:*

$$\partial(X^\Gamma) \cong \big\{(x, \{\beta_1, \ldots, \beta_k\}) \in C_k(X) : x \in X^\Gamma,\ k \geqslant 1,\ \beta_1, \ldots, \beta_k$$
$$\text{are distinct local boundary components for } X \text{ at } x,$$
$$\text{and } \Gamma \text{ acts transitively on } \{\beta_1, \ldots, \beta_k\}\big\}.$$

(d) *Now suppose Y is a manifold with corners with an action of Γ, and $f : X \to Y$ is a Γ-equivariant smooth map. Then X^Γ, Y^Γ are objects in $\check{\mathbf{Man}}^{\mathbf{c}}$ by (a), and $f^\Gamma := f|_{X^\Gamma} : X^\Gamma \to Y^\Gamma$ is a morphism in $\check{\mathbf{Man}}^{\mathbf{c}}$.*

Example 5.18. Let $\Gamma = \{1, \sigma\}$ with $\sigma^2 = 1$, so that $\Gamma \cong \mathbb{Z}_2$, and let Γ act on $X = [0, \infty)^2$ by $\sigma : (x_1, x_2) \mapsto (x_2, x_1)$. Then $X^\Gamma = \big\{(x, x) : x \in [0, \infty)\big\} \cong [0, \infty)$, a manifold with corners, and the inclusion $j_{X,\Gamma} : X^\Gamma \to X$ is

$j_{X,\Gamma} : [0,\infty) \to [0,\infty)^2$, $j_{X,\Gamma} : x \mapsto (x,x)$, a smooth, flat embedding, which is not semisimple. We have $\partial X = \partial([0,\infty)^2) \cong [0,\infty) \amalg [0,\infty)$, where Γ acts freely on ∂X by exchanging the two copies of $[0,\infty)$. Hence $(\partial X)^\Gamma = \emptyset$, but $\partial(X^\Gamma)$ is a point $*$, so in this case $(\partial X)^\Gamma \ncong \partial(X^\Gamma)$. Also $C_2(X) = \big\{\big(0, \{\{x_1 = 0\}, \{x_2 = 0\}\}\big)\big\}$ is a single point, which is Γ-invariant, and $C(j_{X,\Gamma}) : C(X^\Gamma) \to C(X)^\Gamma$ identifies $(0, \{\{x = 0\}\}) \in C_1(X^\Gamma) \cong \partial X$ with this point in $C_2(X)^\Gamma$.

If a finite group Γ acts on a manifold with corners X then as in Proposition 5.17(b) we have $C(X)^\Gamma \cong C(X^\Gamma)$, but as in Example 5.18 in general we do not have $(\partial X)^\Gamma \cong \partial(X^\Gamma)$, but only $(\partial X)^\Gamma \subseteq \partial(X^\Gamma)$. Thus for fixed point loci, corners have more functorial behaviour than boundaries.

6   D-spaces with corners

The goal of [35, Chap.s 6 & 7] is to construct a well-behaved 2-category $\mathbf{dMan^c}$ of *d-manifolds with corners*, a derived version of $\mathbf{Man^c}$. It is tempting to define $\mathbf{dMan^c}$ as a 2-subcategory of d-spaces $\mathbf{dSpa}$, but this turns out not to be a good idea. For example, the natural functor $F_{\mathbf{Man^c}}^{\mathbf{dSpa}} : \mathbf{Man^c} \to \mathbf{dSpa}$ is not full, as 1-morphisms $\boldsymbol{f} : F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(X) \to F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(Y)$ correspond to weakly smooth rather than smooth maps $f : X \to Y$, in the notation of §5.1.

Therefore we begin in [35, Chap. 6] by defining a 2-category $\mathbf{dSpa^c}$ of *d-spaces with corners*, and then define $\mathbf{dMan^c}$ in [35, Chap. 7] as a 2-subcategory of $\mathbf{dSpa^c}$. Many properties of manifolds with corners in §5 work for d-spaces with corners, e.g. boundaries ∂X, simple, semisimple and flat maps $f : X \to Y$, decompositions $\partial X = \partial_+^f X \amalg \partial_-^f X$ and semisimple maps $f_+ : \partial_+^f X \to Y$ and $f_- : \partial_-^f X \to \partial Y$ when f is semisimple, and the corner functors $C, \hat{C}$.

6.1   Outline of the definition of the 2-category dSpa$^{\mathbf{c}}$

The definition of the 2-category of d-spaces with corners $\mathbf{dSpa^c}$ in [35, §6.1] is long and complicated. So here we just sketch the main ideas.

Let X be a manifold with corners. Then it has a boundary ∂X with a proper smooth map $i_X : \partial X \to X$. On ∂X we have an exact sequence

$$0 \longrightarrow \mathcal{N}_X \longrightarrow i_X^*(T^*X) \xrightarrow{(\mathrm{d}i_X)^*} T^*(\partial X) \longrightarrow 0, \qquad (6.1)$$

where $\mathcal{N}_X$ is the conormal bundle of ∂X in X. The line bundle $\mathcal{N}_X$ has a natural orientation ω_X induced by outward-pointing normal vectors to ∂X in X.

Thus, for each manifold with corners X we have a quadruple $(X, \partial X, i_X, \omega_X)$. D-spaces with corners are based on this idea. A *d-space with corners* $\mathbf{X}$ is a quadruple $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i_X}, \omega_{\mathbf{X}})$ where $\boldsymbol{X}, \boldsymbol{\partial X}$ are d-spaces, and $\boldsymbol{i_X} : \boldsymbol{\partial X} \to \boldsymbol{X}$ is a proper 1-morphism, and we have an exact sequence in $\mathrm{qcoh}(\underline{\partial X})$:

$$0 \longrightarrow \mathcal{N}_{\mathbf{X}} \xrightarrow{\nu_{\mathbf{X}}} \underline{i}_{\mathbf{X}}^*(\mathcal{F}_X) \xrightarrow{i_{\mathbf{X}}^2} \mathcal{F}_{\partial X} \longrightarrow 0, \qquad (6.2)$$

with $\mathcal{N}_{\mathbf{X}}$ a line bundle, and $\omega_{\mathbf{X}}$ is an orientation on $\mathcal{N}_{\mathbf{X}}$. These $\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i_X}, \omega_{\mathbf{X}}$ must satisfy some complicated conditions in [35, §6.1], that we will not give. They require $\boldsymbol{\partial X}$ to be locally equivalent to a fibre product $\boldsymbol{X} \times_{[0,\infty)} *$ in $\mathbf{dSpa}$.

If $\mathbf{X} = (X, \partial X, i_{\mathbf{X}}, \omega_{\mathbf{X}})$ and $\mathbf{Y} = (Y, \partial Y, i_{\mathbf{Y}}, \omega_{\mathbf{Y}})$ are d-spaces with corners, a 1-*morphism* $f : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{dSpa^c}$ is a 1-morphism $f : X \to Y$ in $\mathbf{dSpa}$ satisfying extra conditions over $\partial X, \partial Y$, which are analogous to the extra conditions for a weakly smooth map of manifolds with corners $f : X \to Y$ to be smooth in Definition 5.3.

If $f : \mathbf{X} \to \mathbf{Y}$ is a 1-morphism in $\mathbf{dSpa^c}$, we can form the C^∞-scheme fibre products $\underline{\partial X} \times_{f \circ \underline{i}_{\mathbf{X}}, \underline{Y}, \underline{i}_{\mathbf{Y}}} \underline{\partial Y}$ and $\underline{X} \times_{\underline{f}, \underline{Y}, \underline{i}_{\mathbf{Y}}} \underline{\partial Y}$. As for S_f, T_f in Definition 5.3, we can define open and closed C^∞-subschemes $\underline{S}_f \subseteq \underline{\partial X} \times_{\underline{Y}} \underline{\partial Y}$ and $\underline{T}_f \subseteq \underline{X} \times_{\underline{Y}} \underline{\partial Y}$, and define C^∞-scheme morphisms $\underline{s}_f : \underline{S}_f \to \underline{\partial X}$, $\underline{t}_f : \underline{T}_f \to \underline{X}$, $\underline{u}_f : \underline{S}_f \to \underline{\partial Y}$ and $\underline{v}_f : \underline{T}_f \to \underline{\partial Y}$ to be the projections from the fibre products. Then $\underline{s}_f, \underline{t}_f$ are étale.

If $f, g : \mathbf{X} \to \mathbf{Y}$ are 1-morphisms in $\mathbf{dSpa^c}$, a 2-*morphism* $\eta : f \Rightarrow g$ in $\mathbf{dSpa^c}$ is a 2-morphism $\eta : f \Rightarrow g$ in $\mathbf{dSpa}$ such that $\underline{S}_f = \underline{S}_g$, $\underline{T}_f = \underline{T}_g$ and extra vanishing conditions hold on η over $\underline{S}_f, \underline{T}_f$. Identity 1- and 2-morphisms in $\mathbf{dSpa^c}$, and the compositions of 1- and 2-morphisms in $\mathbf{dSpa^c}$, are all given by identities and compositions in $\mathbf{dSpa}$.

A d-space with corners $\mathbf{X} = (X, \partial X, i_{\mathbf{X}}, \omega_{\mathbf{X}})$ is called a *d-space with boundary* if $i_{\mathbf{X}} : \partial X \to X$ is injective, and a *d-space without boundary* if $\partial X = \emptyset$. We write $\mathbf{dSpa^b}$ for the full 2-subcategory of d-spaces with boundary, and $\mathbf{d\bar{S}pa}$ for the full 2-subcategory of d-spaces without boundary, in $\mathbf{dSpa^c}$. There is an isomorphism of 2-categories $F_{\mathbf{dSpa}}^{\mathbf{dSpa^c}} : \mathbf{dSpa} \to \mathbf{d\bar{S}pa}$ mapping $X \mapsto \mathbf{X} = (X, \emptyset, \emptyset, \emptyset)$ on objects, $f \mapsto f$ on 1-morphisms and $\eta \mapsto \eta$ on 2-morphisms. So we can consider d-spaces to be examples of d-spaces with corners.

Remark 6.1. If X is a manifold with corners then the orientation ω_X on $\mathcal{N}_X$ is determined uniquely by $X, \partial X, i_X$. But there are examples of d-spaces with corners $\mathbf{X} = (X, \partial X, i_{\mathbf{X}}, \omega_{\mathbf{X}})$ in which $\omega_{\mathbf{X}}$ is not determined by $X, \partial X, i_{\mathbf{X}}$, and really is extra data. We include $\omega_{\mathbf{X}}$ in the definition so that orientations of d-manifolds with corners behave well in relation to boundaries. If we had omitted $\omega_{\mathbf{X}}$ from the definition, then there would exist examples of oriented d-manifolds with corners $\mathbf{X}$ such that $\partial \mathbf{X}$ is not orientable.

For each d-space with corners $\mathbf{X} = (X, \partial X, i_{\mathbf{X}}, \omega_{\mathbf{X}})$, in [35, §6.2] we define a d-space with corners $\partial \mathbf{X} = (\partial X, \partial^2 X, i_{\partial \mathbf{X}}, \omega_{\partial \mathbf{X}})$ called the *boundary* of $\mathbf{X}$, and show that $i_{\mathbf{X}} : \partial \mathbf{X} \to \mathbf{X}$ is a 1-morphism in $\mathbf{dSpa^c}$. Motivated by (5.2) when $k = 2$, the d-space $\partial^2 X$ in ∂X is given by

$$\partial^2 X \simeq \left(\partial X \times_{i_{\mathbf{X}}, X, i_{\mathbf{X}}} \partial X \right) \setminus \Delta_{\partial X}(\partial X), \tag{6.3}$$

where $\Delta_{\partial X} : \partial X \to \partial X \times_X \partial X$ is the diagonal 1-morphism. The 1-morphism $i_{\partial \mathbf{X}} : \partial^2 X \to \partial X$ is projection to the first factor in the fibre product. There is a natural isomorphism $\mathcal{N}_{\partial \mathbf{X}} \cong \underline{i}_{\mathbf{X}}^*(\mathcal{N}_{\mathbf{X}})$, and the orientation $\omega_{\partial \mathbf{X}}$ on $\mathcal{N}_{\partial \mathbf{X}}$ is defined to correspond to the orientation $\underline{i}_{\mathbf{X}}^*(\omega_{\mathbf{X}})$ on $\underline{i}_{\mathbf{X}}^*(\mathcal{N}_{\mathbf{X}})$.

6.2 Simple, semisimple and flat 1-morphisms

In [35, §6.3] we generalize the material on simple, semisimple, and flat maps of manifolds with corners in §5.2 to d-spaces with corners. Here are the analogues

of Definition 5.4(a)–(c) and Proposition 5.6.

Definition 6.2. Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a 1-morphism of d-spaces with corners.

(a) We call $\boldsymbol{f}$ *simple* if $\underline{s}_{\boldsymbol{f}} : \underline{S}_{\boldsymbol{f}} \to \underline{\partial X}$ is bijective.

(b) We call $\boldsymbol{f}$ *semisimple* if $\underline{s}_{\boldsymbol{f}} : \underline{S}_{\boldsymbol{f}} \to \underline{\partial X}$ is injective.

(c) We call $\boldsymbol{f}$ *flat* if $\underline{T}_{\boldsymbol{f}} = \emptyset$.

Theorem 6.3. *Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a semisimple 1-morphism of d-spaces with corners. Then there exists a natural decomposition $\partial \mathbf{X} = \partial_+^{\boldsymbol{f}} \mathbf{X} \amalg \partial_-^{\boldsymbol{f}} \mathbf{X}$ with $\partial_\pm^{\boldsymbol{f}} \mathbf{X}$ open and closed in $\partial \mathbf{X}$, such that:*

(a) *Define $\boldsymbol{f}_+ = \boldsymbol{f} \circ \boldsymbol{i}_{\mathbf{X}}|_{\partial_+^{\boldsymbol{f}} \mathbf{X}} : \partial_+^{\boldsymbol{f}} \mathbf{X} \to \mathbf{Y}$. Then $\boldsymbol{f}_+$ is semisimple. If $\boldsymbol{f}$ is flat then $\boldsymbol{f}_+$ is also flat.*

(b) *There exists a unique, semisimple 1-morphism $\boldsymbol{f}_- : \partial_-^{\boldsymbol{f}} \mathbf{X} \to \partial \mathbf{Y}$ with $\boldsymbol{f} \circ \boldsymbol{i}_{\mathbf{X}}|_{\partial_-^{\boldsymbol{f}} \mathbf{X}} = \boldsymbol{i}_{\mathbf{Y}} \circ \boldsymbol{f}_-$. If $\boldsymbol{f}$ is simple then $\partial_+^{\boldsymbol{f}} \mathbf{X} = \emptyset$, $\partial_-^{\boldsymbol{f}} \mathbf{X} = \partial \mathbf{X}$, and $\boldsymbol{f}_- : \partial \mathbf{X} \to \partial \mathbf{Y}$ is also simple. If $\boldsymbol{f}$ is flat then $\boldsymbol{f}_-$ is flat, and the following diagram is 2-Cartesian in $\mathbf{dSpa^c}$:*

$$
\begin{array}{ccc}
\partial_-^{\boldsymbol{f}} \mathbf{X} & \xrightarrow{\hspace{4cm}} & \partial \mathbf{Y} \\
{\scriptstyle \boldsymbol{i}_{\mathbf{X}}|_{\partial_-^{\boldsymbol{f}} \mathbf{X}}} \downarrow \quad {\scriptstyle \boldsymbol{f}_- \quad \mathrm{id}_{\boldsymbol{i}_{\mathbf{Y}} \circ \boldsymbol{f}} \Nearrow} & & \downarrow {\scriptstyle \boldsymbol{i}_{\mathbf{Y}}} \\
\mathbf{X} & \xrightarrow[\hspace{4cm}]{\boldsymbol{f}} & \mathbf{Y}.
\end{array}
\qquad (6.4)
$$

(c) *Let $\boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ be another 1-morphism, and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a 2-morphism in $\mathbf{dSpa^c}$. Then $\boldsymbol{g}$ is also semisimple, with $\partial_-^{\boldsymbol{g}} \mathbf{X} = \partial_-^{\boldsymbol{f}} \mathbf{X}$. If $\boldsymbol{f}$ is simple, or flat, then $\boldsymbol{g}$ is simple, or flat, respectively. Part (b) defines 1-morphisms $\boldsymbol{f}_-, \boldsymbol{g}_- : \partial_-^{\boldsymbol{f}} \mathbf{X} \to \partial \mathbf{Y}$. There is a unique 2-morphism $\eta_- : \boldsymbol{f}_- \Rightarrow \boldsymbol{g}_-$ in $\mathbf{dSpa^c}$ such that $\mathrm{id}_{\boldsymbol{i}_{\mathbf{Y}}} * \eta_- = \eta * \mathrm{id}_{\boldsymbol{i}_{\mathbf{X}}|_{\partial_-^{\boldsymbol{f}} \mathbf{X}}} : \boldsymbol{i}_{\mathbf{Y}} \circ \boldsymbol{f}_- \Rightarrow \boldsymbol{i}_{\mathbf{Y}} \circ \boldsymbol{g}_-$.*

We also show that the maps $\boldsymbol{f} \mapsto \boldsymbol{f}_-$, $\eta \mapsto \eta_-$ in Theorem 6.3 are functorial, in that they commute with compositions of 1- and 2-morphisms, and take identities to identities. For simple 1-morphisms, this implies:

Corollary 6.4. *Write $\mathbf{dSpa^c_{si}}$ for the 2-subcategory of $\mathbf{dSpa^c}$ with arbitrary objects and 2-morphisms, but only simple 1-morphisms. Then there is a strict 2-functor $\partial : \mathbf{dSpa^c_{si}} \to \mathbf{dSpa^c_{si}}$ mapping $\mathbf{X} \mapsto \partial \mathbf{X}$ on objects, $\boldsymbol{f} \mapsto \boldsymbol{f}_-$ on (simple) 1-morphisms, and $\eta \mapsto \eta_-$ on 2-morphisms.*

Thus, boundaries in $\mathbf{dSpa^c}$ have strong functoriality properties.

Remark 6.5. According to the general philosophy of working in 2-categories, when one constructs an object with some property in a 2-category, it is usually unique only up to equivalence. When one constructs a 1-morphism with some property in a 2-category, it is usually unique only up to 2-isomorphism. When

one considers diagrams of 1-morphisms in a 2-category, they usually commute only up to (specified) 2-isomorphisms.

From this point of view, Theorem 6.3(b) looks unnatural, as it gives a 1-morphism $\boldsymbol{f}_-$ which is unique, not just up to 2-isomorphism, and a 1-morphism diagram (6.4) which commutes strictly, not just up to 2-isomorphism.

In fact, this unnaturalness pervades our treatment of boundaries. In our definition of d-space with corners $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i_X}, \omega_\mathbf{X})$, the conditions on the 1-morphism $\boldsymbol{i_X} : \boldsymbol{\partial X} \to \boldsymbol{X}$ depend on $\boldsymbol{\partial X}$ up to 1-isomorphism in $\mathbf{dSpa}$, rather than up to equivalence, and depend on $\boldsymbol{i_X}$ up to equality, not just up to 2-isomorphism. Boundaries $\partial\mathbf{X}$ are natural up to 1-isomorphism in $\mathbf{dSpa^c}$, not up to equivalence, and 1-morphisms $\boldsymbol{i_X} : \partial\mathbf{X} \to \mathbf{X}$ natural up to equality.

The author chose this definition of $\mathbf{dSpa^c}$ for its (comparative!) simplicity. In defining objects $\mathbf{X}, \mathbf{Y}$, 1-morphisms $\boldsymbol{f}$, and 2-morphisms η in $\mathbf{dSpa^c}$, we must impose extra conditions, and possibly include extra data, over $\partial\mathbf{X}, \partial\mathbf{Y}$. If these conditions/extra data are imposed weakly, up to equivalence of objects or 2-isomorphism of 1-morphisms, things rapidly become very complicated and unwieldy. For instance, 1-morphisms in $\mathbf{dSpa^c}$ would comprise not just a 1-morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathbf{dSpa}$, but also extra 2-morphism data over $\underline{S}_{\boldsymbol{f}}, \underline{T}_{\boldsymbol{f}}$.

So as a matter of policy, we generally do constructions involving boundaries or corners in $\mathbf{dSpa^c}$ strictly, up to 1-isomorphism of objects, and equality of 1-morphisms. One advantage of this is that 1-morphisms $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ and 2-morphisms $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ in $\mathbf{dSpa^c}$ are special examples of 1- and 2-morphisms in $\mathbf{dSpa}$ of the underlying d-spaces $\boldsymbol{X}, \boldsymbol{Y}$, rather than also containing further data over $\partial\mathbf{X}, \partial\mathbf{Y}$. Another advantage is that boundaries in $\mathbf{dSpa^c}$ behave in a strictly functorial way, as in Corollary 6.4, rather than weakly functorial.

6.3   Manifolds with corners as d-spaces with corners

In [35, §6.4] we define a (2-)functor $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}} : \mathbf{Man^c} \to \mathbf{dSpa^c}$ from manifolds with corners to d-spaces with corners.

Definition 6.6. Let X be a manifold with corners. Then the boundary ∂X is a manifold with corners, with a smooth map $i_{\partial X} : \partial X \to X$. We will define a d-space with corners $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i_X}, \omega_\mathbf{X})$. Set $\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i_X} = F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(X, \partial X, i_X)$. Then the conormal bundle $\mathcal{N}_\mathbf{X}$ in (6.2) is the lift to the C^∞-scheme $\underline{\partial X}$ of the conormal line bundle $\mathcal{N}_X$ of ∂X in X, as in (6.1). Let $\omega_\mathbf{X}$ be the orientation on $\mathcal{N}_\mathbf{X}$ corresponding to that on $\mathcal{N}_X$ induced by outward-pointing normal vectors to ∂X in X. Then $\mathbf{X}$ is a d-space with corners. Set $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X) = \mathbf{X}$.

Let $f : X \to Y$ be a morphism in $\mathbf{Man^c}$, and set $\mathbf{X}, \mathbf{Y} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X, Y)$. Write $\boldsymbol{f} = F_{\mathbf{Man^c}}^{\mathbf{dSpa}}(f) : \boldsymbol{X} \to \boldsymbol{Y}$, as a 1-morphism of d-spaces. Then $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ is a 1-morphism of d-spaces with corners. Define $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(f) = \boldsymbol{f}$.

The only 2-morphisms in $\mathbf{Man^c}$, regarded as a 2-category, are identity 2-morphisms $\mathrm{id}_f : f \Rightarrow f$ for smooth $f : X \to Y$. We define $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(\mathrm{id}_f) = \mathrm{id}_{\boldsymbol{f}}$.

Define $F_{\mathbf{Man}}^{\mathbf{d\bar{S}pa}} : \mathbf{Man} \to \mathbf{d\bar{S}pa}$ and $F_{\mathbf{Man^b}}^{\mathbf{dSpa^b}} : \mathbf{Man^b} \to \mathbf{dSpa^b}$ to be the restrictions of $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ to the subcategories $\mathbf{Man}, \mathbf{Man^b} \subset \mathbf{Man^c}$.

Write $\bar{\mathbf{Man}}, \bar{\mathbf{Man}}^{\mathbf{b}}, \bar{\mathbf{Man}}^{\mathbf{c}}$ for the full 2-subcategories of objects $\mathbf{X}$ in $\mathbf{dSpa^c}$ equivalent to $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X)$ for some manifold X without boundary, or with boundary, or with corners, respectively. Then $\bar{\mathbf{Man}} \subset \mathbf{d\bar{S}pa}$, $\bar{\mathbf{Man}}^{\mathbf{b}} \subset \mathbf{dSpa^b}$ and $\bar{\mathbf{Man}}^{\mathbf{c}} \subset \mathbf{dSpa^c}$. When we say that a d-space with corners $\mathbf{X}$ *is a manifold*, we mean that $\mathbf{X} \in \bar{\mathbf{Man}}^{\mathbf{c}}$.

In [35, §6.4] we show that $F_{\mathbf{Man}}^{\mathbf{d\bar{S}pa}} : \mathbf{Man} \to \mathbf{d\bar{S}pa}$, $F_{\mathbf{Man^b}}^{\mathbf{dSpa^b}} : \mathbf{Man^b} \to \mathbf{dSpa^b}$ and $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}} : \mathbf{Man^c} \to \mathbf{dSpa^c}$ are full and faithful strict 2-functors. We also prove that if X is a manifold with corners, then there is a natural 1-isomorphism $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(\partial X) \cong \partial F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X)$, and if $f : X \to Y$ is a smooth map of manifolds with corners and $\boldsymbol{f} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(f)$, then f is simple, semisimple or flat in $\mathbf{Man^c}$ if and only if $\boldsymbol{f}$ is simple, semisimple or flat in $\mathbf{dSpa^c}$, respectively.

6.4 Equivalences, and gluing by equivalences

In [35, §6.5 & §6.6] we discuss *equivalences* in $\mathbf{dSpa^c}$. First we characterize when a 1-morphism $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{dSpa^c}$ is an equivalence, in terms of the underlying 1-morphism in $\mathbf{dSpa}$:

Proposition 6.7. (a) *Suppose $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ is an equivalence in $\mathbf{dSpa^c}$. Then $\boldsymbol{f}$ is simple and flat, and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is an equivalence in $\mathbf{dSpa}$, where $\mathbf{X} = (\boldsymbol{X}, \partial\boldsymbol{X}, i_{\mathbf{X}}, \omega_{\mathbf{X}})$ and $\mathbf{Y} = (\boldsymbol{Y}, \partial\boldsymbol{Y}, i_{\mathbf{Y}}, \omega_{\mathbf{Y}})$. Also $\boldsymbol{f}_- : \partial\mathbf{X} \to \partial\mathbf{Y}$ in Theorem 6.3(b) is an equivalence in $\mathbf{dSpa^c}$.*

(b) *Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a simple, flat 1-morphism in $\mathbf{dSpa^c}$ with $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ an equivalence in $\mathbf{dSpa}$. Then $\boldsymbol{f}$ is an equivalence in $\mathbf{dSpa^c}$.*

Then we consider gluing d-spaces with corners by equivalences, as for d-spaces in §3.2. The story is the same. Here is the analogue of Definition 3.4:

Definition 6.8. Let $\mathbf{X} = (\boldsymbol{X}, \partial\boldsymbol{X}, i_{\mathbf{X}}, \omega_{\mathbf{X}})$ be a d-space with corners. Suppose $\boldsymbol{U} \subseteq \boldsymbol{X}$ is an open d-subspace in $\mathbf{dSpa}$. Define $\partial\boldsymbol{U} = i_{\mathbf{X}}^{-1}(\boldsymbol{U})$, as an open d-subspace of $\partial\boldsymbol{X}$, and $i_{\mathbf{U}} : \partial\boldsymbol{U} \to \boldsymbol{U}$ by $i_{\mathbf{U}} = i_{\mathbf{X}}|_{\partial\boldsymbol{U}}$. Then $\underline{\partial U} \subseteq \underline{\partial X}$ is an open C^∞-subscheme, and the conormal bundle of $\partial\boldsymbol{U}$ in $\boldsymbol{U}$ is $\mathcal{N}_{\mathbf{U}} = \mathcal{N}_{\mathbf{X}}|_{\underline{\partial U}}$ in qcoh$(\underline{\partial U})$. Define an orientation $\omega_{\mathbf{U}}$ on $\mathcal{N}_{\mathbf{U}}$ by $\omega_{\mathbf{U}} = \omega_{\mathbf{X}}|_{\underline{\partial U}}$. Write $\mathbf{U} = (\boldsymbol{U}, \partial\boldsymbol{U}, i_{\mathbf{U}}, \omega_{\mathbf{U}})$. Then $\mathbf{U}$ is a d-space with corners. We call $\mathbf{U}$ an *open d-subspace* of $\mathbf{X}$. An *open cover* of $\mathbf{X}$ is a family $\{\mathbf{U}_a : a \in A\}$ of open d-subspaces $\mathbf{U}_a$ of $\mathbf{X}$ with $\underline{X} = \bigcup_{a \in A} \underline{U}_a$.

Theorem 6.9. *Proposition 3.5 and Theorems 3.6 and 3.7 hold without change in the 2-category $\mathbf{dSpa^c}$ of d-spaces with corners.*

6.5 Corners and the corner functors

In [35, §6.7] we extend the material of §5.3 on corners and the corner functors from $\mathbf{Man^c}$ to $\mathbf{dSpa^c}$. The next theorem summarizes our results.

Theorem 6.10. (a) *Let* $\mathbf{X}$ *be a d-space with corners. Then for each* $k = 0, 1, \ldots$, *we can define a d-space with corners* $C_k(\mathbf{X})$ *called the* k-**corners** *of* $\mathbf{X}$, *and a 1-morphism* $\mathbf{\Pi}_{\mathbf{X}}^k : C_k(\mathbf{X}) \to \mathbf{X}$ *in* $\mathbf{dSpa^c}$. *It has topological space*

$$
\begin{aligned}
C_k(X) = \big\{ (x, \{x_1', \ldots, x_k'\}) : x \in X, \ \ x_1', \ldots, x_k' \in \boldsymbol{\partial X}, \\
\boldsymbol{i}_{\mathbf{X}}(x_a') = x, \ \ a = 1, \ldots, k, \ \ x_1', \ldots, x_k' \ are \ distinct \big\}.
\end{aligned}
\tag{6.5}
$$

There is a natural, free action of the symmetric group S_k *on* $\partial^k \mathbf{X}$, *and a 1-isomorphism* $C_k(\mathbf{X}) \cong \partial^k \mathbf{X} / S_k$. *We have 1-isomorphisms* $C_0(\mathbf{X}) \cong \mathbf{X}$ *and* $C_1(\mathbf{X}) \cong \partial \mathbf{X}$ *in* $\mathbf{dSpa^c}$. *Write* $C(\mathbf{X}) = \coprod_{k=0}^{\infty} C_k(\mathbf{X})$ *and* $\mathbf{\Pi}_{\mathbf{X}} = \coprod_{k=0}^{\infty} \mathbf{\Pi}_{\mathbf{X}}^k$, *so that* $C(\mathbf{X})$ *is a d-space with corners and* $\mathbf{\Pi}_{\mathbf{X}} : C(\mathbf{X}) \to \mathbf{X}$ *is a 1-morphism.*

(b) *Let* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *be a 1-morphism of d-spaces with corners. Then there is a unique 1-morphism* $C(\boldsymbol{f}) : C(\mathbf{X}) \to C(\mathbf{Y})$ *in* $\mathbf{dSpa^c}$ *such that* $\mathbf{\Pi}_{\mathbf{Y}} \circ C(\boldsymbol{f}) = \boldsymbol{f} \circ \mathbf{\Pi}_{\mathbf{X}} : C(\mathbf{X}) \to \mathbf{Y}$, *and* $C(\boldsymbol{f})$ *acts on points as in* (6.5) *by*

$$
\begin{aligned}
C(f) : \big(x, \{x_1', \ldots, x_k'\} \big) \longmapsto \big(y, \{y_1', \ldots, y_l'\} \big), \quad where \\
\{y_1', \ldots, y_l'\} = \{ y' : (x_i', y') \in \underline{S}_{\boldsymbol{f}}, \ some \ i = 1, \ldots, k \}.
\end{aligned}
$$

For all $k, l \geqslant 0$, *write* $C_k^{\boldsymbol{f}, l}(\mathbf{X}) = C_k(\mathbf{X}) \cap C(\boldsymbol{f})^{-1}(C_l(\mathbf{Y}))$, *so that* $C_k^{\boldsymbol{f}, l}(\mathbf{X})$ *is open and closed in* $C_k(\mathbf{X})$ *with* $C_k(\mathbf{X}) = \coprod_{l=0}^{\infty} C_k^{\boldsymbol{f}, l}(\mathbf{X})$, *and write* $C_k^l(\boldsymbol{f}) = C(\boldsymbol{f})|_{C_k^{\boldsymbol{f}, l}(\mathbf{X})}$, *so that* $C_k^l(\boldsymbol{f}) : C_k^{\boldsymbol{f}, l}(\mathbf{X}) \to C_l(\mathbf{Y})$ *is a 1-morphism in* $\mathbf{dSpa^c}$.

(c) *Let* $\boldsymbol{f}, \boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ *be 1-morphisms and* $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ *a 2-morphism in* $\mathbf{dSpa^c}$. *Then there exists a unique 2-morphism* $C(\eta) : C(\boldsymbol{f}) \Rightarrow C(\boldsymbol{g})$ *in* $\mathbf{dSpa^c}$, *where* $C(\boldsymbol{f}), C(\boldsymbol{g})$ *are as in* **(b)**, *such that*

$$
\mathrm{id}_{\mathbf{\Pi}_{\mathbf{Y}}} * C(\eta) = \eta * \mathrm{id}_{\mathbf{\Pi}_{\mathbf{X}}} : \mathbf{\Pi}_{\mathbf{Y}} \circ C(\boldsymbol{f}) = \boldsymbol{f} \circ \mathbf{\Pi}_{\mathbf{X}} \Longrightarrow \mathbf{\Pi}_{\mathbf{Y}} \circ C(\boldsymbol{g}) = \boldsymbol{g} \circ \mathbf{\Pi}_{\mathbf{X}}.
$$

(d) *Define* $C : \mathbf{dSpa^c} \to \mathbf{dSpa^c}$ *by* $C : \mathbf{X} \mapsto C(\mathbf{X})$ *on objects,* $C : \boldsymbol{f} \mapsto C(\boldsymbol{f})$ *on 1-morphisms, and* $C : \eta \mapsto C(\eta)$ *on 2-morphisms, where* $C(\mathbf{X}), C(\boldsymbol{f}), C(\eta)$ *are as in* **(a)**–**(c)** *above. Then* C *is a strict 2-functor, called a* **corner functor**.

(e) *Let* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *be semisimple. Then* $C(\boldsymbol{f})$ *maps* $C_k(\mathbf{X}) \to \coprod_{l=0}^{k} C_l(\mathbf{Y})$ *for all* $k \geqslant 0$. *The natural 1-isomorphisms* $C_1(\mathbf{X}) \cong \partial \mathbf{X}, C_0(\mathbf{Y}) \cong \mathbf{Y}, C_1(\mathbf{Y}) \cong \partial \mathbf{Y}$ *identify* $C_1^{\boldsymbol{f}, 0}(\mathbf{X}) \cong \partial_+^{\boldsymbol{f}} \mathbf{X}, C_1^{\boldsymbol{f}, 1}(\mathbf{X}) \cong \partial_-^{\boldsymbol{f}} \mathbf{X}, C_1^0(\boldsymbol{f}) \cong \boldsymbol{f}_+$ *and* $C_1^1(\boldsymbol{f}) \cong \boldsymbol{f}_-$.
 If $\boldsymbol{f}$ *is simple then* $C(\boldsymbol{f})$ *maps* $C_k(\mathbf{X}) \to C_k(\mathbf{Y})$ *for all* $k \geqslant 0$.

(f) *Analogues of* **(b)**–**(d)** *also hold for a second corner functor* $\hat{C} : \mathbf{dSpa^c} \to \mathbf{dSpa^c}$, *which acts on objects by* $\hat{C} : \mathbf{X} \mapsto C(\mathbf{X})$ *in* **(a)**, *and for 1-morphisms* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *in* **(b)**, $\hat{C}(\boldsymbol{f}) : C(\mathbf{X}) \to C(\mathbf{Y})$ *acts on points by*

$$
\begin{aligned}
\hat{C}(f) : \big(x, \{x_1', \ldots, x_k'\} \big) \longmapsto \big(y, \{y_1', \ldots, y_l'\} \big), \quad where \\
\{y_1', \ldots, y_l'\} = \{ y' : (x_i', y') \in \underline{S}_{\boldsymbol{f}}, \ some \ i = 1, \ldots, k \} \cup \{ y' : (x, y') \in \underline{T}_{\boldsymbol{f}} \}.
\end{aligned}
$$

If $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *is flat then* $\hat{C}(\boldsymbol{f}) = C(\boldsymbol{f})$.

The comments of Remark 6.5 also apply to Theorem 6.10: our construction characterizes $C_k(\mathbf{X})$ up to 1-isomorphism in $\mathbf{dSpa^c}$, not just up to equivalence,

the 1-morphisms $C(\boldsymbol{f}), \hat{C}(\boldsymbol{f})$ are characterized up to equality, not just up to 2-isomorphism, and in $\boldsymbol{\Pi_Y} \circ C(\boldsymbol{f}) = \boldsymbol{f} \circ \boldsymbol{\Pi_X}$ we require the 1-morphisms to be equal, not just 2-isomorphic. This may seem unnatural from a 2-category point of view, but it has the advantage that corners are strictly 2-functorial rather than weakly 2-functorial.

6.6 Fibre products in dSpa$^{\mathbf{c}}$

In [35, §6.8–§6.9] we study *fibre products* in $\mathbf{dSpa^c}$. Here the situation is more complex than for d-spaces. As in §3.2, all fibre products exist in $\mathbf{dSpa}$, but this fails for $\mathbf{dSpa^c}$. The problem is that in a fibre product $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g}, \mathbf{Z}, \boldsymbol{h}} \mathbf{Y}$ in $\mathbf{dSpa^c}$, the boundary $\partial \mathbf{W}$ depends in a complicated way on $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \partial \mathbf{X}, \partial \mathbf{Y}, \partial \mathbf{Z}$, and sometimes there is no good candidate for $\partial \mathbf{W}$. Here is an example.

Example 6.11. Let $X = Y = [0, \infty) \times \mathbb{R}$ and $Z = [0, \infty)^2 \times \mathbb{R}$, as manifolds with corners, and define smooth maps $g : X \to Z$ and $h : Y \to Z$ by $g(u, v) = (u, u, v)$ and $h(u, v) = (u, e^v u, v)$. Set $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \boldsymbol{g}, \boldsymbol{h} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(X, Y, Z, g, h)$.

In [35, §6.8.6] we show that no fibre product $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g}, \mathbf{Z}, \boldsymbol{h}} \mathbf{Y}$ exists in $\mathbf{dSpa^c}$. We do this by showing that $\partial \mathbf{W}$ would have to have exactly one point, lying over $(0, 0) \in X$ and $(0, 0) \in Y$, which is the only point in $X \times_Z Y$ where normal vectors to $\partial X, \partial Y$ in X, Y project under dg, dh to parallel vectors in TZ. But this would contradict other properties of $\partial \mathbf{W}$.

So, we would like to find useful sufficient conditions for existence of fibre products $\mathbf{X} \times_{\boldsymbol{g}, \mathbf{Z}, \boldsymbol{h}} \mathbf{Y}$ in $\mathbf{dSpa^c}$; and these conditions should be wholly to do with boundaries, since we already know that fibre products exist in $\mathbf{dSpa}$. In [35, §6.8.1] we define two such sufficient conditions on $\boldsymbol{g}, \boldsymbol{h}$, called *b-transversality* and *c-transversality*.

Definition 6.12. Let $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ and $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ be 1-morphisms in $\mathbf{dSpa^c}$. As in §6.1 we have line bundles $\mathcal{N}_{\mathbf{X}}, \mathcal{N}_{\mathbf{Z}}$ over the C^∞-schemes $\underline{\partial X}, \underline{\partial Z}$, and a C^∞-subscheme $\underline{S}_{\boldsymbol{g}} \subseteq \underline{\partial X} \times_{\underline{Z}} \underline{\partial Z}$. As in [35, §7.1], there is a natural isomorphism $\lambda_{\boldsymbol{g}} : \underline{u}_{\boldsymbol{g}}^*(\mathcal{N}_{\mathbf{Z}}) \to \underline{s}_{\boldsymbol{f}}^*(\mathcal{N}_{\mathbf{X}})$ in qcoh$(\underline{S}_{\boldsymbol{g}})$. The same holds for $\boldsymbol{h}$.

We say that $\boldsymbol{g}, \boldsymbol{h}$ are *b-transverse* if whenever $x \in \underline{X}$ and $y \in \underline{Y}$ with $\underline{g}(x) = \underline{h}(y) = z \in \underline{Z}$, the following morphism in qcoh$(\underline{*})$ is injective:

$$\bigoplus_{(x', z') \in \underline{S}_{\boldsymbol{g}} : \underline{i}_{\mathbf{X}}(x') = x} \lambda_{\boldsymbol{g}}|_{(x', z')} \oplus \bigoplus_{(y', z') \in \underline{S}_{\boldsymbol{h}} : \underline{i}_{\mathbf{Y}}(y') = y} \lambda_{\boldsymbol{h}}|_{(y', z')} :$$

$$\bigoplus_{z' \in \underline{i}_{\mathbf{Z}}^{-1}(z)} \mathcal{N}_{\mathbf{Z}}|_{z'} \longrightarrow \bigoplus_{x' \in \underline{i}_{\mathbf{X}}^{-1}(x)} \mathcal{N}_{\mathbf{X}}|_{x'} \oplus \bigoplus_{y' \in \underline{i}_{\mathbf{Y}}^{-1}(y)} \mathcal{N}_{\mathbf{Y}}|_{y'}.$$

Roughly speaking, this says that the corners of $\mathbf{X}, \mathbf{Y}$ are transverse to the corners of $\mathbf{Z}$. In Example 6.11, this condition fails at $x = 0 \in \underline{X}$ and $y = 0 \in \underline{Y}$, so $\boldsymbol{g}, \boldsymbol{h}$ are not b-transverse.

We call $\boldsymbol{g}, \boldsymbol{h}$ *c-transverse* if the following two conditions hold, using the notation of Theorem 6.10:

(a) whenever there are points in $C_j(\mathbf{X}), C_k(\mathbf{Y}), C_l(\mathbf{Z})$ with

$$C(\boldsymbol{g})(x, \{x_1', \ldots, x_j'\}) = C(\boldsymbol{h})(y, \{y_1', \ldots, y_k'\}) = (z, \{z_1', \ldots, z_l'\}),$$

we have either $j + k > l$ or $j = k = l = 0$; and

(b) whenever there are points in $C_j(\mathbf{X}), C_k(\mathbf{Y}), C_l(\mathbf{Z})$ with

$$\hat{C}(\boldsymbol{g})(x, \{x_1', \ldots, x_j'\}) = \hat{C}(\boldsymbol{h})(y, \{y_1', \ldots, y_k'\}) = (z, \{z_1', \ldots, z_l'\}),$$

we have $j + k \geqslant l$.

Here b-transversality is a continuous condition on $\boldsymbol{g}, \boldsymbol{h}$, and c-transversality is a discrete condition. Also c-transversality implies b-transversality (though this is not obvious). Part (a) corresponds to the condition in Definition 5.12 for transverse g, h in $\mathbf{Man^c}$ to be strongly transverse. We can show:

Lemma 6.13. *Let* $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ *and* $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ *be 1-morphisms in* $\mathbf{dSpa^c}$. *The following are sufficient conditions for* $\boldsymbol{g}, \boldsymbol{h}$ *to be c-transverse, and hence b-transverse:*

(i) $\boldsymbol{g}$ *or* $\boldsymbol{h}$ *is semisimple and flat; or*

(ii) $\mathbf{Z}$ *is a d-space without boundary.*

We summarize the main results of [35, §6.8] on fibre products in $\mathbf{dSpa^c}$:

Theorem 6.14. (a) *All b-transverse fibre products exist in* $\mathbf{dSpa^c}$.

(b) *The 2-functor* $F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}$ *of §6.3 takes transverse fibre products in* $\mathbf{Man^c}$ *to b-transverse fibre products in* $\mathbf{dSpa^c}$. *That is, if*

$$
\begin{array}{ccc}
W & \xrightarrow{\quad f \quad} & Y \\
{\scriptstyle e}\downarrow & & \downarrow{\scriptstyle h} \\
X & \xrightarrow{\quad g \quad} & Z
\end{array}
$$

is a Cartesian square in $\mathbf{Man^c}$ *with* g, h *transverse, and* $\mathbf{W}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{g},$ $\boldsymbol{h} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(W, X, Y, Z, e, f, g, h),$ *then*

$$
\begin{array}{ccc}
\mathbf{W} & \xrightarrow{\quad \boldsymbol{f} \quad} & \mathbf{Y} \\
{\scriptstyle \boldsymbol{e}}\downarrow & \quad \mathrm{id}_{\boldsymbol{g} \circ \boldsymbol{e}}\Uparrow & \downarrow{\scriptstyle \boldsymbol{h}} \\
\mathbf{X} & \xrightarrow{\quad \boldsymbol{g} \quad} & \mathbf{Z}
\end{array}
$$

is 2-Cartesian in $\mathbf{dSpa^c}$, *with* $\boldsymbol{g}, \boldsymbol{h}$ *b-transverse. If also* g, h *are strongly transverse in* $\mathbf{Man^c}$, *then* $\boldsymbol{g}, \boldsymbol{h}$ *are c-transverse in* $\mathbf{dSpa^c}$.

(c) *Suppose we are given a 2-Cartesian diagram in* $\mathbf{dSpa^c}$:

$$
\begin{array}{ccc}
\mathbf{W} & \xrightarrow{\quad \boldsymbol{f} \quad} & \mathbf{Y} \\
{\scriptstyle \boldsymbol{e}}\downarrow & \quad \eta\Uparrow & \downarrow{\scriptstyle \boldsymbol{h}} \\
\mathbf{X} & \xrightarrow{\quad \boldsymbol{g} \quad} & \mathbf{Z},
\end{array}
$$

with $\boldsymbol{g}, \boldsymbol{h}$ c-transverse. Then the following are also 2-Cartesian in $\mathbf{dSpa^c}$:

$$
\begin{array}{ccc}
C(\mathbf{W}) & \xrightarrow{C(\boldsymbol{f})} & C(\mathbf{Y}) \\
{\scriptstyle C(\boldsymbol{e})}\downarrow \quad {\scriptstyle C(\eta)\Uparrow} & & \downarrow{\scriptstyle C(\boldsymbol{h})} \\
C(\mathbf{X}) & \xrightarrow[C(\boldsymbol{g})]{} & C(\mathbf{Z}),
\end{array}
\tag{6.6}
$$

$$
\begin{array}{ccc}
C(\mathbf{W}) & \xrightarrow{\hat{C}(\boldsymbol{f})} & C(\mathbf{Y}) \\
{\scriptstyle \hat{C}(\boldsymbol{e})}\downarrow \quad {\scriptstyle \hat{C}(\eta)\Uparrow} & & \downarrow{\scriptstyle \hat{C}(\boldsymbol{h})} \\
C(\mathbf{X}) & \xrightarrow[\hat{C}(\boldsymbol{g})]{} & C(\mathbf{Z}).
\end{array}
\tag{6.7}
$$

Also (6.6)–(6.7) preserve gradings, in that they relate points in $C_i(\mathbf{W}), C_j(\mathbf{X}),$ $C_k(\mathbf{Y}), C_k(\mathbf{Z})$ with $i = j + k - l$. Hence (6.6) implies equivalences in $\mathbf{dSpa^c}$:

$$
C_i(\mathbf{W}) \simeq \coprod_{j,k,l\geqslant 0: i=j+k-l} C_j^{\boldsymbol{g},l}(\mathbf{X}) \times_{C_j^l(\boldsymbol{g}),C_l(\mathbf{Z}),C_k^l(\boldsymbol{h})} C_k^{\boldsymbol{h},l}(\mathbf{Y}),
\tag{6.8}
$$

$$
\partial\mathbf{W} \simeq \coprod_{j,k,l\geqslant 0: j+k=l+1} C_j^{\boldsymbol{g},l}(\mathbf{X}) \times_{C_j^l(\boldsymbol{g}),C_l(\mathbf{Z}),C_k^l(\boldsymbol{h})} C_k^{\boldsymbol{h},l}(\mathbf{Y}).
\tag{6.9}
$$

Part (a) takes some work to prove. For fibre products in $\mathbf{dSpa}$, as in §3.3, we gave an explicit global construction. But for fibre products in $\mathbf{dSpa^c}$, we first prove that local fibre products $\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ exist in $\mathbf{dSpa^c}$ near each $x \in \mathbf{X}$, $y \in \mathbf{Y}$ with $\boldsymbol{g}(x) = \boldsymbol{h}(y) \in \mathbf{Z}$, and then we use the results of §6.4 to glue these local fibre products by equivalences to get a global fibre product.

For general b-transverse fibre products $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}$ in $\mathbf{dSpa^c}$, the description of $\partial\mathbf{W}$ can be complicated. For c-transverse fibre products, we do at least have a (still complicated) explicit formula (6.9) for $\partial\mathbf{W}$. Here are some cases when this formula simplifies, an analogue of Proposition 5.14.

Proposition 6.15. *Let $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ and $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ be 1-morphisms of d-spaces with corners. Then:*

(a) *If $\partial\mathbf{Z} = \emptyset$ then there is an equivalence*

$$
\partial(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}) \simeq (\partial\mathbf{X} \times_{\boldsymbol{g}\circ i_{\mathbf{X}},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}) \amalg (\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}\circ i_{\mathbf{Y}}} \partial\mathbf{Y}).
\tag{6.10}
$$

(b) *If $\boldsymbol{g}$ is semisimple and flat then there is an equivalence*

$$
\partial(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}) \simeq (\partial_+^{\boldsymbol{g}}\mathbf{X} \times_{\boldsymbol{g}_+,\mathbf{Z},\boldsymbol{h}} \mathbf{Y}) \amalg (\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}\circ i_{\mathbf{Y}}} \partial\mathbf{Y}).
\tag{6.11}
$$

(c) *If both $\boldsymbol{g}$ and $\boldsymbol{h}$ are semisimple and flat then there is an equivalence*

$$
\begin{aligned}
\partial(\mathbf{X} \times_{\boldsymbol{g},\mathbf{Z},\boldsymbol{h}} \mathbf{Y}) \simeq {} & (\partial_+^{\boldsymbol{g}}\mathbf{X} \times_{\boldsymbol{g}_+,\mathbf{Z},\boldsymbol{h}_+} \partial_+^{\boldsymbol{h}}\mathbf{Y}) \\
& \amalg (\partial_-^{\boldsymbol{g}}\mathbf{X} \times_{\boldsymbol{g}_-,\partial\mathbf{Z},\boldsymbol{h}_-} \partial_-^{\boldsymbol{h}}\mathbf{Y}).
\end{aligned}
\tag{6.12}
$$

Here all fibre products in (6.10)–(6.12) are c-transverse, and so exist.

6.7 Fixed point loci in d-spaces with corners

Section 3.4 discussed the fixed d-subspace $\boldsymbol{X}^\Gamma$ of a finite group Γ acting on a d-space $\boldsymbol{X}$, and §5.6 considered fixed point loci X^Γ of a finite group Γ acting on a manifold with corners X. In [35, §6.10] we generalize these to d-spaces with corners. Here is the analogue of Theorem 3.10.

Theorem 6.16. *Let $\mathbf{X}$ be a d-space with corners, Γ a finite group, and $\boldsymbol{r} : \Gamma \to$ Aut($\mathbf{X}$) an action of Γ on $\mathbf{X}$ by 1-isomorphisms. Then we can define a d-space with corners $\mathbf{X}^\Gamma$ called the **fixed d-subspace of Γ in $\mathbf{X}$**, with an inclusion 1-morphism $\boldsymbol{j}_{\mathbf{X},\Gamma} : \mathbf{X}^\Gamma \to \mathbf{X}$. It has the following properties:*

(a) *Let $\mathbf{X}, \Gamma, \boldsymbol{r}$ and $\boldsymbol{j}_{\mathbf{X},\Gamma} : \mathbf{X}^\Gamma \to \mathbf{X}$ be as above. Suppose $\boldsymbol{f} : \mathbf{W} \to \mathbf{X}$ is a 1-morphism in $\mathbf{dSpa^c}$. Then $\boldsymbol{f}$ factorizes as $\boldsymbol{f} = \boldsymbol{j}_{\mathbf{X},\Gamma} \circ \boldsymbol{g}$ for some 1-morphism $\boldsymbol{g} : \mathbf{W} \to \mathbf{X}^\Gamma$ in $\mathbf{dSpa^c}$, which must be unique, if and only if $\boldsymbol{r}(\gamma) \circ \boldsymbol{f} = \boldsymbol{f}$ for all $\gamma \in \Gamma$.*

(b) *Suppose $\mathbf{X}, \mathbf{Y}$ are d-spaces with corners, Γ is a finite group, $\boldsymbol{r} : \Gamma \to$ Aut($\mathbf{X}$), $\boldsymbol{s} : \Gamma \to$ Aut($\mathbf{Y}$) are actions of Γ on $\mathbf{X}, \mathbf{Y}$, and $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ is a Γ-equivariant 1-morphism in $\mathbf{dSpa^c}$, that is, $\boldsymbol{f} \circ \boldsymbol{r}(\gamma) = \boldsymbol{s}(\gamma) \circ \boldsymbol{f}$ for all $\gamma \in \Gamma$. Then there exists a unique 1-morphism $\boldsymbol{f}^\Gamma : \mathbf{X}^\Gamma \to \mathbf{Y}^\Gamma$ such that $\boldsymbol{j}_{\mathbf{Y},\Gamma} \circ \boldsymbol{f}^\Gamma = \boldsymbol{f} \circ \boldsymbol{j}_{\mathbf{X},\Gamma}$.*

(c) *Let $\boldsymbol{f}, \boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ be Γ-equivariant 1-morphisms as in (b), and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ be a Γ-equivariant 2-morphism, that is, $\eta * \mathrm{id}_{\boldsymbol{r}(\gamma)} = \mathrm{id}_{\boldsymbol{s}(\gamma)} * \eta$ for all $\gamma \in \Gamma$. Then there exists a unique 2-morphism $\eta^\Gamma : \boldsymbol{f}^\Gamma \Rightarrow \boldsymbol{g}^\Gamma$ such that $\mathrm{id}_{\boldsymbol{j}_{\mathbf{Y},\Gamma}} * \eta^\Gamma = \eta * \mathrm{id}_{\boldsymbol{j}_{\mathbf{X},\Gamma}}$.*

Note that (a) is a universal property that determines $\boldsymbol{X}^\Gamma, \boldsymbol{j}_{\boldsymbol{X},\Gamma}$ up to canonical 1-isomorphism.

As for manifolds with corners in §5.6, in general $\partial(\mathbf{X}^\Gamma) \not\simeq (\partial\mathbf{X})^\Gamma$, so fixed point loci do not commute with boundaries. But the following analogue of Proposition 5.17(b) shows that fixed point loci do commute with corners.

Proposition 6.17. *Let $\mathbf{X}$ be a d-space with corners, Γ a finite group, and $\boldsymbol{r} : \Gamma \to$ Aut($\mathbf{X}$) an action of Γ on $\mathbf{X}$. Applying the corner functor C of §6.5 gives an action $C(\boldsymbol{r}) : \Gamma \to$ Aut($C(\mathbf{X})$). Hence Theorem 6.16 defines fixed d-subspaces $\mathbf{X}^\Gamma, C(\mathbf{X})^\Gamma$ and inclusion 1-morphisms $\boldsymbol{j}_{\mathbf{X},\Gamma} : \mathbf{X}^\Gamma \to \mathbf{X}$, $\boldsymbol{j}_{C(\mathbf{X}),\Gamma} : C(\mathbf{X})^\Gamma \to C(\mathbf{X})$. Applying C to $\boldsymbol{j}_{\mathbf{X},\Gamma}$ also gives $C(\boldsymbol{j}_{\mathbf{X},\Gamma}) : C(\mathbf{X}^\Gamma) \to C(\mathbf{X})$.*

Then there exists a unique equivalence $\boldsymbol{k}_{\mathbf{X},\Gamma} : C(\mathbf{X}^\Gamma) \to C(\mathbf{X})^\Gamma$ in $\mathbf{dSpa^c}$ such that $C(\boldsymbol{j}_{\mathbf{X},\Gamma}) = \boldsymbol{j}_{C(\mathbf{X}),\Gamma} \circ \boldsymbol{k}_{\mathbf{X},\Gamma}$.

We will use fixed d-subspaces $\mathbf{X}^\Gamma$ in §13.7 below to describe orbifold strata $\boldsymbol{\mathcal{X}}^\Gamma$ of quotient d-stacks with corners $\boldsymbol{\mathcal{X}} = [\mathbf{X}/G]$. If $\mathbf{X}$ is a d-manifold with corners, as in §7, then in general the fixed d-subspaces $\mathbf{X}^\Gamma$ are disjoint unions of d-manifolds with corners of different dimensions, that is, $\mathbf{X}^\Gamma$ lies in $\mathbf{d\check{M}an^c}$.

7 D-manifolds with corners

We can now define the 2-categories $\mathbf{dMan^b}$ of *d-manifolds with boundary* and $\mathbf{dMan^c}$ of *d-manifolds with corners*, which are derived versions of manifolds with boundary and with corners, following [35, Chap. 7].

7.1 The definition of d-manifolds with corners

In §4.1 we defined a d-manifold to be a d-space covered by principal open d-submanifolds of fixed dimension, where Proposition 4.2 gave three equivalent definitions of principal d-manifolds, the first as a fibre product $\boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ in $\mathbf{dSpa}$ with $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \hat{\mathbf{Man}}$, and the third as a fibre product $\boldsymbol{V} \times_{\boldsymbol{s, E, 0}} \boldsymbol{V}$ in $\mathbf{dSpa}$, where V is a manifold, $E \to V$ a vector bundle, and $s \in C^\infty(E)$.

When we pass to d-spaces and d-manifolds with corners in [35, §7.1], the analogues of Proposition 4.2(a)–(c) are no longer equivalent. So we have to choose which of them gives the best idea of principal d-manifold with corners. Defining principal d-manifolds with corners to be fibre products $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ in $\mathbf{dSpa^c}$ with $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \bar{\mathbf{Man}}^{\mathbf{c}}$ is unsatisfactory, since as in §6.6 fibre products $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ may not exist in $\mathbf{dSpa^c}$. So instead we define principal d-manifolds with corners to be fibre products $\mathbf{V} \times_{\boldsymbol{s, E, 0}} \mathbf{V}$ in $\mathbf{dSpa^c}$.

Definition 7.1. A d-space with corners $\mathbf{W}$ is called a *principal d-manifold with corners* if is equivalent in $\mathbf{dSpa^c}$ to a fibre product $\mathbf{V} \times_{\boldsymbol{s, E, 0}} \mathbf{V}$, where V is a manifold with corners, $E \to V$ is a vector bundle, $s : V \to E$ is a smooth section of E, $0 : V \to E$ is the zero section, and $\mathbf{V}, \mathbf{E}, \boldsymbol{s}, \mathbf{0} = F_{\mathbf{Man^c}}^{\mathbf{dSpa^c}}(V, E, s, 0)$. Note that $s, 0 : V \to E$ are simple, flat smooth maps in $\mathbf{Man^c}$, so $\boldsymbol{s}, \mathbf{0} : \mathbf{V} \to \mathbf{E}$ are simple, flat 1-morphisms in $\mathbf{dSpa^c}$, and thus $\boldsymbol{s}, \mathbf{0}$ are b-transverse by Lemma 6.13(a), and the fibre product $\mathbf{V} \times_{\boldsymbol{s, E, 0}} \mathbf{V}$ exists in $\mathbf{dSpa^c}$ by Theorem 6.14(a).

If $\mathbf{W} \simeq \mathbf{V} \times_{\boldsymbol{s, E, 0}} \mathbf{V}$ then the virtual cotangent sheaf $T^* \boldsymbol{W}$ of the d-space $\boldsymbol{W}$ is a virtual vector bundle with rank $T^* \boldsymbol{W} = \dim V - \operatorname{rank} E$. Hence, if $\mathbf{W} \neq \emptyset$ then the integer $\dim V - \operatorname{rank} E$ depends only on $\boldsymbol{W}$ up to equivalence in $\mathbf{dSpa}$, and is independent of the choice of V, E, s with $\mathbf{W} \simeq \mathbf{V} \times_{\boldsymbol{s, E, 0}} \mathbf{V}$. Define the *virtual dimension* vdim $\mathbf{W}$ to be vdim $\mathbf{W} = \operatorname{rank} T^* \boldsymbol{W} = \dim V - \operatorname{rank} E$.

A d-space with corners $\mathbf{X}$ is called a *d-manifold with corners of virtual dimension* $n \in \mathbb{Z}$, written vdim $\mathbf{X} = n$, if $\mathbf{X}$ can be covered by open d-subspaces $\mathbf{W}$ which are principal d-manifolds with corners with vdim $\mathbf{W} = n$. A d-manifold with corners $\mathbf{X}$ is called a *d-manifold with boundary* if it is a d-space with boundary, and a *d-manifold without boundary* if it is a d-space without boundary.

Write $\mathbf{d\bar{M}an}, \mathbf{dMan^b}, \mathbf{dMan^c}$ for the full 2-subcategories of d-manifolds without boundary, and d-manifolds with boundary, and d-manifolds with corners in $\mathbf{dSpa^c}$, respectively. The 2-functor $F_{\mathbf{dSpa}}^{\mathbf{dSpa^c}} : \mathbf{dSpa} \to \mathbf{dSpa^c}$ in §6.1 is an isomorphism of 2-categories $\mathbf{dSpa} \to \mathbf{d\bar{S}pa}$, and its restriction to $\mathbf{dMan} \subset \mathbf{dSpa}$ gives an isomorphism of 2-categories $F_{\mathbf{dMan}}^{\mathbf{dMan^c}} : \mathbf{dMan} \to \mathbf{d\bar{M}an} \subset \mathbf{dMan^c}$. So we may as well identify $\mathbf{dMan}$ with its image $\mathbf{d\bar{M}an}$, and consider d-manifolds in §4 as examples of d-manifolds with corners.

If $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i_X}, \omega_{\mathbf{X}})$ is a d-manifold with corners, then the virtual cotangent sheaf $T^*\boldsymbol{X}$ of the d-space $\boldsymbol{X}$ from Definition 4.10 is a virtual vector bundle on $\underline{X}$, of rank vdim $\mathbf{X}$. We will call $T^*\boldsymbol{X} \in \mathrm{vvect}(\underline{X})$ the *virtual cotangent bundle* of $\mathbf{X}$, and also write it $T^*\mathbf{X}$.

Much of §6 on d-spaces with corners applies immediately to d-manifolds with corners. If $\mathbf{X}$ is a d-manifold with corners with vdim $\mathbf{X} = n$ then the boundary $\partial\mathbf{X}$ as a d-space with corners from §6.1 is a d-manifold with corners, with vdim $\partial\mathbf{X} = n - 1$. The material on simple, semisimple, and flat 1-morphisms in $\mathbf{dSpa^c}$ in §6.2 also holds in $\mathbf{dMan^c}$. The functor $F^{\mathbf{dSpa^c}}_{\mathbf{Man^c}} :$ $\mathbf{Man^c} \to \mathbf{dSpa^c}$ in §6.3 maps to $\mathbf{dMan^c} \subset \mathbf{dSpa^c}$, so we write $F^{\mathbf{dMan^c}}_{\mathbf{Man^c}} = F^{\mathbf{dSpa^c}}_{\mathbf{Man^c}} : \mathbf{Man^c} \to \mathbf{dMan^c}$. The 2-categories $\mathbf{\bar{M}an}, \mathbf{\bar{M}an^b}, \mathbf{\bar{M}an^c}$ in Definition 6.6 are 2-subcategories of $\mathbf{d\bar{M}an}, \mathbf{dMan^b}, \mathbf{dMan^c}$, respectively. When we say that a d-manifold with corners $\mathbf{X}$ *is a manifold*, we mean that $\mathbf{X} \in \mathbf{\bar{M}an^c}$.

In §6.4, if we make a d-space with corners $\mathbf{Y}$ by gluing together d-manifolds with corners $\mathbf{X}_i$ for $i \in I$ by equivalences, then $\mathbf{Y}$ is a d-manifold with corners with vdim $\mathbf{Y} = n$ provided vdim $\mathbf{X}_i = n$ for all $i \in I$.

In §6.5, if $\mathbf{X}$ is a d-manifold with corners with vdim $\mathbf{X} = n$ then the k-corners $C_k(\mathbf{X})$ is a d-manifold with corners, with vdim $C_k(\mathbf{X}) = n - k$. Note however that $C(\mathbf{X}) = \coprod_{k=0}^{\infty} C_k(\mathbf{X})$ in Theorem 6.10 is in general *not* a d-manifold with corners, but only a disjoint union of d-manifolds with corners with different dimensions. As for $\mathbf{\check{M}an^c}$ in §5.3, define $\mathbf{d\check{M}an^c}$ to be the full 2-subcategory of $\mathbf{X}$ in $\mathbf{dSpa^c}$ which may be written as a disjoint union $\mathbf{X} = \coprod_{n \in \mathbb{Z}} \mathbf{X}_n$ for $\mathbf{X}_n \in \mathbf{dMan^c}$ with vdim $\mathbf{X}_n = n$, where we allow $\mathbf{X}_n = \emptyset$. We call such $\mathbf{X}$ a *d-manifold with corners of mixed dimension*. Then $C, \hat{C}$ in Theorem 6.10 restrict to strict 2-functors $C, \hat{C} : \mathbf{dMan^c} \to \mathbf{d\check{M}an^c}$.

Here are some examples. The fibre products we give all exist in $\mathbf{dMan^c}$ by results in §7.5 below.

Example 7.2. (i) Let $\mathbf{X}$ be the fibre product $[\boldsymbol{0}, \boldsymbol{\infty}) \times_{\boldsymbol{i}, \mathbb{R}, \boldsymbol{0}} *$ in $\mathbf{dMan^c}$, where $i : [0, \infty) \hookrightarrow \mathbb{R}$ is the inclusion. Then $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i_X}, \omega_{\mathbf{X}})$ is 'a point with point boundary', of virtual dimension 0, and its boundary $\partial\mathbf{X}$ is an 'obstructed point', a point with obstruction space $\mathbb{R}$, of virtual dimension -1.

The conormal bundle $\mathcal{N}_{\mathbf{X}}$ of $\boldsymbol{\partial X}$ in $\boldsymbol{X}$ is the obstruction space $\mathbb{R}$ of $\boldsymbol{\partial X}$. In this case, the orientation $\omega_{\mathbf{X}}$ on $\mathcal{N}_{\mathbf{X}}$ cannot be determined from $\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i_X}$, in fact, there is an automorphism of $\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i_X}$ which reverses the orientation of $\mathcal{N}_{\mathbf{X}}$. So $\omega_{\mathbf{X}}$ really is extra data. We include $\omega_{\mathbf{X}}$ in the definition of d-manifolds with corners to ensure that orientations of d-manifolds with corners are well-behaved. If we omitted $\omega_{\mathbf{X}}$ from the definition, there would exist oriented d-manifolds with corners $\mathbf{X}$ whose boundaries $\partial\mathbf{X}$ are not orientable.

(ii) The fibre product $[\boldsymbol{0}, \boldsymbol{\infty}) \times_{\boldsymbol{i}, [\boldsymbol{0}, \boldsymbol{\infty}), \boldsymbol{0}} *$ is a point $*$ without boundary. The only difference with **(i)** is that we have replaced the target $\mathbb{R}$ with $[\boldsymbol{0}, \boldsymbol{\infty})$, adding a boundary. So in a fibre product $\mathbf{W} = \mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ in $\mathbf{dMan^c}$, the boundary of $\mathbf{Z}$ affects the boundary of $\mathbf{W}$. This does not happen for fibre products in $\mathbf{Man^c}$.

(iii) Let $\mathbf{X}'$ be the fibre product $[\boldsymbol{0}, \boldsymbol{\infty}) \times_{\boldsymbol{i}, \mathbb{R}, \boldsymbol{i}} (-\boldsymbol{\infty}, \boldsymbol{0}]$ in $\mathbf{dMan^c}$, that is, the derived intersection of submanifolds $[0, \infty), (-\infty, 0]$ in $\mathbb{R}$. Topologically, $\mathbf{X}'$ is

just the point $\{0\}$, but as a d-manifold with corners $\mathbf{X}'$ has virtual dimension 1. The boundary $\partial\mathbf{X}'$ is the disjoint union of two copies of $\mathbf{X}$ in **(i)**. The C^∞-scheme $\underline{X}'$ in $\mathbf{X}'$ is the spectrum of the C^∞-ring $C^\infty\big([0,\infty)^2\big)/(x+y)$, which is infinite-dimensional, although its topological space is a point.

7.2 'Standard model' d-manifolds with corners

In Examples 4.4 and 4.5 of §4.2, we defined 'standard model' d-manifolds $\boldsymbol{S}_{V,E,s}$ and 1-morphisms $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$. In [35, §7.1–§7.2] we show that all this extends to d-manifolds with corners in a straightforward way.

Example 7.3. Let V be a manifold with corners, $E \to V$ a vector bundle, and $s : V \to E$ a smooth section of E. We will write down an explicit principal d-manifold with corners $\mathbf{S} = (\boldsymbol{S}, \partial\boldsymbol{S}, \boldsymbol{i}_\mathbf{S}, \omega_\mathbf{S})$.

Define a vector bundle $E_\partial \to \partial V$ by $E_\partial = i_V^*(E)$, and a section $s_\partial : \partial V \to E_\partial$ by $s_\partial = i_V^*(s)$. Define d-spaces $\boldsymbol{S} = \boldsymbol{S}_{V,E,s}$ and $\partial\boldsymbol{S} = \boldsymbol{S}_{\partial V,E_\partial,s_\partial}$ from the triples V, E, s and $\partial V, E_\partial, s_\partial$ exactly as in Example 4.4, although now $V, \partial V$ have corners. Define a 1-morphism $\boldsymbol{i}_\mathbf{S} : \partial\boldsymbol{S} \to \boldsymbol{S}$ in $\mathbf{dSpa}$ to be the 'standard model' 1-morphism $\boldsymbol{S}_{i_V,\mathrm{id}_{E_\partial}} : \boldsymbol{S}_{\partial V,E_\partial,s_\partial} \to \boldsymbol{S}_{V,E,s}$ from Example 4.5.

Comparing the analogues of (6.1) for $i_V : \partial V \to V$ and (6.2) for $\boldsymbol{i}_\mathbf{S} : \partial\boldsymbol{S} \to \boldsymbol{S}$, we see that the conormal bundle $\mathcal{N}_\mathbf{S}$ of $\partial\boldsymbol{S}$ in $\boldsymbol{S}$ is canonically isomorphic to the lift to $\underline{\partial S} \subseteq \underline{\partial V}$ of the conormal bundle $\mathcal{N}_V$ of ∂V in V. Define $\omega_\mathbf{S}$ to be the orientation on $\mathcal{N}_\mathbf{S}$ induced by the orientation on $\mathcal{N}_V$ by outward-pointing normal vectors to ∂V in V. Then $\mathbf{S} = (\boldsymbol{S}, \partial\boldsymbol{S}, \boldsymbol{i}_\mathbf{S}, \omega_\mathbf{S})$ is a d-space with corners. It is equivalent to $\mathbf{V} \times_{\boldsymbol{s},\mathbf{E},\mathbf{0}} \mathbf{V}$ in Definition 7.1, and so is a principal d-manifold with corners. We call $\mathbf{S}$ the *standard model* of (V, E, s), and write it $\mathbf{S}_{V,E,s}$.

There is a natural 1-isomorphism $\partial\mathbf{S}_{V,E,s} \cong \mathbf{S}_{\partial V,E_\partial,s_\partial}$ in $\mathbf{dMan^c}$.

Example 7.4. Let V, W be manifolds with corners, $E \to V$, $F \to W$ be vector bundles, and $s : V \to E$, $t : W \to F$ be smooth sections. Then Example 7.3 defines 'standard model' principal d-manifolds with corners $\mathbf{S}_{V,E,s}, \mathbf{S}_{W,F,t}$, with underlying d-spaces $\boldsymbol{S}_{V,E,s}, \boldsymbol{S}_{W,F,t}$. Suppose $f : V \to W$ is a smooth map, and $\hat{f} : E \to f^*(F)$ is a morphism of vector bundles on V satisfying $\hat{f} \circ s = f^*(t) + O(s^2)$ in $C^\infty(f^*(F))$, where $f^*(t) = t \circ f$, and $O(s^2)$ is as §4.2. Define a 1-morphism $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ in $\mathbf{dSpa}$ using $f, \hat{f}$ exactly as in Example 4.5. Then $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ is a 1-morphism in $\mathbf{dMan^c}$, which we call a 'standard model' 1-morphism.

Suppose $\tilde{V} \subseteq V$ is open, with inclusion $i_{\tilde{V}} : \tilde{V} \to V$. Write $\tilde{E} = E|_{\tilde{V}} = i_{\tilde{V}}^*(E)$ and $\tilde{s} = s|_{\tilde{V}}$. Define $\boldsymbol{i}_{\tilde{V},V} = \mathbf{S}_{i_{\tilde{V}},\mathrm{id}_{\tilde{E}}} : \mathbf{S}_{\tilde{V},\tilde{E},\tilde{s}} \to \mathbf{S}_{V,E,s}$. If $s^{-1}(0) \subseteq \tilde{V}$ then $\boldsymbol{i}_{\tilde{V},V}$ is a 1-isomorphism, with inverse $\boldsymbol{i}_{\tilde{V},V}^{-1}$.

In [35, §7.2 & §7.3] we prove analogues of Theorems 4.7 and 4.8:

Theorem 7.5. *Let $\mathbf{X}$ be a d-manifold with corners, and $x \in \mathbf{X}$. Then there exists an open neighbourhood $\mathbf{U}$ of x in $\mathbf{X}$ and an equivalence $\mathbf{U} \simeq \mathbf{S}_{V,E,s}$ in $\mathbf{dMan^c}$ for some manifold with corners V, vector bundle $E \to V$ and smooth section $s : V \to E$ which identifies $x \in \mathbf{U}$ with a point $v \in S^k(V) \subseteq V$, where*

$S^k(V)$ *is as in* §5.1, *such that* $s(v) = \mathrm{d}s|_{S^k(V)}(v) = 0$. *Furthermore,* V, E, s *and* k *are determined up to non-canonical isomorphism near* v *by* $\mathbf{X}$ *near* x.

Theorem 7.6. *Let* V, W *be manifolds with corners,* $E \to V$, $F \to W$ *be vector bundles, and* $s : V \to E$, $t : W \to F$ *be smooth sections. Suppose* $\boldsymbol{g} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ *is a 1-morphism in* $\mathbf{dMan^c}$. *Then there exist an open neighbourhood* $\tilde{V}$ *of* $s^{-1}(0)$ *in* V, *a smooth map* $f : \tilde{V} \to W$, *and a morphism of vector bundles* $\hat{f} : \tilde{E} \to f^*(F)$ *with* $\hat{f} \circ \tilde{s} = f^*(t)$, *where* $\tilde{E} = E|_{\tilde{V}}$, $\tilde{s} = s|_{\tilde{V}}$, *such that* $\boldsymbol{g} = \mathbf{S}_{f,\hat{f}} \circ \boldsymbol{i}_{\tilde{V},V}^{-1}$, *using the notation of Examples* 7.3 *and* 7.4.

7.3 Equivalences in dMan$^{\mathbf{c}}$, and gluing by equivalences

In [35, §7.4] we study equivalences and gluing in $\mathbf{dMan^c}$, as for $\mathbf{dMan}$ in §4.4. Here are the analogues of Definition 4.14 and Theorems 4.15–4.17.

Definition 7.7. Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a 1-morphism in $\mathbf{dMan^c}$. We call $\boldsymbol{f}$ *étale* if it is a *local equivalence*, that is, if for each $x \in \mathbf{X}$ there exist open $x \in \mathbf{U} \subseteq \mathbf{X}$ and $\boldsymbol{f}(x) \in \mathbf{V} \subseteq \mathbf{Y}$ such that $\boldsymbol{f}(\mathbf{U}) = \mathbf{V}$ and $\boldsymbol{f}|_{\mathbf{U}} : \mathbf{U} \to \mathbf{V}$ is an equivalence.

Theorem 7.8. *Suppose* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *is a 1-morphism of d-manifolds with corners. Then the following are equivalent:*

  (i) $\boldsymbol{f}$ *is étale;*

  (ii) $\boldsymbol{f}$ *is simple and flat, in the sense of* §6.2, *and* $\Omega_{\boldsymbol{f}} : \underline{f}^*(T^*\mathbf{Y}) \to T^*\mathbf{X}$ *is an equivalence in* $\mathrm{vqcoh}(\underline{X})$; *and*

  (iii) $\boldsymbol{f}$ *is simple and flat, and* (4.3) *is a split short exact sequence in* $\mathrm{qcoh}(\underline{X})$.

If in addition $f : X \to Y$ *is a bijection, then* $\boldsymbol{f}$ *is an equivalence in* $\mathbf{dMan^c}$.

Theorem 7.9. *Let* V, W *be manifolds with corners,* $E \to V$, $F \to W$ *be vector bundles,* $s : V \to E$, $t : W \to F$ *be smooth sections,* $f : V \to W$ *be smooth, and* $\hat{f} : E \to f^*(F)$ *be a morphism of vector bundles on* V *with* $\hat{f} \circ s = f^*(t) + O(s^2)$. *Then Examples* 7.3 *and* 7.4 *define principal d-manifolds with corners* $\mathbf{S}_{V,E,s}, \mathbf{S}_{W,F,t}$ *and a 1-morphism* $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$. *This* $\mathbf{S}_{f,\hat{f}}$ *is étale if and only if* f *is simple and flat near* $s^{-1}(0) \subseteq V$, *in the sense of* §5.2, *and for each* $v \in V$ *with* $s(v) = 0$ *and* $w = f(v) \in W$, *equation* (4.4) *is exact. Also* $\mathbf{S}_{f,\hat{f}}$ *is an equivalence if and only if in addition* $f|_{s^{-1}(0)} : s^{-1}(0) \to t^{-1}(0)$ *is a bijection, where* $s^{-1}(0) = \{v \in V : s(v) = 0\}$, $t^{-1}(0) = \{w \in W : t(w) = 0\}$.

Theorem 7.10. *Suppose we are given the following data:*

  (a) *an integer* n;

  (b) *a Hausdorff, second countable topological space* X;

  (c) *an indexing set* I, *and a total order* $<$ *on* I;

  (d) *for each* i *in* I, *a manifold with corners* V_i, *a vector bundle* $E_i \to V_i$ *with* $\dim V_i - \mathrm{rank}\, E_i = n$, *a smooth section* $s_i : V_i \to E_i$, *and a homeomorphism* $\psi_i : X_i \to \hat{X}_i$, *where* $X_i = \{v_i \in V_i : s_i(v_i) = 0\}$ *and* $\hat{X}_i \subseteq X$ *is open; and*

(e) *for all $i < j$ in I, an open submanifold $V_{ij} \subseteq V_i$, a simple, flat map $e_{ij} : V_{ij} \to V_j$, and a morphism of vector bundles $\hat{e}_{ij} : E_i|_{V_{ij}} \to e_{ij}^*(E_j)$.*

Let this data satisfy the conditions:

(i) $X = \bigcup_{i \in I} \hat{X}_i$;

(ii) *if $i < j$ in I then $\hat{e}_{ij} \circ s_i|_{V_{ij}} = e_{ij}^*(s_j)$, and $\psi_i(X_i \cap V_{ij}) = \hat{X}_i \cap \hat{X}_j$, and $\psi_i|_{X_i \cap V_{ij}} = \psi_j \circ e_{ij}|_{X_i \cap V_{ij}}$, and if $v_i \in V_i$ with $s_i(v_i) = 0$ and $v_j = e_{ij}(v_i)$ then the following sequence of vector spaces is exact:*

$$0 \longrightarrow T_{v_i}V_i \xrightarrow{\mathrm{d}s_i(v_i) \oplus \mathrm{d}e_{ij}(v_i)} E_i|_{v_i} \oplus T_{v_j}V_j \xrightarrow{\hat{e}_{ij}(v_i) \oplus -\mathrm{d}s_j(v_j)} E_j|_{v_j} \longrightarrow 0;$$

(iii) *if $i < j < k$ in I then $e_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} = e_{jk} \circ e_{ij}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} + O(s_i^2)$ and $\hat{e}_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} = e_{ij}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})}^*(\hat{e}_{jk}) \circ \hat{e}_{ij}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} + O(s_i).$*

Then there exist a d-manifold with corners $\mathbf{X}$ with $\mathrm{vdim}\,\mathbf{X} = n$ and topological space X, and a 1-morphism $\boldsymbol{\psi}_i : \mathbf{S}_{V_i,E_i,s_i} \to \mathbf{X}$ in $\mathbf{dMan^c}$ with underlying continuous map ψ_i which is an equivalence with the open d-submanifold $\hat{\mathbf{X}}_i \subseteq \mathbf{X}$ corresponding to $\hat{X}_i \subseteq X$ for all $i \in I$, such that for all $i < j$ in I there exists a 2-morphism $\eta_{ij} : \boldsymbol{\psi}_j \circ \mathbf{S}_{e_{ij},\hat{e}_{ij}} \Rightarrow \boldsymbol{\psi}_i \circ \boldsymbol{i}_{V_{ij},V_i}$, where $\mathbf{S}_{e_{ij},\hat{e}_{ij}} : \mathbf{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}} \to \mathbf{S}_{V_j,E_j,s_j}$ and $\boldsymbol{i}_{V_{ij},V_i} : \mathbf{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}} \to \mathbf{S}_{V_i,E_i,s_i}$ are as in Example 7.4. This $\mathbf{X}$ is unique up to equivalence in $\mathbf{dMan^c}$.

Suppose also that Y is a manifold with corners, and $g_i : V_i \to Y$ are smooth maps for all $i \in I$, and $g_j \circ e_{ij} = g_i|_{V_{ij}} + O(s_i^2)$ for all $i < j$ in I. Then there exist a 1-morphism $\boldsymbol{h} : \mathbf{X} \to \mathbf{Y}$ unique up to 2-isomorphism, where $\mathbf{Y} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(Y) = \mathbf{S}_{Y,0,0}$, and 2-morphisms $\zeta_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow \mathbf{S}_{g_i,0}$ for all $i \in I$. Here $\mathbf{S}_{Y,0,0}$ is from Example 7.3 with vector bundle E and section s both zero, and $\mathbf{S}_{g_i,0} : \mathbf{S}_{V_i,E_i,s_i} \to \mathbf{S}_{Y,0,0} = \mathbf{Y}$ is from Example 7.4 with $\hat{g}_i = 0$.

We can use Theorem 7.10 as a tool to prove the existence of d-manifold with corner structures on spaces coming from other areas of geometry.

7.4 Submersions, immersions and embeddings

In §4.5 we defined two kinds of submersions (submersions and w-submersions), immersions, and embeddings for d-manifolds. In §5.2 we defined two kinds of submersions (submersions and s-submersions), and three kinds of immersions (immersions, s- and sf-immersions), and embeddings for manifolds with corners. In [35, §7.5], we combine both alternatives for d-manifolds with corners, giving four types of submersions, and six types of immersions and embeddings.

Definition 7.11. Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a 1-morphism in $\mathbf{dMan^c}$. As in §4.3 and §7.1, $T^*\mathbf{X}, \boldsymbol{f}^*(T^*\mathbf{Y})$ are virtual vector bundles on $\underline{X}$ of ranks $\mathrm{vdim}\,\mathbf{X}, \mathrm{vdim}\,\mathbf{Y}$, and $\Omega_{\boldsymbol{f}} : \underline{\boldsymbol{f}}^*(T^*\mathbf{Y}) \to T^*\mathbf{X}$ is a 1-morphism in $\mathrm{vvect}(\underline{X})$. Also we have 1-morphisms $C(\boldsymbol{f}), \hat{C}(\boldsymbol{f}) : C(\mathbf{X}) \to C(\mathbf{Y})$ in $\mathbf{dMan^c} \subset \mathbf{dSpa^c}$ as in §6.5 and §7.1, so we can form $\Omega_{C(\boldsymbol{f})} : \underline{C(f)}^*(T^*C(\mathbf{Y})) \to T^*C(\mathbf{X})$ and $\Omega_{\hat{C}(\boldsymbol{f})} : \underline{\hat{C}(f)}^*(T^*C(\mathbf{Y})) \to T^*C(\mathbf{X})$. Then:

(a) We call $\boldsymbol{f}$ a *w-submersion* if $\boldsymbol{f}$ is semisimple and flat and $\Omega_{\boldsymbol{f}}$ is weakly injective. We call $\boldsymbol{f}$ an *sw-submersion* if it is also simple.

(b) We call $\boldsymbol{f}$ a *submersion* if $\boldsymbol{f}$ is semisimple and flat and $\Omega_{C(\boldsymbol{f})}$ is injective. We call $\boldsymbol{f}$ an *s-submersion* if it is also simple.

(c) We call $\boldsymbol{f}$ a *w-immersion* if $\Omega_{\boldsymbol{f}}$ is weakly surjective. We call $\boldsymbol{f}$ an *sw-immersion*, or *sfw-immersion*, if $\boldsymbol{f}$ is also simple, or simple and flat.

(d) We call $\boldsymbol{f}$ an *immersion* if $\Omega_{\hat{C}(\boldsymbol{f})}$ is surjective. We call $\boldsymbol{f}$ an *s-immersion* if $\boldsymbol{f}$ is also simple, and an *sf-immersion* if $\boldsymbol{f}$ is also simple and flat.

(e) We call $\boldsymbol{f}$ a *w-embedding, sw-embedding, sfw-embedding, embedding, s-embedding*, or *sf-embedding*, if $\boldsymbol{f}$ is a w-immersion, ..., sf-immersion, respectively, and $f : X \to f(X)$ is a homeomorphism, so f is injective.

Here (weakly) injective and (weakly) surjective 1-morphisms in vvect($\underline{X}$) are defined in §4.5.

Parts (c)–(e) enable us to define *d-submanifolds* $\mathbf{X}$ of a d-manifold with corners $\mathbf{Y}$. *Open d-submanifolds* are open d-subspaces $\mathbf{X}$ in $\mathbf{Y}$. For more general d-submanifolds, we call $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ a *w-immersed, sw-immersed, sfw-immersed, immersed, s-immersed, sf-immersed, w-embedded, sw-embedded, sfw-embedded, embedded, s-embedded*, or *sf-embedded d-submanifold* of $\mathbf{Y}$ if $\mathbf{X}, \mathbf{Y}$ are d-manifolds with corners and $\boldsymbol{f}$ is a w-immersion, ..., sf-embedding, respectively.

Here is the analogue of Theorem 4.20, proved in [35, §7.5].

Theorem 7.12. (i) *Any equivalence of d-manifolds with corners is a w-submersion, submersion, ..., sf-embedding.*

(ii) *If $\boldsymbol{f}, \boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ are 2-isomorphic 1-morphisms of d-manifolds with corners then $\boldsymbol{f}$ is a w-submersion, ..., sf-embedding, if and only if $\boldsymbol{g}$ is.*

(iii) *Compositions of w-submersions, ..., sf-embeddings are of the same kind.*

(iv) *The conditions that a 1-morphism $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{dMan^c}$ is any kind of submersion or immersion are local in $\mathbf{X}$ and $\mathbf{Y}$. The conditions that $\boldsymbol{f}$ is any kind of embedding are local in $\mathbf{Y}$, but not in $\mathbf{X}$.*

(v) *Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a submersion in $\mathbf{dMan^c}$. Then $\mathrm{vdim}\,\mathbf{X} \geqslant \mathrm{vdim}\,\mathbf{Y}$, and if $\mathrm{vdim}\,\mathbf{X} = \mathrm{vdim}\,\mathbf{Y}$ then $\boldsymbol{f}$ is étale.*

(vi) *Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be an immersion in $\mathbf{dMan^c}$. Then $\mathrm{vdim}\,\mathbf{X} \leqslant \mathrm{vdim}\,\mathbf{Y}$. If $\boldsymbol{f}$ is an s-immersion and $\mathrm{vdim}\,\mathbf{X} = \mathrm{vdim}\,\mathbf{Y}$ then $\boldsymbol{f}$ is étale.*

(vii) *Let $f : X \to Y$ be a smooth map of manifolds with corners, and $\boldsymbol{f} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(f)$. Then $\boldsymbol{f}$ is a submersion, s-submersion, immersion, s-immersion, sf-immersion, embedding, s-embedding, or sf-embedding, in $\mathbf{dMan^c}$ if and only if f is a submersion, ..., an sf-embedding in $\mathbf{Man^c}$, respectively. Also $\boldsymbol{f}$ is a w-immersion, sw-immersion, sfw-immersion, w-embedding, sw-embedding, or sfw-embedding in $\mathbf{dMan^c}$ if and only if f is an immersion, ..., sf-embedding in $\mathbf{Man^c}$, respectively.*

(viii) *Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a 1-morphism in $\mathbf{dMan^c}$, with $\mathbf{Y}$ a manifold. Then $\boldsymbol{f}$ is a w-submersion if and only if it is semisimple and flat, and $\boldsymbol{f}$ is an sw-submersion if and only if it is simple and flat.*

(ix) *Let $\mathbf{X}, \mathbf{Y}$ be d-manifolds with corners, with $\mathbf{Y}$ a manifold. Then $\boldsymbol{\pi_X} : \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$ is a submersion, and $\boldsymbol{\pi_X}$ is an s-submersion if $\partial \mathbf{Y} = \emptyset$.*

(x) *Suppose $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ is a submersion in $\mathbf{dMan^c}$, and $x \in \mathbf{X}$ with $\boldsymbol{f}(x) = y \in \mathbf{Y}$. Then there exist open d-submanifolds $x \in \mathbf{U} \subseteq \mathbf{X}$ and $y \in \mathbf{V} \subseteq \mathbf{Y}$ with $\boldsymbol{f}(\mathbf{U}) = \mathbf{V}$, a manifold with corners $\mathbf{Z}$, and an equivalence $\boldsymbol{i} : \mathbf{U} \to \mathbf{V} \times \mathbf{Z}$, such that $\boldsymbol{f}|_{\mathbf{U}} : \mathbf{U} \to \mathbf{V}$ is 2-isomorphic to $\boldsymbol{\pi_V} \circ \boldsymbol{i}$, where $\boldsymbol{\pi_V} : \mathbf{V} \times \mathbf{Z} \to \mathbf{V}$ is the projection. If $\boldsymbol{f}$ is an s-submersion then $\partial \mathbf{Z} = \emptyset$.*

(xi) *Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a submersion of d-manifolds with corners, with $\mathbf{Y}$ a manifold with corners. Then $\mathbf{X}$ is a manifold with corners.*

Parts (ix)-(x) are a d-manifold analogue of Proposition 5.7.

7.5 Bd-transversality and fibre products

In [35, §7.6] we extend §4.6 to the corners case. Here are the analogues of Definition 4.21 and Theorems 4.22–4.25:

Definition 7.13. Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be d-manifolds with corners and $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$, $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ be 1-morphisms. We call $\boldsymbol{g}, \boldsymbol{h}$ *bd-transverse* if they are both b-transverse in $\mathbf{dSpa^c}$ in the sense of Definition 6.12, and d-transverse in the sense of Definition 4.21. We call $\boldsymbol{g}, \boldsymbol{h}$ *cd-transverse* if they are both c-transverse in $\mathbf{dSpa^c}$ in the sense of Definition 6.12, and d-transverse. As in §6.6, c-transverse implies b-transverse, so cd-transverse implies bd-transverse.

Theorem 7.14. *Suppose $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are d-manifolds with corners and $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$, $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ are bd-transverse 1-morphisms, and let $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g}, \mathbf{Z}, \boldsymbol{h}} \mathbf{Y}$ be the fibre product in $\mathbf{dSpa^c}$, which exists by Theorem 6.14(a) as $\boldsymbol{g}, \boldsymbol{h}$ are b-transverse. Then $\mathbf{W}$ is a d-manifold with corners, with*

$$\operatorname{vdim} \mathbf{W} = \operatorname{vdim} \mathbf{X} + \operatorname{vdim} \mathbf{Y} - \operatorname{vdim} \mathbf{Z}. \tag{7.1}$$

Hence, all bd-transverse fibre products exist in $\mathbf{dMan^c}$.

Theorem 7.15. *Suppose $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ and $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ are 1-morphisms in $\mathbf{dMan^c}$. The following are sufficient conditions for $\boldsymbol{g}, \boldsymbol{h}$ to be cd-transverse, and hence bd-transverse, so that $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g}, \mathbf{Z}, \boldsymbol{h}} \mathbf{Y}$ is a d-manifold with corners of virtual dimension (7.1):*

(a) *$\mathbf{Z}$ is a manifold without boundary, that is, $\mathbf{Z} \in \bar{\mathbf{M}}\mathbf{an}$; or*

(b) *$\boldsymbol{g}$ or $\boldsymbol{h}$ is a w-submersion.*

Theorem 7.16. *Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be d-manifolds with corners with $\mathbf{Y}$ a manifold, and $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$, $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ be 1-morphisms with $\boldsymbol{g}$ a submersion. Then $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g}, \mathbf{Z}, \boldsymbol{h}} \mathbf{Y}$ is a manifold, with $\dim \mathbf{W} = \operatorname{vdim} \mathbf{X} + \dim \mathbf{Y} - \operatorname{vdim} \mathbf{Z}$.*

Theorem 7.17. (i) *Let* $\mathbf{X}$ *be a d-manifold with corners and* $\boldsymbol{g} : \mathbf{X} \to [\mathbf{0}, \infty)^{\boldsymbol{k}} \times \mathbb{R}^{\boldsymbol{n-k}}$ *a semisimple, flat 1-morphism in* $\mathbf{dMan^c}$*. Then the fibre product* $\mathbf{W} = \mathbf{X} \times_{\boldsymbol{g}, [\mathbf{0}, \infty)^{\boldsymbol{k}} \times \mathbb{R}^{\boldsymbol{n-k}}, \mathbf{0}} *$ *exists in* $\mathbf{dMan^c}$*, and* $\boldsymbol{\pi}_{\mathbf{X}} : \mathbf{W} \to \mathbf{X}$ *is an s-embedding. When* $k = 0$*, any 1-morphism* $\boldsymbol{g} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}$ *is semisimple and flat, and* $\boldsymbol{\pi}_{\mathbf{X}} : \mathbf{W} \to \mathbf{X}$ *is an sf-embedding.*

(ii) *Suppose* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *is an s-immersion of d-manifolds with corners, and* $x \in \mathbf{X}$ *with* $\boldsymbol{f}(x) = y \in \mathbf{Y}$*. Then there exist open d-submanifolds* $x \in \mathbf{U} \subseteq \mathbf{X}$ *and* $y \in \mathbf{V} \subseteq \mathbf{Y}$ *with* $\boldsymbol{f}(\mathbf{U}) \subseteq \mathbf{V}$ *and a semisimple, flat 1-morphism* $\boldsymbol{g} : \mathbf{V} \to [\mathbf{0}, \infty)^{\boldsymbol{k}} \times \mathbb{R}^{\boldsymbol{n-k}}$ *with* $\boldsymbol{g}(y) = 0$*, where* $n = \mathrm{vdim}\, \mathbf{Y} - \mathrm{vdim}\, \mathbf{X} \geqslant 0$ *and* $0 \leqslant k \leqslant n$*, fitting into a 2-Cartesian square in* $\mathbf{dMan^c}$ *:*

$$
\begin{array}{ccc}
\mathbf{U} & \xrightarrow{\;\;\;\;\;\;\;\;\boldsymbol{\pi}\;\;\;\;\;\;\;\;} & * \\
{\scriptstyle \boldsymbol{f}|_{\mathbf{U}}}\big\downarrow & \quad\quad \Uparrow \quad\quad & \big\downarrow {\scriptstyle \mathbf{0}} \\
\mathbf{V} & \xrightarrow[\;\;\;\;\;\;\;\;\boldsymbol{g}\;\;\;\;\;\;\;\;]{} & [\mathbf{0}, \infty)^{\boldsymbol{k}} \times \mathbb{R}^{\boldsymbol{n-k}}.
\end{array}
$$

If $\boldsymbol{f}$ *is an sf-immersion then* $k = 0$*. If* $\boldsymbol{f}$ *is an s- or sf-embedding then we may take* $\mathbf{U} = \boldsymbol{f}^{-1}(\mathbf{V})$*.*

For ordinary manifolds, a submanifold X in Y may be described locally either as the image of an embedding $X \hookrightarrow Y$, or equivalently as the zeroes of a submersion $Y \to \mathbb{R}^n$, where $n = \dim Y - \dim X$. Theorem 7.17 is an analogue of this for d-manifolds with corners. It should be compared with Proposition 5.8 for manifolds with corners.

7.6 Embedding d-manifolds with corners into manifolds

Section 4.7 discussed embeddings of d-manifolds $\boldsymbol{X}$ into manifolds Y. Our two major results were Theorem 4.29, which gave necessary and sufficient conditions on $\boldsymbol{X}$ for existence of embeddings $\boldsymbol{f} : \boldsymbol{X} \hookrightarrow \mathbb{R}^{\boldsymbol{n}}$ for $n \gg 0$, and Theorem 4.32, which showed that if an embedding $\boldsymbol{f} : \boldsymbol{X} \hookrightarrow \boldsymbol{Y}$ exists with $\boldsymbol{X}$ a d-manifold and $\boldsymbol{Y} = F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$, then $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$ for open $V \subseteq Y$, so $\boldsymbol{X}$ is a principal d-manifold.

In [35, §7.7] we generalize these to d-manifolds with corners. As in §7.4, we have three kinds of embeddings in $\mathbf{dMan^c}$, embeddings, s-embeddings and sf-embeddings. The analogue of Theorem 4.29 naturally holds for embeddings:

Theorem 7.18. *Let* $\mathbf{X}$ *be a d-manifold with corners. Then there exist immersions and/or embeddings* $\boldsymbol{f} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}$ *for some* $n \gg 0$ *if and only if there is an upper bound for* $\dim T_x^* \underline{X}$ *for all* $x \in \underline{X}$*. If there is such an upper bound, then immersions* $\boldsymbol{f} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}$ *exist provided* $n \geqslant 2 \dim T_x^* \underline{X}$ *for all* $x \in \underline{X}$*, and embeddings* $\boldsymbol{f} : \mathbf{X} \to \mathbb{R}^{\boldsymbol{n}}$ *exist provided* $n \geqslant 2 \dim T_x^* \underline{X} + 1$ *for all* $x \in \underline{X}$*. For embeddings we may also choose* $\boldsymbol{f}$ *with* $f(X)$ *closed in* $\mathbb{R}^n$*.*

Example 4.30 shows the hypotheses of Theorem 7.18 need not hold, so there exist d-manifolds with corners $\mathbf{X}$ with no embedding into $\mathbb{R}^n$, or into any manifold with corners. The analogue of Theorem 4.32 holds for sf-embeddings:

Theorem 7.19. *Let $\mathbf{X}$ be a d-manifold with corners, Y a manifold with corners, and $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ an sf-embedding, in the sense of Definition 7.11. Then there exist an open subset V in Y with $\boldsymbol{f}(\mathbf{X}) \subseteq \mathbf{V}$, a vector bundle $E \to V$, and a smooth section $s : V \to E$ of E fitting into a 2-Cartesian diagram in $\mathbf{dMan^c}$, where $0 : V \to E$ is the zero section and $\mathbf{Y}, \mathbf{V}, \mathbf{E}, \boldsymbol{s}, \mathbf{0} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(Y, V, E, s, 0)$:*

$$
\begin{array}{ccc}
\mathbf{X} & \xrightarrow{\ \ \boldsymbol{f}\ \ } & \mathbf{V} \\
\downarrow{\scriptstyle \boldsymbol{f}} & \quad \Uparrow \quad {\scriptstyle \boldsymbol{s}} & \downarrow{\scriptstyle \mathbf{0}} \\
\mathbf{V} & \xrightarrow{\ \ \boldsymbol{s}\ \ } & \mathbf{E}.
\end{array}
$$

Hence $\mathbf{X}$ is equivalent to the 'standard model' $\mathbf{S}_{V,E,s}$ of Example 7.3, and is a principal d-manifold with corners.

Note that, unlike the d-manifolds case in §4.7, we cannot immediately combine Theorems 7.18 and 7.19: we have first to bridge the gap between embeddings and sf-embeddings. For d-manifolds with boundary, we can do this.

Theorem 7.20. *Let $\mathbf{X}$ be a d-manifold with boundary. Then there exist sf-immersions and/or sf-embeddings $\boldsymbol{f} : \mathbf{X} \to [0, \infty) \times \mathbb{R}^{n-1}$ for some $n \gg 0$ if and only if $\dim T_x^*\underline{X}$ is bounded above for all $x \in \underline{X}$. Such an upper bound always exists if $\mathbf{X}$ is compact. If there is such an upper bound, then sf-immersions $\boldsymbol{f} : \mathbf{X} \to [0, \infty) \times \mathbb{R}^{n-1}$ exist provided $n \geqslant 2 \dim T_x^*\underline{X} + 1$ for all $x \in \underline{X}$, and sf-embeddings $\boldsymbol{f} : \mathbf{X} \to [0, \infty) \times \mathbb{R}^{n-1}$ exist provided $n \geqslant 2 \dim T_x^*\underline{X} + 2$ for all $x \in \underline{X}$. For sf-embeddings we may also choose $\boldsymbol{f}$ with $f(X)$ closed in $[0, \infty) \times \mathbb{R}^{n-1}$.*

Combining Theorems 7.19 and 7.20 shows that a d-manifold with boundary $\mathbf{X}$ is principal if and only if $\dim T_x^*\underline{X}$ is bounded above.

Since (nice) d-manifolds with boundary can be embedded into $[0, \infty) \times \mathbb{R}^{n-1}$ for $n \gg 0$, one might guess that (nice) d-manifolds with corners can be embedded into $[0, \infty)^k \times \mathbb{R}^{n-k}$ for $n \gg k \gg 0$. However, this is not true even for manifolds with corners, as the following example from [35, §5.7] shows:

Example 7.21. Consider the *teardrop* $T = \left\{(x, y) \in \mathbb{R}^2 : x \geqslant 0,\ y^2 \leqslant x^2 - x^4\right\}$, shown in Figure 7.1. It is a compact 2-manifold with corners.
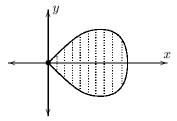
Figure 7.1: The teardrop, a 2-manifold with corners.

Suppose that $f : T \to [0, \infty)^k \times \mathbb{R}^{n-k}$ is an sf-embedding. As f is simple and flat, it maps $S^j(T) \hookrightarrow S^j\left([0, \infty)^k \times \mathbb{R}^{n-k}\right)$ for $j = 0, 1, 2$, in the notation of

§5.1. The connected components of $S^j\big([0,\infty)^k \times \mathbb{R}^{n-k}\big)$ correspond to subsets $I \subseteq \{1,\ldots,k\}$ with $|I| = j$, with the component corresponding to I given by the equations $x_i = 0$ for $i \in I$ and $x_a > 0$ for $a \in \{1,\ldots,k\} \setminus I$. As $(0,0) \in S^2(T)$, we see that $f(0,0)$ lies in the component of $S^2\big([0,\infty)^k \times \mathbb{R}^{n-k}\big)$ given by $x_a = x_b = 0$ for $1 \leqslant a < b \leqslant k$.

Considering local models for f near $(0,0) \in T$, we see that f must map the two ends of $S^1(T)$ at $(0,0)$ into different connected components $x_a = 0$ and $x_b = 0$ of $S^1\big([0,\infty)^k \times \mathbb{R}^{n-k}\big)$. However, $S^1(T) \cong (0,1)$ is connected, so f maps $S^1(T)$ into a single connected component of $S^1\big([0,\infty)^k \times \mathbb{R}^{n-k}\big)$, a contradiction. Hence there do not exist sf-embeddings $f : T \to [0,\infty)^k \times \mathbb{R}^{n-k}$ for any n, k.

Here are necessary and sufficient conditions for existence of sf-embeddings from a d-manifold with corners $\mathbf{X}$ into a manifold with corners Y.

Theorem 7.22. *Let $\mathbf{X}$ be a d-manifold with corners. Then there exist a manifold with corners Y and an sf-embedding $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$, where $\mathbf{Y} = F_{\mathbf{Man^c}}^{\mathbf{dMan^c}}(Y)$, if and only if $\dim T_x^*\underline{X} + |i_{\mathbf{X}}^{-1}(x)|$ is bounded above for all $x \in \underline{X}$. If such an upper bound exists, then we may take Y to be an embedded n-dimensional submanifold of $\mathbb{R}^n$ for any n with $n \geqslant 2\big(\dim T_x^*\underline{X} + |i_{\mathbf{X}}^{-1}(x)|\big) + 1$ for all $x \in \underline{X}$.*

Such an upper bound always exists if $\mathbf{X}$ is compact. Thus, every compact d-manifold with corners admits an sf-embedding into a manifold with corners.

The idea of the proof of Theorem 7.22 is that we first choose an embedding $\boldsymbol{g} : \mathbf{X} \to \mathbb{R}^n$ using Theorem 7.18, and then show that we can choose a submanifold $Y \subseteq \mathbb{R}^n$ which is the set of points in an open neighbourhood U of $g(X)$ in $\mathbb{R}^n$ satisfying local transverse inequalities of the form $c_i(x) \geqslant 0$ for $i = 1,\ldots,k$, where $c_i : U \to \mathbb{R}$ are local smooth functions which lift under $\boldsymbol{g}$ to local boundary defining functions for $\partial\mathbf{X}$.

Combining Theorems 7.19 and 7.22 yields:

Corollary 7.23. *Let $\mathbf{X}$ be a d-manifold with corners. Then $\mathbf{X}$ is principal, that is, $\mathbf{X}$ is equivalent in $\mathbf{dMan^c}$ to some $\mathbf{S}_{V,E,s}$ in Example 7.3, if and only if $\dim T_x^*\underline{X}$ and $|i_{\mathbf{X}}^{-1}(x)|$ are bounded above for all $x \in \underline{X}$. This holds automatically if $\mathbf{X}$ is compact.*

7.7 Orientations

In [35, §7.8] we study orientations on d-manifolds with corners, following the d-manifold case in §4.8. Here is the analogue of Definition 4.35:

Definition 7.24. Let $\mathbf{X}$ be a d-manifold with corners. Then the virtual cotangent bundle $T^*\mathbf{X} = (\mathcal{E}_X, \mathcal{F}_X, \phi_X)$ is a virtual vector bundle on $\underline{X}$, so Theorem 4.34 gives a line bundle $\mathcal{L}_{T^*\mathbf{X}}$ on $\underline{X}$. We call $\mathcal{L}_{T^*\mathbf{X}}$ the *orientation line bundle* of $\mathbf{X}$.

An *orientation* ω on $\mathbf{X}$ is an orientation on $\mathcal{L}_{T^*\mathbf{X}}$, in the sense of Definition 4.35. An *oriented d-manifold with corners* is a pair $(\mathbf{X}, \omega)$ where $\mathbf{X}$ is a d-manifold with corners and ω an orientation on $\mathbf{X}$. Usually we refer to $\mathbf{X}$ as

an oriented d-manifold, leaving ω implicit. We also write $-\mathbf{X}$ for $\mathbf{X}$ with the opposite orientation, that is, $\mathbf{X}$ is short for $(\mathbf{X}, \omega)$ and $-\mathbf{X}$ short for $(\mathbf{X}, -\omega)$.

Example 4.36, Theorem 4.37 and Proposition 4.38 now extend to d-manifolds with corners without change. We can also orient boundaries of oriented d-manifolds with corners. Theorem 7.25 is the main reason for including the data $\omega_{\mathbf{X}}$ in a d-manifold with corners $\mathbf{X} = (X, \boldsymbol{\partial} X, i_{\mathbf{X}}, \omega_{\mathbf{X}})$.

Theorem 7.25. *Let* $\mathbf{X}$ *be a d-manifold with corners. Then* $\partial\mathbf{X}$ *is also a d-manifold with corners, so we have orientation line bundles* $\mathcal{L}_{T^*\mathbf{X}}$ *on* $\underline{X}$ *and* $\mathcal{L}_{T^*(\partial\mathbf{X})}$ *on* $\underline{\partial X}$. *There is a canonical isomorphism of line bundles on* $\underline{\partial X}$:

$$\Psi : \mathcal{L}_{T^*(\partial\mathbf{X})} \longrightarrow \underline{i}_{\mathbf{X}}^*(\mathcal{L}_{T^*\mathbf{X}}) \otimes \mathcal{N}_{\mathbf{X}}^*, \tag{7.2}$$

where $\mathcal{N}_{\mathbf{X}}$ *is the conormal bundle of* $\boldsymbol{\partial} X$ *in* X *from* §6.1.

Now $\mathcal{N}_{\mathbf{X}}$ *comes with an orientation* $\omega_{\mathbf{X}}$ *in* $\mathbf{X} = (X, \boldsymbol{\partial} X, i_{\mathbf{X}}, \omega_{\mathbf{X}})$. *Hence, if* $\mathbf{X}$ *is an oriented d-manifold with corners, then* $\partial\mathbf{X}$ *also has a natural orientation, by combining the orientations on* $\mathcal{L}_{T^*\mathbf{X}}$ *and* $\mathcal{N}_{\mathbf{X}}^*$ *to get an orientation on* $\mathcal{L}_{T^*(\partial\mathbf{X})}$ *using* (7.2).

As for Proposition 4.38, natural equivalences of d-manifolds with corners generally extend to natural equivalences of oriented d-manifolds with corners, with some sign depending on the orientation conventions. Here are two such results, which include signs in Theorem 6.3(b) and Proposition 6.15.

Proposition 7.26. *Suppose* $\mathbf{X}, \mathbf{Y}$ *are oriented d-manifolds with corners, and* $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ *is a semisimple, flat 1-morphism. Then the following holds in oriented d-manifolds with corners, with fibre products cd-transverse:*

$$\partial_-^{\boldsymbol{f}}\mathbf{X} \simeq \partial\mathbf{Y} \times_{i_{\mathbf{Y}}, \mathbf{Y}, \boldsymbol{f}} \mathbf{X} \simeq (-1)^{\mathrm{vdim}\,\mathbf{X} + \mathrm{vdim}\,\mathbf{Y}} \mathbf{X} \times_{\boldsymbol{f}, \mathbf{Y}, i_{\mathbf{Y}}} \partial\mathbf{Y}.$$

If $\boldsymbol{f}$ *is also simple then* $\partial_-^{\boldsymbol{f}}\mathbf{X} = \partial\mathbf{X}$.

Proposition 7.27. *Let* $\boldsymbol{g} : \mathbf{X} \to \mathbf{Z}$ *and* $\boldsymbol{h} : \mathbf{Y} \to \mathbf{Z}$ *be 1-morphisms of oriented d-manifolds with corners. Then the following hold in oriented d-manifolds with corners, where all the fibre products are cd-transverse, and so exist:*

(a) *If* $\mathbf{Z}$ *is a manifold without boundary then there is an equivalence*

$$\partial\big(\mathbf{X} \times_{\boldsymbol{g}, \mathbf{Z}, \boldsymbol{h}} \mathbf{Y}\big) \simeq \big(\partial\mathbf{X} \times_{\boldsymbol{g} \circ i_{\mathbf{X}}, \mathbf{Z}, \boldsymbol{h}} \mathbf{Y}\big) \amalg (-1)^{\mathrm{vdim}\,\mathbf{X} + \dim\mathbf{Z}} \big(\mathbf{X} \times_{\boldsymbol{g}, \mathbf{Z}, \boldsymbol{h} \circ i_{\mathbf{Y}}} \partial\mathbf{Y}\big).$$

(b) *If* $\boldsymbol{g}$ *is a w-submersion then there is an equivalence*

$$\partial\big(\mathbf{X} \times_{\boldsymbol{g}, \mathbf{Z}, \boldsymbol{h}} \mathbf{Y}\big) \simeq \big(\partial_+^{\boldsymbol{g}}\mathbf{X} \times_{\boldsymbol{g}_+, \mathbf{Z}, \boldsymbol{h}} \mathbf{Y}\big) \amalg (-1)^{\mathrm{vdim}\,\mathbf{X} + \mathrm{vdim}\,\mathbf{Z}} \big(\mathbf{X} \times_{\boldsymbol{g}, \mathbf{Z}, \boldsymbol{h} \circ i_{\mathbf{Y}}} \partial\mathbf{Y}\big).$$

(c) *If both* $\boldsymbol{g}$ *and* $\boldsymbol{h}$ *are w-submersions then there is an equivalence*

$$\begin{aligned}\partial\big(\mathbf{X} \times_{\boldsymbol{g}, \mathbf{Z}, \boldsymbol{h}} \mathbf{Y}\big) \simeq{} &\big(\partial_+^{\boldsymbol{g}}\mathbf{X} \times_{\boldsymbol{g}_+, \mathbf{Z}, \boldsymbol{h}} \mathbf{Y}\big) \\ &\amalg (-1)^{\mathrm{vdim}\,\mathbf{X} + \mathrm{vdim}\,\mathbf{Z}} \big(\mathbf{X} \times_{\boldsymbol{g}, \mathbf{Z}, \boldsymbol{h}_+} \partial_+^{\boldsymbol{h}}\mathbf{Y}\big) \amalg \big(\partial_-^{\boldsymbol{g}}\mathbf{X} \times_{\boldsymbol{g}_-, \partial\mathbf{Z}, \boldsymbol{h}_-} \partial_-^{\boldsymbol{h}}\mathbf{Y}\big).\end{aligned}$$

8 Deligne–Mumford C^∞-stacks

Next we discuss *Deligne–Mumford C^∞-stacks*, which are related to C^∞-schemes in the same way that Deligne–Mumford stacks in algebraic geometry are related to schemes, and will be the foundation of our theories of orbifolds, d-stacks and d-orbifolds. C^∞-stacks were introduced by the author in [33, §7–§11].

8.1 C^∞-stacks

The next few definitions assume a lot of standard material from stack theory, which is summarized in [33, §7].

Definition 8.1. Define a *Grothendieck topology $\mathcal{J}$* on the category $\mathbf{C^\infty Sch}$ of C^∞-schemes to have coverings $\{\underline{i}_a : \underline{U}_a \to \underline{U}\}_{a \in A}$ where $V_a = i_a(U_a)$ is open in U with $\underline{i}_a : \underline{U}_a \to (V_a, \mathcal{O}_U|_{V_a})$ an isomorphism for all $a \in A$, and $U = \bigcup_{a \in A} V_a$. Up to isomorphisms of the $\underline{U}_a$, the coverings $\{\underline{i}_a : \underline{U}_a \to \underline{U}\}_{a \in A}$ of $\underline{U}$ correspond exactly to open covers $\{V_a : a \in A\}$ of U. Then $(\mathbf{C^\infty Sch}, \mathcal{J})$ is a *site*.

The stacks on $(\mathbf{C^\infty Sch}, \mathcal{J})$ form a 2-category $\mathbf{Sta}_{(\mathbf{C^\infty Sch}, \mathcal{J})}$, with all 2-morphisms invertible. As the site $(\mathbf{C^\infty Sch}, \mathcal{J})$ is subcanonical, there is a natural, fully faithful functor $\mathbf{C^\infty Sch} \to \mathbf{Sta}_{(\mathbf{C^\infty Sch}, \mathcal{J})}$, defined explicitly below, which we write as $\underline{X} \mapsto \bar{X}$ on objects and $\underline{f} \mapsto \bar{f}$ on morphisms. A *C^∞-stack* is a stack $\mathcal{X}$ on $(\mathbf{C^\infty Sch}, \mathcal{J})$ such that the diagonal 1-morphism $\Delta_\mathcal{X} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable, and there exists a surjective 1-morphism $\Pi : \bar{U} \to \mathcal{X}$ called an *atlas* for some C^∞-scheme $\underline{U}$. Write $\mathbf{C^\infty Sta}$ for the full 2-subcategory of C^∞-stacks in $\mathbf{Sta}_{(\mathbf{C^\infty Sch}, \mathcal{J})}$. The functor $\mathbf{C^\infty Sch} \to \mathbf{Sta}_{(\mathbf{C^\infty Sch}, \mathcal{J})}$ above maps into $\mathbf{C^\infty Sta}$, so we also write it as $F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}} : \mathbf{C^\infty Sch} \to \mathbf{C^\infty Sta}$.

Formally, a C^∞-stack is a category $\mathcal{X}$ with a functor $p_\mathcal{X} : \mathcal{X} \to \mathbf{C^\infty Sch}$, where $\mathcal{X}, p_\mathcal{X}$ must satisfy many complicated conditions, including sheaf-like conditions for all open covers in $\mathbf{C^\infty Sch}$. A *1-morphism* $f : \mathcal{X} \to \mathcal{Y}$ of C^∞-stacks is a functor $f : \mathcal{X} \to \mathcal{Y}$ with $p_\mathcal{Y} \circ f = p_\mathcal{X} : \mathcal{X} \to \mathbf{C^\infty Sch}$. Given 1-morphisms $f, g : \mathcal{X} \to \mathcal{Y}$, a *2-morphism* $\eta : f \Rightarrow g$ is an isomorphism of functors $\eta : f \Rightarrow g$ with $\mathrm{id}_{p_\mathcal{Y}} * \eta = \mathrm{id}_{p_\mathcal{X}} : p_\mathcal{Y} \circ f \Rightarrow p_\mathcal{Y} \circ g$.

If $\underline{X}$ is a C^∞-scheme, the corresponding C^∞-stack $\bar{X} = F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}}(\underline{X})$ is the category with objects $(\underline{U}, \underline{u})$ for $\underline{u} : \underline{U} \to \underline{X}$ a morphism in $\mathbf{C^\infty Sch}$, and morphisms $\underline{h} : (\underline{U}, \underline{u}) \to (\underline{V}, \underline{v})$ for $\underline{h} : \underline{U} \to \underline{V}$ a morphism in $\mathbf{C^\infty Sch}$ with $\underline{v} \circ \underline{h} = \underline{u}$. The functor $p_{\bar{X}} : \bar{X} \to \mathbf{C^\infty Sch}$ maps $p_{\bar{X}} : (\underline{U}, \underline{u}) \mapsto \underline{U}$ and $p_{\bar{X}} : \underline{h} \mapsto \underline{h}$.

If $\underline{f} : \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes, the corresponding 1-morphism $\bar{f} = F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}}(\underline{f}) : \bar{X} \to \bar{Y}$ maps $\bar{f} : (\underline{U}, \underline{u}) \mapsto (\underline{U}, \underline{f} \circ \underline{u})$ on objects $(\underline{U}, \underline{u})$ and $\bar{f} : \underline{h} \mapsto \underline{h}$ on morphisms $\underline{h}$ in $\bar{X}$. This defines a functor $\bar{f} : \bar{X} \to \bar{Y}$ with $p_{\bar{Y}} \circ \bar{f} = p_{\bar{X}} : \mathcal{X} \to \mathbf{C^\infty Sch}$, so $\bar{f}$ is a 1-morphism $\bar{f} : \bar{X} \to \bar{Y}$ in $\mathbf{C^\infty Sta}$.

We define some classes of morphisms of C^∞-schemes:

Definition 8.2. Let $\underline{f} : \underline{X} \to \underline{Y}$ be a morphism in $\mathbf{C^\infty Sch}$. Then:

- We call $\underline{f}$ an *open embedding* if it is an isomorphism with an open C^∞-subscheme of $\underline{Y}$.

- We call f *étale* if it is a local isomorphism (in the Zariski topology).

- We call f *proper* if $f : X \to Y$ is a proper map of topological spaces, that is, if $S \subseteq Y$ is compact then $f^{-1}(S) \subseteq X$ is compact.

- We call f *universally closed* if whenever $g : \underline{W} \to \underline{Y}$ is a morphism then $\pi_W : X \times_{f,Y,g} W \to W$ is a closed map of topological spaces.

Each one is invariant under base change and local in the target in $(\mathbf{C}^\infty\mathbf{Sch}, \mathcal{J})$. Thus, they are also defined for representable 1-morphisms of C^∞-stacks.

Definition 8.3. Let $\mathcal{X}$ be a C^∞-stack. We say that $\mathcal{X}$ is *separated* if the diagonal 1-morphism $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is universally closed. If $\mathcal{X} \simeq \bar{\underline{X}}$ for some C^∞-scheme $\underline{X}$ then $\mathcal{X}$ is separated if and only if $\underline{X}$ is separated (Hausdorff).

Definition 8.4. Let $\mathcal{X}$ be a C^∞-stack. A C^∞-*substack* $\mathcal{Y}$ in $\mathcal{X}$ is a strictly full subcategory $\mathcal{Y}$ in $\mathcal{X}$ such that $p_{\mathcal{Y}} := p_{\mathcal{X}}|_{\mathcal{Y}} : \mathcal{Y} \to \mathbf{C}^\infty\mathbf{Sch}$ is also a C^∞-stack. It has a natural inclusion 1-morphism $i_{\mathcal{Y}} : \mathcal{Y} \hookrightarrow \mathcal{X}$. We call $\mathcal{Y}$ an *open C^∞-substack* of $\mathcal{X}$ if $i_{\mathcal{Y}}$ is a representable open embedding. An *open cover* $\{\mathcal{Y}_a : a \in A\}$ of $\mathcal{X}$ is a family of open C^∞-substacks $\mathcal{Y}_a$ in $\mathcal{X}$ with $\coprod_{a \in A} i_{\mathcal{Y}_a} : \coprod_{a \in A} \mathcal{Y}_a \to \mathcal{X}$ surjective.

8.2 Topological spaces of C^∞-stacks

By [33, §8.4], a C^∞-stack $\mathcal{X}$ has an underlying topological space $\mathcal{X}_{\mathrm{top}}$.

Definition 8.5. Let $\mathcal{X}$ be a C^∞-stack. Write $\underline{*}$ for the point $\mathrm{Spec}\,\mathbb{R}$ in $\mathbf{C}^\infty\mathbf{Sch}$, and $\bar{\underline{*}}$ for the associated point in $\mathbf{C}^\infty\mathbf{Sta}$. Define $\mathcal{X}_{\mathrm{top}}$ to be the set of 2-isomorphism classes $[x]$ of 1-morphisms $x : \bar{\underline{*}} \to \mathcal{X}$. If $\mathcal{U} \subseteq \mathcal{X}$ is an open C^∞-substack then any 1-morphism $x : \bar{\underline{*}} \to \mathcal{U}$ is also a 1-morphism $x : \bar{\underline{*}} \to \mathcal{X}$, and $\mathcal{U}_{\mathrm{top}}$ is a subset of $\mathcal{X}_{\mathrm{top}}$. Define $\mathcal{T}_{\mathcal{X}_{\mathrm{top}}} = \{\mathcal{U}_{\mathrm{top}} : \mathcal{U} \subseteq \mathcal{X} \text{ is an open } C^\infty\text{-substack}$ in $\mathcal{X}\}$. Then $\mathcal{T}_{\mathcal{X}_{\mathrm{top}}}$ is a set of subsets of $\mathcal{X}_{\mathrm{top}}$ which is a topology on $\mathcal{X}_{\mathrm{top}}$, so $(\mathcal{X}_{\mathrm{top}}, \mathcal{T}_{\mathcal{X}_{\mathrm{top}}})$ is a topological space, which we call the *underlying topological space* of $\mathcal{X}$, and usually write as $\mathcal{X}_{\mathrm{top}}$. If $\underline{X} = (X, \mathcal{O}_X)$ is a C^∞-scheme, so that $\bar{X}$ is a C^∞-stack, then $\bar{X}_{\mathrm{top}}$ is naturally homeomorphic to X.

If $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of C^∞-stacks then there is a natural continuous map $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ defined by $f_{\mathrm{top}}([x]) = [f \circ x]$. If $f, g : \mathcal{X} \to \mathcal{Y}$ are 1-morphisms and $\eta : f \Rightarrow g$ is a 2-morphism then $f_{\mathrm{top}} = g_{\mathrm{top}}$. Mapping $\mathcal{X} \mapsto \mathcal{X}_{\mathrm{top}}$, $f \mapsto f_{\mathrm{top}}$ and 2-morphisms to identities defines a 2-functor $F_{\mathbf{C}^\infty\mathbf{Sta}}^{\mathbf{Top}} : \mathbf{C}^\infty\mathbf{Sta} \to \mathbf{Top}$, where the category of topological spaces $\mathbf{Top}$ is regarded as a 2-category with only identity 2-morphisms.

Definition 8.6. Let $\mathcal{X}$ be a C^∞-stack, and $[x] \in \mathcal{X}_{\mathrm{top}}$. Pick a representative x for $[x]$, so that $x : \bar{\underline{*}} \to \mathcal{X}$ is a 1-morphism. Define the *orbifold group* (or *isotropy group*, or *stabilizer group*) $\mathrm{Iso}([x])$ or $\mathrm{Iso}_{\mathcal{X}}([x])$ of $[x]$ to be the group of 2-morphisms $\eta : x \Rightarrow x$. It is independent of the choice of $x \in [x]$ up to isomorphism, which is canonical up to conjugation in $\mathrm{Iso}_{\mathcal{X}}([x])$.

If $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of C^∞-stacks and $[x] \in \mathcal{X}_{\mathrm{top}}$ with $f_{\mathrm{top}}([x]) = [y] \in \mathcal{Y}_{\mathrm{top}}$, for $y = f \circ x$, then we define a group morphism $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to$

$\mathrm{Iso}_{\mathcal{Y}}([y])$ by $f_*(\eta) = \mathrm{id}_f * \eta$. It is independent of choices of $x \in [x]$, $y \in [y]$ up to conjugation in $\mathrm{Iso}_{\mathcal{X}}([x]), \mathrm{Iso}_{\mathcal{Y}}([y])$.

8.3 Strongly representable 1-morphisms

Strongly representable 1-morphisms, discussed in [33, §8.6], will be important in the definitions of orbifolds, d-stacks, and d-orbifolds with corners.

Definition 8.7. Let $\mathcal{Y}, \mathcal{Z}$ be C^∞-stacks, and $g : \mathcal{Y} \to \mathcal{Z}$ a 1-morphism. Then $\mathcal{Y}, \mathcal{Z}$ are categories with functors $p_{\mathcal{Y}} : \mathcal{Y} \to \mathbf{C^\infty Sch}$, $p_{\mathcal{Z}} : \mathcal{Z} \to \mathbf{C^\infty Sch}$, and $g : \mathcal{Y} \to \mathcal{Z}$ is a functor with $p_{\mathcal{Z}} \circ g = p_{\mathcal{Y}}$.

We call g *strongly representable* if whenever $A \in \mathcal{Y}$ with $p_{\mathcal{Y}}(A) = \underline{U} \in \mathbf{C^\infty Sch}$, so that $B = g(A) \in \mathcal{Z}$ with $p_{\mathcal{Z}}(B) = \underline{U}$, and $b : B \to B'$ is an isomorphism in $\mathcal{Z}$ with $p_{\mathcal{Z}}(B') = \underline{U}$ and $p_{\mathcal{Z}}(b) = \underline{\mathrm{id}}_U$, then there exist a unique object A' and isomorphism $a : A \to A'$ in $\mathcal{Y}$ with $g(A') = B'$ and $g(a) = b$.
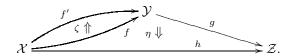
Here are two important properties of strongly representable 1-morphisms. The first says that we may replace a representable 1-morphism $g : \mathcal{Y} \to \mathcal{Z}$ with a strongly representable 1-morphism $g' : \mathcal{Y}' \to \mathcal{Z}$ with $\mathcal{Y}' \simeq \mathcal{Y}$.

Proposition 8.8. (a) *Let $g : \mathcal{Y} \to \mathcal{Z}$ be a strongly representable 1-morphism of C^∞-stacks. Then g is representable.*

(b) *Suppose $g : \mathcal{Y} \to \mathcal{Z}$ is a representable 1-morphism of C^∞-stacks. Then there exist a C^∞-stack $\mathcal{Y}'$, an equivalence $i : \mathcal{Y} \to \mathcal{Y}'$, and a strongly representable 1-morphism $g' : \mathcal{Y}' \to \mathcal{Z}$ with $g = g' \circ i$. Also $\mathcal{Y}'$ is unique up to canonical 1-isomorphism in $\mathbf{C^\infty Sta}$.*

The second says that for some 2-commutative diagrams involving strongly representable morphisms, we can require the diagrams to commute *up to equality*, not just up to 2-isomorphism.

Proposition 8.9. *Suppose $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are C^∞-stacks, $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$, $h : \mathcal{X} \to \mathcal{Z}$ are 1-morphisms with g strongly representable, and $\eta : g \circ f \Rightarrow h$ is a 2-morphism in $\mathbf{C^\infty Sta}$. Then as in the diagram below there exist a 1-morphism $f' : \mathcal{X} \to \mathcal{Y}$ with $g \circ f' = h$, and a 2-morphism $\zeta : f \Rightarrow f'$ with $\mathrm{id}_g * \zeta = \eta$, and f', ζ are unique under these conditions.*

We will use strongly representable 1-morphisms to define orbifolds, d-stacks, and d-orbifolds with corners so that boundaries behave in a strictly functorial rather than weakly functorial way, as for d-spaces with corners in Remark 6.5. Here is an explicit construction of fibre products $\mathcal{X} \times_{g,\mathcal{Z},h} \mathcal{Y}$ in $\mathbf{C^\infty Sta}$ when g is strongly representable, yielding a strictly commutative 2-Cartesian square.

Proposition 8.10. *Let $g : \mathcal{X} \to \mathcal{Z}$ and $h : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of C^∞-stacks with g strongly representable. Define a category $\mathcal{W}$ to have objects pairs (A, B) for $A \in \mathcal{X}$, $B \in \mathcal{Y}$ with $g(A) = h(B)$ in $\mathcal{Z}$, so that $p_\mathcal{X}(A) = p_\mathcal{Y}(B)$ in $\mathbf{C}^\infty\mathbf{Sch}$, and morphisms pairs $(a, b) : (A, B) \to (A', B')$ with $a : A \to A'$, $b : B \to B'$ morphisms in $\mathcal{X}, \mathcal{Y}$ with $p_\mathcal{X}(a) = p_\mathcal{Y}(b)$ in $\mathbf{C}^\infty\mathbf{Sch}$.*

Define functors $p_\mathcal{W} : \mathcal{W} \to \mathbf{C}^\infty\mathbf{Sch}$, $e : \mathcal{W} \to \mathcal{X}$, $f : \mathcal{W} \to \mathcal{Y}$ by $p_\mathcal{W} : (A, B) \mapsto p_\mathcal{X}(A) = p_\mathcal{Y}(B)$, $e : (A, B) \mapsto A$, $f : (A, B) \mapsto B$ on objects and $p_\mathcal{W} : (a, b) \mapsto p_\mathcal{X}(a) = p_\mathcal{Y}(b)$, $e : (a, b) \mapsto a$, $f : (a, b) \mapsto b$ on morphisms. Then $\mathcal{W}$ is a C^∞-stack and $e : \mathcal{W} \to \mathcal{X}$, $f : \mathcal{W} \to \mathcal{Y}$ are 1-morphisms, with f strongly representable, and $g \circ e = h \circ f$. Furthermore, the following diagram in $\mathbf{C}^\infty\mathbf{Sta}$ is 2-Cartesian:

$$
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\quad f \quad} & \mathcal{Y} \\
{\scriptstyle e}\Big\downarrow & \quad \mathrm{id}_{g \circ e} \Uparrow \quad & \Big\downarrow{\scriptstyle h} \\
\mathcal{X} & \xrightarrow[\quad g \quad]{} & \mathcal{Z}.
\end{array}
$$

If also h is strongly representable, then e is strongly representable.

8.4   Quotient C^∞-stacks

An important class of examples of C^∞-stacks $\mathcal{X}$ are *quotient C^∞-stacks* $[\underline{X}/G]$, for $\underline{X}$ a C^∞-scheme acted on by a finite group G. The next three examples define quotient C^∞-stacks $[\underline{X}/G]$, quotient 1-morphisms $[\underline{f}, \rho] : [\underline{X}/G] \to [\underline{Y}/H]$, and quotient 2-morphisms $[\delta] : [\underline{f}, \rho] \Rightarrow [\underline{g}, \sigma]$.

In fact Examples 8.11–8.13 are simplifications of more complicated definitions given in [33, §9.1]. The construction of [33, §9.1] gives equivalent C^∞-stacks $[\underline{X}/G]$, but has the advantage of being strictly functorial, that is, quotient 1-morphisms compose as $[\underline{g}, \sigma] \circ [\underline{f}, \rho] = [\underline{g} \circ \underline{f}, \sigma \circ \rho]$, whereas in Example 8.12 we only have a 2-isomorphism $[\underline{g}, \sigma] \circ [\underline{f}, \rho] \cong [\underline{g} \circ \underline{f}, \sigma \circ \rho]$. We will occasionally assume this strict functoriality below, for instance, in Definition 11.26.

Example 8.11. Let $\underline{X}$ be a separated C^∞-scheme, G a finite group, and $\underline{r} : G \to \mathrm{Aut}(\underline{X})$ an action of G on $\underline{X}$ by isomorphisms. We will define the *quotient C^∞-stack* $\mathcal{X} = [\underline{X}/G]$.

Define a category $\mathcal{X}$ to have objects quintuples $(\underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v})$, where $\underline{T}, \underline{U}$ are C^∞-schemes, $\underline{t} : G \to \mathrm{Aut}(\underline{T})$ is a free action of G on $\underline{T}$ by isomorphisms, $\underline{u} : \underline{T} \to \underline{X}$ is a morphism with $\underline{u} \circ \underline{t}(\gamma) = \underline{r}(\gamma) \circ \underline{u} : \underline{T} \to \underline{X}$ for all $\gamma \in G$, and $\underline{v} : \underline{T} \to \underline{U}$ is a morphism which makes $\underline{T}$ into a principal G-bundle over $\underline{U}$, that is, $\underline{v}$ is proper, étale and surjective, and its fibres are G-orbits in $\underline{T}$ under $\underline{t}$.

A morphism $(\underline{a}, \underline{b}) : (\underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v}) \to (\underline{T}', \underline{U}', \underline{t}', \underline{u}', \underline{v}')$ in $\mathcal{X}$ is a pair of morphisms $\underline{a} : \underline{U} \to \underline{U}'$ and $\underline{b} : \underline{T} \to \underline{T}'$ such that $\underline{b} \circ \underline{t}(\gamma) = \underline{t}'(\gamma) \circ \underline{b}$ for $\gamma \in G$, and $\underline{u} = \underline{u}' \circ \underline{b}$, and $\underline{a} \circ \underline{v} = \underline{v}' \circ \underline{b}$. Composition is $(\underline{c}, \underline{d}) \circ (\underline{a}, \underline{b}) = (\underline{c} \circ \underline{a}, \underline{d} \circ \underline{b})$, and identities are $\mathrm{id}_{(\underline{T}, \dots, \underline{v})} = (\underline{\mathrm{id}}_U, \underline{\mathrm{id}}_T)$.

This defines the category $\mathcal{X}$. The functor $p_\mathcal{X} : \mathcal{X} \to \mathbf{C}^\infty\mathbf{Sch}$ acts by $p_\mathcal{X} : (\underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v}) \mapsto \underline{U}$ on objects, and $p_\mathcal{X} : (\underline{a}, \underline{b}) \mapsto \underline{a}$ on morphisms. Then $\mathcal{X}$ is a C^∞-stack, which we write as $[\underline{X}/G]$.

Example 8.12. Let $\underline{X}, \underline{Y}$ be separated C^∞-schemes acted on by finite groups G, H with actions $\underline{r} : G \to \mathrm{Aut}(\underline{X})$, $\underline{s} : H \to \mathrm{Aut}(\underline{Y})$, so that we have quotient

C^∞-stacks $[\underline{X}/G]$ and $[\underline{Y}/H]$ as in Example 8.11. Suppose we have morphisms $\underline{f} : \underline{X} \to \underline{Y}$ of C^∞-schemes and $\rho : G \to H$ of groups, with $\underline{f} \circ \underline{r}(\gamma) = \underline{s}(\rho(\gamma)) \circ \underline{f}$ for all $\gamma \in G$. Define a functor $[\underline{f}, \rho] : [\underline{X}/G] \to [\underline{Y}/H]$ on objects in $[\underline{X}/G]$ by

$$[\underline{f}, \rho] : (\underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v}) \longmapsto \big((\underline{T} \times H)/G, \underline{U}, \tilde{\underline{t}}, \tilde{\underline{u}}, \tilde{\underline{v}}\big).$$

Here for each $\delta \in H$, write $L_\delta, R_\delta : H \to H$ for left and right multiplication by δ. Then to define $(\underline{T} \times H)/G$, each $\gamma \in G$ acts by $\underline{r}(\gamma) \times R_{\rho(\gamma)^{-1}} : \underline{T} \times H \to \underline{T} \times H$. For each $\delta \in H$, the morphism $\tilde{\underline{t}}(\delta) : (\underline{T} \times H)/G \to (\underline{T} \times H)/G$ is induced by the morphism $\underline{\mathrm{id}}_{\underline{T}} \times L_\delta : \underline{T} \times H \to \underline{T} \times H$. The morphisms $\tilde{\underline{u}} : (\underline{T} \times H)/G \to \underline{Y}$ and $\tilde{\underline{v}} : (\underline{T} \times H)/G \to \underline{U}$ are induced by $\underline{f} \circ \underline{u} \circ \underline{\pi}_{\underline{T}} : \underline{T} \times H \to \underline{Y}$ and $\underline{v} \circ \underline{\pi}_{\underline{T}} : \underline{T} \times H \to \underline{U}$.

On morphisms $(\underline{a}, \underline{b}) : (\underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v}) \to (\underline{T}', \underline{U}', \underline{t}', \underline{u}', \underline{v}')$ in $[\underline{X}/G]$, define $[\underline{f}, \rho]$ to map $(\underline{a}, \underline{b}) \mapsto (\underline{a}, \tilde{\underline{b}})$, where $\tilde{\underline{b}} : (\underline{T} \times H)/G \to (\underline{T}' \times H)/G$ is induced by $\underline{b} \times \mathrm{id}_H : \underline{T} \times H \to \underline{T}' \times H$. Then $[\underline{f}, \rho] : [\underline{X}/G] \to [\underline{Y}/H]$ is a 1-morphism of C^∞-stacks, which we call a *quotient 1-morphism*.

If $\rho : G \to H$ is injective, then $[\underline{f}, \rho] : [\underline{X}/G] \to [\underline{Y}/H]$ is representable.

Example 8.13. Let $[\underline{f}, \rho] : [\underline{X}/G] \to [\underline{Y}/H]$ and $[\underline{g}, \sigma] : [\underline{X}/G] \to [\underline{Y}/H]$ be quotient 1-morphisms, so that $\underline{f}, \underline{g} : \underline{X} \to \underline{Y}$ and $\rho, \sigma : G \to H$ are morphisms. Suppose $\delta \in H$ satisfies $\sigma(\gamma) = \delta \, \rho(\gamma) \, \delta^{-1}$ for all $\gamma \in G$, and $\underline{g} = \underline{s}(\delta) \circ \underline{f}$.

For each object $(\underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v})$ in $[\underline{X}/G]$, define an isomorphism in $[\underline{Y}/H]$:

$$[\delta]\big((\underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v})\big) = (\underline{\mathrm{id}}_{\underline{U}}, \underline{i}_\delta) : [\underline{f}, \rho]\big((\underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v})\big) = \big((\underline{T} \times H)/_{\underline{r} \times R_{\rho^{-1}}} G, \underline{U}, \tilde{\underline{t}}, \tilde{\underline{u}}, \tilde{\underline{v}}\big)$$
$$\longrightarrow [\underline{g}, \sigma]\big((\underline{T}, \underline{U}, \underline{t}, \underline{u}, \underline{v})\big) = \big((\underline{T} \times H)/_{\underline{r} \times R_{\sigma^{-1}}} G, \underline{U}, \dot{\underline{t}}, \dot{\underline{u}}, \dot{\underline{v}}\big),$$

where $\underline{i}_\delta : (\underline{T} \times H)/_{\underline{r} \times R_{\rho^{-1}}} G \to (\underline{T} \times H)/_{\underline{r} \times R_{\sigma^{-1}}} G$ is induced $\underline{\mathrm{id}}_{\underline{T}} \times R_{\delta^{-1}} : \underline{T} \times H \to \underline{T} \times H$. Then $[\delta] : [\underline{f}, \rho] \Rightarrow [\underline{g}, \sigma]$ is a natural isomorphism of functors, and a 2-morphism of C^∞-stacks, which we call a *quotient 2-morphism*.

8.5 Deligne–Mumford C^∞-stacks

Deligne–Mumford stacks in algebraic geometry are locally modelled on quotient stacks $[X/G]$ for X an affine scheme and G a finite group. This motivates:

Definition 8.14. A *Deligne–Mumford C^∞-stack* is a C^∞-stack $\mathcal{X}$ which admits an open cover $\{\mathcal{Y}_a : a \in A\}$ with each $\mathcal{Y}_a$ equivalent to a quotient stack $[\underline{U}_a/G_a]$ in Example 8.11 for $\underline{U}_a$ an affine C^∞-scheme and G_a a finite group. We call $\mathcal{X}$ *locally fair* if it has such an open cover with each $\underline{U}_a$ a fair affine C^∞-scheme.

We call a Deligne–Mumford C^∞-stack $\mathcal{X}$ *second countable, compact, locally compact,* or *paracompact*, if the underlying topological space $\mathcal{X}_{\mathrm{top}}$ from §8.2 is second countable, compact, locally compact, or paracompact, respectively.

Write $\mathbf{DMC^\infty Sta}, \mathbf{DMC^\infty Sta^{lf}}, \mathbf{DMC^\infty Sta^{lf}_{ssc}}$ for the full 2-subcategories of Deligne–Mumford C^∞-stacks, and locally fair Deligne–Mumford C^∞-stacks, and separated, second countable, locally fair Deligne–Mumford C^∞-stacks in $\mathbf{C^\infty Sta}$, respectively.

If $\mathcal{X}$ is a Deligne–Mumford C^∞-stack then $\mathrm{Iso}_{\mathcal{X}}([x])$ is finite for all $[x]$ in $\mathcal{X}_{\mathrm{top}}$. If $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of Deligne–Mumford C^∞-stacks then f is representable if and only if the morphisms of orbifold groups $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$ from Definition 8.6 are injective for all $[x] \in \mathcal{X}_{\mathrm{top}}$ with $f_{\mathrm{top}}([x]) = [x] \in \mathcal{Y}_{\mathrm{top}}$. From [33, §8–§9], we have:

Theorem 8.15. (a) *All fibre products exist in the 2-category* $\mathbf{C^\infty Sta}$.

(b) $\mathbf{DMC^\infty Sta}, \mathbf{DMC^\infty Sta^{lf}}$ *and* $\mathbf{DMC^\infty Sta^{lf}_{ssc}}$ *are closed under fibre products and under taking open C^∞-substacks in* $\mathbf{C^\infty Sta}$.

Proposition 8.16. *Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack and $[x] \in \mathcal{X}_{\mathrm{top}}$, so that $\mathrm{Iso}_{\mathcal{X}}([x]) \cong H$ for some finite group H. Then there exists an open C^∞-substack $\mathcal{U}$ in $\mathcal{X}$ with $[x] \in \mathcal{U}_{\mathrm{top}} \subseteq \mathcal{X}_{\mathrm{top}}$ and an equivalence $\mathcal{U} \simeq [\underline{Y}/H]$, where $\underline{Y} = (Y, \mathcal{O}_Y)$ is an affine C^∞-scheme with an action of H, and $[x] \in \mathcal{U}_{\mathrm{top}} \cong Y/H$ corresponds to a fixed point y of H in Y.*

Theorem 8.17. *Suppose $\mathcal{X}$ is a Deligne–Mumford C^∞-stack with $\mathrm{Iso}_{\mathcal{X}}([x]) \cong \{1\}$ for all $[x] \in \mathcal{X}_{\mathrm{top}}$. Then $\mathcal{X}$ is equivalent to $\underline{\bar{X}}$ for some C^∞-scheme $\underline{X}$.*
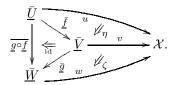
In conventional algebraic geometry, a stack with all orbifold groups trivial is (equivalent to) an *algebraic space*, but may not be a scheme, so the category of algebraic spaces is larger than the category of schemes. Here algebraic spaces are spaces which are locally isomorphic to schemes in the étale topology, but not necessarily locally isomorphic to schemes in the Zariski topology.

In contrast, as Theorem 8.17 shows, in C^∞-algebraic geometry there is no difference between C^∞-schemes and C^∞-algebraic spaces. This is because in C^∞-geometry the Zariski topology is already fine enough, as in Remark 2.9(iii), so we gain no extra generality by passing to the étale topology.

8.6 Quasicoherent sheaves on C^∞-stacks

In [33, §10] we study sheaves on Deligne–Mumford C^∞-stacks.

Definition 8.18. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Define a category $\mathcal{C}_{\mathcal{X}}$ to have objects pairs $(\underline{U}, u)$ where $\underline{U}$ is a C^∞-scheme and $u : \underline{\bar{U}} \to \mathcal{X}$ is an étale 1-morphism, and morphisms $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ where $\underline{f} : \underline{U} \to \underline{V}$ is an étale morphism of C^∞-schemes, and $\eta : u \Rightarrow v \circ \bar{f}$ is a 2-isomorphism. If $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ and $(\underline{g}, \zeta) : (\underline{V}, v) \to (\underline{W}, w)$ are morphisms in $\mathcal{C}_{\mathcal{X}}$ then we define the composition $(\underline{g}, \zeta) \circ (\underline{f}, \eta)$ to be $(\underline{g} \circ \underline{f}, \theta) : (\underline{U}, u) \to (\underline{W}, w)$, where θ is the composition of 2-morphisms across the diagram:

$$
\begin{array}{ccc}
\bar{U} & & \\
\underline{g \circ f} \downarrow \quad \bar{f} \quad u & \Lleftarrow_{\mathrm{id}} \quad \bar{V} \xrightarrow{v} & \mathcal{X}. \\
\bar{W} & \bar{g} \quad w \quad \Lleftarrow_{\zeta} &
\end{array}
$$

Define an $\mathcal{O}_{\mathcal{X}}$-*module* $\mathcal{E}$ to assign an $\mathcal{O}_U$-module $\mathcal{E}(\underline{U}, u)$ on $\underline{U} = (U, \mathcal{O}_U)$ for all objects $(\underline{U}, u)$ in $\mathcal{C}_{\mathcal{X}}$, and an isomorphism $\mathcal{E}_{(\underline{f}, \eta)} : \underline{f}^*(\mathcal{E}(\underline{V}, v)) \to \mathcal{E}(\underline{U}, u)$ for

all morphisms $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ in $\mathcal{C}_\mathcal{X}$, such that for all $(\underline{f}, \eta), (\underline{g}, \zeta), (\underline{g} \circ \underline{f}, \theta)$ as above the following diagram of isomorphisms of $\mathcal{O}_U$-modules commutes:

$$
\begin{array}{c}
(\underline{g} \circ \underline{f})^* \big(\mathcal{E}(\underline{W}, w) \big) \xrightarrow{\quad \mathcal{E}_{(\underline{g} \circ \underline{f}, \theta)} \quad} \mathcal{E}(\underline{U}, u), \\
I_{\underline{f}, \underline{g}}(\mathcal{E}(\underline{W}, w)) \searrow \quad \underline{f}^* \big(\underline{g}^*(\mathcal{E}(\underline{W}, w)) \big) \xrightarrow{\underline{f}^*(\mathcal{E}_{(\underline{g}, \zeta)})} \underline{f}^* \big(\mathcal{E}(\underline{V}, v) \big) \nearrow \mathcal{E}_{(\underline{f}, \eta)}
\end{array}
\tag{8.1}
$$

for $I_{\underline{f}, \underline{g}}(\mathcal{E}(\underline{W}, w))$ as in Remark 2.17.

A *morphism of $\mathcal{O}_\mathcal{X}$-modules* $\phi : \mathcal{E} \to \mathcal{F}$ assigns a morphism of $\mathcal{O}_U$-modules $\phi(\underline{U}, u) : \mathcal{E}(\underline{U}, u) \to \mathcal{F}(\underline{U}, u)$ for each object $(\underline{U}, u)$ in $\mathcal{C}_\mathcal{X}$, such that for all morphisms $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ in $\mathcal{C}_\mathcal{X}$ the following commutes:

$$
\begin{array}{ccc}
\underline{f}^* \big(\mathcal{E}(\underline{V}, v) \big) & \xrightarrow{\mathcal{E}_{(\underline{f}, \eta)}} & \mathcal{E}(\underline{U}, u) \\
{\scriptstyle \underline{f}^*(\phi(\underline{V}, v))} \downarrow & & \downarrow {\scriptstyle \phi(\underline{U}, u)} \\
\underline{f}^* \big(\mathcal{F}(\underline{V}, v) \big) & \xrightarrow{\mathcal{F}_{(\underline{f}, \eta)}} & \mathcal{F}(\underline{U}, u).
\end{array}
$$

We call $\mathcal{E}$ *quasicoherent*, or a *vector bundle of rank n*, if $\mathcal{E}(\underline{U}, u)$ is quasicoherent, or a vector bundle of rank n, respectively, for all $(\underline{U}, u) \in \mathcal{C}_\mathcal{X}$. Write $\mathcal{O}_\mathcal{X}$-mod for the category of $\mathcal{O}_\mathcal{X}$-modules, and $\mathrm{qcoh}(\mathcal{X})$, $\mathrm{vect}(\mathcal{X})$ for the full subcategories of quasicoherent sheaves and vector bundles, respectively. Then $\mathcal{O}_\mathcal{X}$-mod is an abelian category, and $\mathrm{qcoh}(\mathcal{X})$ an abelian subcategory of $\mathcal{O}_\mathcal{X}$-mod. If $\mathcal{X}$ is locally fair then $\mathrm{qcoh}(\mathcal{X}) = \mathcal{O}_\mathcal{X}$-mod.

Note that vector bundles $\mathcal{E}$ on $\mathcal{X}$ are locally trivial in the étale topology, but need not be locally trivial in the Zariski topology. In particular, the orbifold groups $\mathrm{Iso}_\mathcal{X}([x])$ of $\mathcal{X}$ can act nontrivially on the fibres $\mathcal{E}|_x$ of $\mathcal{E}$.

As in [33, §10.5], as well as sheaves of $\mathcal{O}_\mathcal{X}$-modules, we can define other kinds of sheaves on Deligne–Mumford C^∞-stacks $\mathcal{X}$ by the same method. In particular, to define d-stacks in §10, we will need *sheaves of abelian groups* and *sheaves of C^∞-rings* on Deligne–Mumford C^∞-stacks.

Example 8.19. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Define a quasicoherent sheaf $\mathcal{O}_\mathcal{X}$ on $\mathcal{X}$ called the *structure sheaf* of $\mathcal{X}$ by $\mathcal{O}_\mathcal{X}(\underline{U}, u) = \mathcal{O}_U$ for all objects $(\underline{U}, u)$ in $\mathcal{C}_\mathcal{X}$, and $(\mathcal{O}_\mathcal{X})_{(\underline{f}, \eta)} : \underline{f}^*(\mathcal{O}_V) \to \mathcal{O}_U$ is the natural isomorphism for all morphisms $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ in $\mathcal{C}_\mathcal{X}$.

We may also consider $\mathcal{O}_\mathcal{X}$ as a sheaf of C^∞-rings on $\mathcal{X}$.

Example 8.20. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Define an $\mathcal{O}_\mathcal{X}$-module $T^*\mathcal{X}$ called the *cotangent sheaf* of $\mathcal{X}$ by $(T^*\mathcal{X})(\underline{U}, u) = T^*\underline{U}$ for all objects $(\underline{U}, u)$ in $\mathcal{C}_\mathcal{X}$ and $(T^*\mathcal{X})_{(\underline{f}, \eta)} = \Omega_{\underline{f}} : \underline{f}^*(T^*\underline{V}) \to T^*\underline{U}$ for all morphisms $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ in $\mathcal{C}_\mathcal{X}$, where $T^*\underline{U}$ and $\Omega_{\underline{f}}$ are as in §2.4.

Example 8.21. Let $\underline{X}$ be a C^∞-scheme. Then $\mathcal{X} = \bar{\underline{X}}$ is a Deligne–Mumford C^∞-stack. We will define an *inclusion functor* $\mathcal{I}_{\underline{X}} : \mathcal{O}_X$-mod $\to \mathcal{O}_\mathcal{X}$-mod. Let $\mathcal{E}$ be an object in $\mathcal{O}_X$-mod. If $(\underline{U}, u)$ is an object in $\mathcal{C}_\mathcal{X}$ then $u : \bar{\underline{U}} \to \mathcal{X} = \bar{\underline{X}}$ is 2-isomorphic to $\bar{\underline{u}} : \bar{\underline{U}} \to \bar{\underline{X}}$ for some unique morphism $\underline{u} : \underline{U} \to \underline{X}$. Define

$\mathcal{E}'(\underline{U}, u) = \underline{u}^*(\mathcal{E})$. If $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ is a morphism in $\mathcal{C}_{\mathcal{X}}$ and $\underline{u}, \underline{v}$ are associated to u, v as above, so that $\underline{u} = \underline{v} \circ \underline{f}$, then define

$$\mathcal{E}'_{(\underline{f},\eta)} = I_{\underline{f},\underline{v}}(\mathcal{E})^{-1} : \underline{f}^*(\mathcal{E}'(\underline{V}, v)) = \underline{f}^*\big(\underline{v}^*(\mathcal{E})\big) \longrightarrow (\underline{v} \circ \underline{f})^*(\mathcal{E}) = \mathcal{E}'(\underline{U}, u).$$

Then (8.1) commutes for all $(\underline{f}, \eta), (\underline{g}, \zeta)$, so $\mathcal{E}'$ is an $\mathcal{O}_{\mathcal{X}}$-module.

If $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism of $\mathcal{O}_X$-modules then we define a morphism $\phi' : \mathcal{E}' \to \mathcal{F}'$ in $\mathcal{O}_{\mathcal{X}}$-mod by $\phi'(\underline{U}, u) = \underline{u}^*(\phi)$ for $\underline{u}$ associated to u as above. Then defining $\mathcal{I}_{\underline{X}} : \mathcal{E} \mapsto \mathcal{E}'$, $\mathcal{I}_{\underline{X}} : \phi \mapsto \phi'$ gives a functor $\mathcal{O}_X$-mod $\to \mathcal{O}_{\mathcal{X}}$-mod, which induces equivalences between the categories $\mathcal{O}_X$-mod, qcoh($\underline{X}$) defined in §2.4 and $\mathcal{O}_{\mathcal{X}}$-mod, qcoh($\mathcal{X}$) above.

Definition 8.22. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of Deligne–Mumford C^∞-stacks, and $\mathcal{F}$ be an $\mathcal{O}_{\mathcal{Y}}$-module. A *pullback* of $\mathcal{F}$ to $\mathcal{X}$ is an $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{E}$, together with the following data: if $\underline{U}, \underline{V}$ are C^∞-schemes and $u : \underline{U} \to \mathcal{X}$ and $v : \underline{V} \to \mathcal{Y}$ are étale 1-morphisms, then there is a C^∞-scheme $\underline{W}$ and morphisms $\underline{\pi}_{\underline{U}} : \underline{W} \to \underline{U}$, $\underline{\pi}_{\underline{V}} : \underline{W} \to \underline{V}$ giving a 2-Cartesian diagram:

$$\begin{array}{ccc} \underline{W} & \xrightarrow{\ \underline{\pi}_{\underline{V}}\ } & \underline{V} \\ {\scriptstyle \underline{\pi}_{\underline{U}}} \downarrow & {\scriptstyle \zeta} \Uparrow \quad {\scriptstyle f \circ u} & \downarrow {\scriptstyle v} \\ \underline{U} & \xrightarrow{\qquad} & \mathcal{Y}. \end{array} \qquad (8.2)$$

Then an isomorphism $i(\mathcal{F}, f, u, v, \zeta) : \underline{\pi}_{\underline{U}}^*\big(\mathcal{E}(\underline{U}, u)\big) \to \underline{\pi}_{\underline{V}}^*\big(\mathcal{F}(\underline{V}, v)\big)$ of $\mathcal{O}_W$-modules should be given, which is functorial in $(\underline{U}, u)$ in $\mathcal{C}_{\mathcal{X}}$ and $(\underline{V}, v)$ in $\mathcal{C}_{\mathcal{Y}}$ and the 2-isomorphism ζ in (8.2). We usually write pullbacks $\mathcal{E}$ as $f^*(\mathcal{F})$. Pullbacks $f^*(\mathcal{F})$ exist, and are unique up to unique isomorphism. Using the Axiom of Choice, we choose a pullback $f^*(\mathcal{F})$ for all such $f, \mathcal{F}$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism, and $\phi : \mathcal{E} \to \mathcal{F}$ be a morphism in $\mathcal{O}_{\mathcal{Y}}$-mod. Then $f^*(\mathcal{E}), f^*(\mathcal{F}) \in \mathcal{O}_{\mathcal{X}}$-mod. The *pullback morphism* $f^*(\phi) : f^*(\mathcal{E}) \to f^*(\mathcal{F})$ is the unique morphism in $\mathcal{O}_{\mathcal{X}}$-mod such that whenever $u : \underline{U} \to \mathcal{X}$, $v : \underline{V} \to \mathcal{Y}$, $\underline{W}, \underline{\pi}_{\underline{U}}, \underline{\pi}_{\underline{V}}$ are as above, the following diagram in $\mathcal{O}_W$-mod commutes:

$$\begin{array}{ccc} \underline{\pi}_{\underline{U}}^*\big(f^*(\mathcal{E})(\underline{U}, u)\big) & \xrightarrow{\ i(\mathcal{E}, f, u, v, \zeta)\ } & \underline{\pi}_{\underline{V}}^*\big(\mathcal{E}(\underline{V}, v)\big) \\ {\scriptstyle \pi_{\underline{U}}^*(f^*(\phi)(\underline{U}, u))} \downarrow & & \downarrow {\scriptstyle \pi_{\underline{V}}^*(\phi(\underline{V}, v))} \\ \underline{\pi}_{\underline{U}}^*\big(f^*(\mathcal{F})(\underline{U}, u)\big) & \xrightarrow{\ i(\mathcal{F}, f, u, v, \zeta)\ } & \underline{\pi}_{\underline{V}}^*\big(\mathcal{F}(\underline{V}, v)\big). \end{array}$$

This defines a right exact functor $f^* : \mathcal{O}_{\mathcal{Y}}$-mod $\to \mathcal{O}_{\mathcal{X}}$-mod, which also maps qcoh($\mathcal{Y}$) $\to$ qcoh($\mathcal{X}$).

Let $f, g : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms of Deligne–Mumford C^∞-stacks, $\eta : f \Rightarrow g$ a 2-morphism, and $\mathcal{E} \in \mathcal{O}_{\mathcal{Y}}$-mod. Then we have $\mathcal{O}_{\mathcal{X}}$-modules $f^*(\mathcal{E}), g^*(\mathcal{E})$. Define $\eta^*(\mathcal{E}) : f^*(\mathcal{E}) \to g^*(\mathcal{E})$ to be the unique isomorphism such that whenever $\underline{U}, \underline{V}, \underline{W}, u, v, \underline{\pi}_{\underline{U}}, \underline{\pi}_{\underline{V}}$ are as above, so that we have 2-Cartesian diagrams

$$\begin{array}{ccc} \underline{W} & \xrightarrow{\qquad} & \underline{V} \\ {\scriptstyle \underline{\pi}_{\underline{U}}} \downarrow & {\scriptstyle \zeta \odot (\eta * \mathrm{id}_{u \circ \underline{\pi}_{\underline{U}}})} \Uparrow \quad {\scriptstyle \underline{\pi}_{\underline{V}}} & \downarrow {\scriptstyle v} \\ \underline{U} & \xrightarrow[{\scriptstyle f \circ u}]{} & \mathcal{Y}, \end{array} \qquad \begin{array}{ccc} \underline{W} & \xrightarrow{\qquad} & \underline{V} \\ {\scriptstyle \underline{\pi}_{\underline{U}}} \downarrow & {\scriptstyle \zeta} \Uparrow \quad {\scriptstyle \underline{\pi}_{\underline{V}}} & \downarrow {\scriptstyle v} \\ \underline{U} & \xrightarrow[{\scriptstyle g \circ u}]{} & \mathcal{Y}, \end{array}$$

as in (8.2), then we have commuting isomorphisms of $\mathcal{O}_W$-modules:

$$
\begin{array}{ccc}
\underline{\pi}_U^*\big(f^*(\mathcal{E})(\underline{U},u)\big) & \xrightarrow{\;i(\mathcal{E},f,u,v,\zeta\odot(\eta*\mathrm{id}_{u\circ\underline{\pi}_U}))\;} & \\
\underline{\pi}_U^*((\eta^*(\mathcal{E}))(\underline{U},u))\Big\downarrow & & \underline{\pi}_V^*\big(\mathcal{E}(\underline{V},v)\big). \\
\underline{\pi}_U^*\big(g^*(\mathcal{E})(\underline{U},u)\big) & \xrightarrow[\;i(\mathcal{E},g,u,v,\zeta)\;]{} &
\end{array}
$$

This defines a natural isomorphism $\eta^* : f^* \Rightarrow g^*$.

As in Remark 2.17, if $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are 1-morphisms of Deligne–Mumford C^∞-stacks and $\mathcal{E} \in \mathcal{O}_{\mathcal{Z}}$-mod, then we have a canonical isomorphism $I_{f,g}(\mathcal{E}) : (g \circ f)^*(\mathcal{E}) \to f^*(g^*(\mathcal{E}))$. If $\mathcal{X}$ is a Deligne–Mumford C^∞-stack and $\mathcal{E} \in \mathcal{O}_{\mathcal{X}}$-mod, we have a canonical isomorphism $\delta_{\mathcal{X}}(\mathcal{E}) : \mathrm{id}_{\mathcal{X}}^*(\mathcal{E}) \to \mathcal{E}$. These $I_{f,g}, \delta_{\mathcal{X}}$ have the same properties as in the C^∞-scheme case.

In a similar way, we can define pullbacks $f^{-1}(\mathcal{E})$ for sheaves of abelian groups and of C^∞-rings $\mathcal{E}$ on $\mathcal{Y}$, and corresponding isomorphisms $I_{f,g}(\mathcal{E}), \delta_{\mathcal{X}}(\mathcal{E})$.

Example 8.23. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of Deligne–Mumford C^∞-stacks. Then Example 8.19 defines sheaves of C^∞-rings $\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{Y}}$ on $\mathcal{X}, \mathcal{Y}$, so as in Definition 8.22 we have a pullback sheaf $f^{-1}(\mathcal{O}_{\mathcal{Y}})$ of C^∞-rings on $\mathcal{X}$. There is a natural morphism $f^\sharp : f^{-1}(\mathcal{O}_{\mathcal{Y}}) \to \mathcal{O}_{\mathcal{X}}$ of sheaves of C^∞-rings on $\mathcal{X}$, characterized by the following property: for all $(\underline{U},u),(\underline{V},v),\underline{W},\zeta$ as in Definition 8.22, the following diagram of sheaves of C^∞-rings on W commutes:

$$
\begin{array}{ccc}
\pi_U^{-1}\big(f^{-1}(\mathcal{O}_{\mathcal{Y}})(\underline{U},u)\big) & \xrightarrow{\;\pi_U^{-1}(f^\sharp(\underline{U},u))\;} & \pi_U^{-1}\big((\mathcal{O}_{\mathcal{X}})(\underline{U},u)\big) = \pi_U^{-1}(\mathcal{O}_U) \\
\cong\Big\downarrow i(\mathcal{O}_{\mathcal{Y}},f,u,v,\zeta) & & \pi_U^\sharp\Big\downarrow\cong \\
\pi_V^{-1}\big(\mathcal{O}_{\mathcal{Y}}(\underline{V},v)\big) = \pi_V^{-1}(\mathcal{O}_V) & \xrightarrow[\;\pi_V^\sharp\;]{} & \mathcal{O}_W,
\end{array}
$$

where $\underline{\pi}_U = (\pi_U, \pi_U^\sharp)$ and $\underline{\pi}_V = (\pi_V, \pi_V^\sharp)$.

Definition 8.24. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of Deligne–Mumford C^∞-stacks. Then $f^*(T^*\mathcal{Y}), T^*\mathcal{X}$ are $\mathcal{O}_{\mathcal{X}}$-modules, by Example 8.20 and Definition 8.22. Define $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ to be the unique morphism characterized as follows. Let $u : \bar{U} \to \mathcal{X}$, $v : \bar{V} \to \mathcal{Y}$, $\underline{W}, \underline{\pi}_U, \underline{\pi}_V$ be as in Definition 8.22, with (8.2) 2-Cartesian. Then the following diagram commutes in $\mathcal{O}_W$-mod:

$$
\begin{array}{ccc}
\underline{\pi}_U^*\big(f^*(T^*\mathcal{Y})(\underline{U},u)\big) & \xrightarrow{\;i(T^*\mathcal{Y},f,u,v,\zeta)\;} \underline{\pi}_V^*\big((T^*\mathcal{Y})(\underline{V},v)\big) = \underline{\pi}_V^*(T^*\underline{V}) \\
\pi_U^*(\Omega_f(\underline{U},u))\Big\downarrow & \Omega_{\underline{\pi}_V}\Big\downarrow \\
\underline{\pi}_U^*\big((T^*\mathcal{X})(\underline{U},u)\big) & \xrightarrow[\;(T^*\mathcal{X})_{(\underline{\pi}_U,\mathrm{id}_{u\circ\underline{\pi}_U})}\;]{} (T^*\mathcal{X})(\underline{W}, u\circ\underline{\pi}_U) = T^*\underline{W}.
\end{array}
$$

Here [33, Th. 10.15] is the analogue of Theorem 2.21.

Theorem 8.25. (a) *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of Deligne–Mumford C^∞-stacks. Then $\Omega_{g\circ f} = \Omega_f \circ f^*(\Omega_g) \circ I_{f,g}(T^*\mathcal{Z})$.*

(b) *Let $f,g : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms of Deligne–Mumford C^∞-stacks and $\eta : f \Rightarrow g$ a 2-morphism. Then $\Omega_f = \Omega_g \circ \eta^*(T^*\mathcal{Y}) : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$.*

(c) *Suppose* $\mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$ *are locally fair Deligne–Mumford* C^∞*-stacks with a 2-Cartesian square*

$$
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\ f\ } & \mathcal{Y} \\
{\scriptstyle e}\downarrow & {\scriptstyle \eta\Uparrow} & \downarrow{\scriptstyle h} \\
\mathcal{X} & \xrightarrow[\ g\ ]{} & \mathcal{Z}
\end{array}
$$

in $\mathbf{DMC^\infty Sta^{lf}}$, *so that* $\mathcal{W} \simeq \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$. *Then the following is exact in* $\mathrm{qcoh}(\mathcal{W})$:

$$
(g \circ e)^*(T^*\mathcal{Z}) \xrightarrow{\substack{e^*(\Omega_g)\circ I_{e,g}(T^*\mathcal{Z})\oplus \\ -f^*(\Omega_h)\circ I_{f,h}(T^*\mathcal{Z})\circ\eta^*(T^*\mathcal{Z})}} \begin{array}{c} e^*(T^*\mathcal{X})\oplus \\ f^*(T^*\mathcal{Y}) \end{array} \xrightarrow{\ \Omega_e\oplus\Omega_f\ } T^*\mathcal{W} \longrightarrow 0.
$$

8.7 Orbifold strata of Deligne–Mumford C^∞-stacks

Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack, and Γ a finite group. In [33, §11.1] we define six different notions of *orbifold strata* of $\mathcal{X}$, which are Deligne–Mumford C^∞-stacks written $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$, and open C^∞-substacks $\mathcal{X}_\circ^\Gamma \subseteq \mathcal{X}^\Gamma$, $\tilde{\mathcal{X}}_\circ^\Gamma \subseteq \tilde{\mathcal{X}}^\Gamma$, $\hat{\mathcal{X}}_\circ^\Gamma \subseteq \hat{\mathcal{X}}^\Gamma$. The points and orbifold groups of $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}_\circ^\Gamma$ are given by:

(i) Points of $\mathcal{X}^\Gamma$ are isomorphism classes $[x, \rho]$, where $[x] \in \mathcal{X}_{\mathrm{top}}$ and $\rho : \Gamma \to \mathrm{Iso}_\mathcal{X}([x])$ is an injective morphism, and $\mathrm{Iso}_{\mathcal{X}^\Gamma}([x, \rho])$ is the centralizer of $\rho(\Gamma)$ in $\mathrm{Iso}_\mathcal{X}([x])$. Points of $\mathcal{X}_\circ^\Gamma \subseteq \mathcal{X}^\Gamma$ are $[x, \rho]$ with ρ an isomorphism, and $\mathrm{Iso}_{\mathcal{X}_\circ^\Gamma}([x, \rho]) \cong C(\Gamma)$, the centre of Γ.

(ii) Points of $\tilde{\mathcal{X}}^\Gamma$ are pairs $[x, \Delta]$, where $[x] \in \mathcal{X}_{\mathrm{top}}$ and $\Delta \subseteq \mathrm{Iso}_\mathcal{X}([x])$ is isomorphic to Γ, and $\mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x, \Delta])$ is the normalizer of Δ in $\mathrm{Iso}_\mathcal{X}([x])$. Points of $\tilde{\mathcal{X}}_\circ^\Gamma \subseteq \tilde{\mathcal{X}}^\Gamma$ are $[x, \Delta]$ with $\Delta = \mathrm{Iso}_\mathcal{X}([x])$, and $\mathrm{Iso}_{\tilde{\mathcal{X}}_\circ^\Gamma}([x, \Delta]) \cong \Gamma$.

(iii) Points $[x, \Delta]$ of $\hat{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}_\circ^\Gamma$ are the same as for $\tilde{\mathcal{X}}^\Gamma, \tilde{\mathcal{X}}_\circ^\Gamma$, but with orbifold groups $\mathrm{Iso}_{\hat{\mathcal{X}}^\Gamma}([x, \Delta]) \cong \mathrm{Iso}_{\tilde{\mathcal{X}}^\Gamma}([x, \Delta])/\Delta$ and $\mathrm{Iso}_{\hat{\mathcal{X}}_\circ^\Gamma}([x, \Delta]) \cong \{1\}$.

Since the C^∞-stack $\hat{\mathcal{X}}_\circ^\Gamma$ has trivial orbifold groups, it is (equivalent to) a C^∞-scheme. That is, there is a genuine C^∞-scheme $\underline{\hat{X}}_\circ^\Gamma$, unique up to isomorphism in $\mathbf{C^\infty Sch}$, such that $\hat{\mathcal{X}}_\circ^\Gamma \simeq \underline{\bar{\hat{X}}}_\circ^\Gamma$ in $\mathbf{C^\infty Sta}$.

There are 1-morphisms $O^\Gamma(\mathcal{X}), \ldots, \hat{\Pi}_\circ^\Gamma(\mathcal{X})$ forming a strictly commutative diagram, where the columns are inclusions of open C^∞-substacks:

$$
\begin{array}{ccccc}
\mathrm{Aut}(\Gamma)\,\circlearrowleft\mathcal{X}_\circ^\Gamma & \xrightarrow{\ \tilde{\Pi}_\circ^\Gamma(\mathcal{X})\ } & \tilde{\mathcal{X}}_\circ^\Gamma & \xrightarrow{\ \hat{\Pi}_\circ^\Gamma(\mathcal{X})\ } & \hat{\mathcal{X}}_\circ^\Gamma \simeq \underline{\bar{\hat{X}}}_\circ^\Gamma \\
{\scriptstyle\subset}\Big\downarrow{\scriptstyle O_\circ^\Gamma(\mathcal{X})} & \mathcal{X} & {\scriptstyle\subset}\Big\downarrow & & \Big\downarrow{\scriptstyle\subset} \\
\mathrm{Aut}(\Gamma)\,\circlearrowleft\mathcal{X}^\Gamma & \xrightarrow[\ \tilde{\Pi}^\Gamma(\mathcal{X})\ ]{} & \tilde{\mathcal{X}}^\Gamma & \xrightarrow[\ \hat{\Pi}^\Gamma(\mathcal{X})\ ]{} & \hat{\mathcal{X}}^\Gamma.
\end{array} \qquad (8.3)
$$

Also $\mathrm{Aut}(\Gamma)$ acts on $\mathcal{X}^\Gamma, \mathcal{X}_\circ^\Gamma$, with $\tilde{\mathcal{X}}^\Gamma \simeq [\mathcal{X}^\Gamma/\mathrm{Aut}(\Gamma)]$, $\tilde{\mathcal{X}}_\circ^\Gamma \simeq [\mathcal{X}_\circ^\Gamma/\mathrm{Aut}(\Gamma)]$. The topological space $\mathcal{X}_{\mathrm{top}}$ of $\mathcal{X}$ from §8.2 has stratifications

$$
\mathcal{X}_{\mathrm{top}} \cong \coprod_{\substack{\text{iso. classes of} \\ \text{finite groups } \Gamma}} \mathcal{X}_{\circ,\mathrm{top}}^\Gamma/\Gamma \cong \coprod_\Gamma \tilde{\mathcal{X}}_{\circ,\mathrm{top}}^\Gamma \cong \coprod_\Gamma \hat{\mathcal{X}}_{\circ,\mathrm{top}}^\Gamma,
$$

which is why $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}^\Gamma_\circ$ are called orbifold strata. The 1-morphisms $O^\Gamma(\mathcal{X})$, $\tilde{O}^\Gamma(\mathcal{X})$ in (8.3) are proper, and $\hat{\Pi}^\Gamma(\mathcal{X})_{\text{top}} : \tilde{\mathcal{X}}^\Gamma_{\text{top}} \to \hat{\mathcal{X}}^\Gamma_{\text{top}}$ is a homeomorphism of topological spaces. Hence, if $\mathcal{X}$ is compact then $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$ are also compact.

Example 8.26. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. The *inertia stack* $\mathcal{I}_{\mathcal{X}}$ of $\mathcal{X}$ is the fibre product $\mathcal{I}_{\mathcal{X}} = \mathcal{X} \times_{\Delta_{\mathcal{X}}, \mathcal{X} \times \mathcal{X}, \Delta_{\mathcal{X}}} \mathcal{X}$, where $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is the diagonal 1-morphism. One can show there is an equivalence

$$\mathcal{I}_{\mathcal{X}} \simeq \coprod_{k \geqslant 1} \mathcal{X}^{\mathbb{Z}_k}.$$

Points of $\mathcal{I}_{\mathcal{X}}$ are isomorphism classes $[x, \eta]$, where $[x] \in \mathcal{X}_{\text{top}}$ and $\eta \in \text{Iso}_{\mathcal{X}}([x])$. Each such $\eta \in \text{Iso}_{\mathcal{X}}([x])$ has some finite order $k \geqslant 1$, and generates an injective morphism $\rho : \mathbb{Z}_k \to \text{Iso}_{\mathcal{X}}([x])$ mapping $\rho : a \mapsto \eta^a$. We may identify $\mathcal{X}^{\mathbb{Z}_k}$ with the open and closed C^∞-substack of $[x, \eta]$ in $\mathcal{I}_{\mathcal{X}}$ for which η has order k.

Orbifold strata $\mathcal{X}^\Gamma$ are strongly functorial for representable 1-morphisms and their 2-morphisms. That is, if $f : \mathcal{X} \to \mathcal{Y}$ is a representable 1-morphism of Deligne–Mumford C^∞-stacks, we define a unique representable 1-morphism $f^\Gamma : \mathcal{X}^\Gamma \to \mathcal{Y}^\Gamma$ with $O^\Gamma(\mathcal{Y}) \circ f^\Gamma = f \circ O^\Gamma(\mathcal{X})$. If $f, g : \mathcal{X} \to \mathcal{Y}$ are representable and $\eta : f \Rightarrow g$ is a 2-morphism, we define a unique 2-morphism $\eta^\Gamma : f^\Gamma \Rightarrow g^\Gamma$ with $\text{id}_{O^\Gamma(\mathcal{Y})} * \eta^\Gamma = \eta * \text{id}_{O^\Gamma(\mathcal{X})}$. These f^Γ, η^Γ are compatible with compositions of 1- and 2-morphisms, and identities, in the obvious way. Orbifold strata $\tilde{\mathcal{X}}^\Gamma$ have the same kind of functorial behaviour, and $\hat{\mathcal{X}}^\Gamma$ have a weaker functorial behaviour, in that $\hat{f}^\Gamma$ is only natural up to 2-isomorphism.

For $f : \mathcal{X} \to \mathcal{Y}$ and Γ as above, the restriction $f^\Gamma|_{\mathcal{X}^\Gamma_\circ}$ need not map $\mathcal{X}^\Gamma_\circ \to \mathcal{Y}^\Gamma_\circ$, but only $\mathcal{X}^\Gamma_\circ \to \mathcal{Y}^\Gamma$. So we do not define 1-morphisms $f^\Gamma_\circ : \mathcal{X}^\Gamma_\circ \to \mathcal{Y}^\Gamma_\circ$. The same applies for the actions $\tilde{f}^\Gamma, \hat{f}^\Gamma$ of f on orbifold strata $\tilde{\mathcal{X}}^\Gamma_\circ, \hat{\mathcal{X}}^\Gamma_\circ$.

In [33, §11.3] we describe the orbifold strata of a quotient C^∞-stack $[\underline{X}/G]$.

Theorem 8.27. *Suppose $\underline{X}$ is a separated C^∞-scheme and G a finite group acting on $\underline{X}$ by isomorphisms, and write $\mathcal{X} = [\underline{X}/G]$ for the quotient C^∞-stack from Example 8.11, which is a Deligne–Mumford C^∞-stack. Let Γ be a finite*

group. Then there are equivalences of C^∞-stacks

$$\mathcal{X}^\Gamma \simeq \coprod_{\substack{\text{conjugacy classes } [\rho] \text{ of injective} \\ \text{group morphisms } \rho : \Gamma \to G}} \big[\underline{X}^{\rho(\Gamma)} / \{ g \in G : g\rho(\gamma) = \rho(\gamma)g \ \forall \gamma \in \Gamma \} \big], \qquad (8.4)$$

$$\mathcal{X}_\circ^\Gamma \simeq \coprod_{\substack{\text{conjugacy classes } [\rho] \text{ of injective} \\ \text{group morphisms } \rho : \Gamma \to G}} \big[\underline{X}_\circ^{\rho(\Gamma)} / \{ g \in G : g\rho(\gamma) = \rho(\gamma)g \ \forall \gamma \in \Gamma \} \big], \qquad (8.5)$$

$$\tilde{\mathcal{X}}^\Gamma \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta \subseteq G \text{ with } \Delta \cong \Gamma} \big[\underline{X}^\Delta / \{ g \in G : \Delta = g\Delta g^{-1} \} \big], \qquad (8.6)$$

$$\tilde{\mathcal{X}}_\circ^\Gamma \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta \subseteq G \text{ with } \Delta \cong \Gamma} \big[\underline{X}_\circ^\Delta / \{ g \in G : \Delta = g\Delta g^{-1} \} \big]. \qquad (8.7)$$

$$\hat{\mathcal{X}}^\Gamma \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta \subseteq G \text{ with } \Delta \cong \Gamma} \big[\underline{X}^\Delta / (\{ g \in G : \Delta = g\Delta g^{-1} \}/\Delta) \big], \qquad (8.8)$$

$$\hat{\mathcal{X}}_\circ^\Gamma \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta \subseteq G \text{ with } \Delta \cong \Gamma} \big[\underline{X}_\circ^\Delta / (\{ g \in G : \Delta = g\Delta g^{-1} \}/\Delta) \big]. \qquad (8.9)$$

Here for each subgroup $\Delta \subseteq G$, we write $\underline{X}^\Delta$ for the closed C^∞-subscheme in $\underline{X}$ fixed by Δ in G, and $\underline{X}_\circ^\Delta$ for the open C^∞-subscheme in $\underline{X}^\Delta$ of points in $\underline{X}$ whose stabilizer group in G is exactly Δ. In (8.4)–(8.5), morphisms $\rho, \rho' : \Gamma \to G$ are conjugate if $\rho' = \mathrm{Ad}(g) \circ \rho$ for some $g \in G$, and subgroups $\Delta, \Delta' \subseteq G$ are conjugate if $\Delta = g\Delta' g^{-1}$ for some $g \in G$. In (8.4)–(8.9) we sum over one representative ρ or Δ for each conjugacy class.

Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack and Γ a finite group, so that as above we have an orbifold stratum $\mathcal{X}^\Gamma$ with a 1-morphism $O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \to \mathcal{X}$. Let $\mathcal{E}$ be a quasicoherent sheaf on $\mathcal{X}$, so that $\mathcal{E}^\Gamma := O^\Gamma(\mathcal{X})^*(\mathcal{E})$ is a quasicoherent sheaf on $\mathcal{X}^\Gamma$. In [33, §11.4] we show that there is a natural representation of Γ on $\mathcal{E}^\Gamma$ by isomorphisms. Also the action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{X}^\Gamma$ lifts naturally to $\mathcal{E}^\Gamma$, so that $\mathrm{Aut}(\Gamma) \ltimes \Gamma$ acts equivariantly on $\mathcal{E}^\Gamma$.

Write $R_0, \ldots, R_k$ for the irreducible representations of Γ over $\mathbb{R}$ (that is, we choose one representative R_i in each isomorphism class of irreducible representations), with $R_0 = \mathbb{R}$ the trivial representation. Then the Γ-representation on $\mathcal{E}^\Gamma$ induces a splitting

$$\mathcal{E}^\Gamma \cong \bigoplus_{i=0}^k \mathcal{E}_i^\Gamma \otimes R_i \quad \text{for } \mathcal{E}_0^\Gamma, \ldots, \mathcal{E}_k^\Gamma \in \mathrm{qcoh}(\mathcal{X}^\Gamma). \qquad (8.10)$$

We will be interested in splitting $\mathcal{E}^\Gamma$ into *trivial* and *nontrivial* representations of Γ, denoted by subscripts 'tr' and 'nt'. So we write

$$\mathcal{E}^\Gamma = \mathcal{E}_{\mathrm{tr}}^\Gamma \oplus \mathcal{E}_{\mathrm{nt}}^\Gamma, \qquad (8.11)$$

where $\mathcal{E}_{\mathrm{tr}}^\Gamma, \mathcal{E}_{\mathrm{nt}}^\Gamma$ are the subsheaves of $\mathcal{E}^\Gamma$ corresponding to the factors $\mathcal{E}_0^\Gamma \otimes R_0$ and $\bigoplus_{i=1}^k \mathcal{E}_i^\Gamma \otimes R_i$ respectively. The same applies for the orbifold stratum $\mathcal{X}_\circ^\Gamma \subseteq \mathcal{X}^\Gamma$.

We also have an orbifold stratum $\tilde{\mathcal{X}}^\Gamma$ with a 1-morphism $\tilde{O}^\Gamma(\mathcal{X}) : \tilde{\mathcal{X}}^\Gamma \to \mathcal{X}$, so that $\tilde{\mathcal{E}}^\Gamma := \tilde{O}^\Gamma(\mathcal{X})^*(\mathcal{E})$ is a quasicoherent sheaf on $\tilde{\mathcal{X}}^\Gamma$. In general there is no

natural Γ-representation on $\tilde{\mathcal{E}}^\Gamma$, as the quotient by $\mathrm{Aut}(\Gamma)$ in $\tilde{\mathcal{X}}^\Gamma \simeq [\mathcal{X}^\Gamma / \mathrm{Aut}(\Gamma)]$ does not preserve the Γ-action. However, we do have a natural splitting

$$\tilde{\mathcal{E}}^\Gamma = \tilde{\mathcal{E}}^\Gamma_{\mathrm{tr}} \oplus \tilde{\mathcal{E}}^\Gamma_{\mathrm{nt}} \tag{8.12}$$

corresponding to (8.11). The same applies for $\tilde{\mathcal{X}}^\Gamma_\circ \subseteq \tilde{\mathcal{X}}^\Gamma$.

As in (8.3), for the orbifold stratum $\hat{\mathcal{X}}^\Gamma$ we do not have a natural 1-morphism $\hat{\mathcal{X}}^\Gamma \to \mathcal{X}$, so we cannot just pull $\mathcal{E}$ back to $\hat{\mathcal{X}}^\Gamma$. Instead, we push $\tilde{\mathcal{E}}^\Gamma$ down to $\hat{\mathcal{X}}^\Gamma$ along the 1-morphism $\hat{\Pi}^\Gamma : \tilde{\mathcal{X}}^\Gamma \to \hat{\mathcal{X}}^\Gamma$. It turns out that in the splitting (8.12), the push down $\hat{\Pi}^\Gamma_*(\tilde{\mathcal{E}}^\Gamma_{\mathrm{nt}})$ is zero, since $\hat{\Pi}^\Gamma$ has fibre $[\underline{*}/\Gamma]$, and $\hat{\Pi}^\Gamma_*$ essentially takes Γ-equivariant parts. So we define $\hat{\mathcal{E}}^\Gamma_{\mathrm{tr}} = \hat{\Pi}^\Gamma_*(\tilde{\mathcal{E}}^\Gamma_{\mathrm{tr}})$, a quasicoherent sheaf on $\hat{\mathcal{X}}^\Gamma$. The same applies for $\hat{\mathcal{X}}^\Gamma_\circ \subseteq \hat{\mathcal{X}}^\Gamma$.

When passing to orbifold strata, it is often natural to restrict to the trivial parts $\mathcal{E}^\Gamma_{\mathrm{tr}}, \tilde{\mathcal{E}}^\Gamma_{\mathrm{tr}}, \hat{\mathcal{E}}^\Gamma_{\mathrm{tr}}$ of the pullbacks of $\mathcal{E}$. The next theorem illustrates this.

Theorem 8.28. *Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack and Γ a finite group, so that we have a 1-morphism $O^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \to \mathcal{X}$. As in Example 8.20 we have cotangent sheaves $T^*\mathcal{X}, T^*(\mathcal{X}^\Gamma)$ and a morphism $\Omega_{O^\Gamma(\mathcal{X})} : O^\Gamma(\mathcal{X})^*(T^*\mathcal{X}) \to T^*(\mathcal{X}^\Gamma)$ in $\mathrm{qcoh}(\mathcal{X}^\Gamma)$. But $O^\Gamma(\mathcal{X})^*(T^*\mathcal{X}) = (T^*\mathcal{X})^\Gamma$, so by (8.11) we have a splitting $(T^*\mathcal{X})^\Gamma = (T^*\mathcal{X})^\Gamma_{\mathrm{tr}} \oplus (T^*\mathcal{X})^\Gamma_{\mathrm{nt}}$. Then $\Omega_{O^\Gamma(\mathcal{X})}|_{(T^*\mathcal{X})^\Gamma_{\mathrm{tr}}} : (T^*\mathcal{X})^\Gamma_{\mathrm{tr}} \to T^*(\mathcal{X}^\Gamma)$ is an isomorphism, and $\Omega_{O^\Gamma(\mathcal{X})}|_{(T^*\mathcal{X})^\Gamma_{\mathrm{nt}}} = 0$.*

Similarly, using $\tilde{O}^\Gamma(\mathcal{X}) : \tilde{\mathcal{X}}^\Gamma \to \mathcal{X}$ and (8.12) for $(\widetilde{T^*\mathcal{X}})^\Gamma$ we find that $\Omega_{\tilde{O}^\Gamma(\mathcal{X})}|_{(\widetilde{T^*\mathcal{X}})^\Gamma_{\mathrm{tr}}} : (\widetilde{T^*\mathcal{X}})^\Gamma_{\mathrm{tr}} \to T^*(\tilde{\mathcal{X}}^\Gamma)$ is an isomorphism, and $\Omega_{\tilde{O}^\Gamma(\mathcal{X})}|_{(\widetilde{T^*\mathcal{X}})^\Gamma_{\mathrm{nt}}} = 0$. Also, there is a natural isomorphism $(\widehat{T^*\mathcal{X}})^\Gamma_{\mathrm{tr}} \cong T^*(\hat{\mathcal{X}}^\Gamma)$ in $\mathrm{qcoh}(\hat{\mathcal{X}}^\Gamma)$.

9 Orbifolds

We now summarize [35, §8.1–§8.4] on orbifolds. There is already a substantial literature on orbifolds, and §9.1 indicates the main milestones in the field, and explains how our definition of orbifolds relates to those by other authors.

9.1 Different ways to define orbifolds

Orbifolds are geometric spaces locally modelled on $\mathbb{R}^n/G$, for $G \subset \mathrm{GL}(n, \mathbb{R})$ a finite group. There are several *nonequivalent* definitions of orbifolds in the literature, which are reviewed in [35, §8.1]. They were first defined by Satake [52] (who called them 'V-manifolds') and Thurston [55, §13]. Satake and Thurston defined an orbifold to be a Hausdorff topological space X with an atlas of charts (U_i, Γ_i, ϕ_i) for $i \in I$, where $\Gamma_i \subset \mathrm{GL}(n, \mathbb{R})$ is a finite subgroup, $U_i \subseteq \mathbb{R}^n$ a Γ_i-invariant open subset, and $\phi_i : U_i/\Gamma_i \to X$ a homeomorphism with an open set in X, compatible on overlaps $\phi_i(U_i/\Gamma_i) \cap \phi_j(U_j/\Gamma_j)$ in X. Smooth maps between orbifolds are continuous maps $f : X \to Y$, which lift locally to equivariant smooth maps on the charts.

There is a problem with this notion of smooth maps: some differential-geometric operations, such as pullbacks of vector bundles by smooth maps, may not be well-defined. To fix this problem, new definitions were needed. Moerdijk

and Pronk [47, 48] defined orbifolds to be *proper étale Lie groupoids* in **Man**. Chen and Ruan [13, §4] gave an alternative theory more in the spirit of [52, 55]. A book on orbifolds in the sense of [13, 47, 48] is Adem, Leida and Ruan [1].

All of [1, 13, 47, 48, 52, 55] regard orbifolds as an ordinary category. But orbifolds are differential-geometric analogues of Deligne–Mumford stacks, which form a 2-category. So it seems natural to define a 2-category of orbifolds **Orb**. Several important geometric constructions need the extra structure of a 2-category to work properly. For example, transverse fibre products exist in the 2-category **Orb**, where they satisfy a universal property involving 2-morphisms, as in §A.4. In the homotopy category Ho(**Orb**), 'transverse fibre products' can be defined as an *ad hoc* geometric construction, but they are not fibre products in the category-theoretic sense, and do not satisfy a universal property.

There are two main routes in the literature for defining a 2-category of orbifolds **Orb**. The first, as in Pronk [51] and Lerman [39, §3.3], is to define orbifolds to be groupoids in **Man** as in [47,48]. But to define 1- and 2-morphisms in **Orb** one must do more work: one makes proper étale Lie groupoids into a 2-category **Gpoid**, and then **Orb** is defined as a (weak) 2-category localization of **Gpoid** at a suitable class of 1-morphisms.

The second route, as in Behrend and Xu [9, §2], Lerman [39, §4] and Metzler [45, §3.5], is to define orbifolds as a class of Deligne–Mumford stacks on the site (**Man**, $\mathcal{J}_{\textbf{Man}}$) of manifolds with Grothendieck topology $\mathcal{J}_{\textbf{Man}}$ coming from open covers. The relationship between the two routes is discussed in [9, 39, 51].

In the 'classical' approaches to orbifolds [1, 13, 47, 48, 52, 55], the objects, orbifolds, have a simple definition, but the smooth maps, or 1- and 2-morphisms, are either badly behaved, or very complicated to define. In contrast, in the 'stacky' approaches to orbifolds [9, 33, 39, 45], the objects are very complicated to define, but 1- and 2-morphisms are well-behaved and easy to define — 1-morphisms are just functors, and 2-morphisms are natural isomorphisms.

Our approach, described in [35, §8.2], is similar to the second route: we define orbifolds to be special examples of Deligne–Mumford C^∞-stacks, so that they are stacks on the site (**C$^\infty$Sch**, $\mathcal{J}$). This will be convenient for our work on d-stacks and d-orbifolds, which are also based on C^∞-stacks.

Definition 9.1. An *orbifold of dimension n* is a separated, second countable Deligne–Mumford C^∞-stack $\mathcal{X}$ such that for every $[x] \in \mathcal{X}_{\text{top}}$ there exist a linear action of $G = \text{Iso}_{\mathcal{X}}([x])$ on $\mathbb{R}^n$, a G-invariant open neighbourhood U of 0 in $\mathbb{R}^n$, and a 1-morphism $i : [\underline{U}/G] \to \mathcal{X}$ which is an equivalence with an open neighbourhood $\mathcal{U} \subseteq \mathcal{X}$ of $[x]$ in $\mathcal{X}$ with $i_{\text{top}}([0]) = [x]$, where $\underline{U} = F_{\textbf{Man}}^{\textbf{C}^\infty\textbf{Sch}}(U)$.

Write **Orb** for the full 2-subcategory of orbifolds in **DMC$^\infty$Sta**. We may refer to 1-morphisms $f : \mathcal{X} \to \mathcal{Y}$ in **Orb** as *smooth maps* of orbifolds. Define a full and faithful functor $F_{\textbf{Man}}^{\textbf{Orb}} : \textbf{Man} \to \textbf{Orb}$ by $F_{\textbf{Man}}^{\textbf{Orb}} = F_{\textbf{C}^\infty\textbf{Sch}}^{\textbf{C}^\infty\textbf{Sta}} \circ F_{\textbf{Man}}^{\textbf{C}^\infty\textbf{Sch}}$.

Here is [33, Th. 9.26 & Cor. 9.27]. Since equivalent (2-)categories are considered to be 'the same', the moral of Theorem 9.2 is that our orbifolds are essentially the same objects as those considered by other recent authors.

Theorem 9.2. *The 2-category* **Orb** *of orbifolds without boundary defined above*

is equivalent to the 2-categories of orbifolds considered as stacks on **Man** *defined in Metzler [45, §3.4] and Lerman [39, §4], and also equivalent as a weak 2-category to the weak 2-categories of orbifolds regarded as proper étale Lie groupoids defined in Pronk [51] and Lerman [39, §3.3].*

Furthermore, the homotopy category Ho(**Orb**) *of* **Orb** *(that is, the category whose objects are objects in* **Orb**, *and whose morphisms are 2-isomorphism classes of 1-morphisms in* **Orb**) *is equivalent to the category of orbifolds regarded as proper étale Lie groupoids defined in Moerdijk [47]. Transverse fibre products in* **Orb** *agree with the corresponding fibre products in* $\mathbf{C^\infty Sta}$.

We define five classes of smooth maps:

Definition 9.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth map (1-morphism) of orbifolds.

(i) We call f *representable* if it acts injectively on orbifold groups, that is, $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}\big(f_{\mathrm{top}}([x])\big)$ is an injective morphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$.

Equivalently, f is representable if it is a representable 1-morphism of C^∞-stacks. This means that whenever $\underline{V}$ is a C^∞-scheme and $\Pi : \underline{V} \to \mathcal{Y}$ is a 1-morphism then the C^∞-stack fibre product $\mathcal{X} \times_{f,\mathcal{Y},\Pi} \underline{V}$ is a C^∞-scheme.

(ii) We call f an *immersion* if it is representable and $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ is a surjective morphism of vector bundles, i.e. has a right inverse in qcoh$(\mathcal{X})$.

(iii) We call f an *embedding* if it is an immersion, and $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}\big(f_{\mathrm{top}}([x])\big)$ is an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ is a homeomorphism with its image (so in particular it is injective).

(iv) We call f a *submersion* if $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ is an injective morphism of vector bundles, i.e. has a left inverse in qcoh$(\mathcal{X})$.

(v) We call f *étale* if it is representable and $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ is an isomorphism, or equivalently, if f is étale as a 1-morphism of C^∞-stacks.

Note that submersions are not required to be representable.

Definition 9.4. An orbifold $\mathcal{X}$ is called *effective* if $\mathcal{X}$ is locally modelled near each $[x] \in \mathcal{X}_{\mathrm{top}}$ on $\mathbb{R}^n/G$, where G acts effectively on $\mathbb{R}^n$, that is, every $1 \neq \gamma \in G$ acts nontrivially on $\mathbb{R}^n$.

In [35, §8.4] we prove a uniqueness property for 2-morphisms of effective orbifolds.

Proposition 9.5. *Let* $\mathcal{X}, \mathcal{Y}$ *be effective orbifolds, and* $f, g : \mathcal{X} \to \mathcal{Y}$ *be 1-morphisms. Suppose that either:*

(i) *f is an embedding, a submersion, étale, or an equivalence;*

(ii) *$f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}\big(f_{\mathrm{top}}([x])\big)$ is surjective for all $[x] \in \mathcal{X}_{\mathrm{top}}$; or*

(iii) *$\mathcal{Y}$ is a manifold.*

Then there exists at most one 2-morphism $\eta : f \Rightarrow g$.

Some authors include effectiveness in their definition of orbifolds. The Satake–Thurston definitions are not as well-behaved for noneffective orbifolds. One reason is that Proposition 9.5 often allows us to treat effective orbifolds as if they were a category rather than a 2-category, that is, one can work in the homotopy category Ho($\mathbf{Orb^{eff}}$) of the full 2-subcategory $\mathbf{Orb^{eff}}$ of effective orbifolds, because genuinely 2-categorical behaviour comes from non-uniqueness of 2-morphisms.

In [35, §8.3] we discuss *vector bundles* on orbifolds. Now an orbifold $\mathcal{X}$ is an example of a Deligne–Mumford C^∞-stack, and in §8.6 we defined a category qcoh($\mathcal{X}$) of quasicoherent sheaves on $\mathcal{X}$, and a full subcategory vect($\mathcal{X}$) of vector bundles on $\mathcal{X}$. Unless we say otherwise, a *vector bundle $\mathcal{E}$ on an orbifold $\mathcal{X}$* will just mean an object in vect($\mathcal{X}$), a special kind of quasicoherent sheaf on $\mathcal{X}$, and a *smooth section s of $\mathcal{E}$* will mean an element of $C^\infty(\mathcal{E})$, that is, a morphism $s : \mathcal{O}_\mathcal{X} \to \mathcal{E}$ in vect($\mathcal{X}$). The cotangent sheaf $T^*\mathcal{X}$ of an n-orbifold $\mathcal{X}$ is a vector bundle on $\mathcal{X}$ of rank n, which we call the *cotangent bundle.*

For some applications below, this point of view on vector bundles is not ideal. If $E \to X$ is a vector bundle on a manifold, then E is itself a manifold (with extra structure), with a submersion $\pi : E \to X$, and a section $s \in C^\infty(E)$ is a smooth map $s : X \to E$ with $\pi \circ s = \mathrm{id}_X$. In §4.1–§4.2 we considered d-space fibre products $\boldsymbol{V} \times_{\boldsymbol{s,E,0}} \boldsymbol{V}$ where $\boldsymbol{V}, \boldsymbol{E}, \boldsymbol{s}, \boldsymbol{0} = F_{\mathbf{Man}}^{\mathbf{dSpa}}(V, E, s, 0)$. For the d-orbifold analogue of this, we would like to regard a vector bundle $\mathcal{E}$ over an orbifold $\mathcal{X}$ as being an orbifold in its own right, rather than just a quasicoherent sheaf, and a section $s \in C^\infty(\mathcal{E})$ as being a 1-morphism $s : \mathcal{X} \to \mathcal{E}$ in $\mathbf{Orb}$.

To get round this, in [35, §8.3] we define a *total space functor* Tot, which to each $\mathcal{E}$ in vect($\mathcal{X}$) associates an orbifold Tot($\mathcal{E}$), called the *total space* of $\mathcal{E}$, and to each section $s \in C^\infty(\mathcal{E})$ associates a 1-morphism Tot(s) : $\mathcal{X} \to \mathrm{Tot}(\mathcal{E})$ in $\mathbf{Orb}$. Then the d-orbifold analogue of $\boldsymbol{V} \times_{\boldsymbol{s,E,0}} \boldsymbol{V}$ in Proposition 4.2(c) is $\boldsymbol{\mathcal{V}} \times_{\boldsymbol{s,\mathcal{E},0}} \boldsymbol{\mathcal{V}}$, where $\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{E}}, \boldsymbol{s}, \boldsymbol{0} = F_{\mathbf{Orb}}^{\mathbf{dSta}}\big(\mathcal{V}, \mathrm{Tot}(\mathcal{E}), \mathrm{Tot}(s), \mathrm{Tot}(0)\big)$.

Many other standard ideas in differential geometry extend simply to orbifolds, such as submanifolds, transverse fibre products, and orientations, and we will generally use these without comment.

9.2 Orbifold strata of orbifolds

Section 8.7 discussed orbifold strata $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}_\circ^\Gamma$ of a Deligne–Mumford C^∞-stack $\mathcal{X}$. In [35, §8.4] we work these ideas out for orbifolds. If $\mathcal{X}$ is an orbifold, then $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}_\circ^\Gamma$ need not be orbifolds, as the next example shows, but are disjoint unions of orbifolds of different dimensions.

Example 9.6. Let the real projective plane $\mathbb{RP}^2$ have homogeneous coordinates $[x_0, x_1, x_2]$, and let $\mathbb{Z}_2 = \{1, \sigma\}$ act on $\mathbb{RP}^2$ by $\sigma : [x_0, x_1, x_2] \mapsto [x_0, x_1, -x_2]$. The fixed point locus of σ in $\mathbb{RP}^2$ is the disjoint union of the circle $\big\{[x_0, x_1, 0] : [x_0, x_1] \in \mathbb{RP}^1\big\}$ and the point $\big\{[0, 0, 1]\big\}$.

Write $\underline{\mathbb{RP}}^2 = F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}(\mathbb{RP}^2)$, and form the quotient orbifold $\mathcal{X} = [\underline{\mathbb{RP}}^2/\mathbb{Z}_2]$. Then (8.4) shows that the orbifold stratum $\mathcal{X}^{\mathbb{Z}_2}$ is the disjoint union of orbifolds

$\mathbb{RP}^1 \times [*/\mathbb{Z}_2]$ and $[*/\mathbb{Z}_2]$ of dimensions 1 and 0, respectively. Note that $\mathcal{X}^{\mathbb{Z}_2}$ is not an orbifold, as it does not have pure dimension, and nor are $\tilde{\mathcal{X}}^{\mathbb{Z}_2}, \ldots, \hat{\mathcal{X}}_\circ^{\mathbb{Z}_2}$.

So that our constructions remain within the world of orbifolds, we will find it useful to define a decomposition $\mathcal{X}^\Gamma = \coprod_{\lambda \in \Lambda_+^\Gamma} \mathcal{X}^{\Gamma,\lambda}$ of $\mathcal{X}^\Gamma$ such that each $\mathcal{X}^{\Gamma,\lambda}$ is an orbifold of dimension $\dim \mathcal{X} - \dim \lambda$.

Definition 9.7. Let Γ be a finite group. Consider representations (V, ρ) of Γ, where V is a finite-dimensional real vector space and $\rho : \Gamma \to \mathrm{Aut}(V)$ a group morphism. We call (V, ρ) *nontrivial* if $V^{\rho(\Gamma)} = \{0\}$. Write $\mathrm{Rep}_{\mathrm{nt}}(\Gamma)$ for the abelian category of nontrivial (V, ρ), and $K_0(\mathrm{Rep}_{\mathrm{nt}}(\Gamma))$ for its Grothendieck group. Then any (V, ρ) in $\mathrm{Rep}_{\mathrm{nt}}(\Gamma)$ has a class $[(V, \rho)]$ in $K_0(\mathrm{Rep}_{\mathrm{nt}}(\Gamma))$. For brevity, we will use the notation $\Lambda^\Gamma = K_0(\mathrm{Rep}_{\mathrm{nt}}(\Gamma))$ and $\Lambda_+^\Gamma = \{[(V, \rho)] : (V, \rho) \in \mathrm{Rep}_{\mathrm{nt}}(\Gamma)\} \subseteq \Lambda^\Gamma$. We think of Λ_+^Γ as the 'positive cone' in Λ^Γ.

By elementary representation theory, up to isomorphism Γ has finitely many irreducible representations. Let $R_0, R_1, \ldots, R_k$ be choices of irreducible representations in these isomorphism classes, with $R_0 = \mathbb{R}$ the trivial irreducible representation, so that $R_1, \ldots, R_k$ are nontrivial. Then Λ^Γ is freely generated over $\mathbb{Z}$ by $[R_1], \ldots, [R_k]$, so that

$$\Lambda^\Gamma = \{a_1[R_1] + \cdots + a_k[R_k] : a_1, \ldots, a_k \in \mathbb{Z}\}, \qquad \text{and}$$
$$\Lambda_+^\Gamma = \{a_1[R_1] + \cdots + a_k[R_k] : a_1, \ldots, a_k \in \mathbb{N}\} \subseteq \Lambda^\Gamma,$$

where $\mathbb{N} = \{0, 1, 2, \ldots\} \subset \mathbb{Z}$. Hence $\Lambda^\Gamma \cong \mathbb{Z}^k$ and $\Lambda_+^\Gamma \cong \mathbb{N}^k$.

Define a group morphism $\dim : \Lambda^\Gamma \to \mathbb{Z}$ by $\dim : a_1[R_1] + \cdots + a_k[R_k] \mapsto a_1 \dim R_1 + \cdots + a_k \dim R_k$, so that $\dim : [(V, \rho)] \mapsto \dim V$. Then $\dim(\Lambda_+^\Gamma) \subseteq \mathbb{N}$.

Now let $\mathcal{X}$ be an orbifold. As in (8.10)–(8.11) we have decompositions $O^\Gamma(\mathcal{X})^*(T^*\mathcal{X}) = (T^*\mathcal{X})_{\mathrm{tr}}^\Gamma \oplus (T^*\mathcal{X})_{\mathrm{nt}}^\Gamma$ with $(T^*\mathcal{X})_{\mathrm{tr}}^\Gamma \cong (T^*\mathcal{X})_0^\Gamma \otimes R_0$ and $(T^*\mathcal{X})_{\mathrm{nt}}^\Gamma \cong \bigoplus_{i=1}^k (T^*\mathcal{X})_i^\Gamma \otimes R_i$, where $(T^*\mathcal{X})_0^\Gamma, \ldots, (T^*\mathcal{X})_k^\Gamma \in \mathrm{qcoh}(\mathcal{X}^\Gamma)$. Since $T^*\mathcal{X}$ is a vector bundle, $O^\Gamma(\mathcal{X})^*(T^*\mathcal{X})$ is a vector bundle, and so the $(T^*\mathcal{X})_i^\Gamma$ are *vector bundles of mixed rank*, that is, locally they are vector bundles, but their ranks may vary on different connected components of $\mathcal{X}^\Gamma$.

For each $\lambda \in \Lambda_+^\Gamma$, define $\mathcal{X}^{\Gamma,\lambda}$ to be the open and closed C^∞-substack in $\mathcal{X}^\Gamma$ with $\mathrm{rank}\big((T^*\mathcal{X})_1^\Gamma\big)[R_1] + \cdots + \mathrm{rank}\big((T^*\mathcal{X})_k^\Gamma\big)[R_k] = \lambda$ in Λ_+^Γ. Then $(T^*\mathcal{X})_{\mathrm{nt}}^\Gamma|_{\mathcal{X}^{\Gamma,\lambda}}$ is a vector bundle of rank $\dim \lambda$, so $(T^*\mathcal{X})_{\mathrm{tr}}^\Gamma|_{\mathcal{X}^{\Gamma,\lambda}}$ is a vector bundle of dimension $\dim \mathcal{X} - \dim \lambda$ on $\mathcal{X}^{\Gamma,\lambda}$. But $(T^*\mathcal{X})_{\mathrm{tr}}^\Gamma \cong T^*(\mathcal{X}^\Gamma)$ by Theorem 8.28. Hence $T^*(\mathcal{X}^{\Gamma,\lambda})$ is a vector bundle of rank $\dim \mathcal{X} - \dim \lambda$. Since $\mathcal{X}^\Gamma$ is a disjoint union of orbifolds of different dimensions, we see that $\mathcal{X}^{\Gamma,\lambda}$ is an orbifold, with $\dim \mathcal{X}^{\Gamma,\lambda} = \dim \mathcal{X} - \dim \lambda$. Then $\mathcal{X}^\Gamma = \coprod_{\lambda \in \Lambda_+^\Gamma} \mathcal{X}^{\Gamma,\lambda}$.

Write $O^{\Gamma,\lambda}(\mathcal{X}) = O^\Gamma(\mathcal{X})|_{\mathcal{X}^{\Gamma,\lambda}} : \mathcal{X}^{\Gamma,\lambda} \to \mathcal{X}$. It is a proper, representable immersion of orbifolds. We interpret $(T^*\mathcal{X})_{\mathrm{nt}}^\Gamma|_{\mathcal{X}^{\Gamma,\lambda}}$ as the *conormal bundle* of $\mathcal{X}^{\Gamma,\lambda}$ in $\mathcal{X}$. It carries a nontrivial Γ-representation of class $\lambda \in \Lambda_+^\Gamma$, so we refer to λ as the *conormal Γ-representation* of $\mathcal{X}^{\Gamma,\lambda}$.

Define $\mathcal{X}_\circ^{\Gamma,\lambda} = \mathcal{X}_\circ^\Gamma \cap \mathcal{X}^{\Gamma,\lambda}$, and $O_\circ^{\Gamma,\lambda}(\mathcal{X}) = O_\circ^\Gamma(\mathcal{X})|_{\mathcal{X}_\circ^{\Gamma,\lambda}} : \mathcal{X}_\circ^{\Gamma,\lambda} \to \mathcal{X}$. Then $\mathcal{X}_\circ^{\Gamma,\lambda}$ is an orbifold with $\dim \mathcal{X}_\circ^{\Gamma,\lambda} = \dim \mathcal{X} - \dim \lambda$, and $\mathcal{X}_\circ^\Gamma = \coprod_{\lambda \in \Lambda_+^\Gamma} \mathcal{X}_\circ^{\Gamma,\lambda}$.

As in §8.7, we have $\tilde{\mathcal{X}}^\Gamma \simeq [\mathcal{X}^\Gamma/\operatorname{Aut}(\Gamma)]$. Now $\operatorname{Aut}(\Gamma)$ acts on the right on $\operatorname{Rep}_{\mathrm{nt}}(\Gamma)$ by $\alpha : (V,\rho) \mapsto (V, \rho \circ \alpha)$ for $\alpha \in \operatorname{Aut}(\Gamma)$, and this induces right actions of $\operatorname{Aut}(\Gamma)$ on $\Lambda^\Gamma = K_0\big(\operatorname{Rep}_{\mathrm{nt}}(\Gamma)\big)$ and $\Lambda_+^\Gamma \subseteq \Lambda^\Gamma$. Write these actions as $\alpha : \lambda \mapsto \lambda \cdot \alpha$. Then the action of $\alpha \in \operatorname{Aut}(\Gamma)$ on $\mathcal{X}^\Gamma$ maps $\mathcal{X}^{\Gamma,\lambda} \to \mathcal{X}^{\Gamma,\lambda\cdot\alpha}$. Write $\Lambda_+^\Gamma/\operatorname{Aut}(\Lambda)$ for the set of $\operatorname{Aut}(\Gamma)$-orbits $\mu = \lambda \cdot \operatorname{Aut}(\Gamma)$ in Λ_+^Γ. The map $\dim : \Lambda^\Gamma \to \mathbb{Z}$ is $\operatorname{Aut}(\Gamma)$-invariant, and so descends to $\dim : \Lambda^\Gamma/\operatorname{Aut}(\Gamma) \to \mathbb{Z}$.

Then $\coprod_{\lambda \in \mu} \mathcal{X}^{\Gamma,\lambda}$ is an open and closed $\operatorname{Aut}(\Gamma)$-invariant C^∞-substack in $\mathcal{X}^\Gamma$ for each $\mu \in \Lambda_+^\Gamma/\operatorname{Aut}(\Lambda)$, so we may define $\tilde{\mathcal{X}}^{\Gamma,\mu} \simeq \big[\big(\coprod_{\lambda \in \mu} \mathcal{X}^{\Gamma,\lambda}\big)/\operatorname{Aut}(\Gamma)\big]$, an open and closed C^∞-substack of $\tilde{\mathcal{X}}^\Gamma \simeq [\mathcal{X}^\Gamma/\operatorname{Aut}(\Gamma)]$. Write $\tilde{\mathcal{X}}_\circ^{\Gamma,\mu} = \tilde{\mathcal{X}}_\circ^\Gamma \cap \tilde{\mathcal{X}}^{\Gamma,\mu}$. Then $\tilde{\mathcal{X}}^{\Gamma,\mu}, \tilde{\mathcal{X}}_\circ^{\Gamma,\mu}$ are orbifolds of dimension $\dim \mathcal{X} - \dim \mu$, with

$$\tilde{\mathcal{X}}^\Gamma = \coprod_{\mu \in \Lambda_+^\Gamma/\operatorname{Aut}(\Gamma)} \tilde{\mathcal{X}}^{\Gamma,\mu} \quad \text{and} \quad \tilde{\mathcal{X}}_\circ^\Gamma = \coprod_{\mu \in \Lambda_+^\Gamma/\operatorname{Aut}(\Gamma)} \tilde{\mathcal{X}}_\circ^{\Gamma,\mu}.$$

Set $\tilde{O}^{\Gamma,\mu}(\mathcal{X}) = \tilde{O}^\Gamma(\mathcal{X})|_{\tilde{\mathcal{X}}^{\Gamma,\mu}} : \tilde{\mathcal{X}}^{\Gamma,\mu} \to \mathcal{X}$ and $\tilde{O}_\circ^{\Gamma,\mu}(\mathcal{X}) = \tilde{O}_\circ^\Gamma(\mathcal{X})|_{\tilde{\mathcal{X}}_\circ^{\Gamma,\mu}} : \tilde{\mathcal{X}}_\circ^{\Gamma,\mu} \to \mathcal{X}$. Then $\tilde{O}^{\Gamma,\mu}(\mathcal{X}), \tilde{O}_\circ^{\Gamma,\mu}(\mathcal{X})$ are representable immersions, with $\tilde{O}^{\Gamma,\mu}(\mathcal{X})$ proper.

The 1-morphism $\hat{\Pi}^\Gamma(\mathcal{X}) : \tilde{\mathcal{X}}^\Gamma \to \hat{\mathcal{X}}^\Gamma$ maps open and closed C^∞-substacks of $\tilde{\mathcal{X}}^\Gamma$ to open and closed C^∞-substacks of $\hat{\mathcal{X}}^\Gamma$. Let $\hat{\mathcal{X}}^{\Gamma,\mu} = \hat{\Pi}^\Gamma(\mathcal{X})(\tilde{\mathcal{X}}^{\Gamma,\mu})$ for each $\mu \in \Lambda_+^\Gamma/\operatorname{Aut}(\Lambda)$, and write $\hat{\mathcal{X}}_\circ^{\Gamma,\mu} = \hat{\mathcal{X}}_\circ^\Gamma \cap \hat{\mathcal{X}}^{\Gamma,\mu}$. Then $\hat{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ are orbifolds of dimension $\dim \mathcal{X} - \dim \mu$, with

$$\hat{\mathcal{X}}^\Gamma = \coprod_{\mu \in \Lambda_+^\Gamma/\operatorname{Aut}(\Gamma)} \hat{\mathcal{X}}^{\Gamma,\mu} \quad \text{and} \quad \hat{\mathcal{X}}_\circ^\Gamma = \coprod_{\mu \in \Lambda_+^\Gamma/\operatorname{Aut}(\Gamma)} \hat{\mathcal{X}}_\circ^{\Gamma,\mu}.$$

If $f : \mathcal{X} \to \mathcal{Y}$ is a representable 1-morphism of Deligne–Mumford C^∞-stacks and Γ a finite group, then as in §8.7 we have a representable 1-morphism of orbifold strata $f^\Gamma : \mathcal{X}^\Gamma \to \mathcal{Y}^\Gamma$. Note that if $\mathcal{X}, \mathcal{Y}$ are orbifolds, then f^Γ need not map $\mathcal{X}^{\Gamma,\lambda} \to \mathcal{Y}^{\Gamma,\lambda}$, or map $\mathcal{X}_\circ^\Gamma \to \mathcal{Y}_\circ^\Gamma$. The analogue applies for $\tilde{f}^\Gamma, \hat{f}^\Gamma$.

Some important properties of orbifolds can be characterized by the vanishing of certain orbifold strata $\mathcal{X}^{\Gamma,\lambda}$. For example:

- An orbifold $\mathcal{X}$ is *locally orientable* if and only if $\mathcal{X}^{\mathbb{Z}_2,\lambda} = \emptyset$ for all odd $\lambda \in \Lambda_+^{\mathbb{Z}_2} \cong \mathbb{N} = \{0, 1, 2, \ldots\}$.

- An orbifold $\mathcal{X}$ is *effective* in the sense of Definition 9.4 if and only if $\mathcal{X}^{\Gamma,0} = \emptyset$ for all nontrivial finite groups Γ.

In [35, §8.4] we consider the question: if $\mathcal{X}$ is an oriented orbifold, can we define orientations on the orbifold strata $\mathcal{X}^{\Gamma,\lambda}, \ldots, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$? Here is an example:

Example 9.8. Let $\mathcal{S}^4 = \big\{(x_1, \ldots, x_5) \in \mathbb{R}^5 : x_1^2 + \cdots + x_5^2 = 1\big\}$, an oriented 4-manifold. Let $G = \{1, \sigma, \tau, \sigma\tau\} \cong \mathbb{Z}_2^2$ act on $\mathcal{S}^4$ preserving orientations by

$$\sigma : (x_1, \ldots, x_5) \longmapsto (x_1, x_2, x_3, -x_4, -x_5),$$
$$\tau : (x_1, \ldots, x_5) \longmapsto (-x_1, -x_2, -x_3, -x_4, x_5),$$
$$\sigma\tau : (x_1, \ldots, x_5) \longmapsto (-x_1, -x_2, -x_3, x_4, -x_5).$$

Then $\mathcal{X} = [\underline{\mathcal{S}}^4/G]$ is an oriented 4-orbifold. The orbifold groups $\mathrm{Iso}_\mathcal{X}([x])$ for $[x] \in \mathcal{X}_{\mathrm{top}}$ are all $\{1\}$ or $\mathbb{Z}_2$. The singular locus of $\mathcal{X}$ is the disjoint union of a copy of $\mathbb{RP}^2$ from the fixed points $\pm(x_1, x_2, x_3, 0, 0)$ of σ, and two isolated points $\{\pm(0,0,0,0,1)\}$ and $\{\pm(0,0,0,1,0)\}$ from the fixed points of τ and $\sigma\tau$.

Identifying $\Lambda_+^{\mathbb{Z}_2}$ and $\Lambda_+^{\mathbb{Z}_2}/\mathrm{Aut}(\mathbb{Z}_2)$ with $\mathbb{N}$, it follows that

$$\mathcal{X}^{\mathbb{Z}_2,2} = \mathcal{X}_\circ^{\mathbb{Z}_2,2} \cong \tilde{\mathcal{X}}^{\mathbb{Z}_2,2} = \tilde{\mathcal{X}}_\circ^{\mathbb{Z}_2,2} \cong \mathbb{RP}^2 \times [\underline{*}/\mathbb{Z}_2], \qquad \hat{\mathcal{X}}^{\mathbb{Z}_2,2} = \hat{\mathcal{X}}_\circ^{\mathbb{Z}_2,2} \cong \mathbb{RP}^2,$$

$$\mathcal{X}^{\mathbb{Z}_2,4} = \mathcal{X}_\circ^{\mathbb{Z}_2,4} \cong \tilde{\mathcal{X}}^{\mathbb{Z}_2,4} = \tilde{\mathcal{X}}_\circ^{\mathbb{Z}_2,4} \cong [\underline{*}/\mathbb{Z}_2]\,\mathrm{II}\,[\underline{*}/\mathbb{Z}_2], \quad \hat{\mathcal{X}}^{\mathbb{Z}_2,4} = \hat{\mathcal{X}}_\circ^{\mathbb{Z}_2,4} \cong *\,\mathrm{II}\,*.$$

Since $\mathbb{RP}^2$ is not orientable, we see that $\mathcal{X}$ is an oriented orbifold, but none of $\mathcal{X}^{\mathbb{Z}_2,2}, \tilde{\mathcal{X}}^{\mathbb{Z}_2,2}, \hat{\mathcal{X}}^{\mathbb{Z}_2,2}, \mathcal{X}_\circ^{\mathbb{Z}_2,2}, \tilde{\mathcal{X}}_\circ^{\mathbb{Z}_2,2}, \hat{\mathcal{X}}_\circ^{\mathbb{Z}_2,2}$ are orientable.

Thus, we can only orient $\mathcal{X}^{\Gamma,\lambda}, \ldots, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ for all oriented orbifolds $\mathcal{X}$ under some conditions on Γ, λ, μ. The next proposition sets out these conditions:

Proposition 9.9. (a) *Suppose Γ is a finite group and (V, ρ) a nontrivial Γ-representation which has no odd-dimensional subrepresentations, and write $\lambda = [(V, \rho)] \in \Lambda_+^\Gamma$. Choose an orientation on V. Then for all oriented orbifolds $\mathcal{X}$ we can define natural orientations on the orbifold strata $\mathcal{X}^{\Gamma,\lambda}, \mathcal{X}_\circ^{\Gamma,\lambda}$.*

If $|\Gamma|$ is odd then all nontrivial Γ-representations are even-dimensional, so we can orient $\mathcal{X}^{\Gamma,\lambda}, \mathcal{X}_\circ^{\Gamma,\lambda}$ for all $\lambda \in \Lambda_+^\Gamma$.

(b) *Let $\Gamma, (V, \rho), \lambda$ be as in **(a)**, and set $\mu = \lambda \cdot \mathrm{Aut}(\Gamma) \in \Lambda_+^\Gamma/\mathrm{Aut}(\Gamma)$. Write H for the subgroup of $\mathrm{Aut}(\Gamma)$ fixing λ in Λ_+^Γ. Then for each $\delta \in H$ there exists an isomorphism of Γ-representations $i_\delta : (V, \rho \circ \delta) \to (V, \rho)$. Suppose $i_\delta : V \to V$ is orientation-preserving for all $\delta \in H$. If $\lambda \in 2\Lambda_+^\Gamma$ this holds automatically.*

*Then for all oriented orbifolds $\mathcal{X}$ we can define orientations on the orbifold strata $\tilde{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}, \tilde{\mathcal{X}}_\circ^{\Gamma,\mu}, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$. For $\tilde{\mathcal{X}}^{\Gamma,\mu}$ this works as $\tilde{\mathcal{X}}^{\Gamma,\mu} \simeq [\mathcal{X}^{\Gamma,\lambda}/H]$, where $\mathcal{X}^{\Gamma,\lambda}$ is oriented by **(a)**, and the H-action on $\mathcal{X}^{\Gamma,\lambda}$ preserves orientations, so the orientation on $\mathcal{X}^{\Gamma,\lambda}$ descends to an orientation on $\tilde{\mathcal{X}}^{\Gamma,\mu} \simeq [\mathcal{X}^{\Gamma,\lambda}/H]$.*

(c) *Suppose that Γ and $\lambda \in \Lambda_+^\Gamma$ do not satisfy the conditions in **(a)**, or Γ and $\mu \in \Lambda_+^\Gamma/\mathrm{Aut}(\Gamma)$ do not satisfy the conditions in **(b)**. Then as in Example 9.8 we can find examples of oriented orbifolds $\mathcal{X}$ such that $\mathcal{X}^{\Gamma,\lambda}, \mathcal{X}_\circ^{\Gamma,\lambda}$ are not orientable, or $\tilde{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}, \tilde{\mathcal{X}}_\circ^{\Gamma,\mu}, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ are not orientable, respectively. That is, the conditions on Γ, λ, μ in **(a)**,**(b)** are necessary as well as sufficient to be able to orient orbifold strata $\mathcal{X}^{\Gamma,\lambda}, \ldots, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ of all oriented orbifolds $\mathcal{X}$.*

Note that Proposition 9.9(a),(b) do not apply in Example 9.8, since the nontrivial representation of $\mathbb{Z}_2$ on $\mathbb{R}^2$ has an odd-dimensional subrepresentation.

10   The 2-category of d-stacks

In [35, Chap. 9] we define and study the 2-category of *d-stacks* **dSta**, which are orbifold versions of d-spaces in §3. Broadly, to go from d-spaces $\boldsymbol{X} = (\underline{X}, \mathcal{O}_X', \mathcal{E}_X, \imath_X, \jmath_X)$ to d-stacks we just replace the C^∞-scheme $\underline{X}$ by a Deligne–Mumford C^∞-stack $\mathcal{X}$.

One might expect that combining the 2-categories **DMC$^\infty$Sta** and **dSpa** should result in a 3-category **dSta**, but in fact a 2-category is sufficient. For 1-morphisms $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in **dSta**, a 2-morphism $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ in **dSta** is a pair (η, η'), where $\eta : f \Rightarrow g$ is a 2-morphism in **C$^\infty$Sta**, and $\eta' : f^*(\mathcal{F}_{\mathcal{Y}}) \to \mathcal{E}_{\mathcal{X}}$ is as for 2-morphisms in **dSpa**. These η, η' do not interact very much.

10.1   The definition of d-stacks

Definition 10.1. A *d-stack* $\boldsymbol{\mathcal{X}}$ is a quintuple $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}})$, where $\mathcal{X}$ is a separated, second countable, locally fair Deligne–Mumford C^∞-stack in the sense of §8, and $\mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}}$ fit into an exact sequence of sheaves of abelian groups on $\mathcal{X}$, in the sense of §8.6

$$\mathcal{E}_{\mathcal{X}} \xrightarrow{\quad \jmath_{\mathcal{X}} \quad} \mathcal{O}'_{\mathcal{X}} \xrightarrow{\quad \imath_{\mathcal{X}} \quad} \mathcal{O}_{\mathcal{X}} \longrightarrow 0,$$

satisfying the conditions:

(a) $\mathcal{O}'_{\mathcal{X}}$ is a sheaf of C^∞-rings on $\mathcal{X}$, and $\imath_{\mathcal{X}} : \mathcal{O}'_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ is a morphism of sheaves of C^∞-rings on $\mathcal{X}$, where $\mathcal{O}_{\mathcal{X}}$ is the structure sheaf of $\mathcal{X}$ as in Example 8.19, such that for all $(\underline{U}, u)$ in $\mathcal{C}_{\mathcal{X}}$, $(U, \mathcal{O}'_{\mathcal{X}}(\underline{U}, u))$ is a C^∞-scheme, and $\imath_{\mathcal{X}}(\underline{U}, u) : \mathcal{O}'_{\mathcal{X}}(\underline{U}, u) \to \mathcal{O}_{\mathcal{X}}(\underline{U}, u) = \mathcal{O}_U$ is a surjective morphism of sheaves of C^∞-rings on U, whose kernel is a sheaf of square zero ideals.

We call $\imath_{\mathcal{X}} : \mathcal{O}'_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ satisfying these conditions a *square zero extension*.

(b) As $\imath_{\mathcal{X}} : \mathcal{O}'_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$ is a square zero extension, its kernel $\mathcal{I}_{\mathcal{X}}$ is a quasi-coherent sheaf on $\mathcal{X}$. We require that $\mathcal{E}_{\mathcal{X}}$ is also a quasicoherent sheaf on $\mathcal{X}$, and $\jmath_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{I}_{\mathcal{X}}$ is a surjective morphism in qcoh$(\mathcal{X})$.

The sheaf of C^∞-rings $\mathcal{O}'_{\mathcal{X}}$ has a sheaf of cotangent modules $\Omega_{\mathcal{O}'_{\mathcal{X}}}$, which is an $\mathcal{O}'_{\mathcal{X}}$-module with exterior derivative d : $\mathcal{O}'_{\mathcal{X}} \to \Omega_{\mathcal{O}'_{\mathcal{X}}}$. Define $\mathcal{F}_{\mathcal{X}} = \Omega_{\mathcal{O}'_{\mathcal{X}}} \otimes_{\mathcal{O}'_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}$ to be the associated $\mathcal{O}_{\mathcal{X}}$-module, a quasicoherent sheaf on $\mathcal{X}$, and set $\psi_{\mathcal{X}} = \Omega_{\imath_{\mathcal{X}}} \otimes \mathrm{id} : \mathcal{F}_{\mathcal{X}} \to T^*\mathcal{X}$, a morphism in qcoh$(\mathcal{X})$. Define $\phi_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{X}}$ to be the composition of morphisms of sheaves of abelian groups on $\mathcal{X}$:

$$\mathcal{E}_{\mathcal{X}} \xrightarrow{\jmath_{\mathcal{X}}} \mathcal{I}_{\mathcal{X}} \xrightarrow{\mathrm{d}|_{\mathcal{I}_{\mathcal{X}}}} \Omega_{\mathcal{O}'_{\mathcal{X}}} \xlongequal{\sim} \Omega_{\mathcal{O}'_{\mathcal{X}}} \otimes_{\mathcal{O}'_{\mathcal{X}}} \mathcal{O}'_{\mathcal{X}} \xrightarrow{\mathrm{id} \otimes \imath_{\mathcal{X}}} \Omega_{\mathcal{O}'_{\mathcal{X}}} \otimes_{\mathcal{O}'_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}} = \mathcal{F}_{\mathcal{X}}.$$

Then $\phi_{\mathcal{X}}$ is a morphism in qcoh$(\mathcal{X})$, and the following sequence is exact:

$$\mathcal{E}_{\mathcal{X}} \xrightarrow{\quad \phi_{\mathcal{X}} \quad} \mathcal{F}_{\mathcal{X}} \xrightarrow{\quad \psi_{\mathcal{X}} \quad} T^*\mathcal{X} \longrightarrow 0. \tag{10.1}$$

The morphism $\phi_{\mathcal{X}} : \mathcal{E}_{\mathcal{X}} \to \mathcal{F}_{\mathcal{X}}$ will be called the *virtual cotangent sheaf* of $\boldsymbol{\mathcal{X}}$. It is a d-stack analogue of the cotangent complex in algebraic geometry.

Let $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ be d-stacks. A 1-*morphism* $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a triple $\boldsymbol{f} = (f, f', f'')$, where $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of C^∞-stacks, $f' : f^{-1}(\mathcal{O}'_{\mathcal{Y}}) \to \mathcal{O}'_{\mathcal{X}}$ a morphism of sheaves of C^∞-rings on $\mathcal{X}$, and $f'' : f^*(\mathcal{E}_{\mathcal{Y}}) \to \mathcal{E}_{\mathcal{X}}$ a morphism in

qcoh($\mathcal{X}$), such that the following diagram of sheaves on $\mathcal{X}$ commutes:

$$
\begin{array}{l}
f^{-1}(\mathcal{E}_{\mathcal{Y}}) \otimes^{\mathrm{id}}_{f^{-1}(\mathcal{O}_{\mathcal{Y}})} f^{-1}(\mathcal{O}_{\mathcal{Y}}) = f^{-1}(\mathcal{E}_{\mathcal{Y}}) \xrightarrow{} f^{-1}(\mathcal{O}'_{\mathcal{Y}}) \xrightarrow{} f^{-1}(\mathcal{O}_{\mathcal{Y}}) \to 0 \\
\quad\quad \downarrow \mathrm{id} \otimes f^{\sharp} \qquad\qquad\qquad {}_{f^{-1}(\jmath_{\mathcal{Y}})} \qquad\quad {}_{f^{-1}(\imath_{\mathcal{Y}})}
\end{array}
$$

Define morphisms $f^2 = \Omega_{f'} \otimes \mathrm{id} : f^*(\mathcal{F}_{\mathcal{Y}}) \to \mathcal{F}_{\mathcal{X}}$ and $f^3 = \Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ in qcoh($\mathcal{X}$). Then the following commutes in qcoh($\mathcal{X}$), with exact rows:

$$
\begin{array}{ccccccc}
f^*(\mathcal{E}_{\mathcal{Y}}) & \xrightarrow{f^*(\phi_{\mathcal{Y}})} & f^*(\mathcal{F}_{\mathcal{Y}}) & \xrightarrow{f^*(\psi_{\mathcal{Y}})} & f^*(T^*\mathcal{Y}) & \longrightarrow & 0 \\
\downarrow f'' & & \downarrow f^2 & & \downarrow f^3 & & \\
\mathcal{E}_{\mathcal{X}} & \xrightarrow{\phi_{\mathcal{X}}} & \mathcal{F}_{\mathcal{X}} & \xrightarrow{\psi_{\mathcal{X}}} & T^*\mathcal{X} & \longrightarrow & 0.
\end{array}
\tag{10.2}
$$

If $\boldsymbol{\mathcal{X}}$ is a d-stack, the *identity 1-morphism* $\mathbf{id}_{\boldsymbol{\mathcal{X}}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}$ is $\mathbf{id}_{\boldsymbol{\mathcal{X}}} = \big(\mathrm{id}_{\mathcal{X}}, \delta_{\mathcal{X}}(\mathcal{O}'_{\mathcal{X}}), \delta_{\mathcal{X}}(\mathcal{E}_{\mathcal{X}})\big)$, with $\delta_{\mathcal{X}}(*)$ the canonical isomorphisms of Definition 8.22.

Let $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ be d-stacks, and $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$, $\boldsymbol{g} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ be 1-morphisms. As in (3.2) define the *composition of 1-morphisms* $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}$ to be

$$
\boldsymbol{g} \circ \boldsymbol{f} = \big(g \circ f, f' \circ f^{-1}(g') \circ I_{f,g}(\mathcal{O}'_{\mathcal{Z}}), f'' \circ f^*(g'') \circ I_{f,g}(\mathcal{E}_{\mathcal{Z}})\big),
$$

where $I_{*,*}(*)$ are the canonical isomorphisms of Definition 8.22.

Let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be 1-morphisms of d-stacks, where $\boldsymbol{f} = (f, f', f'')$ and $\boldsymbol{g} = (g, g', g'')$. A *2-morphism* $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a pair $\boldsymbol{\eta} = (\eta, \eta')$, where $\eta : f \Rightarrow g$ is a 2-morphism in $\mathbf{C}^{\infty}\mathbf{Sta}$ and $\eta' : f^*(\mathcal{F}_{\mathcal{Y}}) \to \mathcal{E}_{\mathcal{X}}$ a morphism in qcoh($\mathcal{X}$), with

$$
\begin{aligned}
g' \circ \eta^{-1}(\mathcal{O}'_{\mathcal{Y}}) &= f' + \kappa_{\mathcal{X}} \circ \jmath_{\mathcal{X}} \circ \eta' \circ \big(\mathrm{id} \otimes (f^{\sharp} \circ f^{-1}(\imath_{\mathcal{Y}}))\big) \circ \big(f^{-1}(\mathrm{d})\big), \\
\text{and} \quad g'' \circ \eta^*(\mathcal{E}_{\mathcal{Y}}) &= f'' + \eta' \circ f^*(\phi_{\mathcal{Y}}).
\end{aligned}
$$

Then $g^2 \circ \eta^*(\mathcal{F}_{\mathcal{Y}}) = f^2 + \phi_{\mathcal{X}} \circ \eta'$ and $g^3 \circ \eta^*(T^*\mathcal{Y}) = f^3$, so (10.2) for $\boldsymbol{f}, \boldsymbol{g}$ combine to give a commuting diagram (except η') in qcoh($\mathcal{X}$), with exact rows:

$$
\begin{array}{ccccccc}
f^*(\mathcal{E}_{\mathcal{Y}}) & \xrightarrow{f^*(\phi_{\mathcal{Y}})} & f^*(\mathcal{F}_{\mathcal{Y}}) & \xrightarrow{f^*(\psi_{\mathcal{Y}})} & f^*(T^*\mathcal{Y}) & \longrightarrow & 0 \\
& & & & & & \\
\end{array}
$$

If $\boldsymbol{f} = (f, f', f'') : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a 1-morphism, the *identity 2-morphism* $\mathbf{id}_{\boldsymbol{f}} : \boldsymbol{f} \Rightarrow \boldsymbol{f}$ is $\mathbf{id}_{\boldsymbol{f}} = (\mathrm{id}_f, 0)$.

Let $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be 1-morphisms and $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$, $\boldsymbol{\zeta} : \boldsymbol{g} \Rightarrow \boldsymbol{h}$ 2-morphisms. Define the *vertical composition of 2-morphisms* $\boldsymbol{\zeta} \odot \boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{h}$ to be

$$
\boldsymbol{\zeta} \odot \boldsymbol{\eta} = \big(\zeta \odot \eta, \zeta' \circ \eta^*(\mathcal{F}_{\mathcal{Y}}) + \eta'\big).
$$

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Suppose $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ are d-stacks, $\boldsymbol{f}, \tilde{\boldsymbol{f}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ and $\boldsymbol{g}, \tilde{\boldsymbol{g}} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ are 1-morphisms, and $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \tilde{\boldsymbol{f}}$, $\boldsymbol{\zeta} : \boldsymbol{g} \Rightarrow \tilde{\boldsymbol{g}}$ are 2-morphisms. Define the *horizontal composition of 2-morphisms* $\boldsymbol{\zeta} * \boldsymbol{\eta} : \boldsymbol{g} \circ \boldsymbol{f} \Rightarrow \tilde{\boldsymbol{g}} \circ \tilde{\boldsymbol{f}}$ to be

$$\boldsymbol{\zeta} * \boldsymbol{\eta} = \big(\zeta * \eta, \big[\eta' \circ f^*(g^2) + f'' \circ f^*(\zeta') + \eta' \circ f^*(\phi_{\mathcal{Y}}) \circ f^*(\zeta')\big] \circ I_{f,g}(\mathcal{F}_{\mathcal{Z}})\big).$$

This completes the definition of the 2-category of d-stacks **dSta**.

Write $\mathbf{DMC^\infty Sta_{ssc}^{lf}}$ for the 2-category of separated, second countable, locally fair Deligne–Mumford C^∞-stacks. Define a strict 2-functor $F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}} : \mathbf{DMC^\infty Sta_{ssc}^{lf}} \to \mathbf{dSta}$ to map objects $\mathcal{X}$ to $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}, 0, \mathrm{id}_{\mathcal{O}_{\mathcal{X}}}, 0)$, to map 1-morphisms f to $\boldsymbol{f} = (f, f^\sharp, 0)$, and to map 2-morphisms η to $\boldsymbol{\eta} = (\eta, 0)$. Write $\mathbf{D\hat{M}C^\infty Sta_{ssc}^{lf}}$ for the full 2-subcategory of $\boldsymbol{\mathcal{X}} \in \mathbf{dSta}$ equivalent to $F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}(\mathcal{X})$ for $\mathcal{X} \in \mathbf{DMC^\infty Sta_{ssc}^{lf}}$. When we say that a d-stack $\boldsymbol{\mathcal{X}}$ *is a C^∞-stack*, we mean that $\boldsymbol{\mathcal{X}} \in \mathbf{D\hat{M}C^\infty Sta_{ssc}^{lf}}$.

Define a strict 2-functor $F_{\mathbf{Orb}}^{\mathbf{dSta}} : \mathbf{Orb} \to \mathbf{dSta}$ by $F_{\mathbf{Orb}}^{\mathbf{dSta}} = F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}|_{\mathbf{Orb}}$, noting that $\mathbf{Orb}$ is a full 2-subcategory of $\mathbf{DMC^\infty Sta_{ssc}^{lf}}$. Write $\mathbf{\hat{O}rb}$ for the full 2-subcategory of objects $\boldsymbol{\mathcal{X}}$ in $\mathbf{dSta}$ equivalent to $F_{\mathbf{Orb}}^{\mathbf{dSta}}(\mathcal{X})$ for some orbifold $\mathcal{X}$. When we say that a d-stack $\boldsymbol{\mathcal{X}}$ *is an orbifold*, we mean that $\boldsymbol{\mathcal{X}} \in \mathbf{\hat{O}rb}$.

Recall from §8.1 that there is a natural (2-)functor $F_{\mathbf{C^\infty Sch}}^{\mathbf{C^\infty Sta}} : \mathbf{C^\infty Sch} \to \mathbf{C^\infty Sta}$ mapping $\underline{X} \mapsto \bar{X}$ on objects and $\underline{f} \mapsto \bar{f}$ on morphisms. Also, if $\underline{X}$ is a C^∞-scheme and $\bar{X}$ the corresponding C^∞-stack then Example 8.21 defines a functor $\mathcal{I}_{\underline{X}} : \mathcal{O}_X\text{-mod} \to \mathcal{O}_{\bar{X}}\text{-mod}$. In the same way, we can define functors from the category of sheaves of abelian groups on X to the category of sheaves of abelian groups on $\bar{X}$, and from the category of sheaves of C^∞-rings on X to the category of sheaves of C^∞-rings on $\bar{X}$, both of which we also denote by $\mathcal{I}_{\underline{X}}$.

With this notation, define a strict 2-functor $F_{\mathbf{dSpa}}^{\mathbf{dSta}} : \mathbf{dSpa} \to \mathbf{dSta}$ to map $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ to $\boldsymbol{\mathcal{X}} = \big(\bar{X}, \mathcal{I}_{\underline{X}}(\mathcal{O}'_X), \mathcal{I}_{\underline{X}}(\mathcal{E}_X), \mathcal{I}_{\underline{X}}(\imath_X), \mathcal{I}_{\underline{X}}(\jmath_X)\big)$ on objects, and to map $\boldsymbol{f} = (\underline{f}, f', f'')$ to $\boldsymbol{\hat{f}} = \big(\bar{f}, \mathcal{I}_{\underline{X}}(f'), \mathcal{I}_{\underline{X}}(f'')\big)$ on 1-morphisms, and to map η to $\boldsymbol{\eta} = \big(\mathrm{id}_{\bar{f}}, \mathcal{I}_{\underline{X}}(\eta)\big)$ on 2-morphisms. Write $\mathbf{d\hat{S}pa}$ for the full 2-subcategory of $\boldsymbol{\mathcal{X}}$ in $\mathbf{dSta}$ equivalent to $F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{X})$ for some $\boldsymbol{X}$ in $\mathbf{dSpa}$.

In [35, §9.2] we prove:

Theorem 10.2. (a) *Definition 10.1 defines a strict 2-category* **dSta**, *in which all 2-morphisms are 2-isomorphisms.*

(b) $F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}, F_{\mathbf{Orb}}^{\mathbf{dSta}}$ *and* $F_{\mathbf{dSpa}}^{\mathbf{dSta}}$ *are full and faithful strict 2-functors. Hence* $\mathbf{DMC^\infty Sta_{ssc}^{lf}}, \mathbf{Orb}, \mathbf{dSpa}$ *and* $\mathbf{D\hat{M}C^\infty Sta_{ssc}^{lf}}, \mathbf{\hat{O}rb}, \mathbf{d\hat{S}pa}$ *are equivalent 2-categories, respectively.*

10.2   D-stacks as quotients of d-spaces

Section 8.4 defined quotient Deligne–Mumford C^∞-stacks $[\underline{X}/G]$, quotient 1-morphisms $[\underline{f}, \rho] : [\underline{X}/G] \to [\underline{Y}/H]$, and quotient 2-morphisms $[\delta] : [\underline{f}, \rho] \Rightarrow [\underline{g}, \sigma]$. In [35, §9.3] we generalize all this to d-stacks. The next two theorems summarize our results.

Theorem 10.3. (i) *Let $\boldsymbol{X}$ be a d-space, G a finite group, and $\boldsymbol{r} : G \to \mathrm{Aut}(\boldsymbol{X})$ a (strict) action of G on $\boldsymbol{X}$ by 1-isomorphisms. Then we can define a **quotient d-stack** $\boldsymbol{\mathcal{X}} = [\boldsymbol{X}/G]$, which is natural up to 1-isomorphism in $\mathbf{dSta}$. The underlying C^∞-stack $\mathcal{X}$ is $[\underline{X}/G]$ from Example 8.11.*

(ii) *Let $\boldsymbol{X}, \boldsymbol{Y}$ be d-spaces, G, H finite groups, and $\boldsymbol{r} : G \to \mathrm{Aut}(\boldsymbol{X})$, $\boldsymbol{s} : H \to \mathrm{Aut}(\boldsymbol{Y})$ be actions of G, H on $\boldsymbol{X}, \boldsymbol{Y}$, so that by **(i)** we have quotient d-stacks $\boldsymbol{\mathcal{X}} = [\boldsymbol{X}/G]$ and $\boldsymbol{\mathcal{Y}} = [\boldsymbol{Y}/H]$. Suppose $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism in $\mathbf{dSpa}$ and $\rho : G \to H$ is a group morphism, satisfying $\boldsymbol{f} \circ \boldsymbol{r}(\gamma) = \boldsymbol{s}(\rho(\gamma)) \circ \boldsymbol{f}$ for all $\gamma \in G$ (this is an equality of 1-morphisms in $\mathbf{dSpa}$, not just a 2-isomorphism). Then we can define a **quotient 1-morphism** $\tilde{\boldsymbol{f}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in $\mathbf{dSta}$, which we will also write as $[\boldsymbol{f}, \rho] : [\boldsymbol{X}/G] \to [\boldsymbol{Y}/H]$.*

(iii) *Let $\tilde{\boldsymbol{f}} = [\boldsymbol{f}, \rho] : [\boldsymbol{X}/G] \to [\boldsymbol{Y}/H]$ and $\tilde{\boldsymbol{g}} = [\boldsymbol{g}, \sigma] : [\boldsymbol{X}/G] \to [\boldsymbol{Y}/H]$ be two quotient 1-morphisms as in **(ii)**. Suppose $\delta \in H$ satisfies $\delta^{-1}\,\sigma(\gamma) = \rho(\gamma)\,\delta^{-1}$ for all $\gamma \in G$, and $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{s}(\delta^{-1}) \circ \boldsymbol{g}$ is a 2-morphism in $\mathbf{dSpa}$ such that $\eta * \mathrm{id}_{\boldsymbol{r}(\gamma)} = \mathrm{id}_{\boldsymbol{s}(\sigma(\gamma))} * \eta$ for all $\gamma \in G$, using the diagram:*

$$
\begin{array}{ccc}
\boldsymbol{f} \circ \boldsymbol{r}(\gamma) & = & \boldsymbol{s}(\rho(\gamma)) \circ \boldsymbol{f} \\
\Big\Vert {\scriptstyle \eta * \mathrm{id}_{\boldsymbol{r}(\gamma)}} & & {\scriptstyle \mathrm{id}_{\boldsymbol{s}(\sigma(\gamma))} * \eta} \Big\Vert \\
\boldsymbol{s}(\delta^{-1}) \circ \boldsymbol{g} \circ \boldsymbol{r}(\gamma) = \boldsymbol{s}(\delta^{-1}) \circ \boldsymbol{s}(\sigma(\gamma)) \circ \boldsymbol{g} & = & \boldsymbol{s}(\rho(\gamma)) \circ \boldsymbol{s}(\delta^{-1}) \circ \boldsymbol{g}.
\end{array}
$$

*Then we can define a **quotient 2-morphism** $\zeta : \tilde{\boldsymbol{f}} \Rightarrow \tilde{\boldsymbol{g}}$ in $\mathbf{dSta}$, which we also write as $[\eta, \delta] : [\boldsymbol{f}, \rho] \Rightarrow [\boldsymbol{g}, \sigma]$.*

Theorem 10.4. (a) *Let $\boldsymbol{\mathcal{X}}$ be a d-stack and $[x] \in \boldsymbol{\mathcal{X}}_{\mathrm{top}}$, and write $G = \mathrm{Iso}_{\boldsymbol{\mathcal{X}}}([x])$. Then there exist a quotient d-stack $[\boldsymbol{U}/G]$, as in Theorem 10.3(i), and an equivalence $\boldsymbol{i} : [\boldsymbol{U}/G] \to \boldsymbol{\mathcal{X}}$ with an open d-substack $\boldsymbol{\mathcal{U}}$ in $\boldsymbol{\mathcal{X}}$, with $i_{\mathrm{top}} : [u] \mapsto [x] \in \boldsymbol{\mathcal{U}}_{\mathrm{top}} \subseteq \boldsymbol{\mathcal{X}}_{\mathrm{top}}$ for some fixed point u of G in U.*

(b) *Let $\tilde{\boldsymbol{f}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be a 1-morphism in $\mathbf{dSta}$, and $[x] \in \boldsymbol{\mathcal{X}}_{\mathrm{top}}$ with $\tilde{f}_{\mathrm{top}} : [x] \mapsto [y] \in \boldsymbol{\mathcal{Y}}_{\mathrm{top}}$, and write $G = \mathrm{Iso}_{\boldsymbol{\mathcal{X}}}([x])$ and $H = \mathrm{Iso}_{\boldsymbol{\mathcal{Y}}}([y])$. Part **(a)** gives 1-morphisms $\boldsymbol{i} : [\boldsymbol{U}/G] \to \boldsymbol{\mathcal{X}}$, $\boldsymbol{j} : [\boldsymbol{V}/H] \to \boldsymbol{\mathcal{Y}}$ which are equivalences with open $\boldsymbol{\mathcal{U}} \subseteq \boldsymbol{\mathcal{X}}$, $\boldsymbol{\mathcal{V}} \subseteq \boldsymbol{\mathcal{Y}}$, such that $i_{\mathrm{top}} : [u] \mapsto [x] \in \boldsymbol{\mathcal{U}}_{\mathrm{top}} \subseteq \boldsymbol{\mathcal{X}}_{\mathrm{top}}$, $j_{\mathrm{top}} : [v] \mapsto [y] \in \boldsymbol{\mathcal{V}}_{\mathrm{top}} \subseteq \boldsymbol{\mathcal{Y}}_{\mathrm{top}}$ for u, v fixed points of G, H in U, V.*

Then there exist a G-invariant open d-subspace $\boldsymbol{U}'$ of u in $\boldsymbol{U}$ and a quotient 1-morphism $[\boldsymbol{f}, \rho] : [\boldsymbol{U}'/G] \to [\boldsymbol{V}/H]$, as in Theorem 10.3(ii), such that $\boldsymbol{f}(u) = v$, and $\rho : G \to H$ is $\tilde{f}_* : \mathrm{Iso}_{\boldsymbol{\mathcal{X}}}([x]) \to \mathrm{Iso}_{\boldsymbol{\mathcal{Y}}}([y])$, fitting into a 2-commutative diagram:

$$
\begin{array}{ccc}
[\boldsymbol{U}'/G] & \xrightarrow{\ [\boldsymbol{f}, \rho]\ } & [\boldsymbol{V}/H] \\
{\scriptstyle \boldsymbol{i}|_{[\boldsymbol{U}'/G]}} \Big\downarrow & {\scriptstyle \zeta \Uparrow} \quad {\scriptstyle \tilde{\boldsymbol{f}}} & \Big\downarrow {\scriptstyle \boldsymbol{j}} \\
\boldsymbol{\mathcal{X}} & \xrightarrow[\ \ \tilde{\boldsymbol{f}}\ \ ]{} & \boldsymbol{\mathcal{Y}}.
\end{array}
$$

(c) *Let $\tilde{\boldsymbol{f}}, \tilde{\boldsymbol{g}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be 1-morphisms in $\mathbf{dSta}$ and $\boldsymbol{\eta} : \tilde{\boldsymbol{f}} \Rightarrow \tilde{\boldsymbol{g}}$ a 2-morphism, let $[x] \in \boldsymbol{\mathcal{X}}_{\mathrm{top}}$ with $\tilde{f}_{\mathrm{top}} : [x] \mapsto [y] \in \boldsymbol{\mathcal{Y}}_{\mathrm{top}}$, and write $G = \mathrm{Iso}_{\boldsymbol{\mathcal{X}}}([x])$ and $H = \mathrm{Iso}_{\boldsymbol{\mathcal{Y}}}([y])$. Part **(a)** gives $\boldsymbol{i} : [\boldsymbol{U}/G] \to \boldsymbol{\mathcal{X}}$, $\boldsymbol{j} : [\boldsymbol{V}/H] \to \boldsymbol{\mathcal{Y}}$ which are equivalences with open $\boldsymbol{\mathcal{U}} \subseteq \boldsymbol{\mathcal{X}}$, $\boldsymbol{\mathcal{V}} \subseteq \boldsymbol{\mathcal{Y}}$ and map $i_{\mathrm{top}} : [u] \mapsto [x]$, $j_{\mathrm{top}} : [v] \mapsto [y]$ for u, v fixed points of G, H.*

By making U' smaller, we can take the same U' in **(b)** *for both $\tilde{f}, \tilde{g}$. Thus part* **(b)** *gives a G-invariant open $U' \subseteq U$, quotient 1-morphisms $[f, \rho]$: $[U'/G] \to [V/H]$ and $[g, \sigma] : [U'/G] \to [V/H]$ with $f(u) = g(u) = v$ and $\rho = \tilde{f}_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$, $\sigma = \tilde{g}_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$, and 2-morphisms $\zeta : \tilde{f} \circ i|_{[U'/G]} \Rightarrow j \circ [f, \rho]$, $\theta : \tilde{g} \circ i|_{[U'/G]} \Rightarrow j \circ [g, \sigma]$.*

Then there exist a G-invariant open neighbourhood U'' of u in U' and a quotient 2-morphism $[\lambda, \delta] : [f|_{U''}, \rho] \Rightarrow [g|_{U''}, \sigma]$, as in Theorem 10.3(iii), such that the following diagram of 2-morphisms in **dSta** *commutes:*

$$
\begin{array}{ccc}
\tilde{f} \circ i|_{[U''/G]} & \overset{\eta * \mathrm{id}_{i|_{[U''/G]}}}{=\!=\!=\!=\!=\!=\!=\!=\!\Longrightarrow} & \tilde{g} \circ i|_{[U''/G]} \\[4pt]
\Big\Downarrow \zeta|_{[U''/G]} & & \theta|_{[U''/G]} \Big\Downarrow \\[4pt]
j \circ [f|_{U''}, \rho] & \overset{\mathrm{id}_j * [\lambda, \delta]}{=\!=\!=\!=\!=\!=\!\Longrightarrow} & j \circ [g|_{U''}, \sigma].
\end{array}
$$

Effectively, this says that d-stacks and their 1-morphisms and 2-morphisms are Zariski locally modelled on quotient d-stacks, quotient 1-morphisms, and quotient 2-morphisms, up to equivalence in **dSta**.

In [35, §9.2] we define when a 1-morphism of d-stacks $f : \mathcal{X} \to \mathcal{Y}$ is *étale*. Essentially, f is étale if it is an equivalence locally in the étale topology. It implies that the C^∞-stack 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ in f is étale, and so representable.

We can characterize étale 1-morphisms in **dSta** using Theorem 10.4: a 1-morphism $\tilde{f} : \mathcal{X} \to \mathcal{Y}$ in **dSta** is étale if and only if for all $[f, \rho] : [U'/G] \to [V/H]$ in Theorem 10.4(b), $f : U' \to V$ is an étale 1-morphism in **dSpa** (that is, a local equivalence in the Zariski topology), and $\rho : G \to H$ is injective.

10.3 Gluing d-stacks by equivalences

Section 3.2 discussed gluing d-spaces by equivalences in **dSpa**. In [35, §9.4] we generalize this to **dSta**. Here are the analogues of Definition 3.4, Proposition 3.5, and Theorems 3.6 and 3.7.

Definition 10.5. Let $\mathcal{X} = (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}})$ be a d-stack. Suppose $\mathcal{U} \subseteq \mathcal{X}$ is an open C^∞-substack, in the Zariski topology, with inclusion 1-morphism $i_{\mathcal{U}} : \mathcal{U} \to \mathcal{X}$. Then $\mathcal{U} = \big(\mathcal{U}, i_{\mathcal{U}}^{-1}(\mathcal{O}'_{\mathcal{X}}), i_{\mathcal{U}}^*(\mathcal{E}_{\mathcal{X}}), i_{\mathcal{U}}^{\sharp} \circ i_{\mathcal{U}}^{-1}(\imath_{\mathcal{X}}), i_{\mathcal{U}}^*(\jmath_{\mathcal{X}})\big)$ is a d-stack, where $i_{\mathcal{U}}^{\sharp} : i_{\mathcal{U}}^{-1}(\mathcal{O}_{\mathcal{X}}) \to \mathcal{O}_{\mathcal{U}}$ is as in Example 8.23, and is an isomorphism as $i_{\mathcal{U}}$ is étale. We call $\mathcal{U}$ an *open d-substack* of $\mathcal{X}$. An *open cover* of a d-stack $\mathcal{X}$ is a family $\{\mathcal{U}_a : a \in A\}$ of open d-substacks $\mathcal{U}_a$ of $\mathcal{X}$ such that $\{\mathcal{U}_a : a \in A\}$ is an open cover of $\mathcal{X}$, in the Zariski topology.

Proposition 10.6. *Let $\mathcal{X}, \mathcal{Y}$ be d-stacks, $\mathcal{U}, \mathcal{V} \subseteq \mathcal{X}$ be open d-substacks with $\mathcal{X} = \mathcal{U} \cup \mathcal{V}$, $f : \mathcal{U} \to \mathcal{Y}$ and $g : \mathcal{V} \to \mathcal{Y}$ be 1-morphisms, and $\eta : f|_{\mathcal{U} \cap \mathcal{V}} \Rightarrow g|_{\mathcal{U} \cap \mathcal{V}}$ a 2-morphism. Then there exist a 1-morphism $h : \mathcal{X} \to \mathcal{Y}$ and 2-morphisms $\zeta : h|_{\mathcal{U}} \Rightarrow f$, $\theta : h|_{\mathcal{V}} \Rightarrow g$ in* **dSta** *such that $\theta|_{\mathcal{U} \cap \mathcal{V}} = \eta \odot \zeta|_{\mathcal{U} \cap \mathcal{V}} : h|_{\mathcal{U} \cap \mathcal{V}} \Rightarrow g|_{\mathcal{U} \cap \mathcal{V}}$. This h is unique up to 2-isomorphism.*

Theorem 10.7. *Suppose $\mathcal{X}, \mathcal{Y}$ are d-stacks, $\mathcal{U} \subseteq \mathcal{X}$, $\mathcal{V} \subseteq \mathcal{Y}$ are open d-substacks, and $f : \mathcal{U} \to \mathcal{V}$ is an equivalence in* **dSta**. *At the level of topological*

spaces, we have open $\mathcal{U}_{\mathrm{top}} \subseteq \mathcal{X}_{\mathrm{top}}$, $\mathcal{V}_{\mathrm{top}} \subseteq \mathcal{Y}_{\mathrm{top}}$ with a homeomorphism $f_{\mathrm{top}} : \mathcal{U}_{\mathrm{top}} \to \mathcal{V}_{\mathrm{top}}$, so we can form the quotient topological space $\mathcal{Z}_{\mathrm{top}} := \mathcal{X}_{\mathrm{top}} \amalg_{f_{\mathrm{top}}} \mathcal{Y}_{\mathrm{top}} = (\mathcal{X}_{\mathrm{top}} \amalg \mathcal{Y}_{\mathrm{top}})/ \sim$, where the equivalence relation $\sim$ on $\mathcal{X}_{\mathrm{top}} \amalg \mathcal{Y}_{\mathrm{top}}$ identifies $[u] \in \mathcal{U}_{\mathrm{top}} \subseteq \mathcal{X}_{\mathrm{top}}$ with $f_{\mathrm{top}}([u]) \in \mathcal{V}_{\mathrm{top}} \subseteq \mathcal{Y}_{\mathrm{top}}$.

Suppose $\mathcal{Z}_{\mathrm{top}}$ is Hausdorff. Then there exist a d-stack $\mathcal{Z}$, open d-substacks $\hat{\mathcal{X}}, \hat{\mathcal{Y}}$ in $\mathcal{Z}$ with $\mathcal{Z} = \hat{\mathcal{X}} \cup \hat{\mathcal{Y}}$, equivalences $g : \mathcal{X} \to \hat{\mathcal{X}}$ and $h : \mathcal{Y} \to \hat{\mathcal{Y}}$ such that $g|_{\mathcal{U}}$ and $h|_{\mathcal{V}}$ are both equivalences with $\hat{\mathcal{X}} \cap \hat{\mathcal{Y}}$, and a 2-morphism $\eta : g|_{\mathcal{U}} \Rightarrow h \circ f$. Furthermore, $\mathcal{Z}$ is independent of choices up to equivalence.

Theorem 10.8. *Suppose I is an indexing set, and $<$ is a total order on I, and $\mathcal{X}_i$ for $i \in I$ are d-stacks, and for all $i < j$ in I we are given open d-substacks $\mathcal{U}_{ij} \subseteq \mathcal{X}_i$, $\mathcal{U}_{ji} \subseteq \mathcal{X}_j$ and an equivalence $e_{ij} : \mathcal{U}_{ij} \to \mathcal{U}_{ji}$, satisfying the following properties:*

(a) *For all $i < j < k$ in I we have a 2-commutative diagram*

for some η_{ijk}, where all three 1-morphisms are equivalences; and

(b) *For all $i < j < k < l$ in I the components η_{ijk} in $\boldsymbol{\eta}_{ijk} = (\eta_{ijk}, \eta'_{ijk})$ satisfy*

$$\eta_{ikl} \odot (\mathrm{id}_{f_{kl}} * \eta_{ijk})|_{\mathcal{U}_{ij} \cap \mathcal{U}_{ik} \cap \mathcal{U}_{il}} = \eta_{ijl} \odot (\eta_{jkl} * \mathrm{id}_{f_{ij}})|_{\mathcal{U}_{ij} \cap \mathcal{U}_{ik} \cap \mathcal{U}_{il}}. \quad (10.3)$$

On the level of topological spaces, define the quotient topological space $\mathcal{Y}_{\mathrm{top}} = (\coprod_{i \in I} \mathcal{X}_{i,\mathrm{top}})/ \sim$, where $\sim$ is the equivalence relation generated by $[x_i] \sim [x_j]$ if $[x_i] \in \mathcal{U}_{ij,\mathcal{X}_i,\mathrm{top}} \subseteq \mathcal{X}_{i,\mathrm{top}}$ and $[x_j] \in \mathcal{U}_{ji,\mathrm{top}} \subseteq \mathcal{X}_{j,\mathrm{top}}$ with $e_{ij,\mathrm{top}}([x_i]) = [x_j]$. Suppose $\mathcal{Y}_{\mathrm{top}}$ is Hausdorff and second countable. Then there exist a d-stack $\mathcal{Y}$ and a 1-morphism $f_i : \mathcal{X}_i \to \mathcal{Y}$ which is an equivalence with an open d-substack $\hat{\mathcal{X}}_i \subseteq \mathcal{Y}$ for all $i \in I$, where $\mathcal{Y} = \bigcup_{i \in I} \hat{\mathcal{X}}_i$, such that $f_i|_{\mathcal{U}_{ij}}$ is an equivalence $\mathcal{U}_{ij} \to \hat{\mathcal{X}}_i \cap \hat{\mathcal{X}}_j$ for all $i < j$ in I, and there exists a 2-morphism $\eta_{ij} : f_j \circ e_{ij} \Rightarrow f_i|_{\mathcal{U}_{ij}}$. The d-stack $\mathcal{Y}$ is unique up to equivalence.

Suppose also that $\mathcal{Z}$ is a d-stack, and $g_i : \mathcal{X}_i \to \mathcal{Z}$ are 1-morphisms for all $i \in I$, and there exist 2-morphisms $\zeta_{ij} : g_j \circ e_{ij} \Rightarrow g_i|_{\mathcal{U}_{ij}}$ for all $i < j$ in I, such that for all $i < j < k$ in I the components ζ_{ij}, η_{ijk} in $\boldsymbol{\zeta}_{ij}, \boldsymbol{\eta}_{ijk}$ satisfy

$$\big(\zeta_{ij}|_{\mathcal{U}_{ij} \cap \mathcal{U}_{ik}}\big) \odot \big(\zeta_{jk} * \mathrm{id}_{e_{ij}}|_{\mathcal{U}_{ij} \cap \mathcal{U}_{ik}}\big) = \big(\zeta_{ik}|_{\mathcal{U}_{ij} \cap \mathcal{U}_{ik}}\big) \odot \big(\mathrm{id}_{g_k} * \eta_{ijk}|_{\mathcal{U}_{ij} \cap \mathcal{U}_{ik}}\big). \quad (10.4)$$

Then there exist a 1-morphism $h : \mathcal{Y} \to \mathcal{Z}$ and 2-morphisms $\zeta_i : h \circ f_i \Rightarrow g_i$ for all $i \in I$. The 1-morphism h is unique up to 2-isomorphism.

Remark 10.9. Note that in Proposition 3.5 for d-spaces, h is independent of η up to 2-isomorphism, but in Proposition 10.6 for d-stacks, h may depend on $\boldsymbol{\eta}$. Similarly, in Theorem 3.7 for d-spaces, we impose no conditions on 2-morphisms η_{ijk} on quadruple overlaps or ζ_{ij} on triple overlaps, but in Theorem 10.8 for d-stacks, we do impose extra conditions (10.3) on the 2-morphisms $\boldsymbol{\eta}_{ijk}$

on quadruple overlaps and (10.4) on the 2-morphisms $\boldsymbol{\zeta}_{ij}$ on triple overlaps. Thus, the d-stack versions of these results are weaker.

The reason for this is that 2-morphisms $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ of d-space 1-morphisms $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \rightarrow \boldsymbol{Y}$ are morphisms $\eta : \underline{f}^*(\mathcal{F}_Y) \rightarrow \mathcal{E}_X$ in $\mathrm{qcoh}(\underline{X})$. We can interpolate between such morphisms using partitions of unity on $\underline{X}$, and in Remark 3.8 we explained why this enables us to prove $\boldsymbol{h}$ is independent of η in Proposition 3.5, and to do without overlap conditions on η_{ijk}, ζ_{ij} in Theorem 3.7.

In contrast, for 2-morphisms $\boldsymbol{\eta} = (\eta, \eta') : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ in $\mathbf{dSta}$, the C^∞-stack 2-morphisms $\eta : f \Rightarrow g$ are discrete objects, and we cannot join them using partitions of unity. So $\boldsymbol{h}$ may depend on η in Proposition 10.6, and we need overlap conditions on the components η_{ijk}, ζ_{ij} in $\boldsymbol{\eta}_{ijk}, \boldsymbol{\zeta}_{ij}$ in Theorem 10.8.

If $f, g : \mathcal{X} \rightarrow \mathcal{Y}$ are 1-morphisms of Deligne–Mumford C^∞-stacks, we can make extra assumptions on $\mathcal{X}, \mathcal{Y}$ or f, g which imply that there is at most one 2-morphism $\eta : f \Rightarrow g$, as in Proposition 9.5 for orbifolds. Such assumptions can make (10.3) or (10.4) hold automatically, as both sides of (10.3) or (10.4) are 2-morphisms $f \Rightarrow g$. So, for instance, if the C^∞-stacks $\mathcal{X}_i$ are all effective then (10.3) holds, and if the d-stack $\boldsymbol{\mathcal{Z}}$ is a d-space then (10.4) holds.

10.4 Fibre products of d-stacks

Section 3.3 discussed fibre products of d-spaces. In [35, §9.5] we generalize this to d-stacks. Here is the analogue of Theorem 3.9:

Theorem 10.10. (a) *All fibre products exist in the 2-category* $\mathbf{dSta}$.

(b) *The 2-functor* $F_{\mathbf{dSpa}}^{\mathbf{dSta}} : \mathbf{dSpa} \rightarrow \mathbf{dSta}$ *preserves fibre products.*

(c) *Let* $g : \mathcal{X} \rightarrow \mathcal{Z}$ *and* $h : \mathcal{Y} \rightarrow \mathcal{Z}$ *be smooth maps (1-morphisms) of orbifolds, and write* $\boldsymbol{\mathcal{X}} = F_{\mathbf{Orb}}^{\mathbf{dSta}}(\mathcal{X})$, *and similarly for* $\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}, \boldsymbol{g}, \boldsymbol{h}$. *If* g, h *are transverse, so that a fibre product* $\mathcal{X} \times_{g, \mathcal{Z}, h} \mathcal{Y}$ *exists in* $\mathbf{Orb}$, *then the fibre product* $\boldsymbol{\mathcal{X}} \times_{\boldsymbol{g}, \boldsymbol{\mathcal{Z}}, \boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ *in* $\mathbf{dSta}$ *is equivalent in* $\mathbf{dSta}$ *to* $F_{\mathbf{Orb}}^{\mathbf{dSta}}(\mathcal{X} \times_{g, \mathcal{Z}, h} \mathcal{Y})$. *If* g, h *are not transverse then* $\boldsymbol{\mathcal{X}} \times_{\boldsymbol{g}, \boldsymbol{\mathcal{Z}}, \boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ *exists in* $\mathbf{dSta}$, *but is not an orbifold.*
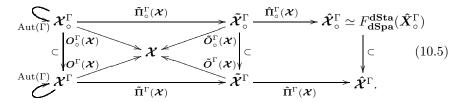
As for d-spaces, we prove (a) by explicitly constructing a d-stack $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{X}} \times_{\boldsymbol{g}, \boldsymbol{\mathcal{Z}}, \boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ and showing it satisfies the universal property to be a fibre product in the 2-category $\mathbf{dSta}$. The proof follows that of Theorem 3.9 closely, inserting extra terms for 2-morphisms of C^∞-stacks.

10.5 Orbifold strata of d-stacks

Section 8.7 discussed orbifold strata of Deligne–Mumford C^∞-stacks. In [35, §9.6] we generalize this to d-stacks. The next theorems summarize the results.

Theorem 10.11. *Let* $\boldsymbol{\mathcal{X}}$ *be a d-stack, and* Γ *a finite group. Then we can define d-stacks* $\boldsymbol{\mathcal{X}}^\Gamma, \tilde{\boldsymbol{\mathcal{X}}}^\Gamma, \hat{\boldsymbol{\mathcal{X}}}^\Gamma$, *and open d-substacks* $\boldsymbol{\mathcal{X}}_\circ^\Gamma \subseteq \boldsymbol{\mathcal{X}}^\Gamma$, $\tilde{\boldsymbol{\mathcal{X}}}_\circ^\Gamma \subseteq \tilde{\boldsymbol{\mathcal{X}}}^\Gamma$, $\hat{\boldsymbol{\mathcal{X}}}_\circ^\Gamma \subseteq \hat{\boldsymbol{\mathcal{X}}}^\Gamma$, *all natural up to 1-isomorphism in* $\mathbf{dSta}$, *a d-space* $\hat{\boldsymbol{X}}_\circ^\Gamma$ *natural up to 1-isomorphism in* $\mathbf{dSpa}$, *and 1-morphisms* $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}}), \tilde{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}}), \ldots$ *fitting into a*

strictly commutative diagram in **dSta**:

$$
\begin{array}{ccccc}
\mathcal{X}_\circ^\Gamma & \xrightarrow{\ \tilde{\mathbf{\Pi}}_\circ^\Gamma(\boldsymbol{x})\ } & \tilde{\mathcal{X}}_\circ^\Gamma & \xrightarrow{\ \hat{\mathbf{\Pi}}_\circ^\Gamma(\boldsymbol{x})\ } & \hat{\mathcal{X}}_\circ^\Gamma \simeq F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\hat{\boldsymbol{X}}_\circ^\Gamma)
\end{array}
\tag{10.5}
$$

We will call $\boldsymbol{\mathcal{X}}^\Gamma, \tilde{\boldsymbol{\mathcal{X}}}^\Gamma, \hat{\boldsymbol{\mathcal{X}}}^\Gamma, \boldsymbol{\mathcal{X}}_\circ^\Gamma, \tilde{\boldsymbol{\mathcal{X}}}_\circ^\Gamma, \hat{\boldsymbol{\mathcal{X}}}_\circ^\Gamma, \hat{\boldsymbol{X}}_\circ^\Gamma$ *the **orbifold strata** of* $\boldsymbol{\mathcal{X}}$.

The underlying C^∞-stacks of $\boldsymbol{\mathcal{X}}^\Gamma, \ldots, \hat{\boldsymbol{\mathcal{X}}}_\circ^\Gamma$ are the orbifold strata $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}_\circ^\Gamma$ from §8.7 of the C^∞-stack $\mathcal{X}$ in $\boldsymbol{\mathcal{X}}$. The C^∞-stack 1-morphisms underlying the d-stack 1-morphisms in (10.5) are those given in (8.3).

Theorem 10.12. (a) *Let* $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ *be d-stacks,* Γ *a finite group, and* $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *a representable 1-morphism in* **dSta**, *that is, the underlying* C^∞-*stack 1-morphism* $f : \mathcal{X} \to \mathcal{Y}$ *is representable. Then there is a unique representable 1-morphism* $\boldsymbol{f}^\Gamma : \boldsymbol{\mathcal{X}}^\Gamma \to \boldsymbol{\mathcal{Y}}^\Gamma$ *in* **dSta** *with* $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{Y}}) \circ \boldsymbol{f}^\Gamma = \boldsymbol{f} \circ \boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})$. *Here* $\boldsymbol{\mathcal{X}}^\Gamma, \boldsymbol{\mathcal{Y}}^\Gamma, \boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}}), \boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{Y}})$ *are as in Theorem* 10.11.

(b) *Let* $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *be representable 1-morphisms and* $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ *a 2-morphism in* **dSta**, *and* $\boldsymbol{f}^\Gamma, \boldsymbol{g}^\Gamma : \boldsymbol{\mathcal{X}}^\Gamma \to \boldsymbol{\mathcal{Y}}^\Gamma$ *be as in* **(a)**. *Then there is a unique 2-morphism* $\boldsymbol{\eta}^\Gamma : \boldsymbol{f}^\Gamma \Rightarrow \boldsymbol{g}^\Gamma$ *in* **dSta** *with* $\mathrm{id}_{\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{Y}})} * \boldsymbol{\eta}^\Gamma = \boldsymbol{\eta} * \mathrm{id}_{\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})}$.

(c) *Write* **dSta$^{\mathbf{re}}$** *for the 2-subcategory of* **dSta** *with only representable 1-morphisms. Then mapping* $\boldsymbol{\mathcal{X}} \mapsto F^\Gamma(\boldsymbol{\mathcal{X}}) = \boldsymbol{\mathcal{X}}^\Gamma$ *on objects,* $\boldsymbol{f} \mapsto F^\Gamma(\boldsymbol{f}) = \boldsymbol{f}^\Gamma$ *on (representable) 1-morphisms, and* $\boldsymbol{\eta} \mapsto F^\Gamma(\boldsymbol{\eta}) = \boldsymbol{\eta}^\Gamma$ *on 2-morphisms defines a strict 2-functor* $F^\Gamma : \mathbf{dSta^{re}} \to \mathbf{dSta^{re}}$.

(d) *Analogues of* **(a)**–**(c)** *hold for the orbifold strata* $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma$, *yielding a strict 2-functor* $\tilde{F}^\Gamma : \mathbf{dSta^{re}} \to \mathbf{dSta^{re}}$. *Weaker analogues of* **(a)**–**(c)** *also hold for the orbifold strata* $\hat{\boldsymbol{\mathcal{X}}}^\Gamma$. *In* **(a)**, *the 1-morphism* $\hat{\boldsymbol{f}}^\Gamma : \hat{\boldsymbol{\mathcal{X}}}^\Gamma \to \hat{\boldsymbol{\mathcal{Y}}}^\Gamma$ *is natural only up to 2-isomorphism, and in* **(c)** *we get a weak 2-functor* $\hat{F}^\Gamma : \mathbf{dSta^{re}} \to \mathbf{dSta^{re}}$.

Since equivalences in **dSta** are automatically representable, and (strict or weak) 2-functors take equivalences to equivalences, we deduce:

Corollary 10.13. *Suppose* $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ *are equivalent d-stacks, and* Γ *is a finite group. Then* $\boldsymbol{\mathcal{X}}^\Gamma$ *and* $\boldsymbol{\mathcal{Y}}^\Gamma$ *are equivalent in* **dSta**, *and similarly for* $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma, \hat{\boldsymbol{\mathcal{X}}}^\Gamma, \boldsymbol{\mathcal{X}}_\circ^\Gamma, \tilde{\boldsymbol{\mathcal{X}}}_\circ^\Gamma,$ $\hat{\boldsymbol{\mathcal{X}}}_\circ^\Gamma$ *and* $\tilde{\boldsymbol{\mathcal{Y}}}^\Gamma, \hat{\boldsymbol{\mathcal{Y}}}^\Gamma, \boldsymbol{\mathcal{Y}}_\circ^\Gamma, \tilde{\boldsymbol{\mathcal{Y}}}_\circ^\Gamma, \hat{\boldsymbol{\mathcal{Y}}}_\circ^\Gamma$. *Also* $\hat{\boldsymbol{X}}_\circ^\Gamma, \hat{Y}_\circ^\Gamma$ *are equivalent in* **dSpa**.

Here are the d-stack analogues of Theorems 8.27 and 8.28:

Theorem 10.14. *Let* $\boldsymbol{X}$ *be a d-space and* G *a finite group acting on* $\boldsymbol{X}$ *by 1-isomorphisms, and write* $\boldsymbol{\mathcal{X}} = [\boldsymbol{X}/G]$ *for the quotient d-stack, from Theorem 10.3. Let* Γ *be a finite group. Then there are equivalences of d-stacks*

$$
\boldsymbol{\mathcal{X}}^\Gamma \simeq \coprod_{\substack{\text{conjugacy classes } [\rho] \text{ of injective} \\ \text{group morphisms } \rho : \Gamma \to G}} \left[\boldsymbol{X}^{\rho(\Gamma)} / \{ g \in G : g\rho(\gamma) = \rho(\gamma)g \ \forall \gamma \in \Gamma \} \right], \tag{10.6}
$$

$$\mathcal{X}_\circ^\Gamma \simeq \coprod_{\substack{\text{conjugacy classes } [\rho] \text{ of injective}\\ \text{group morphisms } \rho:\Gamma\to G}} \big[\,\boldsymbol{X}_\circ^{\rho(\Gamma)}/\{g\in G:g\rho(\gamma)=\rho(\gamma)g\ \forall\gamma\in\Gamma\}\big], \quad (10.7)$$

$$\tilde{\mathcal{X}}^\Gamma \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta\subseteq G \text{ with } \Delta\cong\Gamma} \big[\,\boldsymbol{X}^\Delta/\{g\in G:\Delta=g\Delta g^{-1}\}\big], \quad (10.8)$$

$$\tilde{\mathcal{X}}_\circ^\Gamma \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta\subseteq G \text{ with } \Delta\cong\Gamma} \big[\,\boldsymbol{X}_\circ^\Delta/\{g\in G:\Delta=g\Delta g^{-1}\}\big], \quad (10.9)$$

$$\hat{\mathcal{X}}^\Gamma \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta\subseteq G \text{ with } \Delta\cong\Gamma} \big[\,\boldsymbol{X}^\Delta/(\{g\in G:\Delta=g\Delta g^{-1}\}/\Delta)\big], \quad (10.10)$$

$$\hat{\mathcal{X}}_\circ^\Gamma \simeq \coprod_{\text{conjugacy classes } [\Delta] \text{ of subgroups } \Delta\subseteq G \text{ with } \Delta\cong\Gamma} \big[\,\boldsymbol{X}_\circ^\Delta/(\{g\in G:\Delta=g\Delta g^{-1}\}/\Delta)\big]. \quad (10.11)$$

Here for each subgroup $\Delta\subseteq G$, we write $\boldsymbol{X}^\Delta$ for the closed d-subspace in $\boldsymbol{X}$ fixed by Δ in G, as in §3.4, and $\boldsymbol{X}_\circ^\Delta$ for the open d-subspace in $\boldsymbol{X}^\Delta$ of points in $\boldsymbol{X}$ whose stabilizer group in G is exactly Δ. In (10.6)–(10.7), morphisms $\rho,\rho':\Gamma\to G$ are conjugate if $\rho'=\mathrm{Ad}(g)\circ\rho$ for some $g\in G$, and subgroups $\Delta,\Delta'\subseteq G$ are conjugate if $\Delta=g\Delta'g^{-1}$ for some $g\in G$. In (10.6)–(10.11) we sum over one representative ρ or Δ for each conjugacy class.

Theorem 10.15. *Let $\boldsymbol{\mathcal{X}}$ be a d-stack and Γ a finite group, so that Theorem 10.11 gives a d-stack $\boldsymbol{\mathcal{X}}^\Gamma$ and a 1-morphism $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}}):\boldsymbol{\mathcal{X}}^\Gamma\to\boldsymbol{\mathcal{X}}$. Equation (10.2) for $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})$ becomes:*

$$
\begin{array}{ccccccc}
\substack{O^\Gamma(\mathcal{X})^*(\mathcal{E}_\mathcal{X})=\\ (\mathcal{E}_\mathcal{X})_{\mathrm{tr}}^\Gamma\oplus(\mathcal{E}_\mathcal{X})_{\mathrm{nt}}^\Gamma} & \xrightarrow{\ O^\Gamma(\mathcal{X})^*(\phi_\mathcal{X})\ } & \substack{O^\Gamma(\mathcal{X})^*(\mathcal{F}_\mathcal{X})=\\ (\mathcal{F}_\mathcal{X})_{\mathrm{tr}}^\Gamma\oplus(\mathcal{F}_\mathcal{X})_{\mathrm{nt}}^\Gamma} & \xrightarrow{\ O^\Gamma(\mathcal{X})^*(\psi_\mathcal{X})\ } & \substack{O^\Gamma(\mathcal{X})^*(T^*\mathcal{X})=\\ (T^*\mathcal{X})_{\mathrm{tr}}^\Gamma\oplus(T^*\mathcal{X})_{\mathrm{nt}}^\Gamma} & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle O^\Gamma(\mathcal{X})''} & & \Big\downarrow{\scriptstyle O^\Gamma(\mathcal{X})^2} & & \Big\downarrow{\substack{O^\Gamma(\mathcal{X})^3=\\ \Omega_{O^\Gamma(\mathcal{X})}}} & & \\
\mathcal{E}_{\mathcal{X}^\Gamma} & \xrightarrow{\ \phi_{\mathcal{X}^\Gamma}\ } & \mathcal{F}_{\mathcal{X}^\Gamma} & \xrightarrow{\ \psi_{\mathcal{X}^\Gamma}\ } & T^*(\mathcal{X}^\Gamma) & \longrightarrow & 0.
\end{array}
\quad (10.12)
$$

Then the columns $O^\Gamma(\mathcal{X})''$, $O^\Gamma(\mathcal{X})^2$, $O^\Gamma(\mathcal{X})^3$ of (10.12) are isomorphisms when restricted to the 'trivial' summands $(\mathcal{E}_\mathcal{X})_{\mathrm{tr}}^\Gamma,(\mathcal{F}_\mathcal{X})_{\mathrm{tr}}^\Gamma,(T^*\mathcal{X})_{\mathrm{tr}}^\Gamma$, and are zero when restricted to the 'nontrivial' summands $(\mathcal{E}_\mathcal{X})_{\mathrm{nt}}^\Gamma,(\mathcal{F}_\mathcal{X})_{\mathrm{nt}}^\Gamma,(T^*\mathcal{X})_{\mathrm{nt}}^\Gamma$. In particular, this implies that the virtual cotangent sheaf $\phi_{\mathcal{X}^\Gamma}:\mathcal{E}_{\mathcal{X}^\Gamma}\to\mathcal{F}_{\mathcal{X}^\Gamma}$ of $\boldsymbol{\mathcal{X}}^\Gamma$ is 1-isomorphic in $\mathrm{vqcoh}(\mathcal{X}^\Gamma)$ to $(\phi_\mathcal{X})_{\mathrm{tr}}^\Gamma:(\mathcal{E}_\mathcal{X})_{\mathrm{tr}}^\Gamma\to(\mathcal{F}_\mathcal{X})_{\mathrm{tr}}^\Gamma$, the 'trivial' part of the pullback to $\mathcal{X}^\Gamma$ of the virtual cotangent sheaf $\phi_\mathcal{X}:\mathcal{E}_\mathcal{X}\to\mathcal{F}_\mathcal{X}$ of $\boldsymbol{\mathcal{X}}$.

The analogous results also hold for $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma,\hat{\boldsymbol{\mathcal{X}}}^\Gamma,\boldsymbol{\mathcal{X}}_\circ^\Gamma,\tilde{\boldsymbol{\mathcal{X}}}_\circ^\Gamma$ and $\hat{\boldsymbol{\mathcal{X}}}_\circ^\Gamma$.

11 The 2-category of d-orbifolds

In [35, Chap. 10] we discuss *d-orbifolds*, orbifold versions of d-manifolds. They are related to *Kuranishi spaces* (without boundary) in the work of Fukaya, Oh, Ohta and Ono [19, 20] on symplectic geometry. As we explain briefly in §16, and in more detail in [35, §14.3], although Kuranishi spaces are similar to d-orbifolds in many ways, the theory of Kuranishi spaces in [19, 20] is incomplete

— for instance, there is no notion of morphism of Kuranishi spaces, so they do not form a category. We argue in [35, §14.3] that the 'right' way to define Kuranishi spaces is as d-orbifolds, or d-orbifolds with corners.

11.1 Definition of d-orbifolds

In §4.3 we discussed *virtual quasicoherent sheaves* and *virtual vector bundles* on C^∞-schemes $\underline{X}$. The next remark, drawn from [35, §10.1.1], explains how these generalize to Deligne–Mumford C^∞-stacks $\mathcal{X}$.

Remark 11.1. In the C^∞-stack analogue of Definition 4.9, the 2-categories vqcoh$(\mathcal{X})$ and vvect$(\mathcal{X})$ for a Deligne–Mumford C^∞-stack $\mathcal{X}$ are defined exactly as for C^∞-schemes. For $\mathcal{X} \neq \emptyset$, virtual vector bundles $(\mathcal{E}^\bullet, \phi)$ have a well-defined *rank* rank$(\mathcal{E}^\bullet, \phi) \in \mathbb{Z}$. If $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of Deligne–Mumford C^∞-stacks then pullback f^* defines strict 2-functors $f^* : \text{vqcoh}(\mathcal{Y}) \to \text{vqcoh}(\mathcal{X})$ and $f^* : \text{vvect}(\mathcal{Y}) \to \text{vvect}(\mathcal{X})$, as for C^∞-schemes. If $f, g : \mathcal{X} \to \mathcal{Y}$ are 1-morphisms and $\eta : f \Rightarrow g$ a 2-morphism then $\eta^* : f^* \Rightarrow g^*$ is a 2-natural transformation.

In the d-stack version of Definition 4.10, we define the *virtual cotangent sheaf* $T^*\mathcal{X}$ of a d-stack $\mathcal{X}$ to be the morphism $\phi_\mathcal{X} : \mathcal{E}_\mathcal{X} \to \mathcal{F}_\mathcal{X}$ in qcoh$(\mathcal{X})$ from Definition 10.1. If $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a 1-morphism in **dSta** then $\Omega_{\boldsymbol{f}} := (f'', f^2)$ is a 1-morphism $f^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$ in vqcoh$(\mathcal{X})$. If $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ are 1-morphisms and $\boldsymbol{\eta} = (\eta, \eta') : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism in **dSta**, then we have 1-morphisms $\Omega_{\boldsymbol{f}} : f^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$, $\Omega_{\boldsymbol{g}} : g^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$, and $\eta^*(T^*\boldsymbol{\mathcal{Y}}) : f^*(T^*\boldsymbol{\mathcal{Y}}) \to g^*(T^*\boldsymbol{\mathcal{Y}})$ in qcoh$(\mathcal{X})$, and $\eta' : \Omega_{\boldsymbol{f}} \Rightarrow \Omega_{\boldsymbol{g}} \circ \eta^*(T^*\boldsymbol{\mathcal{Y}})$ is a 2-morphism in vqcoh$(\mathcal{X})$.

We can now define d-orbifolds.

Definition 11.2. A d-stack $\boldsymbol{\mathcal{W}}$ is called a *principal d-orbifold* if is equivalent in **dSta** to a fibre product $\boldsymbol{\mathcal{X}} \times_{\boldsymbol{g}, \boldsymbol{\mathcal{Z}}, \boldsymbol{h}} \boldsymbol{\mathcal{Y}}$ with $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}} \in \hat{\textbf{Orb}}$. If $\boldsymbol{\mathcal{W}}$ is a nonempty principal d-orbifold then as in Proposition 4.11, the virtual cotangent sheaf $T^*\boldsymbol{\mathcal{W}}$ is a virtual vector bundle on $\mathcal{W}$, in the sense of Remark 11.1. We define the *virtual dimension* of $\boldsymbol{\mathcal{W}}$ to be vdim $\boldsymbol{\mathcal{W}} = \text{rank}\, T^*\boldsymbol{\mathcal{W}} \in \mathbb{Z}$. If $\boldsymbol{\mathcal{W}} \simeq \boldsymbol{\mathcal{X}} \times_{\boldsymbol{\mathcal{Z}}} \boldsymbol{\mathcal{Y}}$ for orbifolds $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ then vdim $\boldsymbol{\mathcal{W}} = \dim \mathcal{X} + \dim \mathcal{Y} - \dim \mathcal{Z}$.

A d-stack $\boldsymbol{\mathcal{X}}$ is called a *d-orbifold* (*without boundary*) *of virtual dimension* $n \in \mathbb{Z}$, written vdim $\boldsymbol{\mathcal{X}} = n$, if $\boldsymbol{\mathcal{X}}$ can be covered by open d-substacks $\boldsymbol{\mathcal{W}}$ which are principal d-orbifolds with vdim $\boldsymbol{\mathcal{W}} = n$. The virtual cotangent sheaf $T^*\boldsymbol{\mathcal{X}} = (\mathcal{E}_\mathcal{X}, \mathcal{F}_\mathcal{X}, \phi_\mathcal{X})$ of $\boldsymbol{\mathcal{X}}$ is a virtual vector bundle of rank vdim $\boldsymbol{\mathcal{X}} = n$, so we call it the *virtual cotangent bundle* of $\boldsymbol{\mathcal{X}}$.

Let **dOrb** be the full 2-subcategory of d-orbifolds in **dSta**. The 2-functor $F_{\textbf{Orb}}^{\textbf{dSta}} : \textbf{Orb} \to \textbf{dSta}$ in Definition 10.1 maps into **dOrb**, and we will write $F_{\textbf{Orb}}^{\textbf{dOrb}} = F_{\textbf{Orb}}^{\textbf{dSta}} : \textbf{Orb} \to \textbf{dOrb}$. Also $\hat{\textbf{Orb}}$ is a 2-subcategory of **dOrb**. We say that a d-orbifold $\boldsymbol{\mathcal{X}}$ *is an orbifold* if it lies in $\hat{\textbf{Orb}}$. The 2-functor $F_{\textbf{dSpa}}^{\textbf{dSta}}$ maps **dMan** $\to$ **dOrb**, and we will write $F_{\textbf{dMan}}^{\textbf{dOrb}} = F_{\textbf{dSpa}}^{\textbf{dSta}}|_{\textbf{dMan}} : \textbf{dMan} \to \textbf{dOrb}$. Then $F_{\textbf{dMan}}^{\textbf{dOrb}} \circ F_{\textbf{Man}}^{\textbf{dMan}} = F_{\textbf{Orb}}^{\textbf{dOrb}} \circ F_{\textbf{Man}}^{\textbf{Orb}} : \textbf{Man} \to \textbf{dOrb}$.

Write $\hat{\textbf{dMan}}$ for the full 2-subcategory of objects $\boldsymbol{\mathcal{X}}$ in **dOrb** equivalent to $F_{\textbf{dMan}}^{\textbf{dOrb}}(\boldsymbol{X})$ for some d-manifold $\boldsymbol{X}$. When we say that a d-orbifold $\boldsymbol{\mathcal{X}}$ *is a d-manifold*, we mean that $\boldsymbol{\mathcal{X}} \in \hat{\textbf{dMan}}$.

The orbifold analogue of Proposition 4.2 holds. Using Theorem 8.17 we can deduce:

Lemma 11.3. *Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. Then $\boldsymbol{\mathcal{X}}$ is a d-manifold, that is, $\boldsymbol{\mathcal{X}}$ is equivalent to $F_{\mathbf{dMan}}^{\mathbf{dOrb}}(\boldsymbol{X})$ for some d-manifold $\boldsymbol{X}$, if and only if $\mathrm{Iso}_{\mathcal{X}}([x]) \cong \{1\}$ for all $[x]$ in $\mathcal{X}_{\mathrm{top}}$.*

11.2 Local properties of d-orbifolds

Following Examples 4.4 and 4.5, we define 'standard model' d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ and 1-morphisms $\boldsymbol{\mathcal{S}}_{f,\hat{f}}$.

Example 11.4. Let $\mathcal{V}$ be an orbifold, $\mathcal{E} \in \mathrm{vect}(\mathcal{V})$ a vector bundle on $\mathcal{V}$ as in §8.6, and $s \in C^{\infty}(\mathcal{E})$ a smooth section, that is, $s : \mathcal{O}_{\mathcal{V}} \to \mathcal{E}$ is a morphism in $\mathrm{vect}(\mathcal{V})$. We will define a principal d-orbifold $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} = (\mathcal{S}, \mathcal{O}'_{\mathcal{S}}, \mathcal{E}_{\mathcal{S}}, \imath_{\mathcal{S}}, \jmath_{\mathcal{S}})$, which we call a 'standard model' d-orbifold.

Let the Deligne–Mumford C^{∞}-stack $\mathcal{S}$ be the C^{∞}-substack in $\mathcal{V}$ defined by the equation $s = 0$, so that informally $\mathcal{S} = s^{-1}(0) \subset \mathcal{V}$. Explicitly, as in §8, a C^{∞}-stack $\mathcal{V}$ consists of a category $\mathcal{V}$ and a functor $p_{\mathcal{V}} : \mathcal{V} \to \mathbf{C^{\infty}Sch}$, where there is a 1-1 correspondence between objects u in $\mathcal{V}$ with $p_{\mathcal{V}}(u) = \underline{U}$ in $\mathbf{C^{\infty}Sch}$ and 1-morphisms $\tilde{u} : \underline{U} \to \mathcal{V}$ in $\mathbf{C^{\infty}Sta}$. Define $\mathcal{S}$ to be the full subcategory of objects u in $\mathcal{V}$ such that the morphism $\tilde{u}^{*}(s) : \tilde{u}^{*}(\mathcal{O}_{\mathcal{V}}) \to \tilde{u}^{*}(\mathcal{E})$ in $\mathrm{qcoh}(\underline{U})$ is zero, and define $p_{\mathcal{S}} = p_{\mathcal{V}}|_{\mathcal{S}} : \mathcal{S} \to \mathbf{C^{\infty}Sch}$.

Since $i_{\mathcal{V}} : \mathcal{S} \to \mathcal{V}$ is the inclusion of a C^{∞}-substack, $i_{\mathcal{V}}^{\sharp} : i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}}) \to \mathcal{O}_{\mathcal{S}}$ is a surjective morphism of sheaves of C^{∞}-rings on $\mathcal{S}$. Write $\mathcal{I}_s$ for the kernel of $i_{\mathcal{V}}^{\sharp}$, as a sheaf of ideals in $i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})$, and $\mathcal{I}_s^2$ for the corresponding sheaf of squared ideals, and $\mathcal{O}'_{\mathcal{S}} = i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})/\mathcal{I}_s^2$ for the quotient sheaf of C^{∞}-rings, and $\imath_{\mathcal{S}} : \mathcal{O}'_{\mathcal{S}} \to \mathcal{O}_{\mathcal{S}}$ for the natural projection $i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})/\mathcal{I}_s^2 \twoheadrightarrow i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})/\mathcal{I}_s \cong \mathcal{O}_{\mathcal{S}}$ induced by the inclusion $\mathcal{I}_s^2 \subseteq \mathcal{I}_s$.

Write $\mathcal{E}^{*} \in \mathrm{vect}(\mathcal{V})$ for the dual vector bundle of $\mathcal{E}$, and set $\mathcal{E}_{\mathcal{S}} = i_{\mathcal{V}}^{*}(\mathcal{E}^{*})$. There is a natural, surjective morphism $\jmath_{\mathcal{S}} : \mathcal{E}_{\mathcal{S}} \to \mathcal{I}_{\mathcal{S}} = \mathcal{I}_s/\mathcal{I}_s^2$ in $\mathrm{qcoh}(\mathcal{S})$ which locally maps $\alpha + (\mathcal{I}_s \cdot C^{\infty}(\mathcal{E}^{*})) \mapsto \alpha \cdot s + \mathcal{I}_s^2$. Then $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} = (\mathcal{S}, \mathcal{O}'_{\mathcal{S}}, \mathcal{E}_{\mathcal{S}}, \imath_{\mathcal{S}}, \jmath_{\mathcal{S}})$ is a d-stack. As in the d-manifold case, we can show that $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is equivalent in $\mathbf{dSta}$ to $\boldsymbol{\mathcal{V}} \times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}} \boldsymbol{\mathcal{V}}$, where $\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{E}}, \boldsymbol{s}, \boldsymbol{0} = F_{\mathbf{Orb}}^{\mathbf{dSta}}\big(\mathcal{V}, \mathrm{Tot}(\mathcal{E}), \mathrm{Tot}(s), \mathrm{Tot}(0)\big)$, using the notation of §9.1. Thus $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is a principal d-orbifold. Every principal d-orbifold $\boldsymbol{\mathcal{W}}$ is equivalent in $\mathbf{dSta}$ to some $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$.

Sometimes it is useful to take $\mathcal{V}$ to be an *effective* orbifold, as in §9.1.

Example 11.5. Let $\mathcal{V}, \mathcal{W}$ be orbifolds, $\mathcal{E}, \mathcal{F}$ be vector bundles on $\mathcal{V}, \mathcal{W}$, and $s \in C^{\infty}(\mathcal{E})$, $t \in C^{\infty}(\mathcal{F})$ be smooth sections, so that Example 11.4 defines 'standard model' principal d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}, \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$. Write $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} = \boldsymbol{\mathcal{S}} = (\mathcal{S}, \mathcal{O}'_{\mathcal{S}}, \mathcal{E}_{\mathcal{S}}, \imath_{\mathcal{S}}, \jmath_{\mathcal{S}})$ and $\boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t} = \boldsymbol{\mathcal{T}} = (\mathcal{T}, \mathcal{O}'_{\mathcal{T}}, \mathcal{E}_{\mathcal{T}}, \imath_{\mathcal{T}}, \jmath_{\mathcal{T}})$. Suppose $f : \mathcal{V} \to \mathcal{W}$ is a 1-morphism, and $\hat{f} : \mathcal{E} \to f^{*}(\mathcal{F})$ is a morphism in $\mathrm{vect}(\mathcal{V})$ satisfying

$$\hat{f} \circ s = f^{*}(t). \tag{11.1}$$

We will define a 1-morphism $\boldsymbol{g} = (g, g', g'') : \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{T}}$ in $\mathbf{dSta}$, which we write as $\boldsymbol{\mathcal{S}}_{f,\hat{f}} : \boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W},\mathcal{F},t}$, and call a 'standard model' 1-morphism.

As in Example 11.4, $\mathcal{V}, \mathcal{W}$ are categories, $\mathcal{S} \subseteq \mathcal{V}$, $\mathcal{T} \subseteq \mathcal{W}$ are full subcategories, and $f : \mathcal{V} \to \mathcal{W}$ is a functor. Using (11.1) one can show that $f(\mathcal{S}) \subseteq \mathcal{T} \subseteq \mathcal{W}$. Define $g = f|_{\mathcal{S}} : \mathcal{S} \to \mathcal{T}$. Then $g : \mathcal{S} \to \mathcal{T}$ is a 1-morphism of Deligne–Mumford C^{∞}-stacks, with $i_{\mathcal{W}} \circ g = f \circ i_{\mathcal{V}} : \mathcal{S} \to \mathcal{W}$.

To define $g' : g^{-1}(\mathcal{O}'_{\mathcal{T}}) \to \mathcal{O}'_{\mathcal{S}}$, consider the commutative diagram:

$$
\begin{array}{ccccccccc}
g^{-1}(\mathcal{I}_t^2) & \longrightarrow & g^{-1}(i_{\mathcal{W}}^{-1}(\mathcal{O}_{\mathcal{W}})) & \longrightarrow & g^{-1}(\mathcal{O}'_{\mathcal{T}}) = g^{-1}(i_{\mathcal{W}}^{-1}(\mathcal{O}_{\mathcal{W}})/\mathcal{I}_t^2) & \longrightarrow & 0 \\
\Big\downarrow & & \Big\downarrow {\scriptstyle i_{\mathcal{V}}^{-1}(f^\sharp) \circ I_{i_{\mathcal{V}},f}(\mathcal{O}_{\mathcal{W}}) \circ} & & \Big\downarrow {\scriptstyle g'} & & \\
& & {\scriptstyle I_{g,i_{\mathcal{W}}}(\mathcal{O}_{\mathcal{W}})^{-1}} & & & & \\
\mathcal{I}_s^2 & \longrightarrow & i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}}) & \longrightarrow & \mathcal{O}'_{\mathcal{S}} = i_{\mathcal{V}}^{-1}(\mathcal{O}_{\mathcal{V}})/\mathcal{I}_s^2 & \longrightarrow & 0.
\end{array}
$$

The rows are exact. Using (11.1), we see the central column maps $g^{-1}(\mathcal{I}_t) \to \mathcal{I}_s$, and so maps $g^{-1}(\mathcal{I}_t^2) \to \mathcal{I}_s^2$, and the left column exists. Thus by exactness there is a unique morphism g' making the diagram commute.

We have $\mathcal{E}_{\mathcal{S}} = i_{\mathcal{V}}^*(\mathcal{E}^*)$ and $\mathcal{E}_{\mathcal{T}} = i_{\mathcal{W}}^*(\mathcal{F}^*)$, and $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ induces $\hat{f}^* : f^*(\mathcal{F}^*) \to \mathcal{E}^*$. Define $g'' = i_{\mathcal{V}}^*(\hat{f}^*) \circ I_{i_{\mathcal{V}},f}(\mathcal{F}^*) \circ I_{g,i_{\mathcal{W}}}(\mathcal{F}^*)^{-1} : g^*(\mathcal{E}_{\mathcal{T}}) \to \mathcal{E}_{\mathcal{S}}$ in $\mathrm{qcoh}(\mathcal{S})$. Then $\boldsymbol{g} = (g, g', g'') : \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{T}}$ is a 1-morphism in $\mathbf{dSta}$, which we write as $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{S}_{\mathcal{W},\mathcal{F},t}$.

Suppose now that $\tilde{\mathcal{V}} \subseteq \mathcal{V}$ is open, with inclusion 1-morphism $i_{\tilde{\mathcal{V}}} : \tilde{\mathcal{V}} \to \mathcal{V}$. Write $\tilde{\mathcal{E}} = \mathcal{E}|_{\tilde{\mathcal{V}}} = i_{\tilde{\mathcal{V}}}^*(\mathcal{E})$ and $\tilde{s} = s|_{\tilde{\mathcal{V}}}$. Define $\boldsymbol{i}_{\tilde{\mathcal{V}},\mathcal{V}} = \boldsymbol{S}_{i_{\tilde{\mathcal{V}}},\mathrm{id}_{\tilde{\mathcal{E}}}} : \boldsymbol{S}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}} \to \boldsymbol{S}_{\mathcal{V},\mathcal{E},s}$. If $s^{-1}(0) \subseteq \tilde{\mathcal{V}}$ then $\boldsymbol{i}_{\tilde{\mathcal{V}},\mathcal{V}} : \boldsymbol{S}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}} \to \boldsymbol{S}_{\mathcal{V},\mathcal{E},s}$ is a 1-isomorphism.

We do not define 'standard model' 2-morphisms in $\mathbf{dOrb}$, as in Example 4.6 for d-manifolds, to avoid inconvenience in combining the $O(s), O(s^2)$ notation with 2-morphisms of orbifolds. But see Example 11.9 below for a different form of 'standard model' 2-morphism.

Any d-orbifold $\boldsymbol{\mathcal{X}}$ is locally equivalent near a point $[x]$ to a principal d-orbifold, and so to a standard model d-orbifold $\boldsymbol{S}_{\mathcal{V},\mathcal{E},s}$. The next theorem, the analogue of Theorem 4.7, shows that $\mathcal{V}, \mathcal{E}, s$ are locally determined essentially uniquely if $\dim \mathcal{V}$ is chosen to be minimal (which corresponds to the condition $\mathrm{d}s(v) = 0$).

Theorem 11.6. *Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold, and $[x] \in \mathcal{X}_{\mathrm{top}}$. Then there exists an open neighbourhood $\boldsymbol{\mathcal{U}}$ of $[x]$ in $\boldsymbol{\mathcal{X}}$ and an equivalence $\boldsymbol{\mathcal{U}} \simeq \boldsymbol{S}_{\mathcal{V},\mathcal{E},s}$ in $\mathbf{dOrb}$ for $\boldsymbol{S}_{\mathcal{V},\mathcal{E},s}$ as in Example 11.4, such that the equivalence identifies $[x]$ with $[v] \in \mathcal{V}_{\mathrm{top}}$ with $s(v) = \mathrm{d}s(v) = 0$. Furthermore, $\mathcal{V}, \mathcal{E}, s$ are determined up to non-canonical equivalence near $[v]$ by $\boldsymbol{\mathcal{X}}$ near $[x]$. In fact, they depend only on the C^{∞}-stack $\mathcal{X}$, the point $[x] \in \mathcal{X}_{\mathrm{top}}$, and the representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ on the finite-dimensional vector space $\mathrm{Ker}\big(x^*(\phi_{\mathcal{X}}) : x^*(\mathcal{E}_{\mathcal{X}}) \to x^*(\mathcal{F}_{\mathcal{X}})\big)$.*

In a d-orbifold $\boldsymbol{\mathcal{X}} = (\mathcal{X}, \mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}})$, we think of $\mathcal{X}$ as 'classical' and $\mathcal{O}'_{\mathcal{X}}, \mathcal{E}_{\mathcal{X}}, \imath_{\mathcal{X}}, \jmath_{\mathcal{X}}$ as 'derived'. The extra information in the 'derived' data is like a vector bundle $\mathcal{E}$ over $\mathcal{X}$. A vector bundle $\mathcal{E}$ on a Deligne–Mumford C^{∞}-stack $\mathcal{X}$ is determined locally near $[x] \in \mathcal{X}_{\mathrm{top}}$ by the representation of $\mathrm{Iso}_{\mathcal{X}}([x])$ on the fibre $x^*(\mathcal{E})$ of $\mathcal{E}$ at $[x]$. Thus, it is reasonable that $\boldsymbol{\mathcal{X}}$ should be determined up to equivalence near $[x]$ by $\mathcal{X}$ and a representation of $\mathrm{Iso}_{\mathcal{X}}([x])$.

Here are alternative forms of 'standard model' d-orbifolds, 1-morphisms and 2-morphisms, using the quotient d-stack notation of §10.2.

Example 11.7. Let V be a manifold, $E \to V$ a vector bundle, Γ a finite group acting smoothly on V, E preserving the vector bundle structure, and $s : V \to E$ a smooth, Γ-equivariant section of E. Write the Γ-actions on V, E as $r(\gamma) : V \to V$ and $\hat{r}(\gamma) : E \to r(\gamma)^*(E)$ for $\gamma \in \Gamma$. Then Examples 4.4 and 4.5 give an explicit principal d-manifold $\boldsymbol{S}_{V,E,s}$, and 1-morphisms $\boldsymbol{S}_{r(\gamma),\hat{r}(\gamma)} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{V,E,s}$ for $\gamma \in \Gamma$ which are an action of Γ on $\boldsymbol{S}_{V,E,s}$. Hence Theorem 10.3(i) gives a quotient d-stack $[\boldsymbol{S}_{V,E,s}/\Gamma]$.

In fact $[\boldsymbol{S}_{V,E,s}/\Gamma] \simeq \boldsymbol{\mathcal{S}}_{\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{s}}$ for $\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \tilde{s}$ defined using V, E, s, Γ, with $\tilde{\mathcal{V}} = [\underline{V}/\Gamma]$. Thus, $[\boldsymbol{S}_{V,E,s}/\Gamma]$ is a principal d-orbifold. But not all principal d-orbifolds $\boldsymbol{\mathcal{W}}$ have $\boldsymbol{\mathcal{W}} \simeq [\boldsymbol{S}_{V,E,s}/\Gamma]$, as not all orbifolds $\mathcal{V}$ have $\mathcal{V} \simeq [\underline{V}/\Gamma]$ for some manifold V and finite group Γ.

Example 11.8. Let $[\boldsymbol{S}_{V,E,s}/\Gamma], [\boldsymbol{S}_{W,F,t}/\Delta]$ be quotient d-orbifolds as in Example 11.7, where Γ acts on V, E by $q(\gamma) : V \to V$ and $\hat{q}(\gamma) : E \to q(\gamma)^*(E)$ for $\gamma \in \Gamma$, and Δ acts on W, F by $r(\delta) : W \to W$ and $\hat{r}(\delta) : F \to r(\delta)^*(F)$ for $\delta \in \Delta$. Suppose $f : V \to W$ is a smooth map, and $\hat{f} : E \to f^*(F)$ is a morphism of vector bundles on V satisfying $\hat{f} \circ s = f^*(t) + O(s^2)$, as in (4.2), and $\rho : \Gamma \to \Delta$ is a group morphism satisfying $f \circ q(\gamma) = r(\rho(\gamma)) \circ f : V \to W$ and $q(\gamma)^*(\hat{f}) \circ \hat{q}(\gamma) = f^*(\hat{r}(\rho(\gamma))) \circ \hat{f} : E \to (f \circ q(\gamma))^*(F)$ for all $\gamma \in \Gamma$, so that $f, \hat{f}$ are equivariant under Γ, Δ, ρ. Then Example 4.4 defines a 1-morphism $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{V,E,s} \to \boldsymbol{S}_{W,F,t}$ in $\mathbf{dMan}$. The equivariance conditions on $f, \hat{f}$ imply that $\boldsymbol{S}_{f,\hat{f}} \circ \boldsymbol{S}_{q(\gamma),\hat{q}(\gamma)} = \boldsymbol{S}_{r(\rho(\gamma)),\hat{r}(\rho(\gamma))} \circ \boldsymbol{S}_{f,\hat{f}}$ for $\gamma \in \Gamma$. Hence Theorem 10.3(ii) gives a quotient 1-morphism $[\boldsymbol{S}_{f,\hat{f}}, \rho] : [\boldsymbol{S}_{V,E,s}/\Gamma] \to [\boldsymbol{S}_{W,F,t}/\Delta]$.

Example 11.9. Suppose $[\boldsymbol{S}_{f,\hat{f}}, \rho], [\boldsymbol{S}_{g,\hat{g}}, \sigma] : [\boldsymbol{S}_{V,E,s}/\Gamma] \to [\boldsymbol{S}_{W,F,t}/\Delta]$ are two 1-morphisms as in Example 11.8, and write $q, \hat{q}$ for the actions of Γ on V, E and $r, \hat{r}$ for the actions of Δ on W, F. Then $\rho, \sigma : \Gamma \to \Delta$ are group morphisms. Suppose $\delta \in \Delta$ satisfies $\sigma(\gamma) = \delta \, \rho(\gamma) \, \delta^{-1}$ for all $\gamma \in \Gamma$, and $\Lambda : E \to f^*(TW)$ is a morphism of vector bundles on V which satisfies

$$r(\delta^{-1}) \circ g = f + \Lambda \cdot s + O(s^2) \text{ and } g^*(\hat{r}(\delta^{-1})) \circ \hat{g} = \hat{f} + \Lambda \cdot \mathrm{d}t + O(s), \quad (11.2)$$

$$f^*(\mathrm{d}r(\rho(\gamma))) \circ \Lambda = q(\gamma)^*(\Lambda) \circ \hat{q}(\gamma) : E \longrightarrow (f \circ q(\gamma))^*(TW), \quad \forall \gamma \in \Gamma, \quad (11.3)$$

where $\mathrm{d}r(\rho(\gamma)) : TW \to r(\rho(\gamma))^*(TW)$ is the derivative of $r(\rho(\gamma))$. Here (11.2) is the conditions for Example 4.6 to define a 'standard model' 2-morphism $\boldsymbol{S}_\Lambda : \boldsymbol{S}_{f,\hat{f}} \Rightarrow \boldsymbol{S}_{r(\delta^{-1})\circ g, g^*(\hat{r}(\delta^{-1}))\circ \hat{g}} = \boldsymbol{S}_{r(\delta^{-1}),\hat{r}(\delta^{-1})} \circ \boldsymbol{S}_{g,\hat{g}}$ in $\mathbf{dMan}$. Then (11.3) implies that $\boldsymbol{S}_\Lambda * \mathrm{id}_{\boldsymbol{S}_{q(\gamma),\hat{q}(\gamma)}} = \mathrm{id}_{\boldsymbol{S}_{r(\rho(\gamma)),\hat{r}(\rho(\gamma))}} * \boldsymbol{S}_\Lambda$ for all $\gamma \in \Gamma$. Hence Theorem 10.3(iii) gives a quotient 2-morphism $[\boldsymbol{S}_\Lambda, \delta] : [\boldsymbol{S}_{f,\hat{f}}, \rho] \Rightarrow [\boldsymbol{S}_{g,\hat{g}}, \sigma]$.

Here is an analogue of Theorem 11.6 for the alternative form $[\boldsymbol{S}_{V,E,s}/\Gamma]$.

Proposition 11.10. *A d-stack $\boldsymbol{\mathcal{X}}$ is a d-orbifold of virtual dimension $n \in \mathbb{Z}$ if and only if each $[x] \in \mathcal{X}_{\mathrm{top}}$ has an open neighbourhood $\boldsymbol{\mathcal{U}}$ equivalent to some $[\boldsymbol{S}_{V,E,s}/\Gamma]$ in Example 11.7 with $\dim V - \mathrm{rank}\, E = n$, where $\Gamma = \mathrm{Iso}_{\mathcal{X}}([x])$ and $[x] \in \mathcal{X}_{\mathrm{top}}$ is identified with a fixed point v of Γ in V with $s(v) = 0$ and $\mathrm{d}s(v) = 0$. Furthermore, V, E, s, Γ are determined up to non-canonical isomorphism near v by $\boldsymbol{\mathcal{X}}$ near $[x]$.*

11.3 Equivalences in dOrb, and gluing by equivalences

Next we summarize the results of [35, §10.2], the analogue of §4.4. Section 10.2 discussed étale 1-morphisms in **dSta**. We characterize when 1-morphisms $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ and $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{S}_{\mathcal{W},\mathcal{F},t}$ in **dOrb** are étale, or equivalences.

Theorem 11.11. *Suppose* $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ *is a* 1-*morphism of d-orbifolds, and* $f : \mathcal{X} \to \mathcal{Y}$ *is representable. Then the following are equivalent:*

(i) $\boldsymbol{f}$ *is étale;*

(ii) $\Omega_{\boldsymbol{f}} : f^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$ *is an equivalence in* $\mathrm{vqcoh}(\mathcal{X})$*; and*

(iii) *The following is a split short exact sequence in* $\mathrm{qcoh}(\mathcal{X})$*:*

$$0 \longrightarrow f^*(\mathcal{E}_\mathcal{Y}) \xrightarrow{f'' \oplus -f^*(\phi_\mathcal{Y})} \mathcal{E}_\mathcal{X} \oplus f^*(\mathcal{F}_\mathcal{Y}) \xrightarrow{\phi_\mathcal{X} \oplus f^2} \mathcal{F}_\mathcal{X} \longrightarrow 0. \quad (11.4)$$

If in addition $f_* : \mathrm{Iso}_\mathcal{X}([x]) \to \mathrm{Iso}_\mathcal{Y}(f_{\mathrm{top}}([x]))$ *is an isomorphism for all* $[x] \in \mathcal{X}_{\mathrm{top}}$*, and* $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ *is a bijection, then* $\boldsymbol{f}$ *is an equivalence in* **dOrb**.

Theorem 11.12. *Suppose* $\boldsymbol{S}_{f,\hat{f}} : \boldsymbol{S}_{\mathcal{V},\mathcal{E},s} \to \boldsymbol{S}_{\mathcal{W},\mathcal{F},t}$ *is a 'standard model' 1-morphism, in the notation of Examples* 11.4 *and* 11.5*, with* $f : \mathcal{V} \to \mathcal{W}$ *representable. Then* $\boldsymbol{S}_{f,\hat{f}}$ *is étale if and only if for each* $[v] \in \mathcal{V}_{\mathrm{top}}$ *with* $s(v) = 0$ *and* $[w] = f_{\mathrm{top}}([v]) \in \mathcal{W}_{\mathrm{top}}$*, the following sequence of vector spaces is exact:*

$$0 \longrightarrow T_v\mathcal{V} \xrightarrow{\mathrm{d}s(v) \oplus \mathrm{d}f(v)} \mathcal{E}_v \oplus T_w\mathcal{W} \xrightarrow{\hat{f}(v) \oplus -\mathrm{d}t(w)} \mathcal{F}_w \longrightarrow 0.$$

Also $\boldsymbol{S}_{f,\hat{f}}$ *is an equivalence if and only if in addition* $f_{\mathrm{top}}|_{s^{-1}(0)} : s^{-1}(0) \to t^{-1}(0)$ *is a bijection, where* $s^{-1}(0) = \{[v] \in \mathcal{V}_{\mathrm{top}} : s(v) = 0\}$*,* $t^{-1}(0) = \{[w] \in \mathcal{W}_{\mathrm{top}} : t(w) = 0\}$*, and* $f_* : \mathrm{Iso}_\mathcal{V}([v]) \to \mathrm{Iso}_\mathcal{W}(f_{\mathrm{top}}([v]))$ *is an isomorphism for all* $[v] \in s^{-1}(0) \subseteq \mathcal{V}_{\mathrm{top}}$*.*

Here is an analogue of Theorem 4.17 for d-orbifolds, taken from [35, §10.2]. It is proved by applying Theorem 10.8 to glue together the 'standard model' d-orbifolds $\boldsymbol{S}_{\mathcal{V}_i,\mathcal{E}_i,s_i}$ by equivalences. Now Theorem 10.8 includes extra conditions (10.3)–(10.4) on the 2-morphisms η_{ijk}, ζ_{jk}. But by taking the $\mathcal{V}_i, \mathcal{Y}$ to be effective orbifolds and the g_i to be submersions, the η_{ijk}, ζ_{jk} are unique by Proposition 9.5, and so (10.3)–(10.4) hold automatically.

Theorem 11.13. *Suppose we are given the following data:*

(a) *an integer* n*;*

(b) *a Hausdorff, second countable topological space* X*;*

(c) *an indexing set* I*, and a total order* $<$ *on* I*;*

(d) *for each* i *in* I*, an effective orbifold* $\mathcal{V}_i$ *in the sense of Definition* 9.4*, a vector bundle* $\mathcal{E}_i$ *on* $\mathcal{V}_i$ *with* $\dim \mathcal{V}_i - \mathrm{rank}\,\mathcal{E}_i = n$*, a section* $s_i \in C^\infty(\mathcal{E}_i)$*, and a homeomorphism* $\psi_i : s_i^{-1}(0) \to \hat{X}_i$*, where* $s_i^{-1}(0) = \{[v_i] \in \mathcal{V}_{i,\mathrm{top}} : s_i(v_i) = 0\}$ *and* $\hat{X}_i \subseteq X$ *is open; and*

(e) *for all* $i < j$ *in* I*, an open suborbifold* $\mathcal{V}_{ij} \subseteq \mathcal{V}_i$*, a 1-morphism* $e_{ij} : \mathcal{V}_{ij} \to \mathcal{V}_j$*, and a morphism of vector bundles* $\hat{e}_{ij} : \mathcal{E}_i|_{\mathcal{V}_{ij}} \to e_{ij}^*(\mathcal{E}_j)$*.*

Let this data satisfy the conditions:

(i) $X = \bigcup_{i \in I} \hat{X}_i$;

(ii) *if $i < j$ in I then $(e_{ij})_* : \mathrm{Iso}_{\mathcal{V}_{ij}}([v]) \to \mathrm{Iso}_{\mathcal{V}_j}(e_{ij,\mathrm{top}}([v]))$ is an isomorphism for all $[v] \in \mathcal{V}_{ij,\mathrm{top}}$, and $\hat{e}_{ij} \circ s_i|_{\mathcal{V}_{ij}} = e_{ij}^*(s_j) \circ \iota_{ij}$ where $\iota_{ij} : \mathcal{O}_{\mathcal{V}_{ij}} \to e_{ij}^*(\mathcal{O}_{\mathcal{V}_j})$ is the natural isomorphism, and $\psi_i(s_i|_{\mathcal{V}_{ij}}^{-1}(0)) = \hat{X}_i \cap \hat{X}_j$, and $\psi_i|_{s_i|_{\mathcal{V}_{ij}}^{-1}(0)} = \psi_j \circ e_{ij,\mathrm{top}}|_{s_i|_{\mathcal{V}_{ij}}^{-1}(0)}$, and if $[v_i] \in \mathcal{V}_{ij,\mathrm{top}}$ with $s_i(v_i) = 0$ and $[v_j] = e_{ij,\mathrm{top}}([v_i])$ then the following sequence is exact:*

$$0 \longrightarrow T_{v_i}\mathcal{V}_i \xrightarrow{\mathrm{d}s_i(v_i) \oplus \mathrm{d}e_{ij}(v_i)} \mathcal{E}_i|_{v_i} \oplus T_{v_j}\mathcal{V}_j \xrightarrow{\hat{e}_{ij}(v_i) \oplus -\mathrm{d}s_j(v_j)} \mathcal{E}_j|_{v_j} \longrightarrow 0;$$

(iii) *if $i < j < k$ in I then there exists a 2-morphism $\eta_{ijk} : e_{jk} \circ e_{ij}|_{\mathcal{V}_{ik} \cap e_{ij}^{-1}(\mathcal{V}_{jk})} \Rightarrow e_{ik}|_{\mathcal{V}_{ik} \cap e_{ij}^{-1}(\mathcal{V}_{jk})}$ in **Orb** with*

$$\hat{e}_{ik}|_{\mathcal{V}_{ik} \cap e_{ij}^{-1}(\mathcal{V}_{jk})} = \eta_{ijk}^*(\mathcal{E}_k) \circ I_{e_{ij},e_{jk}}(\mathcal{E}_k)^{-1} \circ e_{ij}|_{\mathcal{V}_{ik} \cap e_{ij}^{-1}(\mathcal{V}_{jk})}^*(\hat{e}_{jk}) \circ \hat{e}_{ij}|_{\mathcal{V}_{ik} \cap e_{ij}^{-1}(\mathcal{V}_{jk})}.$$

Note that η_{ijk} is unique by Proposition 9.5.

*Then there exist a d-orbifold $\boldsymbol{\mathcal{X}}$ with $\mathrm{vdim}\,\boldsymbol{\mathcal{X}} = n$ and underlying topological space $\boldsymbol{\mathcal{X}}_{\mathrm{top}} \cong X$, and a 1-morphism $\boldsymbol{\psi}_i : \boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i} \to \boldsymbol{\mathcal{X}}$ with underlying continuous map ψ_i which is an equivalence with the open d-suborbifold $\hat{\boldsymbol{\mathcal{X}}}_i \subseteq \boldsymbol{\mathcal{X}}$ corresponding to $\hat{X}_i \subseteq X$ for all $i \in I$, such that for all $i < j$ in I there exists a 2-morphism $\boldsymbol{\eta}_{ij} : \boldsymbol{\psi}_j \circ \boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}} \Rightarrow \boldsymbol{\psi}_i \circ \boldsymbol{i}_{\mathcal{V}_{ij},\mathcal{V}_i}$, where $\boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}} : \boldsymbol{\mathcal{S}}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},s_i|_{\mathcal{V}_{ij}}} \to \boldsymbol{\mathcal{S}}_{\mathcal{V}_j,\mathcal{E}_j,s_j}$ and $\boldsymbol{i}_{\mathcal{V}_{ij},\mathcal{V}_i} : \boldsymbol{\mathcal{S}}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},s_i|_{\mathcal{V}_{ij}}} \to \boldsymbol{\mathcal{S}}_{\mathcal{V}_i,\mathcal{E}_i,s_i}$, using the notation of Examples 11.4 and 11.5. This d-orbifold $\boldsymbol{\mathcal{X}}$ is unique up to equivalence in **dOrb**.*

*Suppose also that $\mathcal{Y}$ is an effective orbifold, and $g_i : \mathcal{V}_i \to \mathcal{Y}$ are submersions for all $i \in I$, and there are 2-morphisms $\zeta_{ij} : g_j \circ e_{ij} \Rightarrow g_i|_{\mathcal{V}_{ij}}$ in **Orb** for all $i < j$ in I. Then there exist a 1-morphism $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in **dOrb** unique up to 2-isomorphism, where $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y}) = \boldsymbol{\mathcal{S}}_{\mathcal{Y},0,0}$, and 2-morphisms $\boldsymbol{\zeta}_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow \boldsymbol{\mathcal{S}}_{g_i,0}$ for all $i \in I$.*

Here is another version of the same result using the alternative form of 'standard model' d-orbifolds in §11.1.

Theorem 11.14. *Suppose we are given the following data:*

(a) *an integer n;*

(b) *a Hausdorff, second countable topological space X;*

(c) *an indexing set I, and a total order $<$ on I;*

(d) *for each i in I, a manifold V_i, a vector bundle $E_i \to V_i$ with $\dim V_i - \mathrm{rank}\,E_i = n$, a finite group Γ_i, smooth, locally effective actions $r_i(\gamma) : V_i \to V_i$, $\hat{r}_i(\gamma) : E_i \to r(\gamma)^*(E_i)$ of Γ_i on V_i, E_i for $\gamma \in \Gamma_i$, a smooth, Γ_i-equivariant section $s_i : V_i \to E_i$, and a homeomorphism $\psi_i : X_i \to \hat{X}_i$, where $X_i = \{v_i \in V_i : s_i(v_i) = 0\}/\Gamma_i$ and $\hat{X}_i \subseteq X$ is an open set; and*

(e) *for all $i < j$ in I, an open submanifold $V_{ij} \subseteq V_i$, invariant under Γ_i, a group morphism $\rho_{ij} : \Gamma_i \to \Gamma_j$, a smooth map $e_{ij} : V_{ij} \to V_j$, and a morphism of vector bundles $\hat{e}_{ij} : E_i|_{V_{ij}} \to e_{ij}^*(E_j)$.*

Let this data satisfy the conditions:

(i) $X = \bigcup_{i \in I} \hat{X}_i$;

(ii) *if $i < j$ in I then $\hat{e}_{ij} \circ s_i|_{V_{ij}} = e_{ij}^*(s_j) + O(s_i^2)$, and for all $\gamma \in \Gamma$ we have*

$$e_{ij} \circ r_i(\gamma) = r_j(\rho_{ij}(\gamma)) \circ e_{ij} : V_{ij} \longrightarrow V_j,$$
$$r_i(\gamma)^*(\hat{e}_{ij}) \circ \hat{r}_i(\gamma) = e_{ij}^*(\hat{r}_j(\rho_{ij}(\gamma))) \circ \hat{e}_{ij} : E_i|_{V_{ij}} \longrightarrow (e_{ij} \circ r_i(\gamma))^*(E_j),$$

and $\psi_i(X_i \cap (V_{ij}/\Gamma_i)) = \hat{X}_i \cap \hat{X}_j$, and $\psi_i|_{X_i \cap V_{ij}/\Gamma_i} = \psi_j \circ (e_{ij})_*|_{X_i \cap V_{ij}/\Gamma_j}$, and if $v_i \in V_{ij}$ with $s_i(v_i) = 0$ and $v_j = e_{ij}(v_i)$ then $\rho|_{\mathrm{Stab}_{\Gamma_i}(v_i)} : \mathrm{Stab}_{\Gamma_i}(v_i) \to \mathrm{Stab}_{\Gamma_j}(v_j)$ is an isomorphism, and the following sequence of vector spaces is exact:

$$0 \longrightarrow T_{v_i} V_i \xrightarrow{\mathrm{d}s_i(v_i) \oplus \mathrm{d}e_{ij}(v_i)} E_i|_{v_i} \oplus T_{v_j} V_j \xrightarrow{\hat{e}_{ij}(v_i) \oplus -\mathrm{d}s_j(v_j)} E_j|_{v_j} \longrightarrow 0;$$

(iii) *if $i < j < k$ in I then there exists $\gamma_{ijk} \in \Gamma_k$ satisfying*

$$\rho_{ik}(\gamma) = \gamma_{ijk}\, \rho_{jk}(\rho_{ij}(\gamma))\, \gamma_{ijk}^{-1} \quad \text{for all } \gamma \in \Gamma_i,$$
$$e_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} = r_k(\gamma_{ijk}) \circ e_{jk} \circ e_{ij}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})}, \quad \text{and}$$
$$\hat{e}_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} = \left(e_{ij}^*(e_{jk}^*(\hat{r}_k(\gamma_{ijk}))) \circ e_{ij}^*(\hat{e}_{jk}) \circ \hat{e}_{ij}\right)|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})}.$$

Then there exist a d-orbifold $\boldsymbol{\mathcal{X}}$ with $\mathrm{vdim}\,\boldsymbol{\mathcal{X}} = n$ and underlying topological space $\mathcal{X}_{\mathrm{top}} \cong X$, and a 1-morphism $\boldsymbol{\psi}_i : [\boldsymbol{S}_{V_i, E_i, s_i}/\Gamma_i] \to \boldsymbol{\mathcal{X}}$ with underlying continuous map ψ_i which is an equivalence with the open d-suborbifold $\hat{\boldsymbol{\mathcal{X}}}_i \subseteq \boldsymbol{\mathcal{X}}$ corresponding to $\hat{X}_i \subseteq X$ for all $i \in I$, such that for all $i < j$ in I there exists a 2-morphism $\boldsymbol{\eta}_{ij} : \boldsymbol{\psi}_j \circ [\boldsymbol{S}_{e_{ij}, \hat{e}_{ij}}, \rho_{ij}] \Rightarrow \boldsymbol{\psi}_i \circ [\boldsymbol{i}_{V_{ij}, V_i}, \mathrm{id}_{\Gamma_i}]$, where $[\boldsymbol{S}_{V_i, E_i, s_i}/\Gamma_i]$ is as in Example 11.7, and $[\boldsymbol{S}_{e_{ij}, \hat{e}_{ij}}, \rho_{ij}] : [\boldsymbol{S}_{V_{ij}, E_i|_{V_{ij}}, s_i|_{V_{ij}}}/\Gamma_i] \to [\boldsymbol{S}_{V_j, E_j, s_j}/\Gamma_j]$ and $[\boldsymbol{i}_{V_{ij}, V_i}, \mathrm{id}_{\Gamma_i}] : [\boldsymbol{S}_{V_{ij}, E_i|_{V_{ij}}, s_i|_{V_{ij}}}/\Gamma_i] \to [\boldsymbol{S}_{V_i, E_i, s_i}/\Gamma_i]$ as in Example 11.8. This d-orbifold $\boldsymbol{\mathcal{X}}$ is unique up to equivalence in $\mathbf{dOrb}$.

Suppose also that Y is a manifold, and $g_i : V_i \to Y$ are smooth maps for all $i \in I$ with $g_i \circ r_i(\gamma) = g_i$ for all $\gamma \in \Gamma_i$, and $g_j \circ e_{ij} = g_i|_{V_{ij}}$ for all $i < j$ in I. Then there exist a 1-morphism $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ unique up to 2-isomorphism, where $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Man}}^{\mathbf{dOrb}}(Y) = [\boldsymbol{S}_{Y,0,0}/\{1\}]$, and 2-morphisms $\boldsymbol{\zeta}_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow [\boldsymbol{S}_{g_i,0}, \pi_{\{1\}}]$ for all $i \in I$. Here $[\boldsymbol{S}_{Y,0,0}/\{1\}]$ is from Example 11.7 with E, s both zero and $\Gamma = \{1\}$, and $[\boldsymbol{S}_{g_i,0}, \pi_{\{1\}}] : [\boldsymbol{S}_{V_i, E_i, s_i}/\Gamma_i] \to [\boldsymbol{S}_{Y,0,0}/\{1\}] = \boldsymbol{\mathcal{Y}}$ is from Example 11.8 with $\hat{g}_i = 0$ and $\rho = \pi_{\{1\}} : \Gamma_i \to \{1\}$.

The importance of Theorems 11.13 and 11.14 is that all the ingredients are described wholly in differential-geometric or topological terms. So we can use these theorems as tools to prove the existence of d-orbifold structures on spaces coming from other areas of geometry, such as moduli spaces of J-holomorphic curves. The theorems are used to define functors to d-orbifolds from other geometric structures, as discussed in §16.

11.4 Submersions, immersions, and embeddings

Section 4.5 discussed (w-)submersions, (w-)immersions, and (w-)embeddings for d-manifolds. Following [35, §10.3], here are the analogues for d-orbifolds.

Definition 11.15. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack, so that as in Remark 11.1 we have a 2-category $\mathrm{vvect}(\mathcal{X})$ of virtual vector bundles on $\mathcal{X}$. We define when a 1-morphism $f^\bullet : (\mathcal{E}^\bullet, \phi) \to (\mathcal{F}^\bullet, \psi)$ in $\mathrm{vvect}(\mathcal{X})$ is *weakly injective*, *injective*, *weakly surjective* or *surjective* exactly as in Definition 4.18.

Let $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be a 1-morphism of d-orbifolds. Then $\Omega_{\boldsymbol{f}} : f^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$ is a 1-morphism in $\mathrm{vvect}(\mathcal{X})$.

(a) We call $\boldsymbol{f}$ a *w-submersion* if $\Omega_{\boldsymbol{f}}$ is weakly injective.

(b) We call $\boldsymbol{f}$ a *submersion* if $\Omega_{\boldsymbol{f}}$ is injective.

(c) We call $\boldsymbol{f}$ a *w-immersion* if $f : \mathcal{X} \to \mathcal{Y}$ is representable, i.e. $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}(f_{\mathrm{top}}([x]))$ is injective for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $\Omega_{\boldsymbol{f}}$ is weakly surjective.

(d) We call $\boldsymbol{f}$ an *immersion* if $f : \mathcal{X} \to \mathcal{Y}$ is representable and $\Omega_{\boldsymbol{f}}$ is surjective.

(e) We call $\boldsymbol{f}$ a *w-embedding* or *embedding* if it is a w-immersion or immersion, respectively, and $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}(f_{\mathrm{top}}([x]))$ is an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ is a homeomorphism with its image, so in particular f_{top} is injective.

Parts (c)–(e) enable us to define *d-suborbifolds* of d-orbifolds. *Open d-suborbifolds* are (Zariski) open d-substacks of a d-orbifold. For more general d-suborbifolds, we call $\boldsymbol{i} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ a *w-immersed d-suborbifold*, or *immersed d-suborbifold*, or *w-embedded d-suborbifold*, or *embedded d-suborbifold* of $\boldsymbol{\mathcal{Y}}$, if $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ are d-orbifolds and $\boldsymbol{i}$ is a w-immersion, ..., embedding, respectively.

Theorem 4.20 in §4.5 holds with orbifolds and d-orbifolds in place of manifolds and d-manifolds, except part (v), when we need also to assume $f : \mathcal{X} \to \mathcal{Y}$ representable to deduce $\boldsymbol{f}$ is étale, and part (x), which is false for d-orbifolds (in the Zariski topology, at least).

11.5 D-transversality and fibre products

Section 4.6 discussed d-transversality and fibre products for d-manifolds. This is extended to d-orbifolds in [35, §10.4], with little essential change. Here are the analogues of Definition 4.21 and Theorems 4.22–4.25.

Definition 11.16. Let $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ be d-orbifolds and $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}, \boldsymbol{h} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ be 1-morphisms. Let $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{X}} \times_{g,\mathcal{Z},h} \boldsymbol{\mathcal{Y}}$ be the C^∞-stack fibre product, and write $e : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{X}}$, $f : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{Y}}$ for the projection 1-morphisms, and $\eta : g \circ e \Rightarrow h \circ f$ for the 2-morphism from the fibre product. Consider the morphism

$$\alpha = \begin{pmatrix} e^*(g'') \circ I_{e,g}(\mathcal{E}_{\mathcal{Z}}) \\ -f^*(h'') \circ I_{f,h}(\mathcal{E}_{\mathcal{Z}}) \circ \eta^*(\mathcal{E}_{\mathcal{Z}}) \\ (g \circ e)^*(\phi_{\mathcal{Z}}) \end{pmatrix} : (g \circ e)^*(\mathcal{E}_{\mathcal{Z}}) \longrightarrow \\ e^*(\mathcal{E}_{\mathcal{X}}) \oplus f^*(\mathcal{E}_{\mathcal{Y}}) \oplus (g \circ e)^*(\mathcal{F}_{\mathcal{Z}})$$

in $\mathrm{qcoh}(\mathcal{W})$. We call $\boldsymbol{g}, \boldsymbol{h}$ *d-transverse* if α has a left inverse.

Theorem 11.17. *Suppose* $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ *are d-orbifolds and* $g : \mathcal{X} \to \mathcal{Z}, h : \mathcal{Y} \to \mathcal{Z}$ *are d-transverse 1-morphisms, and let* $\mathcal{W} = \mathcal{X} \times_{g,\mathcal{Z},h} \mathcal{Y}$ *be the d-stack fibre product, which exists by Theorem* 10.10(a). *Then* $\mathcal{W}$ *is a d-orbifold, with*

$$\mathrm{vdim}\,\mathcal{W} = \mathrm{vdim}\,\mathcal{X} + \mathrm{vdim}\,\mathcal{Y} - \mathrm{vdim}\,\mathcal{Z}. \qquad (11.5)$$

Theorem 11.18. *Suppose* $g : \mathcal{X} \to \mathcal{Z}, h : \mathcal{Y} \to \mathcal{Z}$ *are 1-morphisms of d-orbifolds. The following are sufficient conditions for* g, h *to be d-transverse, so that* $\mathcal{W} = \mathcal{X} \times_{g,\mathcal{Z},h} \mathcal{Y}$ *is a d-orbifold of virtual dimension* (11.5):

  (a) $\mathcal{Z}$ *is an orbifold, that is,* $\mathcal{Z} \in \hat{\mathbf{Orb}}$; *or*

  (b) g *or* h *is a w-submersion.*

Theorem 11.19. *Let* $\mathcal{X}, \mathcal{Z}$ *be d-orbifolds,* $\mathcal{Y}$ *an orbifold, and* $g : \mathcal{X} \to \mathcal{Z}$, $h : \mathcal{Y} \to \mathcal{Z}$ *be 1-morphisms with* g *a submersion. Then* $\mathcal{W} = \mathcal{X} \times_{g,\mathcal{Z},h} \mathcal{Y}$ *is an orbifold, with* $\dim \mathcal{W} = \mathrm{vdim}\,\mathcal{X} + \dim \mathcal{Y} - \mathrm{vdim}\,\mathcal{Z}$.

Theorem 11.20. (i) *Let* $\rho : G \to H$ *be a morphism of finite groups, and* H *act linearly on* $\mathbb{R}^n$. *Then as in* §10.2 *we have quotient d-orbifolds* $[*/G]$, $[\mathbb{R}^n/H]$ *and a quotient 1-morphism* $[\mathbf{0}, \rho] : [*/G] \to [\mathbb{R}^n/H]$. *Suppose* $\mathcal{X}$ *is a d-orbifold and* $g : \mathcal{X} \to [\mathbb{R}^n/H]$ *a 1-morphism in* $\mathbf{dOrb}$. *Then the fibre product* $\mathcal{W} = \mathcal{X} \times_{g,[\mathbb{R}^n/H],[\mathbf{0},\rho]} [*/G]$ *exists in* $\mathbf{dOrb}$ *by Theorem* 11.18(a). *The projection* $\boldsymbol{\pi}_{\mathcal{X}} : \mathcal{W} \to \mathcal{X}$ *is an immersion if* ρ *is injective, and an embedding if* ρ *is an isomorphism.*

(ii) *Suppose* $\boldsymbol{f} : \mathcal{X} \to \mathcal{Y}$ *is an immersion of d-orbifolds, and* $[x] \in \mathcal{X}_{\mathrm{top}}$ *with* $f_{\mathrm{top}}([x]) = [y] \in \mathcal{Y}_{\mathrm{top}}$. *Write* $\rho : G \to H$ *for* $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}([y])$. *Then* ρ *is injective, and there exist open neighbourhoods* $\mathcal{U} \subseteq \mathcal{X}$ *and* $\mathcal{V} \subseteq \mathcal{Y}$ *of* $[x], [y]$ *with* $\boldsymbol{f}(\mathcal{U}) \subseteq \mathcal{V}$, *a linear action of* H *on* $\mathbb{R}^n$ *where* $n = \mathrm{vdim}\,\mathcal{Y} - \mathrm{vdim}\,\mathcal{X} \geqslant 0$, *and a 1-morphism* $\boldsymbol{g} : \mathcal{V} \to [\mathbb{R}^n/H]$ *with* $g_{\mathrm{top}}([y]) = [0]$, *fitting into a 2-Cartesian square in* $\mathbf{dOrb}$:

If $\boldsymbol{f}$ *is an embedding then* ρ *is an isomorphism, and we may take* $\mathcal{U} = \boldsymbol{f}^{-1}(\mathcal{V})$.

11.6   Embedding d-orbifolds into orbifolds

Section 4.7 discussed embeddings of d-manifolds into manifolds. Theorem 4.29 gave necessary and sufficient conditions for the existence of embeddings $\boldsymbol{f} :$ $\boldsymbol{X} \to \mathbb{R}^n$ for any d-manifold $\boldsymbol{X}$, and Theorem 4.32 showed that if a d-manifold $\boldsymbol{X}$ has an embedding $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ for a manifold Y then $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$ for open $f(X) \subset V \subseteq Y$. Combining these proves that large classes of d-manifolds — all compact d-manifolds, for instance — are principal d-manifolds.

    In [35, §10.5] we consider how to generalize all this to d-orbifolds. The proof of Theorem 4.32 extends to (d-)orbifolds, giving:

Theorem 11.21. *Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold, $\mathcal{Y}$ an orbifold, and $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ an embedding, in the sense of Definition* 11.15. *Then there exist an open suborbifold $\mathcal{V} \subseteq \mathcal{Y}$ with $\boldsymbol{f}(\boldsymbol{\mathcal{X}}) \subseteq \mathcal{V}$, a vector bundle $\mathcal{E}$ on $\mathcal{V}$, and a smooth section $s \in C^\infty(\mathcal{E})$ fitting into a 2-Cartesian diagram in* **dOrb**, *where $\boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{E}}, \boldsymbol{s}, \boldsymbol{0} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}\big(\mathcal{Y}, \mathcal{V}, \mathrm{Tot}(\mathcal{E}), \mathrm{Tot}(s), \mathrm{Tot}(0)\big)$, in the notation of* §9.1:

$$
\begin{array}{ccc}
\boldsymbol{\mathcal{X}} & \xrightarrow{\quad\quad} & \boldsymbol{\mathcal{V}} \\
{\scriptstyle\boldsymbol{f}}\big\downarrow \quad {\scriptstyle f} & \Nearrow & {\scriptstyle\boldsymbol{0}}\big\downarrow \\
\boldsymbol{\mathcal{V}} & \xrightarrow[\quad s \quad]{} & \boldsymbol{\mathcal{E}}.
\end{array}
$$

Hence $\boldsymbol{\mathcal{X}}$ is equivalent to the 'standard model' d-orbifold $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ of Example 11.4, *and is a principal d-orbifold.*

However, we do not presently have a good analogue of Theorem 4.29 for d-orbifolds, so we cannot state useful necessary and sufficient conditions for when a d-orbifold $\boldsymbol{\mathcal{X}}$ can be embedded into an orbifold, or is a principal d-orbifold.

11.7 Orientations of d-orbifolds

Section 4.8 discusses orientations on d-manifolds. As in [35, §10.6], all this material generalizes easily to d-orbifolds, so we will give few details.

If $\mathcal{X}$ is a Deligne–Mumford C^∞-stack and $(\mathcal{E}^\bullet, \phi)$ a virtual vector bundle on $\mathcal{X}$, then we define a line bundle $\mathcal{L}_{(\mathcal{E}^\bullet, \phi)}$ on $\mathcal{X}$ called the *orientation line bundle* of $(\mathcal{E}^\bullet, \phi)$. It has functorial properties as in Theorem 4.34(a)–(f). If $\boldsymbol{\mathcal{X}}$ is a d-orbifold, the virtual cotangent bundle $T^*\boldsymbol{\mathcal{X}} = (\mathcal{E}_\mathcal{X}, \mathcal{F}_\mathcal{X}, \phi_\mathcal{X})$ is a virtual vector bundle on $\boldsymbol{\mathcal{X}}$. We define an *orientation* ω on $\boldsymbol{\mathcal{X}}$ to be an orientation on the orientation line bundle $\mathcal{L}_{T^*\boldsymbol{\mathcal{X}}}$. The analogues of Theorem 4.37 and Proposition 4.38 hold for d-orbifolds.

One difference between (d-)manifolds and (d-)orbifolds is that line bundles $\mathcal{L}$ on Deligne–Mumford C^∞-stacks $\mathcal{X}$ (such as orientation line bundles) need only be locally trivial in the étale topology, not in the Zariski topology. Because of this, orbifolds and d-orbifolds need not be (Zariski) locally orientable. For example, the orbifold $[\underline{\mathbb{R}}^{2n+1}/\{\pm 1\}]$ is not locally orientable near 0.

11.8 Orbifold strata of d-orbifolds

Section 8.7 discussed the *orbifold strata* $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \ldots, \hat{\mathcal{X}}_\circ^\Gamma$ of a Deligne–Mumford C^∞-stack $\mathcal{X}$. When $\mathcal{X}$ is an orbifold, §9.2 explained that $\mathcal{X}^\Gamma$ decomposes as $\mathcal{X}^\Gamma = \coprod_{\lambda \in \Lambda_+^\Gamma} \mathcal{X}^{\Gamma,\lambda}$, where each $\mathcal{X}^{\Gamma,\lambda}$ is an orbifold of dimension $\dim \mathcal{X} - \dim \lambda$, and similarly for $\tilde{\mathcal{X}}^\Gamma, \ldots, \hat{\mathcal{X}}_\circ^\Gamma$. Section 10.5 discussed the orbifold strata $\boldsymbol{\mathcal{X}}^\Gamma, \tilde{\boldsymbol{\mathcal{X}}}^\Gamma, \ldots, \hat{\boldsymbol{\mathcal{X}}}_\circ^\Gamma$ of a d-stack $\boldsymbol{\mathcal{X}}$. In [35, §10.7] we show that for a d-orbifold $\boldsymbol{\mathcal{X}}$, the orbifold strata decompose as $\boldsymbol{\mathcal{X}}^\Gamma = \coprod_{\lambda \in \Lambda^\Gamma} \boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$, where $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ is a d-orbifold of virtual dimension $\mathrm{vdim}\, \boldsymbol{\mathcal{X}} - \dim \lambda$, and similarly for $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma, \ldots, \hat{\boldsymbol{\mathcal{X}}}_\circ^\Gamma$.

Definition 11.22. Let Γ be a finite group, and use the notation $\mathrm{Rep}_{\mathrm{nt}}(\Gamma)$, $\Lambda^\Gamma = K_0\big(\mathrm{Rep}_{\mathrm{nt}}(\Gamma)\big)$, $\Lambda_+^\Gamma \subseteq \Lambda^\Gamma$ and $\dim : \Lambda^\Gamma \to \mathbb{Z}$ of Definition 9.7. Let

$R_0, R_1, \ldots, R_k$ be the irreducible Γ-representations up to isomorphism, with $R_0 = \mathbb{R}$ the trivial representation, so that $\Lambda^\Gamma \cong \mathbb{Z}^k$ and $\Lambda^\Gamma_+ \cong \mathbb{N}^k$.

Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold. Theorem 10.11 gives a d-stack $\boldsymbol{\mathcal{X}}^\Gamma$ and a 1-morphism $\boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}}) : \boldsymbol{\mathcal{X}}^\Gamma \to \boldsymbol{\mathcal{X}}$. The virtual cotangent bundle of $\boldsymbol{\mathcal{X}}$ is $T^*\boldsymbol{\mathcal{X}} = (\mathcal{E}_{\mathcal{X}}, \mathcal{F}_{\mathcal{X}}, \phi_{\mathcal{X}})$, a virtual vector bundle of rank vdim $\boldsymbol{\mathcal{X}}$ on $\mathcal{X}$. So $O^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}}) = \big(O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}}), O^\Gamma(\mathcal{X})^*(\mathcal{F}_{\mathcal{X}}), O^\Gamma(\mathcal{X})^*(\phi_{\mathcal{X}})\big)$ is a virtual vector bundle on $\mathcal{X}^\Gamma$. As in §8.7, $O^\Gamma(\mathcal{X})^*(\mathcal{E}_{\mathcal{X}}), O^\Gamma(\mathcal{X})^*(\mathcal{F}_{\mathcal{X}})$ have natural Γ-representations inducing decompositions of the form (8.10)–(8.11), and $O^\Gamma(\mathcal{X})^*(\phi_{\mathcal{X}})$ is Γ-equivariant and so preserves these splittings. Hence we have decompositions in $\mathrm{vqcoh}(\mathcal{X}^\Gamma)$:

$$O^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}}) \cong \bigoplus_{i=0}^k (T^*\boldsymbol{\mathcal{X}})^\Gamma_i \otimes R_i \text{ for } (T^*\boldsymbol{\mathcal{X}})^\Gamma_i \in \mathrm{vqcoh}(\mathcal{X}^\Gamma),$$
$$\text{and } O^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}}) = (T^*\boldsymbol{\mathcal{X}})^\Gamma_{\mathrm{tr}} \oplus (T^*\boldsymbol{\mathcal{X}})^\Gamma_{\mathrm{nt}}, \text{ with} \tag{11.6}$$
$$(T^*\boldsymbol{\mathcal{X}})^\Gamma_{\mathrm{tr}} \cong (T^*\boldsymbol{\mathcal{X}})^\Gamma_0 \otimes R_0 \text{ and } (T^*\boldsymbol{\mathcal{X}})^\Gamma_{\mathrm{nt}} \cong \bigoplus_{i=1}^k (T^*\boldsymbol{\mathcal{X}})^\Gamma_i \otimes R_i.$$

Also Theorem 10.15 shows that $T^*(\boldsymbol{\mathcal{X}}^\Gamma) \cong (T^*\boldsymbol{\mathcal{X}})^\Gamma_{\mathrm{tr}}$.

As $O^\Gamma(\mathcal{X})^*(T^*\boldsymbol{\mathcal{X}})$ is a virtual vector bundle, (11.6) implies the $(T^*\boldsymbol{\mathcal{X}})^\Gamma_i$ are *virtual vector bundles of mixed rank*, whose ranks may vary on different connected components of $\mathcal{X}^\Gamma$. For each $\lambda \in \Lambda^\Gamma$, define $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ to be the open and closed d-substack in $\boldsymbol{\mathcal{X}}^\Gamma$ with $\mathrm{rank}\big((T^*\boldsymbol{\mathcal{X}})^\Gamma_1\big)[R_1] + \cdots + \mathrm{rank}\big((T^*\boldsymbol{\mathcal{X}})^\Gamma_k\big)[R_k] = \lambda$ in Λ^Γ. Then $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ is a d-orbifold, with vdim $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda} = $ vdim $\boldsymbol{\mathcal{X}} - \dim \lambda$. Also we have a decomposition $\boldsymbol{\mathcal{X}}^\Gamma = \coprod_{\lambda \in \Lambda^\Gamma} \boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ in $\mathbf{dSta}$.

Note that in the d-orbifold case $\dim \lambda$ may be negative, so we can have vdim $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda} > $ vdim $\boldsymbol{\mathcal{X}}$. This is counterintuitive: the (w-immersed) d-suborbifold $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ has larger dimension than the d-orbifold $\boldsymbol{\mathcal{X}}$ that contains it.

Write $\boldsymbol{O}^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}}) = \boldsymbol{O}^\Gamma(\boldsymbol{\mathcal{X}})|_{\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}} : \boldsymbol{\mathcal{X}}^{\Gamma,\lambda} \to \boldsymbol{\mathcal{X}}$. Then $\boldsymbol{O}^{\Gamma,\lambda}(\boldsymbol{\mathcal{X}})$ is a proper w-immersion of d-orbifolds, in the sense of §11.4. Define $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ = \boldsymbol{\mathcal{X}}^\Gamma_\circ \cap \boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$, and $\boldsymbol{O}^{\Gamma,\lambda}_\circ(\boldsymbol{\mathcal{X}}) = \boldsymbol{O}^\Gamma_\circ(\boldsymbol{\mathcal{X}})|_{\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ} : \boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ \to \boldsymbol{\mathcal{X}}$. Then $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ$ is a d-orbifold with vdim $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ = $ vdim $\boldsymbol{\mathcal{X}} - \dim \lambda$, and $\boldsymbol{\mathcal{X}}^\Gamma_\circ = \coprod_{\lambda \in \Lambda^\Gamma} \boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ$.

As for $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \ldots, \hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$ in §9.2, for each $\mu \in \Lambda^\Gamma/\mathrm{Aut}(\Gamma)$ we define $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu} \simeq \big[\big(\coprod_{\lambda \in \mu} \boldsymbol{\mathcal{X}}^{\Gamma,\lambda}\big)/\mathrm{Aut}(\Gamma)\big]$ in $\tilde{\boldsymbol{\mathcal{X}}}^\Gamma \simeq [\boldsymbol{\mathcal{X}}^\Gamma/\mathrm{Aut}(\Gamma)]$, and $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ = \tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ \cap \tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$, and $\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu} = \hat{\boldsymbol{\Pi}}^\Gamma(\boldsymbol{\mathcal{X}})(\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu})$, and $\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ = \hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ \cap \hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$. Then $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \ldots, \hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$ are d-orbifolds with vdim $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu} = \cdots = $ vdim $\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ = $ vdim $\boldsymbol{\mathcal{X}} - \dim \mu$, with

$$\tilde{\boldsymbol{\mathcal{X}}}^\Gamma = \coprod_\mu \tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \ \ \tilde{\boldsymbol{\mathcal{X}}}^\Gamma_\circ = \coprod_\mu \tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ, \ \ \hat{\boldsymbol{\mathcal{X}}}^\Gamma = \coprod_\mu \hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \ \ \hat{\boldsymbol{\mathcal{X}}}^\Gamma_\circ = \coprod_\mu \hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ.$$

Also $\hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$ is a d-manifold, that is, it lies in $\mathbf{d\hat{M}an}$.

In [35, §10.7] we also consider the question: if $\boldsymbol{\mathcal{X}}$ is an oriented d-orbifold, under what conditions on Γ, λ, μ do the orbifold strata $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \ldots, \hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}_\circ$ have natural orientations? Here is the analogue of Proposition 9.9:

Proposition 11.23. (a) *Let Γ be a finite group with $|\Gamma|$ odd, and $\boldsymbol{\mathcal{X}}$ an oriented d-orbifold. Then we may define orientations on $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \boldsymbol{\mathcal{X}}^{\Gamma,\lambda}_\circ$ for all $\lambda \in \Lambda^\Gamma$.*

(b) *Let Γ be a finite group with $|\Gamma|$ odd, $\lambda \in \Lambda^\Gamma$ and $\mu = \lambda \cdot \mathrm{Aut}(\Gamma)$ in $\Lambda^\Gamma/\mathrm{Aut}(\Gamma)$. We may write $\lambda = [(V^+, \rho^+)] - [(V^-, \rho^-)]$ for nontrivial Γ-representations $(V^\pm, \rho^\pm)$ with no common subrepresentation, and then $(V^\pm, \rho^\pm)$*

are unique up to isomorphism. Define H to be the subgroup of $\mathrm{Aut}(\Gamma)$ fixing λ in Λ^Γ. Then for each $\delta \in H$ there exist isomorphisms of Γ-representations $i_\delta^\pm : (V^\pm, \rho^\pm \circ \delta) \to (V^\pm, \rho^\pm)$. Suppose $i_\delta^+ \oplus i_\delta^- : V^+ \oplus V^- \to V^+ \oplus V^-$ is orientation-preserving for all $\delta \in H$. If $\lambda \in 2\Lambda^\Gamma$ this holds automatically.

Then for all oriented d-orbifolds $\boldsymbol{\mathcal{X}}$ we can define orientations on the orbifold strata $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \tilde{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}, \hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}$. For $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}$ this works as $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu} \simeq [\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}/H]$, where $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ is oriented by **(a)***, and the H-action on $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ preserves orientations, so the orientation on $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}$ descends to an orientation on $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu} \simeq [\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}/H]$.*

(c) *Suppose that Γ and $\lambda \in \Lambda^\Gamma$ do not satisfy the conditions in* **(a)** *(i.e. $|\Gamma|$ is even), or Γ and $\mu \in \Lambda^\Gamma/\mathrm{Aut}(\Gamma)$ do not satisfy the conditions in* **(b)***. Then we can find examples of oriented d-orbifolds $\boldsymbol{\mathcal{X}}$ such that $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \boldsymbol{\mathcal{X}}_\circ^{\Gamma,\lambda}$ are not orientable, or $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \tilde{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}, \hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}$ are not orientable, respectively. That is, the conditions on Γ, λ, μ in* **(a)**,**(b)** *are necessary as well as sufficient to be able to orient orbifold strata $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \ldots, \hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}$ of all oriented d-orbifolds $\boldsymbol{\mathcal{X}}$.*

Note that Proposition 11.23 for d-orbifolds is weaker than Proposition 9.9 for orbifolds. That is, if Γ is a finite group with $|\Gamma|$ even then for some choices of λ, μ we can orient $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \ldots, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ for all oriented orbifolds $\mathcal{X}$, but we cannot orient $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \ldots, \hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}$ for all oriented d-orbifolds $\boldsymbol{\mathcal{X}}$.

11.9 Kuranishi neighbourhoods, good coordinate systems

We now explain the main ideas of [35, §10.8], which are based on parallel material about Kuranishi spaces due to Fukaya, Oh, Ohta and Ono [19, 20].

Definition 11.24. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. A *type A Kuranishi neighbourhood* on $\boldsymbol{\mathcal{X}}$ is a quintuple $(V, E, \Gamma, s, \boldsymbol{\psi})$ where V is a manifold, $E \to V$ a vector bundle, Γ a finite group acting smoothly and locally effectively on V, E preserving the vector bundle structure, and $s : V \to E$ a smooth, Γ-equivariant section of E. Write the Γ-actions on V, E as $r(\gamma) : V \to V$ and $\hat{r}(\gamma) : E \to r(\gamma)^*(E)$ for $\gamma \in \Gamma$. Then Example 11.7 defines a principal d-orbifold $[\boldsymbol{S}_{V,E,s}/\Gamma]$. We require that $\boldsymbol{\psi} : [\boldsymbol{S}_{V,E,s}/\Gamma] \to \boldsymbol{\mathcal{X}}$ is a 1-morphism of d-orbifolds which is an equivalence with a nonempty open d-suborbifold $\boldsymbol{\psi}([\boldsymbol{S}_{V,E,s}/\Gamma]) \subseteq \boldsymbol{\mathcal{X}}$.

Definition 11.25. Suppose $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i), (V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$ are type A Kuranishi neighbourhoods on a d-orbifold $\boldsymbol{\mathcal{X}}$, with

$$\emptyset \neq \boldsymbol{\psi}_i([\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i]) \cap \boldsymbol{\psi}_j([\boldsymbol{S}_{V_j,E_j,s_j}/\Gamma_j]) \subseteq \boldsymbol{\mathcal{X}}.$$

A *type A coordinate change from* $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i)$ *to* $(V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$ is a quintuple $(V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij})$, where:

(a) $\emptyset \neq V_{ij} \subseteq V_i$ is a Γ_i-invariant open submanifold, with

$$\boldsymbol{\psi}_i\big([\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]\big) = \boldsymbol{\psi}_i([\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i]) \cap \boldsymbol{\psi}_j([\boldsymbol{S}_{V_j,E_j,s_j}/\Gamma_j]) \subseteq \boldsymbol{\mathcal{X}}.$$

(b) $\rho_{ij} : \Gamma_i \to \Gamma_j$ is an injective group morphism.

(c) $e_{ij} : V_{ij} \to V_j$ is an embedding of manifolds with $e_{ij} \circ r_i(\gamma) = r_j(\rho_{ij}(\gamma)) \circ e_{ij} : V_{ij} \to V_j$ for all $\gamma \in \Gamma_i$. If $v_i, v_i' \in V_{ij}$ and $\delta \in \Gamma_j$ with $r_j(\delta) \circ e_{ij}(v_i') = e_{ij}(v_i)$, then there exists $\gamma \in \Gamma_i$ with $\rho_{ij}(\gamma) = \delta$ and $r_i(\gamma)(v_i') = v_i$.

(d) $\hat{e}_{ij} : E_i|_{V_{ij}} \to e_{ij}^*(E_j)$ is an embedding of vector bundles (that is, $\hat{e}_{ij}$ has a left inverse), such that $\hat{e}_{ij} \circ s_i|_{V_{ij}} = e_{ij}^*(s_j)$ and $r_i(\gamma)^*(\hat{e}_{ij}) \circ \hat{r}_i(\gamma) = e_{ij}^*(\hat{r}_j(\rho_{ij}(\gamma))) \circ \hat{e}_{ij} : E_i|_{V_{ij}} \to (e_{ij} \circ r_i(\gamma))^*(E_j)$ for all $\gamma \in \Gamma_i$. Thus Example 11.8 defines a quotient 1-morphism

$$[\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}] : [\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i] \longrightarrow [\boldsymbol{S}_{V_j,E_j,s_j}/\Gamma_j], \qquad (11.7)$$

where $[\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]$ is an open d-suborbifold in $[\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i]$.

(e) If $v_i \in V_{ij}$ with $s_i(v_i) = 0$ and $v_j = e_{ij}(v_i) \in V_j$ then the following linear map is an isomorphism:

$$\big(\mathrm{d}s_j(v_j)\big)_* : \big(T_{v_j}V_j\big)/\big(\mathrm{d}e_{ij}(v_i)[T_{v_i}V_i]\big) \to \big(E_j|_{v_j}\big)/\big(\hat{e}_{ij}(v_i)[E_i|_{v_i}]\big).$$

Theorem 11.12 then implies that $[\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}]$ in (11.7) is an equivalence with an open d-suborbifold of $[\boldsymbol{S}_{V_j,E_j,s_j}/\Gamma_j]$.

(f) $\boldsymbol{\eta}_{ij} : \boldsymbol{\psi}_j \circ [\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}] \Rightarrow \boldsymbol{\psi}_i|_{[\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]}$ is a 2-morphism in $\mathbf{dOrb}$.

(g) The quotient topological space $V_i \amalg_{V_{ij}} V_j = (V_i \amalg V_j)/\sim$ is Hausdorff, where the equivalence relation $\sim$ identifies $v \in V_{ij} \subseteq V_i$ with $e_{ij}(v) \in V_j$.

Definition 11.26. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. A *type A good coordinate system* on $\boldsymbol{\mathcal{X}}$ consists of the following data satisfying conditions (a)–(e):

(a) We are given a countable indexing set I, and a total order $<$ on I making $(I, <)$ into a well-ordered set.

(b) For each $i \in I$ we are given a Kuranishi neighbourhood $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i)$ of type A on $\boldsymbol{\mathcal{X}}$. Write $\boldsymbol{\mathcal{X}}_i = \boldsymbol{\psi}_i([\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i])$, so that $\boldsymbol{\mathcal{X}}_i \subseteq \boldsymbol{\mathcal{X}}$ is an open d-suborbifold, and $\boldsymbol{\psi}_i : [\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i] \to \boldsymbol{\mathcal{X}}_i$ is an equivalence. We require that $\bigcup_{i \in I} \boldsymbol{\mathcal{X}}_i = \boldsymbol{\mathcal{X}}$, so that $\{\boldsymbol{\mathcal{X}}_i : i \in I\}$ is an open cover of $\boldsymbol{\mathcal{X}}$.

(c) For all $i < j$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \neq \emptyset$ we are given a type A coordinate change $(V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij})$ from $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i)$ to $(V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$.

(d) For all $i < j < k$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k \neq \emptyset$, we are given $\gamma_{ijk} \in \Gamma_k$ satisfying $\rho_{ik}(\gamma) = \gamma_{ijk}\, \rho_{jk}(\rho_{ij}(\gamma))\, \gamma_{ijk}^{-1}$ for all $\gamma \in \Gamma_i$, and

$$\begin{aligned}
e_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} &= r_k(\gamma_{ijk}) \circ e_{jk} \circ e_{ij}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})}, \\
\hat{e}_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} &= \big(e_{ij}^*(e_{jk}^*(\hat{r}_k(\gamma_{ijk}))) \circ e_{ij}^*(\hat{e}_{jk}) \circ \hat{e}_{ij}\big)|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})}.
\end{aligned} \qquad (11.8)$$

Combining the first equation of (11.8) with Definition 11.25(c) for e_{ik} and Γ_i acting effectively on $V_{ik} \cap e_{ij}^{-1}(V_{jk})$ shows that γ_{ijk} is unique. Example 11.9 with $\delta = \gamma_{ijk}$ and $\Lambda = 0$ then gives a 2-morphism in $\mathbf{dOrb}$:

$$\boldsymbol{\eta}_{ijk} = [\boldsymbol{S}_0, \gamma_{ijk}] : [\boldsymbol{S}_{e_{jk},\hat{e}_{jk}}, \rho_{jk}] \circ [\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}]|_{[\boldsymbol{S}_{V_{ik} \cap e_{ij}^{-1}(V_{jk}),E_i,s_i}/\Gamma_i]}$$

$$\Longrightarrow [\boldsymbol{S}_{e_{ik},\hat{e}_{ik}}, \rho_{ik}]|_{[\boldsymbol{S}_{V_{ik} \cap e_{ij}^{-1}(V_{jk}),E_i,s_i}/\Gamma_i]}.$$

(e) For all $i < j < k$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_k \neq \emptyset$ and $\boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k \neq \emptyset$, we require that if $v_i \in V_{ik}$, $v_j \in V_{jk}$ and $\delta \in \Gamma_k$ with $e_{jk}(v_j) = r_k(\delta) \circ e_{ik}(v_i)$ in V_k, then $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \cap \boldsymbol{\mathcal{X}}_k \neq \emptyset$, and $v_i \in V_{ij}$, and there exists $\gamma \in \Gamma_j$ with $\rho_{jk}(\gamma) = \delta\,\gamma_{ijk}$ and $v_j = r_j(\gamma) \circ e_{ij}(v_i)$.

Suppose now that Y is a manifold, and $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a 1-morphism in **dOrb**, where $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Man}}^{\mathbf{dOrb}}(Y)$. A *type A good coordinate system for* $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ consists of a type A good coordinate system $\big(I, <, \dots, \gamma_{ijk}\big)$ for $\boldsymbol{\mathcal{X}}$ as in (a)–(e) above, together with the following data satisfying conditions (f)–(g):

(f) For each $i \in I$, we are given a smooth map $g_i : V_i \to Y$ with $g_i \circ r_i(\gamma) = g_i$ for all $\gamma \in \Gamma_i$, so that Example 11.8 defines a quotient 1-morphism

$$[\boldsymbol{S}_{g_i,0}, \pi] : [\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i] \longrightarrow [\boldsymbol{S}_{Y,0,0}/\{1\}] = \boldsymbol{\mathcal{Y}},$$

where $\pi : \Gamma_i \to \{1\}$ is the projection. We are given a 2-morphism $\boldsymbol{\zeta}_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow [\boldsymbol{S}_{g_i,0}, \pi]$ in **dOrb**. Sometimes we require g_i to be a submersion.

(g) For all $i < j$ in I with $\boldsymbol{\mathcal{X}}_i \cap \boldsymbol{\mathcal{X}}_j \neq \emptyset$, we require that $g_j \circ e_{ij} = g_i|_{V_{ij}}$. This implies that

$$[\boldsymbol{S}_{g_j,0}, \pi] \circ [\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}] = [\boldsymbol{S}_{g_i,0}, \pi]\big|_{[\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i]} :$$

$$[\boldsymbol{S}_{V_{ij},E_i|_{V_{ij}},s_i|_{V_{ij}}}/\Gamma_i] \longrightarrow [\boldsymbol{S}_{Y,0,0}/\{1\}] = \boldsymbol{\mathcal{Y}}.$$

Here is the main result of [35, §10.8], which is proved in [35, App. D].

Theorem 11.27. *Suppose* $\boldsymbol{\mathcal{X}}$ *is a d-orbifold. Then there exists a type A good coordinate system* $\big(I, <, (V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i), (V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij}), \gamma_{ijk}\big)$ *for* $\boldsymbol{\mathcal{X}}$*. If* $\boldsymbol{\mathcal{X}}$ *is compact, we may take* I *to be finite. If* $\{\boldsymbol{\mathcal{U}}_j : j \in J\}$ *is an open cover of* $\boldsymbol{\mathcal{X}}$*, we may take* $\boldsymbol{\mathcal{X}}_i = \boldsymbol{\psi}_i([\boldsymbol{S}_{V_i,E_i,s_i}/\Gamma_i]) \subseteq \boldsymbol{\mathcal{U}}_{j_i}$ *for each* $i \in I$ *and some* $j_i \in J$.

Now let Y *be a manifold and* $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}} = F_{\mathbf{Man}}^{\mathbf{dOrb}}(Y)$ *a 1-morphism in* **dOrb***. Then all the above extends to type A good coordinate systems for* $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$*, and we may take the* g_i *in Definition* 11.26(f) *to be submersions.*

In [35, §10.8] we also give 'type B' versions of Definitions 11.24–11.26 and Theorem 11.27 using the standard model d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ and 1-morphisms $\boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}}$ of Examples 11.4 and 11.5 in place of $[\boldsymbol{S}_{V,E,s}/\Gamma]$ and $[\boldsymbol{S}_{e_{ij},\hat{e}_{ij}}, \rho_{ij}]$ from Examples 11.7 and 11.8.

Observe that Definition 11.26 is similar to the hypotheses of Theorem 11.14. Given a good coordinate system $I, <, (V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i), \dots$ on $\boldsymbol{\mathcal{X}}$, Theorem 11.14 reconstructs $\boldsymbol{\mathcal{X}}$ up to equivalence in **dOrb** from the data $I, <, V_i, E_i, \Gamma_i, s_i, V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \gamma_{ijk}$. Thus, we can regard Theorem 11.27 as a kind of converse to Theorem 11.14. Combining the two, we see that every d-orbifold $\boldsymbol{\mathcal{X}}$ can be described up to equivalence by a collection of differential-geometric data $I, <, V_i, \dots, \gamma_{ijk}$. The 'type B' version of Theorem 11.27 is a kind of converse to Theorem 11.13.

Fukaya and Ono [20, §5] and Fukaya, Oh, Ohta and Ono [19, §A1] define *Kuranishi spaces*, the geometric structure they put on moduli spaces of

J-holomorphic curves in symplectic geometry. We argue in [35, §14.3] that their definition is not really satisfactory, and that the 'right' way to define Kuranishi spaces is as d-orbifolds, or d-orbifolds with corners.

A *Kuranishi space* in [19, §A1] is a topological space X with a cover by 'Kuranishi neighbourhoods' (V, E, Γ, s, ψ), which are as in Definition 11.24 except that ψ is a homeomorphism with an open set in X, rather than an equivalence with an open d-suborbifold. On overlaps between (images of) Kuranishi neighbourhoods in X we are given 'coordinate changes', roughly as in Definition 11.25 except for the 2-morphisms $\boldsymbol{\eta}_{ij}$. Fukaya et al. define 'good coordinate systems' for Kuranishi spaces, roughly as in Definition 11.26. They state without proof in [19, Lem. A1.11] that good coordinate systems exist for any (compact) Kuranishi space, the analogue of Theorem 11.27.

Good coordinate systems are used in [19, 20] in some kinds of proof involving Kuranishi spaces, in particular, in the construction of virtual classes and virtual chains. The proofs involve choosing data (such as a multi-valued perturbation of s_i) on each Kuranishi neighbourhood $(V_i, E_i, \Gamma_i, s_i, \psi_i)$, by induction on i in I in the order $<$, where the data must satisfy compatibility conditions with coordinate changes $(V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij})$.

In fact we have already met the problem good coordinate systems are designed to solve in §11.6: in contrast to the d-manifold case, we do not have useful criteria for when a d-orbifold $\boldsymbol{\mathcal{X}}$ is principal. The parallel issue for Kuranishi spaces is that we cannot cover a general Kuranishi space $\boldsymbol{\mathcal{X}}$ with a single Kuranishi neighbourhood (V, E, Γ, s, ψ). So we cover (compact) $\boldsymbol{\mathcal{X}}$ with (finitely) many Kuranishi neighbourhoods $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ with particularly well-behaved coordinate changes on overlaps, and then carry out the construction we want on each $(V_i, E_i, \Gamma_i, s_i, \psi_i)$, compatibly with coordinate changes.

The material above is used in [35, §14.3] to explain the relations between d-orbifolds and Kuranishi spaces. As for Kuranishi spaces, it is also helpful for some proofs involving d-orbifolds, for instance, in constructing virtual classes for compact oriented d-orbifolds, and in studying d-orbifold bordism.

11.10 Semieffective and effective d-orbifolds

In [35, §10.9] we define *semieffective* and *effective* d-orbifolds, which are related to the notion of effective orbifold in Definition 9.4.

Definition 11.28. Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold. For $[x] \in \mathcal{X}_{\mathrm{top}}$, so that $x : \underline{\bar{\ast}} \to \boldsymbol{\mathcal{X}}$ is a C^∞-stack 1-morphism, applying pullback x^* to (10.1) gives an exact sequence in $\mathrm{qcoh}(\underline{\bar{\ast}})$, where $K_{[x]} = \mathrm{Ker}(x^*(\phi_\mathcal{X}))$:

$$ 0 \longrightarrow K_{[x]} \longrightarrow x^*(\mathcal{E}_\mathcal{X}) \xrightarrow{x^*(\phi_\mathcal{X})} x^*(\mathcal{F}_\mathcal{X}) \xrightarrow{x^*(\psi_\mathcal{X})} x^*(T^*\mathcal{X}) \cong T_x^*\mathcal{X} \longrightarrow 0. $$

We may think of this as an exact sequence of real vector spaces, where $K_{[x]}, T_x^*\mathcal{X}$ are finite-dimensional with $\dim T_x^*\mathcal{X} - \dim K_{[x]} = \mathrm{vdim}\,\boldsymbol{\mathcal{X}}$.

The orbifold group $\mathrm{Iso}_\mathcal{X}([x])$ is the group of 2-morphisms $\eta : x \Rightarrow x$. Definition 8.22 defines isomorphisms $\eta^*(\mathcal{E}_\mathcal{X}) : x^*(\mathcal{E}_\mathcal{X}) \to x^*(\mathcal{E}_\mathcal{X})$ in $\mathrm{qcoh}(\underline{\bar{\ast}})$,

which make $x^*(\mathcal{E}_\mathcal{X})$ into a representation of $\mathrm{Iso}_\mathcal{X}([x])$. The same holds for $x^*(\mathcal{F}_\mathcal{X}), x^*(T^*\mathcal{X})$, and $x^*(\phi_\mathcal{X}), x^*(\psi_\mathcal{X})$ are equivariant. Hence $K_{[x]}, T_x^*\mathcal{X}$ are also $\mathrm{Iso}_\mathcal{X}([x])$-representations.

We call $\boldsymbol{\mathcal{X}}$ a *semieffective d-orbifold* if $K_{[x]}$ is a trivial representation of $\mathrm{Iso}_\mathcal{X}([x])$ for all $[x] \in \mathcal{X}_{\mathrm{top}}$. We call $\boldsymbol{\mathcal{X}}$ an *effective d-orbifold* if it is semieffective, and $T_x^*\mathcal{X}$ is an effective representation of $\mathrm{Iso}_\mathcal{X}([x])$ for all $[x] \in \mathcal{X}_{\mathrm{top}}$.

That is, $\boldsymbol{\mathcal{X}}$ is semieffective if the orbifold groups $\mathrm{Iso}_\mathcal{X}([x])$ act trivially on the obstruction spaces of $\boldsymbol{\mathcal{X}}$, and effective if the $\mathrm{Iso}_\mathcal{X}([x])$ also act effectively on the tangent spaces of $\boldsymbol{\mathcal{X}}$. One useful property of (semi)effective d-orbifolds is that generic perturbations of semieffective (or effective) d-orbifolds are (effective) orbifolds. We state this for 'standard model' d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$.

Proposition 11.29. *Let $\mathcal{V}$ be an orbifold, $\mathcal{E}$ a vector bundle on $\mathcal{V}$, and $s \in C^\infty(\mathcal{E})$, and let $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ be as in Example* 11.4. *Suppose $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is a semieffective d-orbifold. Then for any generic perturbation $\tilde{s}$ of s in $C^\infty(\mathcal{E})$ with $\tilde{s} - s$ sufficiently small in C^1 locally on $\mathcal{V}$, the d-orbifold $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},\tilde{s}}$ is an orbifold, that is, it lies in $\hat{\mathbf{Orb}} \subset \mathbf{dOrb}$. If $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ is an effective d-orbifold, then $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},\tilde{s}}$ is an effective orbifold.*

Here are some other good properties of (semi)effective d-orbifolds:

- If $\mathcal{X}$ is an orbifold then $\boldsymbol{\mathcal{X}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{X})$ is a semieffective d-orbifold, and if $\mathcal{X}$ is effective then $\boldsymbol{\mathcal{X}}$ is effective.

- Let $\boldsymbol{\mathcal{X}}$ be a semieffective d-orbifold, Γ a finite group, and $\lambda \in \Lambda^\Gamma$. Then the orbifold stratum $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda} = \emptyset$ unless $\lambda \in \Lambda_+^\Gamma \subset \Lambda^\Gamma$. If $\boldsymbol{\mathcal{X}}$ is effective then $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda} = \emptyset$ unless $\lambda = [R]$ for R an effective Γ-representation.

- If $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ are (semi)effective d-orbifolds, then the product $\boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{Y}}$ is also (semi)effective. More generally, any fibre product $\boldsymbol{\mathcal{X}} \times_{\boldsymbol{\mathcal{Z}}} \boldsymbol{\mathcal{Y}}$ in $\mathbf{dOrb}$ with $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ (semi)effective and $\boldsymbol{\mathcal{Z}}$ a manifold is also (semi)effective.

- Proposition 11.23 says that if $\boldsymbol{\mathcal{X}}$ is an oriented d-orbifold, then when $|\Gamma|$ is odd we can define orientations on the orbifold strata $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \boldsymbol{\mathcal{X}}_\circ^{\Gamma,\lambda}$, and under extra conditions on μ we can also orient $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \tilde{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}, \hat{\boldsymbol{\mathcal{X}}}^{\Gamma,\mu}, \hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}$.

  For general d-orbifolds $\boldsymbol{\mathcal{X}}$, this is the best we can do. But for semieffective d-orbifolds $\boldsymbol{\mathcal{X}}$ the analogue of Proposition 9.9 for orbifolds holds. This is stronger, as it orients $\boldsymbol{\mathcal{X}}^{\Gamma,\lambda}, \ldots, \hat{\boldsymbol{\mathcal{X}}}_\circ^{\Gamma,\mu}$ under weaker conditions on Γ, λ, μ, which allow $|\Gamma|$ even for some λ, μ.

12   Orbifolds with corners

In [35, §8.5–§8.9] we discuss 2-categories $\mathbf{Orb^b}$ and $\mathbf{Orb^c}$ of *orbifolds with boundary* and *orbifolds with corners*, which are orbifold versions of manifolds with boundary and with corners in §5. This is new material, and the author knows of no other foundational work on orbifolds with corners.

12.1 The definition of orbifolds with corners

Definition 12.1. An *orbifold with corners* $\mathcal{X}$ of dimension $n \geqslant 0$ is a triple $\mathcal{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathcal{X}})$ where $\mathcal{X}, \partial\mathcal{X}$ are separated, second countable Deligne–Mumford C^∞-stacks, and $i_{\mathcal{X}} : \partial\mathcal{X} \to \mathcal{X}$ is a proper, strongly representable 1-morphism of C^∞-stacks, in the sense of §8.3, such that for each $[x] \in \mathcal{X}_{\mathrm{top}}$ there exists a 2-Cartesian diagram in $\mathbf{C^\infty Sta}$:

$$
\begin{array}{ccc}
\partial\bar{U} & \xrightarrow{\ \ u_\partial\ \ } & \partial\mathcal{X} \\
{\scriptstyle\bar{i}_U}\downarrow & \mathrm{id}\Uparrow & \downarrow{\scriptstyle i_{\mathcal{X}}} \\
\bar{U} & \xrightarrow[\ \ u\ \ ]{} & \mathcal{X}.
\end{array}
$$

Here U is an n-manifold with corners, so that $i_U : \partial U \to U$ is smooth, and $\underline{U}, \underline{\partial U}, \underline{i}_U = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(U, \partial U, i_U)$, and u, u_∂ are étale 1-morphisms, and $u_{\mathrm{top}}([p]) = [x]$ for some $p \in U$. We call $\mathcal{X}$ an *orbifold with boundary*, or an *orbifold without boundary*, if the above condition holds with U a manifold with boundary, or a manifold without boundary, respectively, for each $[x] \in \mathcal{X}_{\mathrm{top}}$.

Now suppose $\mathcal{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathcal{X}})$ and $\mathcal{Y} = (\mathcal{Y}, \partial\mathcal{Y}, i_{\mathcal{Y}})$ are orbifolds with corners. A 1-*morphism* $f : \mathcal{X} \to \mathcal{Y}$, or *smooth map*, is a 1-morphism of C^∞-stacks $f : \mathcal{X} \to \mathcal{Y}$ such that for each $[x] \in \mathcal{X}_{\mathrm{top}}$ with $f_{\mathrm{top}}([x]) = [y] \in \mathcal{Y}_{\mathrm{top}}$ there exists a 2-commutative diagram in $\mathbf{C^\infty Sta}$:

$$
\begin{array}{ccc}
\bar{U} & \xrightarrow{\ \ u\ \ } & \mathcal{X} \\
{\scriptstyle\bar{h}}\downarrow & \Downarrow\eta & \downarrow{\scriptstyle f} \\
\bar{V} & \xrightarrow[\ \ v\ \ ]{} & \mathcal{Y}.
\end{array}
$$

Here U, V are manifolds with corners, $h : U \to V$ is a smooth map, $\underline{U}, \underline{V}, \underline{h} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(U, V, h)$, and u, v are étale, and $u_{\mathrm{top}}([p]) = [x]$ for some $p \in U$.

Let $f, g : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms of orbifolds with corners. A 2-*morphism* $\eta : f \Rightarrow g$ is a 2-morphism of 1-morphisms $f, g : \mathcal{X} \to \mathcal{Y}$ in $\mathbf{C^\infty Sta}$.

Composition of 1-*morphisms* $g \circ f$, *identity* 1-*morphisms* $\mathrm{id}_{\mathcal{X}}$, *vertical and horizontal composition of* 2-*morphisms* $\zeta \odot \eta$, $\zeta \ast \eta$, *and identity* 2-*morphisms* for orbifolds with corners, are all given by the corresponding compositions and identities in $\mathbf{C^\infty Sta}$. This defines the 2-category $\mathbf{Orb^c}$ of orbifolds with corners. Write $\mathbf{Orb^b}$ and $\mathbf{\dot{O}rb}$ for the full 2-subcategories of orbifolds with boundary, and orbifolds without boundary, in $\mathbf{Orb^c}$.

If $\mathcal{X}$ is an orbifold in the sense of Definition 9.1, then $\mathcal{X} = (\mathcal{X}, \emptyset, \emptyset)$ is an orbifold without boundary in this sense, and vice versa. Thus the 2-functor $F_{\mathbf{Orb}}^{\mathbf{Orb^c}} : \mathbf{Orb} \to \mathbf{Orb^c}$ mapping $\mathcal{X} \mapsto \mathcal{X} = (\mathcal{X}, \emptyset, \emptyset)$ on objects, $f \mapsto f$ on 1-morphisms, and $\eta \mapsto \eta$ on 2-morphisms, is an isomorphism of 2-categories $\mathbf{Orb} \to \mathbf{\dot{O}rb}$.

Define $F_{\mathbf{Man^c}}^{\mathbf{Orb^c}} : \mathbf{Man^c} \to \mathbf{Orb^c}$ by $F_{\mathbf{Man^c}}^{\mathbf{Orb^c}} : X \mapsto \mathcal{X} = (\bar{X}, \partial\bar{X}, \bar{i}_X)$ on objects X in $\mathbf{Man^c}$, where $\underline{X}, \underline{\partial X}, \underline{i}_X = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(X, \partial X, i_X)$, and $F_{\mathbf{Man^c}}^{\mathbf{Orb^c}} : f \mapsto \underline{f}$ on morphisms $f : X \to Y$ in $\mathbf{Man^c}$, where $\underline{f} = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(f)$. Then $F_{\mathbf{Man^c}}^{\mathbf{Orb^c}}$ is a full and faithful strict 2-functor.

Let $\mathcal{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathcal{X}})$ be an orbifold with corners, and $\mathcal{V} \subseteq \mathcal{X}$ an open C^∞-substack. Define $\partial\mathcal{V} = i_{\mathcal{X}}^{-1}(\mathcal{V})$, as an open C^∞-substack of $\partial\mathcal{X}$, and $i_{\mathcal{V}} : \partial\mathcal{V} \to \mathcal{V}$

by $i_{\mathcal{V}} = i_{\mathcal{X}}|_{\partial\mathcal{V}}$. Then $\mathcal{V} = (\mathcal{V}, \partial\mathcal{V}, i_{\mathcal{V}})$ is an orbifold with corners. We call $\mathcal{V}$ an *open suborbifold* of $\mathcal{X}$. An *open cover* of $\mathcal{X}$ is a family $\{\mathcal{V}_a : a \in A\}$ of open suborbifolds $\mathcal{V}_a$ of $\mathcal{X}$ with $\mathcal{X} = \bigcup_{a \in A} \mathcal{V}_a$.

Example 12.2. Suppose X is a manifold with corners, G a finite group, and $r : G \to \mathrm{Aut}(X)$ an action of G on X by diffeomorphisms. Since $r(\gamma) : X \to X$ is simple for each $\gamma \in G$, as in §5.2 we have $r_-(\gamma) : \partial X \to \partial X$, which is also a diffeomorphism. Then $r_- : G \to \mathrm{Aut}(\partial X)$ is an action of G on ∂X, and $i_X : \partial X \to X$ is G-equivariant. Set $\underline{X}, \underline{\partial X}, \underline{i}_X, \underline{r}, \underline{r}_- = F_{\mathbf{Man^c}}^{\mathbf{C^\infty Sch}}(X, \partial X, i_X, r, r_-)$. Then $\underline{X}, \underline{\partial X}$ are C^∞-schemes with G-actions $\underline{r}, \underline{r}_-$, and $\underline{i}_X : \underline{\partial X} \to \underline{X}$ is G-equivariant, so Examples 8.11 and 8.12 define Deligne–Mumford C^∞-stacks $[\underline{X}/G], [\underline{\partial X}/G]$ and a 1-morphism $[\underline{i}_X, \mathrm{id}_G] : [\underline{\partial X}/G] \to [\underline{X}/G]$, which turns out to be strongly representable. One can show that $\mathcal{X} = \big([\underline{X}/G], [\underline{\partial X}/G], [\underline{i}_X, \mathrm{id}_G]\big)$ is an orbifold with corners, which we will write as $[X/G]$.

Remark 12.3. (a) We could have defined $\mathbf{Orb^c}$ equivalently and more simply as a (non-full) 2-subcategory of $\mathbf{DMC^\infty Sta}$, so that an orbifold with corners would be a C^∞-stack $\mathcal{X}$ rather than a triple $\mathcal{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathcal{X}})$. We chose the set-up of Definition 12.1 partly for its compatibility with the definitions of d-stacks and d-orbifolds with corners $\boldsymbol{\mathcal{X}} = (\boldsymbol{\mathcal{X}}, \boldsymbol{\partial\mathcal{X}}, \boldsymbol{i_{\mathcal{X}}}, \boldsymbol{\omega_{\mathcal{X}}})$ in §13–§14, and partly because, to make several important constructions more functorial, it is useful to have a particular choice of boundary $\partial\mathcal{X}$ for $\mathcal{X}$ already made.

(b) In Remark 6.5 we noted that boundaries in $\mathbf{dSpa^c}$ are strictly functorial. One sign of this is that for a semisimple 1-morphism $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{dSpa^c}$, the 1-morphism $\boldsymbol{f}_- : \partial_-^{\boldsymbol{f}}\mathbf{X} \to \partial\mathbf{Y}$ is unique, not just unique up to 2-isomorphism, with an equality of 1-morphisms $\boldsymbol{f} \circ \boldsymbol{i_X}|_{\partial_-^{\boldsymbol{f}}\mathbf{X}} = \boldsymbol{i_Y} \circ \boldsymbol{f}_-$, not just a 2-isomorphism. By the general philosophy of 2-categories, this may seem unnatural.

We will arrange that boundaries in $\mathbf{Orb^c}$ and also in $\mathbf{dSta^c}, \mathbf{dOrb^c}$ are strictly functorial in the same way. This is our reason for taking $i_{\mathcal{X}} : \partial\mathcal{X} \to \mathcal{X}$ in Definition 12.1 to be *strongly representable*, in the sense of §8.3. Proposition 8.8(b) shows that this is no real restriction: $i_{\mathcal{X}} : \partial\mathcal{X} \to \mathcal{X}$ is naturally representable, and we can make it strongly representable by replacing $\partial\mathcal{X}$ by an equivalent C^∞-stack. Then Proposition 8.9 applied to $i_{\mathcal{Y}} : \partial\mathcal{Y} \to \mathcal{Y}$ is what we need to show that a semisimple 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\mathbf{dOrb^c}$ lifts to a unique 1-morphism $f_- : \partial_-^f\mathcal{X} \to \partial\mathcal{Y}$ with $f \circ i_{\mathcal{X}}|_{\partial_-^f\mathcal{X}} = i_{\mathcal{Y}} \circ f_-$.

(c) An orbifold with corners $\mathcal{X}$ of dimension n is locally modelled near each point $[x] \in \mathcal{X}_{\mathrm{top}}$ on $\big([0,\infty)^k \times \mathbb{R}^{n-k}\big)/G$ near 0, where G is a finite group acting linearly on $\mathbb{R}^n$ preserving the subset $[0,\infty)^k \times \mathbb{R}^{n-k}$. Note that G is allowed to permute the coordinates $x_1, \ldots, x_k$ in $[0,\infty)^k$. So, for example, we allow 2-dimensional orbifolds with corners modelled on $[0,\infty)^2/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \langle\sigma\rangle$ acts on $[0,\infty)^2$ by $\sigma : (x_1, x_2) \mapsto (x_2, x_1)$.

This implies that the 1-morphism $i_{\mathcal{X}} : \partial\mathcal{X} \to \mathcal{X}$ induces morphisms of orbifold groups $(i_{\mathcal{X}})_* : \mathrm{Iso}_{\partial\mathcal{X}}([x']) \to \mathrm{Iso}_{\mathcal{X}}([x])$ which are injective (so that $i_{\mathcal{X}}$ is representable), but need not be isomorphisms. We will call an orbifold with corners $\mathcal{X}$ *straight* if the morphisms $(i_{\mathcal{X}})_* : \mathrm{Iso}_{\partial\mathcal{X}}([x']) \to \mathrm{Iso}_{\mathcal{X}}([x])$ are isomorphisms for all $[x'] \in \partial\mathcal{X}_{\mathrm{top}}$ with $i_{\mathcal{X},\mathrm{top}}([x']) = [x]$. That is, straight orbifolds

with corners are locally modelled on $[0, \infty)^k \times (\mathbb{R}^{n-k}/G)$. Orbifolds with boundary, with $k = 0$ or 1, are automatically straight. Boundaries of orbifold strata behave better for straight orbifolds with corners.

In §9.1 we explained that a vector bundle $\mathcal{E}$ on an orbifold $\mathcal{X}$ is a vector bundle on $\mathcal{X}$ as a Deligne–Mumford C^∞-stack, in the sense of §8.6. But sometimes it is convenient to regard $\mathcal{E}$ as an orbifold in its own right, so we define a 'total space functor' mapping vector bundles $\mathcal{E}$ to orbifolds $\mathrm{Tot}(\mathcal{E})$.

In the same way, if $\mathcal{X} = (\mathcal{X}, \partial\mathcal{X}, i_\mathcal{X})$ is an orbifold with corners, in [35, §8.5] we define a *vector bundle $\mathcal{E}$ on $\mathcal{X}$* to be a vector bundle on $\mathcal{X}$ as a Deligne–Mumford C^∞-stack. To regard $\mathcal{E}$ as an orbifold with corners in its own right, we define a 'total space functor' $\mathrm{Tot}^{\mathbf{c}} : \mathrm{vect}(\mathcal{X}) \to \mathbf{Orb^c}$, which maps a vector bundle $\mathcal{E}$ on $\mathcal{X}$ to an orbifold with corners $\mathrm{Tot}^{\mathbf{c}}(\mathcal{E})$, and maps a section $s \in C^\infty(\mathcal{E})$ to a simple, flat 1-morphism $\mathrm{Tot}^{\mathbf{c}}(s) : \mathcal{X} \to \mathrm{Tot}^{\mathbf{c}}(\mathcal{E})$ in $\mathbf{Orb^c}$.

Definition 12.4. An orbifold with corners $\mathcal{X}$ is called *effective* if $\mathcal{X}$ is locally modelled near each $[x] \in \mathcal{X}_{\mathrm{top}}$ on $([0, \infty)^k \times \mathbb{R}^{n-k})/G$, where G acts effectively on $\mathbb{R}^n$ preserving $[0, \infty)^k \times \mathbb{R}^{n-k}$, that is, every $1 \neq \gamma \in G$ acts nontrivially.

The analogue of Proposition 9.5 holds for effective orbifolds with corners.

12.2 Boundaries of orbifolds with corners, and simple, semisimple and flat 1-morphisms

In [35, §8.6] we define boundaries of orbifolds with corners.

Definition 12.5. Let $\mathcal{X} = (\mathcal{X}, \partial\mathcal{X}, i_\mathcal{X})$ be an orbifold with corners. We will define an orbifold with corners $\partial\mathcal{X} = (\partial\mathcal{X}, \partial^2\mathcal{X}, i_{\partial\mathcal{X}})$, called the *boundary* of $\mathcal{X}$, such that $i_\mathcal{X} : \partial\mathcal{X} \to \mathcal{X}$ is a 1-morphism in $\mathbf{Orb^c}$. Here $\partial\mathcal{X}$ and $i_\mathcal{X}$ are given in $\mathcal{X}$, so the new data we have to construct is $\partial^2\mathcal{X}, i_{\partial\mathcal{X}}$.

As $i_\mathcal{X} : \partial\mathcal{X} \to \mathcal{X}$ is strongly representable by Definition 12.1, Proposition 8.10 defines an explicit fibre product $\partial\mathcal{X} \times_{i_\mathcal{X}, \mathcal{X}, i_\mathcal{X}} \partial\mathcal{X}$ with strongly representable projection morphisms $\pi_1, \pi_2 : \partial\mathcal{X} \times_\mathcal{X} \partial\mathcal{X} \to \partial\mathcal{X}$ such that $i_\mathcal{X} \circ \pi_1 = i_\mathcal{X} \circ \pi_2$. We will use this explicit fibre product throughout. There is a unique diagonal 1-morphism $\Delta_{\partial\mathcal{X}} : \partial\mathcal{X} \to \partial\mathcal{X} \times_\mathcal{X} \partial\mathcal{X}$ with $\pi_1 \circ \Delta_{\partial\mathcal{X}} = \pi_2 \circ \Delta_{\partial\mathcal{X}} = \mathrm{id}_{\partial\mathcal{X}}$. It is an equivalence with an open and closed C^∞-substack $\Delta_{\partial\mathcal{X}}(\partial\mathcal{X}) \subseteq \partial\mathcal{X} \times_\mathcal{X} \partial\mathcal{X}$. Define $\partial^2\mathcal{X} = \partial\mathcal{X} \times_\mathcal{X} \partial\mathcal{X} \setminus \Delta_{\partial\mathcal{X}}(\partial\mathcal{X})$. Then $\partial^2\mathcal{X}$ is also an open and closed C^∞-substack in $\partial\mathcal{X} \times_\mathcal{X} \partial\mathcal{X}$. Define $i_{\partial\mathcal{X}} = \pi_1|_{\partial^2\mathcal{X}} : \partial^2\mathcal{X} \to \partial\mathcal{X}$. Then $\partial\mathcal{X} = (\partial\mathcal{X}, \partial^2\mathcal{X}, i_{\partial\mathcal{X}})$ is an orbifold with corners, with $\dim(\partial\mathcal{X}) = \dim\mathcal{X} - 1$. Also $i_\mathcal{X} : \partial\mathcal{X} \to \mathcal{X}$ in $\mathcal{X}$ is a 1-morphism $i_\mathcal{X} : \partial\mathcal{X} \to \mathcal{X}$ in $\mathbf{Orb^c}$.

Here is the orbifold analogue of parts of §5.1–§5.2.

Definition 12.6. Let $\mathcal{X} = (\mathcal{X}, \partial\mathcal{X}, i_\mathcal{X})$ and $\mathcal{Y} = (\mathcal{Y}, \partial\mathcal{Y}, i_\mathcal{Y})$ be orbifolds with corners, and $f : \mathcal{X} \to \mathcal{Y}$ a 1-morphism in $\mathbf{Orb^c}$. Consider the C^∞-stack fibre products $\partial\mathcal{X} \times_{f \circ i_\mathcal{X}, \mathcal{Y}, i_\mathcal{Y}} \partial\mathcal{Y}$ and $\mathcal{X} \times_{f, \mathcal{Y}, i_\mathcal{Y}} \partial\mathcal{Y}$. Since $i_\mathcal{Y}$ is strongly representable, we may define these using the explicit construction of Proposition 8.10.

The topological space $(\partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y})_{\text{top}}$ associated to the C^∞-stack $\partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}$ may be written explicitly as

$$(\partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y})_{\text{top}} \cong \big\{ [x',y'] : x' : \underline{\bar{*}} \to \partial\mathcal{X} \text{ and } y' : \underline{\bar{*}} \to \partial\mathcal{Y} \text{ are}$$
$$\text{1-morphisms with } f \circ i_{\mathcal{X}} \circ x' = i_{\mathcal{Y}} \circ y' : \underline{\bar{*}} \to \mathcal{Y} \big\}, \tag{12.1}$$

where $[x',y']$ in (12.1) denotes the $\sim$-equivalence class of pairs (x',y'), with $(x',y') \sim (\tilde{x}',\tilde{y}')$ if there exist 2-morphisms $\eta : x' \Rightarrow \tilde{x}'$ and $\zeta : y' \Rightarrow \tilde{y}'$ with $\mathrm{id}_{f \circ i_{\mathcal{X}}} * \eta = \mathrm{id}_{i_{\mathcal{Y}}} * \zeta$. There is a natural open and closed C^∞-substack $\mathcal{S}_f \subseteq \partial\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}$, the analogue of S_f in §5.1, such that $[x',y']$ in (12.1) lies in $\mathcal{S}_{f,\text{top}}$ if and only if we can complete the following commutative diagram in $\mathrm{qcoh}(\underline{\bar{*}})$ with morphisms '$-\!\!\to$' as shown:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (y')^*(\mathcal{N}_{\mathcal{Y}}) & \xrightarrow{(y')^*(\nu_{\mathcal{Y}})} & (y')^* \circ i_{\mathcal{Y}}^*(T^*\mathcal{Y}) & \xrightarrow{(y')^*(\Omega_{i_{\mathcal{Y}}})} & (y')^*(T^*(\partial\mathcal{Y})) & \longrightarrow & 0 \\
& & \cong \big\downarrow & & \big\downarrow{\scriptstyle I_{x',i_{\mathcal{X}}}(T^*\mathcal{X}) \circ (i_{\mathcal{X}} \circ x')^*(\Omega_f) \circ} & & \big\downarrow & & \\
& & & & {\scriptstyle I_{i_{\mathcal{X}} \circ x',f}(T^*\mathcal{Y}) \circ I_{y',i_{\mathcal{Y}}}(T^*\mathcal{Y})^{-1}} & & & & \\
0 & \longrightarrow & (x')^*(\mathcal{N}_{\mathcal{X}}) & \xrightarrow[(x')^*(\nu_{\mathcal{X}})]{} & (x')^* \circ i_{\mathcal{X}}^*(T^*\mathcal{X}) & \xrightarrow[(x')^*(\Omega_{i_{\mathcal{X}}})]{} & (x')^*(T^*(\partial\mathcal{X})) & \longrightarrow & 0,
\end{array}
$$

where $\mathcal{N}_{\mathcal{X}}, \mathcal{N}_{\mathcal{Y}}$ are the *conormal line bundles* of $\partial\mathcal{X}, \partial\mathcal{Y}$ in $\mathcal{X}, \mathcal{Y}$.

Similarly, the topological space $(\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y})_{\text{top}}$ may be written explicitly as

$$(\mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y})_{\text{top}} \cong \big\{ [x,y'] : x : \underline{\bar{*}} \to \mathcal{X} \text{ and } y' : \underline{\bar{*}} \to \partial\mathcal{Y} \text{ are}$$
$$\text{1-morphisms with } f \circ x = i_{\mathcal{Y}} \circ y' : \underline{\bar{*}} \to \mathcal{Y} \big\}, \tag{12.2}$$

where $[x,y']$ in (12.2) denotes the $\approx$-equivalence class of (x,y'), with $(x,y') \approx (\tilde{x},\tilde{y}')$ if there exist $\eta : x \Rightarrow \tilde{x}$ and $\zeta : y' \Rightarrow \tilde{y}'$ with $\mathrm{id}_f * \eta = \mathrm{id}_{i_{\mathcal{Y}}} * \zeta$. There is a natural open and closed C^∞-substack $\mathcal{T}_f \subseteq \mathcal{X} \times_{\mathcal{Y}} \partial\mathcal{Y}$, the analogue of T_f in §5.1, such that $[x,y']$ in (12.2) lies in $\mathcal{T}_{f,\text{top}}$ if and only if we can complete the following commutative diagram in $\mathrm{qcoh}(\underline{\bar{*}})$:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & (y')^*(\mathcal{N}_{\mathcal{Y}}) & \xrightarrow[(y')^*(\nu_{\mathcal{Y}})]{} & (y')^* \circ i_{\mathcal{Y}}^*(T^*\mathcal{Y}) & \xrightarrow[(y')^*(\Omega_{i_{\mathcal{Y}}})]{} & (y')^*(T^*(\partial\mathcal{Y})) \longrightarrow 0 \\
& {\scriptstyle x^*(\Omega_f) \circ I_{x,f}(T^*\mathcal{Y}) \circ I_{y',i_{\mathcal{Y}}}(T^*\mathcal{Y})^{-1}} \big\downarrow & & & & & \\
& & & & x^*(T^*\mathcal{X}). & &
\end{array}
$$

Define $s_f = \pi_{\partial\mathcal{X}}|_{\mathcal{S}_f} : \mathcal{S}_f \to \partial\mathcal{X}$, $u_f = \pi_{\partial\mathcal{Y}}|_{\mathcal{S}_f} : \mathcal{S}_f \to \partial\mathcal{Y}$, $t_f = \pi_{\mathcal{X}}|_{\mathcal{T}_f} : \mathcal{T}_f \to \mathcal{X}$, and $v_f = \pi_{\partial\mathcal{Y}}|_{\mathcal{T}_f} : \mathcal{T}_f \to \partial\mathcal{Y}$. Then s_f, t_f are proper, étale 1-morphisms. We call f *simple* if $s_f : \mathcal{S}_f \to \partial\mathcal{X}$ is an equivalence, and we call f *semisimple* if $s_f : \mathcal{S}_f \to \partial\mathcal{X}$ is injective as a 1-morphism of Deligne–Mumford C^∞-stacks, and we call f *flat* if $\mathcal{T}_f = \emptyset$. Simple implies semisimple.

The condition that $i_{\mathcal{X}}$ is strongly representable in Definition 12.1 is essential in constructing f_-, η_- in parts (b),(c) of the next theorem.

Theorem 12.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a semisimple 1-morphism of orbifolds with corners. Then there is a natural decomposition $\partial\mathcal{X} = \partial_+^f \mathcal{X} \amalg \partial_-^f \mathcal{X}$, where $\partial_\pm^f \mathcal{X}$ are open and closed suborbifolds in $\partial\mathcal{X}$, such that:*

(a) *Define $f_+ = f \circ i_{\mathcal{X}}|_{\partial^f_+ \mathcal{X}} : \partial^f_+ \mathcal{X} \to \mathcal{Y}$. Then f_+ is semisimple. If f is flat then f_+ is also flat.*

(b) *There exists a unique, semisimple 1-morphism $f_- : \partial^f_- \mathcal{X} \to \partial \mathcal{Y}$ in $\mathbf{Orb^c}$ with $f \circ i_{\mathcal{X}}|_{\partial^f_- \mathcal{X}} = i_{\mathcal{Y}} \circ f_-$. If f is simple then $\partial^f_+ \mathcal{X} = \emptyset$, $\partial^f_- \mathcal{X} = \partial \mathcal{X}$ and $f_- : \partial \mathcal{X} \to \partial \mathcal{Y}$ is simple. If f is flat then f_- is flat.*

(c) *Let $g : \mathcal{X} \to \mathcal{Y}$ be another 1-morphism and $\eta : f \Rightarrow g$ a 2-morphism in $\mathbf{Orb^c}$. Then g is also semisimple, with $\partial^g_- \mathcal{X} = \partial^f_- \mathcal{X}$. If f is simple, or flat, then g is too. Part **(b)** defines 1-morphisms $f_-, g_- : \partial^f_- \mathcal{X} \to \partial \mathcal{Y}$. There is a unique 2-morphism $\eta_- : f_- \Rightarrow g_-$ in $\mathbf{Orb^c}$ such that*

$$\mathrm{id}_{i_{\mathcal{Y}}} * \eta_- = \eta * \mathrm{id}_{i_{\mathcal{X}}|_{\partial^f_- \mathcal{X}}} : f \circ i_{\mathcal{X}}|_{\partial^f_- \mathcal{X}} = i_{\mathcal{Y}} \circ f_- \Longrightarrow g \circ i_{\mathcal{X}}|_{\partial^f_- \mathcal{X}} = i_{\mathcal{Y}} \circ g_-.$$

12.3 Corners $C_k(\mathcal{X})$ and the corner functors $C, \hat{C}$

In [35, §8.7] we extend §5.3 to orbifolds. Here is the orbifold analogue of the category $\mathbf{\check{M}an^c}$ in Definition 5.10.

Definition 12.8. We will define a 2-category $\mathbf{\check{O}rb^c}$ whose objects are disjoint unions $\coprod_{m=0}^{\infty} \mathcal{X}_m$, where $\mathcal{X}_m$ is a (possibly empty) orbifold with corners of dimension m. In more detail, objects of $\mathbf{\check{O}rb^c}$ are triples $\mathcal{X} = (\mathcal{X}, \partial \mathcal{X}, i_{\mathcal{X}})$ with $i_{\mathcal{X}} : \partial \mathcal{X} \to \mathcal{X}$ a strongly representable 1-morphism of Deligne–Mumford C^∞-stacks, such that there exists a decomposition $\mathcal{X} = \coprod_{m=0}^{\infty} \mathcal{X}_m$ with each $\mathcal{X}_m \subseteq \mathcal{X}$ an open and closed C^∞-substack, for which $\mathcal{X}_m := \left(\mathcal{X}_m, i_{\mathcal{X}}^{-1}(\mathcal{X}_m), i_{\mathcal{X}}|_{i_{\mathcal{X}}^{-1}(\mathcal{X}_m)}\right)$ is an orbifold with corners of dimension m.

A 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\mathbf{\check{O}rb^c}$ is a 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\mathbf{C^\infty Sta}$ such that $f|_{\mathcal{X}_m \cap f^{-1}(\mathcal{Y}_n)} : \left(\mathcal{X}_m \cap f^{-1}(\mathcal{Y}_n)\right) \to \mathcal{Y}_n$ is a 1-morphism in $\mathbf{Orb^c}$ for all $m, n \geqslant 0$. For 1-morphisms $f, g : \mathcal{X} \to \mathcal{Y}$, a 2-morphism $\eta : f \Rightarrow g$ is a 2-morphism $\eta : f \Rightarrow g$ in $\mathbf{C^\infty Sta}$. Then $\mathbf{Orb^c}$ is a full 2-subcategory of $\mathbf{\check{O}rb^c}$.

The next theorem summarizes our results on corners functors in $\mathbf{Orb^c}$.

Theorem 12.9. (a) *Suppose $\mathcal{X}$ is an orbifold with corners. Then for each $k = 0, 1, \ldots, \dim \mathcal{X}$ we can define an orbifold with corners $C_k(\mathcal{X})$ of dimension $\dim \mathcal{X} - k$ called the k-**corners** of $\mathcal{X}$, and a 1-morphism $\Pi_{\mathcal{X}}^k : C_k(\mathcal{X}) \to \mathcal{X}$ in $\mathbf{Orb^c}$. It has topological space*

$$C_k(\mathcal{X})_{\mathrm{top}} \cong \big\{ [x, \{x'_1, \ldots, x'_k\}] : x : \underline{\bar{*}} \to \mathcal{X},\ x'_i : \underline{\bar{*}} \to \partial \mathcal{X} \text{ are 1-morphisms} \atop \text{with } x'_1, \ldots, x'_k \text{ distinct and } x = i_{\mathcal{X}} \circ x'_1 = \cdots = i_{\mathcal{X}} \circ x'_k \big\}. \tag{12.3}$$

There is a natural action of the symmetric group S_k on $\partial^k \mathcal{X}$ by 1-isomorphisms, and an equivalence $C_k(\mathcal{X}) \simeq \partial^k \mathcal{X}/S_k$. We have 1-isomorphisms $C_0(\mathcal{X}) \cong \mathcal{X}$ and $C_1(\mathcal{X}) \cong \partial \mathcal{X}$ in $\mathbf{Orb^c}$. Write $C(\mathcal{X}) = \coprod_{k=0}^{\dim \mathcal{X}} C_k(\mathcal{X})$ and $\Pi_{\mathcal{X}} = \coprod_{k=0}^{\dim \mathcal{X}} \Pi_{\mathcal{X}}^k$, so that $C(\mathcal{X})$ is an object and $\Pi_{\mathcal{X}} : C(\mathcal{X}) \to \mathcal{X}$ a 1-morphism in $\mathbf{\check{O}rb^c}$.

(b) *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of orbifolds with corners. Then there is a unique 1-morphism $C(f) : C(\mathcal{X}) \to C(\mathcal{Y})$ in $\check{\mathbf{Orb}}^{\mathbf{c}}$ such that $\Pi_{\mathcal{Y}} \circ C(f) = f \circ \Pi_{\mathcal{X}} : C(\mathcal{X}) \to \mathcal{Y}$, and $C(f)$ acts on points as in (12.3) by*

$$
\begin{aligned}
C(f)_{\mathrm{top}} : \big[x, \{x_1', \ldots, x_k'\}\big] &\longmapsto \big[y, \{y_1', \ldots, y_l'\}\big], \quad \text{where } y = f \circ x, \\
\text{and} \quad \{y_1', \ldots, y_l'\} &= \big\{y' : [x_i', y'] \in \mathcal{S}_{f,\mathrm{top}}, \text{ some } i = 1, \ldots, k\big\},
\end{aligned} \tag{12.4}
$$

where $\mathcal{S}_f$ is as in Definition 12.6.

For all $k, l \geqslant 0$, write $C_k^{f,l}(\mathcal{X}) = C_k(\mathcal{X}) \cap C(f)^{-1}(C_l(\mathcal{Y}))$, so that $C_k^{f,l}(\mathcal{X})$ is open and closed in $C_k(\mathcal{X})$ with $C_k(\mathcal{X}) = \coprod_{l=0}^{\dim \mathcal{Y}} C_k^{f,l}(\mathcal{X})$, and write $C_k^l(f) = C(f)|_{C_k^{f,l}(\mathcal{X})}$, so that $C_k^l(f) : C_k^{f,l}(\mathcal{X}) \to C_l(\mathcal{Y})$ is a 1-morphism in $\mathbf{Orb}^{\mathbf{c}}$.

(c) *Let $f, g : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms and $\eta : f \Rightarrow g$ a 2-morphism in $\mathbf{Orb}^{\mathbf{c}}$. Then there exists a unique 2-morphism $C(\eta) : C(f) \Rightarrow C(g)$ in $\check{\mathbf{Orb}}^{\mathbf{c}}$, where $C(f), C(g)$ are as in (b), such that*

$$
\mathrm{id}_{\Pi_{\mathcal{Y}}} * C(\eta) = \eta * \mathrm{id}_{\Pi_{\mathcal{X}}} : \Pi_{\mathcal{Y}} \circ C(f) = f \circ \Pi_{\mathcal{X}} \Longrightarrow \Pi_{\mathcal{Y}} \circ C(g) = g \circ \Pi_{\mathcal{X}}.
$$

(d) *Define $C : \mathbf{Orb}^{\mathbf{c}} \to \check{\mathbf{Orb}}^{\mathbf{c}}$ by $C : \mathcal{X} \mapsto C(\mathcal{X})$ on objects, $C : f \mapsto C(f)$ on 1-morphisms, and $C : \eta \mapsto C(\eta)$ on 2-morphisms, where $C(\mathcal{X}), C(f), C(\eta)$ are as in (a)–(c) above. Then C is a strict 2-functor, called a* **corner functor***.*

(e) *Let $f : \mathcal{X} \to \mathcal{Y}$ be semisimple. Then $C(f)$ maps $C_k(\mathcal{X}) \to \coprod_{l=0}^k C_l(\mathcal{Y})$ for all $k \geqslant 0$. The natural 1-isomorphisms $C_1(\mathcal{X}) \cong \partial\mathcal{X}$, $C_0(\mathcal{Y}) \cong \mathcal{Y}$, $C_1(\mathcal{Y}) \cong \partial\mathcal{Y}$ identify $C_1^{f,0}(\mathcal{X}) \cong \partial_+^f \mathcal{X}$, $C_1^{f,1}(\mathcal{X}) \cong \partial_-^f \mathcal{X}$, $C_1^0(f) \cong f_+$ and $C_1^1(f) \cong f_-$.*

If f is simple then $C(f)$ maps $C_k(\mathcal{X}) \to C_k(\mathcal{Y})$ for all $k \geqslant 0$.

(f) *Analogues of (b)–(d) also hold for a second corner functor $\hat{C} : \mathbf{Orb}^{\mathbf{c}} \to \check{\mathbf{Orb}}^{\mathbf{c}}$, which acts on objects by $\hat{C} : \mathcal{X} \mapsto C(\mathcal{X})$ in (a), and for 1-morphisms $f : \mathcal{X} \to \mathcal{Y}$ in (b), $\hat{C}(f) : C(\mathcal{X}) \to C(\mathcal{Y})$ acts on points by*

$$
\begin{aligned}
\hat{C}(f)_{\mathrm{top}} : \big[x, \{x_1', \ldots, x_k'\}\big] &\longmapsto \big[y, \{y_1', \ldots, y_l'\}\big], \quad \text{where } y = f \circ x, \\
\{y_1', \ldots, y_l'\} = \big\{y' : [x_i', y'] &\in \mathcal{S}_{f,\mathrm{top}}, \; i = 1, \ldots, k\big\} \cup \big\{y' : [x, y'] \in \mathcal{T}_{f,\mathrm{top}}\big\}.
\end{aligned}
$$

If f is flat then $\hat{C}(f) = C(f)$.

Example 12.10. Suppose $\mathcal{X}$ is a quotient $[X/G]$ as in Example 12.2, where X is a manifold with corners and G is a finite group. Then the action $r : G \to \mathrm{Aut}(X)$ lifts to $C(r) : G \to \mathrm{Aut}(C(X))$, and there is an equivalence $C([X/G]) \simeq [C(X)/G]$ in $\check{\mathbf{Orb}}^{\mathbf{c}}$, where to define $[C(X)/G]$ we note that Example 12.2 also works with X in $\check{\mathbf{Man}}^{\mathbf{c}}$ rather than $\mathbf{Man}^{\mathbf{c}}$, yielding $[X/G] \in \check{\mathbf{Orb}}^{\mathbf{c}}$.

Section 5.2 defined (s-)submersions, (s- or sf-)immersions and (s- or sf-) embeddings in $\mathbf{Man}^{\mathbf{c}}$. Section 9.1 defined submersions, immersions and embeddings in $\mathbf{Orb}$. We combine the two definitions.

Definition 12.11. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of orbifolds with corners.

(i) We call f a *submersion* if $\Omega_{C(f)} : C(f)^*(T^*C(\mathcal{Y})) \to T^*C(\mathcal{X})$ is an injective morphism of vector bundles, i.e. has a left inverse in $\mathrm{qcoh}(C(\mathcal{X}))$, and f is semisimple and flat. We call f an *s-submersion* if f is also simple.

(ii) We call f an *immersion* if it is representable and $\Omega_f : f^*(T^*\mathcal{Y}) \to T^*\mathcal{X}$ is a surjective morphism of vector bundles, i.e. has a right inverse in qcoh($\mathcal{X}$). We call f an *s-immersion* if f is also simple, and an *sf-immersion* if f is also simple and flat.

(iii) We call f an *embedding, s-embedding,* or *sf-embedding*, if it is an immersion, s-immersion, or sf-immersion, respectively, and $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}\big(f_{\mathrm{top}}([x])\big)$ is an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ is a homeomorphism with its image (so in particular it is injective).

Then submersions, ..., sf-embeddings in $\mathbf{Orb^c}$ are étale locally modelled on submersions, ..., sf-embeddings in $\mathbf{Man^c}$.

12.4 Transversality and fibre products

Section 5.4 discussed transversality and fibre products for manifolds with corners. In [35, §8.8] we generalize this to orbifolds with corners.

Definition 12.12. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be orbifolds with corners and $g : \mathcal{X} \to \mathcal{Z}$, $h : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms. Then as in §13 we have 1-morphisms $C(g) : C(\mathcal{X}) \to C(\mathcal{Z})$ and $C(h) : C(\mathcal{Y}) \to C(\mathcal{Z})$ in $\check{\mathbf{Orb}}^{\mathbf{c}}$, and hence 1-morphisms $C(g) : C(\mathcal{X}) \to C(\mathcal{Z})$ and $C(h) : C(\mathcal{Y}) \to C(\mathcal{Z})$ in $\mathbf{C^\infty Sta}$. We call g, h *transverse* if the following holds. Suppose $x : \bar{\bar{*}} \to C(\mathcal{X})$ and $y : \bar{\bar{*}} \to C(\mathcal{Y})$ are 1-morphisms in $\mathbf{C^\infty Sta}$, and $\eta : C(g) \circ x \Rightarrow C(h) \circ y$ a 2-morphism. Then the following morphism in qcoh($\bar{\bar{*}}$) should be injective:

$$\big(x^*(\Omega_{C(g)}) \circ I_{x,C(g)}(T^*C(\mathcal{Z}))\big) \oplus \big(y^*(\Omega_{C(h)}) \circ I_{y,C(h)}(T^*C(\mathcal{Z})) \circ \eta^*(T^*C(\mathcal{Z}))\big) :$$
$$(C(g) \circ x)^*(T^*C(\mathcal{Z})) \longrightarrow x^*(T^*C(\mathcal{X})) \oplus y^*(T^*C(\mathcal{Y})).$$

Now identify $C_k(\mathcal{X})_{\mathrm{top}} \subseteq C(\mathcal{X})_{\mathrm{top}}$ with the right hand of (12.3), and similarly for $C(\mathcal{Y})_{\mathrm{top}}, C(\mathcal{Z})_{\mathrm{top}}$. Then $C(g)_{\mathrm{top}}, C(h)_{\mathrm{top}}$ act as in (12.4). We call g, h *strongly transverse* if they are transverse, and whenever there are points in $C_j(\mathcal{X})_{\mathrm{top}}, C_k(\mathcal{Y})_{\mathrm{top}}, C_l(\mathcal{Z})_{\mathrm{top}}$ with

$$C(g)_{\mathrm{top}}\big([x, \{x'_1, \dots, x'_j\}]\big) = C(h)_{\mathrm{top}}\big([y, \{y'_1, \dots, y'_k\}]\big) = [z, \{z'_1, \dots, z'_l\}],$$

we have either $j + k > l$ or $j = k = l = 0$.

One can show that g, h are (strongly) transverse if and only if they are étale locally equivalent to (strongly) transverse smooth maps in $\mathbf{Man^c}$.

Here is the analogue of Theorem 5.13:
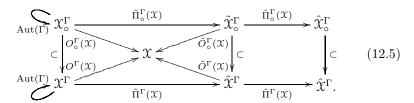
Theorem 12.13. *Suppose $g : \mathcal{X} \to \mathcal{Z}$ and $h : \mathcal{Y} \to \mathcal{Z}$ are transverse 1-morphisms in $\mathbf{Orb^c}$. Then a fibre product $\mathcal{W} = \mathcal{X} \times_{g, \mathcal{Z}, h} \mathcal{Y}$ exists in the 2-category $\mathbf{Orb^c}$.*

Proposition 5.14 and Theorem 5.15 also extend to $\mathbf{Orb^c}$, with equivalences natural up to 2-isomorphism rather than canonical diffeomorphisms.

12.5 Orbifold strata of orbifolds with corners

Sections 8.7 and 9.2 discussed orbifold strata of Deligne–Mumford C^∞-stacks and orbifolds, respectively. In [35, §8.9] we extend this to orbifolds with corners. This is also related to the material on fixed points of finite group actions on manifolds with corners in §5.6.

Theorem 12.14. *Let $\mathfrak{X}$ be an orbifold with corners, and Γ a finite group. Then we can define objects $\mathfrak{X}^\Gamma, \tilde{\mathfrak{X}}^\Gamma, \hat{\mathfrak{X}}^\Gamma$ in $\check{\mathbf{Orb}}^{\mathbf{c}}$, and open subobjects $\mathfrak{X}^\Gamma_\circ \subseteq \mathfrak{X}^\Gamma$, $\tilde{\mathfrak{X}}^\Gamma_\circ \subseteq \tilde{\mathfrak{X}}^\Gamma$, $\hat{\mathfrak{X}}^\Gamma_\circ \subseteq \hat{\mathfrak{X}}^\Gamma$, all natural up to 1-isomorphism in $\check{\mathbf{Orb}}^{\mathbf{c}}$, and 1-morphisms $O^\Gamma(\mathfrak{X}), \tilde{\Pi}^\Gamma(\mathfrak{X}), \ldots$ fitting into a strictly commutative diagram in $\check{\mathbf{Orb}}^{\mathbf{c}}$:*

$$(12.5)$$

The underlying C^∞-stacks of $\mathfrak{X}^\Gamma, \ldots, \hat{\mathfrak{X}}^\Gamma_\circ$ are the orbifold strata $\mathcal{X}^\Gamma, \ldots, \hat{\mathcal{X}}^\Gamma_\circ$ from §8.7 of the C^∞-stack $\mathcal{X}$ in $\mathfrak{X}$, and the 1-morphisms in (12.5), as C^∞-stack 1-morphisms, are those given in (8.3).

Use the notation of Definition 9.7. Then there are natural decompositions

$$\mathfrak{X}^\Gamma = \coprod_{\lambda \in \Lambda^\Gamma_+} \mathfrak{X}^{\Gamma,\lambda}, \quad \tilde{\mathfrak{X}}^\Gamma = \coprod_{\mu \in \Lambda^\Gamma_+ / \operatorname{Aut}(\Gamma)} \tilde{\mathfrak{X}}^{\Gamma,\mu}, \quad \hat{\mathfrak{X}}^\Gamma = \coprod_{\mu \in \Lambda^\Gamma_+ / \operatorname{Aut}(\Gamma)} \hat{\mathfrak{X}}^{\Gamma,\mu},$$

$$\mathfrak{X}^\Gamma_\circ = \coprod_{\lambda \in \Lambda^\Gamma_+} \mathfrak{X}^{\Gamma,\lambda}_\circ, \quad \tilde{\mathfrak{X}}^\Gamma_\circ = \coprod_{\mu \in \Lambda^\Gamma_+ / \operatorname{Aut}(\Gamma)} \tilde{\mathfrak{X}}^{\Gamma,\mu}_\circ, \quad \hat{\mathfrak{X}}^\Gamma_\circ = \coprod_{\mu \in \Lambda^\Gamma_+ / \operatorname{Aut}(\Gamma)} \hat{\mathfrak{X}}^{\Gamma,\mu}_\circ,$$

*where $\mathfrak{X}^{\Gamma,\lambda}, \ldots, \hat{\mathfrak{X}}^{\Gamma,\mu}_\circ$ are orbifolds with corners, open and closed in $\mathfrak{X}^\Gamma, \ldots, \hat{\mathfrak{X}}^\Gamma_\circ$, and of dimensions $\dim \mathfrak{X} - \dim \lambda, \dim \mathfrak{X} - \dim \mu$. All of $\mathfrak{X}^\Gamma, \tilde{\mathfrak{X}}^\Gamma, \hat{\mathfrak{X}}^\Gamma, \mathfrak{X}^\Gamma_\circ, \tilde{\mathfrak{X}}^\Gamma_\circ, \hat{\mathfrak{X}}^\Gamma_\circ, \mathfrak{X}^{\Gamma,\lambda}, \tilde{\mathfrak{X}}^{\Gamma,\mu}, \hat{\mathfrak{X}}^{\Gamma,\mu}, \mathfrak{X}^{\Gamma,\lambda}_\circ, \tilde{\mathfrak{X}}^{\Gamma,\mu}_\circ, \hat{\mathfrak{X}}^{\Gamma,\mu}_\circ$ will be called **orbifold strata** of $\mathfrak{X}$.*

The definitions of $\mathfrak{X}^\Gamma, \tilde{\mathfrak{X}}^\Gamma, \ldots, \hat{\mathfrak{X}}^\Gamma_\circ$ also make sense if $\mathfrak{X}$ lies in $\check{\mathbf{Orb}}^{\mathbf{c}}$ rather than $\mathbf{Orb}^{\mathbf{c}}$. We will not use notation $\mathfrak{X}^{\Gamma,\lambda}, \ldots, \hat{\mathfrak{X}}^{\Gamma,\mu}_\circ$ for $\mathfrak{X} \in \check{\mathbf{Orb}}^{\mathbf{c}} \setminus \mathbf{Orb}^{\mathbf{c}}$.

As for Deligne–Mumford C^∞-stacks in §8.7, orbifold strata $\mathfrak{X}^\Gamma$ are strongly functorial for representable 1-morphisms in $\mathbf{Orb}^{\mathbf{c}}$ and their 2-morphisms. That is, if $f : \mathfrak{X} \to \mathfrak{Y}$ is a representable 1-morphism in $\mathbf{Orb}^{\mathbf{c}}$, there is a unique representable 1-morphism $f^\Gamma : \mathfrak{X}^\Gamma \to \mathfrak{Y}^\Gamma$ in $\check{\mathbf{Orb}}^{\mathbf{c}}$ with $O^\Gamma(\mathfrak{Y}) \circ f^\Gamma = f \circ O^\Gamma(\mathfrak{X})$, which is just the 1-morphism f^Γ from §8.7 for the C^∞-stack 1-morphism $f : \mathcal{X} \to \mathcal{Y}$. Note however that f^Γ need not map $\mathfrak{X}^{\Gamma,\lambda} \to \mathfrak{Y}^{\Gamma,\lambda}$ for $\lambda \in \Lambda^\Gamma_+$.

If $f, g : \mathfrak{X} \to \mathfrak{Y}$ are representable and $\eta : f \Rightarrow g$ is a 2-morphism in $\mathbf{Orb}^{\mathbf{c}}$, there is a unique 2-morphism $\eta^\Gamma : f^\Gamma \Rightarrow g^\Gamma$ in $\check{\mathbf{Orb}}^{\mathbf{c}}$ with $\operatorname{id}_{O^\Gamma(\mathfrak{Y})} * \eta^\Gamma = \eta * \operatorname{id}_{O^\Gamma(\mathfrak{X})}$, which is just the C^∞-stack 2-morphism η^Γ from §8.7. These f^Γ, η^Γ are compatible with compositions of 1- and 2-morphisms, and identities, in the obvious way. Orbifold strata $\tilde{\mathfrak{X}}^\Gamma$ have the same strong functorial behaviour, and orbifold strata $\hat{\mathfrak{X}}^\Gamma$ a weaker functorial behaviour.

We also investigate the relationship between orbifold strata and corners.

Theorem 12.15. *Let $\mathfrak{X}$ be an orbifold with corners, and Γ a finite group. The corners $C(\mathfrak{X})$ lie in $\check{\mathbf{Orb}}^{\mathbf{c}}$ as in §12.3, so we have orbifold strata $\mathfrak{X}^{\Gamma}, C(\mathfrak{X})^{\Gamma}$ and 1-morphisms $O^{\Gamma}(\mathfrak{X}) : \mathfrak{X}^{\Gamma} \to \mathfrak{X}$, $O^{\Gamma}(C(\mathfrak{X})) : C(\mathfrak{X})^{\Gamma} \to C(\mathfrak{X})$. Applying the corner functor C from §12.3 gives a 1-morphism $C(O^{\Gamma}(\mathfrak{X})) : C(\mathfrak{X}^{\Gamma}) \to C(\mathfrak{X})$. Then there exists a unique equivalence $K^{\Gamma}(\mathfrak{X}) : C(\mathfrak{X}^{\Gamma}) \to C(\mathfrak{X})^{\Gamma}$ such that $O^{\Gamma}(C(\mathfrak{X})) \circ K^{\Gamma}(\mathfrak{X}) = C(O^{\Gamma}(\mathfrak{X})) : C(\mathfrak{X}^{\Gamma}) \to C(\mathfrak{X})$. It restricts to an equivalence $K_{\circ}^{\Gamma}(\mathfrak{X}) := K^{\Gamma}(\mathfrak{X})|_{C(\mathfrak{X}_{\circ}^{\Gamma})} : C(\mathfrak{X}_{\circ}^{\Gamma}) \to C(\mathfrak{X})_{\circ}^{\Gamma}$.*

Similarly, there is a unique equivalence $\tilde{K}^{\Gamma}(\mathfrak{X}) : C(\tilde{\mathfrak{X}}^{\Gamma}) \to \widetilde{C(\mathfrak{X})}^{\Gamma}$ with $\tilde{O}^{\Gamma}(C(\mathfrak{X})) \circ \tilde{K}^{\Gamma}(\mathfrak{X}) = C(\tilde{O}^{\Gamma}(\mathfrak{X}))$ and $\tilde{\Pi}^{\Gamma}(C(\mathfrak{X})) \circ K^{\Gamma}(\mathfrak{X}) = \tilde{K}^{\Gamma}(\mathfrak{X}) \circ C(\tilde{\Pi}^{\Gamma}(\mathfrak{X}))$. There is an equivalence $\hat{K}^{\Gamma}(\mathfrak{X}) : C(\hat{\mathfrak{X}}^{\Gamma}) \to \widehat{C(\mathfrak{X})}^{\Gamma}$, unique up to 2-isomorphism, with a 2-morphism $\hat{\Pi}^{\Gamma}(C(\mathfrak{X})) \circ \tilde{K}^{\Gamma}(\mathfrak{X}) \Rightarrow \hat{K}^{\Gamma}(\mathfrak{X}) \circ C(\hat{\Pi}^{\Gamma}(\mathfrak{X}))$. They both restrict to equivalences $\tilde{K}_{\circ}^{\Gamma}(\mathfrak{X}) : C(\tilde{\mathfrak{X}}_{\circ}^{\Gamma}) \to \widetilde{C(\mathfrak{X})}_{\circ}^{\Gamma}$ and $\hat{K}_{\circ}^{\Gamma}(\mathfrak{X}) : C(\hat{\mathfrak{X}}_{\circ}^{\Gamma}) \to \widehat{C(\mathfrak{X})}_{\circ}^{\Gamma}$.

Here is an example:

Example 12.16. Let $\mathbb{Z}_2 = \{1, \sigma\}$ with $\sigma^2 = 1$ act on $X = [0, \infty)^2$ by $\sigma : (x_1, x_2) \mapsto (x_2, x_1)$. Then $\mathfrak{X} = \big[[0, \infty)^2 / \mathbb{Z}_2\big]$ is an orbifold with corners. We have $\partial \mathfrak{X} \cong [0, \infty)$ and $\partial^2 \mathfrak{X} \cong *$, so that $C_2(\mathfrak{X}) \simeq [*/S_2] = [*/\mathbb{Z}_2]$. Hence $C(\mathfrak{X}) = C_0(\mathfrak{X}) \amalg C_1(\mathfrak{X}) \amalg C_2(\mathfrak{X})$ with $C_0(\mathfrak{X}) \simeq \big[[0, \infty)^2 / \mathbb{Z}_2\big]$, $C_1(\mathfrak{X}) \simeq [0, \infty)$ and $C_2(\mathfrak{X}) \simeq [*/\mathbb{Z}_2]$. The orbifold strata $\mathfrak{X}^{\Gamma}, \ldots, \hat{\mathfrak{X}}_{\circ}^{\Gamma}$ are given by

$$\mathfrak{X}^{\mathbb{Z}_2} = \mathfrak{X}_{\circ}^{\mathbb{Z}_2} \simeq \tilde{\mathfrak{X}}^{\mathbb{Z}_2} = \tilde{\mathfrak{X}}_{\circ}^{\mathbb{Z}_2} \simeq [0, \infty) \times [*/\mathbb{Z}_2], \qquad \hat{\mathfrak{X}}^{\mathbb{Z}_2} = \hat{\mathfrak{X}}_{\circ}^{\mathbb{Z}_2} \simeq [0, \infty).$$

Therefore

$$C_0(\mathfrak{X}^{\mathbb{Z}_2}) \simeq [0, \infty) \times [*/\mathbb{Z}_2], \qquad C_1(\mathfrak{X}^{\mathbb{Z}_2}) \simeq [*/\mathbb{Z}_2], \qquad C_2(\mathfrak{X}^{\mathbb{Z}_2}) = \emptyset,$$
$$C_0(\mathfrak{X})^{\mathbb{Z}_2} \simeq [0, \infty) \times [*/\mathbb{Z}_2], \qquad C_1(\mathfrak{X})^{\mathbb{Z}_2} = \emptyset, \qquad C_2(\mathfrak{X})^{\mathbb{Z}_2} \simeq [*/\mathbb{Z}_2].$$

We see from this that $K^{\mathbb{Z}_2}(\mathfrak{X}) : C(\mathfrak{X}^{\mathbb{Z}_2}) \to C(\mathfrak{X})^{\mathbb{Z}_2}$ identifies $C_1(\mathfrak{X}^{\mathbb{Z}_2})$ with $C_2(\mathfrak{X})^{\mathbb{Z}_2}$, so $K^{\Gamma}(\mathfrak{X})$ need not map $C_k(\mathfrak{X}^{\Gamma})$ to $C_k(\mathfrak{X})^{\Gamma}$ for $k > 0$. The same applies to $\tilde{K}^{\Gamma}(\mathfrak{X}), \hat{K}^{\Gamma}(\mathfrak{X})$.

The construction of $K^{\Gamma}(\mathfrak{X})$ in Theorem 12.15 implies that it maps $C_k(\mathfrak{X}^{\Gamma})$ into $\coprod_{l \geqslant k} C_l(\mathfrak{X})^{\Gamma}$ for $k > 0$. This implies that $C_1(\mathfrak{X})^{\Gamma} \simeq (\partial\mathfrak{X})^{\Gamma}$ is equivalent to an open and closed subobject of $C_1(\mathfrak{X}^{\Gamma}) \simeq \partial(\mathfrak{X}^{\Gamma})$. Hence we can choose a 1-morphism $J^{\Gamma}(\mathfrak{X}) : (\partial\mathfrak{X})^{\Gamma} \to \partial(\mathfrak{X}^{\Gamma})$ identified with a quasi-inverse for $K^{\Gamma}(\mathfrak{X})|_{\ldots} : K^{\Gamma}(\mathfrak{X})^{-1}(C_1(\mathfrak{X})^{\Gamma}) \to C_1(\mathfrak{X})^{\Gamma}$ by the equivalences $C_1(\mathfrak{X})^{\Gamma} \simeq (\partial\mathfrak{X})^{\Gamma}$ and $C_1(\mathfrak{X}^{\Gamma}) \simeq \partial(\mathfrak{X}^{\Gamma})$, and $J^{\Gamma}(\mathfrak{X})$ is an equivalence between $(\partial\mathfrak{X})^{\Gamma}$ and an open and closed subobject of $\partial(\mathfrak{X}^{\Gamma})$. We then deduce:

Corollary 12.17. *Let $\mathfrak{X}$ be an orbifold with corners, and Γ a finite group. Then there exist 1-morphisms $J^{\Gamma}(\mathfrak{X}) : (\partial\mathfrak{X})^{\Gamma} \to \partial(\mathfrak{X}^{\Gamma})$, $\tilde{J}^{\Gamma}(\mathfrak{X}) : (\widetilde{\partial\mathfrak{X}})^{\Gamma} \to \partial(\tilde{\mathfrak{X}}^{\Gamma})$, $\hat{J}^{\Gamma}(\mathfrak{X}) : (\widehat{\partial\mathfrak{X}})^{\Gamma} \to \partial(\hat{\mathfrak{X}}^{\Gamma})$ in $\check{\mathbf{Orb}}^{\mathbf{c}}$, natural up to 2-isomorphism, such that $J^{\Gamma}(\mathfrak{X})$ is an equivalence from $(\partial\mathfrak{X})^{\Gamma}$ to an open and closed subobject of $\partial(\mathfrak{X}^{\Gamma})$, and similarly for $\tilde{J}^{\Gamma}(\mathfrak{X}), \hat{J}^{\Gamma}(\mathfrak{X})$.*

For $\lambda \in \Lambda_{+}^{\Gamma}$, $\mu \in \Lambda_{+}^{\Gamma} / \operatorname{Aut}(\Lambda)$ these restrict to 1-morphisms $J^{\Gamma, \lambda}(\mathfrak{X}) : (\partial\mathfrak{X})^{\Gamma, \lambda} \to \partial(\mathfrak{X}^{\Gamma, \lambda})$, $\tilde{J}^{\Gamma, \mu}(\mathfrak{X}) : (\widetilde{\partial\mathfrak{X}})^{\Gamma, \mu} \to \partial(\tilde{\mathfrak{X}}^{\Gamma, \mu})$, $\hat{J}^{\Gamma, \mu}(\mathfrak{X}) : (\widehat{\partial\mathfrak{X}})^{\Gamma, \mu} \to \partial(\hat{\mathfrak{X}}^{\Gamma, \mu})$

in **Orb^c**, *which are equivalences with open and closed suborbifolds. Hence, if* $\mathcal{X}^{\Gamma,\lambda} = \emptyset$ *then* $(\partial\mathcal{X})^{\Gamma,\lambda} = \emptyset$, *and similarly for* $\tilde{\mathcal{X}}^{\Gamma,\mu}, (\widetilde{\partial\mathcal{X}})^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}, (\widehat{\partial\mathcal{X}})^{\Gamma,\mu}.$

As in Remark 12.3(c), an orbifold with corners $\mathcal{X}$ is called *straight* if $(i_\mathcal{X})_* :$ $\mathrm{Iso}_{\partial\mathcal{X}}([x']) \to \mathrm{Iso}_\mathcal{X}([x])$ is an isomorphism for all $[x'] \in \partial\mathcal{X}_{\mathrm{top}}$ with $i_{\mathcal{X},\mathrm{top}}([x']) =$ $[x]$. If $\mathcal{X}$ is straight then $K^\Gamma(\mathcal{X})$ in Theorem 12.15 is an equivalence $C_k(\mathcal{X}^\Gamma) \to$ $C_k(\mathcal{X})^\Gamma$ for all $k \geqslant 0$, and so $J^\Gamma(\mathcal{X})$ in Corollary 12.17 is an equivalence $(\partial\mathcal{X})^\Gamma \to$ $\partial(\mathcal{X}^\Gamma)$. The same applies for $\tilde{J}^\Gamma(\mathcal{X}), \hat{J}^\Gamma(\mathcal{X}), \tilde{K}^\Gamma(\mathcal{X}), \hat{K}^\Gamma(\mathcal{X})$.

Proposition 9.9 on orientations of orbifold strata $\mathcal{X}^{\Gamma,\lambda}, \ldots, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ of oriented orbifolds $\mathcal{X}$ also holds without change for orbifolds with corners $\mathcal{X}$.

13 D-stacks with corners

In [35, Chap. 11] we define and discuss the 2-category **dSta^c** of *d-stacks with corners*. There are few new issues here: almost all the material just combines ideas we have seen already on d-spaces with corners from §6, and on d-stacks from §10, and on orbifolds with corners from §12. So we will be brief.

13.1 Outline of the definition of the 2-category dSta^c

The definition of the 2-category **dSta^c** in [35, §11.1] is long and complicated. So as for **dSpa^c** in §6.1, we will just sketch the main ideas.

A *d-stack with corners* is a quadruple $\boldsymbol{\mathcal{X}} = (\boldsymbol{\mathcal{X}}, \partial\boldsymbol{\mathcal{X}}, \boldsymbol{i_\mathcal{X}}, \omega_\mathcal{X})$, where $\boldsymbol{\mathcal{X}}, \partial\boldsymbol{\mathcal{X}}$ are d-stacks and $\boldsymbol{i_\mathcal{X}} : \partial\boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}$ is a 1-morphism of d-stacks with $i_\mathcal{X} : \partial\mathcal{X} \to \mathcal{X}$ a proper, strongly representable 1-morphism of Deligne–Mumford C^∞-stacks, as in §8.3. We should have an exact sequence in qcoh$(\partial\mathcal{X})$:

$$0 \longrightarrow \mathcal{N}_\mathcal{X} \xrightarrow{\ \nu_\mathcal{X}\ } i_\mathcal{X}^*(\mathcal{F}_\mathcal{X}) \xrightarrow{\ i_\mathcal{X}^2\ } \mathcal{F}_{\partial\mathcal{X}} \longrightarrow 0, \qquad (13.1)$$

where $\mathcal{N}_\mathcal{X}$ is a line bundle on $\partial\mathcal{X}$, the *conormal bundle* of $\partial\boldsymbol{\mathcal{X}}$ in $\boldsymbol{\mathcal{X}}$, and $\omega_\mathcal{X}$ is an orientation on $\mathcal{N}_\mathcal{X}$. These $\boldsymbol{\mathcal{X}}, \partial\boldsymbol{\mathcal{X}}, \boldsymbol{i_\mathcal{X}}, \omega_\mathcal{X}$ must satisfy some complicated conditions in [35, §11.1], that we will not give. They require $\partial\boldsymbol{\mathcal{X}}$ to be locally equivalent to a fibre product $\boldsymbol{\mathcal{X}} \times_{[0,\infty)} *$ in **dSta**.

If $\boldsymbol{\mathcal{X}} = (\boldsymbol{\mathcal{X}}, \partial\boldsymbol{\mathcal{X}}, \boldsymbol{i_\mathcal{X}}, \omega_\mathcal{X})$ and $\boldsymbol{\mathcal{Y}} = (\boldsymbol{\mathcal{Y}}, \partial\boldsymbol{\mathcal{Y}}, \boldsymbol{i_\mathcal{Y}}, \omega_\mathcal{Y})$ are d-stacks with corners, a 1-*morphism* $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in **dSta^c** is a 1-morphism $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in **dSta** satisfying extra conditions over $\partial\boldsymbol{\mathcal{X}}, \partial\boldsymbol{\mathcal{Y}}$. If $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be 1-morphisms in **dSta^c**, so $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ are 1-morphisms in **dSta**, a 2-morphism $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ in **dSta^c** is a 2-morphism $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ in **dSta** satisfying extra conditions over $\partial\boldsymbol{\mathcal{X}}, \partial\boldsymbol{\mathcal{Y}}$. In both cases, 1- and 2-morphisms in **dSta^c** are étale locally modelled on 1- and 2-morphisms in **dSpa^c**. Identity 1- and 2-morphisms in **dSta^c**, and the compositions of 1- and 2-morphisms in **dSta^c**, are all given by identities and compositions in **dSta**.

A d-stack with corners $\boldsymbol{\mathcal{X}} = (\boldsymbol{\mathcal{X}}, \partial\boldsymbol{\mathcal{X}}, \boldsymbol{i_\mathcal{X}}, \omega_\mathcal{X})$ is called a *d-stack with boundary* if $i_\mathcal{X} : \partial\mathcal{X} \to \mathcal{X}$ is injective as a representable 1-morphism of C^∞-stacks,

and a *d-stack without boundary* if $\partial \mathcal{X} = \emptyset$. We write $\mathbf{dSta^b}$ for the full 2-subcategory of d-stacks with boundary, and $\mathbf{d\bar{S}ta}$ for the full 2-subcategory of d-stacks without boundary, in $\mathbf{dSta^c}$. There is an isomorphism of 2-categories $F_{\mathbf{dSta}}^{\mathbf{dSta^c}} : \mathbf{dSta} \to \mathbf{d\bar{S}ta}$ mapping $\mathcal{X} \mapsto \mathfrak{X} = (\mathcal{X}, \emptyset, \emptyset, \emptyset)$ on objects, $\boldsymbol{f} \mapsto \boldsymbol{f}$ on 1-morphisms and $\boldsymbol{\eta} \mapsto \boldsymbol{\eta}$ on 2-morphisms. So we can consider d-stacks to be examples of d-stacks with corners.

Define a strict 2-functor $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}} : \mathbf{dSpa^c} \to \mathbf{dSta^c}$ as follows. If $\mathbf{X} = (X, \partial X, i_{\mathbf{X}}, \omega_{\mathbf{X}})$ is an object in $\mathbf{dSpa^c}$, set $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}(\mathbf{X}) = \mathfrak{X} = (\mathcal{X}, \partial \mathcal{X}, i_{\mathfrak{X}}, \omega_{\mathfrak{X}})$, where $\mathcal{X}, \partial \mathcal{X}, i_{\mathfrak{X}} = F_{\mathbf{dSpa}}^{\mathbf{dSta}}(X, \partial X, i_{\mathbf{X}})$. Then comparing equations (6.2) and (13.1), we find there is a natural isomorphism of line bundles $\mathcal{N}_{\mathfrak{X}} \cong \mathcal{I}_{\partial X}(\mathcal{N}_{\mathbf{X}})$, where $\mathcal{I}_{\partial X} : \mathrm{qcoh}(\partial X) \to \mathrm{qcoh}(\partial \bar{X})$ is the equivalence of categories from Example 8.21. We define $\omega_{\mathfrak{X}}$ to be the orientation on $\mathcal{N}_{\mathfrak{X}}$ identified with the orientation $\mathcal{I}_{\partial X}(\omega_{\mathbf{X}})$ on $\mathcal{I}_{\partial X}(\mathcal{N}_{\mathbf{X}})$ by this isomorphism. On 1- and 2-morphisms $\boldsymbol{f}, \eta$ in $\mathbf{dSpa^c}$, we define $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}(\boldsymbol{f}) = F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\boldsymbol{f})$ and $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}(\eta) = F_{\mathbf{dSpa}}^{\mathbf{dSta}}(\eta)$.

Write $\mathbf{d\hat{S}pa^c}$ for the full 2-subcategory of objects $\mathfrak{X}$ in $\mathbf{dSta^c}$ equivalent to $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}(\mathbf{X})$ for some d-space with corners $\mathbf{X}$. When we say that a d-stack with corners $\mathfrak{X}$ *is a d-space*, we mean that $\mathfrak{X} \in \mathbf{d\hat{S}pa^c}$.

Define a strict 2-functor $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}} : \mathbf{Orb^c} \to \mathbf{dSta^c}$ as follows. If $\mathcal{X} = (\mathcal{X}, \partial \mathcal{X}, i_{\mathcal{X}})$ is an orbifold with corners, as in §12.1, define $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}(\mathcal{X}) = \mathfrak{X} = (\mathcal{X}, \partial \mathcal{X}, i_{\mathfrak{X}}, \omega_{\mathfrak{X}})$, where $\mathcal{X}, \partial \mathcal{X}, i_{\mathfrak{X}} = F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}(\mathcal{X}, \partial \mathcal{X}, i_{\mathcal{X}})$. Then $\mathcal{N}_{\mathfrak{X}}$ in (13.1) is isomorphic to the conormal line bundle of $\partial \mathcal{X}$ in $\mathcal{X}$, and we define $\omega_{\mathfrak{X}}$ to be the orientation on $\mathcal{N}_{\mathfrak{X}}$ induced by 'outward-pointing' normal vectors to $\partial \mathcal{X}$ in $\mathcal{X}$. Then $\mathfrak{X} = (\mathcal{X}, \partial \mathcal{X}, i_{\mathfrak{X}}, \omega_{\mathfrak{X}})$ is a d-orbifold with corners. On 1- and 2-morphisms f, η in $\mathbf{Orb^c}$, we define $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}(f) = F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}(f)$ and $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}(\eta) = F_{\mathbf{C^\infty Sta}}^{\mathbf{dSta}}(\eta)$.

Write $\mathbf{\bar{O}rb}, \mathbf{\bar{O}rb^b}, \mathbf{\bar{O}rb^c}$ for the full 2-subcategories of objects $\mathfrak{X}$ in $\mathbf{dSta^c}$ equivalent to $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}(\mathcal{X})$ for some orbifold $\mathcal{X}$ without boundary, or with boundary, or with corners, respectively. Then $\mathbf{\bar{O}rb} \subset \mathbf{d\bar{S}ta}$, $\mathbf{\bar{O}rb^b} \subset \mathbf{dSta^b}$ and $\mathbf{\bar{O}rb^c} \subset \mathbf{dSta^c}$. When we say that a d-stack with corners $\mathfrak{X}$ *is an orbifold*, we mean that $\mathfrak{X} \in \mathbf{\bar{O}rb^c}$.

Remark 13.1. As discussed for orbifolds with corners in Remark 12.3(b), in a d-stack with corners $\mathfrak{X} = (\mathcal{X}, \partial \mathcal{X}, i_{\mathfrak{X}}, \omega_{\mathfrak{X}})$ we require $i_{\mathfrak{X}} : \partial \mathcal{X} \to \mathcal{X}$ to be *strongly representable*, in the sense of §8.3, so that we can make boundaries and corners in $\mathbf{dSta^c}$ strictly functorial, as in Remark 6.5 for $\mathbf{dSpa^c}$.

For each d-stack with corners $\mathfrak{X} = (\mathcal{X}, \partial \mathcal{X}, i_{\mathfrak{X}}, \omega_{\mathfrak{X}})$, in [35, §11.3] we define a d-stack with corners $\partial \mathfrak{X} = (\partial \mathcal{X}, \partial^2 \mathcal{X}, i_{\partial \mathfrak{X}}, \omega_{\partial \mathfrak{X}})$ called the *boundary* of $\mathfrak{X}$, and show that $i_{\mathfrak{X}} : \partial \mathfrak{X} \to \mathfrak{X}$ is a 1-morphism in $\mathbf{dSta^c}$. As for d-spaces with corners in (6.3), the d-stack $\partial^2 \mathcal{X}$ in $\partial \mathfrak{X}$ satisfies

$$\partial^2 \mathcal{X} \simeq (\partial \mathcal{X} \times_{i_{\mathfrak{X}}, \mathcal{X}, i_{\mathfrak{X}}} \partial \mathcal{X}) \setminus \Delta_{\partial \mathcal{X}}(\partial \mathcal{X}),$$

where $\Delta_{\partial \mathcal{X}} : \partial \mathcal{X} \to \partial \mathcal{X} \times_{\mathcal{X}} \partial \mathcal{X}$ is the diagonal 1-morphism. The 1-morphism $i_{\partial \mathfrak{X}} : \partial^2 \mathcal{X} \to \partial \mathcal{X}$ is projection to the first factor in the fibre product. There is a natural isomorphism $\mathcal{N}_{\partial \mathfrak{X}} \cong i_{\mathfrak{X}}^*(\mathcal{N}_{\mathfrak{X}})$, and the orientation $\omega_{\partial \mathfrak{X}}$ on $\mathcal{N}_{\partial \mathfrak{X}}$ corresponds to the orientation $i_{\mathfrak{X}}^*(\omega_{\mathfrak{X}})$ on $i_{\mathfrak{X}}^*(\mathcal{N}_{\mathfrak{X}})$.

13.2 D-stacks with corners as quotients of d-spaces

Section 10.2 discussed quotient d-stacks $[\boldsymbol{X}/G]$, for $\boldsymbol{X}$ a d-space and $\boldsymbol{r} : G \to \mathrm{Aut}(\boldsymbol{X})$ an action of G on $\boldsymbol{X}$ by 1-isomorphisms. In [35, §11.2] we extend this to d-spaces with corners and d-stacks with corners, and prove:

Theorem 13.2. *Theorems* 10.3 *and* 10.4 *hold unchanged in* $\mathbf{dSta^c}$.

Here if $\mathbf{X} = (\boldsymbol{X}, \boldsymbol{\partial X}, \boldsymbol{i_X}, \omega_\mathbf{X})$ is a d-space with corners and $\boldsymbol{r} : G \to \mathrm{Aut}(\mathbf{X})$ an action of G on $\mathbf{X}$ then each $\boldsymbol{r}(\gamma) : \mathbf{X} \to \mathbf{X}$ for $\gamma \in G$ is simple, so Theorem 6.3(b) gives a lift $\boldsymbol{r}_-(\gamma) : \boldsymbol{\partial X} \to \boldsymbol{\partial X}$, defining an action $\boldsymbol{r}_- : G \to \mathrm{Aut}(\boldsymbol{\partial X})$ of G on $\boldsymbol{\partial X}$. Then $\boldsymbol{r} : G \to \mathrm{Aut}(\boldsymbol{X})$ and $\boldsymbol{r}_- : G \to \mathrm{Aut}(\boldsymbol{\partial X})$ are actions of G on the d-spaces $\boldsymbol{X}, \boldsymbol{\partial X}$, and $\boldsymbol{i_X} : \boldsymbol{\partial X} \to \boldsymbol{X}$ is G-equivariant. So Theorem 10.3(a),(b) give quotient d-stacks $[\boldsymbol{X}/G], [\boldsymbol{\partial X}/G]$ and a quotient 1-morphism $[\boldsymbol{i_X}, \mathrm{id}_G] : [\boldsymbol{\partial X}/G] \to [\boldsymbol{X}/G]$. The quotient d-stack with corners $[\mathbf{X}/G]$ given by the analogue of Theorem 10.3 is defined to be $[\mathbf{X}/G] = \big([\boldsymbol{X}/G], [\boldsymbol{\partial X}/G], [\boldsymbol{i_X}, \mathrm{id}_G], \omega_{[\mathbf{X}/G]}\big)$, for a natural orientation $\omega_{[\mathbf{X}/G]}$ on $\mathcal{N}_{[\mathbf{X}/G]}$ constructed from $\omega_\mathbf{X}$.

In [35, §11.4] we define when a 1-morphism of d-stacks with corners $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ is *étale*. Essentially, $\boldsymbol{f}$ is étale if it is an equivalence locally in the étale topology. It implies that the C^∞-stack 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\boldsymbol{f}$ is étale, and so representable. As for d-stacks in §10.2, we can characterize étale 1-morphisms in $\mathbf{dSta^c}$ using the corners analogue of Theorem 10.4(b) and the definition of étale 1-morphisms in $\mathbf{dSpa^c}$ as (Zariski) local equivalences.

13.3 Simple, semisimple and flat 1-morphisms

In [35, §11.3] we generalize §6.2 to d-stacks with corners. Here is the analogue of Definition 12.6.

Definition 13.3. Let $\mathbf{X} = (\mathcal{X}, \boldsymbol{\partial X}, \boldsymbol{i_X}, \omega_\mathbf{X})$ and $\mathbf{Y} = (\mathcal{Y}, \boldsymbol{\partial Y}, \boldsymbol{i_Y}, \omega_\mathbf{Y})$ be d-stacks with corners, and $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ a 1-morphism in $\mathbf{dSta^c}$. Consider the C^∞-stack fibre products $\partial\mathcal{X} \times_{f \circ i_\mathcal{X}, \mathcal{Y}, i_\mathcal{Y}} \partial\mathcal{Y}$ and $\mathcal{X} \times_{f, \mathcal{Y}, i_\mathcal{Y}} \partial\mathcal{Y}$. Since $i_\mathcal{Y}$ is strongly representable, we may define these using the construction of Proposition 8.10.

As in (12.1), we may write $(\partial\mathcal{X} \times_\mathcal{Y} \partial\mathcal{Y})_{\mathrm{top}}$ explicitly as

$$(\partial\mathcal{X} \times_\mathcal{Y} \partial\mathcal{Y})_{\mathrm{top}} \cong \big\{ [x', y'] : x' : \underline{\bar{*}} \to \partial\mathcal{X} \text{ and } y' : \underline{\bar{*}} \to \partial\mathcal{Y} \text{ are} \tag{13.2}$$
$$\text{1-morphisms with } f \circ i_\mathcal{X} \circ x' = i_\mathcal{Y} \circ y' : \underline{\bar{*}} \to \mathcal{Y} \big\},$$

where $[x', y']$ in (13.2) denotes the $\sim$-equivalence class of pairs (x', y'), with $(x', y') \sim (\tilde{x}', \tilde{y}')$ if there exist 2-morphisms $\eta : x' \Rightarrow \tilde{x}'$ and $\zeta : y' \Rightarrow \tilde{y}'$ with $\mathrm{id}_{f \circ i_\mathcal{X}} * \eta = \mathrm{id}_{i_\mathcal{Y}} * \zeta$. There is a natural open and closed C^∞-substack $\mathcal{S}_{\boldsymbol{f}} \subseteq \partial\mathcal{X} \times_\mathcal{Y} \partial\mathcal{Y}$ such that $[x', y']$ in (13.2) lies in $\mathcal{S}_{\boldsymbol{f}, \mathrm{top}}$ if and only if we can

complete the following commutative diagram in qcoh($\underline{\bar{*}}$) with morphisms '$-\dashrightarrow$':

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (y')^*(\mathcal{N}_{\mathcal{Y}}) & \xrightarrow{(y')^*(\nu_{\mathcal{Y}})} & (y')^* \circ i_{\mathcal{Y}}^*(\mathcal{F}_{\mathcal{Y}}) & \xrightarrow{(y')^*(i_{\mathcal{Y}}^2)} & (y')^*(\mathcal{F}_{\partial \mathcal{Y}}) & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \begin{array}{l} I_{x',i_{\mathcal{X}}}(\mathcal{F}_{\mathcal{X}}) \circ (i_{\mathcal{X}} \circ x')^*(f^2) \circ \\ I_{i_{\mathcal{X}} \circ x',f}(\mathcal{F}_{\mathcal{Y}}) \circ I_{y',i_{\mathcal{Y}}}(\mathcal{F}_{\mathcal{Y}})^{-1} \end{array}} & & \Big\downarrow & & \\
0 & \longrightarrow & (x')^*(\mathcal{N}_{\mathcal{X}}) & \xrightarrow[(x')^*(\nu_{\mathcal{X}})]{} & (x')^* \circ i_{\mathcal{X}}^*(\mathcal{F}_{\mathcal{X}}) & \xrightarrow[(x')^*(i_{\mathcal{X}}^2)]{} & (x')^*(\mathcal{F}_{\partial \mathcal{X}}) & \longrightarrow & 0.
\end{array}
$$

Similarly, as in (12.2) we may write $(\mathcal{X} \times_{\mathcal{Y}} \partial \mathcal{Y})_{\mathrm{top}}$ explicitly as

$$
\begin{aligned}
(\mathcal{X} \times_{\mathcal{Y}} \partial \mathcal{Y})_{\mathrm{top}} \cong \big\{ [x, y'] : x : \underline{\bar{*}} \to \mathcal{X} \text{ and } y' : \underline{\bar{*}} \to \partial \mathcal{Y} \text{ are} \\
\text{1-morphisms with } f \circ x = i_{\mathcal{Y}} \circ y' : \underline{\bar{*}} \to \mathcal{Y} \big\},
\end{aligned} \tag{13.3}
$$

where $[x, y']$ in (13.3) denotes the $\approx$-equivalence class of (x, y'), with $(x, y') \approx (\tilde{x}, \tilde{y}')$ if there exist $\eta : x \Rightarrow \tilde{x}$ and $\zeta : y' \Rightarrow \tilde{y}'$ with $\mathrm{id}_f * \eta = \mathrm{id}_{i_{\mathcal{Y}}} * \zeta$. There is a natural open and closed C^∞-substack $\mathcal{T}_f \subseteq \mathcal{X} \times_{\mathcal{Y}} \partial \mathcal{Y}$ with $[x, y']$ in (13.3) lies in $\mathcal{T}_{f,\mathrm{top}}$ if and only if we can complete the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (y')^*(\mathcal{N}_{\mathcal{Y}}) & \xrightarrow[(y')^*(\nu_{\mathcal{Y}})]{} & (y')^* \circ i_{\mathcal{Y}}^*(\mathcal{F}_{\mathcal{Y}}) & \xrightarrow[(y')^*(i_{\mathcal{Y}}^2)]{} & (y')^*(\mathcal{F}_{\partial \mathcal{Y}}) & \longrightarrow & 0 \\
& & & & \Big\downarrow{\scriptstyle x^*(f^2) \circ I_{x,f}(\mathcal{F}_{\mathcal{Y}}) \circ I_{y',i_{\mathcal{Y}}}(\mathcal{F}_{\mathcal{Y}})^{-1}} & & & & \\
& & & & x^*(\mathcal{F}_{\mathcal{X}}). & & & &
\end{array}
$$

Define $s_f = \pi_{\partial \mathcal{X}}|_{\mathcal{S}_f} : \mathcal{S}_f \to \partial \mathcal{X}$, $u_f = \pi_{\partial \mathcal{Y}}|_{\mathcal{S}_f} : \mathcal{S}_f \to \partial \mathcal{Y}$, $t_f = \pi_{\mathcal{X}}|_{\mathcal{T}_f} : \mathcal{T}_f \to \mathcal{X}$, and $v_f = \pi_{\partial \mathcal{Y}}|_{\mathcal{T}_f} : \mathcal{T}_f \to \partial \mathcal{Y}$. Then s_f, t_f are proper, étale 1-morphisms. We call f *simple* if $s_f : \mathcal{S}_f \to \partial \mathcal{X}$ is an equivalence, and we call f *semisimple* if $s_f : \mathcal{S}_f \to \partial \mathcal{X}$ is injective, as a 1-morphism of Deligne–Mumford C^∞-stacks, and we call f *flat* if $\mathcal{T}_f = \emptyset$. Simple implies semisimple.

Theorem 13.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a semisimple 1-morphism of d-stacks with corners. Then there exists a natural decomposition $\partial \mathcal{X} = \partial_+^f \mathcal{X} \amalg \partial_-^f \mathcal{X}$ with $\partial_\pm^f \mathcal{X}$ open and closed in $\partial \mathcal{X}$, such that:*

(a) *Define $f_+ = f \circ i_{\mathcal{X}}|_{\partial_+^f \mathcal{X}} : \partial_+^f \mathcal{X} \to \mathcal{Y}$. Then f_+ is semisimple. If f is flat then f_+ is also flat.*

(b) *There exists a unique, semisimple 1-morphism $f_- : \partial_-^f \mathcal{X} \to \partial \mathcal{Y}$ with $f \circ i_{\mathcal{X}}|_{\partial_-^f \mathcal{X}} = i_{\mathcal{Y}} \circ f_-$. If f is simple then $\partial_+^f \mathcal{X} = \emptyset$, $\partial_-^f \mathcal{X} = \partial \mathcal{X}$, and $f_- : \partial \mathcal{X} \to \partial \mathcal{Y}$ is also simple. If f is flat then f_- is flat, and the following diagram is 2-Cartesian in $\mathbf{dSta^c}$:*

$$
\begin{array}{ccc}
\partial_-^f \mathcal{X} & \xrightarrow{\quad\quad\quad\quad\quad\quad} & \partial \mathcal{Y} \\
{\scriptstyle i_{\mathcal{X}}|_{\partial_-^f \mathcal{X}}}\Big\downarrow \quad {\scriptstyle f_-} \quad {\scriptstyle \mathrm{id}_{i_{\mathcal{Y}} \circ f}} \nwarrow & & \Big\downarrow{\scriptstyle i_{\mathcal{Y}}} \\
\mathcal{X} & \xrightarrow[\quad\quad\quad f \quad\quad\quad]{} & \mathcal{Y}.
\end{array}
$$

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(c) *Let $g : \mathfrak{X} \to \mathcal{Y}$ be another 1-morphism and $\eta : f \Rightarrow g$ a 2-morphism in* **dSta$^{\mathbf{c}}$**. *Then g is also semisimple, with $\partial_-^g \mathfrak{X} = \partial_-^f \mathfrak{X}$. If f is simple, or flat, then g is simple, or flat, respectively. Part* **(b)** *defines 1-morphisms $f_-, g_- : \partial_-^f \mathfrak{X} \to \partial \mathcal{Y}$. There is a unique 2-morphism $\eta_- : f_- \Rightarrow g_-$ in* **dSpa$^{\mathbf{c}}$** *such that* $\mathrm{id}_{i_{\mathcal{Y}}} * \eta_- = \eta * \mathrm{id}_{i_{\mathfrak{X}}|_{\partial_-^f \mathfrak{X}}} : i_{\mathcal{Y}} \circ f_- \Rightarrow i_{\mathcal{Y}} \circ g_-$.

13.4   Equivalences in dSta$^{\mathbf{c}}$, and gluing by equivalences

Sections 3.2, 6.4 and 10.3 discussed equivalences and gluing for d-spaces, d-spaces with corners, and d-stacks. In [35, §11.4] we generalize these to **dSta$^{\mathbf{c}}$**.

Proposition 13.5. (a) *Suppose $f : \mathfrak{X} \to \mathcal{Y}$ is an equivalence in* **dSta$^{\mathbf{c}}$**. *Then f is simple and flat, and $f : \mathcal{X} \to \mathcal{Y}$ is an equivalence in* **dSta**, *where $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}}, \omega_{\mathfrak{X}})$ and $\mathcal{Y} = (\mathcal{Y}, \partial\mathcal{Y}, i_{\mathcal{Y}}, \omega_{\mathcal{Y}})$. Also $f_- : \partial\mathfrak{X} \to \partial\mathcal{Y}$ in Theorem 13.4(b) is an equivalence in* **dSta$^{\mathbf{c}}$**.

(b) *Let $f : \mathfrak{X} \to \mathcal{Y}$ be a simple, flat 1-morphism in* **dSta$^{\mathbf{c}}$** *with $f : \mathcal{X} \to \mathcal{Y}$ an equivalence in* **dSta**. *Then f is an equivalence in* **dSta$^{\mathbf{c}}$**.

Here is the analogue of Definition 10.5:

Definition 13.6. Let $\mathfrak{X} = (\mathcal{X}, \partial\mathcal{X}, i_{\mathfrak{X}}, \omega_{\mathfrak{X}})$ be a d-stack with corners. Suppose $\mathcal{V} \subseteq \mathcal{X}$ is an open d-substack in **dSta**. Define $\partial\mathcal{V} = i_{\mathfrak{X}}^{-1}(\mathcal{V})$, as an open d-substack of $\partial\mathcal{X}$, and $i_{\mathcal{V}} : \partial\mathcal{V} \to \mathcal{V}$ by $i_{\mathcal{V}} = i_{\mathfrak{X}}|_{\partial\mathcal{V}}$. Then $\partial\mathcal{V} \subseteq \partial\mathcal{X}$ is open, and the conormal bundle of $\partial\mathcal{V}$ in $\mathcal{V}$ is $\mathcal{N}_{\mathcal{V}} = \mathcal{N}_{\mathfrak{X}}|_{\partial\mathcal{V}}$ in qcoh$(\partial\mathcal{V})$. Define an orientation $\omega_{\mathcal{V}}$ on $\mathcal{N}_{\mathcal{V}}$ by $\omega_{\mathcal{V}} = \omega_{\mathfrak{X}}|_{\partial\mathcal{V}}$. Write $\mathcal{V} = (\mathcal{V}, \partial\mathcal{V}, i_{\mathcal{V}}, \omega_{\mathcal{V}})$. Then $\mathcal{V}$ is a d-stack with corners. We call $\mathcal{V}$ an *open d-substack* of $\mathfrak{X}$. An *open cover* of $\mathfrak{X}$ is a family $\{\mathcal{V}_a : a \in A\}$ of open d-substacks $\mathcal{V}_a$ of $\mathfrak{X}$ with $\mathcal{X} = \bigcup_{a \in A} \mathcal{V}_a$.

Theorem 13.7. *Proposition* 10.6 *and Theorems* 10.7 *and* 10.8 *hold without change in the 2-category* **dSta$^{\mathbf{c}}$** *of d-stacks with corners.*

13.5   Corners $C_k(\mathfrak{X})$, and the corner functors $C, \hat{C}$

In [35, §11.5] we generalize the material of §5.3, §6.5, and §12.3 to d-stacks with corners. Here are the main results.

Theorem 13.8. (a) *Let $\mathfrak{X}$ be a d-stack with corners. Then for each $k \geqslant 0$ we can define a d-stack with corners $C_k(\mathfrak{X})$ called the k-**corners** of $\mathfrak{X}$, and a 1-morphism $\mathbf{\Pi}_{\mathfrak{X}}^k : C_k(\mathfrak{X}) \to \mathfrak{X}$, such that $C_k(\mathfrak{X})$ is equivalent to a quotient d-stack $[\partial^k \mathfrak{X}/S_k]$ for a natural action of S_k on $\partial^k \mathfrak{X}$ by 1-isomorphisms. The construction of $C_k(\mathfrak{X})$ is unique up to canonical 1-isomorphism.*

We can describe the topological space $C_k(\mathfrak{X})_{\mathrm{top}}$ as follows. Consider pairs $(x, \{x'_1, \ldots, x'_k\})$, where $x : \underline{\overline{*}} \to \mathcal{X}$ and $x'_i : \underline{\overline{*}} \to \partial\mathcal{X}$ for $i = 1, \ldots, k$ are 1-morphisms in **C$^{\infty}$Sta** *with $x'_1, \ldots, x'_k$ distinct and $x = i_{\mathfrak{X}} \circ x'_1 = \cdots = i_{\mathfrak{X}} \circ x'_k$. Define an equivalence relation $\approx$ on such pairs by $(x, \{x'_1, \ldots, x'_k\}) \approx (\tilde{x}, \{\tilde{x}'_1, \ldots, \tilde{x}'_k\})$ if there exist $\sigma \in S_k$ and 2-morphisms $\eta : x \Rightarrow \tilde{x}$ and $\eta'_i : x'_i \Rightarrow$*

$\tilde{x}'_{\sigma(i)}$ for $i = 1, \ldots, k$ with $\eta = \mathrm{id}_{i_{\mathbf{X}}} * \eta'_1 = \cdots = \mathrm{id}_{i_{\mathbf{X}}} * \eta'_k$. Write $[x, \{x'_1, \ldots, x'_k\}]$ for the $\approx$-equivalence class of $(x, \{x'_1, \ldots, x'_k\})$. Then

$$
\begin{aligned}
C_k(\mathcal{X})_{\mathrm{top}} \cong \big\{ &[x, \{x'_1, \ldots, x'_k\}] : x : \underline{\bar{*}} \to \mathcal{X}, \ x'_i : \underline{\bar{*}} \to \partial\mathcal{X} \ \textit{1-morphisms} \\
&\textit{with } x'_1, \ldots, x'_k \textit{ distinct and } x = i_{\mathbf{X}} \circ x'_1 = \cdots = i_{\mathbf{X}} \circ x'_k \big\}.
\end{aligned}
\tag{13.4}
$$

We have 1-isomorphisms $C_0(\mathbf{X}) \cong \mathbf{X}$ and $C_1(\mathbf{X}) \cong \partial\mathbf{X}$. We write $C(\mathbf{X}) = \coprod_{k \geqslant 0} C_k(\mathbf{X})$, so that $C(\mathbf{X})$ is a d-stack with corners, called the **corners** of $\mathbf{X}$.
(b) Let $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ be a 1-morphism of d-stacks with corners. Then there are unique 1-morphisms $C(\boldsymbol{f}) : C(\mathbf{X}) \to C(\mathbf{Y})$ and $\hat{C}(\boldsymbol{f}) : C(\mathbf{X}) \to C(\mathbf{Y})$ in $\mathbf{dSta^c}$ such that $\boldsymbol{\Pi_Y} \circ C(\boldsymbol{f}) = \boldsymbol{f} \circ \boldsymbol{\Pi_X} = \boldsymbol{\Pi_Y} \circ \hat{C}(\boldsymbol{f}) : C(\mathbf{X}) \to \mathbf{Y}$, with maps $C(f)_{\mathrm{top}} : C(\mathcal{X})_{\mathrm{top}} \to C(\mathcal{Y})_{\mathrm{top}}$, $\hat{C}(f)_{\mathrm{top}} : C(\mathcal{X})_{\mathrm{top}} \to C(\mathcal{Y})_{\mathrm{top}}$ characterized as follows. Identify $C_k(\mathcal{X})_{\mathrm{top}} \subseteq C(\mathcal{X})_{\mathrm{top}}$ with the right hand side of (13.4), and similarly for $C_l(\mathcal{Y})_{\mathrm{top}}$, and identify $\mathcal{S}_{\boldsymbol{f},\mathrm{top}}, \mathcal{T}_{\boldsymbol{f},\mathrm{top}}$ with subsets of (13.2)–(13.3) as in §13.3. Then $C(f)_{\mathrm{top}}$ and $\hat{C}(f)_{\mathrm{top}}$ act by

$$
\begin{aligned}
&C(f)_{\mathrm{top}} : [x, \{x'_1, \ldots, x'_k\}] \longmapsto [y, \{y'_1, \ldots, y'_l\}], \quad \textit{where } y = f \circ x, \\
&\{y'_1, \ldots, y'_l\} = \big\{ y' : [x'_i, y'] \in \mathcal{S}_{\boldsymbol{f},\mathrm{top}}, \ \textit{some } i = 1, \ldots, k \big\}, \ \textit{and}
\end{aligned}
\tag{13.5}
$$

$$
\begin{aligned}
&\hat{C}(f)_{\mathrm{top}} : [x, \{x'_1, \ldots, x'_k\}] \longmapsto [y, \{y'_1, \ldots, y'_l\}], \quad \textit{where } y = f \circ x, \\
&\{y'_1, \ldots, y'_l\} = \big\{ y' : [x'_i, y'] \in \mathcal{S}_{\boldsymbol{f},\mathrm{top}}, \ i = 1, \ldots, k \big\} \cup \big\{ y' : [x, y'] \in \mathcal{T}_{\boldsymbol{f},\mathrm{top}} \big\}.
\end{aligned}
\tag{13.6}
$$

For all $k, l \geqslant 0$, write $C_k^{\boldsymbol{f},l}(\mathbf{X}) = C_k(\mathbf{X}) \cap C(\boldsymbol{f})^{-1}(C_l(\mathbf{Y}))$, so that $C_k^{\boldsymbol{f},l}(\mathbf{X})$ is an open and closed d-substack of $C_k(\mathbf{X})$ with $C_k(\mathbf{X}) = \coprod_{l=0}^{\infty} C_k^{\boldsymbol{f},l}(\mathbf{X})$, and write $C_k^l(\boldsymbol{f}) = C(\boldsymbol{f})|_{C_k^{\boldsymbol{f},l}(\mathbf{X})} : C_k^{\boldsymbol{f},l}(\mathbf{X}) \to C_l(\mathbf{Y})$. If $\boldsymbol{f}$ is simple then $C(\boldsymbol{f})$ maps $C_k(\mathbf{X}) \to C_k(\mathbf{Y})$ for all $k \geqslant 0$. If $\boldsymbol{f}$ is flat then $C(\boldsymbol{f}) = \hat{C}(\boldsymbol{f})$.
(c) Let $\boldsymbol{f}, \boldsymbol{g} : \mathbf{X} \to \mathbf{Y}$ be 1-morphisms and $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ a 2-morphism in $\mathbf{dSta^c}$. Then there exist unique 2-morphisms $C(\boldsymbol{\eta}) : C(\boldsymbol{f}) \Rightarrow C(\boldsymbol{g})$, $\hat{C}(\boldsymbol{\eta}) : \hat{C}(\boldsymbol{f}) \Rightarrow \hat{C}(\boldsymbol{g})$ in $\mathbf{dSta^c}$, where $C(\boldsymbol{f}), C(\boldsymbol{g}), \hat{C}(\boldsymbol{f}), \hat{C}(\boldsymbol{g})$ are as in **(b)**, such that

$$
\mathrm{id}_{\boldsymbol{\Pi_Y}} * C(\boldsymbol{\eta}) = \boldsymbol{\eta} * \mathrm{id}_{\boldsymbol{\Pi_X}} : \boldsymbol{\Pi_Y} \circ C(\boldsymbol{f}) = \boldsymbol{f} \circ \boldsymbol{\Pi_X} \Longrightarrow \boldsymbol{\Pi_Y} \circ C(\boldsymbol{g}) = \boldsymbol{g} \circ \boldsymbol{\Pi_X},
$$

$$
\mathrm{id}_{\boldsymbol{\Pi_Y}} * \hat{C}(\boldsymbol{\eta}) = \boldsymbol{\eta} * \mathrm{id}_{\boldsymbol{\Pi_X}} : \boldsymbol{\Pi_Y} \circ \hat{C}(\boldsymbol{f}) = \boldsymbol{f} \circ \boldsymbol{\Pi_X} \Longrightarrow \boldsymbol{\Pi_Y} \circ \hat{C}(\boldsymbol{g}) = \boldsymbol{g} \circ \boldsymbol{\Pi_X}.
$$

If $\boldsymbol{f}, \boldsymbol{g}$ are flat then $C(\boldsymbol{\eta}) = \hat{C}(\boldsymbol{\eta})$.
(d) Define $C : \mathbf{dSta^c} \to \mathbf{dSta^c}$ by $C : \mathbf{X} \mapsto C(\mathbf{X})$, $C : \boldsymbol{f} \mapsto C(\boldsymbol{f})$, $C : \boldsymbol{\eta} \mapsto C(\boldsymbol{\eta})$ on objects, 1- and 2-morphisms, where $C(\mathbf{X}), C(\boldsymbol{f}), C(\boldsymbol{\eta})$ are as in **(a)**–**(c)** above. Similarly, define $\hat{C} : \mathbf{dSta^c} \to \mathbf{dSta^c}$ by $\hat{C} : \mathbf{X} \mapsto C(\mathbf{X})$, $\hat{C} : \boldsymbol{f} \mapsto \hat{C}(\boldsymbol{f})$, $\hat{C} : \boldsymbol{\eta} \mapsto \hat{C}(\boldsymbol{\eta})$. Then $C, \hat{C}$ are strict 2-functors, called **corner functors**.

13.6 Fibre products in dSta$^{\mathbf{c}}$

In [35, §11.6] we generalize §6.6 and §10.4 to d-stacks with corners. Here are the analogues of Definition 6.12, Lemma 6.13 and Theorem 6.14:

Definition 13.9. Let $g : \mathcal{X} \to \mathcal{Z}$ and $h : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms in $\mathbf{dSta^c}$. As in §13.1 we have line bundles $\mathcal{N}_\mathcal{X}, \mathcal{N}_\mathcal{Z}$ over the C^∞-stacks $\partial\mathcal{X}, \partial\mathcal{Z}$, and §13.3 defines a C^∞-substack $\mathcal{S}_g \subseteq \partial\mathcal{X} \times_\mathcal{Z} \partial\mathcal{Z}$. As in [35, §11.1], there is a natural isomorphism $\lambda_g : u_g^*(\mathcal{N}_\mathcal{Z}) \to s_f^*(\mathcal{N}_\mathcal{X})$ in $\mathrm{qcoh}(\mathcal{S}_g)$. The same holds for h.

We call g, h *b-transverse* if the following holds. Suppose $x : \underline{\bar{*}} \to \mathcal{X}$ and $y : \underline{\bar{*}} \to \mathcal{Y}$ are 1-morphisms in $\mathbf{C^\infty Sta}$, and $\eta : g \circ x \Rightarrow h \circ y$ is a 2-morphism. Since $i_\mathcal{X} : \partial\mathcal{X} \to \mathcal{X}$ is finite and strongly representable, there are finitely many 1-morphisms $x' : \underline{\bar{*}} \to \partial\mathcal{X}$ with $x = i_\mathcal{X} \circ x'$. Write these x' as $x'_1, \ldots, x'_j$. Similarly, write $y'_1, \ldots, y'_k$ for the 1-morphisms $y' : \underline{\bar{*}} \to \partial\mathcal{Y}$ with $y = i_\mathcal{Y} \circ y'$. Write $z = g \circ x$ and $\tilde{z} = h \circ y$, so that $z, \tilde{z} : \underline{\bar{*}} \to \mathcal{Z}$ and $\eta : z \Rightarrow \tilde{z}$. Write $z'_1, \ldots, z'_l$ for the 1-morphisms $z' : \underline{\bar{*}} \to \partial\mathcal{Z}$ with $z = i_\mathcal{Z} \circ z'$. Then by Proposition 8.9, for each $c = 1, \ldots, l$ there are unique $\tilde{z}'_c : \underline{\bar{*}} \to \partial\mathcal{Z}$ and $\eta'_c : z'_c \Rightarrow \tilde{z}'_c$ with $i_\mathcal{Z} \circ \tilde{z}'_c = \tilde{z}$ and $\mathrm{id}_{i_\mathcal{Z}} * \eta'_c = \eta$.

Definition 13.3 defined $\mathcal{S}_g \subseteq \partial\mathcal{X} \times_\mathcal{Z} \partial\mathcal{Z}$ in terms of points $[x', z']$ in (13.2); write $(x', z') : \underline{\bar{*}} \to \mathcal{S}_g$ for the corresponding 1-morphisms. Then we require that for all such x, y, η, the following morphism in $\mathrm{qcoh}(\underline{\bar{*}})$ is injective:

$$\bigoplus_{\substack{a=1,\ldots,j,\ c=1,\ldots,l:\ [x'_a,z'_c]\in\mathcal{S}_{g,\mathrm{top}}}} I_{(x'_a,z'_c),s_g}(\mathcal{N}_\mathcal{X})^{-1} \circ (x'_a, z'_c)^*(\lambda_g) \circ I_{(x'_a,z'_c),u_g}(\mathcal{N}_\mathcal{Z}) \oplus$$

$$\bigoplus_{\substack{b=1,\ldots,k,\ c=1,\ldots,l:\ [y'_b,\tilde{z}'_c]\in\mathcal{S}_{h,\mathrm{top}}}} I_{(y'_b,\tilde{z}'_c),s_h}(\mathcal{N}_\mathcal{Y})^{-1} \circ (y'_b, \tilde{z}'_c)^*(\lambda_h) \circ I_{(y'_b,\tilde{z}'_c),u_h}(\mathcal{N}_\mathcal{Z}) \circ (\eta'_c)^*(\mathcal{N}_\mathcal{Z}) :$$

$$\bigoplus_{c=1}^{l} (z'_c)^*(\mathcal{N}_\mathcal{Z}) \longrightarrow \bigoplus_{a=1}^{j} (x'_a)^*(\mathcal{N}_\mathcal{X}) \oplus \bigoplus_{b=1}^{k} (y'_b)^*(\mathcal{N}_\mathcal{Y}).$$

We call g, h *c-transverse* if the following holds. Identify $C_k(\mathcal{X})_{\mathrm{top}} \subseteq C(\mathcal{X})_{\mathrm{top}}$ with the right hand of (13.4), and similarly for $C(\mathcal{Y})_{\mathrm{top}}, C(\mathcal{Z})_{\mathrm{top}}$. Then $C(g)_{\mathrm{top}}$, $C(h)_{\mathrm{top}}, \hat{C}(g)_{\mathrm{top}}, \hat{C}(h)_{\mathrm{top}}$ act as in (13.5)–(13.6). We require that:

(a) whenever there are points in $C_j(\mathcal{X})_{\mathrm{top}}, C_k(\mathcal{Y})_{\mathrm{top}}, C_l(\mathcal{Z})_{\mathrm{top}}$ with

$$C(g)_{\mathrm{top}}\big([x, \{x'_1, \ldots, x'_j\}]\big) = C(h)_{\mathrm{top}}\big([y, \{y'_1, \ldots, y'_k\}]\big) = [z, \{z'_1, \ldots, z'_l\}],$$

we have either $j + k > l$ or $j = k = l = 0$; and

(b) whenever there are points in $C_j(\mathcal{X})_{\mathrm{top}}, C_k(\mathcal{Y})_{\mathrm{top}}, C_l(\mathcal{Z})_{\mathrm{top}}$ with

$$\hat{C}(g)_{\mathrm{top}}\big([x, \{x'_1, \ldots, x'_j\}]\big) = \hat{C}(h)_{\mathrm{top}}\big([y, \{y'_1, \ldots, y'_k\}]\big) = [z, \{z'_1, \ldots, z'_l\}],$$

we have $j + k \geqslant l$.

Then g, h c-transverse implies g, h b-transverse.

Lemma 13.10. *Let $g : \mathcal{X} \to \mathcal{Z}$ and $h : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms in $\mathbf{dSta^c}$. The following are sufficient conditions for g, h to be c-transverse, and hence b-transverse:*

(i) *g or h is semisimple and flat; or*

(ii) *$\mathcal{Z}$ is a d-stack without boundary.*

Theorem 13.11. (a) *All b-transverse fibre products exist in* $\mathbf{dSta^c}$.

(b) *The 2-functor* $F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}} : \mathbf{dSpa^c} \to \mathbf{dSta^c}$ *of §13.1 takes b- and c-transverse fibre products in* $\mathbf{dSpa^c}$ *to b- and c-transverse fibre products in* $\mathbf{dSta^c}$.

(c) *The 2-functor* $F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}$ *of §13.1 takes transverse fibre products in* $\mathbf{Orb^c}$ *to b-transverse fibre products in* $\mathbf{dSta^c}$. *That is, if*

$$
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\quad f \quad} & \mathcal{Y} \\
{\scriptstyle e}\downarrow & {\scriptstyle \eta \Uparrow} & \downarrow{\scriptstyle h} \\
\mathcal{X} & \xrightarrow{\quad g \quad} & \mathcal{Z}
\end{array}
$$

is a 2-Cartesian square in $\mathbf{Orb^c}$ *with* g, h *transverse, and* $\mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}, e, f, g, h, \eta = F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}(\mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}, e, f, g, h, \eta)$, *then*

$$
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\quad f \quad} & \mathcal{Y} \\
{\scriptstyle e}\downarrow & {\scriptstyle \eta \Uparrow} & \downarrow{\scriptstyle h} \\
\mathcal{X} & \xrightarrow{\quad g \quad} & \mathcal{Z}
\end{array}
$$

is 2-Cartesian in $\mathbf{dSta^c}$, *with* g, h *b-transverse. If also* g, h *are strongly transverse in* $\mathbf{Orb^c}$, *then* g, h *are c-transverse in* $\mathbf{dSta^c}$.

(d) *Suppose we are given a 2-Cartesian diagram in* $\mathbf{dSta^c}$:

$$
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\quad f \quad} & \mathcal{Y} \\
{\scriptstyle e}\downarrow & {\scriptstyle \eta \Uparrow} & \downarrow{\scriptstyle h} \\
\mathcal{X} & \xrightarrow{\quad g \quad} & \mathcal{Z},
\end{array}
$$

with g, h *c-transverse. Then the following are also 2-Cartesian in* $\mathbf{dSta^c}$:

$$
\begin{array}{ccc}
C(\mathcal{W}) & \xrightarrow{\quad C(f) \quad} & C(\mathcal{Y}) \\
{\scriptstyle C(e)}\downarrow & {\scriptstyle C(\eta) \Uparrow} & \downarrow{\scriptstyle C(h)} \\
C(\mathcal{X}) & \xrightarrow{\quad C(g) \quad} & C(\mathcal{Z}),
\end{array}
\tag{13.7}
$$

$$
\begin{array}{ccc}
C(\mathcal{W}) & \xrightarrow{\quad \hat{C}(f) \quad} & C(\mathcal{Y}) \\
{\scriptstyle \hat{C}(e)}\downarrow & {\scriptstyle \hat{C}(\eta) \Uparrow} & \downarrow{\scriptstyle \hat{C}(h)} \\
C(\mathcal{X}) & \xrightarrow{\quad \hat{C}(g) \quad} & C(\mathcal{Z}).
\end{array}
\tag{13.8}
$$

Also (13.7)–(13.8) *preserve gradings, in that they relate points in* $C_i(\mathcal{W}), C_j(\mathcal{X})$, $C_k(\mathcal{Y}), C_k(\mathcal{Z})$ *with* $i = j + k - l$. *Hence* (13.7) *implies equivalences in* $\mathbf{dSta^c}$:
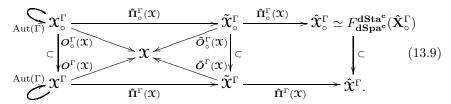
$$
C_i(\mathcal{W}) \simeq \coprod_{j,k,l \geqslant 0 : i = j+k-l} C_j^{g,l}(\mathcal{X}) \times_{C_j^l(g), C_l(\mathcal{Z}), C_k^l(h)} C_k^{h,l}(\mathcal{Y}),
$$

$$
\partial\mathcal{W} \simeq \coprod_{j,k,l \geqslant 0 : j+k=l+1} C_j^{g,l}(\mathcal{X}) \times_{C_j^l(g), C_l(\mathcal{Z}), C_k^l(h)} C_k^{h,l}(\mathcal{Y}).
$$

The analogue of Proposition 6.15 also holds in $\mathbf{dSta^c}$.

13.7   Orbifold strata of d-stacks with corners

In [35, §11.7] we combine material in §10.5 and §12.5 on orbifold strata of d-stacks and of orbifolds with corners. It is also related to §6.7 on fixed loci in d-spaces with corners. Here is the analogue of Theorem 10.11.

Theorem 13.12. *Let $\mathcal{X}$ be a d-stack with corners, and Γ a finite group. Then we can define d-stacks with corners $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma$, and open d-substacks $\mathcal{X}_\circ^\Gamma \subseteq \mathcal{X}^\Gamma$, $\tilde{\mathcal{X}}_\circ^\Gamma \subseteq \tilde{\mathcal{X}}^\Gamma$, $\hat{\mathcal{X}}_\circ^\Gamma \subseteq \hat{\mathcal{X}}^\Gamma$, all natural up to 1-isomorphism in $\mathbf{dSta^c}$, a d-space with corners $\hat{\mathbf{X}}_\circ^\Gamma$ natural up to 1-isomorphism in $\mathbf{dSpa^c}$, and 1-morphisms $\mathbf{O}^\Gamma(\mathcal{X}), \tilde{\mathbf{\Pi}}^\Gamma(\mathcal{X}), \dots$ fitting into a strictly commutative diagram in $\mathbf{dSta^c}$:*

$$(13.9)$$

*We will call $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma, \mathcal{X}_\circ^\Gamma, \tilde{\mathcal{X}}_\circ^\Gamma, \hat{\mathcal{X}}_\circ^\Gamma, \hat{\mathbf{X}}_\circ^\Gamma$ the **orbifold strata** of $\mathcal{X}$.*

The underlying d-stacks of $\mathcal{X}^\Gamma, \dots, \hat{\mathcal{X}}_\circ^\Gamma$ are the orbifold strata $\boldsymbol{\mathcal{X}}^\Gamma, \dots, \hat{\boldsymbol{\mathcal{X}}}_\circ^\Gamma$ from §10.5 of the d-stack $\boldsymbol{\mathcal{X}}$ in $\mathcal{X}$. The 1-morphisms (13.9), as 1-morphisms in $\mathbf{dSta}$, are those given in (10.5).

The rest of §10.5 also extends to $\mathbf{dSta^c}$:

Theorem 13.13. *Theorems 10.12, 10.14, 10.15 and Corollary 10.13 hold without change in $\mathbf{dSta^c}, \mathbf{dSpa^c}$ rather than $\mathbf{dSta}, \mathbf{dSpa}$.*

Here are analogues of Theorem 12.15 and Corollary 12.17.

Theorem 13.14. *Let $\mathcal{X}$ be a d-stack with corners, and Γ a finite group. The corners $C(\mathcal{X})$ from §13.5 lie in $\mathbf{dSta^c}$, so Theorem 13.12 gives orbifold strata $\mathcal{X}^\Gamma, C(\mathcal{X})^\Gamma$ and 1-morphisms $\mathbf{O}^\Gamma(\mathcal{X}) : \mathcal{X}^\Gamma \to \mathcal{X}$, $\mathbf{O}^\Gamma(C(\mathcal{X})) : C(\mathcal{X})^\Gamma \to C(\mathcal{X})$. Applying the corner functor C from §13.5 gives a 1-morphism $C(\mathbf{O}^\Gamma(\mathcal{X})) : C(\mathcal{X}^\Gamma) \to C(\mathcal{X})$. There exists a unique equivalence $\mathbf{K}^\Gamma(\mathcal{X}) : C(\mathcal{X}^\Gamma) \to C(\mathcal{X})^\Gamma$ in $\mathbf{dSta^c}$ with $\mathbf{O}^\Gamma(C(\mathcal{X})) \circ \mathbf{K}^\Gamma(\mathcal{X}) = C(\mathbf{O}^\Gamma(\mathcal{X})) : C(\mathcal{X}^\Gamma) \to C(\mathcal{X})$. It restricts to an equivalence $\mathbf{K}_\circ^\Gamma(\mathcal{X}) := \mathbf{K}^\Gamma(\mathcal{X})|_{C(\mathcal{X}_\circ^\Gamma)} : C(\mathcal{X}_\circ^\Gamma) \to C(\mathcal{X})_\circ^\Gamma$.*

Similarly, there is a unique equivalence $\tilde{\mathbf{K}}^\Gamma(\mathcal{X}) : C(\tilde{\mathcal{X}}^\Gamma) \to \widetilde{C(\mathcal{X})}^\Gamma$ with $\tilde{\mathbf{O}}^\Gamma(C(\mathcal{X})) \circ \tilde{\mathbf{K}}^\Gamma(\mathcal{X}) = C(\tilde{\mathbf{O}}^\Gamma(\mathcal{X}))$ and $\tilde{\mathbf{\Pi}}^\Gamma(C(\mathcal{X})) \circ \mathbf{K}^\Gamma(\mathcal{X}) = \tilde{\mathbf{K}}^\Gamma(\mathcal{X}) \circ C(\tilde{\mathbf{\Pi}}^\Gamma(\mathcal{X}))$. There is an equivalence $\hat{\mathbf{K}}^\Gamma(\mathcal{X}) : C(\hat{\mathcal{X}}^\Gamma) \to \widehat{C(\mathcal{X})}^\Gamma$, unique up to 2-isomorphism, with a 2-morphism $\hat{\mathbf{\Pi}}^\Gamma(C(\mathcal{X})) \circ \tilde{\mathbf{K}}^\Gamma(\mathcal{X}) \Rightarrow \hat{\mathbf{K}}^\Gamma(\mathcal{X}) \circ C(\hat{\mathbf{\Pi}}^\Gamma(\mathcal{X}))$. They restrict to equivalences $\tilde{\mathbf{K}}_\circ^\Gamma(\mathcal{X}) : C(\tilde{\mathcal{X}}_\circ^\Gamma) \to \widetilde{C(\mathcal{X})}_\circ^\Gamma$ and $\hat{\mathbf{K}}_\circ^\Gamma(\mathcal{X}) : C(\hat{\mathcal{X}}_\circ^\Gamma) \to \widehat{C(\mathcal{X})}_\circ^\Gamma$.

Corollary 13.15. *Let $\mathcal{X}$ be a d-stack with corners, and Γ a finite group. Then there exist 1-morphisms $\mathbf{J}^\Gamma(\mathcal{X}) : (\partial\mathcal{X})^\Gamma \to \partial(\mathcal{X}^\Gamma)$, $\tilde{\mathbf{J}}^\Gamma(\mathcal{X}) : (\widetilde{\partial\mathcal{X}})^\Gamma \to \partial(\tilde{\mathcal{X}}^\Gamma)$, $\hat{\mathbf{J}}^\Gamma(\mathcal{X}) : (\widehat{\partial\mathcal{X}})^\Gamma \to \partial(\hat{\mathcal{X}}^\Gamma)$ in $\mathbf{dSta^c}$, natural up to 2-isomorphism, such that $\mathbf{J}^\Gamma(\mathcal{X})$ is an equivalence from $(\partial\mathcal{X})^\Gamma$ to an open and closed d-substack of $\partial(\mathcal{X}^\Gamma)$, and similarly for $\tilde{\mathbf{J}}^\Gamma(\mathcal{X}), \hat{\mathbf{J}}^\Gamma(\mathcal{X})$.*

A d-stack with corners $\boldsymbol{\mathcal{X}}$ is called *straight* if $(i_{\boldsymbol{\mathcal{X}}})_* : \mathrm{Iso}_{\partial\mathcal{X}}([x']) \to \mathrm{Iso}_{\mathcal{X}}([x])$ is an isomorphism for all $[x'] \in \partial\mathcal{X}_{\mathrm{top}}$ with $i_{\mathcal{X},\mathrm{top}}([x']) = [x]$. D-stacks with boundary are automatically straight. If $\boldsymbol{\mathcal{X}}$ is straight then $\partial\boldsymbol{\mathcal{X}}$ is straight, so by induction $\partial^k\boldsymbol{\mathcal{X}}$ is also straight for all $k \geqslant 0$. If $\boldsymbol{\mathcal{X}}$ is straight then $\boldsymbol{K}^\Gamma(\boldsymbol{\mathcal{X}})$ in Theorem 13.14 is an equivalence $C_k(\boldsymbol{\mathcal{X}}^\Gamma) \to C_k(\boldsymbol{\mathcal{X}})^\Gamma$ for all $k \geqslant 0$, and so $\boldsymbol{J}^\Gamma(\boldsymbol{\mathcal{X}})$ in Corollary 13.15 is an equivalence $(\partial\boldsymbol{\mathcal{X}})^\Gamma \to \partial(\boldsymbol{\mathcal{X}}^\Gamma)$. The same applies for $\tilde{\boldsymbol{J}}^\Gamma(\boldsymbol{\mathcal{X}}), \hat{\boldsymbol{J}}^\Gamma(\boldsymbol{\mathcal{X}}), \tilde{\boldsymbol{K}}^\Gamma(\boldsymbol{\mathcal{X}}), \hat{\boldsymbol{K}}^\Gamma(\boldsymbol{\mathcal{X}})$.

14 D-orbifolds with corners

In [35, Chap. 12] we discuss the 2-category $\mathbf{dOrb^c}$ of *d-orbifolds with corners*. Again, there are few new issues here: almost all the material combines ideas we have seen already on d-manifolds with corners from §7, on orbifolds with corners from §12, and on d-stacks with corners from §13. So we will be brief.

As we explain in §16 and [35, §14.3], d-orbifolds with corners are related to Kuranishi spaces in the work of Fukaya, Oh, Ohta and Ono [19].

14.1 Definition of d-orbifolds with corners

In [35, §12.1] we define d-orbifolds with corners, following §7.1 and §11.1.

Definition 14.1. A d-stack with corners $\boldsymbol{\mathcal{W}}$ is called a *principal d-orbifold with corners* if is equivalent in $\mathbf{dSta^c}$ to a fibre product $\boldsymbol{\mathcal{V}} \times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}} \boldsymbol{\mathcal{V}}$, where $\mathcal{V}$ is an orbifold with corners, $\mathcal{E}$ is a vector bundle on $\mathcal{V}$, $s \in C^\infty(\mathcal{E})$, and $\boldsymbol{\mathcal{V}}, \boldsymbol{\mathcal{E}}, \boldsymbol{s}, \boldsymbol{0} = F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}\big(\mathcal{V}, \mathrm{Tot}^c(\mathcal{E}), \mathrm{Tot}^c(s), \mathrm{Tot}^c(0)\big)$, for Tot^c as in §12.1. Note that $\mathrm{Tot}^c(s), \mathrm{Tot}^c(0) : \mathcal{V} \to \mathrm{Tot}^c(\mathcal{E})$ are simple, flat 1-morphisms in $\mathbf{Orb^c}$, so $\boldsymbol{s}, \boldsymbol{0} : \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{E}}$ are simple, flat 1-morphisms in $\mathbf{dSta^c}$. Thus $\boldsymbol{s}, \boldsymbol{0}$ are b-transverse by Lemma 13.10(i), and $\boldsymbol{\mathcal{V}} \times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}} \boldsymbol{\mathcal{V}}$ exists in $\mathbf{dSta^c}$ by Theorem 13.11(a).

If $\boldsymbol{\mathcal{W}}$ is a nonempty principal d-orbifold with corners, then $T^*\boldsymbol{\mathcal{W}}$ is a virtual vector bundle. We define the *virtual dimension* of $\boldsymbol{\mathcal{W}}$ to be $\mathrm{vdim}\,\boldsymbol{\mathcal{W}} = \mathrm{rank}\,T^*\boldsymbol{\mathcal{W}} \in \mathbb{Z}$. If $\boldsymbol{\mathcal{W}} \simeq \boldsymbol{\mathcal{V}} \times_{\boldsymbol{s},\boldsymbol{\mathcal{E}},\boldsymbol{0}} \boldsymbol{\mathcal{V}}$ then $\mathrm{vdim}\,\boldsymbol{\mathcal{W}} = \dim\mathcal{V} - \mathrm{rank}\,\mathcal{E}$.

A d-stack with corners $\boldsymbol{\mathcal{X}}$ is called a *d-orbifold with corners of virtual dimension* $n \in \mathbb{Z}$, written $\mathrm{vdim}\,\boldsymbol{\mathcal{X}} = n$, if $\boldsymbol{\mathcal{X}}$ can be covered by open d-substacks $\boldsymbol{\mathcal{W}}$ which are principal d-orbifolds with corners with $\mathrm{vdim}\,\boldsymbol{\mathcal{W}} = n$. A d-orbifold with corners $\boldsymbol{\mathcal{X}}$ is called a *d-orbifold with boundary* if it is a d-stack with boundary, and a *d-orbifold without boundary* if it is a d-stack without boundary.

Write $\mathbf{d\bar{O}rb}, \mathbf{dOrb^b}, \mathbf{dOrb^c}$ for the full 2-subcategories of d-orbifolds without boundary, with boundary, and with corners, in $\mathbf{dSta^c}$, respectively. Then $\bar{\mathbf{O}}\mathbf{rb}, \bar{\mathbf{O}}\mathbf{rb^b}, \bar{\mathbf{O}}\mathbf{rb^c}$ in §13.1 are full 2-subcategories of $\mathbf{d\bar{O}rb}, \mathbf{dOrb^b}, \mathbf{dOrb^c}$. When we say that a d-orbifold with corners $\boldsymbol{\mathcal{X}}$ *is an orbifold*, we mean that $\boldsymbol{\mathcal{X}}$
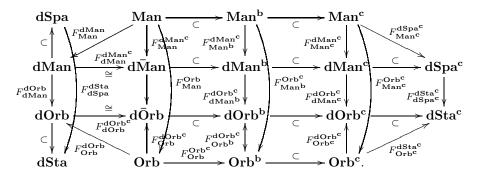
lies in $\mathbf{\bar{O}rb^c}$. Define full and faithful strict 2-functors

$$F_{\mathbf{dOrb}}^{\mathbf{dOrb^c}} : \mathbf{dOrb} \to \mathbf{d\bar{O}rb} \subset \mathbf{dOrb^c}, \quad F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}} : \mathbf{Orb^c} \to \mathbf{dOrb^c},$$

$$F_{\mathbf{Orb^b}}^{\mathbf{dOrb^c}} : \mathbf{Orb^b} \to \mathbf{dOrb^b} \subset \mathbf{dOrb^c}, \quad F_{\mathbf{Orb}}^{\mathbf{dOrb^c}} : \mathbf{Orb} \to \mathbf{d\bar{O}rb} \subset \mathbf{dOrb^c},$$

$$F_{\mathbf{dMan^c}}^{\mathbf{dOrb^c}} : \mathbf{dMan^c} \to \mathbf{dOrb^c}, \qquad F_{\mathbf{dMan^b}}^{\mathbf{dOrb^c}} : \mathbf{dMan^b} \to \mathbf{dOrb^b} \subset \mathbf{dOrb^c},$$

$$\text{and} \qquad F_{\mathbf{dMan}}^{\mathbf{dOrb^c}} : \mathbf{dMan} \to \mathbf{d\bar{O}rb} \subset \mathbf{dOrb^c}, \qquad \text{by}$$

$$F_{\mathbf{dOrb}}^{\mathbf{dOrb^c}} = F_{\mathbf{dSta^c}}^{\mathbf{dSta^c}}|_{\mathbf{dOrb}}, \quad F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}} = F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}, \qquad F_{\mathbf{Orb^b}}^{\mathbf{dOrb^c}} = F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}|_{\mathbf{Orb^b}},$$

$$F_{\mathbf{Orb}}^{\mathbf{Orb^c}} = F_{\mathbf{dSta}}^{\mathbf{dSta^c}} \circ F_{\mathbf{Orb}}^{\mathbf{dSta}}, \quad F_{\mathbf{dMan^c}}^{\mathbf{dOrb^c}} = F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}|_{\mathbf{dMan^c}}, \quad F_{\mathbf{dMan^b}}^{\mathbf{dOrb^c}} = F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}|_{\mathbf{dMan^b}},$$

$$\text{and} \qquad F_{\mathbf{dMan}}^{\mathbf{dOrb^c}} = F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}} \circ F_{\mathbf{dMan}}^{\mathbf{dMan^c}} = F_{\mathbf{dOrb}}^{\mathbf{dOrb^c}} \circ F_{\mathbf{dMan}}^{\mathbf{dOrb}},$$

where $F_{\mathbf{Orb}}^{\mathbf{Orb^c}}, F_{\mathbf{Orb}}^{\mathbf{dSta}}, F_{\mathbf{dSta}}^{\mathbf{dSta^c}}, F_{\mathbf{dSpa^c}}^{\mathbf{dSta^c}}, F_{\mathbf{dMan}}^{\mathbf{dMan^c}}, F_{\mathbf{dMan}}^{\mathbf{dOrb}}, F_{\mathbf{Orb^c}}^{\mathbf{dSta^c}}$ are as in §7.1, §11.1, §12.1, and §13.1. Here $F_{\mathbf{dOrb}}^{\mathbf{dOrb^c}} : \mathbf{dOrb} \to \mathbf{d\bar{O}rb}$ is an isomorphism of 2-categories. So we may as well identify $\mathbf{dOrb}$ with its image $\mathbf{d\bar{O}rb}$, and consider d-orbifolds in §11 as examples of d-orbifolds with corners.

Write $\mathbf{d\hat{M}an^c}$ for the full 2-subcategory of objects $\boldsymbol{\mathcal{X}}$ in $\mathbf{dOrb^c}$ equivalent to $F_{\mathbf{dMan^c}}^{\mathbf{dOrb^c}}(\mathbf{X})$ for some d-manifold with corners $\mathbf{X}$. When we say that a d-orbifold with corners $\boldsymbol{\mathcal{X}}$ *is a d-manifold*, we mean that $\boldsymbol{\mathcal{X}} \in \mathbf{d\hat{M}an^c}$.

These 2-categories lie in a commutative diagram:

If $\boldsymbol{\mathcal{X}} = (\boldsymbol{\mathcal{X}}, \partial\boldsymbol{\mathcal{X}}, i_{\boldsymbol{\mathcal{X}}}, \omega_{\boldsymbol{\mathcal{X}}})$ is a d-orbifold with corners, then the virtual cotangent sheaf $T^*\boldsymbol{\mathcal{X}}$ of the d-stack $\boldsymbol{\mathcal{X}}$ from Remark 11.1 is a virtual vector bundle on $\mathcal{X}$, of rank vdim $\boldsymbol{\mathcal{X}}$. We will call $T^*\boldsymbol{\mathcal{X}} \in \text{vvect}(\mathcal{X})$ the *virtual cotangent bundle* of $\boldsymbol{\mathcal{X}}$, and also write it $T^*\boldsymbol{\mathcal{X}}$.

Here is the analogue of Lemma 11.3:

Lemma 14.2. *Let $\boldsymbol{\mathcal{X}}$ be a d-orbifold with corners. Then $\boldsymbol{\mathcal{X}}$ is a d-manifold, that is, $\boldsymbol{\mathcal{X}} \simeq F_{\mathbf{dMan^c}}^{\mathbf{dOrb^c}}(\mathbf{X})$ for some d-manifold with corners $\mathbf{X}$, if and only if $\mathrm{Iso}_{\mathcal{X}}([x]) \cong \{1\}$ for all $[x]$ in $\mathcal{X}_{\mathrm{top}}$.*

D-orbifolds with corners are preserved by boundaries and corners.

Proposition 14.3. *Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold with corners. Then $\partial\boldsymbol{\mathcal{X}}$ in §13.1 and $C_k(\boldsymbol{\mathcal{X}})$ in §13.5 are d-orbifolds with corners, with $\mathrm{vdim}\,\partial\boldsymbol{\mathcal{X}} = \mathrm{vdim}\,\boldsymbol{\mathcal{X}} - 1$ and $\mathrm{vdim}\,C_k(\boldsymbol{\mathcal{X}}) = \mathrm{vdim}\,\boldsymbol{\mathcal{X}} - k$ for all $k \geqslant 0$.*

Definition 14.4. As for $\mathbf{d\check{M}an^c}$ in §7.1, define $\mathbf{d\check{O}rb^c}$ to be the full 2-subcategory of $\mathcal{X}$ in $\mathbf{dSta^c}$ which may be written as a disjoint union $\mathcal{X} = \coprod_{n \in \mathbb{Z}} \mathcal{X}_n$ for $\mathcal{X}_n \in \mathbf{dOrb^c}$ with $\mathrm{vdim}\,\mathcal{X}_n = n$, where we allow $\mathcal{X}_n = \emptyset$. Then $\mathbf{dOrb^c} \subset \mathbf{d\check{O}rb^c} \subset \mathbf{dSta^c}$, and the corner functors $C, \hat{C} : \mathbf{dSta^c} \to \mathbf{dSta^c}$ in §13.5 restrict to strict 2-functors $C, \hat{C} : \mathbf{d\check{O}rb^c} \to \mathbf{d\check{O}rb^c}$.

14.2 Local properties of d-orbifolds with corners

In [35, §12.2] we combine §7.2 and §11.2. Here are analogues of Examples 7.3, 7.4 and Theorem 7.5, and of Examples 11.4, 11.5 and Theorem 11.6.

Example 14.5. Let $\mathcal{V} = (\mathcal{V}, \partial\mathcal{V}, i_\mathcal{V})$ be an orbifold with corners, $\mathcal{E}$ a vector bundle on $\mathcal{V}$ as in §12.1, and $s \in C^\infty(\mathcal{E})$. We will define an explicit principal d-orbifold with corners $\mathbf{S} = (\boldsymbol{\mathcal{S}}, \boldsymbol{\partial\mathcal{S}}, \boldsymbol{i_S}, \omega_\mathbf{S})$

Define a vector bundle $\mathcal{E}_\partial$ on $\partial\mathcal{V}$ by $\mathcal{E}_\partial = i_\mathcal{V}^*(\mathcal{E})$, and a section $s_\partial = i_\mathcal{V}^*(s) \in C^\infty(\mathcal{E}_\partial)$. Define d-stacks $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}_{\mathcal{V}, \mathcal{E}, s}$ and $\boldsymbol{\partial\mathcal{S}} = \boldsymbol{\mathcal{S}}_{\partial\mathcal{V}, \mathcal{E}_\partial, s_\partial}$ from the triples $\mathcal{V}, \mathcal{E}, s$ and $\partial\mathcal{V}, \mathcal{E}_\partial, s_\partial$ exactly as in Example 11.4, although now $\mathcal{V}, \partial\mathcal{V}$ have corners. Define a 1-morphism $\boldsymbol{i_S} : \boldsymbol{\partial\mathcal{S}} \to \boldsymbol{S}$ in $\mathbf{dSta}$ to be the 'standard model' 1-morphism $\boldsymbol{\mathcal{S}}_{i_\mathcal{V}, \mathrm{id}_{\mathcal{E}_\partial}} : \boldsymbol{\mathcal{S}}_{\partial\mathcal{V}, \mathcal{E}_\partial, s_\partial} \to \boldsymbol{\mathcal{S}}_{\mathcal{V}, \mathcal{E}, s}$ from Example 11.5.

As in Example 7.3, the conormal bundle $\mathcal{N}_\mathbf{S}$ of $\boldsymbol{\partial\mathcal{S}}$ in $\boldsymbol{\mathcal{S}}$ is canonically isomorphic to the lift to $\partial\mathcal{S} \subseteq \partial\mathcal{V}$ of the conormal bundle $\mathcal{N}_\mathcal{V}$ of $\partial\mathcal{V}$ in $\mathcal{V}$. Define $\omega_\mathbf{S}$ to be the orientation on $\mathcal{N}_\mathbf{S}$ induced by the orientation on $\mathcal{N}_\mathcal{V}$ by outward-pointing normal vectors to $\partial\mathcal{V}$ in $\mathcal{V}$. Then $\mathbf{S} = (\boldsymbol{\mathcal{S}}, \boldsymbol{\partial\mathcal{S}}, \boldsymbol{i_S}, \omega_\mathbf{S})$ is a d-stack with corners. It is equivalent in $\mathbf{dSta^c}$ to $\mathcal{V} \times_{s, \mathcal{E}, 0} \mathcal{V}$ in Definition 14.1. We call $\mathbf{S}$ a 'standard model' d-orbifold with corners, and write it $\mathbf{S}_{\mathcal{V}, \mathcal{E}, s}$.

Every principal d-orbifold with corners $\mathcal{W}$ is equivalent in $\mathbf{dOrb^c}$ to some $\mathbf{S}_{\mathcal{V}, \mathcal{E}, s}$. Sometimes it is useful to take $\mathcal{V}$ to be an *effective* orbifold with corners, as in §12.1. There is a natural 1-isomorphism $\partial\mathbf{S}_{\mathcal{V}, \mathcal{E}, s} \cong \mathbf{S}_{\partial\mathcal{V}, \mathcal{E}_\partial, s_\partial}$ in $\mathbf{dOrb^c}$.

Example 14.6. Let $\mathcal{V}, \mathcal{W}$ be orbifolds with corners, $\mathcal{E}, \mathcal{F}$ be vector bundles on $\mathcal{V}, \mathcal{W}$, and $s \in C^\infty(\mathcal{E})$, $t \in C^\infty(\mathcal{F})$, so that Example 14.5 defines d-orbifolds with corners $\mathbf{S}_{\mathcal{V}, \mathcal{E}, s}, \mathbf{S}_{\mathcal{W}, \mathcal{F}, t}$. Suppose $f : \mathcal{V} \to \mathcal{W}$ is a 1-morphism in $\mathbf{Orb^c}$, and $\hat{f} : \mathcal{E} \to f^*(\mathcal{F})$ is a morphism in $\mathrm{vect}(\mathcal{V})$ satisfying $\hat{f} \circ s = f^*(t)$, as in (11.1).

The d-stacks $\boldsymbol{\mathcal{S}}_{\mathcal{V}, \mathcal{E}, s}, \boldsymbol{\mathcal{S}}_{\mathcal{W}, \mathcal{F}, t}$ in $\mathbf{S}_{\mathcal{V}, \mathcal{E}, s}, \mathbf{S}_{\mathcal{W}, \mathcal{F}, t}$ are defined as for 'standard model' d-orbifolds $\boldsymbol{\mathcal{S}}_{\mathcal{V}, \mathcal{E}, s}$ in Example 11.4. Thus we can follow Example 11.5 to define a 1-morphism $\mathbf{S}_{f, \hat{f}} : \boldsymbol{\mathcal{S}}_{\mathcal{V}, \mathcal{E}, s} \to \boldsymbol{\mathcal{S}}_{\mathcal{W}, \mathcal{F}, t}$ in $\mathbf{dSta}$. Then $\mathbf{S}_{f, \hat{f}} : \mathbf{S}_{\mathcal{V}, \mathcal{E}, s} \to \mathbf{S}_{\mathcal{W}, \mathcal{F}, t}$ is a 1-morphism in $\mathbf{dOrb^c}$. We call it a 'standard model' 1-morphism.

Suppose now that $\tilde{\mathcal{V}} \subseteq \mathcal{V}$ is open, with inclusion 1-morphism $i_{\tilde{\mathcal{V}}} : \tilde{\mathcal{V}} \to \mathcal{V}$. Write $\tilde{\mathcal{E}} = \mathcal{E}|_{\tilde{\mathcal{V}}} = i_{\tilde{\mathcal{V}}}^*(\mathcal{E})$ and $\tilde{s} = s|_{\tilde{\mathcal{V}}}$. Define $\boldsymbol{i_{\tilde{\mathcal{V}}, \mathcal{V}}} = \mathbf{S}_{i_{\tilde{\mathcal{V}}}, \mathrm{id}_{\tilde{\mathcal{E}}}} : \mathbf{S}_{\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \tilde{s}} \to \mathbf{S}_{\mathcal{V}, \mathcal{E}, s}$. If $s^{-1}(0) \subseteq \tilde{\mathcal{V}}$ then $\boldsymbol{i_{\tilde{\mathcal{V}}, \mathcal{V}}} : \mathbf{S}_{\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \tilde{s}} \to \mathbf{S}_{\mathcal{V}, \mathcal{E}, s}$ is a 1-isomorphism.

Theorem 14.7. *Let $\mathcal{X}$ be a d-orbifold with corners, and $[x] \in \mathcal{X}_{\mathrm{top}}$. Then there exists an open neighbourhood $\mathcal{U}$ of $[x]$ in $\mathcal{X}$ and an equivalence $\mathcal{U} \simeq \mathbf{S}_{\mathcal{V}, \mathcal{E}, s}$ in $\mathbf{dOrb^c}$ for some orbifold with corners $\mathcal{V}$, vector bundle $\mathcal{E}$ over $\mathcal{V}$ and $s \in C^\infty(\mathcal{E})$ which identifies $[x] \in \mathcal{U}_{\mathrm{top}}$ with a point $[v] \in S^k(\mathcal{V})_{\mathrm{top}} \subseteq \mathcal{V}_{\mathrm{top}}$ such that $s(v) = \mathrm{d}s|_{S^k(\mathcal{V})}(v) = 0$, where $S^k(\mathcal{V}) \subseteq \mathcal{V}$ is the locally closed C^∞-substack*

of $[v] \in \mathcal{V}_{\text{top}}$ such that $\underline{\overline{*}} \times_{v,\mathcal{V},i_\mathcal{V}} \partial\mathcal{V}$ is k points, for $k \geqslant 0$. Furthermore, $\mathcal{V},\mathcal{E},s,k$ are determined up to non-canonical equivalence near $[v]$ by $\mathfrak{X}$ near $[x]$.

As in Examples 11.7–11.8 for d-orbifolds, we can combine the 'standard model' d-manifolds with corners $\mathbf{S}_{V,E,s}$ and 1-morphisms $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ of Examples 7.3–7.4 with quotient d-stacks with corners of §13.2 to define an alternative form of 'standard model' d-orbifolds with corners $[\mathbf{S}_{V,E,s}/\Gamma]$ and 'standard model' 1-morphisms $[\mathbf{S}_{f,\hat{f}},\rho] : [\mathbf{S}_{V,E,s}/\Gamma] \to [\mathbf{S}_{W,F,t}/\Delta]$.

14.3 Equivalences in dOrb$^\mathbf{c}$, and gluing by equivalences

In [35, §12.3] we combine §7.3 and §11.3. Here are the analogues of Theorems 11.11–11.14. Étale 1-morphisms in $\mathbf{dSta^c}$ were discussed in §13.2.

Theorem 14.8. *Suppose $\boldsymbol{f} : \mathfrak{X} \to \mathcal{Y}$ is a 1-morphism of d-orbifolds with corners, and $f : \mathcal{X} \to \mathcal{Y}$ is representable. Then the following are equivalent:*

(i) *$\boldsymbol{f}$ is étale;*

(ii) *$\boldsymbol{f}$ is simple and flat, in the sense of §13.3, and $\Omega_{\boldsymbol{f}} : f^*(T^*\mathcal{Y}) \to T^*\mathfrak{X}$ is an equivalence in $\text{vqcoh}(\mathcal{X})$; and*

(iii) *$\boldsymbol{f}$ is simple and flat, and (11.4) is a split short exact sequence in $\text{qcoh}(\mathcal{X})$.*

If in addition $f_* : \text{Iso}_\mathcal{X}([x]) \to \text{Iso}_\mathcal{Y}(f_{\text{top}}([x]))$ is an isomorphism for all $[x] \in \mathcal{X}_{\text{top}}$, and $f_{\text{top}} : \mathcal{X}_{\text{top}} \to \mathcal{Y}_{\text{top}}$ is a bijection, then $\boldsymbol{f}$ is an equivalence in $\mathbf{dOrb^c}$.

Theorem 14.9. *Suppose $\mathbf{S}_{f,\hat{f}} : \mathbf{S}_{\mathcal{V},\mathcal{E},s} \to \mathbf{S}_{\mathcal{W},\mathcal{F},t}$ is a 'standard model' 1-morphism in $\mathbf{dOrb^c}$, in the notation of Examples 14.5 and 14.6, with $f : \mathcal{V} \to \mathcal{W}$ representable. Then $\mathbf{S}_{f,\hat{f}}$ is étale if and only if f is simple and flat near $s^{-1}(0) \subseteq \mathcal{V}$, in the sense of §12.2, and for each $[v] \in \mathcal{V}_{\text{top}}$ with $s(v) = 0$ and $[w] = f_{\text{top}}([v]) \in \mathcal{W}_{\text{top}}$, the following sequence is exact:*

$$0 \longrightarrow T_v\mathcal{V} \xrightarrow{\mathrm{d}s(v) \oplus \mathrm{d}f(v)} \mathcal{E}_v \oplus T_w\mathcal{W} \xrightarrow{\hat{f}(v) \oplus -\mathrm{d}t(w)} \mathcal{F}_w \longrightarrow 0.$$

Also $\mathbf{S}_{f,\hat{f}}$ is an equivalence if and only if in addition $f_{\text{top}}|_{s^{-1}(0)} : s^{-1}(0) \to t^{-1}(0)$ is a bijection, where $s^{-1}(0) = \{[v] \in \mathcal{V}_{\text{top}} : s(v) = 0\}$, $t^{-1}(0) = \{[w] \in \mathcal{W}_{\text{top}} : t(w) = 0\}$, and $f_* : \text{Iso}_\mathcal{V}([v]) \to \text{Iso}_\mathcal{W}(f_{\text{top}}([v]))$ is an isomorphism for all $[v] \in s^{-1}(0) \subseteq \mathcal{V}_{\text{top}}$.

Theorem 14.10. *Suppose we are given the following data:*

(a) *an integer n;*

(b) *a Hausdorff, second countable topological space X;*

(c) *an indexing set I, and a total order $<$ on I;*

(d) *for each i in I, an effective orbifold with corners $\mathcal{V}_i$, a vector bundle $\mathcal{E}_i$ on $\mathcal{V}_i$ with $\dim \mathcal{V}_i - \text{rank}\,\mathcal{E}_i = n$, a section $s_i \in C^\infty(\mathcal{E}_i)$, and a homeomorphism $\psi_i : s_i^{-1}(0) \to \hat{X}_i$, where $s_i^{-1}(0) = \{[v_i] \in \mathcal{V}_{i,\text{top}} : s_i(v_i) = 0\}$ and $\hat{X}_i \subseteq X$ is open; and*

(e) *for all $i < j$ in I, an open suborbifold $\mathcal{V}_{ij} \subseteq \mathcal{V}_i$, a simple, flat 1-morphism $e_{ij} : \mathcal{V}_{ij} \to \mathcal{V}_j$, and a morphism of vector bundles $\hat{e}_{ij} : \mathcal{E}_i|_{\mathcal{V}_{ij}} \to e_{ij}^*(\mathcal{E}_j)$.*

Let this data satisfy the conditions:

(i) $X = \bigcup_{i \in I} \hat{X}_i$;

(ii) *if $i < j$ in I then $(e_{ij})_* : \mathrm{Iso}_{\mathcal{V}_{ij}}([v]) \to \mathrm{Iso}_{\mathcal{V}_j}(e_{ij,\mathrm{top}}([v]))$ is an isomorphism for all $[v] \in \mathcal{V}_{ij,\mathrm{top}}$, and $\hat{e}_{ij} \circ s_i|_{\mathcal{V}_{ij}} = e_{ij}^*(s_j) \circ \iota_{ij}$ where $\iota_{ij} : \mathcal{O}_{\mathcal{V}_{ij}} \to e_{ij}^*(\mathcal{O}_{\mathcal{V}_j})$ is the natural isomorphism, and $\psi_i(s_i|_{\mathcal{V}_{ij}}^{-1}(0)) = \hat{X}_i \cap \hat{X}_j$, and $\psi_i|_{s_i|_{\mathcal{V}_{ij}}^{-1}(0)} = \psi_j \circ e_{ij,\mathrm{top}}|_{s_i|_{\mathcal{V}_{ij}}^{-1}(0)}$, and if $[v_i] \in \mathcal{V}_{ij,\mathrm{top}}$ with $s_i(v_i) = 0$ and $[v_j] = e_{ij,\mathrm{top}}([v_i])$ then the following sequence is exact:*

$$0 \longrightarrow T_{v_i}\mathcal{V}_i \xrightarrow{\mathrm{d}s_i(v_i) \oplus \mathrm{d}e_{ij}(v_i)} \mathcal{E}_i|_{v_i} \oplus T_{v_j}\mathcal{V}_j \xrightarrow{\hat{e}_{ij}(v_i) \oplus -\mathrm{d}s_j(v_j)} \mathcal{E}_j|_{v_j} \longrightarrow 0;$$

(iii) *if $i < j < k$ in I then there exists a 2-morphism $\eta_{ijk} : e_{jk} \circ e_{ij}|_{\mathcal{V}_{ik} \cap e_{ij}^{-1}(\mathcal{V}_{jk})} \Rightarrow e_{ik}|_{\mathcal{V}_{ik} \cap e_{ij}^{-1}(\mathcal{V}_{jk})}$ in $\mathbf{Orb^c}$ with*

$$\hat{e}_{ik}|_{\mathcal{V}_{ik} \cap e_{ij}^{-1}(\mathcal{V}_{jk})} = \eta_{ijk}^*(\mathcal{E}_k) \circ I_{e_{ij},e_{jk}}(\mathcal{E}_k)^{-1} \circ e_{ij}|_{\mathcal{V}_{ik} \cap e_{ij}^{-1}(\mathcal{V}_{jk})}^*(\hat{e}_{jk}) \circ \hat{e}_{ij}|_{\mathcal{V}_{ik} \cap e_{ij}^{-1}(\mathcal{V}_{jk})}.$$

Note that η_{ijk} is unique by the corners analogue of Proposition 9.5(ii).

    Then there exist a d-orbifold with corners $\boldsymbol{\mathcal{X}}$ with $\mathrm{vdim}\,\boldsymbol{\mathcal{X}} = n$ and underlying topological space $\mathcal{X}_{\mathrm{top}} \cong X$, and a 1-morphism $\boldsymbol{\psi}_i : \boldsymbol{S}_{\mathcal{V}_i,\mathcal{E}_i,s_i} \to \boldsymbol{\mathcal{X}}$ in $\mathbf{dOrb^c}$ with underlying continuous map ψ_i which is an equivalence with the open d-suborbifold $\hat{\boldsymbol{\mathcal{X}}}_i \subseteq \boldsymbol{\mathcal{X}}$ corresponding to $\hat{X}_i \subseteq X$ for all $i \in I$, such that for all $i < j$ in I there exists a 2-morphism $\boldsymbol{\eta}_{ij} : \boldsymbol{\psi}_j \circ \boldsymbol{S}_{e_{ij},\hat{e}_{ij}} \Rightarrow \boldsymbol{\psi}_i \circ \boldsymbol{i}_{\mathcal{V}_{ij},\mathcal{V}_i}$, where $\boldsymbol{S}_{e_{ij},\hat{e}_{ij}} : \boldsymbol{S}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},s_i|_{\mathcal{V}_{ij}}} \to \boldsymbol{S}_{\mathcal{V}_j,\mathcal{E}_j,s_j}$ and $\boldsymbol{i}_{\mathcal{V}_{ij},\mathcal{V}_i} : \boldsymbol{S}_{\mathcal{V}_{ij},\mathcal{E}_i|_{\mathcal{V}_{ij}},s_i|_{\mathcal{V}_{ij}}} \to \boldsymbol{S}_{\mathcal{V}_i,\mathcal{E}_i,s_i}$ are as in Examples 14.5–14.6. This $\boldsymbol{\mathcal{X}}$ is unique up to equivalence in $\mathbf{dOrb^c}$.

    Suppose also that $\mathcal{Y}$ is an effective orbifold with corners, and $g_i : \mathcal{V}_i \to \mathcal{Y}$ are submersions for all $i \in I$, and there are 2-morphisms $\zeta_{ij} : g_j \circ e_{ij} \Rightarrow g_i|_{\mathcal{V}_{ij}}$ in $\mathbf{Orb^c}$ for all $i < j$ in I. Then there exist a 1-morphism $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in $\mathbf{dOrb^c}$ unique up to 2-isomorphism, where $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}}(\mathcal{Y}) = \boldsymbol{S}_{\mathcal{Y},0,0}$, and 2-morphisms $\boldsymbol{\zeta}_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow \boldsymbol{S}_{g_i,0}$ for all $i \in I$.

Theorem 14.11. *Suppose we are given the following data:*

(a) *an integer n;*

(b) *a Hausdorff, second countable topological space X;*

(c) *an indexing set I, and a total order $<$ on I;*

(d) *for each i in I, a manifold with corners V_i, a vector bundle $E_i \to V_i$ with $\dim V_i - \mathrm{rank}\,E_i = n$, a finite group Γ_i, smooth, locally effective actions $r_i(\gamma) : V_i \to V_i$, $\hat{r}_i(\gamma) : E_i \to r(\gamma)^*(E_i)$ of Γ_i on V_i, E_i for $\gamma \in \Gamma_i$, a smooth, Γ_i-equivariant section $s_i : V_i \to E_i$, and a homeomorphism $\psi_i : X_i \to \hat{X}_i$, where $X_i = \{v_i \in V_i : s_i(v_i) = 0\}/\Gamma_i$ and $\hat{X}_i \subseteq X$ is an open set; and*

(e) *for all $i < j$ in I, an open submanifold $V_{ij} \subseteq V_i$, invariant under Γ_i, a group morphism $\rho_{ij} : \Gamma_i \to \Gamma_j$, a simple, flat, smooth map $e_{ij} : V_{ij} \to V_j$, and a morphism of vector bundles $\hat{e}_{ij} : E_i|_{V_{ij}} \to e_{ij}^*(E_j)$.*

Let this data satisfy the conditions:

(i) $X = \bigcup_{i \in I} \hat{X}_i$;

(ii) *if $i < j$ in I then $\hat{e}_{ij} \circ s_i|_{V_{ij}} = e_{ij}^*(s_j) + O(s_i^2)$, and for all $\gamma \in \Gamma$ we have*

$$e_{ij} \circ r_i(\gamma) = r_j(\rho_{ij}(\gamma)) \circ e_{ij} : V_{ij} \longrightarrow V_j,$$
$$r_i(\gamma)^*(\hat{e}_{ij}) \circ \hat{r}_i(\gamma) = e_{ij}^*(\hat{r}_j(\rho_{ij}(\gamma))) \circ \hat{e}_{ij} : E_i|_{V_{ij}} \longrightarrow (e_{ij} \circ r_i(\gamma))^*(E_j),$$

and $\psi_i(X_i \cap (V_{ij}/\Gamma_i)) = \hat{X}_i \cap \hat{X}_j$, and $\psi_i|_{X_i \cap V_{ij}/\Gamma_i} = \psi_j \circ (e_{ij})_*|_{X_i \cap V_{ij}/\Gamma_j}$, and if $v_i \in V_{ij}$ with $s_i(v_i) = 0$ and $v_j = e_{ij}(v_i)$ then $\rho|_{\mathrm{Stab}_{\Gamma_i}(v_i)} : \mathrm{Stab}_{\Gamma_i}(v_i) \to \mathrm{Stab}_{\Gamma_j}(v_j)$ is an isomorphism, and the following sequence of vector spaces is exact:

$$0 \longrightarrow T_{v_i} V_i \xrightarrow{\mathrm{d}s_i(v_i) \oplus \mathrm{d}e_{ij}(v_i)} E_i|_{v_i} \oplus T_{v_j} V_j \xrightarrow{\hat{e}_{ij}(v_i) \oplus -\mathrm{d}s_j(v_j)} E_j|_{v_j} \longrightarrow 0;$$

(iii) *if $i < j < k$ in I then there exists $\gamma_{ijk} \in \Gamma_k$ satisfying*

$$\rho_{ik}(\gamma) = \gamma_{ijk} \, \rho_{jk}(\rho_{ij}(\gamma)) \, \gamma_{ijk}^{-1} \quad \textit{for all } \gamma \in \Gamma_i,$$
$$e_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} = r_k(\gamma_{ijk}) \circ e_{jk} \circ e_{ij}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})}, \quad \textit{and}$$
$$\hat{e}_{ik}|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})} = \big(e_{ij}^*(e_{jk}^*(\hat{r}_k(\gamma_{ijk}))) \circ e_{ij}^*(\hat{e}_{jk}) \circ \hat{e}_{ij}\big)|_{V_{ik} \cap e_{ij}^{-1}(V_{jk})}.$$

Then there exist a d-orbifold with corners $\boldsymbol{\mathcal{X}}$ with $\mathrm{vdim}\,\boldsymbol{\mathcal{X}} = n$ and underlying topological space $\boldsymbol{\mathcal{X}}_{\mathrm{top}} \cong X$, and a 1-morphism $\boldsymbol{\psi}_i : [\mathbf{S}_{V_i, E_i, s_i}/\Gamma_i] \to \boldsymbol{\mathcal{X}}$ in $\mathbf{dOrb^c}$ with underlying continuous map ψ_i which is an equivalence with the open d-suborbifold $\hat{\boldsymbol{\mathcal{X}}}_i \subseteq \boldsymbol{\mathcal{X}}$ corresponding to $\hat{X}_i \subseteq X$ for all $i \in I$, such that for all $i < j$ in I there exists a 2-morphism $\boldsymbol{\eta}_{ij} : \boldsymbol{\psi}_j \circ [\mathbf{S}_{e_{ij}, \hat{e}_{ij}}, \rho_{ij}] \Rightarrow \boldsymbol{\psi}_i \circ [\boldsymbol{i}_{V_{ij}, V_i}, \mathrm{id}_{\Gamma_i}],$ where $[\mathbf{S}_{e_{ij}, \hat{e}_{ij}}, \rho_{ij}] : [\mathbf{S}_{V_{ij}, E_i|_{V_{ij}}, s_i|_{V_{ij}}}/\Gamma_i] \to [\mathbf{S}_{V_j, E_j, s_j}/\Gamma_j]$ and $[\boldsymbol{i}_{V_{ij}, V_i}, \mathrm{id}_{\Gamma_i}] : [\mathbf{S}_{V_{ij}, E_i|_{V_{ij}}, s_i|_{V_{ij}}}/\Gamma_i] \to [\mathbf{S}_{V_i, E_i, s_i}/\Gamma_j]$ combine the notation of Examples 7.3–7.4 and §13.2. This $\boldsymbol{\mathcal{X}}$ is unique up to equivalence in $\mathbf{dOrb^c}$.

Suppose also that Y is a manifold with corners, and $g_i : V_i \to Y$ are smooth maps for all $i \in I$ with $g_i \circ r_i(\gamma) = g_i$ for all $\gamma \in \Gamma_i$, and $g_j \circ e_{ij} = g_i|_{V_{ij}}$ for all $i < j$ in I. Then there exist a 1-morphism $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in $\mathbf{dOrb^c}$ unique up to 2-isomorphism, where $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Man^c}}^{\mathbf{dOrb^c}}(Y) = [\mathbf{S}_{Y,0,0}/\{1\}]$, and 2-morphisms $\boldsymbol{\zeta}_i : \boldsymbol{h} \circ \boldsymbol{\psi}_i \Rightarrow [\mathbf{S}_{g_i,0}, \pi_{\{1\}}]$ for all $i \in I$. Here $[\mathbf{S}_{g_i,0}, \pi_{\{1\}}] : [\mathbf{S}_{V_i, E_i, s_i}/\Gamma_i] \to [\mathbf{S}_{Y,0,0}/\{1\}] = \boldsymbol{\mathcal{Y}}$ with $\hat{g}_i = 0$ and $\rho = \pi_{\{1\}} : \Gamma_i \to \{1\}$.

We can use Theorems 14.10 and 14.11 to prove the existence of d-orbifold with corners structures on spaces coming from other areas of geometry, such as moduli spaces of J-holomorphic curves.

14.4 Submersions, immersions and embeddings

In [35, §12.4] we extend §7.4 and §11.4 to d-orbifolds with corners.

Definition 14.12. Let $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ be a 1-morphism in $\mathbf{dOrb^c}$. Then $T^*\boldsymbol{\mathcal{X}}$ and $f^*(T^*\boldsymbol{\mathcal{Y}})$ are virtual vector bundles on $\mathcal{X}$ of ranks $\mathrm{vdim}\,\boldsymbol{\mathcal{X}}, \mathrm{vdim}\,\boldsymbol{\mathcal{Y}}$, and $\Omega_{\boldsymbol{f}} : f^*(T^*\boldsymbol{\mathcal{Y}}) \to T^*\boldsymbol{\mathcal{X}}$ is a 1-morphism in $\mathrm{vvect}(\mathcal{X})$, as in Remark 11.1 and Definition 14.1. 'Weakly injective', ..., below are as in Definition 11.15.

(a) We call $\boldsymbol{f}$ a *w-submersion* if $\boldsymbol{f}$ is semisimple and flat and $\Omega_{\boldsymbol{f}}$ is weakly injective. We call $\boldsymbol{f}$ an *sw-submersion* if it is also simple.

(b) We call $\boldsymbol{f}$ a *submersion* if $\boldsymbol{f}$ is semisimple and flat and $\Omega_{C(\boldsymbol{f})}$ is injective, for $C(\boldsymbol{f})$ as in §13.5. We call $\boldsymbol{f}$ an *s-submersion* if it is also simple.

(c) We call $\boldsymbol{f}$ a *w-immersion* if $f : \mathcal{X} \to \mathcal{Y}$ is representable and $\Omega_{\boldsymbol{f}}$ is weakly surjective. We call $\boldsymbol{f}$ an *sw-immersion*, or *sfw-immersion*, if $\boldsymbol{f}$ is also simple, or simple and flat.

(d) We call $\boldsymbol{f}$ an *immersion* if $f : \mathcal{X} \to \mathcal{Y}$ is representable and $\Omega_{\hat{C}(\boldsymbol{f})}$ is surjective, for $\hat{C}(\boldsymbol{f})$ as in §13.5. We call $\boldsymbol{f}$ an *s-immersion* if $\boldsymbol{f}$ is also simple, and an *sf-immersion* if $\boldsymbol{f}$ is also simple and flat.

(e) We call $\boldsymbol{f}$ a *w-embedding, sw-embedding, sfw-embedding, embedding, s-embedding*, or *sf-embedding*, if $\boldsymbol{f}$ is a w-immersion, ..., sf-immersion, respectively, and $f_* : \mathrm{Iso}_{\mathcal{X}}([x]) \to \mathrm{Iso}_{\mathcal{Y}}(f_{\mathrm{top}}([x]))$ is an isomorphism for all $[x] \in \mathcal{X}_{\mathrm{top}}$, and $f_{\mathrm{top}} : \mathcal{X}_{\mathrm{top}} \to \mathcal{Y}_{\mathrm{top}}$ is a homeomorphism with its image, so in particular f_{top} is injective.

Parts (c)–(e) enable us to define *d-suborbifolds* $\boldsymbol{\mathcal{X}}$ of a d-orbifold with corners $\boldsymbol{\mathcal{Y}}$. *Open d-suborbifolds* are open d-substacks $\boldsymbol{\mathcal{X}}$ in $\boldsymbol{\mathcal{Y}}$. For more general d-suborbifolds, we call $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ a *w-immersed, sw-immersed, sfw-immersed, immersed, s-immersed, sf-immersed, w-embedded, sw-embedded, sfw-embedded, embedded, s-embedded*, or *sf-embedded suborbifold* of $\boldsymbol{\mathcal{Y}}$ if $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}$ are d-orbifolds with corners and $\boldsymbol{f}$ is a w-immersion, ..., sf-embedding, respectively.

Theorem 7.12 in §7.4 holds with orbifolds and d-orbifolds with corners in place of manifolds and d-manifolds with corners, except part (v), when we need also to assume $f : \mathcal{X} \to \mathcal{Y}$ representable to deduce $\boldsymbol{f}$ is étale, and part (x), which is false for d-orbifolds with corners (in the Zariski topology, at least).

14.5 Bd-transversality and fibre products

In [35, §12.5] we generalize §7.5 and §11.5 to $\mathbf{dOrb^c}$. Here are the analogues of Definition 7.13 and Theorems 7.14–7.17.

Definition 14.13. Let $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \boldsymbol{\mathcal{Z}}$ be d-orbifolds with corners and $\boldsymbol{g} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}$, $\boldsymbol{h} : \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{Z}}$ be 1-morphisms. We call $\boldsymbol{g}, \boldsymbol{h}$ *bd-transverse* if they are both b-transverse in $\mathbf{dSta^c}$ in the sense of Definition 13.9, and d-transverse in the sense of Definition 11.16. We call $\boldsymbol{g}, \boldsymbol{h}$ *cd-transverse* if they are both c-transverse in $\mathbf{dSta^c}$ in the sense of Definition 13.9, and d-transverse. As in §13.6, c-transverse implies b-transverse, so cd-transverse implies bd-transverse.

Theorem 14.14. *Suppose* $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ *are d-orbifolds with corners and* $\boldsymbol{g} : \mathcal{X} \to \mathcal{Z}$, $\boldsymbol{h} : \mathcal{Y} \to \mathcal{Z}$ *are bd-transverse 1-morphisms, and let* $\mathcal{W} = \mathcal{X} \times_{\boldsymbol{g}, \mathcal{Z}, \boldsymbol{h}} \mathcal{Y}$ *be the fibre product in* $\mathbf{dSta^c}$, *which exists by Theorem 13.11(a) as* $\boldsymbol{g}, \boldsymbol{h}$ *are b-transverse. Then* $\mathcal{W}$ *is a d-orbifold with corners, with*

$$\operatorname{vdim} \mathcal{W} = \operatorname{vdim} \mathcal{X} + \operatorname{vdim} \mathcal{Y} - \operatorname{vdim} \mathcal{Z}. \tag{14.1}$$

Hence, all bd-transverse fibre products exist in $\mathbf{dOrb^c}$.

Theorem 14.15. *Suppose* $\boldsymbol{g} : \mathcal{X} \to \mathcal{Z}$ *and* $\boldsymbol{h} : \mathcal{Y} \to \mathcal{Z}$ *are 1-morphisms in* $\mathbf{dOrb^c}$. *The following are sufficient conditions for* $\boldsymbol{g}, \boldsymbol{h}$ *to be cd-transverse, and hence bd-transverse, so that* $\mathcal{W} = \mathcal{X} \times_{\boldsymbol{g}, \mathcal{Z}, \boldsymbol{h}} \mathcal{Y}$ *is a d-orbifold with corners of virtual dimension* (14.1):

- **(a)** $\mathcal{Z}$ *is an orbifold without boundary, that is,* $\mathcal{Z} \in \bar{\mathbf{Orb}}$; *or*
- **(b)** $\boldsymbol{g}$ *or* $\boldsymbol{h}$ *is a w-submersion.*

Theorem 14.16. *Let* $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ *be d-orbifolds with corners with* $\mathcal{Y}$ *an orbifold, and* $\boldsymbol{g} : \mathcal{X} \to \mathcal{Z}$, $\boldsymbol{h} : \mathcal{Y} \to \mathcal{Z}$ *be 1-morphisms with* $\boldsymbol{g}$ *a submersion. Then* $\mathcal{W} = \mathcal{X} \times_{\boldsymbol{g}, \mathcal{Z}, \boldsymbol{h}} \mathcal{Y}$ *is an orbifold, with* $\dim \mathcal{W} = \operatorname{vdim} \mathcal{X} + \dim \mathcal{Y} - \operatorname{vdim} \mathcal{Z}$.

Theorem 14.17. (i) *Let* $\rho : G \to H$ *be an injective morphism of finite groups, and* H *act linearly on* $\mathbb{R}^n$ *preserving* $[0, \infty)^k \times \mathbb{R}^{n-k}$. *Then* §13.2 *gives a quotient 1-morphism* $[\boldsymbol{0}, \rho] : [*/G] \to \left[[\boldsymbol{0}, \infty)^{\boldsymbol{k}} \times \mathbb{R}^{\boldsymbol{n-k}}/H \right]$ *in* $\mathbf{dOrb^c}$. *Suppose* $\mathcal{X}$ *is a d-orbifold with corners and* $\boldsymbol{g} : \mathcal{X} \to \left[[\boldsymbol{0}, \infty)^{\boldsymbol{k}} \times \mathbb{R}^{\boldsymbol{n-k}}/H \right]$ *is a semisimple, flat 1-morphism in* $\mathbf{dOrb^c}$. *Then the fibre product* $\mathcal{W} = \mathcal{X} \times_{\boldsymbol{g}, [[0, \infty)^{\boldsymbol{k}} \times \mathbb{R}^{\boldsymbol{n-k}}/H], [\boldsymbol{0}, \rho]} [*/G]$ *exists in* $\mathbf{dOrb^c}$. *The projection* $\boldsymbol{\pi}_{\mathcal{X}} : \mathcal{W} \to \mathcal{X}$ *is an s-immersion, and an s-embedding if* ρ *is an isomorphism.*

When $k = 0$, *any 1-morphism* $\boldsymbol{g} : \mathcal{X} \to [\mathbb{R}^{\boldsymbol{n}}/H]$ *is semisimple and flat, and* $\boldsymbol{\pi}_{\mathcal{X}} : \mathcal{W} \to \mathcal{X}$ *is an sf-immersion, and an sf-embedding if* ρ *is an isomorphism.*

(ii) *Suppose* $\boldsymbol{f} : \mathcal{X} \to \mathcal{Y}$ *is an s-immersion of d-orbifolds with corners, and* $[x] \in \mathcal{X}_{\text{top}}$ *with* $f_{\text{top}}([x]) = [y] \in \mathcal{Y}_{\text{top}}$. *Write* $\rho : G \to H$ *for* $f_* : \operatorname{Iso}_{\mathcal{X}}([x]) \to \operatorname{Iso}_{\mathcal{Y}}([y])$. *Then* ρ *is injective, and there exist open neighbourhoods* $\mathcal{U} \subseteq \mathcal{X}$ *and* $\mathcal{V} \subseteq \mathcal{Y}$ *of* $[x], [y]$ *with* $\boldsymbol{f}(\mathcal{U}) \subseteq \mathcal{V}$, *a linear action of* H *on* $\mathbb{R}^n$ *preserving* $[0, \infty)^k \times \mathbb{R}^{n-k} \subseteq \mathbb{R}^n$ *where* $n = \operatorname{vdim} \mathcal{Y} - \operatorname{vdim} \mathcal{X} \geqslant 0$ *and* $0 \leqslant k \leqslant n$, *and a 1-morphism* $\boldsymbol{g} : \mathcal{V} \to \left[[\boldsymbol{0}, \infty)^{\boldsymbol{k}} \times \mathbb{R}^{\boldsymbol{n-k}}/H \right]$ *with* $g_{\text{top}}([y]) = [0]$, *fitting into a 2-Cartesian square in* $\mathbf{dOrb^c}$:

$$
\begin{array}{ccc}
\mathcal{U} & \longrightarrow & [*/G] \\
{\scriptstyle \boldsymbol{f}|_{\mathcal{U}}} \downarrow \quad & \Uparrow & \quad \downarrow {\scriptstyle [\boldsymbol{0}, \rho]} \\
\mathcal{V} & \underset{\boldsymbol{g}}{\longrightarrow} & \left[[\boldsymbol{0}, \infty)^{\boldsymbol{k}} \times \mathbb{R}^{\boldsymbol{n-k}}/H \right].
\end{array}
$$

If $\boldsymbol{f}$ *is an sf-immersion then* $k = 0$. *If* $\boldsymbol{f}$ *is an s- or sf-embedding then* ρ *is an isomorphism, and we may take* $\mathcal{U} = \boldsymbol{f}^{-1}(\mathcal{V})$.

14.6   Embedding d-orbifolds with corners into orbifolds

Section 4.7 discussed embeddings of d-manifolds $\boldsymbol{X}$ into manifolds Y. Our two main results were Theorem 4.29, which gave necessary and sufficient conditions on $\boldsymbol{X}$ for existence of embeddings $\boldsymbol{f} : \boldsymbol{X} \hookrightarrow \mathbb{R}^n$ for $n \gg 0$, and Theorem 4.32, which showed that if an embedding $\boldsymbol{f} : \boldsymbol{X} \hookrightarrow \boldsymbol{Y}$ exists with $\boldsymbol{X}$ a d-manifold and $\boldsymbol{Y} = F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$, then $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$ for open $V \subseteq Y$.

Section 7.6 generalized §4.7 to d-manifolds with corners, requiring $\boldsymbol{f} : \boldsymbol{X} \hookrightarrow \boldsymbol{Y}$ to be an sf-embedding for the analogue of Theorem 4.32. Section 11.6 explained that while Theorem 4.32 generalizes to d-orbifolds, we do not have a good d-orbifold generalization of Theorem 4.29. Thus, we do not have a useful necessary and sufficient criterion for when a d-orbifold is principal.

As in [35, §12.6], the situation is the same for d-orbifolds with corners as for d-orbifolds. Here is the analogue of Theorem 4.32:

Theorem 14.18. *Suppose $\mathfrak{X}$ is a d-orbifold with corners, $\mathcal{Y}$ an orbifold with corners, and $\boldsymbol{f} : \mathfrak{X} \to \mathcal{Y}$ an sf-embedding, in the sense of §14.4. Then there exist an open suborbifold $\mathcal{V} \subseteq \mathcal{Y}$ with $\boldsymbol{f}(\mathfrak{X}) \subseteq \mathcal{V}$, a vector bundle $\mathcal{E}$ on $\mathcal{V}$, and a section $s \in C^\infty(\mathcal{E})$ fitting into a 2-Cartesian diagram in $\mathbf{dOrb^c}$:*

$$
\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{\quad\boldsymbol{f}\quad} & \mathcal{V} \\
{\scriptstyle \boldsymbol{f}}\downarrow & \Uparrow & \downarrow{\scriptstyle \mathbf{0}} \\
\mathcal{V} & \xrightarrow{\quad\boldsymbol{s}\quad} & \mathcal{E},
\end{array}
$$

where $\mathcal{Y}, \mathcal{V}, \mathcal{E}, \boldsymbol{s}, \mathbf{0} = F_{\mathbf{Orb^c}}^{\mathbf{dOrb^c}}\big(\mathcal{Y}, \mathcal{V}, \mathrm{Tot}^{\mathrm{c}}(\mathcal{E}), \mathrm{Tot}^{\mathrm{c}}(s), \mathrm{Tot}^{\mathrm{c}}(0)\big)$, in the notation of §12.1. Thus $\mathfrak{X}$ is equivalent to the 'standard model' $\boldsymbol{S}_{\mathcal{V},\mathcal{E},s}$ of Example 14.5, and is a principal d-orbifold with corners.

Again, in contrast to d-manifolds with corners, the author does not know useful necessary and sufficient conditions for when a d-orbifold with corners admits an sf-embedding into an orbifold, or is a principal d-orbifold with corners.

14.7   Orientations on d-orbifolds with corners

Section 4.8 discussed orientations on d-manifolds. This was extended to d-manifolds with corners in §7.7, and to d-orbifolds in §11.7. As in [35, §12.7], all this generalizes easily to d-orbifolds with corners, so we will give few details.

If $\mathfrak{X}$ is a d-orbifold with corners, the virtual cotangent bundle $T^*\mathfrak{X}$ is a virtual vector bundle on $\mathcal{X}$. We define an *orientation* ω on $\mathfrak{X}$ to be an orientation on the orientation line bundle $\mathcal{L}_{T^*\mathfrak{X}}$. The analogues of Example 4.36, Theorem 4.37, Proposition 4.38, Theorem 7.25, and Propositions 7.26 and 7.27 all hold for d-orbifolds with corners.

14.8   Orbifold strata of d-orbifolds with corners

Sections 8.7, 9.2, 11.8, 12.5, and 13.7 discussed orbifold strata for Deligne–Mumford C^∞-stacks, orbifolds, d-orbifolds, orbifolds with corners, and d-stacks with corners. In [35, §12.8] we extend this to d-orbifolds with corners.

Let $\mathcal{X}$ be a d-orbifold with corners and Γ a finite group, so that §13.7 gives orbifold strata $\mathcal{X}^\Gamma, \tilde{\mathcal{X}}^\Gamma, \hat{\mathcal{X}}^\Gamma, \mathcal{X}_\circ^\Gamma, \tilde{\mathcal{X}}_\circ^\Gamma, \hat{\mathcal{X}}_\circ^\Gamma$, which are d-stacks with corners. Use the notation $\Lambda^\Gamma, \Lambda^\Gamma / \mathrm{Aut}(\Gamma)$ of Definition 9.7. Exactly as in the d-orbifold case in §11.8, there are natural decompositions

$$\mathcal{X}^\Gamma = \coprod_{\lambda \in \Lambda^\Gamma} \mathcal{X}^{\Gamma,\lambda}, \quad \tilde{\mathcal{X}}^\Gamma = \coprod_{\mu \in \Lambda^\Gamma / \mathrm{Aut}(\Gamma)} \tilde{\mathcal{X}}^{\Gamma,\mu}, \quad \hat{\mathcal{X}}^\Gamma = \coprod_{\mu \in \Lambda^\Gamma / \mathrm{Aut}(\Gamma)} \hat{\mathcal{X}}^{\Gamma,\mu},$$
$$\mathcal{X}_\circ^\Gamma = \coprod_{\lambda \in \Lambda^\Gamma} \mathcal{X}_\circ^{\Gamma,\lambda}, \quad \tilde{\mathcal{X}}_\circ^\Gamma = \coprod_{\mu \in \Lambda^\Gamma / \mathrm{Aut}(\Gamma)} \tilde{\mathcal{X}}_\circ^{\Gamma,\mu}, \quad \hat{\mathcal{X}}_\circ^\Gamma = \coprod_{\mu \in \Lambda^\Gamma / \mathrm{Aut}(\Gamma)} \hat{\mathcal{X}}_\circ^{\Gamma,\mu},$$

where $\mathcal{X}^{\Gamma,\lambda}, \ldots, \tilde{\mathcal{X}}_\circ^{\Gamma,\mu}$ are d-orbifolds with corners with $\mathrm{vdim}\, \mathcal{X}^{\Gamma,\lambda} = \mathrm{vdim}\, \mathcal{X}_\circ^{\Gamma,\lambda} = \mathrm{vdim}\, \mathcal{X} - \dim \lambda$ and $\mathrm{vdim}\, \tilde{\mathcal{X}}^{\Gamma,\mu} = \cdots = \mathrm{vdim}\, \hat{\mathcal{X}}_\circ^{\Gamma,\mu} = \mathrm{vdim}\, \mathcal{X} - \dim \mu$.

The analogue of Proposition 11.23 on orientations of orbifold strata $\mathcal{X}^{\Gamma,\lambda}$, $\ldots, \hat{\mathcal{X}}_\circ^{\Gamma,\mu}$ for oriented d-orbifolds $\mathcal{X}$ also holds for d-orbifolds with corners. In an analogue of Corollary 12.17, we can relate boundaries of orbifold strata to orbifold strata of boundaries:

Proposition 14.19. *Let $\mathcal{X}$ be a d-orbifold with corners, and Γ a finite group. Then Corollary 13.15 gives 1-morphisms $\boldsymbol{J}^\Gamma(\mathcal{X}) : (\partial\mathcal{X})^\Gamma \to \partial(\mathcal{X}^\Gamma)$, $\tilde{\boldsymbol{J}}^\Gamma(\mathcal{X}) : (\partial\tilde{\mathcal{X}})^\Gamma \to \partial(\tilde{\mathcal{X}}^\Gamma)$, $\hat{\boldsymbol{J}}^\Gamma(\mathcal{X}) : (\partial\hat{\mathcal{X}})^\Gamma \to \partial(\hat{\mathcal{X}}^\Gamma)$ in $\mathbf{d\check{O}rb^c}$, which are equivalences with open and closed subobjects in $\partial(\mathcal{X}^\Gamma), \partial(\tilde{\mathcal{X}}^\Gamma), \partial(\hat{\mathcal{X}}^\Gamma)$.*

These restrict to 1-morphisms $\boldsymbol{J}^{\Gamma,\lambda}(\mathcal{X}) : (\partial\mathcal{X})^{\Gamma,\lambda} \to \partial(\mathcal{X}^{\Gamma,\lambda})$ in $\mathbf{dOrb^c}$ for $\lambda \in \Lambda^\Gamma$ and $\tilde{\boldsymbol{J}}^{\Gamma,\mu}(\mathcal{X}) : (\partial\tilde{\mathcal{X}})^{\Gamma,\mu} \to \partial(\tilde{\mathcal{X}}^{\Gamma,\mu})$, $\hat{\boldsymbol{J}}^{\Gamma,\mu}(\mathcal{X}) : (\partial\hat{\mathcal{X}})^{\Gamma,\mu} \to \partial(\hat{\mathcal{X}}^{\Gamma,\mu})$ for $\mu \in \Lambda^\Gamma / \mathrm{Aut}(\Lambda)$, which are equivalences with open and closed d-suborbifolds. Hence, if $\mathcal{X}^{\Gamma,\lambda} = \emptyset$ then $(\partial\mathcal{X})^{\Gamma,\lambda} = \emptyset$, and similarly for $\tilde{\mathcal{X}}^{\Gamma,\mu}, \hat{\mathcal{X}}^{\Gamma,\mu}$.

Now suppose $\mathcal{X}$ is straight, in the sense of §13.7, for instance $\mathcal{X}$ could be a d-orbifold with boundary. Then $\boldsymbol{J}^\Gamma(\mathcal{X}), \ldots, \hat{\boldsymbol{J}}^{\Gamma,\mu}(\mathcal{X})$ are equivalences, so that $(\partial\mathcal{X})^\Gamma \simeq \partial(\mathcal{X}^\Gamma)$, $(\partial\mathcal{X})^{\Gamma,\lambda} \simeq \partial(\mathcal{X}^{\Gamma,\lambda})$, and so on.

14.9 Kuranishi neighbourhoods, good coordinate systems

Section 11.9 defined type A Kuranishi neighbourhoods, coordinate changes, and good coordinate systems, on d-orbifolds. In [35, §12.9] we generalize these to d-orbifolds with corners. The definitions in the corners case are obtained by replacing $\mathbf{Man}, \mathbf{Orb}, \mathbf{dMan}, \mathbf{dOrb}$ by $\mathbf{Man^c}, \mathbf{Orb^c}, \mathbf{dMan^c}, \mathbf{dOrb^c}$ throughout, and making a few other easy changes such as taking the e_{ij} to be sf-embeddings in Definitions 11.25(c). For brevity we will not write the definitions out again, but just indicate the differences.

Definition 14.20. Let $\mathcal{X}$ be a d-orbifold with corners. Define a *type A Kuranishi neighbourhood* $(V, E, \Gamma, s, \boldsymbol{\psi})$ on $\mathcal{X}$ following Definition 11.24, but taking V to be a manifold with corners, and defining the principal d-orbifold with corners $[\mathbf{S}_{V,E,s}/\Gamma]$ by combining Example 7.3 and §13.2, as in §14.2.

If $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i), (V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$ are type A Kuranishi neighbourhoods on $\mathcal{X}$ with $\emptyset \neq \boldsymbol{\psi}_i([\mathbf{S}_{V_i,E_i,s_i}/\Gamma_i]) \cap \boldsymbol{\psi}_j([\mathbf{S}_{V_j,E_j,s_j}/\Gamma_j]) \subseteq \mathcal{X}$, define a *type A coordinate change* $(V_{ij}, e_{ij}, \hat{e}_{ij}, \rho_{ij}, \boldsymbol{\eta}_{ij})$ from $(V_i, E_i, \Gamma_i, s_i, \boldsymbol{\psi}_i)$ to $(V_j, E_j, \Gamma_j, s_j, \boldsymbol{\psi}_j)$ following Definition 11.25, but taking $e_{ij} : V_{ij} \to V_j$ to be an sf-embedding of

manifolds with corners, and defining the quotient 1-morphism $[\mathbf{S}_{e_{ij},\hat{e}_{ij}},\rho_{ij}]$ by combining Example 7.4 and §13.2, as in §14.2.

Define a *type A good coordinate system* on $\boldsymbol{\mathcal{X}}$ following Definition 11.26, defining quotient 2-morphisms $\boldsymbol{\eta}_{ijk} = [0,\gamma_{ijk}]$ in (d) using §13.2. Let Y be a manifold with corners, and $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ a 1-morphism in $\mathbf{dOrb^c}$, where $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Man^c}}^{\mathbf{dOrb^c}}(Y)$. Define a *type A good coordinate system for* $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ following Definition 11.26.

Here is the analogue of Theorem 11.27. It is proved in [35, App. D].

Theorem 14.21. *Suppose $\boldsymbol{\mathcal{X}}$ is a d-orbifold with corners. Then there exists a type A good coordinate system $\big(I,<,(V_i,E_i,\Gamma_i,s_i,\boldsymbol{\psi}_i),(V_{ij},e_{ij},\hat{e}_{ij},\rho_{ij},\boldsymbol{\eta}_{ij}),\gamma_{ijk}\big)$ for $\boldsymbol{\mathcal{X}}$, in the sense of Definition 14.20. If $\boldsymbol{\mathcal{X}}$ is compact, we may take I to be finite. If $\{\boldsymbol{\mathcal{U}}_j : j \in J\}$ is an open cover of $\boldsymbol{\mathcal{X}}$, we may take $\boldsymbol{\mathcal{X}}_i = \boldsymbol{\psi}_i([\mathbf{S}_{V_i,E_i,s_i}/\Gamma_i]) \subseteq \boldsymbol{\mathcal{U}}_{j_i}$ for each $i \in I$ and some $j_i \in J$. If $\boldsymbol{\mathcal{X}}$ is a d-orbifold with boundary, we may take the V_i to be manifolds with boundary.*

Now let Y be a manifold with corners and $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}} = F_{\mathbf{Man^c}}^{\mathbf{dOrb^c}}(Y)$ a semisimple, flat 1-morphism in $\mathbf{dOrb^c}$. Then all the above extends to type A good coordinate systems for $\boldsymbol{h} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$, and we may take the $g_i : V_i \to Y$ to be submersions in $\mathbf{Man^c}$.

We can regard Theorem 14.21 as a kind of converse to Theorem 14.11. Note that we make the extra assumption that $\boldsymbol{h}$ is semisimple and flat in the last part. This happens automatically if Y is without boundary. Since submersions in $\mathbf{Man^c}$ are automatically semisimple and flat, $\boldsymbol{h}$ being semisimple and flat is a necessary condition for the $g_i : V_i \to Y$ to be submersions. In [35, §12.9] we also give 'type B' versions of Definition 14.20 and Theorem 14.21 using the 'standard model' d-orbifolds with corners $\boldsymbol{\mathcal{S}}_{\mathcal{V},\mathcal{E},s}$ and 1-morphisms $\boldsymbol{\mathcal{S}}_{e_{ij},\hat{e}_{ij}}$ of Examples 14.5–14.6 instead of $[\mathbf{S}_{V_i,E_i,s_i}/\Gamma_i]$ and $[\mathbf{S}_{e_{ij},\hat{e}_{ij}},\rho_{ij}]$.

14.10 Semieffective and effective d-orbifolds with corners

Section 11.10 discussed semieffective and effective d-orbifolds. As in [35, §12.10], all this material extends to d-orbifolds with corners essentially unchanged. We define *semieffective* and *effective* d-orbifolds with corners $\boldsymbol{\mathcal{X}}$ following Definition 11.28. The analogues of Proposition 11.29 and the rest of §11.9 then hold, with (d-)orbifolds replaced by (d-)orbifolds with corners throughout.

Proposition 14.22. *Let $\boldsymbol{\mathcal{X}}$ be an effective (or semieffective) d-orbifold with corners. Then $\partial^k\boldsymbol{\mathcal{X}}$ is also effective (or semieffective), for all $k \geqslant 0$.*

However, $\boldsymbol{\mathcal{X}}$ (semi)effective does not imply $C_k(\boldsymbol{\mathcal{X}})$ (semi)effective.

15   D-manifold and d-orbifold bordism

In [35, Chap. 13] we discuss *bordism groups* of manifolds and orbifolds, defined using manifolds, d-manifolds, orbifolds, and d-orbifolds. We can use these to

prove that compact, oriented d-manifolds admit virtual cycles, and so can be used in enumerative invariant problems. The same applies for compact, oriented d-orbifolds, although the direct proof using bordism no longer works.

15.1 Classical bordism groups for manifolds

In [35, §13.1] we review background material on bordism from the literature. Classical bordism groups $MSO_k(Y)$ were defined by Atiyah [5] for topological spaces Y, using continuous maps $f : X \to Y$ for X a compact, oriented manifold. Conner [15, §I] gives a good introduction. We define bordism $B_k(Y)$ only for manifolds Y, using smooth $f : X \to Y$, following Conner's *differential bordism groups* [15, §I.9]. By [15, Th. I.9.1], the natural projection $B_k(Y) \to MSO_k(Y)$ is an isomorphism, so our notion of bordism agrees with the usual definition.

Definition 15.1. Let Y be a manifold without boundary, and $k \in \mathbb{Z}$. Consider pairs (X, f), where X is a compact, oriented manifold without boundary with $\dim X = k$, and $f : X \to Y$ is a smooth map. Define an equivalence relation $\sim$ on such pairs by $(X, f) \sim (X', f')$ if there exists a compact, oriented $(k+1)$-manifold with boundary W, a smooth map $e : W \to Y$, and a diffeomorphism of oriented manifolds $j : -X \amalg X' \to \partial W$, such that $f \amalg f' = e \circ i_W \circ j$, where $-X$ is X with the opposite orientation.

Write $[X, f]$ for the $\sim$-equivalence class (*bordism class*) of a pair (X, f). For each $k \in \mathbb{Z}$, define the kth *bordism group* $B_k(Y)$ of Y to be the set of all such bordism classes $[X, f]$ with $\dim X = k$. We give $B_k(Y)$ the structure of an abelian group, with zero element $0_Y = [\emptyset, \emptyset]$, and addition given by $[X, f] + [X', f'] = [X \amalg X', f \amalg f']$, and additive inverses $-[X, f] = [-X, f]$.

Define $\Pi_{\mathrm{bo}}^{\mathrm{hom}} : B_k(Y) \to H_k(Y; \mathbb{Z})$ by $\Pi_{\mathrm{bo}}^{\mathrm{hom}} : [X, f] \mapsto f_*([X])$, where $H_*(-; \mathbb{Z})$ is singular homology, and $[X] \in H_k(X; \mathbb{Z})$ is the fundamental class.

If Y is oriented and of dimension n, there is a biadditive, associative, super-commutative *intersection product* $\bullet : B_k(Y) \times B_l(Y) \to B_{k+l-n}(Y)$, such that if $[X, f], [X', f']$ are classes in $B_*(Y)$, with f, f' transverse, then the fibre product $X \times_{f, Y, f'} X'$ exists as a compact oriented manifold, and

$$[X, f] \bullet [X', f'] = [X \times_{f, Y, f'} X', f \circ \pi_X].$$

As in [15, §I.5], bordism is a generalized homology theory. Results of Thom, Wall and others in [15, §I.2] compute the bordism groups $B_k(*)$ of the point $*$. This partially determines the bordism groups of general manifolds Y, as there is a spectral sequence $H_i(Y; B_j(*)) \Rightarrow B_{i+j}(Y)$.

15.2 D-manifold bordism groups

In [35, §13.2] we define *d-manifold bordism* by replacing manifolds X in pairs $[X, f]$ in §15.1 by d-manifolds $\boldsymbol{X}$. For simplicity, we identify the 2-category **dMan** of d-manifolds $\boldsymbol{X}$ in §4.1, and the 2-subcategory **dM̄an** of d-manifolds without boundary $\mathbf{X} = (\boldsymbol{X}, \emptyset, \emptyset, \emptyset)$ in **dMan^c** in §7.1, writing both as $\boldsymbol{X}$.

Definition 15.2. Let Y be a manifold without boundary, and $k \in \mathbb{Z}$. Consider pairs $(\boldsymbol{X}, \boldsymbol{f})$, where $\boldsymbol{X} \in \mathbf{dMan}$ is a compact, oriented d-manifold without boundary with vdim $\boldsymbol{X} = k$, and $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism in $\mathbf{dMan}$, where $\boldsymbol{Y} = F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$.

Define an equivalence relation $\sim$ between such pairs by $(\boldsymbol{X}, \boldsymbol{f}) \sim (\boldsymbol{X}', \boldsymbol{f}')$ if there exists a compact, oriented d-manifold with boundary $\boldsymbol{W}$ with vdim $\boldsymbol{W} = k + 1$, a 1-morphism $\boldsymbol{e} : \boldsymbol{W} \to \boldsymbol{Y}$ in $\mathbf{dMan^b}$, an equivalence of oriented d-manifolds $\boldsymbol{j} : -\boldsymbol{X} \amalg \boldsymbol{X}' \to \partial\boldsymbol{W}$, and a 2-morphism $\eta : \boldsymbol{f} \amalg \boldsymbol{f}' \Rightarrow \boldsymbol{e} \circ \boldsymbol{i_W} \circ \boldsymbol{j}$.

Write $[\boldsymbol{X}, \boldsymbol{f}]$ for the $\sim$-equivalence class (*d-bordism class*) of a pair $(\boldsymbol{X}, \boldsymbol{f})$. For each $k \in \mathbb{Z}$, define the kth *d-manifold bordism group*, or *d-bordism group*, $dB_k(Y)$ of Y to be the set of all such d-bordism classes $[\boldsymbol{X}, \boldsymbol{f}]$ with vdim $\boldsymbol{X} = k$. As for $B_k(Y)$, we give $dB_k(Y)$ the structure of an abelian group, with zero element $0_Y = [\emptyset, \emptyset]$, addition $[\boldsymbol{X}, \boldsymbol{f}] + [\boldsymbol{X}', \boldsymbol{f}'] = [\boldsymbol{X} \amalg \boldsymbol{X}', \boldsymbol{f} \amalg \boldsymbol{f}']$, and additive inverses $-[\boldsymbol{X}, \boldsymbol{f}] = [-\boldsymbol{X}, \boldsymbol{f}]$.

If Y is oriented and of dimension n, we define a biadditive, associative, supercommutative *intersection product* $\bullet : dB_k(Y) \times dB_l(Y) \to dB_{k+l-n}(Y)$ by

$$[\boldsymbol{X}, \boldsymbol{f}] \bullet [\boldsymbol{X}', \boldsymbol{f}'] = [\boldsymbol{X} \times_{\boldsymbol{f}, \boldsymbol{Y}, \boldsymbol{f}'} \boldsymbol{X}', \boldsymbol{f} \circ \boldsymbol{\pi_X}]. \tag{15.1}$$

Here $\boldsymbol{X} \times_{\boldsymbol{f}, \boldsymbol{Y}, \boldsymbol{f}'} \boldsymbol{X}'$ exists as a d-manifold by Theorem 4.23(a), and is oriented by Theorem 4.37. Note that we do not need to restrict to $[X, f], [X', f']$ with f, f' transverse as in Definition 15.1. Define a morphism $\Pi_{\mathrm{bo}}^{\mathrm{dbo}} : B_k(Y) \to dB_k(Y)$ for $k \geqslant 0$ by $\Pi_{\mathrm{bo}}^{\mathrm{dbo}} : [X, f] \mapsto \left[F_{\mathbf{Man}}^{\mathbf{dMan}}(X), F_{\mathbf{Man}}^{\mathbf{dMan}}(f) \right]$.

In [35, §13.2] we prove that $B_*(Y)$ and $dB_*(Y)$ are isomorphic. See Spivak [53, Th. 2.6] for the analogous result for Spivak's derived manifolds.

Theorem 15.3. *For any manifold Y, we have $dB_k(Y) = 0$ for $k < 0$, and $\Pi_{\mathrm{bo}}^{\mathrm{dbo}} : B_k(Y) \to dB_k(Y)$ is an isomorphism for $k \geqslant 0$. When Y is oriented, $\Pi_{\mathrm{bo}}^{\mathrm{dbo}}$ identifies the intersection products $\bullet$ on $B_*(Y)$ and $dB_*(Y)$.*

Here is the main idea in the proof of Theorem 15.3. Let $[\boldsymbol{X}, \boldsymbol{f}] \in dB_k(Y)$. By Corollary 4.31 there exists an embedding $\boldsymbol{g} : \boldsymbol{X} \to \mathbb{R}^n$ for $n \gg 0$. Then the direct product $(\boldsymbol{f}, \boldsymbol{g}) : \boldsymbol{X} \to \boldsymbol{Y} \times \mathbb{R}^n$ is also an embedding. Theorem 4.32 shows that there exist an open set $V \subseteq Y \times \mathbb{R}^n$, a vector bundle $E \to V$ and $s \in C^\infty(E)$ such that $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$. Let $\tilde{s} \in C^\infty(E)$ be a small, generic perturbation of s. As $\tilde{s}$ is generic, the graph of $\tilde{s}$ in E intersects the zero section transversely. Hence $\tilde{X} = \tilde{s}^{-1}(0)$ is a k-manifold for $k \geqslant 0$, which is compact and oriented for $\tilde{s} - s$ small, and $\tilde{X} = \emptyset$ for $k < 0$. Set $\tilde{f} = \pi_Y|_{\tilde{X}} : \tilde{X} \to Y$. Then $\Pi_{\mathrm{bo}}^{\mathrm{dbo}}\big([\tilde{X}, \tilde{f}]\big) = [\boldsymbol{X}, \boldsymbol{f}]$, so that $\Pi_{\mathrm{bo}}^{\mathrm{dbo}}$ is surjective. A similar argument for $\boldsymbol{W}, \boldsymbol{e}$ in Definition 15.2 shows that $\Pi_{\mathrm{bo}}^{\mathrm{dbo}}$ is injective.

Theorem 15.3 implies that we may define projections

$$\Pi_{\mathrm{dbo}}^{\mathrm{hom}} : dB_k(Y) \longrightarrow H_k(Y; \mathbb{Z}) \ \text{ by } \ \Pi_{\mathrm{dbo}}^{\mathrm{hom}} = \Pi_{\mathrm{bo}}^{\mathrm{hom}} \circ (\Pi_{\mathrm{bo}}^{\mathrm{dbo}})^{-1}. \tag{15.2}$$

We think of $\Pi_{\mathrm{dbo}}^{\mathrm{hom}}$ as a *virtual class map*. Virtual classes (or virtual cycles, or virtual chains) are used in several areas of geometry to construct enumerative invariants using moduli spaces. In algebraic geometry, Behrend and Fantechi [8]

construct virtual classes for schemes with obstruction theories. In symplectic geometry, there are many versions — see for example Fukaya et al. [20, §6], [19, §A1], Hofer et al. [25], and McDuff [44].

The main message we want to draw from this is that *compact oriented d-manifolds admit virtual classes* (or virtual cycles, or virtual chains, as appropriate). Thus, we can use d-manifolds (and d-orbifolds) as the geometric structure on moduli spaces in enumerative invariant problems such as Gromov–Witten invariants, Lagrangian Floer cohomology, Donaldson–Thomas invariants, ..., as this structure is strong enough to contain all the 'counting' information.

15.3 Classical bordism for orbifolds

In [35, §13.3] we generalize §15.1 to orbifolds. Here we will be brief, much more information is given in [35, §13.3]. We use the notation of §9 on orbifolds **Orb** and §12 on orbifolds with boundary **Orb$^{\mathbf{b}}$** freely. For simplicity we do not distinguish between the 2-categories **Orb** in §9.1 and **Ȯrb** in §12.1.

Definition 15.4. Let $\mathcal{Y}$ be an orbifold, and $k \in \mathbb{Z}$. Consider pairs $(\mathcal{X}, f)$, where $\mathcal{X}$ is a compact, oriented orbifold (without boundary) with $\dim \mathcal{X} = k$, and $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism in **Orb**. Define an equivalence relation $\sim$ between such pairs by $(\mathcal{X}, f) \sim (\mathcal{X}', f')$ if there exists a compact, oriented $(k+1)$-orbifold with boundary $\mathcal{W}$, a 1-morphism $e : \mathcal{W} \to \mathcal{Y}$ in **Orb$^{\mathbf{b}}$**, an orientation-preserving equivalence $j : -\mathcal{X} \amalg \mathcal{X}' \to \partial\mathcal{W}$, and a 2-morphism $\eta : f \amalg f' \Rightarrow e \circ i_{\mathcal{W}} \circ j$ in **Orb$^{\mathbf{b}}$**.

Write $[\mathcal{X}, f]$ for the $\sim$-equivalence class (*bordism class*) of a pair $(\mathcal{X}, f)$. For each $k \in \mathbb{Z}$, define the kth *orbifold bordism group* $B_k^{\mathrm{orb}}(\mathcal{Y})$ of $\mathcal{Y}$ to be the set of all such bordism classes $[\mathcal{X}, f]$ with $\dim \mathcal{X} = k$. It is an abelian group, with zero $0_{\mathcal{Y}} = [\emptyset, \emptyset]$, addition $[\mathcal{X}, f] + [\mathcal{X}', f'] = [\mathcal{X} \amalg \mathcal{X}', f \amalg f']$, and additive inverses $-[\mathcal{X}, f] = [-\mathcal{X}, f]$. If $k < 0$ then $B_k^{\mathrm{orb}}(\mathcal{Y}) = 0$.

Define *effective orbifold bordism* $B_k^{\mathrm{eff}}(\mathcal{Y})$ in the same way, but requiring both orbifolds $\mathcal{X}$ and orbifolds with boundary $\mathcal{W}$ to be effective (as in §9.1 and §12.1) in pairs $(\mathcal{X}, f)$ and the definition of $\sim$.

If $\mathcal{Y}$ is an orbifold, define group morphisms

$$\Pi_{\mathrm{eff}}^{\mathrm{orb}} : B_k^{\mathrm{eff}}(\mathcal{Y}) \longrightarrow B_k^{\mathrm{orb}}(\mathcal{Y}), \ \ \Pi_{\mathrm{orb}}^{\mathrm{hom}} : B_k^{\mathrm{orb}}(\mathcal{Y}) \longrightarrow H_k(\mathcal{Y}_{\mathrm{top}}; \mathbb{Q})$$
$$\text{and} \qquad \Pi_{\mathrm{eff}}^{\mathrm{hom}} : B_k^{\mathrm{eff}}(\mathcal{Y}) \longrightarrow H_k(\mathcal{Y}_{\mathrm{top}}; \mathbb{Z})$$

by $\Pi_{\mathrm{eff}}^{\mathrm{orb}} : [\mathcal{X}, f] \mapsto [\mathcal{X}, f]$ and $\Pi_{\mathrm{orb}}^{\mathrm{hom}}, \Pi_{\mathrm{eff}}^{\mathrm{hom}} : [\mathcal{X}, f] \mapsto (f_{\mathrm{top}})_*([\mathcal{X}])$, where $[\mathcal{X}]$ is the fundamental class of the compact, oriented k-orbifold $\mathcal{X}$, which lies in $H_k(\mathcal{X}_{\mathrm{top}}; \mathbb{Q})$ for general $\mathcal{X}$, and in $H_k(\mathcal{X}_{\mathrm{top}}; \mathbb{Z})$ for effective $\mathcal{X}$. The morphisms $\Pi_{\mathrm{eff}}^{\mathrm{orb}} : B_k^{\mathrm{eff}}(\mathcal{Y}) \to B_k^{\mathrm{orb}}(\mathcal{Y})$ are injective.

Suppose $\mathcal{Y}$ is an oriented orbifold of dimension n which is a manifold, that is, the orbifold groups $\mathrm{Iso}_{\mathcal{Y}}([y])$ are trivial for all $[y] \in \mathcal{Y}_{\mathrm{top}}$. Define biadditive, associative, supercommutative *intersection products* $\bullet : B_k^{\mathrm{orb}}(\mathcal{Y}) \times B_l^{\mathrm{orb}}(\mathcal{Y}) \to B_{k+l-n}^{\mathrm{orb}}(\mathcal{Y})$ and $\bullet : B_k^{\mathrm{eff}}(\mathcal{Y}) \times B_l^{\mathrm{eff}}(\mathcal{Y}) \to B_{k+l-n}^{\mathrm{eff}}(\mathcal{Y})$ as follows. Given classes $[\mathcal{X}, f], [\mathcal{X}', f']$, we perturb f, f' in their bordism classes to make $f : \mathcal{X} \to \mathcal{Y}$ and

$f' : \mathcal{X}' \to \mathcal{Y}$ transverse 1-morphisms, and then as in (15.1) we set

$$[\mathcal{X}, f] \bullet [\mathcal{X}', f'] = [\mathcal{X} \times_{f, \mathcal{Y}, f'} \mathcal{X}', f \circ \pi_{\mathcal{X}}].$$

If $\mathcal{Y}$ is not a manifold then f, f' may not admit transverse perturbations.

Again, orbifold bordism is a generalized homology theory. Results of Dr-uschel [16, 17] and Angel [2–4] give a complete description of the rational effective orbifold bordism ring $B_*^{\mathrm{eff}}(*) \otimes_{\mathbb{Z}} \mathbb{Q}$ when $\mathcal{Y}$ is the point $*$, and some information on the full ring $B_*^{\mathrm{eff}}(*)$. It is much more complicated than bordism $B_*(*)$ for manifolds in §15.1, because of contributions from orbifold strata.

As in §9.2, if $\mathcal{X}$ is an orbifold and Γ a finite group then we may define orbifold strata $\mathcal{X}^{\Gamma, \lambda}$ for $\lambda \in \Lambda_+^{\Gamma}$ and $\tilde{\mathcal{X}}^{\Gamma, \mu}$ for $\mu \in \Lambda_+^{\Gamma} / \operatorname{Aut}(\Gamma)$, which are orbifolds, with proper 1-morphisms $O^{\Gamma, \lambda}(\mathcal{X}) : \mathcal{X}^{\Gamma, \lambda} \to \mathcal{X}$ and $\tilde{O}^{\Gamma, \mu}(\mathcal{X}) : \tilde{\mathcal{X}}^{\Gamma, \mu} \to \mathcal{X}$. Hence, if $\mathcal{X}$ is compact then $\mathcal{X}^{\Gamma, \lambda}, \tilde{\mathcal{X}}^{\Gamma, \mu}$ are compact. If $\mathcal{X}$ is oriented then under extra conditions on Γ, λ, μ, which hold automatically for $\mathcal{X}^{\Gamma, \lambda}$ if $|\Gamma|$ is odd, we can define natural orientations on $\mathcal{X}^{\Gamma, \lambda}, \tilde{\mathcal{X}}^{\Gamma, \mu}$. Using these ideas, under the assumptions on Γ, λ, μ needed to orient $\mathcal{X}^{\Gamma, \lambda}, \tilde{\mathcal{X}}^{\Gamma, \mu}$ we define morphisms

$$\Pi_{\mathrm{orb}}^{\Gamma, \lambda} : B_k^{\mathrm{orb}}(\mathcal{Y}) \to B_{k - \dim \lambda}^{\mathrm{orb}}(\mathcal{Y}) \ \text{ by } \ \Pi_{\mathrm{orb}}^{\Gamma, \lambda} : [\mathcal{X}, f] \mapsto [\mathcal{X}^{\Gamma, \lambda}, f \circ O^{\Gamma, \lambda}(\mathcal{X})], \ (15.3)$$

$$\tilde{\Pi}_{\mathrm{orb}}^{\Gamma, \mu} : B_k^{\mathrm{orb}}(\mathcal{Y}) \to B_{k - \dim \mu}^{\mathrm{orb}}(\mathcal{Y}) \ \text{ by } \ \tilde{\Pi}_{\mathrm{orb}}^{\Gamma, \mu} : [\mathcal{X}, f] \mapsto [\tilde{\mathcal{X}}^{\Gamma, \mu}, f \circ \tilde{O}^{\Gamma, \mu}(\mathcal{X})]. \ (15.4)$$

One moral is that orbifold bordism groups $B_*^{\mathrm{orb}}(\mathcal{Y}), B_*^{\mathrm{eff}}(\mathcal{Y})$ are generally much bigger than manifold bordism groups $B_*(Y)$, because in elements $[\mathcal{X}, f]$ of orbifold bordism groups, extra information is contained in the orbifold strata of $\mathcal{X}$. The morphisms $\Pi_{\mathrm{orb}}^{\Gamma, \lambda}, \tilde{\Pi}_{\mathrm{orb}}^{\Gamma, \mu}$ recover some of this extra information.

15.4 Bordism for d-orbifolds

In [35, §13.4] we combine the ideas of §15.2 and §15.3 to define bordism for d-orbifolds. For simplicity we do not distinguish between the 2-categories **dOrb** in §11.1 and **d$\bar{\text{O}}$rb** $\subset$ **dOrb$^{\mathbf{c}}$** in §14.1.

Definition 15.5. Let $\mathcal{Y}$ be an orbifold, and $k \in \mathbb{Z}$. Consider pairs $(\boldsymbol{\mathcal{X}}, \boldsymbol{f})$, where $\boldsymbol{\mathcal{X}} \in \mathbf{dOrb}$ is a compact, oriented d-orbifold without boundary with vdim $\boldsymbol{\mathcal{X}} = k$, and $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ is a 1-morphism in **dOrb**, where $\boldsymbol{\mathcal{Y}} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{Y})$.

Define an equivalence relation $\sim$ between such pairs by $(\boldsymbol{\mathcal{X}}, \boldsymbol{f}) \sim (\boldsymbol{\mathcal{X}}', \boldsymbol{f}')$ if there exists a compact, oriented d-orbifold with boundary $\boldsymbol{\mathcal{W}}$ with vdim $\boldsymbol{\mathcal{W}} = k + 1$, a 1-morphism $\boldsymbol{e} : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{Y}}$ in **dOrb$^{\mathbf{b}}$**, an equivalence of oriented d-orbifolds $\boldsymbol{j} : -\boldsymbol{\mathcal{X}} \amalg \boldsymbol{\mathcal{X}}' \to \partial \boldsymbol{\mathcal{W}}$, and a 2-morphism $\eta : \boldsymbol{f} \amalg \boldsymbol{f}' \Rightarrow \boldsymbol{e} \circ \boldsymbol{i}_{\boldsymbol{\mathcal{W}}} \circ \boldsymbol{j}$.

Write $[\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$ for the $\sim$-equivalence class (*d-bordism class*) of a pair $(\boldsymbol{\mathcal{X}}, \boldsymbol{f})$. For each $k \in \mathbb{Z}$, define the kth *d-orbifold bordism group* $dB_k^{\mathrm{orb}}(\mathcal{Y})$ of $\mathcal{Y}$ to be the set of all such d-bordism classes $[\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$ with vdim $\boldsymbol{\mathcal{X}} = k$. We give $dB_k^{\mathrm{orb}}(\mathcal{Y})$ the structure of an abelian group, with zero element $0_{\mathcal{Y}} = [\emptyset, \emptyset]$, addition $[\boldsymbol{\mathcal{X}}, \boldsymbol{f}] + [\boldsymbol{\mathcal{X}}', \boldsymbol{f}'] = [\boldsymbol{\mathcal{X}} \amalg \boldsymbol{\mathcal{X}}', \boldsymbol{f} \amalg \boldsymbol{f}']$, and additive inverses $-[\boldsymbol{\mathcal{X}}, \boldsymbol{f}] = [-\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$.

Similarly, define the *semieffective d-orbifold bordism group* $dB_k^{\mathrm{sef}}(\mathcal{Y})$ and the *effective d-orbifold bordism group* $dB_k^{\mathrm{eff}}(\mathcal{Y})$ as above, but taking $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{W}}$ to be semieffective, or effective, respectively, in the sense of §11.10 and §14.10.

If $\mathcal{Y}$ is oriented and of dimension n, we define a biadditive, associative, supercommutative *intersection product* $\bullet : dB_k^{\mathrm{orb}}(\mathcal{Y}) \times dB_l^{\mathrm{orb}}(\mathcal{Y}) \to dB_{k+l-n}^{\mathrm{orb}}(\mathcal{Y})$ by

$$[\boldsymbol{\mathcal{X}}, \boldsymbol{f}] \bullet [\boldsymbol{\mathcal{X}}', \boldsymbol{f}'] = [\boldsymbol{\mathcal{X}} \times_{\boldsymbol{f}, \boldsymbol{\mathcal{Y}}, \boldsymbol{f}'} \boldsymbol{\mathcal{X}}', \boldsymbol{f} \circ \boldsymbol{\pi}_{\boldsymbol{\mathcal{X}}}],$$

as in (15.1). Here $\boldsymbol{\mathcal{X}} \times_{\boldsymbol{f}, \boldsymbol{\mathcal{Y}}, \boldsymbol{f}'} \boldsymbol{\mathcal{X}}'$ exists in **dOrb** by Theorem 11.18(a), and is oriented by the d-orbifold analogue of Theorem 4.37.

If $\mathcal{Y}$ is an orbifold, define group morphisms

$$\Pi_{\mathrm{orb}}^{\mathrm{sef}} : B_k^{\mathrm{orb}}(\mathcal{Y}) \longrightarrow dB_k^{\mathrm{sef}}(\mathcal{Y}), \quad \Pi_{\mathrm{eff}}^{\mathrm{deff}} : B_k^{\mathrm{eff}}(\mathcal{Y}) \longrightarrow dB_k^{\mathrm{eff}}(\mathcal{Y}),$$

$$\Pi_{\mathrm{deff}}^{\mathrm{sef}} : dB_k^{\mathrm{eff}}(\mathcal{Y}) \longrightarrow dB_k^{\mathrm{sef}}(\mathcal{Y}), \quad \Pi_{\mathrm{deff}}^{\mathrm{dorb}} : dB_k^{\mathrm{eff}}(\mathcal{Y}) \longrightarrow dB_k^{\mathrm{orb}}(\mathcal{Y}), \qquad (15.5)$$

$$\text{and} \qquad \Pi_{\mathrm{sef}}^{\mathrm{dorb}} : dB_k^{\mathrm{sef}}(\mathcal{Y}) \longrightarrow dB_k^{\mathrm{orb}}(\mathcal{Y})$$

by $\Pi_{\mathrm{orb}}^{\mathrm{sef}}, \Pi_{\mathrm{eff}}^{\mathrm{deff}} : [\mathcal{X}, f] \mapsto [\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$, where $\boldsymbol{\mathcal{X}}, \boldsymbol{f} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\mathcal{X}, f)$, and $\Pi_{\mathrm{deff}}^{\mathrm{sef}}, \Pi_{\mathrm{deff}}^{\mathrm{dorb}}$, $\Pi_{\mathrm{sef}}^{\mathrm{dorb}} : [\boldsymbol{\mathcal{X}}, \boldsymbol{f}] \mapsto [\boldsymbol{\mathcal{X}}, \boldsymbol{f}]$.

Here is the main result of [35, §13.4], an orbifold analogue of Theorem 15.3. The key idea is that semieffective (or effective) d-orbifolds $\boldsymbol{\mathcal{X}}$ can be perturbed to (effective) orbifolds, as in §11.10; to make this rigorous, we use good coordinate systems on $\boldsymbol{\mathcal{X}}$, as in §11.9.

Theorem 15.6. *For any orbifold $\mathcal{Y}$, the maps $\Pi_{\mathrm{orb}}^{\mathrm{sef}} : B_k^{\mathrm{orb}}(\mathcal{Y}) \to dB_k^{\mathrm{sef}}(\mathcal{Y})$ and $\Pi_{\mathrm{eff}}^{\mathrm{deff}} : B_k^{\mathrm{eff}}(\mathcal{Y}) \to dB_k^{\mathrm{eff}}(\mathcal{Y})$ in (15.5) are isomorphisms for all $k \in \mathbb{Z}$.*

As for (15.2), the theorem implies that we may define projections

$$\Pi_{\mathrm{sef}}^{\mathrm{hom}} : dB_k^{\mathrm{sef}}(\mathcal{Y}) \to H_k(\mathcal{Y}_{\mathrm{top}}; \mathbb{Q}), \quad \Pi_{\mathrm{deff}}^{\mathrm{hom}} : dB_k^{\mathrm{eff}}(\mathcal{Y}) \to H_k(\mathcal{Y}_{\mathrm{top}}; \mathbb{Z})$$

$$\text{by} \quad \Pi_{\mathrm{sef}}^{\mathrm{hom}} = \Pi_{\mathrm{orb}}^{\mathrm{hom}} \circ (\Pi_{\mathrm{orb}}^{\mathrm{sef}})^{-1} \quad \text{and} \quad \Pi_{\mathrm{deff}}^{\mathrm{hom}} = \Pi_{\mathrm{eff}}^{\mathrm{hom}} \circ (\Pi_{\mathrm{eff}}^{\mathrm{deff}})^{-1}.$$

We think of these $\Pi_{\mathrm{sef}}^{\mathrm{hom}}, \Pi_{\mathrm{deff}}^{\mathrm{hom}}$ as *virtual class maps* on $dB_*^{\mathrm{sef}}(\mathcal{Y}), dB_*^{\mathrm{eff}}(\mathcal{Y})$. In fact, with more work, one can also define virtual class maps on $dB_*^{\mathrm{orb}}(\mathcal{Y})$:

$$\Pi_{\mathrm{dorb}}^{\mathrm{hom}} : dB_k^{\mathrm{orb}}(\mathcal{Y}) \longrightarrow H_k(\mathcal{Y}_{\mathrm{top}}; \mathbb{Q}), \qquad (15.6)$$

satisfying $\Pi_{\mathrm{dorb}}^{\mathrm{hom}} \circ \Pi_{\mathrm{sef}}^{\mathrm{dorb}} = \Pi_{\mathrm{sef}}^{\mathrm{hom}}$, for instance following the method of Fukaya et al. [20, §6], [19, §A1] for virtual classes of Kuranishi spaces using 'multisections'.

In future work the author intends to define a virtual chain construction for d-manifolds and d-orbifolds, expressed in terms of new (co)homology theories whose (co)chains are built from d-manifolds or d-orbifolds, as for the 'Kuranishi (co)homology' described in [30, 31].

As in §11.8, if $\boldsymbol{\mathcal{X}}$ is a d-orbifold and Γ a finite group then we may define orbifold strata $\boldsymbol{\mathcal{X}}^{\Gamma, \lambda}$ for $\lambda \in \Lambda^{\Gamma}$ and $\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma, \mu}$ for $\mu \in \Lambda^{\Gamma} / \mathrm{Aut}(\Gamma)$, which are d-orbifolds, with proper 1-morphisms $\boldsymbol{O}^{\Gamma, \lambda}(\boldsymbol{\mathcal{X}}) : \boldsymbol{\mathcal{X}}^{\Gamma, \lambda} \to \boldsymbol{\mathcal{X}}$ and $\tilde{\boldsymbol{O}}^{\Gamma, \mu}(\boldsymbol{\mathcal{X}}) : \tilde{\boldsymbol{\mathcal{X}}}^{\Gamma, \mu} \to \boldsymbol{\mathcal{X}}$. Hence, if $\boldsymbol{\mathcal{X}}$ is compact then $\boldsymbol{\mathcal{X}}^{\Gamma, \lambda}, \tilde{\boldsymbol{\mathcal{X}}}^{\Gamma, \mu}$ are compact. If $\boldsymbol{\mathcal{X}}$ is oriented and Γ is odd then we under extra conditions on μ can define natural orientations on $\boldsymbol{\mathcal{X}}^{\Gamma, \lambda}, \tilde{\boldsymbol{\mathcal{X}}}^{\Gamma, \mu}$. As in (15.3)–(15.4), for such Γ, λ, μ we define morphisms

$$\Pi_{\mathrm{dorb}}^{\Gamma, \lambda} : dB_k^{\mathrm{orb}}(\mathcal{Y}) \to dB_{k-\dim \lambda}^{\mathrm{orb}}(\mathcal{Y}) \text{ by } \Pi_{\mathrm{dorb}}^{\Gamma, \lambda} : [\boldsymbol{\mathcal{X}}, \boldsymbol{f}] \mapsto [\boldsymbol{\mathcal{X}}^{\Gamma, \lambda}, \boldsymbol{f} \circ \boldsymbol{O}^{\Gamma, \lambda}(\boldsymbol{\mathcal{X}})],$$

$$\tilde{\Pi}_{\mathrm{dorb}}^{\Gamma, \mu} : dB_k^{\mathrm{orb}}(\mathcal{Y}) \to dB_{k-\dim \mu}^{\mathrm{orb}}(\mathcal{Y}) \text{ by } \tilde{\Pi}_{\mathrm{dorb}}^{\Gamma, \mu} : [\boldsymbol{\mathcal{X}}, \boldsymbol{f}] \mapsto [\tilde{\boldsymbol{\mathcal{X}}}^{\Gamma, \mu}, \boldsymbol{f} \circ \tilde{\boldsymbol{O}}^{\Gamma, \mu}(\boldsymbol{\mathcal{X}})].$$

We can use these operators $\Pi_{\mathrm{dorb}}^{\Gamma,\lambda}$ to study the d-orbifold bordism ring $dB_*^{\mathrm{orb}}(*)$. Let Γ be a finite group with $|\Gamma|$ odd, and R be a nontrivial Γ-representation. Define an element $[* \times_{\mathbf{0},\mathbf{R},\mathbf{0}} */\Gamma, \boldsymbol{\pi}]$ in $dB_{-\dim R}^{\mathrm{orb}}(*)$, where $\boldsymbol{R} = F_{\mathbf{Man}}^{\mathbf{dMan}}(R)$, and set $\lambda = [R] \in \Lambda_+^{\Gamma}$. Then $\Pi_{\mathrm{dorb}}^{\Gamma,-\lambda}\big([* \times_{\mathbf{0},\mathbf{R},\mathbf{0}} */\Gamma, \boldsymbol{\pi}]\big) \in dB_0^{\mathrm{orb}}(*)$, so $\Pi_{\mathrm{dorb}}^{\mathrm{hom}} \circ \Pi_{\mathrm{dorb}}^{\Gamma,-\lambda}\big([* \times_{\mathbf{0},\mathbf{R},\mathbf{0}} */\Gamma, \boldsymbol{\pi}]\big)$ lies in $H_0^{\mathrm{orb}}(*; \mathbb{Q}) \cong \mathbb{Q}$ by (15.6). Calculation shows that $\Pi_{\mathrm{dorb}}^{\mathrm{hom}} \circ \Pi_{\mathrm{dorb}}^{\Gamma,-\lambda}\big([* \times_{\mathbf{0},\mathbf{R},\mathbf{0}} */\Gamma, \boldsymbol{\pi}]\big)$ is either $|\mathrm{Aut}(\Gamma)|/|\Gamma|$ or 0, depending on λ. In the first case, $[* \times_{\mathbf{0},\mathbf{R},\mathbf{0}} */\Gamma, \boldsymbol{\pi}]$ has infinite order in $dB_{-\dim R}^{\mathrm{orb}}(*)$. Extending this argument, we can show that $dB_{4k}^{\mathrm{orb}}(*)$ has infinite rank for all $k \leqslant 0$. In contrast, $dB_k^{\mathrm{sef}}(*) = dB_k^{\mathrm{eff}}(*) = 0$ for all $k < 0$ by Theorem 15.6.

16 Relation to other classes of spaces

In [35, Chap. 14] we study the relationships between d-manifolds and d-orbifolds and other classes of geometric spaces in the literature. The next theorem summarizes our results:

Theorem 16.1. *We may construct 'truncation functors' from various classes of geometric spaces to d-manifolds and d-orbifolds, as follows:*

(a) *There is a functor $\Pi_{\mathbf{BManFS}}^{\mathbf{dMan}} : \mathbf{BManFS} \to \mathrm{Ho}(\mathbf{dMan})$, where $\mathbf{BManFS}$ is a category whose objects are triples (V, E, s) of a Banach manifold V, Banach vector bundle $E \to V$, and smooth section $s : V \to E$ whose linearization $\mathrm{d}s|_x : T_xV \to E|_x$ is Fredholm with index $n \in \mathbb{Z}$ for each $x \in V$ with $s|_x = 0$, and $\mathrm{Ho}(\mathbf{dMan})$ is the homotopy category of the 2-category of d-manifolds $\mathbf{dMan}$.*

  There is also an orbifold version $\Pi_{\mathbf{BOrbFS}}^{\mathbf{dOrb}} : \mathrm{Ho}(\mathbf{BOrbFS}) \to \mathrm{Ho}(\mathbf{dOrb})$ of this using Banach orbifolds $\mathcal{V}$, and 'corners' versions of both.

(b) *There is a functor $\Pi_{\mathbf{MPolFS}}^{\mathbf{dMan}} : \mathbf{MPolFS} \to \mathrm{Ho}(\mathbf{dMan})$, where $\mathbf{MPolFS}$ is a category whose objects are triples (V, E, s) of an **M-polyfold** without boundary V as in Hofer, Wysocki and Zehnder [22, §3.3], a fillable strong M-polyfold bundle E over V [22, §4.3], and an sc-smooth Fredholm section s of E [22, §4.4] whose linearization $\mathrm{d}s|_x : T_xV \to E|_x$ [22, §4.4] has Fredholm index $n \in \mathbb{Z}$ for all $x \in V$ with $s|_x = 0$.*

  *There is also an orbifold version $\Pi_{\mathbf{PolFS}}^{\mathbf{dOrb}} : \mathrm{Ho}(\mathbf{PolFS}) \to \mathrm{Ho}(\mathbf{dOrb})$ of this using **polyfolds** $\mathcal{V}$, and 'corners' versions of both.*

(c) *Given a d-orbifold with corners $\mathcal{X}$, we can construct a **Kuranishi space** (X, κ) in the sense of Fukaya, Oh, Ohta and Ono [19, §A], with the same underlying topological space X. Conversely, given a Kuranishi space (X, κ), we can construct a d-orbifold with corners $\mathcal{X}'$. Composing the two constructions, $\mathcal{X}$ and $\mathcal{X}'$ are equivalent in $\mathbf{dOrb^c}$.*

  Very roughly speaking, this means that the 'categories' of d-orbifolds with corners, and Kuranishi spaces, are equivalent. However, Fukaya et al. [19] do not define morphisms of Kuranishi spaces, so we have no category of Kuranishi spaces.

(d) *There is a functor* $\Pi^{\mathbf{dMan}}_{\mathbf{SchObs}} : \mathbf{Sch}_{\mathbb{C}}\mathbf{Obs} \to \mathrm{Ho}(\mathbf{dMan})$*, where* $\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}$ *is a category whose objects are triples* $(X, E^\bullet, \phi)$*, for* X *a separated, second countable* $\mathbb{C}$*-scheme and* $\phi : E^\bullet \to \tau_{\geqslant -1}(L_X)$ *a perfect obstruction theory on* X *with constant virtual dimension, in the sense of Behrend and Fantechi* [8]*. We may define a natural orientation on* $\Pi^{\mathbf{dMan}}_{\mathbf{SchObs}}(X, E^\bullet, \phi)$ *for each* $(X, E^\bullet, \phi)$*.*

There is also an orbifold version $\Pi^{\mathbf{dOrb}}_{\mathbf{StaObs}} : \mathrm{Ho}(\mathbf{Sta}_{\mathbb{C}}\mathbf{Obs}) \to \mathrm{Ho}(\mathbf{dOrb})$*, taking* $\mathcal{X}$ *to be a Deligne–Mumford* $\mathbb{C}$*-stack.*

(e) *There is a functor* $\Pi^{\mathbf{dMan}}_{\mathbf{QsDSch}} : \mathrm{Ho}(\mathbf{QsDSch}_{\mathbb{C}}) \longrightarrow \mathrm{Ho}(\mathbf{dMan})$*, where* $\mathbf{QsDSch}_{\mathbb{C}}$ *is the* ∞*-category of separated, second countable, quasi-smooth derived* $\mathbb{C}$*-schemes* X *of constant dimension, as in Toën and Vezzosi* [56–58]*. We may define a natural orientation on* $\Pi^{\mathbf{dMan}}_{\mathbf{QsDSch}}(X)$ *for each* X*.*

There is also an orbifold version $\Pi^{\mathbf{dOrb}}_{\mathbf{QsDSta}} : \mathrm{Ho}(\mathbf{QsDSta}_{\mathbb{C}}) \to \mathrm{Ho}(\mathbf{dOrb})$*, taking* $\mathcal{X}$ *to be a derived Deligne–Mumford* $\mathbb{C}$*-stack.*

(f) (Borisov [11]**)** *There is a natural functor* $\Pi^{\mathbf{dMan}}_{\mathbf{DerMan}} : \mathrm{Ho}(\mathbf{DerMan}^{\mathbf{pd}}_{\mathbf{ft}}) \to \mathrm{Ho}(\mathbf{dMan_{pr}})$ *from the homotopy category of the* ∞*-category* $\mathbf{DerMan}^{\mathbf{pd}}_{\mathbf{ft}}$ *of **derived manifolds** of finite type with pure dimension, in the sense of Spivak* [53]*, to the homotopy category of the full 2-subcategory* $\mathbf{dMan_{pr}}$ *of principal d-manifolds in* $\mathbf{dMan}$*. This functor induces a bijection between isomorphism classes of objects in* $\mathrm{Ho}(\mathbf{DerMan}^{\mathbf{pd}}_{\mathbf{ft}})$ *and* $\mathrm{Ho}(\mathbf{dMan_{pr}})$*. It is full, but not faithful. If* $[\boldsymbol{f}]$ *is a morphism in* $\mathrm{Ho}(\mathbf{DerMan}^{\mathbf{pd}}_{\mathbf{ft}})$*, then* $[\boldsymbol{f}]$ *is an isomorphism if and only if* $\Pi^{\mathbf{dMan}}_{\mathbf{DerMan}}([\boldsymbol{f}])$ *is an isomorphism.*

Here, as in §A.3, if $\mathcal{C}$ is a 2-category (or ∞-category), the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of $\mathcal{C}$ is the category whose objects are objects of $\mathcal{C}$, and whose morphisms are 2-isomorphism classes of 1-morphisms in $\mathcal{C}$. Then equivalences in $\mathcal{C}$ become isomorphisms in $\mathrm{Ho}(\mathcal{C})$, 2-commutative diagrams in $\mathcal{C}$ become commutative diagrams in $\mathrm{Ho}(\mathcal{C})$, and so on.

One moral of Theorem 16.1 is that essentially every geometric structure on moduli spaces which is used to define enumerative invariants, either in differential geometry, or in algebraic geometry over $\mathbb{C}$, has a truncation functor to d-manifolds or d-orbifolds. Combining Theorem 16.1 with proofs from the literature of the existence on moduli spaces of the geometric structures listed in Theorem 16.1, in [35, Chap. 14] we deduce:

Theorem 16.2. (i) *Any solution set of a smooth nonlinear elliptic equation with fixed topological invariants on a compact manifold naturally has the structure of a d-manifold, uniquely up to equivalence in* $\mathbf{dMan}$*.*

For example, let $(M, g), (N, h)$ *be Riemannian manifolds, with* M *compact. Then the family of **harmonic maps** $f : M \to N$ is a d-manifold* $\mathcal{H}_{M,N}$ *with* $\mathrm{vdim}\, \mathcal{H}_{M,N} = 0$*. If* $M = \mathcal{S}^1$*, then* $\mathcal{H}_{M,N}$ *is the moduli space of **parametrized closed geodesics** in* (N, h)*.*

(ii) *Let* (X, ω) *be a compact symplectic manifold of dimension* $2n$*, and* J *an almost complex structure on* X *compatible with* ω*. For* $\beta \in H_2(X, \mathbb{Z})$ *and*

$g, m \geqslant 0$, write $\overline{\mathcal{M}}_{g,m}(X, J, \beta)$ for the moduli space of stable triples $(\Sigma, \vec{z}, u)$ for Σ a genus g prestable Riemann surface with m marked points $\vec{z} = (z_1, \ldots, z_m)$ and $u : \Sigma \to X$ a J-holomorphic map with $[u(\Sigma)] = \beta$ in $H_2(X, \mathbb{Z})$. Using results of Hofer, Wysocki and Zehnder [27] involving their theory of polyfolds, we can make $\overline{\mathcal{M}}_{g,m}(X, J, \beta)$ into a compact, oriented d-orbifold $\overline{\boldsymbol{\mathcal{M}}}_{g,m}(X, J, \beta)$.

(iii) Let (X, ω) be a compact symplectic manifold, J an almost complex structure on X compatible with ω, and Y a compact, embedded Lagrangian submanifold in X. For $\beta \in H_2(X, Y; \mathbb{Z})$ and $k \geqslant 0$, write $\overline{\mathcal{M}}_k(X, Y, J, \beta)$ for the moduli space of J-***holomorphic stable maps*** $(\Sigma, \vec{z}, u)$ to X from a prestable holomorphic disc Σ with k boundary marked points $\vec{z} = (z_1, \ldots, z_k)$, with $u(\partial \Sigma) \subseteq Y$ and $[u(\Sigma)] = \beta$ in $H_2(X, Y; \mathbb{Z})$. Using results of Fukaya, Oh, Ohta and Ono [19, §7–§8] involving their theory of Kuranishi spaces, we can make $\overline{\mathcal{M}}_k(X, Y, J, \beta)$ into a compact d-orbifold with corners $\overline{\boldsymbol{\mathcal{M}}}_k(X, Y, J, \beta)$. Given a relative spin structure for (X, Y), we may define an orientation on $\overline{\boldsymbol{\mathcal{M}}}_k(X, Y, J, \beta)$.

(iv) Let X be a complex projective manifold, and $\overline{\mathcal{M}}_{g,m}(X, \beta)$ the Deligne–Mumford moduli $\mathbb{C}$-stack of stable triples $(\Sigma, \vec{z}, u)$ for Σ a genus g prestable Riemann surface with m marked points $\vec{z} = (z_1, \ldots, z_m)$ and $u : \Sigma \to X$ a morphism with $u_*([\Sigma]) = \beta \in H_2(X; \mathbb{Z})$. Then Behrend [6] defines a perfect obstruction theory on $\overline{\mathcal{M}}_{g,m}(X, \beta)$, so we can make $\overline{\mathcal{M}}_{g,m}(X, \beta)$ into a compact, oriented d-orbifold $\overline{\boldsymbol{\mathcal{M}}}_{g,m}(X, \beta)$.

(v) Let X be a complex algebraic surface, and $\mathcal{M}$ a stable moduli $\mathbb{C}$-scheme of vector bundles or coherent sheaves E on X with fixed Chern character. Then Mochizuki [46] defines a perfect obstruction theory on $\mathcal{M}$, so we can make $\mathcal{M}$ into an oriented d-manifold $\boldsymbol{\mathcal{M}}$.

(vi) Let X be a complex Calabi–Yau 3-fold or smooth Fano 3-fold, and $\mathcal{M}$ a stable moduli $\mathbb{C}$-scheme of coherent sheaves E on X with fixed Hilbert polynomial. Then Thomas [54] defines a perfect obstruction theory on $\mathcal{M}$, so we can make $\mathcal{M}$ into an oriented d-manifold $\boldsymbol{\mathcal{M}}$.

(vii) Let X be a smooth complex projective 3-fold, and $\mathcal{M}$ a moduli $\mathbb{C}$-scheme of 'stable PT pairs' (C, D) in X, where $C \subset X$ is a curve and $D \subset C$ is a divisor. Then Pandharipande and Thomas [50] define a perfect obstruction theory on $\mathcal{M}$, so we can make $\mathcal{M}$ into a compact, oriented d-manifold $\boldsymbol{\mathcal{M}}$.

(ix) Let X be a complex Calabi–Yau 3-fold, and $\mathcal{M}$ a separated moduli $\mathbb{C}$-scheme of simple perfect complexes in the derived category $D^b \operatorname{coh}(X)$. Then Huybrechts and Thomas [28] define a perfect obstruction theory on $\mathcal{M}$, so we can make $\mathcal{M}$ into an oriented d-manifold $\boldsymbol{\mathcal{M}}$.

A consequence of Theorems 16.1 and 16.2 is that we can use d-manifolds and d-orbifolds in many areas of differential and algebraic geometry to do with moduli spaces, enumerative invariants, Floer homology theories, and so on. Working with d-manifolds and d-orbifolds instead of the geometric structures currently in use may lead to new results, or simplifications of known proofs. We discuss some applications in symplectic geometry:

Remark 16.3. (a) Suppose (X, ω) is a compact symplectic manifold, and choose an almost complex structure J on X compatible with ω. Let g, m be non-

negative integers, and $\beta \in H_2(X;\mathbb{Z})$. Consider the moduli space $\overline{\mathcal{M}}_{g,m}(X,\beta)$ of genus g stable J-holomorphic curves $(\Sigma,\vec{z},u)$ with m marked points in homology class β in X. Using Hofer, Wysocki and Zehnder [27], Theorem 16.2(ii) makes $\overline{\mathcal{M}}_{g,m}(X,\beta)$ into a compact, oriented d-orbifold $\overline{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta)$.

There are evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,m}(X,J,\beta) \to X$ for $i = 1,\ldots,m$ mapping $\mathrm{ev}_i : (\Sigma,\vec{z},u) \mapsto u(z_i)$. Also, if $2g + m \geqslant 3$ there is a projection $\pi_{g,m} : \overline{\mathcal{M}}_{g,m}(X,J,\beta) \to \overline{\mathcal{M}}_{g,m}$ mapping $\mathrm{ev}_i : (\Sigma,\vec{z},u) \mapsto \bar{\Sigma}$, where $\overline{\mathcal{M}}_{g,m}$ is the moduli space of Deligne–Mumford stable Riemann surfaces of genus g with m marked points, a compact orbifold of real dimension $2(m + 3g - 3)$, and $\bar{\Sigma}$ is the stabilization of Σ. As in [35, §14.2], these lift naturally to 1-morphisms $\mathbf{ev}_i : \overline{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta) \to \boldsymbol{\mathcal{X}}$ and $\boldsymbol{\pi}_{g,m} : \overline{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta) \to \overline{\boldsymbol{\mathcal{M}}}_{g,m}$ in $\mathbf{dOrb}$, where $\boldsymbol{\mathcal{X}} = F_{\mathbf{Man}}^{\mathbf{dOrb}}(X)$ and $\overline{\boldsymbol{\mathcal{M}}}_{g,m} = F_{\mathbf{Orb}}^{\mathbf{dOrb}}(\overline{\mathcal{M}}_{g,m})$.

Define the *Gromov–Witten d-orbifold bordism invariant* $GW_{g,m}^{\mathrm{dorb}}(X,\omega,\beta)$ by

$$GW_{g,m}^{\mathrm{dorb}}(X,\omega,\beta) =$$
$$\begin{cases} \big[\overline{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta), \mathbf{ev}_1 \times \cdots \times \mathbf{ev}_m\big] \in dB_{2k}^{\mathrm{orb}}(X^m), & 2g+m < 3, \\ \big[\overline{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta), \mathbf{ev}_1 \times \cdots \times \mathbf{ev}_m \times \boldsymbol{\pi}_{g,m}\big] \in dB_{2k}^{\mathrm{orb}}(X^m \times \overline{\mathcal{M}}_{g,m}), & 2g+m \geqslant 3, \end{cases}$$

where $k = c_1(X) \cdot \beta + (n - 3)(1 - g) + m$, with $\dim X = 2n$. As in [35, §14.2], we can prove this is *independent of the choice of almost complex structure J on X*. This is because any two choices J_0, J_1 for J may be joined by a smooth path of almost complex structures J_t, $t \in [0,1]$ compatible with ω. One can then make $\coprod_{t\in[0,1]} \overline{\mathcal{M}}_{g,m}(X,J_t,\beta)$ into a compact oriented d-orbifold with boundary, whose boundary is $\overline{\boldsymbol{\mathcal{M}}}_{g,m}(X,J_1,\beta) \amalg -\overline{\boldsymbol{\mathcal{M}}}_{g,m}(X,J_0,\beta)$, and this defines a bordism between the moduli spaces for J_0 and J_1.

Applying the virtual class maps $\Pi_{\mathrm{dorb}}^{\mathrm{hom}}$ in (15.6) gives homology classes

$$\Pi_{\mathrm{dorb}}^{\mathrm{hom}}\big(GW_{g,m}^{\mathrm{dorb}}(X,\omega,\beta)\big) \in \begin{cases} H_{2k}(X^m;\mathbb{Q}), & 2g + m < 3, \\ H_{2k}\big(X^m \times (\overline{\mathcal{M}}_{g,m})_{\mathrm{top}};\mathbb{Q}\big), & 2g + m \geqslant 3. \end{cases}$$

These homology classes are essentially the *Gromov–Witten invariants* of (X,ω), as in Hofer, Wysocki and Zehnder [27] or Fukaya and Ono [20].

As in §15.4, d-orbifold bordism groups are far larger than homology groups, and can have infinite rank even in negative degrees. Therefore the new invariants $GW_{g,m}^{\mathrm{dorb}}(X,\omega,\beta)$ *contain more information than conventional Gromov–Witten invariants*, in particular, on the orbifold strata of the moduli spaces $\overline{\boldsymbol{\mathcal{M}}}_{g,m}(X,J,\beta)$, which can be recovered by applying the functors $\Pi_{\mathrm{dorb}}^{\Gamma,\lambda}$ of §15.4. Since the $dB_*^{\mathrm{orb}}(-)$ are defined over $\mathbb{Z}$ rather than $\mathbb{Q}$, these new invariants may be good tools for studying *integrality properties* of Gromov–Witten invariants.

(b) Let (X,J) be a projective complex manifold, with Kähler form ω. Consider the moduli space $\overline{\mathcal{M}}_{g,m}(X,\beta)$ of genus g stable J-holomorphic curves with m marked points in class $\beta \in H_2(X;\mathbb{Z})$ in X. As in **(a)**, by constructing a virtual class $[\overline{\mathcal{M}}_{g,m}(X,\beta)]_{\mathrm{virt}}$, we define the *Gromov–Witten invariants* of X.

We can do this in two different ways. In symplectic geometry, by Hofer, Wysocki and Zehnder [27] we can write $\overline{\mathcal{M}}_{g,m}(X,\beta)$ as the zeroes of a Fredholm

section of a polyfold bundle over a polyfold, as in Theorem 16.2(ii). In algebraic geometry, as in Behrend [6] we can make $\overline{\mathcal{M}}_{g,m}(X,\beta)$ into a Deligne–Mumford $\mathbb{C}$-stack with obstruction theory, as in Theorem 16.2(iv). We then apply the virtual class construction for polyfolds [25] or stacks with obstruction theories [8] to define the virtual class, and the Gromov–Witten invariants.

A priori it is not clear that the symplectic and algebraic routes give the same Gromov–Witten invariants (though it is generally accepted that they do). Comparing the two is difficult, as the techniques used are so different.

Now by Theorem 16.1(b),(d) we can make $\overline{\mathcal{M}}_{g,m}(X,\beta)$ into a compact oriented d-orbifold $\overline{\mathcal{M}}_{g,m}(X,J,\beta)_{\text{sym}}$ or $\overline{\mathcal{M}}_{g,m}(X,J,\beta)_{\text{alg}}$ in two different ways, by applying the truncation functors from polyfolds and Deligne–Mumford $\mathbb{C}$-stacks with obstruction theories. Then we could apply a virtual class construction for d-orbifolds, as discussed in §15.4, to define Gromov–Witten invariants.

Thus, d-orbifolds provide a bridge between symplectic and algebraic geometry, and a means to compare symplectic and algebraic Gromov–Witten theory. To use d-orbifolds to prove that symplectic and algebraic Gromov–Witten invariants coincide, we would have to do two things. Firstly, we would have to show that the virtual class constructions for polyfolds and Deligne–Mumford stacks with obstruction theories, yield the same answer as applying the truncation functors to d-orbifolds and then applying the virtual class construction for d-orbifolds. This does not look difficult.

Secondly, we would have to compare the d-orbifolds $\overline{\mathcal{M}}_{g,m}(X,J,\beta)_{\text{sym}}$ and $\overline{\mathcal{M}}_{g,m}(X,J,\beta)_{\text{alg}}$. The author does *not* expect these to be equivalent in **dOrb**. This is because the smooth structure near singular curves depends on a choice of *gluing profile* $\varphi : (0,1] \to [0,\infty)$, in the language of Hofer et al. [26, §4.2], [27, §2.1]. As in [27, §2.1], the gluing profiles used to construct $\overline{\mathcal{M}}_{g,m}(X,J,\beta)_{\text{sym}}$ and $\overline{\mathcal{M}}_{g,m}(X,J,\beta)_{\text{alg}}$ are $\varphi(r) = e^{1/r} - e$ and $\varphi(r) = -\frac{1}{2\pi}\ln r$, respectively. However, the author expects that $\overline{\mathcal{M}}_{g,m}(X,J,\beta)_{\text{sym}}$ and $\overline{\mathcal{M}}_{g,m}(X,J,\beta)_{\text{alg}}$ are naturally bordant, so they define the same Gromov–Witten d-orbifold bordism invariants in **(a)**, and the same Gromov–Witten invariants.

(c) Some areas of symplectic geometry — Lagrangian Floer cohomology, Fukaya categories, contact homology, and Symplectic Field Theory — involve compact moduli spaces $\overline{\mathcal{M}}$ of J-holomorphic curves Σ which either have boundary in a Lagrangian submanifold, or have ends asymptotic to cones on a Reeb orbit. Then $\overline{\mathcal{M}}$ naturally becomes a d-orbifold *with corners* $\overline{\mathcal{M}}$, as in Theorem 16.2(iii) above, where the boundary $\partial\overline{\mathcal{M}}$ parametrizes real codimension one bubbling behaviour of the curves Σ. This situation does not arise in (complex) algebraic geometry, where real codimension one singular behaviour does not occur.

For a moduli space $\overline{\mathcal{M}}$ with corners, we cannot define a virtual class $[\overline{\mathcal{M}}]_{\text{virt}}$ in some homology group $H_*(X)$. Instead, we can only construct a (nonunique, and depending on choices) *virtual chain* $\overline{\mathcal{M}}_{\text{virt}}$ in the chains $\big(C_*(X),\partial\big)$ of a homology theory $H_*(X)$, where $\partial\big(\overline{\mathcal{M}}_{\text{virt}}\big) \in C_{*-1}(X)$ is a virtual chain for $\partial\overline{\mathcal{M}}$. In applications (as in the Lagrangian Floer cohomology of Fukaya et al. [19], for instance), one wants to choose $\overline{\mathcal{M}}_{\text{virt}}$ so that $\partial\big(\overline{\mathcal{M}}_{\text{virt}}\big)$ is compatible with choices of virtual chains for other moduli spaces. This leads to difficult technical

issues about virtual chains, and some ugly homological algebra.

The author believes that these areas of symplectic geometry involving 'moduli spaces with corners' can be made simpler, more elegant, and less technical, by writing them in terms of d-orbifolds with corners. There are two parts to this. Firstly, by developing a well-behaved 2-category $\mathbf{dOrb^c}$ of d-orbifolds with corners, we have made it easy to state precise relationships between boundaries of moduli spaces and other moduli spaces. For example, for the moduli spaces $\overline{\mathcal{M}}_k(X, Y, J, \beta)$ of Theorem 16.2(iii), the Kuranishi space boundary formula of Fukaya et al. [19, Prop. 8.3.3] translates to an equivalence in $\mathbf{dOrb^c}$:

$$\partial \overline{\mathcal{M}}_k(X, Y, J, \beta) \simeq \coprod_{i+j=k, \ \beta=\gamma+\delta} \overline{\mathcal{M}}_{i+1}(X, Y, J, \gamma) \times_{\mathbf{ev}_{i+1}, \boldsymbol{\mathcal{y}}, \mathbf{ev}_{j+1}} \overline{\mathcal{M}}_{j+1}(X, Y, J, \delta),$$

where the fibre products exist in $\mathbf{dOrb^c}$ by Theorem 14.15(a).

Secondly, in future work the author intends to define a virtual chain construction for d-manifolds and d-orbifolds with corners, expressed in terms of new (co)homology theories whose (co)chains are built from d-manifolds or d-orbifolds with corners, as for the 'Kuranishi (co)homology' described in [30, 31]. Defining virtual chains for moduli spaces in this homology theory would be almost trivial, and would not involve perturbing the moduli space. Many issues to do with transversality would also become trivial.

Finally, we note that d-manifolds should not be confused with *differential graded manifolds*, or *dg-manifolds*. This term is used in two senses, in algebraic geometry to mean a special kind of dg-scheme, as in Ciocan-Fontanine and Kapranov [14, Def. 2.5.1], and in differential geometry to mean a supermanifold with extra structure, as in Cattaneo and Schätz [12, Def. 3.6]. In both cases, a dg-manifold $\mathfrak{E}$ is roughly the total space of a graded vector bundle $E^\bullet$ over a manifold V, with a vector field Q of degree 1 satisfying $[Q, Q] = 0$.

For example, if E is a vector bundle over V and $s \in C^\infty(E)$, we can make E into a dg-manifold $\mathfrak{E}$ by giving E the grading -1, and taking Q to be the vector field on E corresponding to s. To this $\mathfrak{E}$ we can associate the d-manifold $\boldsymbol{S}_{V,E,s}$ from Example 4.4. Note that $\boldsymbol{S}_{V,E,s}$ only knows about an infinitesimal neighbourhood of $s^{-1}(0)$ in V, but $\mathfrak{E}$ remembers all of V, E, s.

A   Categories and 2-categories

We now explain the background in category theory we need. Some good references are Behrend et al. [7, App. B], and MacLane [43] for §A.1–§A.2.

A.1   Basics of category theory

For completeness, here are the basic definitions in category theory, as in [43, §I].

Definition A.1. A *category* (or 1-*category*) $\mathcal{C}$ consists of a proper class of *objects* $\mathrm{Obj}(\mathcal{C})$, and for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ a set $\mathrm{Hom}(X, Y)$ of *morphisms* f

from X to Y, written $f : X \to Y$, and for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ a *composition map* $\circ : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$, written $(f, g) \mapsto g \circ f$. Composition must be *associative*, that is, if $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$ are morphisms in $\mathcal{C}$ then $(h \circ g) \circ f = h \circ (g \circ f)$. For each $X \in \mathrm{Obj}(\mathcal{C})$ there must exist an *identity morphism* $\mathrm{id}_X : X \to X$ such that $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$ for all $f : X \to Y$ in $\mathcal{C}$.

A morphism $f : X \to Y$ is an *isomorphism* if there exists $f^{-1} : Y \to X$ with $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$. A category $\mathcal{C}$ is called a *groupoid* if every morphism is an isomorphism. In a (small) groupoid $\mathcal{C}$, for each $X \in \mathrm{Obj}(\mathcal{C})$ the set $\mathrm{Hom}(X, X)$ of morphisms $f : X \to X$ form a group.

If $\mathcal{C}$ is a category, the *opposite category* $\mathcal{C}^{\mathrm{op}}$ is $\mathcal{C}$ with the directions of all morphisms reversed. That is, we define $\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Obj}(\mathcal{C})$, and for all $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$ we define $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$, and for $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$ we define $f \circ_{\mathcal{C}^{\mathrm{op}}} g = g \circ_{\mathcal{C}} f$, and $\mathrm{id}_{\mathcal{C}^{\mathrm{op}}} X = \mathrm{id}_{\mathcal{C}} X$.

Given categories $\mathcal{C}, \mathcal{D}$, the *product category* $\mathcal{C} \times \mathcal{D}$ has objects (W, X) in $\mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{D})$ and morphisms $f \times g : (W, X) \to (Y, Z)$ when $f : W \to Y$ is a morphism in $\mathcal{C}$ and $g : X \to Z$ is a morphism in $\mathcal{D}$, in the obvious way.

We call $\mathcal{D}$ a *subcategory* of $\mathcal{C}$ if $\mathrm{Obj}(\mathcal{D}) \subseteq \mathrm{Obj}(\mathcal{C})$, and $\mathrm{Hom}_{\mathcal{D}}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathrm{Obj}(\mathcal{D})$. We call $\mathcal{D}$ a *full* subcategory if also $\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all X, Y.

Definition A.2. Let $\mathcal{C}, \mathcal{D}$ be categories. A (*covariant*) *functor* $F : \mathcal{C} \to \mathcal{D}$ gives, for all objects X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$, and for all morphisms $f : X \to Y$ in $\mathcal{C}$ a morphism $F(f) : F(X) \to F(Y)$, such that $F(g \circ f) = F(g) \circ F(f)$ for all $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$, and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Obj}(\mathcal{C})$. A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Functors compose in the obvious way. Each category $\mathcal{C}$ has an obvious *identity functor* $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ with $\mathrm{id}_{\mathcal{C}}(X) = X$ and $\mathrm{id}_{\mathcal{C}}(f) = f$ for all X, f. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *full* if the maps $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$, $f \mapsto F(f)$ are surjective for all $X, Y \in \mathrm{Obj}(\mathcal{C})$, and *faithful* if the maps $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ are injective for all $X, Y \in \mathrm{Obj}(\mathcal{C})$.

Let $\mathcal{C}, \mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta : F \Rightarrow G$ gives, for all objects X in $\mathcal{C}$, a morphism $\eta(X) : F(X) \to G(X)$ such that if $f : X \to Y$ is a morphism in $\mathcal{C}$ then $\eta(Y) \circ F(f) = G(f) \circ \eta(X)$ as a morphism $F(X) \to G(Y)$ in $\mathcal{D}$. We call η a *natural isomorphism* if $\eta(X)$ is an isomorphism for all $X \in \mathrm{Obj}(\mathcal{C})$.

An *equivalence* between categories $\mathcal{C}, \mathcal{D}$ consists of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms $\eta : G \circ F \Rightarrow \mathrm{id}_{\mathcal{C}}$, $\zeta : F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$.

It is a fundamental principle of category theory that equivalent categories $\mathcal{C}, \mathcal{D}$ should be thought of as being 'the same', and naturally isomorphic functors $F, G : \mathcal{C} \to \mathcal{D}$ should be thought of as being 'the same'. Note that equivalence of categories $\mathcal{C}, \mathcal{D}$ is much weaker than strict isomorphism: isomorphism classes of objects in $\mathcal{C}$ are naturally in bijection with isomorphism classes of objects in $\mathcal{D}$, but there is no relation between the sizes of the isomorphism classes, so that $\mathcal{C}$ could have many more objects than $\mathcal{D}$, for instance.

A.2 Limits, colimits and fibre products in categories

We shall be interested in various kinds of *limits* and *colimits* in our categories of spaces. These are objects in the category with a universal property with respect to some class of diagrams. For an introduction to limits and colimits in category theory, see MacLane [43, §III]. Here are the basic definitions:

Definition A.3. Let $\mathcal{C}$ be a category. A *diagram* Δ in $\mathcal{C}$ is a class of objects S_i in $\mathcal{C}$ for $i \in I$, and a class of morphisms $\rho_j : S_{b(j)} \to S_{e(j)}$ in $\mathcal{C}$ for $j \in J$, where $b, e : J \to I$. The diagram is called *finite* if I, J are finite sets.

A *limit* of the diagram Δ is an object L in $\mathcal{C}$ and morphisms $\pi_i : L \to S_i$ for $i \in I$ such that $\rho_j \circ \pi_{b(j)} = \pi_{e(j)}$ for all $j \in J$, which has the universal property that given $L' \in \mathcal{C}$ and $\pi'_i : L' \to S_i$ for $i \in I$ with $\rho_j \circ \pi'_{b(j)} = \pi'_{e(j)}$ for all $j \in J$, there is a unique morphism $\lambda : L' \to L$ with $\pi'_i = \pi_i \circ \lambda$ for all $i \in I$.

A *fibre product* is a limit of a diagram $X \xrightarrow{g} Z \xleftarrow{h} Y$. The limit object W is often written $X \times_{g,Z,h} Y$ or $X \times_Z Y$, and the diagram

$$
\begin{array}{ccc}
W & \xrightarrow{\ \pi_Y\ } & Y \\
{\scriptstyle \pi_X}\big\downarrow & & \big\downarrow{\scriptstyle h} \\
X & \xrightarrow{\ g\ } & Z
\end{array}
$$

is called a *Cartesian square* in the category $\mathcal{C}$. A *terminal object* is a limit of the empty diagram.

A *colimit* of the diagram Δ is an object L in $\mathcal{C}$ and morphisms $\lambda_i : S_i \to L$ for $i \in I$ such that $\lambda_{b(j)} = \lambda_{e(j)} \circ \rho_j$ for all $j \in J$, which has the universal property that given $L' \in \mathcal{C}$ and $\lambda'_i : S_i \to L'$ for $i \in I$ with $\lambda'_{b(j)} = \lambda'_{e(j)} \circ \rho_j$ for all $j \in J$, there is a unique morphism $\pi : L \to L'$ with $\lambda'_i = \pi \circ \lambda_i$ for all $i \in I$.

A *pushout* is a colimit of a diagram $X \xleftarrow{e} W \xrightarrow{f} Y$.

If a limit or colimit exists, it is unique up to unique isomorphism in $\mathcal{C}$. We say that *all finite limits*, or *all fibre products exist in* $\mathcal{C}$, if limits exist for all finite diagrams, or for all diagrams $X \xrightarrow{g} Z \xleftarrow{h} Y$ respectively.
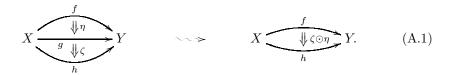
A.3 2-categories

Next we discuss 2-*categories*. A good reference for our purposes is Behrend et al. [7, App. B], and Kelly and Street [37] is also helpful.

Definition A.4. A 2-*category* $\mathcal{C}$ (also called a *strict* 2-*category*) consists of a proper class of *objects* $\mathrm{Obj}(\mathcal{C})$, for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ a category $\mathrm{Hom}(X, Y)$, for all $X \in \mathrm{Obj}(\mathcal{C})$ an object id_X in $\mathrm{Hom}(X, X)$ called the *identity 1-morphism*, and for all $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$ a functor $\mu_{X,Y,Z} : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$. These must satisfy the *identity property*, that $\mu_{X,X,Y}(\mathrm{id}_X, -) = \mu_{X,Y,Y}(-, \mathrm{id}_Y) = \mathrm{id}_{\mathrm{Hom}(X,Y)}$ as functors $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Y)$, and the *associativity property*, that $\mu_{W,Y,Z} \circ (\mu_{W,X,Y} \times \mathrm{id}_{\mathrm{Hom}(Y,Z)}) = \mu_{W,X,Z} \circ (\mathrm{id}_{\mathrm{Hom}(W,X)} \times \mu_{X,Y,Z})$ as functors $\mathrm{Hom}(W, X) \times \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(W, X)$, for all W, X, Y, Z.

Objects f of $\mathrm{Hom}(X,Y)$ are called 1-*morphisms*, written $f : X \to Y$. For 1-morphisms $f, g : X \to Y$, morphisms $\eta \in \mathrm{Hom}_{\mathrm{Hom}(X,Y)}(f,g)$ are called 2-*morphisms*, written $\eta : f \Rightarrow g$. Thus, a 2-category has objects X, and two kinds of morphisms, 1-morphisms $f : X \to Y$ between objects, and 2-morphisms $\eta : f \Rightarrow g$ between 1-morphisms. In many examples, all 2-morphisms are 2-isomorphisms (i.e. have an inverse), so that the categories $\mathrm{Hom}(X,Y)$ are groupoids. Such 2-categories are called (2,1)-*categories*.

This is quite a complicated structure. There are three kinds of composition in a 2-category, satisfying various associativity relations. If $f : X \to Y$ and $g : Y \to Z$ are 1-morphisms then $\mu_{X,Y,Z}(f,g)$ is the *composition of 1-morphisms*, written $g \circ f : X \to Z$. If $f, g, h : X \to Y$ are 1-morphisms and $\eta : f \Rightarrow g$, $\zeta : g \Rightarrow h$ are 2-morphisms then composition of η, ζ in the category $\mathrm{Hom}(X,Y)$ gives the *vertical composition of 2-morphisms* of η, ζ, written $\zeta \odot \eta : f \Rightarrow h$, as a diagram

$$\begin{array}{ccc}
X \xrightarrow{\quad} Y & \rightsquigarrow & X \xrightarrow{\quad} Y. \qquad (\text{A.1})
\end{array}$$

And if $f, \tilde{f} : X \to Y$ and $g, \tilde{g} : Y \to Z$ are 1-morphisms and $\eta : f \Rightarrow \tilde{f}$, $\zeta : g \Rightarrow \tilde{g}$ are 2-morphisms then $\mu_{X,Y,Z}(\eta, \zeta)$ is the *horizontal composition of 2-morphisms*, written $\zeta * \eta : g \circ f \Rightarrow \tilde{g} \circ \tilde{f}$, as a diagram

$$\begin{array}{ccc}
X \xrightarrow{\quad} Y \xrightarrow{\quad} Z & \rightsquigarrow & X \xrightarrow{\quad} Z. \qquad (\text{A.2})
\end{array}$$

There are also two kinds of identity: *identity 1-morphisms* $\mathrm{id}_X : X \to X$ and *identity 2-morphisms* $\mathrm{id}_f : f \Rightarrow f$.

A basic example is the 2-*category of categories* $\mathfrak{Cat}$, with objects categories $\mathcal{C}$, 1-morphisms functors $F : \mathcal{C} \to \mathcal{D}$, and 2-morphisms natural transformations $\eta : F \Rightarrow G$ for functors $F, G : \mathcal{C} \to \mathcal{D}$. Orbifolds naturally form a 2-category, as do Deligne–Mumford stacks and Artin stacks in algebraic geometry.

In a 2-category $\mathcal{C}$, there are three notions of when objects X, Y in $\mathcal{C}$ are 'the same': *equality* $X = Y$, and *isomorphism*, that is we have 1-morphisms $f : X \to Y$, $g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, and *equivalence*, that is we have 1-morphisms $f : X \to Y$, $g : Y \to X$ and 2-isomorphisms $\eta : g \circ f \Rightarrow \mathrm{id}_X$ and $\zeta : f \circ g \Rightarrow \mathrm{id}_Y$. Usually equivalence is the most useful.
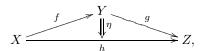
Let $\mathcal{C}$ be a 2-category. The *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of $\mathcal{C}$ is the category whose objects are objects of $\mathcal{C}$, and whose morphisms $[f] : X \to Y$ are 2-isomorphism classes $[f]$ of 1-morphisms $f : X \to Y$ in $\mathcal{C}$. Then equivalences in $\mathcal{C}$ become isomorphisms in $\mathrm{Ho}(\mathcal{C})$, 2-commutative diagrams in $\mathcal{C}$ become commutative diagrams in $\mathrm{Ho}(\mathcal{C})$, and so on.

As in Borceux [10, §7.7], there is also a second kind of 2-category, called a *weak* 2-*category* (or *bicategory*), which we will not define in detail. In a weak 2-category, compositions of 1-morphisms need only be associative up to (specified) 2-isomorphisms. That is, part of the data of a weak 2-category $\mathcal{C}$ is a 2-isomorphism $\alpha(f,g,h) : (h \circ g) \circ f \Rightarrow h \circ (g \circ f)$ for all 1-morphisms $f : W \to X$, $g : X \to Y$, $h : Y \to Z$ in $\mathcal{C}$. A strict 2-category $\mathcal{C}$ can be made into a weak 2-category by putting $\alpha(f,g,h) = \mathrm{id}_{h \circ g \circ f}$ for all f,g,h.

Some categorical constructions naturally yield weak 2-categories rather than strict 2-categories, e.g. the weak 2-categories of orbifolds defined by Pronk [51] and Lerman [39, §3.3] mentioned in §9.1. Every weak 2-category is equivalent as a weak 2-category to a strict 2-category (that is, weak 2-categories can be strictified), so we lose little by working only with strict 2-categories.

A.4   Fibre products in 2-categories

Commutative diagrams in 2-categories should in general only commute *up to (specified)* 2-*isomorphisms*, rather than strictly. Then we say the diagram 2-*commutes*. A simple example of a commutative diagram in a 2-category $\mathcal{C}$ is

$$
\begin{array}{ccc}
 & Y & \\
{}^{f}\nearrow \Downarrow{\eta} & & \searrow{}^{g} \\
X \xrightarrow{\hspace{2cm}} & & Z,
\end{array}
$$

which means that X, Y, Z are objects of $\mathcal{C}$, $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$ are 1-morphisms in $\mathcal{C}$, and $\eta : g \circ f \Rightarrow h$ is a 2-isomorphism.

The generalizations of *limit* and *colimit* to 2-categories turn out to be rather complicated. As in [10, §7] there are many different kinds — 2-limits, bilimits, pseudolimits, lax limits, and weighted limits (or indexed limits), depending on whether one considers diagrams to commute on the nose, up to 2-isomorphism, or up to 2-morphisms, and what kind of universal property one requires. Our definition of *fibre product*, following Behrend et al. [7, Def. B.13], is actually an example of a pseudolimit.

Definition A.5. Let $\mathcal{C}$ be a 2-category and $g : X \to Z$, $h : Y \to Z$ be 1-morphisms in $\mathcal{C}$. A *fibre product* $X \times_Z Y$ in $\mathcal{C}$ consists of an object W, 1-morphisms $e : W \to X$ and $f : W \to Y$ and a 2-isomorphism $\eta : g \circ e \Rightarrow h \circ f$ in $\mathcal{C}$, so that we have a 2-commutative diagram

$$
\begin{array}{ccc}
W & \xrightarrow{\hspace{2cm}} & Y \\
{\scriptstyle e}\downarrow \quad {}^{f}\quad {}_{\eta}\Uparrow & & \downarrow{\scriptstyle h} \\
X & \xrightarrow[g]{\hspace{2cm}} & Z
\end{array}
\tag{A.3}
$$

with the following universal property: suppose $e' : W' \to X$ and $f' : W' \to Y$ are 1-morphisms and $\eta' : g \circ e' \Rightarrow h \circ f'$ is a 2-isomorphism in $\mathcal{C}$. Then there should exist a 1-morphism $b : W' \to W$ and 2-isomorphisms $\zeta : e \circ b \Rightarrow e'$,

$\theta : f \circ b \Rightarrow f'$ such that the following diagram of 2-isomorphisms commutes:

$$
\begin{array}{ccc}
g \circ e \circ b & \xrightarrow{\;\;\eta * \mathrm{id}_b\;\;} & h \circ f \circ b \\
{\scriptstyle \mathrm{id}_g * \zeta} \Big\Downarrow & & \Big\Downarrow {\scriptstyle \mathrm{id}_h * \theta} \\
g \circ e' & \xrightarrow[\;\;\eta'\;\;]{} & h \circ f'.
\end{array}
$$

Furthermore, if $\tilde{b}, \tilde{\zeta}, \tilde{\theta}$ are alternative choices of b, ζ, θ then there should exist a unique 2-isomorphism $\epsilon : \tilde{b} \Rightarrow b$ with

$$\tilde{\zeta} = \zeta \odot (\mathrm{id}_e * \epsilon) \quad \text{and} \quad \tilde{\theta} = \theta \odot (\mathrm{id}_f * \epsilon).$$

We call such a fibre product diagram (A.3) a 2-*Cartesian square*.

If a fibre product $X \times_Z Y$ in $\mathcal{C}$ exists then it is unique up to equivalence in $\mathcal{C}$. If $\mathcal{C}$ is a category, that is, all 2-morphisms are identities $\mathrm{id}_f : f \Rightarrow f$, this definition of fibre products in $\mathcal{C}$ coincides with that in §A.2.

Orbifolds, and stacks in algebraic geometry, form 2-categories, and Definition A.5 is the right way to define fibre products of orbifolds or stacks, as in [7]. One can also define *pushouts* in a 2-category $\mathcal{C}$ in a dual way to Definition A.5, reversing directions of morphisms.

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Glossary of Notation

$B_k(Y)$ classical bordism group of manifold Y, 141

$B_k^{\mathrm{orb}}(\mathcal{Y})$ orbifold bordism group of orbifold $\mathcal{Y}$, 143

$B_k^{\mathrm{eff}}(\mathcal{Y})$ effective orbifold bordism group of orbifold $\mathcal{Y}$, 143

$C, \hat{C} : \mathbf{Man^c} \to \mathbf{\check{M}an^c}$ 'corner functors' for manifolds with corners, 43

$C, \hat{C} : \mathbf{dSpa^c} \to \mathbf{dSpa^c}$ 'corner functors' for d-spaces with corners, 52

$C, \hat{C} : \mathbf{Orb^c} \to \mathbf{\check{O}rb^c}$ 'corner functors' for orbifolds with corners, 117

$C, \hat{C} : \mathbf{dSta^c} \to \mathbf{dSta^c}$ 'corner functors' for d-stacks with corners, 126

$\mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \ldots$ C^∞-rings, 8

$\mathbf{C^\infty Rings}$ category of C^∞-rings, 8

$\mathbf{C^\infty Sch}$ category of C^∞-schemes, 10

$\mathbf{C^\infty Sch_{ssc}^{lf}}$ category of separated, second countable, locally fair C^∞-schemes, 18

$\mathbf{C^\infty Sta}$ 2-category of C^∞-stacks, 68

$dB_k(Y)$ d-manifold bordism group of manifold Y, 142

$dB_k^{\mathrm{orb}}(\mathcal{Y})$ d-orbifold bordism group of orbifold $\mathcal{Y}$, 144

$dB_k^{\mathrm{sef}}(\mathcal{Y})$ semieffective d-orbifold bordism group of orbifold $\mathcal{Y}$, 144

$dB_k^{\mathrm{eff}}(\mathcal{Y})$ effective d-orbifold bordism group of orbifold $\mathcal{Y}$, 144

$\partial_\pm^f X$ sets of decomposition $\partial X = \partial_+^f X \amalg \partial_-^f X$ of boundary ∂X induced by $f : X \to Y$ in $\mathbf{Man^c}$, 42

$\partial_\pm^{\boldsymbol{f}} \mathbf{X}$ sets of decomposition $\partial \mathbf{X} = \partial_+^{\boldsymbol{f}} \mathbf{X} \amalg \partial_-^{\boldsymbol{f}} \mathbf{X}$ of boundary $\partial \mathbf{X}$ induced by 1-morphism $\boldsymbol{f} : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{dSpa^c}$, 49

$\partial_\pm^f \mathcal{X}$ sets of decomposition $\partial \mathcal{X} = \partial_+^f \mathcal{X} \amalg \partial_-^f \mathcal{X}$ of boundary $\partial \mathcal{X}$ induced by 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\mathbf{Orb^c}$, 115

$\partial_\pm^{\boldsymbol{f}} \boldsymbol{\mathcal{X}}$ sets of decomposition $\partial \boldsymbol{\mathcal{X}} = \partial_+^{\boldsymbol{f}} \boldsymbol{\mathcal{X}} \amalg \partial_-^{\boldsymbol{f}} \boldsymbol{\mathcal{X}}$ of boundary $\partial \boldsymbol{\mathcal{X}}$ induced by 1-morphism $\boldsymbol{f} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Y}}$ in $\mathbf{dSta^c}$, 124

$\mathbf{dMan}$ 2-category of d-manifolds, 23

$\mathbf{d\bar{M}an}$ 2-subcategory of d-manifolds with corners equivalent to d-manifolds, 57

$\mathbf{d\hat{M}an}$ 2-subcategory of d-orbifolds equivalent to d-manifolds, 96

$\mathbf{dMan^b}$ 2-category of d-manifolds with boundary, 57

$\mathbf{dMan^c}$ 2-category of d-manifolds with corners, 57

$\mathbf{d\check{M}an^c}$ 2-category of disjoint unions of d-manifolds with corners of different dimensions, 58

$\mathbf{d\hat{M}an^c}$ 2-subcategory of d-orbifolds with corners equivalent to d-manifolds with corners, 131

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