Causal horizons in a bouncing universe

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Abstract

In this article we study the nature of the particle horizon, event horizon and the Hubble radius in a cosmological model which accommodates a cosmological bounce. The nature of the horizons and their variation with time are presented for various models of the universe. The effective role of the Hubble radius in affecting causality in such kind of models is briefly discussed.

1 Introduction

The issue of causality is at the heart of relativistic physics. In cosmology, where it is assumed that we live in an expanding universe modelled on the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, the causal nature of the universe is understood by the properties of the particle horizon and the event horizon. In Big-Bang cosmology it is assumed that there is a finite age of the universe and intuitively one can visualize that during this time light has travelled only a finite region and consequently Big-Bang cosmology predicts a calculable particle horizon for any observer in the present universe. On the other hand if there was no Big-Bang to start with, as in non-singular bouncing cosmological models [1, 2], then it becomes very difficult to say whether a particle horizon exists for any observer. In this article we will show that for non-singular bouncing cosmologies, where a contracting phase precedes the expansion phase there may exist certain conditions which dictate when a particle horizon will exist. Like the particle horizon the event horizon plays an important role in gravitational theories. In general thermodynamic behavior of a system can be linked with the event horizon as is done in black hole physics. Some authors have even tried to associate thermodynamic properties of the universe with the cosmic event horizon [3]. In the present article we will show that some of our toy examples of bouncing cosmologies do have event horizons.

Except the particle and event horizons the Hubble radius also plays an important role [4, 5] in determining the causal structure of the universe. The Hubble radius defines the Hubble sphere and the surface of the Hubble sphere is called the Hubble surface. Many authors use Hubble horizon to describe the boundary of the Hubble sphere although in this article we will not associate the word horizon with the Hubble surface. Later we will see that in bouncing cosmologies the Hubble surface affects the causal structure of spacetime in a very subtle way. A general discussion illuminating the relationship of the particle horizon and the Hubble surface, in expanding spacetime, can be found in Ref. [6]. The referred article addresses various misconceptions related to the actual role of the Hubble surface.

Although the concepts of the particle horizon, event horizon and the Hubble surface are commonly used and well discussed topics in Big-Bang cosmology [7–9] they are rarely discussed in the bouncing universe paradigm where a contracting phase of the universe changes course and starts to expand again giving rise to the expanding universe we observe. The bouncing models are interesting because they do not contain any singularities [1, 2, 10–12]. Once one includes a contracting universe, a universe which does not have any initial time to start, the concept of the particle

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horizon becomes much more involved. Some discussion on the causality issue in bouncing cosmological scenario is included in Ref. [13]. In a bouncing universe the scale-factor of the FLRW metric is not a power law function of time during the bounce and consequently the difference between Hubble radius and the particle horizon distance (if it exists) diverge maximally near the bounce point. In this article we will discuss the general nature of particle horizon, event horizon and the Hubble surface in bouncing cosmological models. The material in the article is presented in the following way. In the next section we describe the preliminary theoretical tools which we will employ throughout the article. In section 3 we quantitatively define the concepts of the horizons and the Hubble radius. In section 4 we specify two toy bounce models where all the horizons exist. In the subsequent section 5 a more realistic bounce model is discussed. We give multiple examples of this model using various parameters specifying the bounce. We discuss about the results obtained in this article in section 6 and finally conclude the paper in the subsequent section.

2 Requirements of a cosmological bounce

In this article we will use the homogeneous and isotropic FLRW spacetime. We will particularly work with the spatially flat FLRW metric and its form in the spherical polar coordinates is given by

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]\,, \tag{1}$$

where r is comoving radial coordinate. We are using units where the velocity of light $c = 1$. Here $a(t)$ is the scale-factor for the FLRW spacetime. The Einstein equation in the cosmological setting can be expressed in terms of the Hubble parameter $H \equiv \dot{a}/a$, where a dot over a quantity specifies a time derivative of that quantity. The relevant equations are

$$H^2 = \frac{\kappa}{3}\rho\,, \tag{2}$$

$$\dot{H} = -\frac{\kappa}{2}(\rho + P)\,, \tag{3}$$

where ρ is the energy density and P is the pressure of matter which pervades the universe. In the above equations $\kappa = 8\pi G$ where $G = 1/M_P^2$, $M_P = 1.2\times 10^{19}$ GeV, being the Planck mass. We are using units where $\hbar = 1$ where $\hbar$ is the reduced Planck constant. We have assumed that the cosmological constant term to be zero. The conservation of energy-momentum leads to another equation

$$\dot{\rho} = -3H(\rho + P)\,. \tag{4}$$

The above set of equations by themselves cannot give proper dynamical solutions as the number of variables, a, $\dot{a}$, ρ and P, exceeds the number of equations. In such a case one assumes a barotropic equation of state as

$$P = \omega\rho\,, \tag{5}$$

where ω is supposed to be constant for a particular kind of matter. The above equations always predicts a spacetime singularity as $t \to 0$ if the matter content of the universe satisfies some energy conditions. One can evade the singularity, by introducing a bouncing universe where the scale-factor remains finite as $t \to 0$, by violating the null energy condition (NEC) in general relativity. Although the above set of equation with the specification of a constant ω makes the system solvable and one can in principle find out the scale-factor as a function of time as a solution, in reality when we apply this theory in the case of a cosmological bounce the program may get inverted. In this inverted way one assumes a particular form of the scale-factor to start with. Then using the same above set of equations one can derive how energy scales with time and the form of the barotropic ratio. In this approach the barotropic ratio may not turn out to be a constant in time. This inverted way of solving the bouncing problem is applied in various cases because it is relatively easier to guess a bouncing scale-factor than to guess the kind of fluid which can produce a cosmological bounce.

If one wants to avoid the Big-Bang singularity near $t = 0$ then one can model a universe where the scale-factor $a(t)$ never becomes zero at $t = 0$. In these models the universe contracts during the time $-\infty < t \leq 0$ and expands during $t \geq 0$ and the cosmological bounce happens at $t = 0$ when $a(t = 0) \neq 0$ and $\dot{a}(t = 0) = 0$. The global minima of the scale-factor is at the bounce point. The universe expands after the bounce, implying $\dot{a}$ becomes positive and increases after the bounce. This second condition implies that $\ddot{a}(t = 0) > 0$. The above conditions can also be specified in terms of the Hubble parameter as:

$$H\big|_{t=0} = 0\,, \quad \dot{H}\Big|_{t=0} > 0\,. \tag{6}$$

The bounce conditions stated above and Eq. (3) immediately shows that in the flat FLRW spacetime, during bounce

$$[\rho + P]_{t=0} < 0\,. \tag{7}$$

The above condition specifies that the NEC has to be violated at the bounce point. The violation of the NEC require exotic matter [14] in the early universe. One may even change the gravitational theory to accommodate a cosmological bounce [15,16]. In this article we will not go into the details of the complexities of theories which generates a cosmological bounce, assuming satisfactory solutions of these difficult issue exists in principle. We will concentrate on the main issue of the article which is related to the causality question in bouncing cosmologies.

3 Particle Horizon, Event Horizon and Hubble Radius in bouncing cosmological models

In the context of, spatially flat, FLRW spacetime let us think of a light ray which travels from a point (t_i, R, θ, ϕ) to $(t_0, 0, \theta, \phi)$. Light travels along a null geodesic and in our particular case we have assumed a null, radial geodesic which serves our purpose. From the line-element we see that for such a null geodesic

$$ds^2 = -dt^2 + a^2(t)dr^2 = 0\,.$$

This above equation gives,

$$\int_{t_i}^{t_0} \frac{dt}{a(t)} = R\,, \tag{8}$$

which gives the comoving distance between the emitter and the observer. In the above the subscript 'i' refers to the time of emission of the light signal from the source, and the subscript '0' refers to the time of reception of the light signal by the observer.

If the time t_0 is the present cosmological time when the observer is observing the universe, the physical distance to the the emitter in the observers frame will be $R_P(t_0) = a(t_0)R$. The regions from where light could reach the observer at $t = t_0$ forms the region which may have any causal effect on the present condition of the universe and regions outside this region can have no causal effect on the present day universe. All the regions from which light has reached the present observable universe is enclosed by the causal horizon for the central observer. The causally connected region is enclosed by a 2-dimensional spacelike spherical surface whose radial physical coordinate at $t = t_0$ is called R_P, and this spacelike surface is called the particle horizon. In the bouncing universe, $t_i \to -\infty$, and we can write,

$$R_P(t_0) \equiv a(t_0) \int_{-\infty}^{t_0} \frac{dt}{a(t)}\,. \tag{9}$$

We will use the above definition of the particle horizon throughout this article.

If there exists a finite distance which light can travel in infinite time in the future, emanating from a spatial point at some time, then an event horizon can exist. The event horizon is defined by a spatial two-dimensional spherical surface, whose radius is given by the finite distance travelled by light in infinite time in the future. The formal mathematical definition (of the radius) of the event horizon is:

$$R_E(t_0) \equiv a(t_0) \int_{t_0}^{\infty} \frac{dt}{a(t)}\,, \tag{10}$$

where again t_0 is the present time of the observer.

The Hubble radius is the radial coordinate of the boundary of the Hubble sphere which is a closed two dimensional spatial surface at any cosmological time. The Hubble radius is defined as:

$$R_H(t_0) \equiv \frac{1}{|H(t_0)|}\,. \tag{11}$$

The center of the Hubble sphere is located at the observers position. The Hubble sphere does not depend upon the past history or the future of the universe. In the expression of R_H we have deliberately used the modulus of H to accommodate a contracting phase of the universe when $H < 0$.

From the definitions of the horizons one can deduce

$$\dot{R}_P = 1 + HR_P\,, \tag{12}$$

$$\dot{R}_E = -1 + HR_E\,. \tag{13}$$

In the Big-Bang paradigm it is seen that R_P always increases superluminally as HR_P is positive definite. In the Big-Bang paradigm the rate of increment of the particle horizon is more than the expansion rate of the universe. The galaxies on the particle horizon are receding with a speed HR_P where as the particle horizon is receding relatively faster and the size of the observable universe increases with time. In the bouncing scenario more interesting things can happen.

In the contracting phase of the universe $H < 0$ and Eq. (12) shows that during this time

$$0 \leq \dot{R}_P < 1\,, \quad \text{or} \quad \dot{R}_P < 0\,.$$

The particle horizon distance can increases with time when $HR_P > -1$ and the opposite can happen when $HR_P < -1$. If $HR_P = -1$ then $\dot{R}_P = 0$ and consequently bouncing cosmologies may have a minimum of the particle horizon distance. When the particle horizon distance increases with time, during the contraction phase, one must have $|H|R_P < 1$ or

$$0 \leq \dot{R}_P < 1\,, \quad \text{if } R_P < R_H\,. \tag{14}$$

Similarly, when the particle horizon distance decreases with time, during the contraction phase, one must have

$$\dot{R}_P < 0\,, \quad \text{if } R_P > R_H\,. \tag{15}$$

The above equations show that during the contraction phase of the universe the particle horizon can increase subluminally or may decrease at any rate. On the other hand

$$\dot{R}_P = 0\,, \quad \text{if } R_P = R_H\,. \tag{16}$$

If the above condition can be maintained for a brief period then during that period the particle horizon size may remain constant. If the above condition turns out to be true at only one instant then the particle horizon will have an extremum at that instant. From the above discussion one can predict some properties related to the particle horizon and the Hubble radius. The points are as follows:

1. If the particle horizon distance tends to a constant, non-singular, value as $t_0 \to -\infty$ then the Hubble radius must be equal to the particle horizon distance as $t_0 \to -\infty$.

2. If $R_P(t_0) \to \infty$ as $t_0 \to -\infty$ then in the initial phase, or the full phase, of contraction one must have $R_P > R_H$.

One can easily prove the above statements. If R_P tends to a constant as t_0 tends to large negative time then $\dot{R}_P \sim 0$ during that very early period and consequently $R_P \sim R_H$ as $t_0 \to -\infty$. The second statement can be proved by noting the fact that if R_P is maximum as $t_0 \to -\infty$ then for later times the particle horizon distance can only decrease. When the particle horizon distance decreases with time one must have $R_P > R_H$.

The way the event horizon grows with time is given in Eq. (13). In the contracting phase, where HR_E is negative definite, it can be easily seen that the event horizon will steeply decrease with time. All the above points will become apparent in various bouncing models in the later part of this article.

4 The two horizons and the Hubble radius in symmetrical toy bounce models

In this section we present some simplified examples where the functional forms of the scale-factor remains same during $-\infty < t < \infty$. The bouncing models presented in this section can be taken as toy examples as in reality the functional form of the scale-factor of evolution changes its form at different times when the matter content of the universe undergoes a qualitative transformation, like the transformation of radiation dominated universe to a universe filled with dust matter. In the present case we will simply ignore the various phases of the universe during its evolution and consider only a bouncing phase which extends in both the temporal directions. In later sections we will present the more realistic cases. The present discussion will show how a contracting phase affects the concept of the horizons discussed in the previous section.

We present two cases, where the scale-factors describing a bouncing universe, are of the following form:

$$a(t) = \begin{cases} a_0 e^{At^2} \\ a_0 \cosh(Bt) \end{cases} \tag{17}$$

where a_0, A and B are positive real valued constants[1]. The functional forms of the scale-factors show that both the bounces are symmetrical in time. It is to be noted that there can be various other forms of scale-factors which can give rise to contraction and bounce. In the later sections of this article we will show some examples where the evolution of the universe is not symmetrical with respect to the bounce point.

The particle horizon for the first case is given by

$$R_P(t_0) = e^{At_0^2} \int_{-\infty}^{t_0} e^{-At^2} dt \,, \tag{18}$$

assuming the scale-factor remained the same till $t \to -\infty$. The integral on the right hand side can be evaluated by using the properties of the Gaussian integrals. After doing the integral one obtains

$$\int_{-\infty}^{t_0} e^{-At^2} dt = \sqrt{\frac{\pi}{A}} \left[1 - \frac{1}{2}\, \mathrm{erfc}(\sqrt{A}\, t_0)\right] , \tag{19}$$

where $\mathrm{erfc}(\sqrt{A}\, t_0)$ is the complimentary error function. To see how the above integral behaves in the extreme cases where $t \to \pm\infty$ one requires the following properties of the complementary error function,

$$\lim_{x \to -\infty} \mathrm{erfc}(x) = 2 \,, \quad \lim_{x \to \infty} \mathrm{erfc}(x) = 0 \,.$$

It must be noted that the above limits saturates near $x = 0$, this information will help us to figure out the behavior of the particle horizon at the extreme limits. With all these information we can now write the expression for particle horizon for the first case as:

$$R_P(t_0) = e^{At_0^2} \sqrt{\frac{\pi}{A}} \left[1 - \frac{1}{2}\, \mathrm{erfc}(\sqrt{A}\, t_0)\right] . \tag{20}$$

[1] Here it must be noted that if the spatial sections of the FLRW universe is assumed to be closed (spherical slicing) then the second scale-factor actually describes a bouncing de Sitter universe where the Hubble parameter is a constant, $H = B$. In the present article we are working with flat FLRW metric and consequently the second scale-factor does not correspond to a de Sitter solution and the Hubble parameter is not a constant.

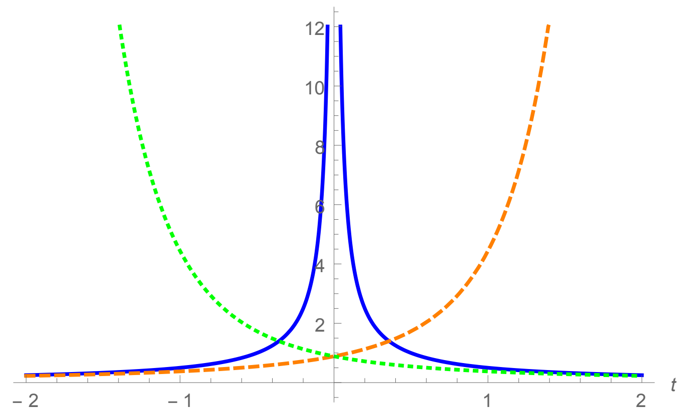

Figure 1: Plot of Hubble radius represented by continuous blue curve, particle horizon radius in orange dashed curve and event horizon radius in green dotted curve for $a = e^{t^2}$ bounce model. Here $a_0 = 1$ sets the scale-factor at the bounce point and A is assumed to have unit magnitude. For more explanation see the text below.

The limits and their properties of the complimentary error function listed above shows that the value of the particle horizon remains finite at both the extremities of the time variable.

The above expression shows that the particle horizon quickly vanishes as one goes back in negative time and the particle horizon increases indefinitely as time evolves, $t \to \infty$. At the bounce moment, $R_P(0) = \frac{1}{2}\sqrt{\frac{\pi}{A}}$. The plot of the particle horizon, represented in orange dashed curve, is shown in Fig. 1. The above result can be interpreted in a simple way. As the observer moves back in time, $t \to -\infty$, the observer does not receive any light from the other parts of the universe from the past as light emitting particles are infinitely distant from the observer. After a long time the observer first receives light from its past, and a causal connection is made. At this moment, t_0, the particle horizon $R_P(t_0)$ comes into account. The amount of light which the observer receives is coming from the closest sources in the past. As time proceeds more and more regions from the past are coming in causal contact with the observer and the process continues forever.

In this case we have $|H| = 2A|t|$ and consequently one can easily see that $R_P(t_0)$ is never proportional to $1/|H(t_0)|$. In general if the universe admits of both an expanding phase and a contracting phase it is better to work with $|H|$ instead of H as the Hubble parameter remains negative during the cosmological contraction. In the present case we see that near $t \to -\infty$ both $R_P(t_0)$ and $1/|H(t_0)|$ vanishes. At $t = 0$, $R_P(0)$ remains finite whereas $1/|H(t_0)| \to \infty$. The nature of the Hubble surface, plotted in the blue, is shown in Fig. 1. The interesting thing to note is the relative behavior of the Hubble surface and particle horizon before the bounce. In this toy model the Hubble radius is always greater than the particle horizon radius before the bounce and consequently the particle horizon size increases with time. Here $R_H > R_P$ and the two distance scales differ maximally near the bounce point $t = 0$. Although near the bouncing point the universe is moving towards a momentarily static universe, where dynamical effects of the expansion can be neglected, $R_H \to \infty$, practically the whole region inside the 2-surface with radius R_H is causally disconnected as R_P is finite at $t = 0$.

It is interesting to note that this toy example of a bouncing universe do admit an event horizon. In this case we have

$$\begin{aligned} R_E(t_0) &= e^{At_0^2}\int_{t_0}^{\infty} e^{-At^2}dt \\ &= \frac{1}{2}\sqrt{\frac{\pi}{A}}\, e^{At_0^2}\, \mathrm{erfc}(\sqrt{A}\, t_0)\,. \end{aligned} \tag{21}$$

The above result shows the existence of finite event horizon distance for a bouncing cosmological model. The behavior of the event horizon is plotted in green in Fig. 1. The plot shows that the event horizon diverges at $t_0 \to -\infty$, but in general it has a finite value for all other times and decreases smoothly as time evolves. As predicted in the last

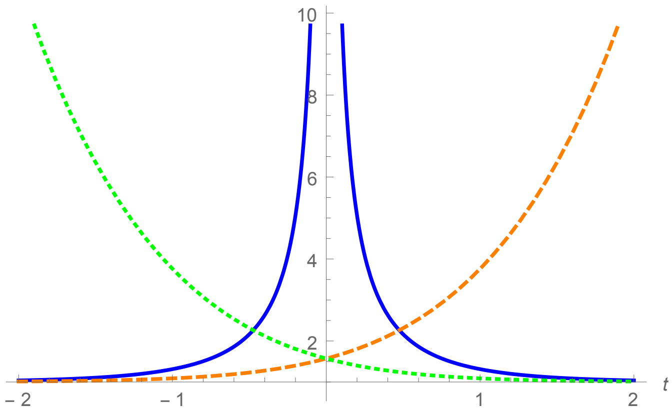

Figure 2: Plot of Hubble radius represented by continuous blue curve, particle horizon radius in orange dashed curve and event horizon radius in green dotted curve for $a = \cosh t$ bounce model. Here $a_0 = 1$ sets the scale-factor at the bounce point and B is assumed to have unit magnitude. For more explanation see the text below.

section, the event horizon steeply decreases during the contracting phase. In this toy model the event horizon and particle horizon has the same radius at the bounce point $t = 0$.

In the second case we see that although the scale-factor is very different when compared to the first case, still the horizon properties remain qualitatively the same. In this case the particle horizon is given by:

$$R_P(t_0) = \frac{2}{B}\cosh(Bt_0)\tan^{-1}(e^{Bt_0})\,. \tag{22}$$

It is seen that the particle horizon asymptotically goes to $1/B$ as $t_0 \to -\infty$, and $R_P(0) = \pi/2B$ which is finite. The particle horizon diverges as time increases in the positive direction as expected. The nature of the particle horizon radius is shown by the orange dashed curve in Fig. 2. The interpretation of this kind of behavior of $R_P(t_0)$ remains the same as before, except the fact that the particle horizon is non-zero in the far past. In the present case $|H(t)| = B|\tanh(Bt)|$ and as a result $1/|H(t)|$ tends to $1/B$ when $t \to \pm\infty$ and it diverges as $t \to 0$. The Hubble radius is shown by the continuous blue line in Fig. 2. In this case also we see that $R_P > R_H$ and the particle horizon size increases with time. The two length scales differ maximally near the bounce point as in the last example. In both the cases we see that at very early times the particle horizon distance and the Hubble radius becomes equal in magnitude. This happens as the particle horizon distance attains a constant value during the very early phase of contraction.

Like the previous example, the present bouncing model also admits of finite event horizon (except at $t \to -\infty$). In the present case,

$$R_E(t_0) = \frac{2}{B}\cosh(Bt_0)\tan^{-1}(e^{-Bt_0})\,. \tag{23}$$

The nature of the event horizon radius is shown by the dotted green curve in Fig. 2. The nature of the event horizons from the two examples are consistent. Both the plots show that the event horizon grows indefinitely as one moves back in time and steeply decreases with time during contraction.

The nature of the horizons as plotted in Fig. 1 and Fig. 2 show an interesting feature. As the bounces are symmetrical in time the particle horizon and the event horizon are symmetrical in time. The particle horizon transforms into the event horizon if one changes the direction of time and vice versa. In both the cases discussed above we notice that the particle horizon radius is an increasing function of time. Once some part of the universe gets causally connected that part always remains so and more regions of the universe gets inside the particle horizon as time increases. In the more realistic bounce model discussed in the next section we will see that this feature of the particle horizon is lost.

5 A more realistic calculation

In this section we assume at least three transformations in the bouncing universe. The first transformation occurs during the contracting phase of the universe. We assume that the contracting universe was dominated by some form of matter which gave rise to a scale-factor whose functional form was given by a power law. A power law scale-factor cannot lead to a cosmological bounce and consequently we assume that at some time $t' < 0$ the nature of the matter content/geometry of the universe changed. During $t' < t < t''$ where $t'' > 0$, the scale-factor of the universe changed from the power law form and the new scale-factor accommodates a bounce at $t = 0$. Ultimately the universe comes out of the bouncing phase at t'' and the scale-factor again transforms to a power law function. This is a simple but realistic description of a cosmological bounce as we know that the scale-factor of the bouncing universe must have changed to a power law form after the bounce[2]. From our understanding of the expanding phase of the universe and symmetry arguments it is natural to think that some power law contraction phase may precede the bouncing phase of the universe. The geometry of the universe changes at t' and t'' and in our simplistic model the change happens instantaneously. In flat FLRW spacetime one can always associate a series of flat spatial hypersurfaces at each instant of cosmic time t. When the nature of the scale-factor changes at some time t_0 it implies that there is a flat 3-dimensional spatial hypersurface which separates spacetime into two regions. For $t < t_0$ the scale-factor on the spatially flat hypersurfaces were given by the old scale-factor, $a_{\rm old}(t)$, and for spatial slices corresponding to $t > 0$ the scale-factor takes the new value, $a_{\rm new}(t)$. The surface at t_0 acts like a junction between two temporal regions with different scale-factors and in general relativity two metrics with different time dependence can only be matched over a hypersurface if the junction conditions are satisfied[3]. For the FLRW metric the junction conditions state that at the junction, which corresponds to a 3-dim spatial hypersurface at time t_0 in the present case,

$$a_{\rm old}(t_0) = a_{\rm new}(t_0)\,, \quad \dot{a}_{\rm old}(t_0) = \dot{a}_{\rm new}(t_0)\,.$$

When ever the geometry of the universe changes at some time we will assume that the above junction conditions are satisfied.

We will assume that our model universe has the three following scale-factors in the three different phases as:

$$a(t) = \begin{cases} c_0(-t)^m\,, & t_i \le t < t'\,, \\ a_0 + b_0 t^{2p}\,, & t' < t < t''\,, \\ d_0 t^n\,, & t'' < t \le \infty\,. \end{cases} \tag{24}$$

In the above equation $t_i(< 0)$ is assumed to be some initial time which will ultimately tend to $-\infty$. The quantities m and n are positive real constants and m is in general not equal to n. The constant p takes positive integer values. Out of the three constants we will assume that $0 < n < 1$ as because in an expanding flat FLRW model this constraint is generally followed. This condition on n predicts that the bouncing universe scenario we are discussing at present does not admit an event horizon as it is well known that the flat FLRW metric does not have any event horizon if the scale-factor has a power law form where n satisfies the above constraint. The coefficients c_0, a_0, b_0 and d_0 are also positive real constants out of which a_0 normalizes the scale-factor at the bounce time. The Hubble radius during the three phases are:

$$R_H(t_0) \equiv \frac{1}{|H(t_0)|} = \begin{cases} \frac{|t_0|}{m}\,, & t_i \le t_0 < t'\,, \\ \left|\frac{a_0 + b_0 t_0^{2p}}{2b_0 p t_0^{2p-1}}\right|\,, & t' < t_0 < t'' \\ \frac{t_0}{n}\,, & t'' < t_0 \le \infty \end{cases} \tag{25}$$

Because of the junction conditions, on the metric, at t' and t'' one can easily verify that $R_H(t_0)$ changes continuously at the junctions. The particle horizon at any time during expansion can be written as:

$$R_P(t_0) = d_0 t_0^n \left[\int_{t_i}^{t'} \frac{dt}{c_0(-t)^m} + \int_{t'}^{t''} \frac{dt}{a_0 + b_0 t^{2p}} + \int_{t''}^{t_0} \frac{dt}{d_0 t^n}\right]\,. \quad (t'' < t_0 \le \infty) \tag{26}$$

[2]In this article we do not consider the case where the bouncing universe culminates in an expanding inflationary universe.

[3]A nice discussion on the junction conditions in general relativity can be found in Ref. [17].

Unlike the case with the toy models of bounce in the present case we have to specify the particle horizons in all the three phases of development of the universe as the scale-factors change during these phases. The particle horizon radius during the power law contraction phase is given by

$$R_P(t_0) = \frac{1}{1-m}[(-t_0)^m(-t_i)^{1-m} + t_0]\,. \quad (t_i \le t_0 < t') \tag{27}$$

Similarly, the particle horizon radius during the bouncing phase is given by

$$R_P(t_0) = (a_0 + b_0 t_0^{2p})\left[\frac{1}{c_0(1-m)}\left((-t_i)^{1-m} - (-t')^{1-m}\right) + \int_{t'}^{t_0} \frac{dt}{a_0 + b_0 t^{2p}}\right]. \quad (t' < t_0 < t'') \tag{28}$$

The expression of the particle horizon distance during the expansion phase is

$$\begin{aligned} R_P(t_0) &= d_0 t_0^n \left[\frac{1}{c_0(1-m)}\left((-t_i)^{1-m} - (-t')^{1-m}\right) + \int_{t'}^{t''} \frac{dt}{a_0 + b_0 t^{2p}}\right. \\ &+ \left.\frac{1}{d_0(1-n)}\left(t_0^{1-n} - t''^{1-n}\right)\right]. \quad (t'' < t_0 \le \infty) \end{aligned} \tag{29}$$

We will specify the the integral containing the term $1/(a_0 + b_0 t^{2p})$ later, at present we concentrate on the junction conditions. Applying the junction conditions at t' we get

$$a_0 = \left(\frac{2p}{m} - 1\right) b_0 t'^{2p}\,, \qquad c_0(-t')^m = \frac{a_0}{1-(m/2p)}\,. \tag{30}$$

Similarly applying the junction conditions at t'' one gets,

$$a_0 = \left(\frac{2p}{n} - 1\right) b_0 t''^{2p}\,, \qquad d_0 t''^n = \frac{a_0}{1-(n/2p)}\,. \tag{31}$$

Comparing the above conditions one easily gets

$$\left(\frac{t''}{t'}\right)^{2p} = \frac{\frac{2p}{m} - 1}{\frac{2p}{n} - 1}\,, \tag{32}$$

which sets a relationship between the matching times and the parameters appearing in the scale-factors. Henceforth whenever we specify the expression of the particle horizon and the Hubble radius we will assume that the constants appearing in those expressions satisfy the above junction conditions.

The expressions of the particle horizon distances, as given in Eqs. (27), (28) and (29), shows that $R_P(t_0)$ for all the phases is finite (as $t_i \to -\infty$) only when $m > 1$. Consequently, the existence of the particle horizon in the realistic case depends upon the value of m.

5.1 Nature of particle horizon

In this subsection we will consider $m > 1$ and assume $t_i \to -\infty$. In this article we will present the results for two values of p. In the first case $p = 1$ and in the second case $p = 2$ both of which gives rise to a symmetric bouncing phase.

5.1.1 The case where $p = 1$

When $p = 1$ one can write the expressions for particle horizon distance as:

$$R_P(t_0) = \begin{cases} \frac{t_0}{1-m}\,, \quad (-\infty \le t_0 < t') \\ (a_0 + b_0 t_0^2)\left[\frac{1}{c_0(m-1)}(-t')^{1-m} - \frac{1}{\sqrt{a_0 b_0}}\arctan(\sqrt{\frac{b_0}{a_0}}t') + \frac{1}{\sqrt{a_0 b_0}}\arctan(\sqrt{\frac{b_0}{a_0}}t_0)\right], \quad (t' < t_0 < t'') \\ d_0 t_0^n\left[\frac{1}{c_0(m-1)}(-t')^{1-m} - \frac{1}{\sqrt{a_0 b_0}}\arctan\left(\sqrt{\frac{b_0}{a_0}}t'\right) + \frac{1}{\sqrt{a_0 b_0}}\arctan\left(\sqrt{\frac{b_0}{a_0}}t''\right)\right. \\ \left. + \frac{1}{d_0(1-n)}\left(t_0^{1-n} - t''^{1-n}\right)\right]. \quad (t'' < t_0 \le \infty) \end{cases} \tag{33}$$

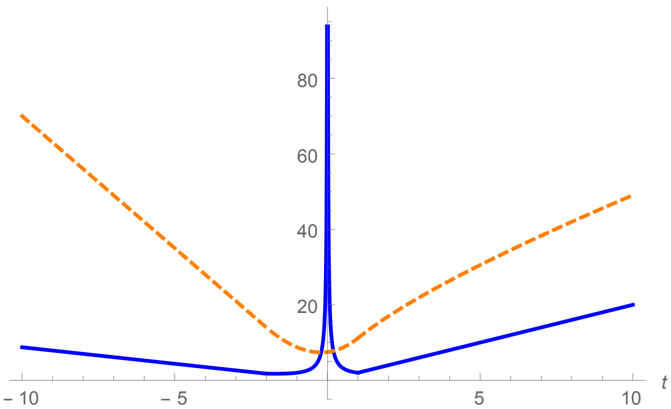

Figure 3: Variation of particle horizon radius in orange dashed line and Hubble radius in blue through bounce for p=1.

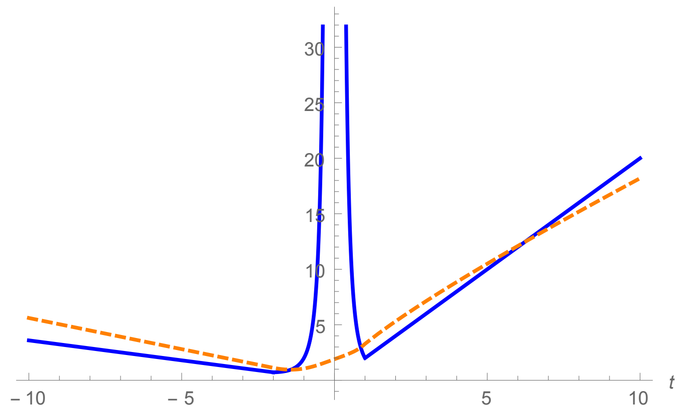

Figure 4: Variation of particle horizon radius in orange dashed line and Hubble radius in blue through bounce for p=2.

Finally we have to choose the constants appearing in the above expressions judiciously such that the junction conditions are satisfied. We take $a_0 = 1$ and $n = 1/2$ assuming a radiation dominated universe just after the bouncing phase. The time instants where the scale-factors change are assumed to be $t' = -2$ and $t'' = 1$ in some units. From the junction conditions one can now easily obtain

$$m = \frac{8}{7}\,,\ b_0 = \frac{1}{3}\,,\ c_0 = \frac{7}{3}2^{-\frac{8}{7}}\,,\ d_0 = \frac{4}{3}\,.$$

Using these values we can write the particle horizon radius at any phase of evolution of the universe. Particle horizon distance at any time during power law contraction is simply given by

$$R_P(t_0) = -7t_0\,. \tag{34}$$

In this phase $t_0 < 0$. Particle horizon distance in the intermediate bouncing phase comes out to be:

$$R_P(t_0) = \left(1 + \frac{t_0^2}{3}\right)\left[6 + \sqrt{3}\Big(\arctan\left(\frac{t_0}{\sqrt{3}}\right) + \arctan\left(\frac{2}{\sqrt{3}}\right)\Big)\right]\,. \tag{35}$$

Finally the particle horizon radius during the expanding phase of the universe is:

$$R_P(t_0) = \frac{4}{3}t_0^{\frac{1}{2}}\left[6 + \sqrt{3}\Big(\arctan\left(\frac{1}{\sqrt{3}}\right) + \arctan\left(\frac{2}{\sqrt{3}}\right)\Big) + \frac{3}{2}(t_0^{1/2} - 1)\right]\,. \tag{36}$$

The particle horizon and the Hubble radius are plotted for the case $p = 1$ in Fig. 3. In the plot the Hubble radius is drawn in blue and the particle horizon is shown by orange dashed curve. The plot shows that most of the time the Hubble sphere is causally connected except very near to the bounce point where the Hubble radius diverge. We have plotted the behavior of the particle horizon distance and the Hubble radius near the bounce point and consequently the plots do not convey the complete information about these distance scales away from the bounce point. As one goes back in time the Hubble radius increases and so does the particle horizon distance. The important feature which comes out of the plot is that the particle horizon decreases initially and then it attains a minimum value near the bounce and increases in the expanding phase.

5.1.2 The case where $p = 2$

In the present case the integral involving the term $1/(a_0 + b_0 t^{2p})$ yields [18]

$$\int \frac{dt}{a_0 + b_0 t^4} = \frac{\alpha}{4\sqrt{2}a_0}\left[\ln\left(\frac{t^2 + \sqrt{2}\alpha t + \alpha^2}{t^2 - \sqrt{2}\alpha t + \alpha^2}\right) + 2\arctan\frac{\sqrt{2}\alpha t}{\alpha^2 - t^2}\right]\,,$$

where $\alpha = (\frac{a_0}{b_0})^{\frac{1}{4}}$. Using the above result we can write the particle horizon distance in the three cases as:

$$R_P(t_0) = \begin{cases} \frac{t_0}{1-m}\,, \qquad (-\infty \leq t < t') \\ (a_0 + b_0 t_0^4)\left[\frac{1}{c_0(m-1)}(-t')^{1-m} + \frac{\alpha}{4\sqrt{2}a_0}\left(\ln\left[\frac{t'^2-\sqrt{2}\alpha t'+\alpha^2}{t'^2+\sqrt{2}\alpha t'+\alpha^2} \times \frac{t_0^2+\sqrt{2}\alpha t+\alpha^2}{t_0^2-\sqrt{2}\alpha t+\alpha^2}\right]\right.\right. \\ \left.\left. +2\arctan\frac{\sqrt{2}\alpha t_0}{\alpha^2-t_0^2} - 2\arctan\frac{\sqrt{2}\alpha t'}{\alpha^2-t'^2}\right)\right]\,, \qquad (t' < t_0 < t'') \\ d_0 t_0^n\left[\frac{1}{c_0(m-1)}(-t')^{1-m} + \frac{\alpha}{4\sqrt{2}a_0}\left(\ln\left[\frac{t'^2-\sqrt{2}\alpha t'+\alpha^2}{t'^2+\sqrt{2}\alpha t'+\alpha^2} \times \frac{t''^2+\sqrt{2}\alpha t''+\alpha^2}{t''^2-\sqrt{2}\alpha t''+\alpha^2}\right]\right.\right. \\ \left.\left. +2\arctan\frac{\sqrt{2}\alpha t''}{\alpha^2-t''^2} - 2\arctan\frac{\sqrt{2}\alpha t'}{\alpha^2-t'^2}\right) + \frac{1}{d_0(1-n)}(t_0^{1-n} - t''^{1-n})\right]\,. \qquad (t'' < t_0 \leq \infty) \end{cases} \tag{37}$$

In the present we set $a_0 = 1$ and $n = 1/2$ as done in the previous case. The time instants where the scale-factors change are assumed to be the same as in the previous case. The junction conditions now predict

$$m = \frac{64}{23}\,,\ b_0 = \frac{1}{7}\,,\ c_0 = \frac{23}{7}2^{-\frac{64}{23}}\,,\ d_0 = \frac{8}{7}\,.$$

The resulting particle horizon distance is plotted in orange dashed curve in Fig. 4. The Hubble radius at each instant is plotted in blue curve. The present plot is qualitatively same as the one in Fig. 3. The only difference between them is that for the case $p = 2$ the radiation dominated universe can have a certain region where the Hubble radius exceeds the particle horizon distance. Both the curves show that the particle horizon follows a smooth curve which has a minima near the bounce point.

In both the above cases we observe that during the contraction phase the Hubble surface lies within the particle horizon. From our discussion in section 3 one can infer that in such cases the particle horizon size must decrease with time. This behavior of the particle horizon is in contrast to the corresponding behavior of the particle horizons in the toy model examples.

6 Discussion

We discuss the important points regarding horizons in bouncing cosmologies in this section. At first we will like to point out the the role of various energy conditions in deciding the fate of the existence of the horizons. We start with the toy examples. In the first case in the toy examples of bounce one can easily show that

$$\rho = \frac{12A^2}{\kappa}t^2 \geq 0\,,\quad \rho + P = -\frac{4A}{\kappa} < 0\,,$$

when $A > 0$. In the second case we have

$$\rho = \frac{3B^2}{\kappa}\tanh^2(Bt) \geq 0\,,\quad \rho + P = -\frac{2B^2}{\kappa}\,\mathrm{sech}^2(Bt) < 0\,.$$

Both the above examples show that in these cases the strong energy condition (SEC), the weak energy condition (WEC) and the NEC are violated during $-\infty < t < \infty$. In these cases all the horizons exist.

In the realistic case of cosmological bounce presented in the paper we see that if the contracting phase prior to the bouncing phase has a power law scale-factor then the particle horizon exists only when $m > 1$. As in general relativity the exponent in the power law is related with the barotropic equation of state ω via

$$\omega = \frac{2-3m}{3m}\,,$$

it is seen that $\omega \geq 0$ only when $m \leq 2/3$, where the equation of state becomes zero at $m = 2/3$. If $m > 2/3$ the barotropic ratio becomes negative. As a consequence it follows that if there is a power law contraction phase before the bouncing phase then the condition $\omega < 0$ ensures the existence of a finite particle horizon. The condition that $m > 1$ translates to

$$-1 < \omega < -\frac{1}{3}\,, \tag{38}$$

which specifies that in such a case $\rho + 3P < 0$ during the contraction phase. This result shows that the particle horizon in such cases can only exist if the SEC is violated during the power law contraction phase. On the other hand $\rho + P > 0$ and the WEC will hold during the contraction phase. Near the bounce point all the above energy conditions will be violated.

In the realistic bounce model we have used various forms of the scale-factors in the various evolutionary phases of the universe. The different metric smoothly transforms from one form to the other because of the junction conditions. The junction conditions and our choice of the bouncing scale-factor combines to produce an interesting effect. It is apparent from Eq. (32) that if one chooses $t'' = 1$, $t' = -1$ and $n = 1/2$ then m also turns out to be 1/2 when p is an integer. The junction condition and symmetric matching times combine to predict a symmetrical evolution of the universe through bounce. If we want to have an asymmetrical evolution of the universe the matching times should be different which will lead to dissimilar values of m and n or one may choose to have $m \neq n$ which will give asymmetric matching times.

In the toy examples we saw the minima of the particle horizons appear near $t \to -\infty$. The particle horizons then grows monotonically. The particle horizon radius displays such a behavior because in both the toy examples the scale-factors at $t \to -\infty$ diverge too fast as one moves back in time and consequently considerable amount of light cannot reach any region of the universe in the first case. In the second case only a small region remains causally connected as $t \to -\infty$. In these cases, the surface defining the particle horizon, at a particular time, have a subluminal velocity of contraction. As a consequence, centrally directed, in-coming, light from outside the particle horizon can overtake this surface and reach the central observer. In this case the particle horizon increases as more light from distant parts of the universe reach the central observer as time increases. In the realistic cases, where particle horizon exists, we observe that the minima of the particle horizon distance is always near the bounce point. In these cases the scale-factor during the contraction phase is given by a power law function of time which diverges as $t \to -\infty$ but this divergence is much milder than the divergences of the scale-factors in the toy models. As one moves back in time a much wider part of the universe seems to be causally connected in the realistic cases. But as in these cases the particle horizon distance exceeds the Hubble radius the surface defining the particle horizon, at a particular time, is radially moving inward in a superluminal way. Photons which are moving radially inward from outside this surface will not be able to reach this surface. As the surface contracts with time the particle horizon distance diminishes with time. It is to be noted that a contracting particle horizon does not imply that causal regions of the universe are moving out of the horizon, it implies that as the universe contracts no new causal regions are entering the particle horizon.

The minima of the particle horizon appears near the bounce point in the more realistic models. This happens when $R_P = R_H$. In the examples given in this paper the condition $R_P = R_H$ happens at an instant and consequently the particle horizon attains a minima at that instant. If the two surfaces, describing the particle horizon and the Hubble surface, remain identical for some period of time then during that period the particle horizon distance will remain constant. This will happen because light cannot enter radially inward into the particle horizon as the surface defining the particle horizon, at a particular time, contracts with the speed of light. More over as the spatial surface defining the particle horizon, at a particular time, is moving inwards with just the velocity of light all the emitters inside this surface move inwards subluminally and the light they emit can reach the observer in due time. In this case neither the particle horizon distance increases nor it decreases with time.

It must be noted that the surface, with definite physical radius, which define the particle horizon or the Hubble surface at a particular time, t, may not define the particle horizon or the Hubble surface at a later time $t' > t$. Another surface, with a different physical radius, will define the particle horizon at t'. As an example, the surface which coincides with the particle horizon at time t, in the case where $R_P < R_H$, contracts with time as its physical radial coordinate shrinks with time. On the other hand the particle horizon distance increases with time. The surface which defines the Hubble sphere at any particular time, t, will contract in time where as the Hubble radius may increase with time.

Although in the Big-Bang paradigm the particle horizon distance and the Hubble radius are proportional to each other when the scale-factor of expansion is given by a power law function in bouncing models this fact does not hold anymore. In the bouncing models the power law expansion phase may accommodate a particle horizon and the relationship between the particle horizon distance and the Hubble radius is much more complex. Although the present authors have not seen any article addressing the issue related to the particle horizons in bouncing models, a recent publication [19] discuss the effect of bouncing models on luminosity distance in cosmology.

If a bouncing universe does not have a finite particle horizon then the only compact surface which determines the causal structure of the universe is the Hubble surface. Although the Hubble radius diverges during the bounce time, it is the only compact 2-dimensional surface which can have any say on the causal structure of the universe during the contracting and expanding phases.

The above points encapsulate the important conclusions drawn from the present work. The above points specify the complex nature of the causality problem encountered in cosmological models which accommodate a cosmological bounce. It is evident from the discussion that there can be various conditions dictating the presence of the particle horizon and the event horizon in bouncing cosmologies. In some cases both may cease to exist. If the contraction phase is dominated by dust and the expansion phase dominated by radiation non of the horizons exist. In some cases only the event horizon may exist. If the contracting phase is dust dominated and the future expanding universe is de Sitter then we expect that event horizon will exist even though the particle horizon does not exist. In some, artificial cases, both may exist as in the case of the toy examples presented in the article.

7 Conclusion

The present article addresses the topic related to causality in general bouncing cosmological models based on the general relativistic flat FLRW solution. Keeping the standard definitions of the particle horizon, event horizon and Hubble radius the present article generalizes their meaning in a bouncing universe which accommodates an infinitely stretched (in time) contraction phase. It is shown that in many bouncing cases the particle horizon may not exist. On the other hand if matter content of the universe, during the contraction phase, violates some energy energy conditions then the particle horizon can exist in some simple models. The realistic models studied in the paper are simple as they involve a single phase change, when the scale-factor changes, during the contraction period. One can model more complex contraction phases in the future but the existence of the particle horizon will mainly depend upon the earliest phase one considers during contraction.

Two toy model bounce scenario is presented at first which illuminates the generalization of the concepts of the horizons in the bouncing scenario. The toy models have a single scale-factor during contraction, bounce and expansion phases. In the examples we show the toy models have both the horizons.

A more realistic model of cosmological bounce, which accommodates three phases of evolution of the universe, is presented in the article. The realistic case exposes the difficulty in calculating the particle horizon distance as the properties of particle horizon becomes dependent on the earliest history of the bouncing models. In the more realistic calculation, presented in the article, the contraction and the expansion phases are guided by a power law scale-factor where as the bouncing scale-factor is assumed to be an even function of time. It is shown that in the more realistic models of bounce there can be some cases where the particle horizon may exist. The criterion for existence of particle horizons depends upon the energy conditions followed by matter during contraction. We have emphasized the special role of the Hubble surface which affects the time evolution of the particle horizon.

The present work shows that the causality problem in bouncing universe is intrinsically related to an understanding of the various phases of the universe during the contraction phase. As our understanding of the contraction phase is purely speculative at present the models we use to figure out the nature of particle horizon remains over simplistic. The present authors believe that although the causality problem in bouncing universe models are far from being solved the present article shows the qualitative and quantitative difficulties one must have to circumvent in the future to produce more meaningful results.

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