ASSIGNMENT I: THE AIYGARI MODEL

Vision: This project teaches you to solve for the *stationary equilibrium* in a neoclassical-style heterogeneous agent model and analyze the economic results.

- **Problem:** The problem consists of
    1. A number of questions (page 2)
    2. A model (page 3 onward, incl. solution tricks)
- **Code:** The problem is designed to be solved with the *GEModelTools* package.
- **Structure:** Your project should consist of
    1. A single self-contained pdf-file with all results
    2. A single Jupyter notebook showing how the results are produced
    3. Well-documented *.py* files
- **Hand-in:** Upload a single zip-file on Absalon (and nothing else)
- **Deadline:** 8th of October 2024
- **Exam:** Your Aiygari-project will be a part of your exam portfolio.
  You can incorporate feedback before handing in the final version.

Questions

1. **Define the stationary equilibrium for the model on the next page**

2. **Solve for the stationary equilibrium with positive tax rates**

   Calibrate the model using the parameter values on the next page. Guess on $(\beta, \mathcal{T}_{ss}, K_{ss}, \varphi)$ to match the following 4 targets in steady state:

   - Asset market clearing
   - The government budget constraint holds
   - A steady state capital-output ratio K_{ss}/Y_{ss} of 3
   - Effective steady state labor supply L_t^{hh} is 1

   Show aggregate quantities and prices

   Show a measure of wealth inequality, and discuss briefly what determines wealth inequality in the model.

   Note: *You might need to change play around with the initial values for the numerical solver when calibrating the steady state*

3. **Illustrate how changes in the tax rates affect households in partial equilibrium**

   Show and discuss how the two tax rates, τ^a, τ^ℓ affect consumption, savings and labor supply of households in *partial* equilibrium (i.e. keeping w and r fixed).

4. **Implement a tax reform in general equilibrium**

   Consider a reform that raises the capital income tax from $\tau^a = 0.1$ to $\tau^a = 0.5$. Assume that the reform is budget neutral such that the overall level of taxes and transfers $\mathcal{T}$ is unchanged. This can be implemented by changing the labor income tax τ^ℓ. Compute the competitive steady state general equilibrium after the reform. Does aggregate »efficiency« (measured in terms of aggregate output Y) increase or decrease? What happens to expected welfare $v_t(z_t, a_{t-1})$ across the income and wealth distribution?

5. **Illustrate welfare effects of transitional dynamics**

   Compute now the transition from the orginal steady state with $\tau^a = 0.1$ to the new steady state with $\tau^a = 0.5$. What happens to expected household welfare across the income and wealth distribution once you take into account the transition?

1. Model

Households. The model has a continuum of infinitely lived households indexed by $i \in [0,1]$. Households are *ex post* heterogeneous in terms of their time-varying stochastic productivity, z_t, and their (end-of-period) savings, a_t. The distribution of households over idiosyncratic states is denoted $\underline{\boldsymbol{D}}_t$ before shocks are realized and $\boldsymbol{D}_t$ afterwards. Households choose to supply labor, ℓ_t, and consumption, c_t. Households are not allowed to borrow. The real interest rate is r_t, the real wage is w_t, and real profits are Π_t. Households receive real lumpsum transfers $\mathcal{T}_t$ from the government. Interest-rate income is taxed with the rate $\tau_t^a \in [0,1]$ and labor income is taxed with the rate $\tau_t^\ell \in [0,1]$.

The household problem is

$$
\begin{aligned}
v_t(z_t, a_{t-1}) &= \max_{c_t, \ell_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} + \beta \mathbb{E}\left[v_{t+1}(z_{t+1}, a_t) \,|\, z_t, a_t\right] \qquad (1) \\
\text{s.t. } a_t + c_t &= (1+\tilde{r}_t)a_{t-1} + \tilde{w}_t \ell_t z_t + \Pi_t + \mathcal{T}_t \\
\log z_{t+1} &= \rho_z \log z_t + \psi_{t+1} \quad, \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi),\ \mathbb{E}[z_t] = 1 \\
a_t &\geq 0
\end{aligned}
$$

where $\tilde{r}_t = (1-\tau_t^a)r_t$ and $\tilde{w}_t = (1-\tau_t^\ell)w_t$. Aggregate quantities are

$$L_t^{hh} = \int \ell_t z_t d\boldsymbol{D}_t \tag{2}$$

$$C_t^{hh} = \int c_t d\boldsymbol{D}_t \tag{3}$$

$$A_t^{hh} = \int a_t d\boldsymbol{D}_t \tag{4}$$

Firms. A representative firm rents capital, K_{t-1}, and hire labor, L_t, to produce goods, with the production function

$$Y_t = K_{t-1}^\alpha L_t^{1-\alpha} \tag{5}$$

Capital depreciates with the rate $\delta \in (0,1)$. The real rental price of capital is r_t^K and the real wage is w_t. Profits are

$$\Pi_t = Y_t - w_t L_t - r_t^K K_{t-1} \tag{6}$$

The law-of-motion for capital is

$$K_t = (1-\delta)K_{t-1} + I_t \tag{7}$$

The households own the representative firm in equal shares.

Government. The government raises taxes from the capital income tax and the labor income tax and transfer this back to the households as a lumpsum transfer. The budget constraint for the government is

$$\begin{aligned} \mathcal{T}_t &= \int \left[\tau_t^a r_t a_{t-1} + \tau_t^\ell w_t \ell_t z_t\right] d\boldsymbol{D}_t \\ &= \tau_t^a r_t A_t^{hh} + \tau_t^\ell w_t L_t^{hh} \end{aligned} \tag{8}$$

Market clearing. Market clearing implies

1. Labor market: $L_t = L_t^{hh}$
2. Goods market: $Y_t = C_t^{hh} + I_t$
3. Asset market: $K_t = A_t^{hh}$

2. Calibration

The parameters and steady state government behavior are as follows:

1. **Preferences and abilities:** $\sigma = 2$, $\nu = 1$
2. **Income:** $\rho_z = 0.96$, $\sigma_\psi = 0.257$
3. **Production:** $\alpha = 0.33$, $\delta = 0.06$
4. **Government:** $\tau_{ss}^a = 0.1$, $\tau_{ss}^\ell = 0.3$

3. Household welfare

You will need to extend the code in order to calculate expected household welfare (or *lifetime utility*). For a household with initial states (z_{t-1}, a_{t-1}) we define this as:

$$v_t(z_t, a_{t-1}) = \max_{c_t, \ell_t} u(c_t) - \nu(\ell_t) + \beta \mathbb{E}\left[v_{t+1}(z_{t+1}, a_t) \,|\, z_t, a_t\right]$$

with $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}, \nu(\ell_t) = \varphi \frac{\ell_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}$.
The provided code solves for the optimal choice of consumption, savings and labor supply c_t^*, a_t^*, ℓ_t^*, so at the end of each iteration we can compute:

$$v_t(z_t, a_{t-1}) = u(c_t^*) - \nu(\ell_t^*) + \beta \mathbb{E}\left[v_{t+1}(z_{t+1}, a_t^*)\right] \tag{9}$$

as the expected welfare at time t. The steps are:

1. Provide an initial guess for $v_t(z_t, a_{t-1})$.
2. Iterate backwards from the initial guess. In each period use the provided EGM code to solve the household's problem to obtain c_t^*, ℓ_t^*, a_t^*.
3. Interpolate $v_{t+1}(z_{t+1}, a_t)$ to $v_{t+1}(z_{t+1}, a_t^*)$ using linear interpolation.
4. Compute the expected continuation value $\mathbb{E}\left[v_{t+1}(z_{t+1}, a_t^*)\right]$ by using the markov transition matrix for z_t.
5. Compute time t expected welfare $v_t(z_t, a_{t-1})$ using eq. (9).

There are multiplie ways to implement this in the code. A straightforward approach is to include v_t as an intertemporal variable in the household problem. You do this by adding it to the list `intertemps_hh` defined in the `settings()` function in `HANCModel.py`. The initial value used for the backwards iteration can be set in `prepare_hh_ss` in `steady_state.py`. The actual computation of $\mathbb{E}\left[v_{t+1}(z_{t+1}, a_t^*)\right]$ and $v_t(z_t, a_{t-1})$ can be included in the function `solve_hh_backwards()` in `household_problem.py`.[1]

4. Transition

In order to find the transition between the steady states following a permanent shock you need to calculate the jacobian around the new steady state, and use `find_transition_path`

[1] For inspiration you can look at the VFI code for lecture 1, which also computes lifetime expected welfare.

`()` on the model associated with the new steady state.[2] You will need to update the following attributes of the model: `shocks`, `unknowns`, `targets`.

Appendix - solving the consumption-saving problem with endogenous labor supply

The provided code solves the standard consumption-saving problem augmented with endogenous labor supply. The code applies the endogenous grid method, which utilize the first-order conditions of the problem. Let us first derive these. The Lagrangian of the problem is:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[u\left((1+\tilde{r}_t)a_{t-1} + \tilde{w}_t \ell_t z_t + \Pi_t + \mathcal{T}_t - a_t \right) - \nu\left(\ell_t\right) \right] + \beta^t \lambda_t \left[a_t - 0 \right]$$

where λ_t the multiplier on the borrowing constraint. The first-order conditions w.r.t savings and labor supply are:

$$u'\left(c_t\right) = \beta(1+\tilde{r}_{t+1})\mathbb{E}_0 u'\left(c_{t+1}\right) + \lambda_t$$

$$u'\left(c_t\right)\tilde{w}_t z_t = \nu'\left(\ell_t\right)$$

If agents are unconstrained ($\lambda_t = 0$) we can invert the Euler equation to obtain c_t, and afterwards use the labor supply FOC to obtain ℓ_t:

$$c_t = \left(\beta \underline{v}_{a,t+1}(z_t, a_t)\right)^{-\frac{1}{\sigma}} \tag{10}$$

$$\ell_t = \left(\frac{\tilde{w}_t z_t}{\varphi}\right)^{\nu} c_t^{-\sigma\nu} \tag{11}$$

where the first equation uses $\underline{v}_{a,t+1}(z_t, a_t)) = (1+\tilde{r}_{t+1})\mathbb{E}_0 u'\left(c_{t+1}\right)$ from the envelope condition. If agents are constrained ($\lambda_t \neq 0$) we set $a_t = 0$, and solve for c_t, ℓ_t using the budget constraint and the labor supply FOC (which still holds with equality). The algorithm is as follows:

[2] See for instance HANCGovModel.ipynb for an example of a transition between different steady states in GEModelTools.

1. Calculate c_t and ℓ_t over end-of-period states from FOCs
2. Construct endogenous grid $m_t = c_t + a_t - \tilde{w}_t \ell_t z_t$
3. Use linear interpolation to find consumption $c^*(z_t, a_{t-1})$ and labor supply $\ell^*(z_t, a_{t-1})$ with $m_t = (1 + \tilde{r}_t)a_{t-1} + \Pi_t + \mathcal{T}_t$
4. Calculate savings $a^*(z_t, a_{t-1}) = (1 + \tilde{r}_t)a_{t-1} + \Pi_t + \mathcal{T}_t + \tilde{w}_t \ell_t^* z_t - c_t^*$
5. If $a^*(z_t, a_{t-1}) < 0$ set $a^*(z_t, a_{t-1}) = 0$ and search for ℓ_t such that $f(\ell_t) \equiv \ell_t - \left(\frac{\tilde{w}_t z_t}{\varphi_i}\right)^{\frac{1}{\nu}} c_t^{-\sigma/\nu} = 0$ holds and $c_t = (1 + \tilde{r}_t)a_{t-1} + \tilde{w}_t \ell_t z_t$. This can be done with a recusive algorithm with an update from step j to step $j+1$ given by

$$c_t^j = (1 + \tilde{r}_t)a_{t-1} + \tilde{w}_t \ell_t^j z_t + \Pi_t + \mathcal{T}_t$$
$$\ell_t^{j+1} = (1 - \vartheta)\, \ell_t^j + \vartheta \left(\frac{\tilde{w}_t z_t}{\varphi_i}\right)^{\frac{1}{\nu}} \left(c_t^j\right)^{-\sigma/\nu}$$

where $0 < \vartheta \leq 1$ is a dampening parameter which helps with convergence. The solution for c_t, ℓ_t can be found by repeatingly using this updating scheme until $\ell_t^{j+1} - \ell_t^j < \epsilon$, where ϵ is the convergence tolerance.