Discrete Mathematics: Homework #3

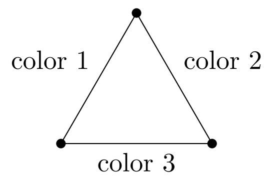
Due on February 15, 2025 at 4:00pm

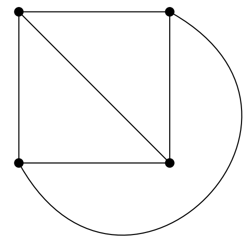
Professor Satish Rao

Zachary Brandt
zbrandt@berkeley.edu

Problem 1: Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.





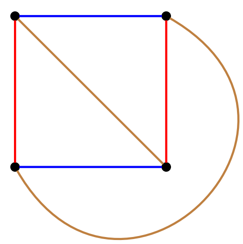
A) Show that the 4 vertex complete graph above can be 3 edge colored. (You may use the numbers $1, 2, 3$ for colors. A figure is shown on the right.)

B) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors.

C) Prove that a tree can be edge colored with d colors where d is the maximum degree of any vertex.

Part A

Below is the 4-vertex graph with edges colored such that only 3 colors are used.



Part B

To prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors I will use the principle of induction.

- *Base case*: When the maximum degree of a graph is $d = 1$ there can only be two vertices and one edge. It therefore only takes $2d - 1 = 2(1) - 1 = 1$ colors to color the edges and the claim holds.
- *Inductive hypothesis*: Assume that any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors for some $1 \leq d \leq k$ where $k \in \mathbb{N}$
- *Inductive step*: For $d = k + 1$, we can show that any graph can be edge colored with $2d - 1$ colors. There exists at least one vertex with degree $k + 1$, which we can remove along with its incident edges. The remaining graph has a maximum degree of k, since the other vertices lost their connection and dropped in degree. By the inductive hypothesis, the remaining graph can be edge colored with $2k - 1$ colors. Coloring the graph with the removed vertex requires $k + 1$ colors (one color for each edge). There are $2(k + 1) - 1 = 2k + 1$ colors at our disposal, which is enough to color each edge uniquely. Therefore, the graph can be colored with $2(k + 1) - 1$ colors. Therefore, any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors.

Part C

The maximum degree of any vertex in a tree must be two, otherwise there would be a cycle. Therefore, d is equal to 2. To be edge colored, any two edges incident to the same vertex must have different colors. Since there are two colors available, it is possible to edge color the tree.

Problem 2: Touring Hypercube

An the lecture, you have seen that if G is a hypercube of dimension n, then

- The vertices of G are the binary strings of length n.
- u and v are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian tour* of a graph (with $n \geq 2$ vertices) is a tour that visits every vertex exactly once.

A) Prove that a hypercube has an Eulerian tour if and only if n is even.

B) Prove that every hypercube has a Hamiltonian tour.

Problem 3: Planarity and Graph Complements

Let $G = (V, E)$ be an undirected graph. We define the complement of G as $\overline{G} = (V, \overline{E})$ where $\overline{E} = \{(i, j) \mid i, j \in V, i \neq j\} - E$; that is, $\overline{G}$ has the same set of vertices as G, but an edge e exists is $\overline{G}$ if and only if it does not exist in G.

A) Suppose G has v vertices and e edges. How many edges does $\overline{G}$ have?

B) Prove that for any graph with at least 13 vertices, G being planar implies that $\overline{G}$ is non-planar.

C) Now consider the converse of the previous part, i.e., for any graph G with at least 13 vertices, if $\overline{G}$ is non-planar, then G is planar. Construct a counterexample to show that the converse does not hold.

Hint: Recall that if a graph contains a copy of K_5, then it is non-planar. Can this fact be used to construct a counterexample?

Problem 4: Modular Practice

Solve the following modular arithmetic equations for x and y. For each subpart, show your work and justify your answers.

A) $9x + 5 \equiv 7 \pmod{13}$.

B) Prove that $3x + 12 \equiv 4 \pmod{21}$ does not have a solution.

C) The system of simultaneous equations $5x + 4y \equiv 0 \pmod 7$ and $2x + y \equiv 4 \pmod 7$.

D) $13^{2023} \equiv x \pmod{12}$.

E) $7^{62} \equiv x \pmod{11}$.

Problem 5: Wilson's Theorem

Wilson's Theorem states the following is true if and only if p is prime:

$$(p-1)! \equiv -1 \pmod{p}.$$

Prove both directions (it holds if AND only if p is prime).
Hint for the if direction: Consider rearranging the terms in $(p-1)! = 1 \cdot 2 \cdot \cdots \cdot (p-1)$ to pair up terms with their inverses, when possible. What terms are left unpaired?
Hint for the only if direction: If p is composite, then it has some prime factor q. What can we say about $(p-1)! \pmod{q}$?

Problem 6: How Many Solutions?

Consider the equation $ax \equiv b \pmod p$ for prime p. In the below three parts, when we discuss solutions, we mean a solution x in the range $\{0, 1, \ldots p-1\}$. In addition, include justification for your answers to all the subparts of this problem.

A) For how many pairs (a, b) does the equation have a unique solution?

B) For how many pairs (a, b) does the equation have no solution?

C) For how many pairs (a, b) does the equation have p solutions?

Now, consider the equation $ax \equiv b \pmod{pq}$ for distinct primes p, q. In the below three parts, when we discuss solutions, we mean a solution x in the range $\{0, 1, \ldots pq-1\}$.

D) If $\gcd(a, pq) = p$, show that there exists a solution if and only if $b = 0 \pmod p$.

E) If $\gcd(a, pq) = p$ and there is a solution x, show that there are exactly p solutions. (Hint: consider how you can generate another solution $x + \_\_\_$)

F) For how many pairs (a, b) are there exactly p solutions?