# Discrete Mathematics: Homework #3

Due on February 15, 2025 at 4:00pm

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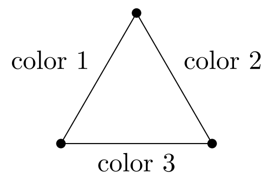
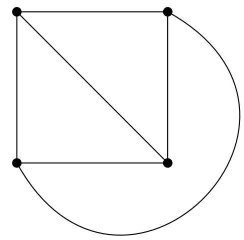
# Problem 1: Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.



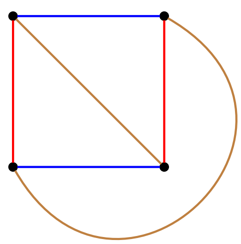
A) Show that the 4 vertex complete graph above can be 3 edge colored. (You may use the numbers $1, 2, 3$ for colors. A figure is shown on the right.)

B) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors.

C) Prove that a tree can be edge colored with $d$ colors where $d$ is the maximum degree of any vertex.

## Part A

Below is the 4-vertex graph with edges colored such that only 3 colors are used.



## Part B

To prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors I will use the principle of induction.

- *Base case*: When the maximum degree of a graph is $d = 1$ there can only be two vertices and one edge. It therefore only takes $2d - 1 = 2(1) - 1 = 1$ colors to color the edges and the claim holds.
- *Inductive hypothesis*: Assume that any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors for some $1 \leq d \leq k$ where $k \in \mathbb{N}$
- *Inductive step*: For $d = k + 1$, we can show that any graph can be edge colored with $2d - 1$ colors. There exists at least one vertex with degree $k + 1$, which we can remove along with its incident edges. The remaining graph has a maximum degree of $k$, since the other vertices lost their connection and dropped in degree. By the inductive hypothesis, the remaining graph can be edge colored with $2k - 1$ colors. Coloring the graph with the removed vertex requires $k + 1$ colors (one color for each edge). There are $2(k + 1) - 1 = 2k + 1$ colors at our disposal, which is enough to color each edge uniquely. Therefore, the graph can be colored with $2(k + 1) - 1$ colors. Therefore, any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors.

## Part C

The maximum degree of any vertex in a tree must be two, otherwise there would be a cycle. Therefore, $d$ is equal to 2. To be edge colored, any two edges incident to the same vertex must have different colors. Since there are two colors available, it is possible to edge color the tree.

# Problem 2: Touring Hypercube

An the lecture, you have seen that if $G$ is a hypercube of dimension $n$, then

- The vertices of $G$ are the binary strings of length $n$.
- $u$ and $v$ are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian tour* of a graph (with $n \geq 2$ vertices) is a tour that visits every vertex exactly once.

A) Prove that a hypercube has an Eulerian tour if and only if $n$ is even.

B) Prove that every hypercube has a Hamiltonian tour.

## Part A

An Eulerian tour is a sequence of edges, that starts and ends on the same vertex, in a graph that uses each edge exactly once. To prove that a hypercube has an Eulerian tour if and only if $n$ is even, I will prove both directions of the biconditional.

*A hypercube has an Eulerian tour if $n$ is even.* To prove this, I will show that a hypercube of even dimension only has vertices of even degree, which would imply that the hypercube has an Eulerian tour, since it is also connected. If $n$ is even, then $G$ is a hypercube of an even dimension, meaning the binary strings that are the vertices of $G$ are also of even length. Since each vertex is incident as many edges as their are one-bit-location differences in its binary string, if $n$ is even then there are also an even number of differences. This is because there are $n$ one-bit-location differences for a binary string, as each digit can either be zero or one. Therefore, since each vertex is incident to an even number of edges when $n$ is even, $G$ has an Eulerian tour.

*$n$ is even if the hypercube has an Eulerian tour.* If a graph $G$ has an Eulerian tour, all its vertices must be of even degree. For a hypercube to have all its vertices of even degree, the binary strings must be of even length, i.e., $n$ must be even. If the binary strings were of odd length, that would mean vertices would be incident to an odd number of edges, and therefore not have an Eulerian tour, leading to a contradiction.

## Part B

To prove that every hypercube has a Hamiltonian tour, I will use the principle of induction on $n$, the dimension of the hypercube.

- *Base case*: For a hypercube $G$ to have at least two vertices, it must be of dimension $n = 1$, as a binary string of length one has 1 one-bit-location difference. $G$ then has a Hamiltonian tour, as traversing the one edge visits both vertices.
- *Inductive hypothesis*: Assume that any hypercube of dimension $1 \leq n \leq k$ has a Hamiltonian tour where $k \in \mathbb{N}$.
- *Inductive step*: For $n = k + 1$, we can show that the hypercube has a Hamiltonian tour. The $k + 1$ dimensional hypercube is composed of two $k$ dimensional hypercubes that, under the inductive hypothesis, each have a Hamiltonian tour. It is then possible to construct a consolidated Hamiltonian tour for the $k + 1$ hypercube by removing the closing step of one of the tours and replacing it with one of the edges that crosses to the other $k$ dimensional hypercube. Therefore, there exists a Hamiltonian tour for every hypercube.

# Problem 3: Planarity and Graph Complements

Let $G = (V, E)$ be an undirected graph. We define the complement of $G$ as $\overline{G} = (V, \overline{E})$ where $\overline{E} = \{(i, j) \mid i, j \in V, i \neq j\} - E$; that is, $\overline{G}$ has the same set of vertices as $G$, but an edge $e$ exists is $\overline{G}$ if and only if it does not exist in $G$.

A) Suppose $G$ has $v$ vertices and $e$ edges. How many edges does $\overline{G}$ have?

B) Prove that for any graph with at least 13 vertices, $G$ being planar implies that $\overline{G}$ is non-planar.

C) Now consider the converse of the previous part, i.e., for any graph $G$ with at least 13 vertices, if $\overline{G}$ is non-planar, then $G$ is planar. Construct a counterexample to show that the converse does not hold.

*Hint: Recall that if a graph contains a copy of $K_5$, then it is non-planar. Can this fact be used to construct a counterexample?*

## Part A

The maximum number of edges $G$ can have is $\frac{v(v-1)}{2}$, because in a complete graph, each vertex can have a maximum degree of $v - 1$. Since there are $v$ vertices and each edge is incident to two vertices, $\frac{v(v-1)}{2}$ is the maximum number of edges $G$ can have. Therefore, $\overline{G}$ has $v(v - 1) - e$ edges.

## Part B

If a graph $G$ is planar, then the following inequality must be true, $e \leq 3v - 6$. If $G$ has at least 13 vertices, then $e \leq 3(13) - 6 = 33$. $\overline{G}$ must therefore have $\frac{13(13-1)}{2} - 33 = 156 \div 2 - 33 = 78 - 33 = 45$ edges. Since $\overline{G}$ must have at least 45 edges, which is greater than 33, $\overline{G}$ must be non-planar.

## Part C

Consider a graph with K5 and all other vertices unconnected. The complement must then also be non planar.

# Problem 4: Modular Practice

Solve the following modular arithmetic equations for $x$ and $y$. For each subpart, show your work and justify your answers.

A) $9x + 5 \equiv 7 \pmod{13}$.

B) Prove that $3x + 12 \equiv 4 \pmod{21}$ does not have a solution.

C) The system of simultaneous equations $5x + 4y \equiv 0 \pmod{7}$ and $2x + y \equiv 4 \pmod{7}$.

D) $13^{2023} \equiv x \pmod{12}$.

E) $7^{62} \equiv x \pmod{11}$.

# Problem 5: Wilson's Theorem

Wilson's Theorem states the following is true if and only if $p$ is prime:

$$(p-1)! \equiv -1 \pmod{p}.$$

Prove both directions (it holds if AND only if $p$ is prime).
Hint for the if direction: Consider rearranging the terms in $(p-1)! = 1 \cdot 2 \cdot \cdots \cdot (p-1)$ to pair up terms with their inverses, when possible. What terms are left unpaired?
Hint for the only if direction: If $p$ is composite, then it has some prime factor $q$. What can we say about $(p-1)! \pmod{q}$?

# Problem 6: How Many Solutions?

Consider the equation $ax \equiv b \pmod{p}$ for prime $p$. In the below three parts, when we discuss solutions, we mean a solution $x$ in the range $\{0, 1, \ldots p-1\}$. In addition, include justification for your answers to all the subparts of this problem.

A) For how many pairs $(a, b)$ does the equation have a unique solution?

B) For how many pairs $(a, b)$ does the equation have no solution?

C) For how many pairs $(a, b)$ does the equation have $p$ solutions?

Now, consider the equation $ax \equiv b \pmod{pq}$ for distinct primes $p, q$. In the below three parts, when we discuss solutions, we mean a solution $x$ in the range $\{0, 1, \ldots pq-1\}$.

D) If $\gcd(a, pq) = p$, show that there exists a solution if and only if $b = 0 \pmod{p}$.

E) If $\gcd(a, pq) = p$ and there is a solution $x$, show that there are exactly $p$ solutions. (Hint: consider how you can generate another solution $x + \_\_\_$)

F) For how many pairs $(a, b)$ are there exactly $p$ solutions?