Discrete Mathematics: Homework #3

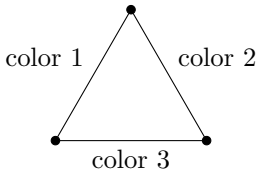
Due on February 15, 2025 at 4:00pm

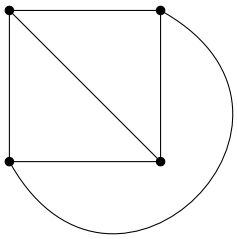
Professor Satish Rao

Zachary Brandt
zbrandt@berkeley.edu

Problem 1: Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.





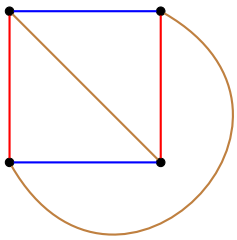
A) Show that the 4 vertex complete graph above can be 3 edge colored. (You may use the numbers $1, 2, 3$ for colors. A figure is shown on the right.)

B) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors.

C) Prove that a tree can be edge colored with d colors where d is the maximum degree of any vertex.

Part A

Below is the 4-vertex graph with edges colored such that only 3 colors are used.



Part B

To prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors I will use the principle of induction on d.

- *Base case*: When the maximum degree of a graph is $d = 1$ there can only be two vertices and one edge. It therefore only takes $2d - 1 = 2(1) - 1 = 1$ colors to color the edges and the claim holds.
- *Inductive hypothesis*: Assume that any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors for some $1 \leq d \leq k$ where $k \in \mathbb{N}$
- *Inductive step*: For $d = k + 1$, we can show that any graph can be edge colored with $2d - 1$ colors. There exists at least one vertex with degree $k + 1$, which we can remove along with its incident edges. The remaining graph has a maximum degree of k, since the other vertices lost their connection and dropped in degree. By the inductive hypothesis, the remaining graph can be edge colored with $2k - 1$ colors. Coloring the graph with the removed vertex requires $k + 1$ colors (one color for each edge). There are $2(k + 1) - 1 = 2k + 1$ colors at our disposal, which is enough to color each edge uniquely. Therefore, the graph can be colored with $2(k + 1) - 1$ colors. Therefore, any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors.

Part C

I prove that a tree can be edge colored with d colors where d is the maximum degree of any vertex using the principle of induction on the number of vertices in the tree, n.

Problem 1 continued on next page...

- *Base case*: When there are two vertices, there is only one edge to color, and so can be colored in d colors, since the maximum degree of either vertex is 1.
- *Inductive hypothesis*: Assume that the claim holds for any tree with vertices $2 \leq n \leq k$.
- *Inductive step*: Remove any one of the leaves of the tree, i.e. a vertex of degree 1 alongisde its edge. By the inductive hypothesis, this tree can be colored with d colors. For the parent of this leaf vertex, its maximum degree is $d - 1$ after removing the edge, and there is at least one color for all the edges such that it can be edge colored, with one extra one as $d - (d - 1) = 1$. Therefore, there is a color remaining for the leaf and edge that was removed to be colored. Therefore the claim has been proven true for all trees with n vertices.

Problem 2: Touring Hypercube

An the lecture, you have seen that if G is a hypercube of dimension n, then

- The vertices of G are the binary strings of length n.
- u and v are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian tour* of a graph (with $n \geq 2$ vertices) is a tour that visits every vertex exactly once.

A) Prove that a hypercube has an Eulerian tour if and only if n is even.

B) Prove that every hypercube has a Hamiltonian tour.

Part A

An Eulerian tour is a sequence of edges, that starts and ends on the same vertex, in a graph that uses each edge exactly once. To prove that a hypercube has an Eulerian tour if and only if n is even, I will prove both directions of the biconditional.

A hypercube has an Eulerian tour if n is even. To prove this, I will show that a hypercube of even dimension only has vertices of even degree, which would imply that the hypercube has an Eulerian tour, since it is also connected. If n is even, then G is a hypercube of an even dimension, meaning the binary strings that are the vertices of G are also of even length. Since each vertex is incident as many edges as their are one-bit-location differences in its binary string, if n is even then there are also an even number of differences. This is because there are n one-bit-location differences for a binary string, as each digit can either be zero or one. Therefore, since each vertex is incident to an even number of edges when n is even, G has an Eulerian tour.

n is even if the hypercube has an Eulerian tour. If a graph G has an Eulerian tour, all its vertices must be of even degree. For a hypercube to have all its vertices of even degree, the binary strings must be of even length, i.e., n must be even. If the binary strings were of odd length, that would mean vertices would be incident to an odd number of edges, and therefore not have an Eulerian tour, leading to a contradiction.

Part B

To prove that every hypercube has a Hamiltonian tour, I will use the principle of induction on n, the dimension of the hypercube.

- *Base case*: For a hypercube G to have at least two vertices, it must be of dimension $n = 1$, as a binary string of length one has 1 one-bit-location difference. G then has a Hamiltonian tour, as traversing the one edge visits both vertices.
- *Inductive hypothesis*: Assume that any hypercube of dimension $1 \leq n \leq k$ has a Hamiltonian tour where $k \in \mathbb{N}$.
- *Inductive step*: For $n = k + 1$, we can show that the hypercube has a Hamiltonian tour. The $k + 1$ dimensional hypercube is composed of two k dimensional hypercubes that, under the inductive hypothesis, each have a Hamiltonian tour. It is then possible to construct a consolidated Hamiltonian tour for the $k + 1$ hypercube by removing the closing step of one of the tours and replacing it with one of the edges that crosses to the other k dimensional hypercube, following that tour backwards until the very last step again, and then finally crossing over again to the starting vertex to finish the Hamiltonian tour. Therefore, there exists a Hamiltonian tour for every hypercube.

Problem 3: Planarity and Graph Complements

Let $G = (V, E)$ be an undirected graph. We define the complement of G as $\overline{G} = (V, \overline{E})$ where $\overline{E} = \{(i, j) \mid i, j \in V, i \neq j\} - E$; that is, $\overline{G}$ has the same set of vertices as G, but an edge e exists is $\overline{G}$ if and only if it does not exist in G.

A) Suppose G has v vertices and e edges. How many edges does $\overline{G}$ have?

B) Prove that for any graph with at least 13 vertices, G being planar implies that $\overline{G}$ is non-planar.

C) Now consider the converse of the previous part, i.e., for any graph G with at least 13 vertices, if $\overline{G}$ is non-planar, then G is planar. Construct a counterexample to show that the converse does not hold.

Hint: Recall that if a graph contains a copy of K_5, then it is non-planar. Can this fact be used to construct a counterexample?

Part A

The maximum number of edges G can have is $\frac{v(v-1)}{2}$, because in a complete graph, each vertex can have a maximum degree of $v - 1$. Since there are v vertices and each edge is incident to two vertices, $\frac{v(v-1)}{2}$ is the maximum number of edges G can have. Therefore, $\overline{G}$ has $v(v-1) - e$ edges.

Part B

If a graph G is planar, then the following inequality must be true, $e \leq 3v - 6$. If G has at least 13 vertices, then $e \leq 3(13) - 6 = 33$. $\overline{G}$ must therefore have $\frac{13(13-1)}{2} - 33 = 156 \div 2 - 33 = 78 - 33 = 45$ edges. Since $\overline{G}$ must have at least 45 edges, which is greater than 33, $\overline{G}$ must be non-planar. After 13 we continue to have more than the $3v-6$ edges allowed in a planar graph, so the complement must not be planar.

Part C

onsider a graph G with K_5 but all other vertices unconnected. The complement must then also then be non planar, as some other group of 5 vertices in the 8 unconnected ones are now connected, also forming K_5.

Problem 4: Modular Practice

Solve the following modular arithmetic equations for x and y. For each subpart, show your work and justify your answers.

A) $9x + 5 \equiv 7 \pmod{13}$.

$$\begin{aligned} 9x + 5 &\equiv 7 \pmod{13} \\ 9x &\equiv 2 \pmod{13} \\ 3 \cdot 9x &\equiv 2 \cdot 3 \pmod{13} \\ x &\equiv 6 \pmod{13} \end{aligned}$$

The multiplicative inverse of 9 modulo 13 is 3, using this I found that x is equivalent to 6 modulo 13.

B) Prove that $3x + 12 \equiv 4 \pmod{21}$ does not have a solution.

$$\begin{aligned} 3x + 12 &\equiv 4 \pmod{21} \\ 3x &\equiv -8 \pmod{21} \\ 3x &\equiv 5 \pmod{21} \end{aligned}$$

Notice first that I can do the negative thing This equivalency has no solution for x. This is because $\gcd(3, 21) = 3$, and 3 does not divide 5. We would need to find a solution for x in an equation of the form $3x = k \cdot 21 + 5$, where $k \in \mathbb{N}$. If 3 and 21 did have a greatest common denominator that divided 5, it would be possible to divide out 3 to see that there exists a k multiple of 21 with an integer remainder. However, this is not the case.

C) The system of simultaneous equations $5x + 4y \equiv 0 \pmod{7}$ and $2x + y \equiv 4 \pmod{7}$.

$$\begin{aligned} 5x + 4y &\equiv 0 \pmod{7} \\ 2x + y &\equiv 4 \pmod{7} \\ -3x &\equiv -16 \pmod{7} \\ 4x &\equiv 5 \pmod{7} \\ x &\equiv 5 \cdot 2 \pmod{7} \\ x &\equiv 10 \pmod{7} \\ x &\equiv 3 \pmod{7} \end{aligned}$$

$$\begin{aligned} 6 + y &\equiv 4 \pmod{7} \\ y &\equiv -2 \pmod{7} \\ y &\equiv 5 \pmod{7} \end{aligned}$$

D) $13^{2023} \equiv x \pmod{12}$.

$$\begin{aligned} 13^{2023} &\equiv x \pmod{12} \\ 13^{2 \cdot 1011 + 1} &\equiv x \pmod{12} \\ (13^2)^{1011} \cdot 13 &\equiv x \pmod{12} \\ 1^{1011} \cdot 1 &\equiv x \pmod{12} \\ 1 &\equiv x \pmod{12} \end{aligned}$$

Problem 4 continued on next page...

E) $7^{62} \equiv x \pmod{11}$.

$$
\begin{aligned}
7^{62} &\equiv x \pmod{11} \\
7^{2 \cdot 31} &\equiv x \pmod{11} \\
(7^2)^{31} &\equiv x \pmod{11} \\
5^{31} &\equiv x \pmod{11} \\
5^{2 \cdot 15 + 1} &\equiv x \pmod{11} \\
(5^2)^{15} \cdot 5 &\equiv x \pmod{11} \\
(3)^{15} \cdot 5 &\equiv x \pmod{11} \\
(3^3)^5 \cdot 5 &\equiv x \pmod{11} \\
(4)^5 \cdot 5 &\equiv x \pmod{11} \\
(4)^{2 \cdot 2 + 1} \cdot 5 &\equiv x \pmod{11} \\
(4^2)^2 \cdot 20 &\equiv x \pmod{11} \\
(5)^2 \cdot 9 &\equiv x \pmod{11} \\
3 \cdot 9 &\equiv x \pmod{11} \\
5 &\equiv x \pmod{11}
\end{aligned}
$$

Problem 5: Wilson's Theorem

Wilson's Theorem states the following is true if and only if p is prime:

$$(p-1)! \equiv -1 \pmod{p}.$$

Prove both directions (it holds if AND only if p is prime).

Hint for the if direction: Consider rearranging the terms in $(p-1)! = 1 \cdot 2 \cdot \cdots \cdot (p-1)$ to pair up terms with their inverses, when possible. What terms are left unpaired?

Hint for the only if direction: If p is composite, then it has some prime factor q. What can we say about $(p-1)! \pmod{q}$?

Solution

For the if direction, we can rearrange the terms in the equivalency to pair up terms with their inverses.

$$\begin{aligned} (p-1)! &\equiv -1 \pmod{p} \\ 1 \cdot 2 \cdot \cdots \cdot (p-1) &\equiv -1 \pmod{p} \end{aligned}$$

Since p is prime, it's greatest common divisor with any other number is 1. Therefore, each number in the product series $1, 2, \ldots, (p-1)$ has a unique multiplicative inverse modulo p. Since every prime number is odd (if there existed an even prime number, it would be divisible by 2, and no longer be prime), there are an even number of terms in the $(p-1)!$ product series. It seems to be that, since there are an even number of terms, and each term has a unique inverse, that the series is equivalent to 1. However, the term 1 is its own inverse, and so is not paired up with any other term. Additionally, $p-1$ is also its own inverse, $(p-1) \cdot (p-1) = p^2 + 2p + 1 \equiv 1 \pmod{p}$. Therefore, the product series of $(p-1)!$ is equivalent to $p-1$. And since p is a multiple of p, this demonstrates the equivalency of $(p-1)!$ and -1.

$$\begin{aligned} 1 \cdot 2 \cdot \cdots \cdot (p-1) &\equiv -1 \pmod{p} \\ p-1 &\equiv -1 \pmod{p} \\ -1 &\equiv -1 \pmod{p} \end{aligned}$$

For the only if direction, assume that p is not prime yet the equivalency remains true. Therefore, if it is not prime, and greater than 1, p is composite, and has some prime factor q. We then know that $(p-1)! \equiv 0 \pmod{q}$, since q is less than p, q is somewhere in the $(p-1)!$ product series, which can then be expressed as some multiple of q, i.e. $(p-1)! = kq$ for some $k \in \mathbb{N}$. The original equivalency can be expressed as $(p-1)! = lp - 1$. But since p is composite, $(p-1)! = kq - 1$, which contradicts what we initially found, that $(p-1)!$ is a multiple of q without any remainder. Therefore, p cannot be composite and must be prime.

Problem 6: How Many Solutions?

Consider the equation $ax \equiv b \pmod{p}$ for prime p. In the below three parts, when we discuss solutions, we mean a solution x in the range $\{0, 1, \ldots p-1\}$. In addition, include justification for your answers to all the subparts of this problem.

A) For how many pairs (a, b) does the equation have a unique solution?

When $a \equiv 0 \pmod{p}$, for any x, the equation will have the same solution where $ax \equiv 0 \equiv b \pmod{p}$. Therefore, for the equation to produce a unique solution for x, $a \not\equiv 0 \pmod{p}$. There are then $p-1$ options for a and p options for b to form pairs, i.e., there are $p(p-1)$ pairs (a, b) for which the equation has unique solutions.

B) For how many pairs (a, b) does the equation have no solution?

The equation has no solutions when $a \equiv 0 \pmod{p}$ but $b \not\equiv 0 \pmod{p}$. Therefore, there is one option for a, 0, and $p-1$ not zero options for b, i.e., there are $p-1$ pairs (a, b) for which the equation has no solutions.

C) For how many pairs (a, b) does the equation have p solutions?

When both a and b are equivalent to 0 modulos p, there aren't any unique solutions but there are p solutions, since all elements in the set $\{0, 1, \ldots p-1\}$ times 0 will be equivalent to 0 modulos p. There is one pair, $(0, 0)$, for which the equation has p solutions.

Now, consider the equation $ax \equiv b \pmod{pq}$ for distinct primes p, q. In the below three parts, when we discuss solutions, we mean a solution x in the range $\{0, 1, \ldots pq-1\}$.

D) If $\gcd(a, pq) = p$, show that there exists a solution if and only if $b = 0 \pmod{p}$.

If $b \equiv 0 \pmod{p}$, then the equation $ax \equiv b \pmod{pq}$ has a solution. Since $\gcd(a, pq) = p$, we can write $a = kp$ for $k \in \mathbb{Z}$. Then the equation becomes $kp \cdot x \equiv b \pmod{pq}$. Since $b \equiv 0 \pmod{p}$, we can write $b = mp$ for some integer m. Thus, the equation simplifies to $kp \cdot x \equiv mp \pmod{pq}$, which reduces to $kx \equiv m \pmod{q}$. Since k and q do not share a greatest common divisor that is not 1, there exists a solution considering $k \equiv k^{-1}m \pmod{q}$.

If we only know that there exists a solution, we need to show that $b \equiv 0 \pmod{p}$. Since a and pq share p as a greatest common divisor, b must also be a multiple of p for $ax \equiv b \pmod{pq}$ to hold. Therefore, $b \equiv 0 \pmod{p}$.

E) If $\gcd(a, pq) = p$ and there is a solution x, show that there are exactly p solutions. (Hint: consider how you can generate another solution $x +$ ___)

We can express a as a multiple k of p in our equation to answer the question

$$\begin{aligned} ax &\equiv b \pmod{pq} \\ kp \cdot x &\equiv b \pmod{pq} \end{aligned}$$

To generate $x +$ ___ solutions, we can add multiples l of q to x, e.g. $kp(x + 2q)$, as each $klpq$ is equivalent to 0 modulos q. We can only add up to the p multiple of q however, at which point the solutions cycle and are identical to earlier ones.

F) For how many pairs (a, b) are there exactly p solutions?

The requirements from the last part are that a and pq share a greatest common divisor of p, and form part D) that $b \equiv 0 \pmod{p}$. b can be expressed as a multiple of p, e.g. kp. k must be less than q, otherwise b becomes equivalent to pq or out of range. Therefore, there are q values b can take on where

Problem 6 continued on next page...

$k \in \{0, 1, \ldots, q-1\}$. Similarly for a, it can be expressed as a multiple of p, e.g. mp. For a to be divisible by p, but also not have pq as a greatest common divisor and stay in the range, m must be in the set $\{1, 2, \ldots, q-1\}$. Therefore, there are $q \cdot (q-1)$ pairs with exactly p solutions.