Macroeconomics: Problem Set #1

Due on September 22, 2024 at 10:00pm

Prof. Barnichon Section 102

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Problem

Politicians seeking to garner support for lower taxes will sometimes argue that lowering taxes will actually lead to an increase in tax revenue because this will increase output sharply. Clearly, if tax rates were 100%, little revenue would be raised since few people would work much. So, at sufficiently high tax rates, this argument is valid. But how high do taxes need to be for this to be the case?

The argument that lowering taxes will raise revenue was famously made by economist Arthur Laffer in a dinner meeting in 1974 with – then – Ford Administration officials Dick Cheney and Donald Rumsfeld. Laffer is said to have illustrated his argument by drawing a curve on his napkin showing how tax revenue rises with the tax rate at low tax rates but eventually falls back down to zero at 100% tax rates. A graph of tax revenue as a function of tax rates has since been called the Laffer curve. Having a tax rate higher than the tax rate that maximizes revenue (i.e., a tax rate on the downward sloping portion of the curve) is called being on the wrong side of the Laffer curve.

In this question, you will solve for the "top of the Laffer curve," i.e., the tax rate that maximizes tax revenue, in a simple model.

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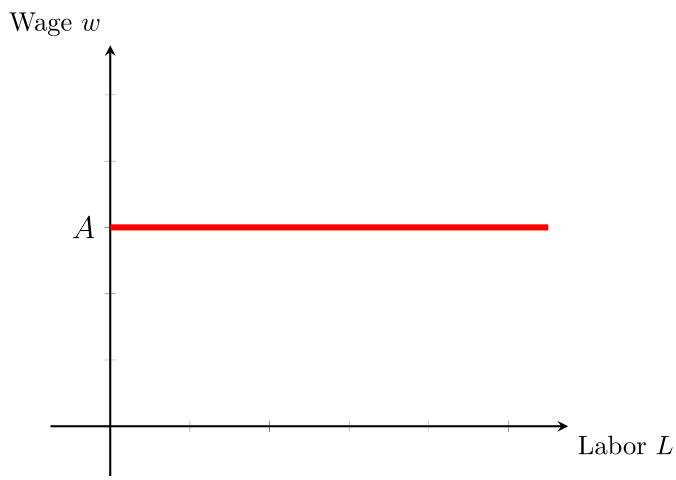
Part A

Suppose output in the economy is produced with only labor (no capital, no land, etc., for simplicity). Suppose the production function is $Y = AL$, where Y denotes total output in the economy, A denotes productivity, and L denotes total labor in the economy. Suppose firms take wages w as given. Derive the labor demand curve in this economy. Plot the labor demand curve in (w, L) space (i.e., with w on the y-axis and L on the x-axis). In one or two sentences, comment on why it makes sense that the labor demand curve takes this form in this model.

Solution



In a competitive market, firms hire until the marginal product of labor is equal to the wage w. This behavior determines the labor demand curve. The derivative of the production function quantifies the marginal product of labor: the marginal change in output with respect to a change in labor.

$$w = \frac{\partial Y}{\partial L} = \frac{\partial (AL)}{\partial L} = A$$

This shows that the wage is equal to the productivity of labor A, which is constant in this economy. The labor demand curve therefore does not depend on L, meaning there are no diminishing returns to labor. This explains why the labor demand curve takes on this horizontal form in this model.

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Part B

Suppose each household's utility function is

$$U = \log(C) - \psi\frac{H^{1+\frac{1}{\eta}}}{1+\eta^{-1}},$$

where C denotes per capita consumption, H denotes per capita hours, and η and ψ are parameters. Suppose that all households are identical. This implies that they will all consume the same amount in equilibrium and supply the same numbers of hours of labor. Each household's budget constraint is

$$C = (1-\tau_l)wH + T,$$

where τ_l denotes the labor income tax in this economy and T denotes a lump sum transfer from the government to the household. For simplicity, we assume that the government redistributes all tax receipts back to the households lump sum. "Lump sum" means that the household takes the amount of transfers it receives as given—i.e., it believes that it can't affect these with its actions. Derive the household's labor supply curve.

Solution

The household maximizes utility subject to the budget constraint. Since the household's consumption is a function of hours, we can eliminate C from the utility function by substitution.

$$U(C) - V(H) = \log((1-\tau_l)wH + T) - \psi\frac{H^{1+\frac{1}{\eta}}}{1+\eta^{-1}}$$

Now we can maximize the utility function with respect to one variable H. To do this, we differentiate the utility function with respect to H and solve for when the result equals zero. This way we can find an equation that implicitly desribes how much labor H the household will supply as a function of the wage w, assuming all other variables are constant.

$$\begin{aligned}
\frac{\partial U}{\partial H} &= \frac{(1-\tau_l)w}{(1-\tau_l)wH+T} - \psi H^{\frac{1}{\eta}} \\
\text{Set equal to zero} \quad 0 &= \frac{(1-\tau_l)w}{(1-\tau_l)wH+T} - \psi H^{\frac{1}{\eta}} \\
\psi H^{\frac{1}{\eta}} &= \frac{(1-\tau_l)w}{(1-\tau_l)wH+T} \\
\psi H^{\frac{1}{\eta}} &= \frac{(1-\tau_l)}{(1-\tau_l)wH+T}\cdot w
\end{aligned}$$

The equation above describes what is true for households optimally trading off the utility of consumption, and the disulity of work. On the left hand side is the marginal disutility of work in utils per hour, $V'(H)$. On the right hand side is the marginal value of extra consumption afforded by working an extra hour. This is equal to the wage w times the marginal value of consumption, $U'(C)$, resulting in utils per hour worked again.

The equation therefore shows that for households to optimize they must set the marginal benefit of working an extra hour to its marginal cost. This is the labor supply curve. It describes how much labor in hours worked H a household will supply as a function of the wage w.

Part C

Suppose that there are N households in the economy. Total tax revenue will then be $\tau_l wHN$, i.e., the tax rate times household labor income (wH) times the number of households. Suppose that each household recieves an equal share of this tax revenue as a lump sum transfer from the government. Use this fact, the household's budget constraint, its labor supply curve, and the labor demand equation to show that hours worked per person in this economy can be expressed as

$$H = (1-\tau_l)^{\frac{\eta}{\eta+1}}\psi^{\frac{-\eta}{\eta+1}}.$$

Plot this curve in $(\tau_l,\, H)$ space for $\eta = 1$ and $\psi = 2$. In one or two sentences, describe the intuition for the shape of this curve.

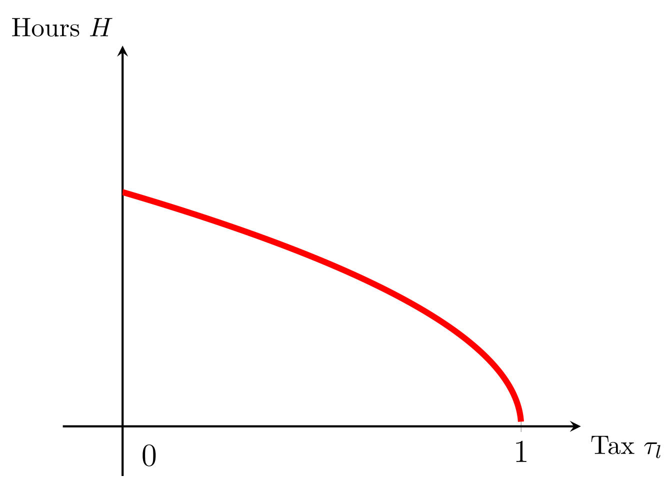
Solution

We can express the lump sum transfers T from the government to the households with the total tax revenue. If each household recieves an equal share of the tax revenue, then the lump sum transfer is simply this total tax revenue divided by the number of households.

$$T = \frac{\tau_l wHN}{N} = \tau_l wH$$

We can substitute this expression for T into the labor supply curve equation from part (b) to find the hours worked per person in this economy as a function of the tax rate τ_l and parameters η and ψ.

$$\begin{aligned}
\psi H^{\frac{1}{\eta}} &= \frac{(1-\tau_l)w}{(1-\tau_l)wH + \tau_l wH} \\
&= \frac{(1-\tau_l)}{(1-\tau_l)H + \tau_l H} \\
&= \frac{1}{H}\cdot\frac{1-\tau_l}{(1+\tau_l)+\tau_l} \\
&= \frac{1}{H}\cdot\frac{1-\tau_l}{1} \\
H\cdot\psi H^{\frac{1}{\eta}} &= 1-\tau_l \\
H^{\frac{\eta}{\eta}}\cdot\psi H^{\frac{1}{\eta}} &= \\
\psi H^{\frac{\eta+1}{\eta}} &= \\
\psi^{\frac{\eta}{\eta+1}} H &= (1-\tau_l)^{\frac{\eta}{\eta+1}} \\
H &= (1-\tau_l)^{\frac{\eta}{\eta+1}}\psi^{\frac{-\eta}{\eta+1}}
\end{aligned}$$



The curve is downward sloping because as the tax rate increases from 0 to 1, the amount of hours worked per person decreases. The intuition is that as the tax rate increases, the wage that households receive decreases for each hour worked, which incentivizes them to work less.

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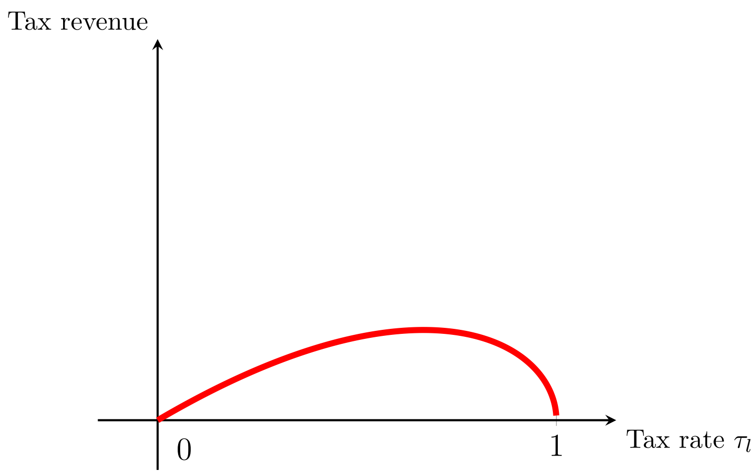
Part D

Recall that tax revenue is $\tau_l wHN$. The Laffer curve is a tax revenue as a function of the tax rate and only exogenous variables and parameters. Use the expression you derived in part (c) as well as the labor demand curve to derive an expression for tax revenue that is a function of only the tax rate τ_l and the exogenous variables (A, N) and parameters (η, ψ). Plot this function assuming that $\eta = 1$, $\psi = 2$, $A = 1$, and $N = 1$.

Solution

Using our expression for hours worked per person from part (c) and the labor demand curve in part (a), we can substitute these into $\tau_l wHN$ to find an expression for tax revenue as a function of τ_l and the exogenous variables and parameters.

$$\begin{aligned}\tau_l wHN &= \tau_l w(1-\tau_l)^{\frac{\eta}{\eta+1}}\psi^{\frac{-\eta}{\eta+1}}N \quad \text{substitute from part (c)}\\ &= \tau_l A(1-\tau_l)^{\frac{\eta}{\eta+1}}\psi^{\frac{-\eta}{\eta+1}}N\end{aligned}$$



This plot demonstrates features of a Laffer curve described in the problem description. From the plot, we can see that tax revenue is minimized at a tax rate of 0 and 1, and that tax revenues rise going up from 0, reach a maximum, and then fall back down again as part of being on the wrong side of the Laffer curve.

Part E

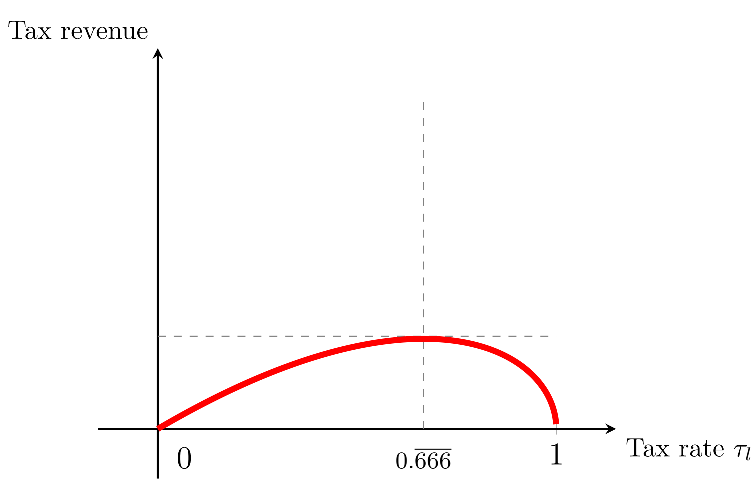
Find the top of the Laffer curve. In other words, derive an expression for the tax rate that yields maximum tax revenue as a function of the exogenous parameters (η, ψ).

Solution

To find the tax rate that maximizes tax revenue, we can take the derivative of the tax revenue function with respect to the tax rate τ_l and set it equal to zero.

$$\begin{aligned}
\frac{\partial(\text{Tax revenue})}{\partial \tau_l} &= \frac{\partial(\tau_l A(1-\tau_l)^{\frac{\eta}{\eta+1}}\psi^{\frac{-\eta}{\eta+1}}N)}{\partial \tau_l} \\
&= A(1-\tau_l)^{\frac{\eta}{\eta+1}}\psi^{\frac{-\eta}{\eta+1}}N - A\tau_l\frac{\eta}{\eta+1}(1-\tau_l)^{\frac{-1}{\eta+1}}\psi^{\frac{-\eta}{\eta+1}}N \\
\text{Set equal to zero} \qquad 0 &= A(1-\tau_l)^{\frac{\eta}{\eta+1}}\psi^{\frac{-\eta}{\eta+1}}N - A\tau_l\frac{\eta}{\eta+1}(1-\tau_l)^{\frac{-1}{\eta+1}}\psi^{\frac{-\eta}{\eta+1}}N \\
&= (1-\tau_l)^{\frac{\eta}{\eta+1}} - \tau_l\frac{\eta}{\eta+1}(1-\tau_l)^{\frac{-1}{\eta+1}} \\
\tau_l\frac{\eta}{\eta+1}(1-\tau_l)^{\frac{-1}{\eta+1}} &= (1-\tau_l)^{\frac{\eta}{\eta+1}} \\
\tau_l\frac{\eta}{\eta+1} &= (1-\tau_l) \\
\eta\tau_l &= (1-\tau_l)(\eta+1) \\
\eta\tau_l &= \eta + 1 - \eta\tau_l - \tau_l \\
2\eta\tau_l + \tau_l &= \eta + 1 \\
\tau_l &= \frac{\eta+1}{2\eta+1}
\end{aligned}$$

The tax rate that maximizes tax revenue is $\frac{\eta+1}{2\eta+1}$, and solves for the top of the Laffer curve. It is only a function of the parameter η. Using our specification from part (d), the tax rate that maximizes tax revenue is $\frac{1+1}{2(1)+1} = \frac{2}{3}$. Taking a look at our plot of tax revenue where $\eta = 1$, $\psi = 2$, $A = 1$, and $N = 1$, we can see that this is the case.



Part F

There is a lively debate among economists about what an appropriate value for the parameter η is. This parameter is called the Frisch elasticity of labor supply. It is the percentage change in hours worked when wages change by 1% but holding fixed consumption. Based on data for middle aged males, many labor economists believe that η is quite small, e.g., $\eta \approx 0.5$. Macroeconomists, however, often point to changes in participation decisions as evidence for high labor supply elasticities, e.g., early retirement, unemployment, etc. Macroeconomists therefore sometimes use values for η that are quite above 1, e.g., $\eta \approx 3$. Calculate the tax rate that yields maximal tax revenue (i.e., the top of the Laffer curve) for the following three values of η: 0.5, 1, 3.

Solution

We can use the formula we derived in part (e) to calculate the tax rate that maximizes tax revenue for the three values of η.

$$\begin{aligned}
\eta = 0.5: &\qquad \tau_l = \frac{0.5+1}{2(0.5)+1} = \frac{1.5}{2} = 0.75 \\
\eta = 1: &\qquad \tau_l = \frac{1+1}{2(1)+1} = \frac{2}{3} = 0.\overline{666} \\
\eta = 3: &\qquad \tau_l = \frac{3+1}{2(3)+1} = \frac{4}{7} \approx 0.571
\end{aligned}$$

From these three examples, higher values of η lead to lower tax rates that maximize tax revenue. This could be because higher values of η imply that households are more sensitive to changes in wages, and therefore will respond with working less as the tax rate increases.