Macroeconomics: Problem Set #3

Due on October 20, 2024 at 10:00pm

Prof. Barnichon Section 104

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Problem 1: Recovering from a War

Consider the basic Solow model with constant technology and constant population. Recall that the key equations of this model are

(1) $Y_t = \bar{A}K_t^{1/3}\bar{L}^{2/3}$

(2) $I_t = \bar{s}Y_t$

(3) $C_t = Y_t - I_t$

(4) $K_{t+1} = K_t + I_t - \bar{d}K_t$

A) Suppose the economy starts off with a capital stock K_0. Using the Solow diagram, explain how the capital stock will evolve over time.

B) Now starting from a point where none of the exogenous variables have changed for a long period of time, suppose that a war occurs that destroys a large part of the capital stock. Using time series plots (i.e., plots with time on the x-axis and the variable being described on the y-axis), describe the evolution of the capital stock and output due to the war in qualitative terms.

C) Recall that with competitive labor and capital markets the wage rate and the rental rate on capital will be given by

$$\frac{2}{3}\bar{A}\frac{K_t^{1/3}}{\bar{L}^{1/3}} = w_t$$
$$\frac{1}{3}\bar{A}\frac{\bar{L}^{2/3}}{K_t^{2/3}} = r_t$$

Using time series plots, describe the evolution of the wage rate and the rental rate on capital that occur due to the war in qualitative terms.

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Part A

Suppose the economy starts off with a capital stock K_0. Using the Solow diagram, explain how the capital stock will evolve over time.

Solution
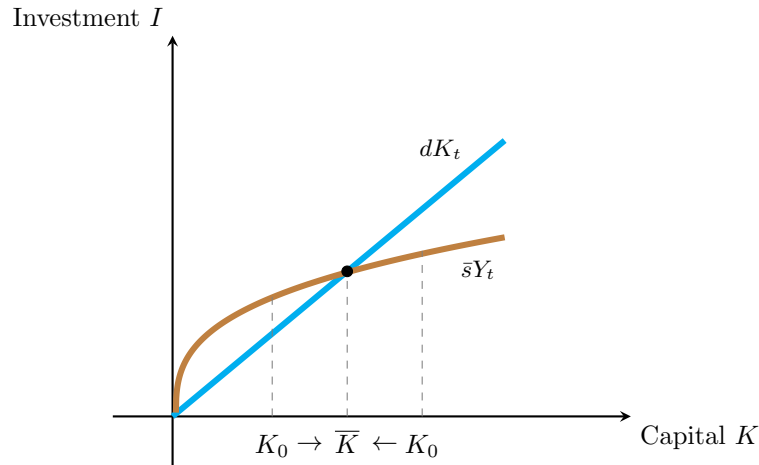
No matter the initial starting conditions of the economy, the capital stock will increase or decrease over time to reach steady state. Steady state is where the investment in capital $\bar{s}Y_t$ is equal to the depreciation of capital $\bar{d}K_t$. At this point the capital stock will remain constant over time. The above Solow diagram shows the direction in which the capital stock will move from K_0.

At steady state $K_{t+1} = K_t = \overline{K}$ and we can use our equations to substitute and solve for $\overline{K}$

$$\overline{K} = \overline{K} + I_t - \overline{dK}$$

$$\text{from (2)} \quad \overline{dK} = \bar{s}\overline{Y}$$

$$\text{from (1)} \quad \overline{dK} = \bar{s}\overline{A}\overline{K}^{1/3}\overline{L}^{2/3}$$

$$\overline{K}^{2/3} = \frac{\bar{s}\overline{A}}{\bar{d}}\overline{L}^{2/3}$$

$$\overline{K} = \left(\frac{\bar{s}\overline{A}}{\bar{d}}\right)^{3/2}\overline{L}$$

The above result confirms that the steady state value of K does not depend on the initial conditions but only on our exogenous model parameters.

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Part B

Now starting from a point where none of the exogenous variables have changed for a long period of time, suppose that a war occurs that destroys a large part of the capital stock. Using time series plots (i.e., plots with time on the x-axis and the variable being described on the y-axis), describe the evolution of the capital stock and output due to the war in qualitative terms.

Solution

If none of the exogenous variables have changed for a long period of time, then output and capital will have reached their equilibrium points. We can solve for the long run, or steady state, when $K_{t+1} = K_t = \overline{K}$. Substituting $\overline{K}$ into (4)

$$\overline{K} = \overline{K} + I_t - \overline{dK}$$
$$\overline{dK} = \overline{s}\overline{Y} \quad \text{from (2)}$$
$$\overline{dK} = \overline{s}\overline{A}\overline{K}^{1/3}\overline{L}^{2/3} \quad \text{from (1)}$$
$$\overline{K}^{2/3} = \frac{\overline{s}\overline{A}}{\overline{d}}\overline{L}^{2/3}$$
$$\overline{K} = \left(\frac{\overline{s}\overline{A}}{\overline{d}}\right)^{3/2}\overline{L}$$

Now, to find the time series plots of Y_t and K_t we need to solve the differential equation $\frac{dK}{dt} = sAK^{1/3}L^{2/3} - dK$. General solutions to this nonlinear differential equation take the form

$$K(t) = \overline{K} - \left(K_0 - \overline{K}\right)e^{-dt}$$

and subsituting in our expression for $\overline{K}$ our plots for output and capital are

$$K(t) = \left(\frac{\overline{s}\overline{A}}{\overline{d}}\right)^{3/2}\overline{L} - \left(K_0 - \left(\frac{\overline{s}\overline{A}}{\overline{d}}\right)^{3/2}\overline{L}\right)e^{-dt}$$
$$Y(t) = \overline{A}(K(t))^{1/3}\overline{L}^{2/3}$$

Part C
Recall that with competitive labor and capital markets the wage rate and the rental rate on capital will be given by

$$\frac{2}{3}\bar{A}\frac{K_t^{1/3}}{\bar{L}^{1/3}} = w_t$$

$$\frac{1}{3}\bar{A}\frac{\bar{L}^{2/3}}{K_t^{2/3}} = r_t$$

Using time series plots, describe the evolution of the wage rate and the rental rate on capital that occur due to the war in qualitative terms.

Solution

First we need to find the long run equilibrium wage rate and rental rate on capital. Substituting $\overline{K}$ into the equations for w_t and r_t we get

$$\overline{w} = \frac{2}{3}\bar{A}\frac{\left(\left(\frac{\overline{sA}}{\overline{d}}\right)^{3/2}\overline{L}\right)^{1/3}}{\bar{L}^{1/3}} = \frac{2}{3}\bar{A}\left(\frac{\overline{sA}}{\overline{d}}\right)^{1/2}$$

$$\overline{r} = \frac{1}{3}\bar{A}\frac{\bar{L}^{2/3}}{\left(\left(\frac{\overline{sA}}{\overline{d}}\right)^{3/2}\overline{L}\right)^{2/3}} = \frac{1}{3}\bar{A}\left(\frac{\overline{d}}{\overline{sA}}\right)$$

We can create the time series plots for the wage rate and the rental rate on capital by substituting in the expression for $K(t)$ into the equations for w_t and r_t

$$w_t = \frac{2}{3}\bar{A}\frac{K_t^{1/3}}{\bar{L}^{1/3}} = \frac{2}{3}\bar{A}\frac{\left(\left(\frac{\overline{sA}}{\overline{d}}\right)^{3/2}\overline{L} - \left(K_0 - \left(\frac{\overline{sA}}{\overline{d}}\right)^{3/2}\overline{L}\right)e^{-dt}\right)^{1/3}}{\bar{L}^{1/3}}$$

$$r_t = \frac{1}{3}\bar{A}\frac{\bar{L}^{2/3}}{K_t^{2/3}} = \frac{1}{3}\bar{A}\frac{\bar{L}^{2/3}}{\left(\left(\frac{\overline{sA}}{\overline{d}}\right)^{3/2}\overline{L} - \left(K_0 - \left(\frac{\overline{sA}}{\overline{d}}\right)^{3/2}\overline{L}\right)e^{-dt}\right)^{2/3}}$$

Problem 2: Misallocation and TFP

One lesson from the Solow model is that the determinants of long-run growth have to be found in total factor productivity A_t. Since A_t is measured as the residual of a growth accounting decomposition, TFP is often referred to as a measure of our ignorance.

A recent insight from the academic literature on economic growth is that TFP can be affected by the allocation of factor inputs. This excercise will introduce you to this idea.

Suppose output is produced using two tasks according to $Y = X_1^\alpha X_2^{1-\alpha}$. The tasks could be management vs. production work, manufacturing vs. services, or private sector work vs. public (e.g., regulatory, judicial, police) work.

One unit of labor can produce one unit of either task, and the economy is endowed with L units of labor. Finally, suppose that the allocation of labor is such that a fraction s of total labor works in the first task, and the fraction $1 - s$ works in the second task.

A) Derive a production function of the form $Y = f(L)$, and derive an expression for TFP of this production function.

B) Draw how TFP depends on the task allocation s (recall $s \in [0, 1]$).

C) What is the output maximizing allocation s^*? What happens to TFP then?

D) In many developing countries, taxes, poor management, information problems, or corruption can lead to a non-optimal allocation of tasks. How can this theory explain that some countries remain poorer than the US?

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Part A
Derive a production function of the form $Y = f(L)$, and derive an expression for TFP of this production function.

Solution
A production function of the form $Y = f(L)$ given the problem specification could be the following

$$Y = (sL)^\alpha((1-s)L)^{1-\alpha}$$
$$Y = s^\alpha(1-s)^{1-\alpha}L$$

Total Factor Productivity is the ratio of output to input, or that portion of growth not explained by the change in inputs. In our case, we can divide output by the change in labor

$$A = \frac{Y}{L} = s^\alpha(1-s)^{1-\alpha}$$

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Part B
Draw how TFP depends on the task allocation s (recall $s \in [0, 1]$).

Solution
My answer... (not sure how to answer this question)

Part C
What is the output maximizing allocation s^*? What happens to TFP then?

Solution
Output is maximized when TFP is maximized. We can find the optimal allocation s^* by taking the derivative of TFP with respect to a change in s and setting the result equal to zero

$$
\begin{aligned}
\frac{dA}{ds} &= \frac{d}{ds}\left(s^\alpha(1-s)^{1-\alpha}\right) \\
&= \alpha s^{\alpha-1}(1-s)^{1-\alpha} - s^\alpha(1-\alpha)(1-s)^{-\alpha} \\
0 &= \alpha s^{\alpha-1}(1-s)^{1-\alpha} - s^\alpha(1-\alpha)(1-s)^{-\alpha} \\
s^\alpha(\alpha-1)(1-s)^{-\alpha} &= \alpha s^{\alpha-1}(1-s)^{1-\alpha} \\
s^\alpha(1-\alpha) &= \alpha s^{\alpha-1}(1-s) \\
s(1-\alpha) &= \alpha(1-s) \\
s - s\alpha &= \alpha - \alpha s \\
s &= \alpha
\end{aligned}
$$

From this we can see the optimal allocation s^* is reached when $s = \alpha$.

Part D
In many developing countries, taxes, poor management, information problems, or corruption can lead to a non-optimal allocation of tasks. How can this theory explain that some countries remain poorer than the US?

Zachary Brandt   Macroeconomics (Prof. Barnichon Section 104): Problem Set #3   Problem 2

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Problem 3: From Land to Fossil Energy

Consider the Malthus model of population growth with $\frac{N_{t+1}}{N_t} = \left(\frac{w_t}{w_s}\right)$.

In the model we saw in class, we had the production function $Y_t = D^\alpha N_t^{1-\alpha}$, where land D was fixed. In that economy, wages are stuck at subsistence levels in the long-run.

But imagine that we discover abundant (underground) fossil fuels so that land is only needed for food, making the land constraint effectively no longer binding so that we can assume that there is always enough land to grow in line with population. Instead, energy becomes a central part of the production process, and the function becomes the function $Y_t = E_t^\gamma L_{y,t}^{1-\gamma}$ with $\gamma < 1$ and $L_{y,t}$ is labor employed in the production of final goods Y.

Extracting energy from the ground requires labor and the production process for energy is $E_t = L_{e,t}$ with $L_{e,t} = sN_t$, where the fraction of the population devoted to energy extraction (s) is fixed. The rest of the population is devoted to production of Y_t, so $L_{y,t} = (1-s)N_t$.

It will be useful to define the term $g = (1-\gamma)\left(\frac{s}{1-s}\right)^\gamma / w_s$. We assume that $g > 1$.

A) Derive the equation for the wage rate w_t in the final goods sector. Given the Malthus population dynamics $\frac{N_{t+1}}{N_t} = \left(\frac{w_t}{w_s}\right)$, what is the population growth rate?

B) What is the economy's growth rate (i.e., the growth rate of Y_t)? What is the per capita growth rate (i.e., the growth rate of Y_t/N_t)?

C) How do workers fare in this economy compared to (i) a Malthusian economy, and (ii) a Solow economy (with constant A_t) that we saw in class?

D) Derive the level of energy extracted at each date t, i.e., derive an expression for E_t as a function of initial conditions N_0 (the population at date 0). Derive the *total* amount of energy extracted since time 0.

E) Fossil energy is in fact in finite supply on Earth. At which date τ will we have exhausted all fossil fuel? Derive an expression for τ as a function of model paramters. What will happen then to growth?

Part A
Derive the equation for the wage rate w_t in the final goods sector. Given the Malthus population dynamics $\frac{N_{t+1}}{N_t} = \left(\frac{w_t}{w_s}\right)$, what is the population growth rate?

Solution
To find the wage rate w_t for labor employed in the final goods sector, we find the marginal product of labor in the production fo these final goods by taking the partial derivative of output with respect to a change in $L_{y,t}$

$$w_t = \frac{\partial Y_t}{\partial L_{y,t}} = \frac{\partial}{\partial L_{y,t}}\left(E_t^{\gamma} L_{y,t}^{1-\gamma}\right)$$
$$= (1-\gamma)L_{e,t}^{\gamma}L_{y,t}^{-\gamma}$$
$$= (1-\gamma)\frac{s^{\gamma}N_t^{\gamma}}{(1-s)^{\gamma}N_t^{\gamma}}$$
$$= (1-\gamma)\left(\frac{s}{1-s}\right)^{\gamma}$$

Given this expression for our w_t, we can find the population growth rate by subsitution

$$\frac{N_{t+1}}{N_t} = \left(\frac{w_t}{w_s}\right)$$
$$\frac{N_{t+1}}{N_t} = \frac{(1-\gamma)\left(\frac{s}{1-s}\right)^{\gamma}}{w_s}$$
$$\frac{N_{t+1}}{N_t} = g$$

So our population growth rate in this economy becomes our term g defined earler in the problem specification. The population as a function of time t is then $N_t = N_0 g^t$ since

$$N_{t+1} = N_t g$$
$$t=0 \quad N_1 = N_0 g$$
$$t=1 \quad N_2 = N_1 g = N_0 g^2$$
$$t=2 \quad N_3 = N_2 g = N_0 g^3$$
$$\dots$$
$$N_t = N_0 g^t$$

Therefore, the first time derivative of N_t, $\frac{dN_t}{dt}$, is the following

$$\frac{dN_t}{dt} = \frac{d}{dt}\left(N_0 e^{t\ln(g)}\right) = \ln(g)N_0 e^{t\ln(g)} = \ln(g)N_t$$

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Part B
What is the economy's growth rate (i.e., the growth rate of Y_t)? What is the per capita growth rate (i.e., the growth rate of Y_t/N_t)?

Solution
To find the economy's growth rate, we find the change in output with respect to a change in time using the first time derivative of Y_t. First, I will simplify the expression for output so that it is only in terms of s, γ, and N_t

$$
\begin{aligned}
Y_t &= E_t^{\gamma} L_{y,t}^{1-\gamma} \\
&= L_{e,t}^{\gamma} L_{y,t}^{1-\gamma} \\
&= (sN_t)^{\gamma}((1-s)N_t)^{1-\gamma} \\
&= s^{\gamma}(1-s)^{1-\gamma} N_t
\end{aligned}
$$

Now to find the growth rate I'll differentiate this expression with respect to a change in time

$$
\begin{aligned}
\frac{dY_t}{dt} &= \frac{d}{dt}\left(s^{\gamma}(1-s)^{1-\gamma} N_t\right) \\
&= s^{\gamma}(1-s)^{1-\gamma} \frac{dN_t}{dt} \\
&= s^{\gamma}(1-s)^{1-\gamma} ln(g) N_t \\
&= ln(g) Y_t
\end{aligned}
$$

From the above expression we can see that the economy's growth rate is proportional to the population growth rate g. To find the per capita growth rate, I'll first divide the expression for output Y_t by population N_t and take the derivative with respect to time

$$
\begin{aligned}
y_t &= \frac{Y_t}{N_t} = s^{\gamma}(1-s)^{1-\gamma} \\
\frac{dy_t}{dt} &= \frac{d}{dt}\left(s^{\gamma}(1-s)^{1-\gamma}\right) \\
\frac{dy_t}{dt} &= 0
\end{aligned}
$$

Since neither s or γ are functions of time, there is no per capita growth rate in this economy.

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Part C
How do workers fare in this economy compared to (i) a Malthusian economy, and (ii) a Solow economy (with constant A_t) that we saw in class?

Solution
Compared to a malthusian economy.... Just like in a Solow economy with constant TFP, there is no per capita growth in this economy

Part D
Derive the level of energy extracted at each date t, i.e., derive an expression for E_t as a function of initial conditions N_0 (the population at date 0). Derive the *total* amount of energy extracted since time 0.

Solution
To derive the level of energy extracted E_t as a function of initial conditions N_0, we can substitute in our expression for N_t in terms of N_0, g, and t

$$E_t = L_{e,t} = sN_t = sN_0g^t$$

To derive the total amount of energy extracted since $t = 0$, we can perform an integral to sum all the infinitesimal changes in energy with respect to a change in time

$$\int_0^t E_t \, dt = \int_0^t sN_0g^t \, dt$$
$$= sN_0 \int_0^t e^{tln(g)} \, dt$$
$$= \frac{sN_0}{ln(g)} \left(e^{tln(g)} - 1 \right)$$
$$= \frac{sN_0}{ln(g)} \left(g^t - 1 \right)$$

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Part E
Fossil energy is in fact in finite supply on Earth. At which date τ will we have exhausted all fossil fuel? Derive an expression for τ as a function of model paramters. What will happen then to growth?

Solution
We can find the time τ when all fossil fuels reserves have been exhausted when $\int_0^\tau E_t \, dt$ is equal to some E_{total}, which encapsulates the Earth's fossil fuel reserves.

$$E_{\text{total}} = \int_0^\tau E_t \, dt$$

$$E_{\text{total}} = \frac{sN_0}{ln(g)} \left(g^\tau - 1 \right)$$

$$\frac{ln(g)}{sN_0} E_{\text{total}} = g^\tau - 1$$

$$g^\tau = \frac{ln(g)}{sN_0} E_{\text{total}} + 1$$

$$\tau ln(g) = ln \left(\frac{ln(g)}{sN_0} E_{\text{total}} + 1 \right)$$

$$\tau = \frac{ln \left(\frac{ln(g)}{sN_0} E_{\text{total}} + 1 \right)}{ln(g)}$$

For times where $t > \tau$, output will be drastically decreased as $E_t = 0$ and should revert back to our Malthusian production model where land once again becomes a binding constraint on output.