Macroeconomics: Problem Set #3

Due on October 20, 2024 at 10:00pm

Prof. Barnichon Section 104

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Problem 1: Recovering from a War

Consider the basic Solow model with constant technology and constant population. Recall that the key equations of this model are

(1) $Y_t = \bar{A}K_t^{1/3}\bar{L}^{2/3}$

(2) $I_t = \bar{s}Y_t$

(3) $C_t = Y_t - I_t$

(4) $K_{t+1} = K_t + I_t - \bar{d}K_t$

A) Suppose the economy starts off with a capital stock K_0. Using the Solow diagram, explain how the capital stock will evolve over time.

B) Now starting from a point where none of the exogenous variables have changed for a long period of time, suppose that a war occurs that destroys a large part of the capital stock. Using time series plots (i.e., plots with time on the x-axis and the variable being described on the y-axis), describe the evolution of the capital stock and output due to the war in qualitative terms.

C) Recall that with competitive labor and capital markets the wage rate and the rental rate on capital will be given by

$$\frac{2}{3}\bar{A}\frac{K_t^{1/3}}{\bar{L}^{1/3}} = w_t$$
$$\frac{1}{3}\bar{A}\frac{\bar{L}^{2/3}}{K_t^{2/3}} = r_t$$

Using time series plots, describe the evolution of the wage rate and the rental rate on capital that occur due to the war in qualitative terms.

Problem 1 continued on next page...

Part A

Suppose the economy starts off with a capital stock K_0. Using the Solow diagram, explain how the capital stock will evolve over time.

Solution

No matter the initial starting conditions of the economy, the capital stock will increase or decrease over time to reach steady state. Steady state is where the investment in capital $\bar{s}Y_t$ is equal to the depreciation of capital $\bar{d}K_t$. At this point the capital stock will remain constant over time. The above Solow diagram shows the direction in which the capital stock will move from K_0.

At steady state $K_{t+1} = K_t = \overline{K}$ and we can use our equations to substitute and solve for $\overline{K}$

$$\begin{aligned}
\overline{K} &= \overline{K} + I_t - \overline{dK} \\
\text{from (2)} \quad \overline{dK} &= \bar{s}\overline{Y} \\
\text{from (1)} \quad \overline{dK} &= \bar{s}\overline{AK}^{1/3}\overline{L}^{2/3} \\
\overline{K}^{2/3} &= \frac{\bar{s}\overline{A}}{\bar{d}}\overline{L}^{2/3} \\
\overline{K} &= \left(\frac{\bar{s}\overline{A}}{\bar{d}}\right)^{3/2}\overline{L}
\end{aligned}$$

The above result confirms that the steady state value of K does not depend on the initial conditions but only on our exogenous model parameters.

Part B

Now starting from a point where none of the exogenous variables have changed for a long period of time, suppose that a war occurs that destroys a large part of the capital stock. Using time series plots (i.e., plots with time on the x-axis and the variable being described on the y-axis), describe the evolution of the capital stock and output due to the war in qualitative terms.

Solution

If none of the exogenous variables have changed for a long period of time, then output and capital will have reached their steady state values of $\overline{Y}$ and $\overline{K}$ respectively. Due to the war that destroys a large part of the capital stock, output and capital will fall briefly to Y_0 and K_0 respectively. Over time, if none of the model parameters have changed, the economy will recover and output and capital will return to their steady state values.

To find these time series plots of Y_t and K_t we need to solve the differential equation $\frac{dK}{dt} = sAK^{1/3}L^{2/3} - dK$ from (4). General solutions to this nonlinear differential equation take the form

$$K_t = \overline{K} - \left(K_0 - \overline{K}\right) e^{-dt}$$

and subsituting in our expression for $\overline{K}$ our time series plots for output and capital are

$$\begin{aligned} K_t &= \left(\frac{\overline{s}\overline{A}}{\overline{d}}\right)^{3/2} \overline{L} - \left(K_0 - \left(\frac{\overline{s}\overline{A}}{\overline{d}}\right)^{3/2} \overline{L}\right) e^{-dt} \\ Y_t &= \overline{A} K_t^{1/3} \overline{L}^{2/3} \quad \text{from (1)} \end{aligned}$$

Two potential plots for Y_t and K_t are shown below. The first plot is of K_t and the second plot is of Y_t. While the exact shape of these plots will depend on the initial conditions and the model parameters, the general shape of the recovery should look the same.

From the above, we can see that the capital stock and output fall sharply to K_0 and Y_0 respectively due to the war. Over time, the economy will recover and both capital and output will approach their steady states as time goes to infinity.

Problem 1 continued on next page. . .

Part C

Recall that with competitive labor and capital markets the wage rate and the rental rate on capital will be given by

$$\frac{2}{3}\bar{A}\frac{K_t^{1/3}}{\bar{L}^{1/3}} = w_t$$
$$\frac{1}{3}\bar{A}\frac{\bar{L}^{2/3}}{K_t^{2/3}} = r_t$$

Using time series plots, describe the evolution of the wage rate and the rental rate on capital that occur due to the war in qualitative terms.

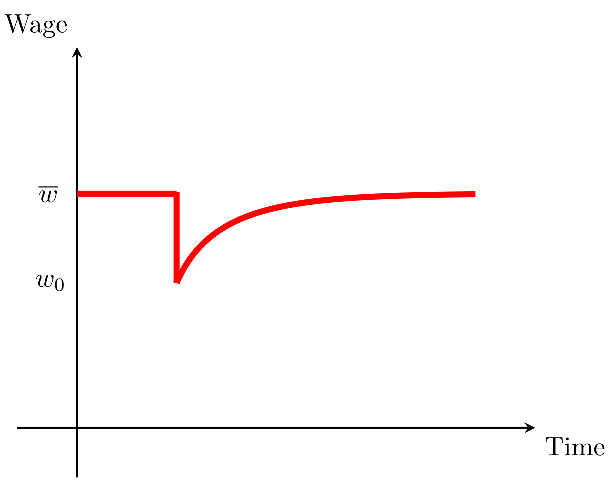
Solution

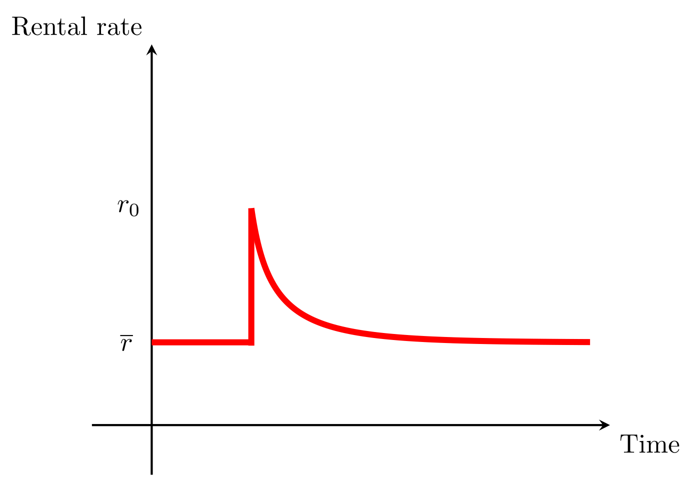
First we need to find the long run equilibrium wage rate and rental rate on capital. Substituting our expression for $\overline{K}$ into the above equations for w_t and r_t we get

$$\overline{w} = \frac{2}{3}\bar{A}\frac{\left(\left(\frac{\bar{s}\overline{A}}{\bar{d}}\right)^{3/2}\overline{L}\right)^{1/3}}{\bar{L}^{1/3}} = \frac{2}{3}\bar{A}\left(\frac{\bar{s}\overline{A}}{\bar{d}}\right)^{1/2}$$

$$\overline{r} = \frac{1}{3}\bar{A}\frac{\bar{L}^{2/3}}{\left(\left(\frac{\bar{s}\overline{A}}{\bar{d}}\right)^{3/2}\overline{L}\right)^{2/3}} = \frac{1}{3}\bar{A}\left(\frac{\bar{d}}{\bar{s}\overline{A}}\right)$$

We can create the time series plots for the wage rate and the rental rate on capital by substituting in our time series expression for K_t from B) into the equations for w_t and r_t and plotting. Two possible plots for the wage rate and rental rate on capital respectively are shown below.





From the above plots we can see that the wage rate will fall sharply to w_0 and the rental rate will spike to r_0 respectively due to the war. This makes sense considering the marginal product of labor depends directly on the capital stock and the marginal product of capital depends inversely on the capital stock. Over time, the wage rate and rental rate will recover to their steady state values as time approaches infinity.

Problem 2: Misallocation and TFP

One lesson from the Solow model is that the determinants of long-run growth have to be found in total factor productivity A_t. Since A_t is measured as the residual of a growth accounting decomposition, TFP is often referred to as a measure of our ignorance.

A recent insight from the academic literature on economic growth is that TFP can be affected by the allocation of factor inputs. This excercise will introduce you to this idea.

Suppose output is produced using two tasks according to $Y = X_1^{\alpha} X_2^{1-\alpha}$. The tasks could be management vs. production work, manufacturing vs. services, or private sector work vs. public (e.g., regulatory, judicial, police) work.

One unit of labor can produce one unit of either task, and the economy is endowed with L units of labor. Finally, suppose that the allocation of labor is such that a fraction s of total labor works in the first task, and the fraction $1 - s$ works in the second task.

A) Derive a production function of the form $Y = f(L)$, and derive an expression for TFP of this production function.

B) Draw how TFP depends on the task allocation s (recall $s \in [0, 1]$).

C) What is the output maximizing allocation s^*? What happens to TFP then?

D) In many developing countries, taxes, poor management, information problems, or corruption can lead to a non-optimal allocation of tasks. How can this 'theory explain that some countries remain poorer than the US?

Problem 2 continued on next page. . .

Part A

Derive a production function of the form $Y = f(L)$, and derive an expression for TFP of this production function.

Solution

A production function of the form $Y = f(L)$ given in the problem specification could be the following

$$Y = (sL)^\alpha((1-s)L)^{1-\alpha}$$
$$Y = s^\alpha(1-s)^{1-\alpha}L$$

The production function is broken into two tasks, X_1 and X_2, where X_1 is the task that requires s units of labor and X_2 is the task that requires $(1-s)$ units of labor. The total output Y is the product of the output of each task.

Total Factor Productivity is the ratio of output to input, or that portion of growth not explained by the change in inputs. In our case, we can divide output by the change in the labor input to find TFP.

$$A = \frac{Y}{L} = s^\alpha(1-s)^{1-\alpha}$$

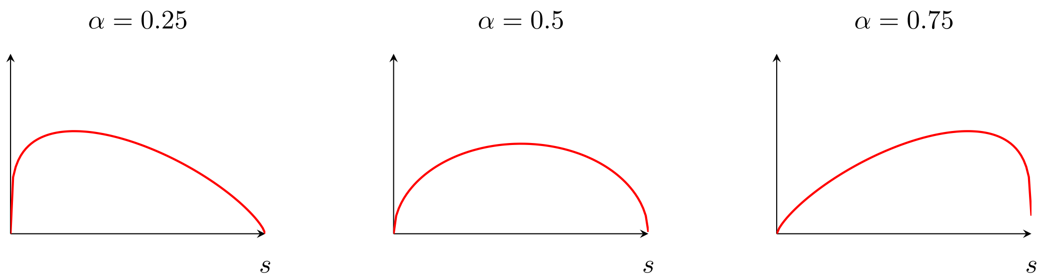
Problem 2 continued on next page. . .

Part B

Draw how TFP depends on the task allocation s (recall $s \in [0, 1]$).

Solution

Below I have plotted three different TFP functions for different values of α. The first plot is for $\alpha = 0.25$, the second plot is for $\alpha = 0.5$, and the third plot is for $\alpha = 0.75$. The general shape of the TFP function is a bell curve with a maximum at $s = \alpha$.



From the above plots we can see that TFP is maximized at some value of s between 0 and 1. The exact value of s that maximizes TFP is α, which is shown in the next part.

Part C

What is the output maximizing allocation s^*? What happens to TFP then?

Solution

Output in this case is maximized when TFP is maximized. We can find the optimal allocation s^* by taking the derivative of TFP with respect to a change in s and setting the result equal to zero

$$\begin{aligned}
\frac{dA}{ds} &= \frac{d}{ds}\left(s^\alpha(1-s)^{1-\alpha}\right) \\
&= \alpha s^{\alpha-1}(1-s)^{1-\alpha} - s^\alpha(1-\alpha)(1-s)^{-\alpha} \\
0 &= \alpha s^{\alpha-1}(1-s)^{1-\alpha} - s^\alpha(1-\alpha)(1-s)^{-\alpha} \\
s^\alpha(\alpha-1)(1-s)^{-\alpha} &= \alpha s^{\alpha-1}(1-s)^{1-\alpha} \\
s^\alpha(1-\alpha) &= \alpha s^{\alpha-1}(1-s) \\
s(1-\alpha) &= \alpha(1-s) \\
s - s\alpha &= \alpha - \alpha s \\
s &= \alpha
\end{aligned}$$

From this we can see the optimal allocation s^* is reached when $s = \alpha$. At this point, TFP is maximized and the economy is producing output at its maximum level given a fixed amount of labor.

Problem 2 continued on next page. . .

Part D

In many developing countries, taxes, poor management, information problems, or corruption can lead to a non-optimal allocation of tasks. How can this theory explain that some countries remain poorer than the US?

Solution

If in these developing countries there are factors that prohibit an efficient task allocation s, then TFP will be lower than it could be. Considering that under the Solow model there is only long run per capita growth with increasing TFP, these countries will remain poorer than the US if there allocation of labor remains inefficient.

Problem 3: From Land to Fossil Energy

Consider the Malthus model of population growth with $\frac{N_{t+1}}{N_t} = (\frac{w_t}{w_s})$.

In the model we saw in class, we had the production function $Y_t = D^\alpha N_t^{1-\alpha}$, where land D was fixed. In that economy, wages are stuck at subsistence levels in the long-run.

But imagine that we discover abundant (underground) fossil fuels so that land is only needed for food, making the land constraint effectively no longer binding so that we can assume that there is always enough land to grow in line with population. Instead, energy becomes a central part of the production process, and the function becomes the function $Y_t = E_t^\gamma L_{y,t}^{1-\gamma}$ with $\gamma < 1$ and $L_{y,t}$ is labor employed in the production of final goods Y.

Extracting energy from the ground requires labor and the production process for energy is $E_t = L_{e,t}$ with $L_{e,t} = sN_t$, where the fraction of the population devoted to energy extraction (s) is fixed. The rest of the population is devoted to production of Y_t, so $L_{y,t} = (1-s)N_t$.

It will be useful to define the term $g = (1-\gamma)(\frac{s}{1-s})^\gamma / w_s$. We assume that $g > 1$.

A) Derive the equation for the wage rate w_t in the final goods sector. Given the Malthus population dynamics $\frac{N_{t+1}}{N_t} = (\frac{w_t}{w_s})$, what is the population growth rate?

B) What is the economy's growth rate (i.e., the growth rate of Y_t)? What is the per capita growth rate (i.e., the growth rate of Y_t/N_t)?

C) How do workers fare in this economy compared to (i) a Malthusian economy, and (ii) a Solow economy (with constant A_t) that we saw in class?

D) Derive the level of energy extracted at each date t, i.e., derive an expression for E_t as a function of initial conditions N_0 (the population at date 0). Derive the *total* amount of energy extracted since time 0.

E) Fossil energy is in fact in finite supply on Earth. At which date τ will we have exhausted all fossil fuel? Derive an expression for τ as a function of model paramters. What will happen then to growth?

Problem 3 continued on next page. . .

Part A

Derive the equation for the wage rate w_t in the final goods sector. Given the Malthus population dynamics $\frac{N_{t+1}}{N_t} = (\frac{w_t}{w_s})$, what is the population growth rate?

Solution

To find the wage rate w_t for labor employed in the final goods sector, we find the marginal product of labor in the production fo these final goods by taking the partial derivative of output with respect to a change in $L_{y,t}$

$$\begin{aligned} w_t = \frac{\partial Y_t}{\partial L_{y,t}} &= \frac{\partial}{\partial L_{y,t}} \left(E_t^\gamma L_{y,t}^{1-\gamma} \right) \\ &= (1-\gamma) L_{e,t}^\gamma L_{y,t}^{-\gamma} \\ &= (1-\gamma) \frac{s^\gamma N_t^\gamma}{(1-s)^\gamma N_t^\gamma} \\ &= (1-\gamma) \left(\frac{s}{1-s} \right)^\gamma \end{aligned}$$

Given this expression for our w_t, we can find the population growth rate by subsitution

$$\begin{aligned} \frac{N_{t+1}}{N_t} &= \left(\frac{w_t}{w_s} \right) \\ \frac{N_{t+1}}{N_t} &= \frac{(1-\gamma)\left(\frac{s}{1-s}\right)^\gamma}{w_s} \\ \frac{N_{t+1}}{N_t} &= g \end{aligned}$$

So our population growth rate in this economy becomes our term g defined earler in the problem specification. The population as a function of time t is then $N_t = N_0 g^t$ since

$$\begin{aligned} & N_{t+1} = N_t g \\ t = 0 \quad & N_1 = N_0 g \\ t = 1 \quad & N_2 = N_1 g = N_0 g^2 \\ t = 2 \quad & N_3 = N_2 g = N_0 g^3 \\ & \dots \\ & N_t = N_0 g^t \end{aligned}$$

Therefore, the first time derivative of N_t, $\frac{dN_t}{dt}$, is the following

$$\frac{dN_t}{dt} = \frac{d}{dt} \left(N_0 e^{tln(g)} \right) = ln(g) N_0 e^{tln(g)} = ln(g) N_t$$

Part B

What is the economy's growth rate (i.e., the growth rate of Y_t)? What is the per capita growth rate (i.e., the growth rate of Y_t/N_t)?

Solution

To find the economy's growth rate, we find the change in output with respect to a change in time using the first time derivative of Y_t. First, I will simplify the expression for output so that it is only in terms of s, γ, and N_t

$$\begin{aligned}
Y_t &= E_t^{\gamma} L_{y,t}^{1-\gamma} \\
&= L_{e,t}^{\gamma} L_{y,t}^{1-\gamma} \\
&= (sN_t)^{\gamma}((1-s)N_t)^{1-\gamma} \\
&= s^{\gamma}(1-s)^{1-\gamma} N_t
\end{aligned}$$

Now to find the growth rate I'll differentiate this expression with respect to a change in time

$$\begin{aligned}
\frac{dY_t}{dt} &= \frac{d}{dt}\left(s^{\gamma}(1-s)^{1-\gamma}N_t\right) \\
&= s^{\gamma}(1-s)^{1-\gamma}\frac{dN_t}{dt} \\
&= s^{\gamma}(1-s)^{1-\gamma} ln(g) N_t \\
&= ln(g) Y_t
\end{aligned}$$

From the above expression we can see that the economy's growth rate is proportional to the population growth rate g. To find the per capita growth rate, I'll first divide the expression for output Y_t by population N_t and take the derivative with respect to time

$$\begin{aligned}
y_t &= \frac{Y_t}{N_t} = s^{\gamma}(1-s)^{1-\gamma} \\
\frac{dy_t}{dt} &= \frac{d}{dt}\left(s^{\gamma}(1-s)^{1-\gamma}\right) \\
\frac{dy_t}{dt} &= 0
\end{aligned}$$

Since neither s or γ are functions of time, there is no per capita growth rate in this economy.

Problem 3 continued on next page...

Part C

How do workers fare in this economy compared to (i) a Malthusian economy, and (ii) a Solow economy (with constant A_t) that we saw in class?

Solution

Compared to a malthusian economy.... Just like in a Solow economy with constant TFP, there is no per capita growth in this economy

Part D

Derive the level of energy extracted at each date t, i.e., derive an expression for E_t as a function of initial conditions N_0 (the population at date 0). Derive the *total* amount of energy extracted since time 0.

Solution

To derive the level of energy extracted E_t as a function of initial conditions N_0, we can substitute in our expression for N_t in terms of N_0, g, and t

$$E_t = L_{e,t} = sN_t = sN_0 g^t$$

To derive the total amount of energy extracted since $t = 0$, we can perform an integral to sum all the infinitesimal changes in energy with respect to a change in time

$$\begin{aligned}
\int_0^t E_t \, dt &= \int_0^t sN_0 g^t \, dt \\
&= sN_0 \int_0^t e^{tln(g)} \, dt \\
&= \frac{sN_0}{ln(g)} \left(e^{tln(g)} - 1 \right) \\
&= \frac{sN_0}{ln(g)} \left(g^t - 1 \right)
\end{aligned}$$

Problem 3 continued on next page. . .

Part E

Fossil energy is in fact in finite supply on Earth. At which date τ will we have exhausted all fossil fuel? Derive an expression for τ as a function of model paramters. What will happen then to growth?

Solution

We can find the time τ when all fossil fuels reserves have been exhausted when $\int_0^\tau E_t\,dt$ is equal to some E_{total}, which encapsulates the Earth's fossil fuel reserves.

$$\begin{aligned}
E_{\text{total}} &= \int_0^\tau E_t\,dt \\
E_{\text{total}} &= \frac{sN_0}{ln(g)}\left(g^\tau - 1\right) \\
\frac{ln(g)}{sN_0}E_{\text{total}} &= g^\tau - 1 \\
g^\tau &= \frac{ln(g)}{sN_0}E_{\text{total}} + 1 \\
\tau ln(g) &= ln\left(\frac{ln(g)}{sN_0}E_{\text{total}} + 1\right) \\
\tau &= \frac{ln\left(\frac{ln(g)}{sN_0}E_{\text{total}} + 1\right)}{ln(g)}
\end{aligned}$$

For times where $t > \tau$, output will be drastically decreased as $E_t = 0$ and should revert back to our Malthusian production model where land once again becomes a binding constraint on output.