ECON C103: Problem Set #1

Due on January 30, 2026 at 11:59pm

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Problem 1

A group of people met and some of them shook each-other's hands. Prove that the number of people who shook others' hands an odd number of times is, in fact, even.

Solution

Proof. This situtation can be represented in the form of a graph, where people are vertices and handshakes are edges. The degree of any vertex corresponds to the number of handshakes a person made. The sum of the degrees of all vertices add up to twice the number of edges there are in the graph. This is because each new edge increases the degrees of two vertices, since an edge must connect two vertices.

$$\sum_{v \in V} \deg v = 2|E|$$

Since the right-hand side of the equation, two times the number of vertices, is a multiple of two, the number is even. The left-hand side sum of the above equation can be decomposed into a sum over all the vertices with odd degrees and a sum over all the vertices with even degrees.

$$\sum_{v_{\mathrm{odd}} \in V} \deg v + \sum_{v_{\mathrm{even}} \in V} \deg v = 2|E|$$

The sum of degrees over all even-degree vertices is naturally even. The sum of degrees over all odd-degree vertices must then also be even since $2|E|$ is an even number. For $\sum \deg v_{\mathrm{odd}}$ to be even, there must be an even number of odd-degree vertices. Since this formulation corresponds one-to-one with people and handshakes, this shows that the number of people in a group who shake others' hands an odd number of times is even. □

Problem 2

Let X be a nonempty set, and let 2^X denote the set of all subsets of X. Show that there is no onto (aka surjective) function $f : X \mapsto 2^X$. (The function f is surjective if for every set $A \in 2^X$ there is an $x \in X$ such that $f(x) = A$.)

Solution

Proof. Assume there exists an onto function that maps elements in X to those in 2^X. There then must be a set S in 2^X that contains those elements in X that are not in their corresponding subsets in 2^X.

$$S = \{x \in X : x \notin f(x)\}$$

Continuing from the assumption, there then must be some element s in X where $f(s) = S$. s must either be in S or not in S. If s is in S, then that violates the restriction that $s \notin f(s)$, and therefore s cannot be in S since $f(s) = S$. If s is not in S, that then means that $s \notin f(s)$, which is what defines elements in S. So there is a contradiction leading from the original assumption, and therefore there does not exist an onto function $f : X \mapsto 2^X$. $\square$

Problem 3

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a twice-differentiable function. Show that $\frac{\partial^2}{\partial x \partial y} f \geq 0$ if and only if for every $x, x', y, y' \in \mathbb{R}$:

$$x' > x \text{ and } y' > y \Rightarrow f(x', y') - f(x, y') \geq f(x', y) - f(x, y).$$

Solution

Problem 4

A pair $G = (V, E)$ is a *directed graph* if V is a set and $E \subset V \times V$. The graph G has a cycle if there exists an integer $n \geq 1$ and elements $v_1, \dots, v_n \in V$ such that $(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1) \in E$. Show that if V is a non-empty finite set and for all $v \in V$ there is $v' \in V$ such that $(v, v') \in E$, then G has a cycle.

Solution

Problem 5

Consider an $n \times n$ chessboard (the figure below shows an example with $n = 8$).

Show that if n is a power of 2 it is possible to cover the board with L-shaped trominoes (consisting of three squares each, shown in blue) so that all but one of the squares on the board is covered.

Solution