ECON C103: Problem Set #2

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Problem 1

Prove Nash's theorem that every finite game (meaning a finite set of players and a finite set of actions) has at least one Nash equilibrium, possibly in mixed strategies (Hint: follow the steps in the notes).

Solution

I'll begin the proof by first establishing some preliminaries about the finite game.

- There are N players.
- Given an action profile a, let player i's payoff be $\pi_i(a)$
- A mixed strategy, σ_i, for player i is a lottery over A_i, the choices available to the player. Let $\Delta(A_i)$ be the set of such lotteries, and let $\Sigma = \Pi_{i=1}^{N}\Delta(A_i)$ be the set of strategy profiles.
- Given a profile of mixed strategies σ_{-i}, player i's expected utility from action $a \in A_i$ is

$$U_i(a) = \mathbb{E}_{\sigma_{-i}}[\pi_i(a, a_{-i})]$$

- Let $BR_i(\sigma_{-i}) \subset \Delta(A_i)$ be the set of mixed strategies for i that are best responses to σ_{-i}, where $\sigma_i \in BR_i(\sigma_{-i})$ if and only if $U_i(a) \geq U_i(a')$ for all a in the support of σ_i, i.e., those actions played with non-zero probability, and all $a' \in A_i$.

A correspondence $B : \Sigma \rightrightarrows \Sigma$ is

$$B(\sigma) = \Pi_{i=1}^{N} BR_i(\sigma_{-i})$$

A Nash equilibrium is just a strategy profile σ such that $\sigma \in B(\sigma)$. That is, I'll show that there exists a fixed point of the mapping B, i.e., a $\sigma \in B(\sigma)$. To prove this, I'll show that the best-response correspondence B satisfies the conditions of Kakutani's fixed-point theorem:

1. That Σ is compact, convex, and non-empty.
2. That $B(\sigma)$ is non-empty for every $\sigma \in \Sigma$.
3. That $B(\sigma)$ is convex-valued, i.e., for all σ, $B(\sigma)$ is a convex set.
4. That B has a closed graph.

Part One

Since $\Sigma = \Pi_{i=1}^{N}\Delta(A_i)$, and each A_i is finite, each $\Delta(A_i)$ is the set of all probability distributions over the actions in A_i, i.e.,

$$\Delta(A_i) = \{p \in \mathbb{R}^{|A_i|} : p_a \geq 0, \sum_a p_a = 1\}$$

which is closed and bounded in $\mathbb{R}^{|\mathbb{A}_\beth|}$. The set is convex since every element is a linear combination of the choices in A_i where the coefficients are between 0 and 1. The product of compact sets is compact, so Σ is compact. The product of convex sets is convex, so Σ is convex. The product of non-empty sets is non-empty, so Σ is non-empty. □

Part Two

By definition, $BR_i(\sigma_{-i}) = \text{argmax}\{U_i(\sigma_i, \sigma_{-i}) | \sigma_i \in \Delta(A_i)\}$. By the Weierstrass theorem, since $\Delta(A_i)$ is non-empty and compact, and the utility function is continuous in σ_i, there exists an optimal solution to the optimization problem that defines the best-response function, and therefore $B(\sigma)$ is non-empty for every $\sigma \in \Sigma$ as a product of non-empty sets. $\square$

Part Three

For $B(\sigma)$ to be convex-valued, the $BR_i(\sigma_{-i})$ must be shown to be convex. Let $\sigma_i, \sigma_i' \in BR_i(\sigma_{-i})$ and let $\lambda \in [0,1]$. Consider the mixed strategy $\sigma_i'' = \lambda\sigma_i + (1-\lambda)\sigma_i'$. Since expected utility is linear in the mixing probabilities, for any action $a \in A_i$,

$$U_i(\sigma_i'', \sigma_{-i}) = \lambda U_i(\sigma_i, \sigma_{-i}) + (1-\lambda)U_i(\sigma_i', \sigma_{-i})$$

Since both σ_i and σ_i' are best responses, they yield the same expected utility (the maximum possible). Therefore, σ_i'' also yields this maximum expected utility, so $\sigma_i'' \in BR_i(\sigma_{-i})$. Therefore, $BR_i(\sigma_{-i})$ is convex for each i, and $B(\sigma) = \Pi_{i=1}^N BR_i(\sigma_{-i})$ is convex as the product of convex sets. $\square$

Part Four

For the graph of B to be closed, $\{(\sigma, \sigma') : \sigma' \in B(\sigma)\}$, must be a closed subset of $\Sigma \times \Sigma$. Consider a sequence $\{(\sigma^n, \sigma'^n)\}$ in the graph of B such that $(\sigma^n, \sigma'^n) \to (\sigma, \sigma')$. I'll show that $\sigma' \in B(\sigma)$.

Since $\sigma'^n \in B(\sigma^n)$, for each player i, we have $\sigma_i'^n \in BR_i(\sigma_{-i}^n)$. This means that for all actions a in the support of $\sigma_i'^n$ and all $a' \in A_i$,

$$U_i(a, \sigma_{-i}^n) \geq U_i(a', \sigma_{-i}^n)$$

Taking limits as $n \to \infty$, and using the continuity of expected utilities in the strategy profile (which holds because payoffs are finite and the action space is finite), the inequality becomes

$$U_i(a, \sigma_{-i}) \geq U_i(a', \sigma_{-i})$$

for all actions a in the support of σ_i' and all $a' \in A_i$. This means $\sigma_i' \in BR_i(\sigma_{-i})$ for each i, so $\sigma' \in B(\sigma)$. Therefore, the graph of B is closed. $\square$

Since all four conditions of Kakutani's fixed-point theorem are satisfied, there exists a fixed point $\sigma^* \in B(\sigma^*)$, which is precisely a Nash equilibrium. This completes the proof of Nash's theorem.

Problem 2

This is a question about grading and course design. There are N students in a course. Assume for simplicity that an instructor can give each student one of two grades, A or B.

Say that the class is *curved* if there is a number $x \in \{1, 2, \dots, N\}$ such that the instructor must give A's to x students, and B's to the remaining $N - x$ students.

(a) Suppose the grade in the class is based on homework and exams. To be precise there is a weight $\alpha \in (0, 1)$ such the score of student i, denoted by s_i, is given by

$$s_i = \alpha h_i + (1 - \alpha) e_i,$$

where h_i is student i's homework score, and e_i is their exam score. If the class is curved, the x students with the highest grade get A's, and the rest get B's (throughout the question you can assume that there are no ties). Fixing the exam scores, suppose the instructor grades homework more leniently. This means that for some $\delta > 1$, each student's grade goes from h_i to δh_i. What can you say about how the old scores compare to the new scores? What about the new grades? Which students benefit and which are hurt? Be as precise as you can, and prove your answer.

Now, assume instead that each student's score in the course is determined by the amount of time they have to study for the course, t_i, and difficulty of the course, d, so $s_i = s(t_i, d)$ for some function $s : \mathbb{R}^2_+ \to \mathbb{R}_+$ which is increasing in its first argument (time) and decreasing in the second argument (difficulty). [1]

Students like getting A's more than B's, but they also like a class that is not too easy, nor too hard (if the class is to easy you don't learn anything new, and if it's too hard you don't understand anything). We can represent student i's preferences over difficulty levels and grades by a utility function

$$U_i(G, d) = u(G) - (d - d_i)^2$$

which describes their payoff from getting grade $G \in \{A, B\}$ when the class has difficulty level $d \in \mathbb{R}_+$. The parameter d_i represents the ideal difficulty level for student i, if they had no concern about grades (make sure you understand why this is the case).

The instructor wants to choose the difficulty level efficiently, to maximize the sum of student utilities.

(b) If the class is curved, what is the efficient choice of d?

In reality the instructor does not observe $(t_i, d_i)_{i=1}^N$. Instead, each student privately knows their own t_i and d_i. In order to set the level of difficulty for the course the instructor has to run a survey to learn about $\langle d_i \rangle_{i=1}^N$. The instructor is worried about students' incentives to report truthfully.

Assume that t_i and d_i are fixed for each student, and will remain the same regardless of what difficulty level is chosen.[2] However students can report whatever t_i, d_i they want to the instructor.

(c) Suppose that the class is curved as in 2(b). If the instructor wants to implement the efficient choice of d, is it a dominant strategy for all students to report truthfully? If not, in which direction would students like to misreport? Prove your answer.

(d) Now assume, as before, that the class is curved. Assume also that $U_i(G, d) = u(G) - |d - d_i|$. Prove that it is indeed a dominant strategy for students to report truthfully.

[1] $\mathbb{R}_+$ denotes the non-negative real numbers.

[2] In particular, students cannot adjust the time devoted to the course as a function of the course difficulty. This is obviously a simplification of reality. It turns out, however, that it doesn't really affect the conclusions here.

Problem 2 continued on next page. . .

Part A

Suppose the grade in the class is based on homework and exams. To be precise there is a weight $\alpha \in (0, 1)$ such the score of student i, denoted by s_i, is given by

$$s_i = \alpha h_i + (1 - \alpha)e_i,$$

where h_i is student i's homework score, and e_i is their exam score. If the class is curved, the x students with the highest grade get A's, and the rest get B's (throughout the question you can assume that there are no ties). Fixing the exam scores, suppose the instructor grades homework more leniently. This means that for some $\delta > 1$, each student's grade goes from h_i to δh_i. What can you say about how the old scores compare to the new scores? What about the new grades? Which students benefit and which are hurt? Be as precise as you can, and prove your answer.

Solution

All students' scores increase. Specifically, the new score is

$$s_i' = \alpha(\delta h_i) + (1 - \alpha)e_i = \alpha h_i + (1 - \alpha)e_i + \alpha(\delta - 1)h_i = s_i + \alpha(\delta - 1)h_i > s_i$$

The increase in score is $\alpha(\delta - 1)h_i$, which is proportional to the homework score. Consider two students i and j, their pairwise difference is

$$s_i' - s_j' = (s_i - s_j) + \alpha(\delta - 1)(h_i - h_j)$$

If $h_i \geq h_j$, then $\alpha(\delta - 1)(h_i - h_j) \geq 0$, so $s_i' - s_j' \geq s_i - s_j$. Increasing the weight on homework increases the advantage of the student with the higher homework score. Any pair whose homework and exam order agree, e.g., $h_i \geq h_j$ and $e_i \geq e_j$, keeps the same ordering after the homework weight change. If $s_i > s_j$ and $h_i \geq h_j$, then $s_i' > s_j'$. Therefore, only those pairs of students for which homework and exam rankings point in opposing directions can change their relative ordering. The set of top-x students can change only by substituing students who are relatively strong on homework for students who are relatively weak on homework. Students with relatively large homework scores, relative to other students with wimilar exam scores, gain relative and are more likely to move into the top x. Students with relatively small homework scores lose relative rank.

Part B

If the class is curved, what is the efficient choice of d?

Solution

The instructor maximizes the sum of utilities:

$$\max_d \sum_{i=1}^N U_i(G_i, d) = \max_d \sum_{i=1}^N \left[u(G_i) - (d - d_i)^2\right]$$

Since the class is curved, exactly x students receive grade A and $N - x$ students receive grade B, determined by their ranking based on $s(t_i, d)$. For any given d, the grades $\{G_i\}_{i=1}^N$ are determined, so we can focus on minimizing the difficulty cost:

$$\min_d \sum_{i=1}^N (d - d_i)^2$$

Taking the first-order condition:

$$\frac{\partial}{\partial d} \sum_{i=1}^N (d - d_i)^2 = 2 \sum_{i=1}^N (d - d_i) = 0$$

Solving for d:

$$Nd = \sum_{i=1}^N d_i \implies d^* = \frac{1}{N} \sum_{i=1}^N d_i$$

The second-order condition is $2N > 0$, confirming this is a minimum. The efficient difficulty level is the arithmetic mean of all students' ideal difficulty levels: $d^* = \bar{d} = \frac{1}{N} \sum_{i=1}^N d_i$.

Problem 2 continued on next page...

Part C

Suppose that the class is curved as in 2(b). If the instructor wants to implement the efficient choice of d, is it a dominant strategy for all students to report truthfully? If not, in which direction would students like to misreport? Prove your answer.

Solution

The instructor, lacking direct observation of students' d_i's, uses the reported $\hat{d}_1, \ldots, \hat{d}_N$ and sets

$$\hat{d} = \frac{1}{N} \sum_{i=1}^{N} \hat{d}_i$$

Student i's utility is

$$U_i(G_i, \hat{d}) = u(G_i) - (\hat{d} - d_i)^2 = u(G_i) - \left(\frac{\hat{d}_i + \sum_{j \neq i} \hat{d}_j}{N} - d_i \right)^2$$

Minimizing $\left(\frac{\hat{d}_i + \sum_{j \neq i} \hat{d}_j}{N} - d_i \right)^2$ with respect to $\hat{d}_i$ gives the best response for student i to report to the instructor

$$\hat{d}_i^{\text{BR}} = N d_i - \sum_{j \neq i} \hat{d}_j$$

Therefore, truthfully reporting $\hat{d}_i = d_i$ is generally not the best response and not a dominant strategy. A student can move the chosen d closer to their ideal by misreporting their preferences. If the average of the other students' reports is below d_i, student i will report a value larger than d_i, to raise $\hat{d}$, and vice versa.

Part D

Now assume, as before, that the class is curved. Assume also that $U_i(G, d) = u(G) - |d - d_i|$. Prove that it is indeed a dominant strategy for students to report truthfully.

Solution

If $U_i(G, d)$ now equals $u(G) - |d - d_i|$, then the d that maximizes the social objective is the median of the reported $\hat{d}_i$. I'll show that it is a dominant strategy for students to report truthfully in this case.

For a student i, the reports of all other students are $\hat{d}_{-i}$

$$\hat{d}_{-i} = \{\hat{d}_1, \ldots \hat{d}_{i-1}, \hat{d}_{i+1}, \ldots, \hat{d}_N\}$$

Student i's optimization problem is to find the right $\hat{d}_i$ to report to maximize their utility. Since the instructor chooses the median of the reported difficulties, the student i needs to choose $\hat{d}_i$ such that the median is as close as possible to their preferred difficulty d_i. There are three cases to consider:

1. The median of $\hat{d}_{-i}$ is less than d_i: in this case, reporting truthfully weakly dominates since, unlike in the previous part, scaling the magnitude higher in an attempt to move the median to student i's advantage will have no greater effect than reporting d_i since the median is resistant to extreme outliers.
2. The median of $\hat{d}_{-i}$ is equal to d_i: in this case, there is no benefit to not reporting truthfully, since reporting d_i will cement this value as the median and optimize student i's utility.
3. the median of $\hat{d}_{-i}$ is greater than d_i: this case is similar to the first one, except in the opposite direction. Again, reporting truthfully weakly dominates since scaling the magnitude of $\hat{d}_i$ lower will have no greater effect than reporting d_i since the median is resistant to extreme outliers.

So, reporting truthfully is indeed the dominant strategy with this configuration.