

ECON C103: Problem Set #2

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Problem 1

Prove Nash's theorem that every finite game (meaning a finite set of players and a finite set of actions) has at least one Nash equilibrium, possibly in mixed strategies (Hint: follow the steps in the notes).

Solution

I'll begin the proof by first establishing some preliminaries about the finite game.

- There are N players.
- Given an action profile a , let player i 's payoff be $\pi_i(a)$
- A mixed strategy, σ_i , for player i is a lottery over A_i , the choices available to the player. Let $\Delta(A_i)$ be the set of such lotteries, and let $\Sigma = \prod_{i=1}^N \Delta(A_i)$ be the set of strategy profiles.
- Given a profile of mixed strategies σ_{-i} , player i 's expected utility from action $a \in A_i$ is

$$U_i(a) = \mathbb{E}_{\sigma_{-i}}[\pi_i(a, a_{-i})]$$

- Let $BR_i(\sigma_{-i}) \subset \Delta(A_i)$ be the set of mixed strategies for i that are best responses to σ_{-i} , where $\sigma_i \in BR_i(\sigma_{-i})$ if and only if $U_i(a) \geq U_i(a')$ for all a in the support of σ_i , i.e., those actions played with non-zero probability, and all $a' \in A_i$.

A correspondence $B : \Sigma \rightrightarrows \Sigma$ is

$$B(\sigma) = \prod_{i=1}^N BR_i(\sigma_{-i})$$

A Nash equilibrium is just a strategy profile σ such that $\sigma \in B(\sigma)$. That is, I'll show that there exists a fixed point of the mapping B , i.e., a $\sigma \in B(\sigma)$. To prove this, I'll show that the best-response correspondence B satisfies the conditions of Kakutani's fixed-point theorem:

1. That Σ is compact, convex, and non-empty.
2. That $B(\sigma)$ is non-empty for every $\sigma \in \Sigma$.
3. That $B(\sigma)$ is convex-valued, i.e., for all σ , $B(\sigma)$ is a convex set.
4. That B has a closed graph.

Part One

Since $\Sigma = \prod_{i=1}^N \Delta(A_i)$, and each A_i is finite, each $\Delta(A_i)$ is the set of all probability distributions over the actions in A_i , i.e.,

$$\Delta(A_i) = \{p \in \mathbb{R}^{|A_i|} : p_a \geq 0, \sum_a p_a = 1\}$$

which is closed and bounded in $\mathbb{R}^{|A_i|}$. The set is convex since every element is a linear combination of the choices in A_i where the coefficients are between 0 and 1. The product of compact sets is compact, so Σ is compact. The product of convex sets is convex, so Σ is convex. The product of non-empty sets is non-empty, so Σ is non-empty. \square

Part Two

By definition, $BR_i(\sigma_{-i}) = \operatorname{argmax}\{\pi_i(\sigma_i, \sigma_{-i}) | \sigma_i \in \Delta(A_i)\}$. By the Weierstrass theorem, since $\Delta(A_i)$ is non-empty and compact, and the utility function is continuous in σ_i , there exists an optimal solution to the

optimization problem that defines the best-response function, and therefore $B(\sigma)$ is non-empty for every $\sigma \in \Sigma$ as a product of non-empty sets. \square

Part Three

For $B(\sigma)$ to be convex-valued, the $BR_i(\sigma_{-i})$ must be shown to be convex. Let $\sigma_i, \sigma'_i \in BR_i(\sigma_{-i})$ and let $\lambda \in [0, 1]$. Consider the mixed strategy $\sigma''_i = \lambda\sigma_i + (1 - \lambda)\sigma'_i$. Since expected utility is linear in the mixing probabilities, for any action $a \in A_i$,

$$U_i(\sigma''_i, \sigma_{-i}) = \lambda U_i(\sigma_i, \sigma_{-i}) + (1 - \lambda)U_i(\sigma'_i, \sigma_{-i})$$

Since both σ_i and σ'_i are best responses, they yield the same expected utility (the maximum possible). Therefore, σ''_i also yields this maximum expected utility, so $\sigma''_i \in BR_i(\sigma_{-i})$. Therefore, $BR_i(\sigma_{-i})$ is convex for each i , and $B(\sigma) = \prod_{i=1}^N BR_i(\sigma_{-i})$ is convex as the product of convex sets. \square

Part Four

For the graph of B to be closed, $\{(\sigma, \sigma') : \sigma' \in B(\sigma)\}$, must be a closed subset of $\Sigma \times \Sigma$. Consider a sequence $\{(\sigma^n, \sigma'^n)\}$ in the graph of B such that $(\sigma^n, \sigma'^n) \rightarrow (\sigma, \sigma')$. I'll show that $\sigma' \in B(\sigma)$.

Since $\sigma'^n \in B(\sigma^n)$, for each player i , we have $\sigma'^n_i \in BR_i(\sigma^n_{-i})$. This means that for all actions a in the support of σ'^n_i and all $a' \in A_i$,

$$U_i(a, \sigma^n_{-i}) \geq U_i(a', \sigma^n_{-i})$$

Taking limits as $n \rightarrow \infty$, and using the continuity of expected utilities in the strategy profile (which holds because payoffs are finite and the action space is finite), the inequality becomes

$$U_i(a, \sigma_{-i}) \geq U_i(a', \sigma_{-i})$$

for all actions a in the support of σ'_i and all $a' \in A_i$. This means $\sigma'_i \in BR_i(\sigma_{-i})$ for each i , so $\sigma' \in B(\sigma)$. Therefore, the graph of B is closed. \square

Since all four conditions of Kakutani's fixed-point theorem are satisfied, there exists a fixed point $\sigma^* \in B(\sigma^*)$, which is precisely a Nash equilibrium. This completes the proof of Nash's theorem.

Problem 2

This is a question about grading and course design. There are N students in a course. Assume for simplicity that an instructor can give each student one of two grades, A or B .

Say that the class is *curved* if there is a number $x \in \{1, 2, \dots, N\}$ such that the instructor must give A 's to x students, and B 's to the remaining $N - x$ students.

- (a) Suppose the grade in the class is based on homework and exams. To be precise there is a weight $\alpha \in (0, 1)$ such the score of student i , denoted by s_i , is given by

$$s_i = \alpha h_i + (1 - \alpha)e_i,$$

where h_i is student i 's homework score, and e_i is their exam score. If the class is curved, the x students with the highest grade get A 's, and the rest get B 's (throughout the question you can assume that there are no ties). Fixing the exam scores, suppose the instructor grades homework more leniently. This means that for some $\delta > 1$, each student's grade goes from h_i to δh_i . What can you say about how the old scores compare to the new scores? What about the new grades? Which students benefit and which are hurt? Be as precise as you can, and prove your answer.

Now, assume instead that each student's score in the course is determined by the amount of time they have to study for the course, t_i , and difficulty of the course, d , so $s_i = s(t_i, d)$ for some function $s : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ which is increasing in its first argument (time) and decreasing in the second argument (difficulty). ¹

Students like getting A 's more than B 's, but they also like a class that is not too easy, nor too hard (if the class is too easy you don't learn anything new, and if it's too hard you don't understand anything). We can represent student i 's preferences over difficulty levels and grades by a utility function

$$U_i(G, d) = u(G) - (d - d_i)^2$$

which describes their payoff from getting grade $G \in \{A, B\}$ when the class has difficulty level $d \in \mathbb{R}_+$. The parameter d_i represents the ideal difficulty level for student i , if they had no concern about grades (make sure you understand why this is the case).

The instructor wants to choose the difficulty level efficiently, to maximize the sum of student utilities.

- (b) If the class is curved, what is the efficient choice of d ?

In reality the instructor does not observe $(t_i, d_i)_{i=1}^N$. Instead, each student privately knows their own t_i and d_i . In order to set the level of difficulty for the course the instructor has to run a survey to learn about $\langle d_i \rangle_{i=1}^N$. The instructor is worried about students' incentives to report truthfully.

Assume that t_i and d_i are fixed for each student, and will remain the same regardless of what difficulty level is chosen.² However students can report whatever t_i, d_i they want to the instructor.

- (c) Suppose that the class is not curved. Instead, each student whose score, s_i , exceeds some threshold, a , gets an A , and otherwise they get a B . If the instructor wants to implement the efficient choice of d , is it a dominant strategy for all students to report truthfully? If not, in which direction would students like to misreport? Prove your answer.
- (d) Now assume, as before, that the class is curved. Assume also that $U_i(G, d) = u(G) - |d - d_i|$. Prove that it is indeed a dominant strategy for students to report truthfully.

¹ \mathbb{R}_+ denotes the non-negative real numbers.

²In particular, students cannot adjust the time devoted to the course as a function of the course difficulty. This is obviously a simplification of reality. It turns out, however, that it doesn't really affect the conclusions here.

Part A

Suppose the grade in the class is based on homework and exams. To be precise there is a weight $\alpha \in (0, 1)$ such the score of student i , denoted by s_i , is given by

$$s_i = \alpha h_i + (1 - \alpha)e_i,$$

where h_i is student i 's homework score, and e_i is their exam score. If the class is curved, the x students with the highest grade get A 's, and the rest get B 's (throughout the question you can assume that there are no ties). Fixing the exam scores, suppose the instructor grades homework more leniently. This means that for some $\delta > 1$, each student's grade goes from h_i to δh_i . What can you say about how the old scores compare to the new scores? What about the new grades? Which students benefit and which are hurt? Be as precise as you can, and prove your answer.

Solution

Comparison of scores: All students' scores increase. Specifically, the new score is

$$s'_i = \alpha(\delta h_i) + (1 - \alpha)e_i = \alpha h_i + (1 - \alpha)e_i + \alpha(\delta - 1)h_i = s_i + \alpha(\delta - 1)h_i > s_i$$

since $\alpha, \delta - 1, h_i > 0$. The increase in score is $\alpha(\delta - 1)h_i$, which is proportional to the homework score.

Effect on grades: Since the class is curved, exactly x students receive A 's regardless of the score distribution. Therefore, grades depend only on the *relative ranking* of students, not their absolute scores.

Change in ranking: Consider two students i and j . Originally, student i ranks higher than student j if $s_i > s_j$. After the change, student i ranks higher if

$$\begin{aligned} s'_i > s'_j &\iff \alpha\delta h_i + (1 - \alpha)e_i > \alpha\delta h_j + (1 - \alpha)e_j \\ &\iff \alpha\delta(h_i - h_j) > (1 - \alpha)(e_j - e_i) \end{aligned}$$

The original ranking has $i > j$ when $\alpha(h_i - h_j) > (1 - \alpha)(e_j - e_i)$.

Since $\delta > 1$, the weight on homework differences increases. The ranking is preserved if both sides have the same sign, but can reverse when $h_i - h_j$ and the original ranking have opposite signs.

Who benefits: Student i benefits (or maintains their grade) if their new rank is at least as high as their old rank. This occurs when:

- Student i has a relatively *high homework score* compared to their exam score
- Specifically, students whose homework score relative to the class is better than their exam score relative to the class benefit most
- Students in the top x with strong homework performance definitely keep their A 's
- Students just below the cutoff with strong homework scores may rise above it

Who is hurt: Students with relatively *low homework scores* compared to their exam scores are hurt:

- Students who performed better on exams than homework lose ground in the rankings
- Students at the bottom of the top x (barely getting A 's) with weak homework scores may fall below the cutoff

Proof of ranking change: Consider students i, j where originally $s_i > s_j$ (so i ranks higher). The ranking reverses if $s'_i < s'_j$. This happens when:

$$\begin{aligned} s'_i < s'_j &\iff \alpha\delta(h_i - h_j) < (1 - \alpha)(e_j - e_i) \\ &\text{and originally } \alpha(h_i - h_j) > (1 - \alpha)(e_j - e_i) \end{aligned}$$

This requires $(h_i - h_j) < 0$ (student j has better homework) and $(e_i - e_j) > 0$ (student i has better exam), with the original exam advantage being smaller than the homework disadvantage after scaling by δ . Such reversals are possible, showing that students with strong homework performance benefit at the expense of those with strong exam performance.

Part B

If the class is curved, what is the efficient choice of d ?

Solution

The instructor maximizes the sum of utilities:

$$\max_d \sum_{i=1}^N U_i(G_i, d) = \max_d \sum_{i=1}^N [u(G_i) - (d - d_i)^2]$$

Since the class is curved, exactly x students receive grade A and $N - x$ students receive grade B , determined by their ranking based on $s(t_i, d)$. For any given d , the grades $\{G_i\}_{i=1}^N$ are determined, so we can focus on minimizing the difficulty cost:

$$\min_d \sum_{i=1}^N (d - d_i)^2$$

Taking the first-order condition:

$$\frac{\partial}{\partial d} \sum_{i=1}^N (d - d_i)^2 = 2 \sum_{i=1}^N (d - d_i) = 0$$

Solving for d :

$$Nd = \sum_{i=1}^N d_i \implies d^* = \frac{1}{N} \sum_{i=1}^N d_i$$

The second-order condition is $2N > 0$, confirming this is a minimum.

Answer: The efficient difficulty level is the arithmetic mean of all students' ideal difficulty levels: $d^* = \bar{d} = \frac{1}{N} \sum_{i=1}^N d_i$.

Part C

Suppose that the class is not curved. Instead, each student whose score, s_i , exceeds some threshold, a , gets an A , and otherwise they get a B . If the instructor wants to implement the efficient choice of d , is it a dominant strategy for all students to report truthfully? If not, in which direction would students like to misreport? Prove your answer.

Solution

Claim: It is *not* a dominant strategy to report truthfully. Students have an incentive to misreport in a direction that helps them achieve a better grade.

Proof: From part (b), if students report $\{\tilde{d}_i\}_{i=1}^N$, the instructor chooses $d = \frac{1}{N} \sum_{i=1}^N \tilde{d}_i$.

Student i 's utility is:

$$U_i(G_i, d) = u(G_i) - (d - d_i)^2$$

where $G_i = A$ if $s(t_i, d) \geq a$ and $G_i = B$ otherwise.

Since $s(t_i, d)$ is increasing in t_i and *decreasing* in d , student i gets an A when d is sufficiently low (easier class means higher scores).

Consider two cases:

Case 1: Student i is currently getting a B (i.e., $s(t_i, d^*) < a$ where $d^* = \bar{d}$). This student wants to lower d to increase their score. By reporting $\tilde{d}_i < d_i$, student i pulls the average down to $d' = d^* - \frac{1}{N}(d_i - \tilde{d}_i) < d^*$.

The benefit: If d' is low enough that $s(t_i, d') \geq a$, the student gains $u(A) - u(B) > 0$.

The cost: The misreport changes the difficulty cost by $(d' - d_i)^2 - (d^* - d_i)^2$. For small deviations, this cost is second-order while the grade benefit is discrete.

Case 2: Student i is currently getting an A (i.e., $s(t_i, d^*) \geq a$). If the student is far enough above the threshold, they may want to report \tilde{d}_i closer to d_i to minimize $(d - d_i)^2$ without losing the A .

However, if d_i is far from d^* , student i faces a tradeoff. If $d_i > d^*$ (student prefers harder course), reporting truthfully pulls d up, which might cause them to fall below the threshold if they're close to it.

Direction of misreporting: Students near the threshold a from below have the strongest incentive to report $\tilde{d}_i < d_i$ (reporting a preference for easier coursework than they truly have), since this lowers d and increases their score $s(t_i, d)$, potentially lifting them above the threshold to earn an A .

Conclusion: Truthful reporting is not a dominant strategy. Students with $s(t_i, d^*) < a$ (getting B 's) want to underreport their ideal difficulty to make the course easier and potentially improve their grade.

Part D

Now assume, as before, that the class is curved. Assume also that $U_i(G, d) = u(G) - |d - d_i|$. Prove that it is indeed a dominant strategy for students to report truthfully.

Solution

Claim: Under a curved grading system with utility $U_i(G, d) = u(G) - |d - d_i|$, truthful reporting is a dominant strategy.

Proof: With the absolute value objective, if students report $\{\tilde{d}_i\}_{i=1}^N$, the efficient difficulty level is:

$$d^* = \text{median}\{\tilde{d}_1, \dots, \tilde{d}_N\}$$

(The median minimizes the sum of absolute deviations.)

Student i 's utility is:

$$U_i = u(G_i(d^*)) - |d^* - d_i|$$

where $G_i(d^*)$ depends on whether student i is in the top x students according to $s(t_i, d^*)$.

Key observation: With a curved system, exactly x students get A 's regardless of d^* . Therefore, $\sum_{i=1}^N u(G_i(d^*))$ is constant (equal to $xu(A) + (N - x)u(B)$) for any d^* . However, the *identity* of which students get A 's changes with d^* .

For student i , we need to show that reporting $\tilde{d}_i = d_i$ maximizes their utility regardless of what others report.

Let $d^*(\tilde{d}_i)$ denote the median when student i reports \tilde{d}_i and others report \tilde{d}_{-i} . Student i chooses \tilde{d}_i to maximize:

$$U_i(\tilde{d}_i) = u(G_i(d^*(\tilde{d}_i))) - |d^*(\tilde{d}_i) - d_i|$$

Step 1: Consider the difficulty cost term $|d^*(\tilde{d}_i) - d_i|$. This is minimized when $d^*(\tilde{d}_i) = d_i$. Since the median is weakly increasing in each argument and moves toward student i 's report, the best report from a pure difficulty-cost perspective is $\tilde{d}_i = d_i$.

Step 2: Can student i improve their grade by misreporting? Student i 's grade depends on their ranking: they get an A if they're in the top x according to $s(t_i, d^*)$.

Since $s(t_i, d)$ is decreasing in d , student i might try to manipulate d^* to improve their ranking. However:

- If student i reports $\tilde{d}_i < d_i$, this tends to lower d^* , which increases $s(t_i, d^*)$
- But it also increases $s(t_j, d^*)$ for all other students j
- The ranking is determined by relative performance: whether $s(t_i, d^*) > s(t_j, d^*)$ for at least $N - x$ other students

Step 3: The crucial point is that student i does not know the (t_j, d_j) values of other students. When d changes, all scores $s(t_j, d)$ change simultaneously. Student i cannot predict whether their relative ranking improves or worsens.

More formally: Consider two difficulty levels d' and d'' with $d' < d''$. Student i ranks higher at d' than at d'' if:

$$|\{j : s(t_j, d') \leq s(t_i, d')\}| > |\{j : s(t_j, d'') \leq s(t_i, d'')\}|$$

Whether this holds depends on the (t_j) values and how they interact with the difficulty change, which student i doesn't know.

Step 4: Given this uncertainty about the grade component, the only component student i can definitely influence in their favor is the difficulty cost $|d^* - d_i|$. This is uniquely minimized at $\tilde{d}_i = d_i$.

Conclusion: Reporting $\tilde{d}_i = d_i$ is a dominant strategy because:

1. It minimizes the certain cost $|d^* - d_i|$
2. It does not systematically improve (or worsen) the grade component, since grade depends on relative ranking which is uncertain without knowing others' types
3. Any deviation from d_i strictly increases the difficulty cost without providing a definite benefit to grades

Therefore, truthful reporting $\tilde{d}_i = d_i$ is optimal regardless of others' reports, making it a dominant strategy.

□