

# **ECON C103: Problem Set #1**

Due on January 30, 2026 at 11:59pm

*Professor Quitze Valenzuela-Stookey*

**Zachary Brandt**

## Problem 1

A group of people met and some of them shook each-other's hands. Prove that the number of people who shook others' hands an odd number of times is, in fact, even.

### Solution

*Proof.* This situation can be represented in the form of a graph, where people are vertices and handshakes are edges. The degree of any vertex corresponds to the number of handshakes a person made. The sum of the degrees of all vertices add up to twice the number of edges there are in the graph. This is because each new edge increases the degrees of two vertices, since an edge must connect two vertices.

$$\sum_{v \in V} \deg v = 2|E|$$

Since the right-hand side of the equation, two times the number of vertices, is a multiple of two, the number is even. The left-hand side sum of the above equation can be decomposed into a sum over all the vertices with odd degrees and a sum over all the vertices with even degrees.

$$\sum_{v_{\text{odd}} \in V} \deg v + \sum_{v_{\text{even}} \in V} \deg v = 2|E|$$

The sum of degrees over all even-degree vertices is naturally even. The sum of degrees over all odd-degree vertices must then also be even since  $2|E|$  is an even number. For  $\sum \deg v_{\text{odd}}$  to be even, there must be an even number of odd-degree vertices. Since this formulation corresponds one-to-one with people and handshakes, this shows that the number of people in a group who shake others' hands an odd number of times is even.  $\square$

## Problem 2

Let  $X$  be a nonempty set, and let  $2^X$  denote the set of all subsets of  $X$ . Show that there is no onto (aka surjective) function  $f : X \mapsto 2^X$ . (The function  $f$  is surjective if for every set  $A \in 2^X$  there is an  $x \in X$  such that  $f(x) = A$ .)

### Solution

*Proof.* Assume there exists an onto function that maps elements in  $X$  to those in  $2^X$ . There then must be a set  $S$  in  $2^X$  that contains those elements in  $X$  that are not in their corresponding subsets in  $2^X$ .

$$S = \{x \in X : x \notin f(x)\}$$

Continuing from the assumption, there then must be some element  $s$  in  $X$  where  $f(s) = S$ .  $s$  must either be in  $S$  or not in  $S$ . If  $s$  is in  $S$ , then that violates the restriction that  $s \notin f(s)$ , and therefore  $s$  cannot be in  $S$  since  $f(s) = S$ . If  $s$  is not in  $S$ , that then means that  $s \notin f(s)$ , which is what defines elements in  $S$ . So there is a contradiction leading from the original assumption, and therefore there does not exist an onto function  $f : X \mapsto 2^X$ .  $\square$

## Problem 3

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a twice-differentiable function. Show that  $\frac{\partial^2}{\partial x \partial y} f \geq 0$  if and only if for every  $x, x', y, y' \in \mathbb{R}$ :

$$x' > x \text{ and } y' > y \Rightarrow f(x', y') - f(x, y') \geq f(x', y) - f(x, y).$$

### Solution

*Proof.* To prove the if and only if condition, both directions need to be proven.

$\Rightarrow$  **Forward direction:** Assume that  $\frac{\partial^2}{\partial x \partial y} f \geq 0$  everywhere. By the Fundamental Theorem of Calculus,

$$f'(x', y) - f'(x, y) = \int_x^{x'} \frac{\partial^2 f}{\partial x \partial y}(t, y) dt$$

Since  $\frac{\partial^2 f}{\partial x \partial y} \geq 0$ ,  $f'(x', y) - f'(x, y) \geq 0$ , i.e.,  $f(x', y) - f(x, y)$  is non-decreasing in  $y$ . And since  $y' > y$ , this forward direction is proven to hold.

$$f(x', y') - f(x, y') \geq f(x', y) - f(x, y))$$

$\Leftarrow$  **Reverse direction:** Now assume that the inequality holds for all  $x' > x$  and  $y' > y$ . Define  $x'$  as  $x_0 + h$  and  $y'$  as  $y_0 + k$  where  $h, k > 0$ . The inequality can be rearranged to find partial derivatives in terms of  $y$  as  $k \rightarrow 0^+$ .

$$\begin{aligned} f(x_0 + h, y_0 + k) - f(x_0, y_0 + k) &\geq f(x_0 + h, y_0) - f(x_0, y_0) \\ \frac{f(x_0 + h, y_0 + k) - f(x_0 + h, y_0)}{k} &\geq \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k} \\ \frac{\partial f}{\partial y}(x_0 + h, y_0) &\geq \frac{\partial f}{\partial y}(x_0, y_0) \end{aligned}$$

This shows that  $\frac{\partial f}{\partial y}$  is non-decreasing in  $x$  for fixed  $y$ . Therefore:

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y \partial x} \geq 0$$

So both directions have been shown to hold and the claim has been proven.  $\square$

## Problem 4

A pair  $G = (V, E)$  is a *directed graph* if  $V$  is a set and  $E \subset V \times V$ . The graph  $G$  has a cycle if there exists an integer  $n \geq 1$  and elements  $v_1, \dots, v_n \in V$  such that  $(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1) \in E$ . Show that if  $V$  is a non-empty finite set and for all  $v \in V$  there is  $v' \in V$  such that  $(v, v') \in E$ , then  $G$  has a cycle.

### Solution

*Proof.* Since  $V$  is non-empty and finite, let  $|V| = n$  for some positive integer  $n$ . Pick an arbitrary starting vertex  $v_0 \in V$ . By the given condition, for every vertex there exists an outgoing edge, so we can construct a path:

$$v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \dots$$

where each arrow represents an edge  $(v_i, v_{i+1}) \in E$ . Since  $V$  is finite with  $n$  vertices, any path of length greater than  $n$  must visit at least one vertex more than once by the Pigeonhole Principle.

Consider the path of length  $n + 1$ :  $v_0, v_1, v_2, \dots, v_n, v_{n+1}$ . This path contains  $n + 2$  vertices (counting with repetition). Since there are only  $n$  distinct vertices in  $V$ , at least two vertices in this sequence must be the same.

Let  $v_i$  be the first vertex that appears again in the sequence, and let  $v_j$  be its first reappearance, where  $0 \leq i < j \leq n + 1$ . Then we have:

$$v_i \rightarrow v_{i+1} \rightarrow v_{i+2} \rightarrow \dots \rightarrow v_{j-1} \rightarrow v_j = v_i$$

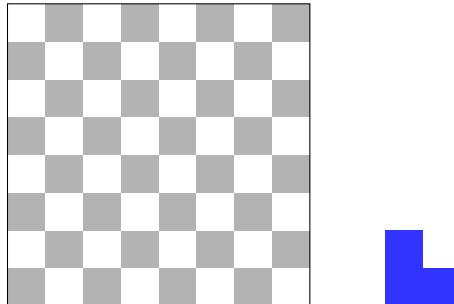
This forms a cycle because:

- $(v_i, v_{i+1}) \in E$
- $(v_{i+1}, v_{i+2}) \in E$
- $\vdots$
- $(v_{j-1}, v_j) \in E$
- Since  $v_j = v_i$ , the edge  $(v_{j-1}, v_j)$  is actually  $(v_{j-1}, v_i)$ , completing the cycle back to  $v_i$

Therefore,  $G$  contains a cycle with vertices  $v_i, v_{i+1}, \dots, v_{j-1}$  and edges  $(v_i, v_{i+1}), (v_{i+1}, v_{i+2}), \dots, (v_{j-1}, v_i)$ , where the cycle length is  $j - i \geq 1$ .  $\square$

## Problem 5

Consider an  $n \times n$  chessboard (the figure below shows an example with  $n = 8$ ).



Show that if  $n$  is a power of 2 it is possible to cover the board with L-shaped trominoes (consisting of three squares each, shown in blue) so that all but one of the squares on the board is covered.

### Solution

*Proof.* Below is an inductive proof on  $m$  to show that it is possible to cover the board with L-shaped trominoes so that all but one of the squares on the board is covered.

- **Base case:** When  $m = 1$ , remove any square from the  $2^m \times 2^m$  square to reveal where the tromino will cover the rest of the board.
- **Inductive hypothesis:** Assume that for all  $1 \leq m \leq k$  that it is possible to cover the board with L-shaped trominoes so that all but one of the squares on the board is covered.
- **Inductive step:** For  $m = k + 1$ , consider the  $2^{k+1} \times 2^{k+1}$  board as four  $2^k \times 2^k$  boards combined together, with each making up one quadrant of the larger board. It is possible to remove one square at random from this larger board, and the removed square will lie in one of the quadrants. By placing a tromino over the center of this larger board such that it covers those quadrants without the removed square this effectively renders each quadrant as a  $2^k \times 2^k$  board with one square missing. By the inductive hypothesis, it's the case that these can be covered with L-shaped trominoes so that all but one of the squares on the board is missing.

Therefore, the entire  $2^{k+1} \times 2^{k+1}$  board can be tiled with L-shaped trominoes such that all but one square on the board is covered. By induction, this proves the claim for all  $n \geq 1$  that are powers of 2.  $\square$