Optimization Models: Homework #2

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Problem 1

Consider the subspace $\mathcal{S} = \text{span}(x^{(1)}, x^{(2)}, x^{(3)})$, where

$$x^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x^{(2)} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad x^{(3)} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \tag{1}$$

i) Find the dimension of $\mathcal{S}$.

ii) Calculate the projection of the point $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ on $\mathcal{S}$.

Solution

The dimension of the subspace $\mathcal{S}$ is 2, because the span is not linearly independent. Vector $x^{(3)}$ is within the span of $x^{(1)}$ and $x^{(2)}$

$$x^{(1)} + x^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = x^{(3)}.$$

The other two vectors span a plane with dimension 2. To calculate the projection of the point $x = (2, 3, 5)$ onto $\mathcal{S}$, I solve a system of equations to find $x^* = \alpha_1 x^{(1)} + \alpha_2 x^{(2)}$. The vector pointing between x and its projection onto the span should be normal to $x^{(1)}$ and $x^{(2)}$:

$$\begin{aligned}
\langle x - x^*, x^{(1)} \rangle &= \langle x - \alpha_1 x^{(1)} - \alpha_2 x^{(2)}, x^{(1)} \rangle = 0 \\
\langle x - x^*, x^{(2)} \rangle &= \langle x - \alpha_1 x^{(1)} - \alpha_2 x^{(2)}, x^{(2)} \rangle = 0 \\
\Rightarrow \langle x, x^{(1)} \rangle &= \alpha_1 \langle x^{(1)}, x^{(1)} \rangle + \alpha_2 \langle x^{(2)}, x^{(1)} \rangle \\
\Rightarrow \langle x, x^{(2)} \rangle &= \alpha_1 \langle x^{(1)}, x^{(2)} \rangle + \alpha_2 \langle x^{(2)}, x^{(2)} \rangle
\end{aligned}$$

which then equals

$$\begin{aligned}
10 &= \alpha_1 \cdot 3 + \alpha_2 \cdot 0 \\
3 &= \alpha_1 \cdot 0 + \alpha_2 \cdot 2
\end{aligned}$$

Solving this system of equations gives $\alpha_1 = \frac{10}{3}$ and $\alpha_2 = \frac{3}{2}$. Therefore, the projection of the point x onto $\mathcal{S}$ is given by

$$x^* = \frac{10}{3} x^{(1)} + \frac{3}{2} x^{(2)} = \left(\frac{11}{6}, \frac{10}{3}, \frac{19}{6} \right).$$

Problem 2

Consider the box $\mathcal{S}_1$ and ball $\mathcal{S}_2$ defined as

$$\mathcal{S}_1 = \{x \in \mathbb{R}^2 \mid -3 \le x_1 \le 3,\ -1 \le x_2 \le 1\}, \quad \mathcal{S}_2 = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \le 2\} \tag{2}$$

Given a point $z \in \mathbb{R}^2$, write an optimization problem in a standard form that finds the projection of z onto the set $\mathcal{S}_1 \cap \mathcal{S}_2$ (i.e., the solution of the optimization problem should correspond to the closest point in $\mathcal{S}_1 \cap \mathcal{S}_2$ to z; note that you do not need to solve the optimization problem).

Solution

$$\begin{aligned}
\arg\min_{\boldsymbol{x}} \quad & \underbrace{\boxed{\|\boldsymbol{x} - \boldsymbol{z}\|_2}}_{f_0(\boldsymbol{x})}, \\
\text{s.t.} \quad & \underbrace{\boxed{x_1 - 3}}_{f_1(\boldsymbol{x})} \le 0, \\
& \underbrace{\boxed{-x_1 - 3}}_{f_2(\boldsymbol{x})} \le 0, \\
& \underbrace{\boxed{x_2 - 1}}_{f_3(\boldsymbol{x})} \le 0, \\
& \underbrace{\boxed{-x_2 - 1}}_{f_4(\boldsymbol{x})} \le 0, \\
& \underbrace{\boxed{x_1^2 + x_2^2 - 2}}_{f_5(\boldsymbol{x})} \le 0.
\end{aligned}$$

where

- $(x_1, x_2) \in \mathbb{R}^2$ is the *decision variable*;
- $f_0 : \mathbb{R}^2 \to \mathbb{R}$ is the *objective function*, or the *Euclidean distance* between x and z;
- f_1 & $f_2 : \mathbb{R}^2 \to \mathbb{R}$ represent the left and right bounds on the box $\mathcal{S}_1$;
- f_3 & $f_4 : \mathbb{R}^2 \to \mathbb{R}$ represent the upper and lower bounds on the box $\mathcal{S}_1$;
- $f_5 : \mathbb{R}^2 \to \mathbb{R}$ represents the constraint on the ball $\mathcal{S}_2$.

Problem 3

A company has n factories. Factory i (for $i = 1, 2, ..., n$) is located at point (a_i, b_i) in the two-dimensional plane $\mathbb{R}^2$. The company wants to locate a warehouse at a point (x_1, x_2) that minimizes

$$\sum_{i=1}^{n}(\text{distance from factory } i \text{ to the warehouse})^2$$

Find all possible values of the point (x_1^*, x_2^*) that satisfy the necessary condition for local optimality.

Problem 4

Given a natural number $k \in \{1, 2, \ldots\}$, a symmetric matrix $P \in \mathbb{R}^{n \times n}$, a vector $q \in \mathbb{R}^n$ and a scalar $r \in \mathbb{R}$, consider the optimization problem

$$\min_{x \in \mathbb{R}^n} \quad (x^\top P x)^k + q^\top x + r \tag{3}$$

Assume that q is a nonzero vector.

i) Calculate the gradient of the function $q^\top x$.

ii) Calculate the gradient of the function $x^\top P x$.

iii) Calculate the gradient of the objective function of the optimization problem (3).

iv) Given a point x^*, write the necessary optimality condition for x^* to be a local minimum of the optimization problem (3).

v) Assume that q is not in the range of P. Prove that the optimization problem (3) cannot have any local minimum (hint: show that the necessary optimality condition has no solution).

vi) Assume that P is invertible. Given a local minimum x^* of the optimization problem (3), show that there is a scalar α such that $x^* = \alpha P^{-1} q$.

vii) Again assume that P is invertible. Solve for α in Part (vi) and calculate it in terms of only the known parameters P, q, r, k (hint: Substitute the formula $x^* = \alpha P^{-1} q$ into the optimality condition and write it in terms of α).