Optimization Models: Homework #4

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Problem 1

Answer the following questions about SVD (note: all calculations should be done by hand):

i) Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix} \tag{1}$$

Show that the columns of A are orthogonal to each other. By using this fact, find a singular value decomposition of A.

ii) Find a singular value decomposition of the matrix

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix} \tag{2}$$

iii) Consider the optimization problem

$$\min_{B \in \mathbb{R}^{3\times 3}} \|C - B\|_F \qquad \text{s.t.} \qquad \text{rank}(B) \leq 2 \tag{3}$$

Find the optimal solution B^* and compute the error $\frac{||C-B^*\|_F^2}{\|C\|_F^2}$.

Solution

Problem 2

An exam with m questions is given to n students. The instructor collects all the grades in an $n \times m$ matrix G where G_{ij} shows the grade of student $i \in \{1, 2, ..., n\}$ for question $j \in \{1, 2...., m\}$. By analyzing the matrix G, the goal is to design a difficulty score for each question that shows the difficulty level of that question. As a naive approach, one may consider the average grade $\frac{\sum_{i=1}^{n} G_{ij}}{n}$ as the difficulty score of question j. To understand the issue with this difficulty score, assume that $n = m = 2$ and G is equal to

$$G = \begin{bmatrix} 50 & 100 \\ 50 & 0 \end{bmatrix} \tag{4}$$

where the minimum and maximum grades for each question are 0 and 100. In this example, both questions have the same average grade of 50. Both students have done poorly on question 1. For question 2, one student got the highest grade possible while the other student got 0 (which may imply that the student was not prepared for that question rather than the question being hard). Question 1 seems to be much harder than question 2 due to the distribution of the grades while the average grades cannot provide any useful information. To address this issue for arbitrary values of n and m, we propose an optimization model for the design of difficulty scores.

i) Consider the optimization problem

$$\min_{B \in \mathbb{R}^{n \times m}} \|G - B\|_F \qquad \text{s.t.} \qquad \text{rank}(B) \leq 1 \tag{5}$$

We decompose the optimal solution B^* as xy^T, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Assume that $x, y \geq 0$ (note: if no student receives a zero score on any question, then it can be proven that x and y are automatically nonnegative vectors). Assume that the error $\frac{||G - B^*\|_F^2}{\|G\|_F^2}$ is small. Explain how a difficulty score can be designed for each question in terms of x and y.

ii) Consider the case with $n = 3$ and $m = 5$, together with the grade matrix

$$G = \begin{bmatrix} 100 & 90 & 100 & 80 & 70 \\ 80 & 70 & 60 & 70 & 80 \\ 60 & 50 & 40 & 50 & 60 \end{bmatrix} \tag{6}$$

Using Part (i), design a difficulty score for each question, and rank the questions from the hardest to the easiest based on their scores (note: you can use a computer code for SVD calculations).

Solution

Problem 3

(Coding) Consider a picture of the Berkeley logo named Berkeley1.png that you can download from bCourses $\rightarrow$ Files $\rightarrow$ HW.

i) Convert the image to a grayscale image, and store its data into a matrix A.

ii) Plot the singular values of A. To do so, draw the point (k, σ_k) in $\mathbb{R}^2$ for $k = 1, ..., \min(m, n)$, where m and n denote the dimensions of A and σ_k is the k^{th} largest singular value of A.

iii) Consider the optimization problem

$$\min_{B \in \mathbb{R}^{m \times n}} \|A - B\|_F, \qquad \text{s.t.} \qquad \text{rank}(B) \leq k \tag{7}$$

and consider the error $e_k = \frac{\|A - B^*\|_F^2}{\|A\|_F^2}$, where B^* is an optimal solution. Solve the above optimization problem in three scenarios of $k = 30$, $k = 80$ and $k = 100$. For each case, report the error e_k and the percentage $\frac{k}{\min(m,n)} \times 100$ (which shows what percentage of the singular values of A is used), and also draw the grayscale image corresponding to B^* (note: you do not need to report the matrix B^* in your homework submission and only the image together with the code is enough).

iv) Redo Parts (i), (ii), and (iii) for the Berkeley campus picture Berkeley2.png that you can download from bCourses $\rightarrow$ Files $\rightarrow$ HW.

Solution

Problem 4

Consider the least-squares problem

$$\min_x \|Ax - y\| \tag{8}$$

where $A \in \mathbb{R}^{2\times 3}$ and $y \in \mathbb{R}^2$. Assume that

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, \qquad y = \begin{bmatrix} 4 \\ 7 \end{bmatrix} \tag{9}$$

i) Show that the set of all solutions is equal to

$$S = \left\{ x^* \in \mathbb{R}^3 \;\middle|\; x_1^* + x_2^* + x_3^* = \frac{18}{5} \right\} \tag{10}$$

ii) Assume that y is perturbed to $y + \Delta y$. Find the set of all solutions of the perturbed least-squares problem $\min_x \|Ax - (y + \Delta y)\|$.

iii) Assume that y is perturbed to $y + \Delta y$, where Δy can take any value in the set $\{\Delta y \mid \|\Delta y\| \leq 1\}$. Find the set of all possibilities for the minimum-norm solution of the perturbed least-squares problem $\min_x \|Ax - (y + \Delta y)\|$ (hint: write the ellipsoidal formula for Δx and compute the semi-axes and lengths of the ellipse; you can use a computer code for SVD calculations).

Solution