Optimization Models: Homework #4

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Zachary Brandt
zbrandt@berkeley.edu

Problem 1

Find all constant coefficients a_0, a_1, a_2, a_3 for which the following problem is a convex optimization:

$$\min_{x\in\mathbb{R}^2} \quad \left(\frac{1}{12}x_1^4 + a_3x_1^3 + a_2x_1^2 + a_1x_1 + a_0\right) + x_2^{18} \tag{1a}$$

$$\text{s.t.} \quad a_2x_1^2 + a_1x_1 + a_0 \leq -10x_2 + 5 \tag{1b}$$

$$a_1x_1 + a_0 = x_2 + 1 \tag{1c}$$

Solution

For the above to be a convex optimization problem, the objective function and the function defining the inequality constraint must be convex, while the function defining the equality constraint must be affine. For the objective function, the 2$^{\text{nd}}$-order convexity condition, $\nabla^2 f(x) \succeq 0$, must hold:

$$\nabla f(x) = \begin{bmatrix} \frac{1}{3}x_1^3 + 3a_3x_1^2 + 2a_2x_1 + a_1 \\ 18x_2^{17} \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} x_1^2 + 6a_3x_1 + 2a_2 & 0 \\ 0 & 306x_2^{16} \end{bmatrix}$$

If the leading principal minors of this matrix are non-negative, then the matrix is positive semidefinite. For the first leading principal minor:

$$\begin{aligned} x_1^2 + 6a_3x_1 + 2a_2 &\geq 0 \\ x_1^2 + 6a_3x_1 + 9a_3^2 - 9a_3^2 + 2a_2 &\geq 0 \\ (x_1 + 3a_3)^2 + (2a_2 - 9a_3^2) &\geq 0 \\ (\text{since } (x_1+3a_3)^2 \geq 0) \quad (2a_2 - 9a_3^2) &\geq 0 \\ \Rightarrow a_2 &\geq \frac{9}{2}a_3^2 \end{aligned}$$

The second leading principal minor is $(x_1^2 + 6a_3x_1 + 2a_2)(306x_2^{16}) \geq 0$, which holds for all $x_2 \in \mathbb{R}$ since $306x_2^{16} \geq 0$. Therefore, for the objective function to be convex:

$$a_2 \geq \frac{9}{2}a_3^2$$

The function defining the inequality constraint must also be convex. This function is $g(x) = a_2x_1^2 + a_1x_1 + a_0 + 10x_2 - 5$. Its Hessian is:

$$\nabla^2 g(x) = \begin{bmatrix} 2a_2 & 0 \\ 0 & 0 \end{bmatrix}$$

For this matrix to be positive semidefinite, $2a_2 \geq 0$. The function defining the equality constraint is $h(x) = a_1x_1 + a_0 - x_2 - 1$, which is affine for all values of a_1 and a_0. Therefore, the optimization problem is convex for:

$$a_2 \geq \frac{9}{2}a_3^2,\ a_0 \in \mathbb{R}, a_1 \in \mathbb{R}, a_3 \in \mathbb{R}$$

Problem 2

Consider the optimization problem:

$$\min_{x \in \mathbb{R}^3} \quad e^{-x_1 - x_2 - x_3} + (x_1 + 2x_2 + 3x_3)^4 \tag{2a}$$
$$\text{s.t.} \quad x_1 - x_2 - x_3 = 1 \tag{2b}$$
$$x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3 \leq 2 \tag{2c}$$
$$0 \leq x_1 \leq 1 \tag{2d}$$
$$0 \leq x_2 \leq 1 \tag{2e}$$
$$0 \leq x_3 \leq 1 \tag{2f}$$

i) Prove that $x = [1 \;\; 0 \;\; 0]^T$ is a feasible point.

ii) Using the Weierstrass theorem, prove that the above optimization problem has a solution.

iii) Prove that every local minimum of the above optimization problem is a global minimum.

Solution

i) For the point $x = [1 \;\; 0 \;\; 0]^T$ is a feasible point, it needs to satisfy all the constraints of the optimization problem. For the equality constraint:

$$\begin{aligned} x_1 - x_2 - x_3 - 1 &= 0 \\ 1 - 0 - 0 - 1 &= 0 \end{aligned}$$

This holds true. For the inequality constraint:

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3 - 2 &\leq 0 \\ 1^2 + 0^2 + 0^2 + 0 + 0 + 0 - 2 = -1 &\leq 0 \end{aligned}$$

This also holds true. Finally, for the bound constraints:

$$\begin{aligned} 0 \leq x_1 \leq 1 \quad &\Rightarrow \quad 0 \leq 1 \leq 1 \\ 0 \leq x_2 \leq 1 \quad &\Rightarrow \quad 0 \leq 0 \leq 1 \\ 0 \leq x_3 \leq 1 \quad &\Rightarrow \quad 0 \leq 0 \leq 1 \end{aligned}$$

All bound constraints hold true as well. Therefore, the point $x = [1 \;\; 0 \;\; 0]^T$ is a feasible point.

ii) The Weierstrass theorem states that a continuous function defined on a compact set attains its minimum and maximum values. In this optimization problem, the objective function is continuous, as it is composed of exponential and polynomial functions, which are both continuous. The feasible set defined by the constraints is closed and bounded, making it compact. Therefore, by the Weierstrass theorem, the optimization problem has a solution.

iii) If the optimization problem is convex, then every local minimum is a global minimum. The objective function must be convex, the functions defining the inequality constraints must be convex, and the functions defining the equality contraints must be affine. The objective function is the sum of the exponential function $e^{-x_1-x_2-x_3}$, which is convex, and the polynomial function $(x_1+2x_2+3x_3)^4$, which is also convex. The function defining the inequality constraint is $x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3$, which is convex. The function defining the equality constraint is $x_1 - x_2 - x_3 - 1$, which is affine. Therefore, the optimization problem is convex, and every local minimum is a global minimum.

Problem 3

Consider a convex function $f(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}$ such that:

- The points $(2, 2)$ and $(-2, 4)$ are both global minima of the function $f(x_1, x_2)$ with the property that $f(2, 2) = 5$.
- $f(0, 0)$ is equal to 10.

Answer the following questions:

i) Prove that the point $(0, 3)$ is a global minimum of the function $f(x_1, x_2)$.

ii) Prove that the point $(0, -1)$ is not a local minimum of the function $f(x_1, x_2)$.

Problem 4

Answer the following questions about coercive functions:

i) Find all coefficients $\alpha_1, ..., \alpha_n$ and $\beta_1, ..., \beta_n$ for which the function

$$\alpha_1 x_1^2 + \cdots + \alpha_n x_n^2 + \cos\left(\beta_1 x_1^3 + \cdots + \beta_n x_n^3\right) \tag{3}$$

is coercive.

ii) Given a constant a, prove that the function $x_1^2 + x_2^2 + a x_1 x_2$ is coercive if $|a| < 2$ and is not coercive if $|a| \geq 2$.