Optimization Models: Homework #3

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Problem 1

Consider an aerial system that moves in $\mathbb{R}^3$ according to the dynamics

$$x(k+1) = Ax(k) + Bu(k), \quad k = 0, 1, 2, 3 \tag{1}$$

where $x(k) \in \mathbb{R}^3$ is the position of the system at time $k \in \{0, 1, 2, 3, 4\}$ and $u(k) \in \mathbb{R}$ is the scalar input applied to the system at time k. Assume that the initial position $x(0)$ is equal to $[0 \;\; 0 \;\; 0]^T$. Given a target position $x_d \in \mathbb{R}^3$, the goal is to design the input sequence $u(0), u(1), u(2), u(3)$ to take the system to the target position x_d at time $k = 4$, i.e., $x(4) = x_d$.

i) Find a matrix $H \in \mathbb{R}^{3\times 4}$ in terms of A and B with the property that

$$x(4) = H \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \end{bmatrix} \tag{2}$$

ii) Assume that

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tag{3}$$

Show that the vector $[1 \;\; 1 \;\; 0]^T$ belongs to $\mathcal{N}(H^T)$ (note: you are allowed to use a calculator to compute H, but you cannot use a calculator or a computer code to study the null space of H^T and the analysis should be done by hand).

iii) Again, consider the system parameters given in (3). By studying the relationship between $\mathcal{N}(H^T)$ and $\mathcal{R}(H)$, prove that there is no sequence of inputs that can take the system to the position $x_d = [1 \;\; 1 \;\; 0]^T$ at time 4.

iv) Again, consider the system parameters given in (3). By finding $\mathcal{N}(H^T)$ and using the relations $\mathcal{N}(H^T) \perp \mathcal{R}(H)$ and $\mathcal{N}(H^T) \oplus \mathcal{R}(H) = \mathbb{R}^3$, show that there exists a sequence of inputs to take the system to the position x_d at time 4 if and only if x_d belongs to the set

$$\{x \in \mathbb{R}^3 \mid x_1 + x_2 = 0\} \tag{4}$$

v) **(Coding)** Now, assume that

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \tag{5}$$

The goal is to find a sequence of inputs such that the total energy $u(0)^2 + u(1)^2 + u(2)^2 + u(3)^2$ is minimized and yet the system arrives at the target position $x_d = [3 \;\; 2 \;\; 2]^T$ at time 4. Formulate this as an optimization problem and write a code in CVX to solve the problem numerically. Plot the optimal trajectory (i.e., plot the optimal values of the points $x(0), ..., x(4)$ in $\mathbb{R}^3$ and then connect each point to the next one (such as $x(1)$ to $x(2)$)).

vi) **(Coding)** Consider the safety set

$$\mathcal{S} = \{x \in \mathbb{R}^3 \mid -3.3 \le x_i \le 3.2, \quad i = 1, 2, 3\} \tag{6}$$

Assume that the state $x(k)$ must always stay in the safety set $\mathcal{S}$ for $k = 0, 1, ..., 4$. Redo Part (v) under this additional constraint and find the optimal input sequence. Compares the optimal trajectories and optimal energies (objective values) obtained in Parts (v) and (vi).

Solution

i) To find a matrix H in terms of A and B, I'll solve for $x(k)$ at times 1, 2, 3, and 4

$$\begin{aligned}
x(1) &= Ax(0) + Bu(0) = Bu(0) \\
x(2) &= Ax(1) + Bu(1) = ABu(0) + Bu(1) \\
x(3) &= Ax(2) + Bu(2) = A^2Bu(0) + ABu(1) + Bu(2) \\
x(4) &= Ax(3) + Bu(3) = A^3Bu(0) + A^2Bu(1) + ABu(2) + Bu(3)
\end{aligned}$$

Therefore, the matrix H is

$$H = \begin{bmatrix} A^3B & A^2B & AB & B \end{bmatrix}$$

ii) To show that the vector $[1 \;\; 1 \;\; 0]^\top$ belongs to $\mathcal{N}(H^\top)$, I'll first compute H using the given A and B matrices

$$AB = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix}$$

$$A^3B = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -8 \\ 1 \end{bmatrix}$$

Therefore, the matrix H is

$$H = \begin{bmatrix} A^3B & A^2B & AB & B \end{bmatrix} = \begin{bmatrix} 8 & 4 & 2 & 1 \\ -8 & -4 & -2 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Now, I'll compute $H^\top$ and multiply it by the vector $[1 \;\; 1 \;\; 0]^\top$

$$H^\top = \begin{bmatrix} 8 & -8 & 1 \\ 4 & -4 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$H^\top \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 8(1) + -8(1) + 1(0) \\ 4(1) + -4(1) + 1(0) \\ 2(1) + -2(1) + 1(0) \\ 1(1) + -1(1) + 1(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the result is the zero vector, the vector $[1 \;\; 1 \;\; 0]^\top$ belongs to $\mathcal{N}(H^\top)$.

iii) There is no sequence of inputs that can take the system to the position $x_d = [1 \;\; 1 \;\; 0]^\top$ at time 4 because x_d is not in the range of H. Since $[1 \;\; 1 \;\; 0]^\top$ is in the null space of $H^\top$, it is orthogonal to every vector in the range of H ($\mathcal{R}(H) \perp \mathcal{N}(H^\top)$). If there were a sequence of inputs that could take the system to the position x_d at time 4, then x_d would be in the range of H and would have to be orthogonal to itself, which is a contradiction since x_d is not the zero vector.

Problem 1 continued on next page...

iv) To show that there exists a sequence of inputs to take the system to the position x_d at time 4 if and only if x_d belongs to the set $\{x \in \mathbb{R}^3 \mid x_1 + x_2 = 0\}$, I'll first find the null space of $H^\top$. From part (ii), $[1, 1, 0]^\top \in \mathcal{N}(H^\top)$. Since H is 3×4 and has rank 2 (the first two rows are linearly independent), $\dim(\mathcal{N}(H^\top)) = n - \text{rank}(H) = 3 - 2 = 1$. So basis for the null space is $\mathcal{N}(H^\top) = \text{span}\{[1, 1, 0]^\top\}$. Using $\mathcal{N}(H^\top) \perp \mathcal{R}(H)$, a vector $x_d \in \mathcal{R}(H)$ is in range if and only if $x_d \perp v$ for all $v \in \mathcal{N}(H^\top)$. Then the following must be true:

$$x_d^\top \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$$

which simplifies to

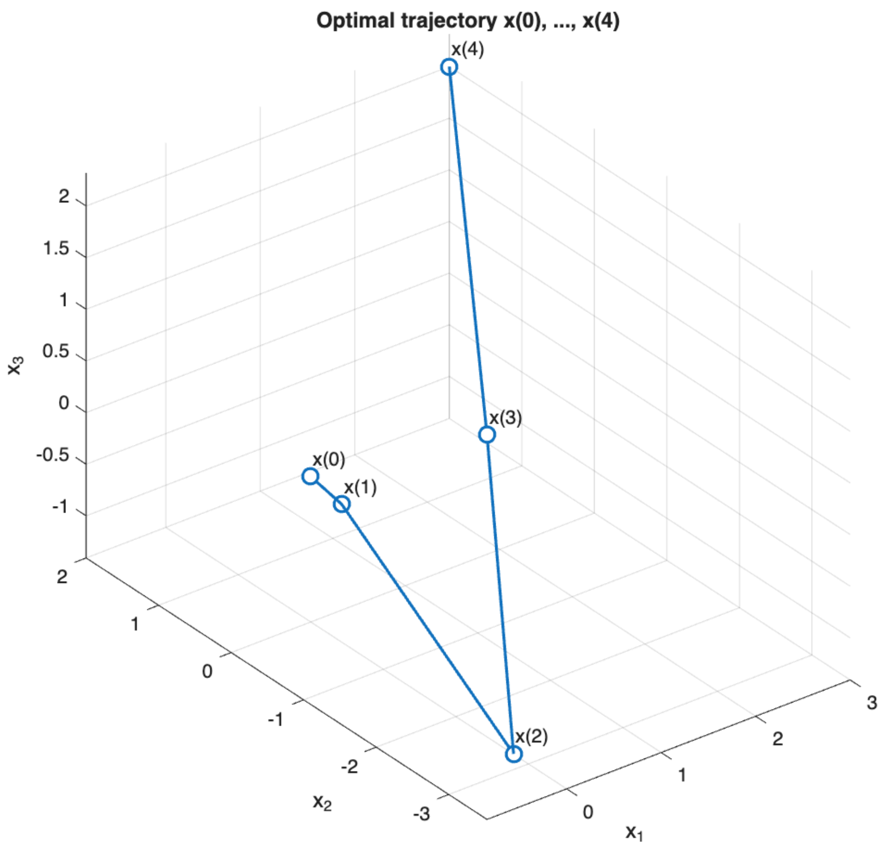
$$x_1 + x_2 = 0$$

where x_1, x_2, x_3 are the components of x_d. Therefore, there exists a sequence of inputs to reach x_d at time 4 if and only if $x_d \in \{x \in \mathbb{R}^3 \mid x_1 + x_2 = 0\}$.

v) Here is the optimization problem for minimizing total energy with $u(0), ..., u(3) \in \mathbb{R}^3$ as arguments

$$\begin{aligned} \min_{\boldsymbol{u}} \quad & u(0)^2 + u(1)^2 + u(2)^2 + u(3)^2, \\ \text{s.t.} \quad & x_d - H\boldsymbol{u} = 0 \end{aligned}$$

where $\boldsymbol{u} = [u(0)\ u(1)\ u(2)\ u(3)]^\top$ and $x_d = [3\ 2\ 2]^\top$. I wrote a script in `problem1v.m` to solve the problem numerically in CVX. The optimal sequence of inputs $\boldsymbol{u}$ is $u(0) = -1.4076$, $u(1) = 2.8804$, $u(2) = 0.7120$, and $u(3) = -4.6848$. The minimized total energy is 32.7323. Below is a plot of the optimal trajectory.

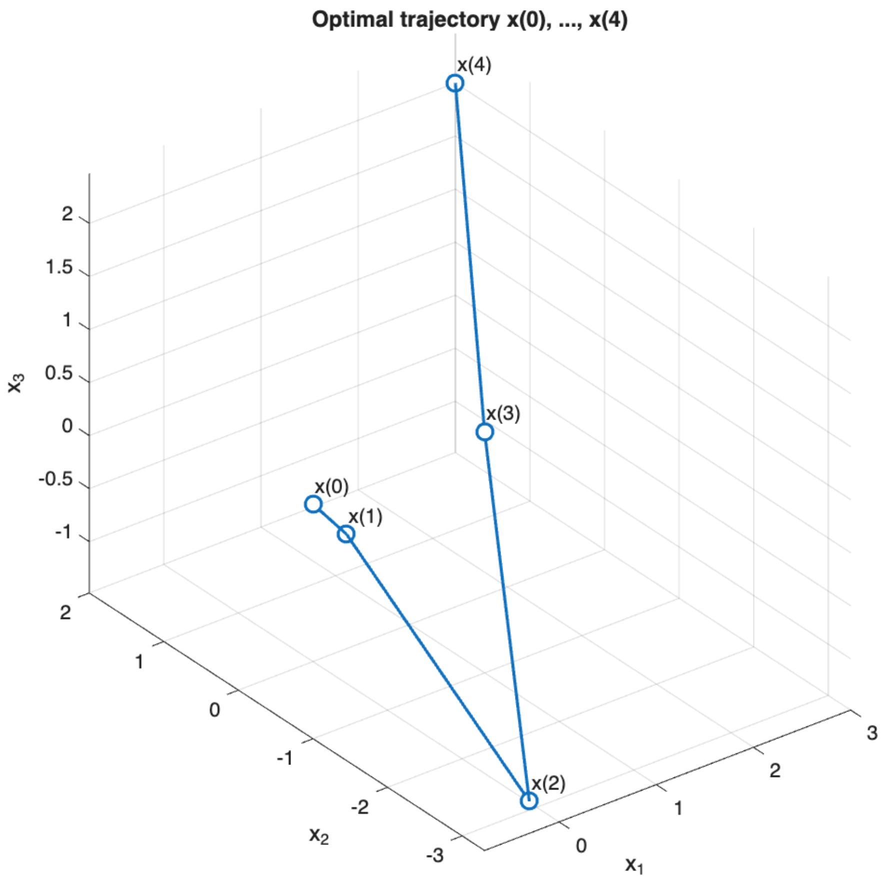


Problem 1 continued on next page...

vi) Here is the optimization problem for minimizing total energy with $u(0), ..., u(3) \in \mathbb{R}^3$ as arguments

$$\begin{aligned} \min_{\boldsymbol{u}} \quad & u(0)^2 + u(1)^2 + u(2)^2 + u(3)^2, \\ \text{s.t.} \quad & x_d - H\boldsymbol{u} = 0, \\ & -3.3 \leq x_i(k) \leq 3.2, \quad i = 1, 2, 3, \quad k = 0, 1, 2, 3, 4 \end{aligned}$$

where $\boldsymbol{u} = [u(0)\ u(1)\ u(2)\ u(3)]^\top$ and $x_d = [3\ 2\ 2]^\top$. I wrote a script in `problem1vi.m` to solve the problem numerically in CVX. The optimal sequence of inputs $\boldsymbol{u}$ is $u(0) = -1.4833$, $u(1) = 3.1833$, $u(2) = 0.3333$, and $u(3) = -4.5333$. The minimized total energy is 32.9961. Below is a plot of the optimal trajectory.



Problem 2

(Coding) Consider the matrix A and vector x^* defined as

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad x^* = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{7}$$

Define $b = Ax^* + v$ where $v \in \mathbb{R}^6$ is some measurement noise. Assume that the user has no access to x^* and aims to learn x^* from the measurement vector b. We consider two different estimators to learn x^*:

$$l_1 \text{ estimator:} \qquad \min_x \|Ax - b\|_1, \tag{8a}$$

$$l_2 \text{ estimator:} \qquad \min_x \|Ax - b\|_2 \tag{8b}$$

Given a solution $\hat{x}$ obtained from any of the above estimators, we define the estimation error $e = \|\hat{x} - x^*\|_2$ (note that the error is always computed with respect to the l_2-norm no matter which estimator is used for obtaining $\hat{x}^*$). Assume that the noise v is in the form

$$v = \begin{bmatrix} t_1 \\ 0 \\ 0 \\ 0 \\ t_2 \\ 0 \end{bmatrix} \tag{9}$$

where t_1 and t_2 are constants that belong to the discrete set $\{-2, -1.9, -1.8, ..., -0.1, 0, 0.1, ..., 1.8, 1.9, 2\}$ (the increment is 0.1).

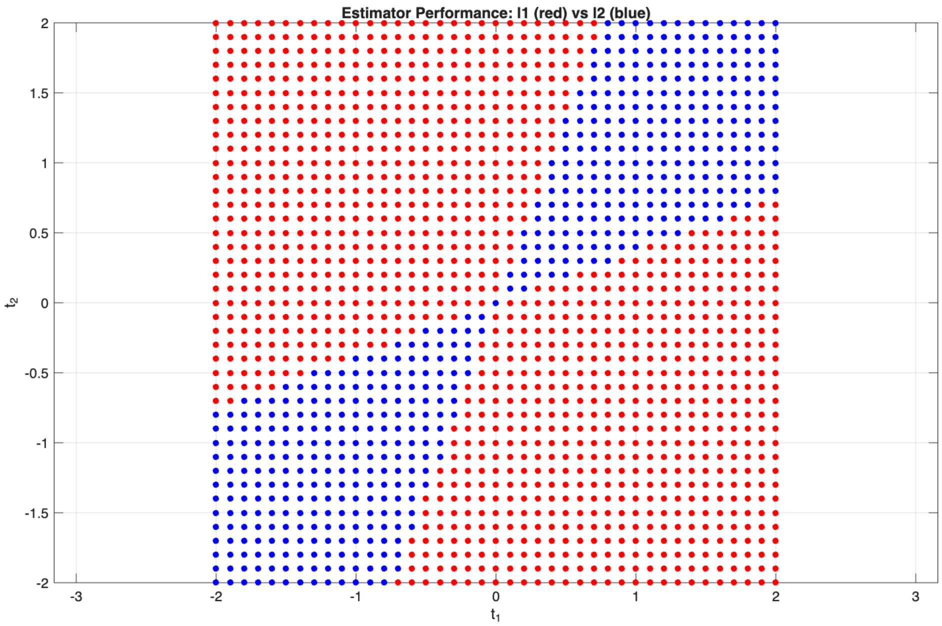
i) For each possible value of the pair (t_1, t_2), solve the l_1 and l_2 estimators in CVX and record the corresponding estimation errors (note: there are 41×41 possibilities for (t_1, t_2)).

ii) Draw a grid in $\mathbb{R}^2$ obtained as follows: For each possible value of the pair (t_1, t_2), we put a symbol in the location (t_1, t_2) in $\mathbb{R}^2$, where the symbol is a small red circle if the l_1 estimator gives the lowest estimation error and is a small blue circle if the l_2 estimator gives the lowest estimation error (note: if the estimation errors for both estimators are the same, use the blue circle). Analyze the plot and report your observations.

Solution

I wrote a script in `problem2.m` to solve the problem numerically in CVX. The script iterates through each possible pair (t_1, t_2), solves both the l_1 and l_2 estimators, and records the estimation errors. For each pair (t_1, t_2), the script then compares the estimation errors and records which estimator gives the lower error. Finally, the script generates a plot in $\mathbb{R}^2$ indicating which estimator performs better for each possible pair (t_1, t_2), where a small red circle indicates that the l_1 estimator gives the lowest estimation error and a small blue circle indicates that the l_2 estimator gives the lowest or the same estimation error as that of the l_1 estimator.

Problem 2 continued on next page. . .



Above is the plot generated by the script. There is a clear pattern in the performance of the two estimators. The l_1 estimator (red circles) performs better in the upper-left quadrant where t_1 is negative and t_2 is positive, and in the lower-right quadrant where t_1 is positive and t_2 is negative. The l_2 estimator (blue circles) dominates in the remaining regions, particularly in the lower-left and upper-right quadrants where t_1 and t_2 have the same sign, as well as in a thin band in the center, around the origin, connecting the two larger regions.

Problem 3

Consider the optimization problem

$$\min_{x\in\mathbb{R}^2} \quad e^{x_1+x_2} + 2x_1^2 + 2x_2^2 - x_1x_2 - \sin(x_1+x_2) \tag{10}$$

By analyzing the gradient and Hessian of the objective function, prove that $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a global minimum of the optimization problem (Hint: A matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite if and only if $a > 0$ and $ac - b^2 > 0$).

Proof

To prove that x^* is a global minimum, the following first-order and second-order conditions must be true:

$$\nabla f(x^*) = 0 \quad \text{and} \quad H_f(x^*) \succ 0,\ \forall x$$

For the first condition, the gradient of the objective function is:

$$\nabla f(x) = \begin{bmatrix} e^{x_1+x_2} + 4x_1 - x_2 - \cos(x_1+x_2) \\ e^{x_1+x_2} + 4x_2 - x_1 - \cos(x_1+x_2) \end{bmatrix}$$

Evaluating at $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$\nabla f(x^*) = \begin{bmatrix} e^0 + 0 - 0 - \cos(0) \\ e^0 + 0 - 0 - \cos(0) \end{bmatrix} = \begin{bmatrix} 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, x^* could be a local minimum. For the second-order condition, the Hessian is:

$$H_f(x) = \begin{bmatrix} e^{x_1+x_2} + 4 + \sin(x_1+x_2) & e^{x_1+x_2} - 1 + \sin(x_1+x_2) \\ e^{x_1+x_2} - 1 + \sin(x_1+x_2) & e^{x_1+x_2} + 4 + \sin(x_1+x_2) \end{bmatrix}$$

For x^* to be a global minimum, $H_f(x^*)$ must be positive definite for all x. $H_f(x^*)$ is positive definite if and only if $a > 0$ and $ac - b^2 > 0$. For the first condition, since $e^{x_1+x_2} > 0$ and $\sin(x_1+x_2) \geq -1$:

$$a = e^{x_1+x_2} + 4 + \sin(x_1+x_2) > 0$$

with the included $+4$. For the second condition:

$$ac - b^2 = (e^{x_1+x_2} + 4 + \sin(x_1+x_2))^2 - (e^{x_1+x_2} - 1 + \sin(x_1+x_2))^2 > 0$$

Substituting $e^{x_1+x_2} + \sin(x_1+x_2) = z$:

$$ac - b^2 = (z+4)^2 - (z-1)^2 = 6z + 15 > 0$$

$$z > -2.5$$

Since $e^{x_1+x_2} > 0$ and $\sin(x_1+x_2) \geq -1$,

$$z = e^{x_1+x_2} + \sin(x_1+x_2) > -1 > -2.5$$

Therefore, $H_f(x^*)$ is positive definite for all x and the second-order condition is true. So $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a global minimum.