Optimization Models: Homework #4

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Problem 1

Find all constant coefficients a_0, a_1, a_2, a_3 for which the following problem is a convex optimization:

$$\min_{x\in\mathbb{R}^2} \quad \left(\frac{1}{12}x_1^4 + a_3x_1^3 + a_2x_1^2 + a_1x_1 + a_0\right) + x_2^{18} \tag{1a}$$

$$\text{s.t.} \quad a_2x_1^2 + a_1x_1 + a_0 \le -10x_2 + 5 \tag{1b}$$

$$a_1x_1 + a_0 = x_2 + 1 \tag{1c}$$

Solution

For the above to be a convex optimization problem, the objective function and the function defining the inequality constraint must be convex, while the function defining the equality constraint must be affine. For the objective function, the 2nd-order convexity condition, $\nabla^2 f(x) \succeq 0$, must hold:

$$\nabla f(x) = \begin{bmatrix} \frac{1}{3}x_1^3 + 3a_3x_1^2 + 2a_2x_1 + a_1 \\ 18x_2^{17} \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} x_1^2 + 6a_3x_1 + 2a_2 & 0 \\ 0 & 306x_2^{16} \end{bmatrix}$$

If the leading principal minors of this matrix are non-negative, then the matrix is positive semidefinite. For the first leading principal minor:

$$\begin{aligned} x_1^2 + 6a_3x_1 + 2a_2 &\ge 0 \\ x_1^2 + 6a_3x_1 + 9a_3^2 - 9a_3^2 + 2a_2 &\ge 0 \\ (x_1 + 3a_3)^2 + (2a_2 - 9a_3^2) &\ge 0 \\ (\text{since } (x_1 + 3a_3)^2 \ge 0) \quad (2a_2 - 9a_3^2) &\ge 0 \\ \Rightarrow \; a_2 &\ge \frac{9}{2}a_3^2 \end{aligned}$$

The second leading principal minor is $(x_1^2 + 6a_3x_1 + 2a_2)(306x_2^{16}) \ge 0$, which holds for all $x_2 \in \mathbb{R}$ since $306x_2^{16} \ge 0$. Therefore, for the objective function to be convex:

$$a_2 \ge \frac{9}{2}a_3^2$$

The function defining the inequality constraint must also be convex. This function is $g(x) = a_2x_1^2 + a_1x_1 + a_0 + 10x_2 - 5$. Its Hessian is:

$$\nabla^2 g(x) = \begin{bmatrix} 2a_2 & 0 \\ 0 & 0 \end{bmatrix}$$

For this matrix to be positive semidefinite, $2a_2 \ge 0$. The function defining the equality constraint is $h(x) = a_1x_1 + a_0 - x_2 - 1 = [a_0 \;\; -1][x_1 \;\; x_2]^\top + (a_0 - 1)$, which is affine for all values of a_1 and a_0. Therefore, the optimization problem is convex for:

$$a_2 \ge \frac{9}{2}a_3^2,\; a_0 \in \mathbb{R}, a_1 \in \mathbb{R}, a_3 \in \mathbb{R}$$

Problem 2

Consider the optimization problem:

$$\min_{x\in\mathbb{R}^3} \quad e^{-x_1-x_2-x_3} + (x_1+2x_2+3x_3)^4 \tag{2a}$$

$$\text{s.t.} \quad x_1 - x_2 - x_3 = 1 \tag{2b}$$

$$x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3 \leq 2 \tag{2c}$$

$$0 \leq x_1 \leq 1 \tag{2d}$$

$$0 \leq x_2 \leq 1 \tag{2e}$$

$$0 \leq x_3 \leq 1 \tag{2f}$$

i) Prove that $x = [1 \;\; 0 \;\; 0]^T$ is a feasible point.

ii) Using the Weierstrass theorem, prove that the above optimization problem has a solution.

iii) Prove that every local minimum of the above optimization problem is a global minimum.

Solution

i) The point $x = [1 \;\; 0 \;\; 0]^\top$ is feasible if each constraint is satisfied at it. The equality constraint is satisfied since

$$x_1 - x_2 - x_3 = 1 - 0 - 0 = 1.$$

The inequality constraint is satisfied since

$$x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3 = 1^2 + 0^2 + 0^2 + 1\cdot 0 + 0 \cdot 0 + 1 \cdot 0 = 1 \leq 2.$$

The remaining constraints are satisfied since

$$0 \leq x_1 = 1 \leq 1, \quad 0 \leq x_2 = 0 \leq 1, \quad 0 \leq x_3 = 0 \leq 1.$$

Therefore, the point $x = [1 \;\; 0 \;\; 0]^\top$ is feasible.

ii) The Weierstrass theorem states that an optimization problem with a continuous objective function has a finite solution if the feasible set is closed and bounded. In terms of this optimization problem, if the functions defining the inequality and equality constraints are continuous, then the feasible set is closed. The functions

$$x_1 - x_2 - x_3 - 1, \quad x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3, \quad -x_1, \quad x_1 - 1, \quad -x_2, \quad x_2 - 1, \quad -x_3, \quad x_3 - 1$$

are all continuous. Therefore, the feasible set is closed. The feasible set is a subset of the box defined by the constraints $0 \leq x_i \leq 1$ for $i = 1, 2, 3$, which is bounded. Therefore, the feasible set is also bounded. Since the objective function

$$e^{-x_1-x_2-x_3} + (x_1+2x_2+3x_3)^4$$

is continuous, the optimization problem has a finite solution.

iii) For a convex optimization problem, any local minimum is also a global minimum. Since the objective function is the sum of a composition of an exponential function with an affine transformation ($a^\top x$ with $a = (-1, -1, -1)$) and a composition of a convex polynomial function ($f''(u) = 12u^2$) with an affine transformation ($b^\top x$ with $b = (1, 2, 3)$), both of which are convex, so the objective function is convex. The inequality constraint can be rewritten as

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq 2$$

The matrix in the quadratic form is positive semidefinite since its eigenvalues are $\lambda_1 = 2$, $\lambda_2 = \frac{1}{2}$, and $\lambda_3 = \frac{1}{2}$, which are all non-negative. Therefore, the function defining the inequality constraint is convex. The equality constraint is affine since it can be written as

$$\begin{bmatrix} 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - 1 = 0.$$

The functions defining the last inequality constraints are also convex since their Hessians are zero matrices. Since the functions defining the inequality constraints are convex and the function defining the equality constraint is affine, the feasible set is convex. Therefore, the optimization problem is convex, and every local minimum is a global minimum.

Problem 3

Consider a convex function $f(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}$ such that:

- The points $(2, 2)$ and $(-2, 4)$ are both global minima of the function $f(x_1, x_2)$ with the property that $f(2, 2) = 5$.
- $f(0, 0)$ is equal to 10.

Answer the following questions:

i) Prove that the point $(0, 3)$ is a global minimum of the function $f(x_1, x_2)$.

ii) Prove that the point $(0, -1)$ is not a local minimum of the function $f(x_1, x_2)$.

Solution

i) The point $(0, 3)$ lies on the line segment between the points $(2, 2)$ and $(-2, 4)$ when $\alpha = \frac{1}{2}$. Since $f(x_1, x_2)$ is convex, for any two points $x, y \in \mathbb{R}^2$ and any $\alpha \in [0, 1]$, $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$. Substituting in $x = (2, 2)$ $y = (-2, 4)$ and $\alpha = \frac{1}{2}$ into the inequality,

$$\begin{aligned} f(\alpha(2,2) + (1-\alpha)(-2,4)) &\leq \frac{1}{2}f(2,2) + \frac{1}{2}f(-2,4) \\ f(0,3) &\leq \frac{1}{2}(5) + \frac{1}{2}(5) \\ f(0,3) &\leq 5 \end{aligned}$$

Since $f(2, 2) = f(-2, 4) = 5$ are both global minima, $f(0, 3) = 5$ and is a global minimum.

ii) There exists a line through the points $(0, -1)$ and $(0, 3)$ that goes through the point $(0, 0)$. When α is equal to $\frac{3}{4}$, the point is on the line $\alpha(0, 3) + (1-\alpha)(0, -1) = (0, 0)$. Since $f(x_1, x_2)$ is a convex function,

$$\begin{aligned} f(0,0) &\leq \frac{3}{4}f(0,-1) + \frac{1}{4}f(0,3) \\ 10 &\leq \frac{3}{4}f(0,-1) + \frac{1}{4}(5) \\ \frac{35}{3} &\leq f(0,-1) \end{aligned}$$

So, $f(0, -1)$ is at least $\frac{35}{3}$, which is greater than $f(0, 3)$. Therefore, $(0, -1)$ cannot be a local minimum of $f(x_1, x_2)$, because for any $\alpha \in [0, 1]$

$$f(\alpha(0,-1) + (1-\alpha)(0,3)) \leq \alpha f(0,-1) + (1-\alpha)f(0,3) \leq f(0,-1)$$

So any point on the line segment between $(0, -1)$ and $(0, 3)$, which lies at a global minimum, has a function value less than that at $(0, -1)$. Any neighborhood around $(0, -1)$ will contain points on this line segment, so $(0, -1)$ cannot be a local minimum.

Problem 4

Answer the following questions about coercive functions:

i) Find all coefficients $\alpha_1, ..., \alpha_n$ and $\beta_1, ..., \beta_n$ for which the function

$$\alpha_1 x_1^2 + \cdots + \alpha_n x_n^2 + \cos\left(\beta_1 x_1^3 + \cdots + \beta_n x_n^3\right) \tag{3}$$

is coercive.

ii) Given a constant a, prove that the function $x_1^2 + x_2^2 + ax_1x_2$ is coercive if $|a| < 2$ and is not coercive if $|a| \geq 2$.

Solution

i) For the function to be coercive, $\lim_{\|x\|\to\infty} f(x) = \infty$ must hold. The cosine term is bounded between -1 and 1, so as $\|x\| \to \infty$, so the coefficients β_i can take on any value. All coefficients α_i must be positive for the quadratic term to grow without bound as $\|x\| \to \infty$ for any x. This is because if any α_i is negative, then along the axis corresponding to that variable, that is, when all other variables are zero, the function will tend to $-\infty$ as that variable tends to ∞. Therefore, the function is coercive if and only if $\alpha_i > 0$ for all $i = 1, \dots, n$.

ii) The function $x_1^2 + x_2^2 + ax_1x_2$ can be rewritten as

$$x_1^2 + x_2^2 + ax_1x_2 = x^\top Ax = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & \frac{a}{2} \\ \frac{a}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The eigenvalues of symmetric matrix A are given by the characteristic polynomial $\det(A - \lambda I) = 0$, which gives eigenvalues $\lambda = 1 \pm \frac{a}{2}$, with minimum eigenvalue of $\lambda_m in = 1 - \frac{|a|}{2}$. If $|a| < 2$, all eigenvalues are strictly positive, and A is positive definite. The function is then lower bounded by $\lambda_{\min}\|x\|_2^2$ since

$$\begin{aligned} x^\top Ax &= x^\top V\Lambda V^\top x \\ &= y^\top \Lambda y \\ &= \sum_{i=1}^n \lambda_i y_i^2 \\ &\geq \lambda_{\min} \sum_{i=1}^n y_i^2 \\ &= \lambda_{\min}\|y\|_2^2 \\ &= \lambda_{\min}\|x\|_2^2 \end{aligned}$$

where V is the orthogonal matrix of eigenvectors and Λ is the diagonal matrix of eigenvalues. Since V is orthogonal, $\|y\|_2 = \|x\|_2$. Therefore, $x^\top Ax \geq \lambda_{\min}\|x\|_2^2 \to \infty$ as $\|x\| \to \infty$ and the function is coercive. If $|a| = 2$, A is linearly dependent and the null space consists of more than just the zero vector. Therefore, there exist non-zero vectors x such that $Ax = 0$, and along those directions the function equals 0 and does not tend to ∞ as $\|x\| \to \infty$, and is not coercive. When $|a| > 2$, the minimum eigenvalue is negative, and there exists a direction along its eigenvector for which the function tends to $-\infty$ as $\|x\| \to \infty$, and the function is not coercive. Therefore, the function is coercive if $|a| < 2$ and is not coercive if $|a| \geq 2$.

Problem 5

Consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & \alpha x_1 + \beta x_2 \\ \text{s.t.} \quad & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1 + 4x_2 \leq 4 \\ & 4x_1 + x_2 \leq 4 \end{aligned} \tag{4}$$

where α and β are two constants. Run the following experiment 200 times:

- Generate a random pair (α, β) where α and β are chosen uniformly from the interval $[-1, 1]$.
- Solve the resulting LP in CVX and record the obtained optimal solution x^*.

By checking how many different solutions you have obtained in these 200 experiments and how many times each solution has appeared, empirically find the vertices of the feasible set and for each obtained vertex calculate the probability that the vertex is a solution when (α, β) is random.

Solution

The vertices of the feasible set are

$$(0,0), \quad (0,1), \quad (1,0), \quad \left(\frac{4}{5}, \frac{4}{5}\right)$$

After one instance of running the experiment 200 times, the following probabilities were observed for each vertex being the optimal solution:

- $(0,0)$: 0.310
- $(0,1)$: 0.235
- $(1,0)$: 0.185
- $\left(\frac{4}{5}, \frac{4}{5}\right)$: 0.270