Optimization Models: Homework #7

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Problem 1

Given constants $\alpha_1, \ldots, \alpha_n$, consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & e^{\left(\sum_{i=1}^n \alpha_i x_i\right)} \\ \text{s.t.} \quad & \left(\|x\|_\infty + 10 \times \|x\|_1\right)^2 \leq 5 \end{aligned} \tag{1}$$

i) Convert the optimization problem (1) to an LP.

ii) Put the obtained LP into the LP standard form.

Solution

i) To convert the optimization problem (1) to an LP, we note that the objective function $e^{\left(\sum_{i=1}^n \alpha_i x_i\right)}$ is monotonically increasing with respect to $\sum_{i=1}^n \alpha_i x_i$. Therefore, we can minimize $\sum_{i=1}^n \alpha_i x_i$ instead. The constraint can be rewritten as:

$$\|x\|_\infty + 10 \times \|x\|_1 \leq \sqrt{5}$$

Let $t = \|x\|_\infty$ and $s = \|x\|_1$. Then we have:

$$\begin{aligned} & t + 10s \leq \sqrt{5} \\ & -t \leq x_i \leq t, \quad i = 1, \ldots, n \\ & s = \sum_{i=1}^n |x_i| \end{aligned} \tag{2}$$

To handle the absolute values in s, we introduce auxiliary variables $u_i \geq 0$ such that $u_i \geq x_i$ and $u_i \geq -x_i$. Thus, we can rewrite s as:

$$s = \mathbf{1}^T u$$

where $\mathbf{1}$ is the vector of all ones and $u = [u_1, \ldots, u_n]^T$. The LP formulation becomes:

$$\begin{aligned} \min_{x,t,s,u} \quad & \sum_{i=1}^n \alpha_i x_i \\ \text{s.t.} \quad & t + 10s \leq \sqrt{5} \\ & -t \leq x_i \leq t, \quad i = 1, \ldots, n \\ & u_i \geq x_i, \quad i = 1, \ldots, n \\ & u_i \geq -x_i, \quad i = 1, \ldots, n \\ & s = \mathbf{1}^T u \end{aligned} \tag{3}$$

ii) To put the obtained LP into the LP standard form, we need to express all inequalities as equalities and ensure all variables are non-negative. We can introduce slack variables for the inequalities:

$$\begin{aligned} \min_{x,t,s,u,\text{slack}} \quad & \sum_{i=1}^n \alpha_i x_i \\ \text{s.t.} \quad & t + 10s + \text{slack}_1 = \sqrt{5} \\ & -t + \text{slack}_{2,i} = x_i, \quad i = 1, \ldots, n \\ & t + \text{slack}_{3,i} = x_i, \quad i = 1, \ldots, n \\ & u_i - x_i + \text{slack}_{4,i} = 0, \quad i = 1, \ldots, n \\ & u_i + x_i + \text{slack}_{5,i} = 0, \quad i = 1, \ldots, n \\ & s - \mathbf{1}^T u + \text{slack}_6 = 0 \\ & x_i, t, s, u_i, \text{slack} \geq 0, \quad i = 1, \ldots, n \end{aligned} \tag{4}$$

Problem 2

(Coding) Consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^{40}} \quad & a^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{5}$$

where $a \in \mathbb{R}^{40}$, $A \in \mathbb{R}^{5 \times 40}$, and $b \in \mathbb{R}^5$. Write code to run the following experiment 10 times:

- Generate a random pair (A, b) where each element of A and b is chosen uniformly from the interval $[-1, 1]$.
- Find all vertices of the LP (5) and report the number of vertices (note: you do not need to report the values of the vertices).

By averaging over the 10 experiments, calculate the expected number of vertices for the feasible set of the LP (5).

Problem 3

Consider the IP

$$
\begin{aligned}
\max_{x\in\mathbb{R}^5} \quad & 5x_1 - 7x_2 + 10x_3 + 3x_4 - x_5 \\
\text{s.t.} \quad & -x_1 - 3x_2 + 3x_3 - x_4 - 2x_5 \leq 0 \\
& 2x_1 - 5x_2 + 3x_3 - 2x_4 - 2x_5 \leq 3 \\
& -x_2 + x_3 + x_4 - x_5 \geq 2 \\
& x_1, x_2, x_3, x_4, x_5 \in \{0, 1, 2, 3\}
\end{aligned} \tag{6}
$$

i) Formulate the LP relaxation of the problem (6).

ii) **(Coding)** Solve the obtained LP in CVX and report its optimal solution.

iii) Calculate the difference between the optimal objective values of the IP (6) and its LP relaxation.

Problem 4

A company supplies goods to three customers, each requiring 30 units, i.e., $d_1 = d_2 = d_3 = 30$. The company has two warehouses. Warehouse 1 has $s_1 = 40$ units and warehouse 2 has $s_2 = 30$ units. The costs of shipping 1 unit from warehouse 1 to customers 1, 2, and 3 are \$20, \$25, and \$20, respectively. The costs of shipping 1 unit from warehouse 2 to customers 1, 2, and 3 are \$10, \$30, and \$40, respectively.
There is a penalty for each unmet unit of demand: the penalties for customers 1, 2, and 3 are \$90, \$80, and \$70, respectively. Since $s_1 + s_2 < d_1 + d_2 + d_3$, define a fictitious warehouse 3 with supply s_3 that models the shortage so that $s_1 + s_2 + s_3 \geq d_1 + d_2 + d_3$. The goal is to minimize the sum of penalties and shipping costs.

i) Formulate the transportation problem as an IP.

ii) Obtain an equivalent LP model of the problem.

iii) **(Coding)** By solving the problem in CVX, report the optimal solution of the transportation problem.

Problem 5

(Coding) Consider a vector $x^* \in \mathbb{R}^{1000}$ with entries defined by

$$x_i^* = \begin{cases} 5 & \text{if } i = 10 \\ -2 & \text{if } i = 20 \\ 7 & \text{if } i = 95 \\ 15 & \text{if } i = 750 \\ -10 & \text{if } i = 920 \\ 0 & \text{if } i \in \{1, \dots, 1000\} \setminus \{10, 20, 95, 750, 920\} \end{cases} \tag{7}$$

This vector has only 5 nonzero entries. Assume x^* is unknown and we aim to find x^* from m linear measurements $y_1, \dots, y_m$ of the form

$$y_i = a_i^\top x^*, \quad i = 1, \dots, m \tag{8}$$

where $a_i \in \mathbb{R}^{1000}$ has entries drawn i.i.d. from a normal distribution. If $m \geq 1000$, we can solve a linear system to find x^*. The goal is to find the smallest m for which we can still learn x^* from the measurements. Based on compressed sensing, consider the convex optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^{1000}} \quad & \|x\|_1 \\ \text{s.t.} \quad & y_i = a_i^\top x, \quad i = 1, \dots, m \end{aligned} \tag{9}$$

Let $\hat{x}$ denote an optimal solution of (9). If the error $\|x^* - \hat{x}\|_1 < 0.01$, we declare that (9) has been successful in finding x^* (up to a small precision error); otherwise, it has failed. Write code in CVX to run the following experiment 10 times for every $m \in \{5, 6, \dots, 98, 99, 100\}$:

- Generate random vectors $a_1, \dots, a_m$, and compute the measurements $y_1, \dots, y_m$.
- Solve the optimization problem (9) and check whether it has been successful in finding x^*.

Define the empirical success probability $p(m)$ as the number of successes divided by 10.

i) Draw the function $p(m)$ for $m = 5, \dots, 100$.

ii) What observation(s) can you make from the function $p(m)$?