Optimization Models: Homework #1

Due on February 12, 2014 at 3:10pm

Zachary Brandt

Problem 1

The Fruit Computer company produces two types of computers: Pear computers and Apricot computers. The following table shows the number of hours and the number of chips needed to make a computer as well as the equipment cost and selling price:

Computer	Labor	Chips	Equipment cost per unit ($)	Selling price ($)
Pear	1 hour	3	55	400
Apricot	2 hours	6	100	900

A total of 3000 chips and 1300 hours of labor are available. The company needs to decide how many computers from each type should be made in order to maximize the profit.
Formulate this problem as an optimization problem.

Solution

$$\min_{\boldsymbol{x}} \quad \underbrace{\boxed{-400x_p - 900x_a + 55x_p + 100x_a}}_{f_0(\boldsymbol{x})},$$

$$\text{s.t.} \quad \underbrace{\boxed{x_p + 2x_a - 1300}}_{f_1(\boldsymbol{x})} \leq 0,$$

$$\underbrace{\boxed{3x_p + 6x_a - 3000}}_{f_2(\boldsymbol{x})} \leq 0,$$

$$\underbrace{\boxed{-x_p, -x_a}}_{f_3(\boldsymbol{x})} \leq 0$$

where

- vector $x \in \mathbb{R}^2$ is the *decision variable*;

- $f_0 : \mathbb{R}^2 \to \mathbb{R}$ is the *objective function*, or *cost*;

- $f_1 : \mathbb{R}^2 \to \mathbb{R}$ represents the *labor-hour constraint*;

- $f_2 : \mathbb{R}^2 \to \mathbb{R}$ represents the *chip constraint*;

- $f_3 : x_p$ and x_a must be non-negative;

Problem 2

The Paradise City Police Department employs 30 police officers. Each officer works 5 days per week. The crime rate fluctuates with the day of the week, so the minimum number of police officers required each day depends on which day of the week it is: Saturday, 24; Sunday, 13; Monday, 15; Tuesday, 25; Wednesday, 26; Thursday, 18; Friday, 22. The police department wants to schedule police officers to minimize the number of officers whose days off are not consecutive.

Formulate this problem as an optimization problem (hint: define the variable x_{ij}, where $i, j \in \{1, 2, \ldots, 7\}$ and $i \neq j$, to be the number of officers who are scheduled to not work on days i and j).

Solution

$$
\min_{x_{ij}} \quad \boxed{\sum_{i=1}^{7} \sum_{j=i+1}^{7} x_{ij}}_{f_0},
$$

$$
\text{s.t.} \quad \sum_{j=1, j \neq 1}^{7} x_{1j} \leq 6,
$$

$$
\sum_{j=1, j \neq 2}^{7} x_{2j} \leq 17,
$$

$$
\sum_{j=1, j \neq 3}^{7} x_{3j} \leq 15,
$$

$$
\sum_{j=1, j \neq 4}^{7} x_{4j} \leq 5,
$$

$$
\sum_{j=1, j \neq 5}^{7} x_{5j} \leq 4,
$$

$$
\sum_{j=1, j \neq 6}^{7} x_{6j} \leq 12,
$$

$$
\sum_{j=1, j \neq 7}^{7} x_{7j} \leq 8,
$$

$$
x_{ij} \geq 0 \text{ and integer}, \quad \forall i, j \in \{1, \ldots, 7\}, i \neq j
$$

where

- $x_{ij} \geq 0$ is an integer variable, where x_{ij} is the number of officers who are scheduled to not work on days i and j;

Problem 3

A taxi company has n taxis available and n customers to be picked up as soon as possible. For every $i, j \in \{1, \ldots, n\}$, if taxi i decides to pick up customer j, the amount of time (delay) to pick up the customer is d_{ij}. Each taxi is allowed to pick up only one customer. The goal is to assign each customer to a taxi so that the total delay (i.e., sum of the delays for all customers) is minimized.

Formulate this assignment problem as an optimization problem.

Solution

$$
\min_{x_{ij}} \quad \underbrace{\boxed{\sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_{ij}}}_{f_0(x_{ij})},
$$

$$
\text{s.t.} \quad \underbrace{\boxed{-1 + \sum_{i=1}^{n} x_{ij} \leq 0 \quad \forall i \in \{1, \ldots, n\}}}_{f_1(x_{ij})},
$$

where

- $x_{ij} \in \{0, 1\}$ is a binary variable, where $x_{ij} = 1$ if taxi i is assigned to customer j and $x_{ij} = 0$ otherwise;

- f_0 : the *objective function*, or *total delay* for all customers;

- f_1 : the *taxi assignment constraint*, each taxi can only pick up one customer;

Problem 4

Consider the set $\mathcal{S}$ defined as

$$\mathcal{S} = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 + 3x_3 = 0, \;\; 3x_1 + 2x_2 + x_3 = 0\}$$

Show that $\mathcal{S}$ is a subspace. Determine its dimension and find a basis for it.

Solution
The set $\mathcal{S}$ is a subset of $\mathbb{R}^3$ if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{S} \Rightarrow \alpha\boldsymbol{x} + \beta\boldsymbol{y} \in \mathcal{S} \text{for all scalars } \alpha, \beta.$$

Setting the equations of the set $\mathcal{S}$ equal to each other:

$$x_1 + 2x_2 + 3x_3 = 3x_1 + 2x_2 + x_3$$
$$-2x_1 + 0 + 2x_3 = 0$$
$$x_1 = x_3$$

Substituting for x_3 into the first equation:

$$x_1 + 2x_2 + 3x_3 = 0$$
$$4x_1 + 2x_2 = 0$$
$$2x_1 + x_2 = 0$$
$$x_2 = -2x_1$$

Therefore, the set $\mathcal{S}$ can be expressed as a line:

$$\mathcal{S} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_3, x_2 = -2x_1\}$$
$$= \{(x_1, -2x_1, x_1) \mid x_1 \in \mathbb{R}\}$$
$$= \{x_1(1, -2, 1) \mid x_1 \in \mathbb{R}\}$$

which is a subspace of $\mathbb{R}^3$. The dimension of this subspace is 1, and a basis for it is given by the vector

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Problem 5

Consider the set $\mathcal{P}$ defined as

$$\mathcal{P} = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 + 3x_3 = 5\}$$

Show that $\mathcal{P}$ is an affine set of dimension 2. To this end, express it as $x^{(0)} + \text{span}(x^{(1)}, x^{(2)})$, where $x^{(0)} \in \mathcal{P}$ and $x^{(1)}, x^{(2)}$ are linearly independent vectors.

Solution

The set $\mathcal{P} \in \mathbb{R}^3$ is affine if there is a subspace V of $\mathbb{R}^3$ and a vector $\boldsymbol{x}^{(0)} \in \mathbb{R}^3$ such that

$$\mathcal{P} = \boldsymbol{x}^{(0)} + V,$$

i.e., a vector $\boldsymbol{x}^{(0)}$ is added to all vectors in the subspace V to obtain $\mathcal{P}$.

$x_1 + 2x_2 + 3x_3 = 5$ defines a plane in $\mathbb{R}^3$, which is an affine set of dimension 2, and constitutes the span of two vectors. The following equation represents the equation as a dot product of a normal vector and an arbitrary position vector on the plane:

$$\boldsymbol{n} \cdot \boldsymbol{r} = \boldsymbol{n} \cdot \boldsymbol{r_0}$$
$$(1, 2, 3) \cdot (x_1, x_2, x_3) = (1, 2, 3) \cdot (0, 1, 1)$$
$$x_1 + 2x_2 + 3x_3 = 5$$

where $\boldsymbol{r_0} = x^{(0)} = (0, 1, 1) \in \mathcal{P}$ is a point on the plane. The normal vector $\boldsymbol{n} = (1, 2, 3)$ is orthogonal to the plane and allows for finding two linearly independent vectors that make up the span:

$$\boldsymbol{n} \cdot \boldsymbol{v} = 0$$
$$(1, 2, 3) \cdot (x_1, x_2, x_3) = 0$$
$$x_1 + 2x_2 + 3x_3 = 0$$

Two vectors that satisfy the equation are $(-1, -1, 1)$ and $(1, -2, 1)$. Therefore, the set $\mathcal{P}$ can be expressed as:

$$\mathcal{P} = \boldsymbol{x}^{(0)} + \text{span}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})$$
$$= (0, 1, 1) + \text{span}((-1, -1, 1), (1, -2, 1)).$$