Optimization Models: Homework #1

Due on September 11, 2025 at 11:59pm

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Problem 1

The Fruit Computer company produces two types of computers: Pear computers and Apricot computers. The following table shows the number of hours and the number of chips needed to make a computer as well as the equipment cost and selling price:

Computer	Labor	Chips	Equipment cost per unit ($)	Selling price ($)
Pear	1 hour	3	55	400
Apricot	2 hours	6	100	900

A total of 3000 chips and 1300 hours of labor are available. The company needs to decide how many computers from each type should be made in order to maximize the profit.

Formulate this problem as an optimization problem.

Solution

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & \underbrace{\boxed{-400x_p - 900x_a + 55x_p + 100x_a}}_{f_0(\boldsymbol{x})}, \\
\text{s.t.} \quad & \underbrace{\boxed{x_p + 2x_a - 1300}}_{f_1(\boldsymbol{x})} \leq 0, \\
& \underbrace{\boxed{3x_p + 6x_a - 3000}}_{f_2(\boldsymbol{x})} \leq 0, \\
& \underbrace{\boxed{-x_p}}_{f_3(\boldsymbol{x})} \leq 0, \\
& \underbrace{\boxed{-x_a}}_{f_4(\boldsymbol{x})} \leq 0
\end{aligned}
$$

where

- $(x_p, x_a) \in \mathbb{Z}^2$ is the *decision variable*;
- $f_0 : \mathbb{R}^2 \to \mathbb{R}$ is the *objective function*, or *cost*;
- $f_1 : \mathbb{R}^2 \to \mathbb{R}$ represents the *labor-hour constraint*;
- $f_2 : \mathbb{R}^2 \to \mathbb{R}$ represents the *chip constraint*;
- f_3 & f_4 : x_p and x_a must be non-negative;

Problem 2

The Paradise City Police Department employs 30 police officers. Each officer works 5 days per week. The crime rate fluctuates with the day of the week, so the minimum number of police officers required each day depends on which day of the week it is: Saturday, 24; Sunday, 13; Monday, 15; Tuesday, 25; Wednesday, 26; Thursday, 18; Friday, 22. The police department wants to schedule police officers to minimize the number of officers whose days off are not consecutive.

Formulate this problem as an optimization problem (hint: define the variable x_{ij}, where $i, j \in \{1, 2, \ldots, 7\}$ and $i \neq j$, to be the number of officers who are scheduled to not work on days i and j).

Solution

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & \underbrace{\boxed{-x_{12} - x_{23} - x_{34} - x_{45} - x_{56} - x_{67} - x_{71}}}_{f_0(\boldsymbol{x})}, \\
\text{s.t.} \quad & -6 + \textstyle\sum_{j=2}^{7} x_{1j} \leq 0, \\
& -17 + \textstyle\sum_{j=1, j\neq 2}^{7} x_{2j} \leq 0, \\
& -15 + \textstyle\sum_{j=1, j\neq 3}^{7} x_{3j} \leq 0, \\
& -5 + \textstyle\sum_{j=1, j\neq 4}^{7} x_{4j} \leq 0, \\
& -4 + \textstyle\sum_{j=1, j\neq 5}^{7} x_{5j} \leq 0, \\
& -12 + \textstyle\sum_{j=1, j\neq 6}^{7} x_{6j} \leq 0, \\
& -8 + \textstyle\sum_{j=1, j\neq 7}^{7} x_{7j} \leq 0, \\
& x_{ij} - 30 \leq 0, \quad \forall i, j \in \{1, 2, \ldots, 7\}, i \neq j \\
& -x_{ij} \leq 0 \quad \forall i, j \in \{1, 2, \ldots, 7\}, i \neq j
\end{aligned}
$$

where

- $x_{ij} \in \mathbb{Z}$ is the number of officers who are scheduled to not work on days i and j, where $i, j \in \{1, 2, \ldots, 7\}$ and $i \neq j$;
- $f_0 : \mathbb{R}^7 \to \mathbb{R}$ is the *objective function*, or the number of officers whose days off are not consecutive;
- The first seven constraints represent the minimum number of officers required each day of the week, where day 1 is Sunday, day 2 is Monday, day 3 is Tuesday, day 4 is Wednesday, day 5 is Thursday, day 6 is Friday, and day 7 is Saturday;
- The eighth constraint prevents the number of officers scheduled to not work on any days i and j from exceeding 30 (the total number of officers employed);
- The ninth constraint prevents the number of officers scheduled to not work on any days i and j from going negative;

Problem 3

A taxi company has n taxis available and n customers to be picked up as soon as possible. For every $i, j \in \{1, \ldots, n\}$, if taxi i decides to pick up customer j, the amount of time (delay) to pick up the customer is d_{ij}. Each taxi is allowed to pick up only one customer. The goal is to assign each customer to a taxi so that the total delay (i.e., sum of the delays for all customers) is minimized.

Formulate this assignment problem as an optimization problem.

Solution

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & \underbrace{\boxed{\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}x_{ij}}}_{f_0(\boldsymbol{x})}, \\
\text{s.t.} \quad & \underbrace{\boxed{-1 + \sum_{i=1}^{n} x_{ij}}}_{f_1(\boldsymbol{x})} \leq 0 \quad \forall i \in \{1, \ldots, n\}, \\
& \underbrace{\boxed{1 - \sum_{i=1}^{n} x_{ij}}}_{f_2(\boldsymbol{x})} \leq 0 \quad \forall i \in \{1, \ldots, n\}, \\
& \underbrace{\boxed{-1 + \sum_{j=1}^{n} x_{ij}}}_{f_3(\boldsymbol{x})} \leq 0 \quad \forall j \in \{1, \ldots, n\}, \\
& \underbrace{\boxed{1 - \sum_{j=1}^{n} x_{ij}}}_{f_4(\boldsymbol{x})} \leq 0 \quad \forall j \in \{1, \ldots, n\}
\end{aligned}
$$

where

- $x_{ij} \in \{0, 1\}$ is a binary variable, where $x_{ij} = 1$ if taxi i is assigned to customer j and $x_{ij} = 0$ otherwise;
- f_0 : the *objective function*, or *total delay* for all customers;
- f_1 & f_2 : the *taxi assignment constraint*, each taxi can only pick up one customer;
- f_3 & f_4 : the *customer assignment constraint*, each customer must be picked up by one taxi;

Problem 4

Consider the set $\mathcal{S}$ defined as

$$\mathcal{S} = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 + 3x_3 = 0, \quad 3x_1 + 2x_2 + x_3 = 0\}$$

Show that $\mathcal{S}$ is a subspace. Determine its dimension and find a basis for it.

Solution

The set $\mathcal{S}$ is a subset of $\mathbb{R}^3$ if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{S} \Rightarrow \alpha \boldsymbol{x} + \beta \boldsymbol{y} \in \mathcal{S} \text{ for all scalars } \alpha, \beta.$$

Setting the equations of the set $\mathcal{S}$ equal to each other:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 3x_1 + 2x_2 + x_3 \\ -2x_1 + 0 + 2x_3 &= 0 \\ x_1 &= x_3 \end{aligned}$$

Substituting for x_3 into the first equation:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 4x_1 + 2x_2 &= 0 \\ 2x_1 + x_2 &= 0 \\ x_2 &= -2x_1 \end{aligned}$$

Therefore, the set $\mathcal{S}$ can be expressed as a line:

$$\begin{aligned} \mathcal{S} &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_3, x_2 = -2x_1\} \\ &= \{(x_1, -2x_1, x_1) \mid x_1 \in \mathbb{R}\} \\ &= \{x_1(1, -2, 1) \mid x_1 \in \mathbb{R}\} \end{aligned}$$

which is a subspace of $\mathbb{R}^3$ since it includes $\mathbf{0}$ when $x_1 = 0$, and any addition &/or multiplication of vectors in $\mathcal{S}$ remains along the line. The dimension of this subspace is 1, and a basis for $\mathcal{S}$ is given by the vector

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Problem 5

Consider the set $\mathcal{P}$ defined as

$$\mathcal{P} = \{x \in \mathbb{R}^3 \mid x_1 + 2x_2 + 3x_3 = 5\}$$

Show that $\mathcal{P}$ is an affine set of dimension 2. To this end, express it as $x^{(0)} + \text{span}(x^{(1)}, x^{(2)})$, where $x^{(0)} \in \mathcal{P}$ and $x^{(1)}, x^{(2)}$ are linearly independent vectors.

Solution

The set $\mathcal{P} \in \mathbb{R}^3$ is affine if there is a subspace V of $\mathbb{R}^3$ and a vector $\boldsymbol{x}^{(0)} \in \mathbb{R}^3$ such that

$$\mathcal{P} = \boldsymbol{x}^{(0)} + V,$$

i.e., a vector $\boldsymbol{x}^{(0)}$ is added to all vectors in the subspace V to obtain $\mathcal{P}$.
$x_1 + 2x_2 + 3x_3 = 5$ defines a plane in $\mathbb{R}^3$, which is an affine set of dimension 2, and constitutes the span of two vectors. The following equation represents the equation as a dot product of a normal vector and an arbitrary position vector on the plane:

$$\begin{aligned} \boldsymbol{n} \cdot \boldsymbol{r} &= \boldsymbol{n} \cdot \boldsymbol{r_0} \\ (1,2,3) \cdot (x_1, x_2, x_3) &= (1,2,3) \cdot (0,1,1) \\ x_1 + 2x_2 + 3x_3 &= 5 \end{aligned}$$

where $\boldsymbol{r_0} = x^{(0)} = (0,1,1) \in \mathcal{P}$ is a point on the plane. The normal vector $\boldsymbol{n} = (1,2,3)$ is orthogonal to the plane and allows for finding two linearly independent vectors that make up the span:

$$\begin{aligned} \boldsymbol{n} \cdot \boldsymbol{v} &= 0 \\ (1,2,3) \cdot (x_1, x_2, x_3) &= 0 \\ x_1 + 2x_2 + 3x_3 &= 0 \end{aligned}$$

Two vectors that satisfy the equation are $(-1,-1,1)$ and $(1,-2,1)$. Therefore, the set $\mathcal{P}$ can be expressed as:

$$\begin{aligned} \mathcal{P} &= \boldsymbol{x}^{(0)} + \text{span}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) \\ &= (0,1,1) + \text{span}((-1,-1,1),(1,-2,1)). \end{aligned}$$