Optimization Models: Homework #3

Due on October 10, 2025 at 11:59pm

Zachary Brandt
zbrandt@berkeley.edu

Problem 1

Consider an aerial system that moves in $\mathbb{R}^3$ according to the dynamics

$$x(k+1) = Ax(k) + Bu(k), \quad k = 0, 1, 2, 3 \tag{1}$$

where $x(k) \in \mathbb{R}^3$ is the position of the system at time $k \in \{0, 1, 2, 3, 4\}$ and $u(k) \in \mathbb{R}$ is the scalar input applied to the system at time k. Assume that the initial position $x(0)$ is equal to $[0 \ \ 0 \ \ 0]^T$. Given a target position $x_d \in \mathbb{R}^3$, the goal is to design the input sequence $u(0), u(1), u(2), u(3)$ to take the system to the target position x_d at time $k = 4$, i.e., $x(4) = x_d$.

i) Find a matrix $H \in \mathbb{R}^{3\times 4}$ in terms of A and B with the property that

$$x(4) = H \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ u(3) \end{bmatrix} \tag{2}$$

ii) Assume that

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tag{3}$$

Show that the vector $[1 \ \ 1 \ \ 0]^T$ belongs to $\mathcal{N}(H^T)$ (note: you are allowed to use a calculator to compute H, but you cannot use a calculator or a computer code to study the null space of H^T and the analysis should be done by hand).

iii) Again, consider the system parameters given in (3). By studying the relationship between $\mathcal{N}(H^T)$ and $\mathcal{R}(H)$, prove that there is no sequence of inputs that can take the system to the position $x_d = [1 \ \ 1 \ \ 0]^T$ at time 4.

iv) Again, consider the system parameters given in (3). By finding $\mathcal{N}(H^T)$ and using the relations $\mathcal{N}(H^T) \perp \mathcal{R}(H)$ and $\mathcal{N}(H^T) \oplus \mathcal{R}(H) = \mathbb{R}^3$, show that there exists a sequence of inputs to take the system to the position x_d at time 4 if and only if x_d belongs to the set

$$\{x \in \mathbb{R}^3 \mid x_1 + x_2 = 0\} \tag{4}$$

v) **(Coding)** Now, assume that

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \tag{5}$$

The goal is to find a sequence of inputs such that the total energy $u(0)^2 + u(1)^2 + u(2)^2 + u(3)^2$ is minimized and yet the system arrives at the target position $x_d = [3 \ \ 2 \ \ 2]^T$ at time 4. Formulate this as an optimization problem and write a code in CVX to solve the problem numerically. Plot the optimal trajectory (i.e., plot the optimal values of the points $x(0), ..., x(4)$ in $\mathbb{R}^3$ and then connect each point to the next one (such as $x(1)$ to $x(2)$)).

vi) **(Coding)** Consider the safety set

$$\mathcal{S} = \{x \in \mathbb{R}^3 \mid -3.3 \le x_i \le 3.2, \quad i = 1, 2, 3\} \tag{6}$$

Assume that the state $x(k)$ must always stay in the safety set $\mathcal{S}$ for $k = 0, 1, ..., 4$. Redo Part (v) under this additional constraint and find the optimal input sequence. Compares the optimal trajectories and optimal energies (objective values) obtained in Parts (v) and (vi).

Problem 2

(Coding) Consider the matrix A and vector x^* defined as

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad x^* = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{7}$$

Define $b = Ax^* + v$ where $v \in \mathbb{R}^6$ is some measurement noise. Assume that the user has no access to x^* and aims to learn x^* from the measurement vector b. We consider two different estimators to learn x^*:

$$l_1 \text{ estimator:} \qquad \min_x \|Ax - b\|_1, \tag{8a}$$

$$l_2 \text{ estimator:} \qquad \min_x \|Ax - b\|_2 \tag{8b}$$

Given a solution $\hat{x}$ obtained from any of the above estimators, we define the estimation error $e = \|\hat{x} - x^*\|_2$ (note that the error is always computed with respect to the l_2-norm no matter which estimator is used for obtaining $\hat{x}^*$). Assume that the noise v is in the form

$$v = \begin{bmatrix} t_1 \\ 0 \\ 0 \\ 0 \\ t_2 \\ 0 \end{bmatrix} \tag{9}$$

where t_1 and t_2 are constants that belong to the discrete set $\{-2, -1.9, -1.8, ..., -0.1, 0, 0.1, ..., 1.8, 1.9, 2\}$ (the increment is 0.1).

i) For each possible value of the pair (t_1, t_2), solve the l_1 and l_2 estimators in CVX and record the corresponding estimation errors (note: there are 41×41 possibilities for (t_1, t_2)).

ii) Draw a grid in $\mathbb{R}^2$ obtained as follows: For each possible value of the pair (t_1, t_2), we put a symbol in the location (t_1, t_2) in $\mathbb{R}^2$, where the symbol is a small red circle if the l_1 estimator gives the lowest estimation error and is a small blue circle if the l_2 estimator gives the lowest estimation error (note: if the estimation errors for both estimators are the same, use the blue circle). Analyze the plot and report your observations.

Problem 3

Consider the optimization problem

$$\min_{x \in \mathbb{R}^2} \quad e^{x_1+x_2} + 2x_1^2 + 2x_2^2 - x_1 x_2 - \sin(x_1 + x_2) \tag{10}$$

By analyzing the gradient and Hessian of the objective function, prove that $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a global minimum of the optimization problem (Hint: A matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite if and only if $a > 0$ and $ac - b^2 > 0$).