CS 70 Spring 2025 | Discrete Mathematics and Probability Theory Rao | HW 11

CS 70 Discrete Mathematics and Probability Theory
Spring 2025 Rao

HW 11

Due: Saturday, 4/12, 4:00 PM
Grace period until Saturday, 4/12, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

Zachary Brandt
zbrandt@berkeley.edu

1 Combined Head Count

Note 19

Suppose you flip a fair coin twice.

(a) What is the sample space Ω generated from these flips?

(b) Define a random variable X to be the number of heads. What is the distribution of X?

(c) Define a random variable Y to be 1 if you get a heads followed by a tails and 0 otherwise. What is the distribution of Y?

(d) Compute the conditional probabilities $\mathbb{P}[Y = i \mid X = j]$ for all i, j.

(e) Define a third random variable $Z = X + Y$. Use the conditional probabilities you computed in part (d) to find the distribution of Z.

Solution:

(a) The sample space Ω generated from two flips is $\{HH, HT, TH, TT\}$.

(b) The distribution of the random variable X is the collection of values $\{(a, \mathbb{P}[X = a]) : a \in \mathscr{A}\}$, which is displayed below:

outcomes ω	value of X (# heads)	probability of occurring
TT	0	$\frac{1}{4}$
HT, TH	1	$\frac{2}{4}$
HH	2	$\frac{1}{4}$

(c) The distribution of the random variable Y is displayed below:

outcomes ω	value of Y (heads, tails)	probability of occurring
TT, TH, HH	0	$\frac{3}{4}$
HT	1	$\frac{1}{4}$

(d) The conditional probabilities are displayed below:

X/Y	0	1	2	X
0	0.25	0.25	0.25	0.75
1	0	0.25	0	0.25
Y	0.25	0.5	0.25	

(e) The distribution of the random variable Z is displayed below:

outcomes ω	value of Z (X+Y)	probability of occurring
TT	0	$\frac{1}{4}$
TH	1	$\frac{1}{4}$
HH, HT	2	$\frac{2}{4}$

2 Testing Model Planes

Note 15

Amin is testing model airplanes. He starts with n model planes which each independently have probability p of flying successfully each time they are flown, where $0 < p < 1$. Each day, he flies every single plane and keeps the ones that fly successfully (i.e. don't crash), throwing away all other models. He repeats this process for many days, where each "day" consists of Amin flying all remaining model planes and throwing away any that crash. Let X_i be the random variable representing how many model planes remain after i days. Note that $X_0 = n$. Justify your answers for each part.

(a) What is the distribution of X_1? That is, what is $\mathbb{P}[X_1 = k]$?

(b) What is the distribution of X_2? That is, what is $\mathbb{P}[X_2 = k]$? Recognize the distribution of X_2 as one of the famous ones and provide its name and parameters.

(c) Repeat the previous part for X_t for arbitrary $t \geq 1$.

(d) What is the probability that at least one model plane still remains (has not crashed yet) after t days? Do not have any summations in your answer.

(e) Considering only the first day of flights, is the event A_1 that the first and second model planes crash independent from the event B_1 that the second and third model planes crash? Recall that two events A and B are independent if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$. Prove your answer using this definition.

(f) Considering only the first day of flights, let A_2 be the event that the first model plane crashes *and* exactly two model planes crash in total. Let B_2 be the event that the second plane crashes on the first day. What must n be equal to in terms of p such that A_2 is independent from B_2? Prove your answer using the definition of independence stated in the previous part.

(g) Are the random variables X_i and X_j, where $i < j$, independent? Recall that two random variables X and Y are independent if $\mathbb{P}[X = k_1 \cap Y = k_2] = \mathbb{P}[X = k_1]\,\mathbb{P}[Y = k_2]$ for all k_1 and k_2. Prove your answer using this definition.

3 Fishy Computations

Note 19

Assume for each part that the random variable can be modelled by a Poisson distribution.

(a) Suppose that on average, a fisherman catches 20 salmon per week. What is the probability that he will catch exactly 7 salmon this week?

(b) Suppose that on average, you go to Fisherman's Wharf twice a year. What is the probability that you will go at most once in 2024?

(c) Suppose that in March, on average, there are 5.7 boats that sail in Laguna Beach per day. What is the probability there will be *at least* 3 boats sailing throughout the *next two days* in Laguna?

(d) Denote $X \sim \text{Pois}(\lambda)$. Prove that

$$\mathbb{E}[X f(X)] = \lambda\, \mathbb{E}[f(X+1)]$$

for any function f.

4 Such High Expectations

Note 19

Suppose X and Y are independently drawn from a Geometric distribution with parameter p. For each of the below subparts, your answer must be simplified (i.e. NOT left in terms of a summation).

(a) Compute $\mathbb{E}[\min(X,Y)]$.

(b) Compute $\mathbb{E}[\max(X,Y)]$.

(c) Compute $\mathbb{P}[X+Y \geq t]$

5 Swaps and Cycles

Note 15

A permutation of n objects is a bijection from $(1,\ldots,n)$ to itself. For example, the permutation $\pi = (2,1,4,3)$ of 4 objects is the mapping $\pi(1) = 2$, $\pi(2) = 1$, $\pi(3) = 4$, and $\pi(4) = 3$. We'll say that a permutation $\pi = (\pi(1),\ldots,\pi(n))$ contains a *swap* if there exist $i, j \in \{1,\ldots,n\}$ so that $\pi(i) = j$ and $\pi(j) = i$, where $i \neq j$. The example above contains two swaps: $(1,2)$ and $(3,4)$.

(a) In terms of n, what is the expected number of swaps in a random permutation?

(b) In the same spirit as above, we'll say that π contains a *k-cycle* if there exist $i_1, \ldots, i_k \in \{1, \ldots, n\}$ with $\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_k) = i_1$. Compute the expectation of the number of k-cycles.