Probability Theory: Homework #7

Due on March 15, 2025 at 6:00pm

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Problem 1: Counting on Graphs + Symmetry

A) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

Solution

hello

- How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

Solution

- How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.

Solution

- How many distinct cycles are there in a complete graph K_n with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1), (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).

Solution

Problem 2: Proofs of the Combinatorial Variety

Note 10 Prove each of the following identities using a combinatorial proof.

- For every positive integer $n > 1$,

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k}.$$

  ### Solution

- For each positive integer m and each positive integer $n > m$,

$$\sum_{a+b+c=m} \binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c} = \binom{3n}{m}.$$

  (Notation: the sum on the left is taken over all triples of nonnegative integers (a, b, c) such that $a + b + c = m$.)

  ### Solution

Problem 3: Strings

Note 10 Show your work/justification for all parts of this problem.

- How many different strings of length 5 can be constructed using the characters A, B, C?

  Solution

- How many different strings of length 5 can be constructed using the characters A, B, C that contain at least one of each character?

  Solution

Problem 4: Unions and Intersections

Note 11 Given:

- X is a countable, non-empty set. For all $i \in X$, A_i is an uncountable set.
- Y is an uncountable set. For all $i \in Y$, B_i is a countable set.

For each of the following, decide if the expression is "Always countable", "Always uncountable", "Sometimes countable, Sometimes uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

- $X \cap Y$

  Solution

- $X \cup Y$

  Solution

- $\bigcup_{i \in X} A_i$

  Solution

- $\bigcap_{i \in X} A_i$

  Solution

- $\bigcup_{i \in Y} B_i$

  Solution

- $\bigcap_{i \in Y} B_i$

  Solution

Problem 5: Count It!

Note 11 For each of the following collections, determine and briefly explain whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

- The integers which divide 8.

  Solution

- The integers which 8 divides.

  Solution

- The functions from $\mathbb{N}$ to $\mathbb{N}$.

  Solution

- The set of strings over the English alphabet. (Note that the strings may be arbitrarily long, but each string has finite length. Also the strings need not be real English words.)

  Solution

- The set of finite-length strings drawn from a countably infinite alphabet, $\mathcal{C}$.

  Solution

- The set of infinite-length strings over the English alphabet.

  Solution