CS 70	Discrete Mathematics and Probability Theory	
Spring 2025	Rao	HW 07

Due: Saturday, 3/15, 4:00 PM
Grace period until Saturday, 3/15, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

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1 Counting on Graphs + Symmetry

Note 10

(a) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

(b) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

(c) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.

(d) How many distinct cycles are there in a complete graph K_n with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1), (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).

Solution:

(a) The number of colorings, without considering rotations, is 6!. For a particular coloring, there are 24 other colorings that can be created, just by rotating the cube. 24 because you can choose any of the 6 sides to start rotating the cube around 4 times, i.e., 6×4. So, the total number of colorings is $\frac{6!}{24} = 30$.

(b) The number of colorings is $n!$ without considering rotations of the string. However, since it possible to rotate the entire string over by one bead at a time to generate an equivalent but different coloring, there are $n! \div n = (n-1)!$ different colorings.

(c) Each edge is determined by a pair of vertices. There are $\binom{n}{2}$ distinct vertex pairs that can then determine an edge. For each pair, there is either an edge, or there isn't an edge. Therefore, there are $2^{\binom{n}{2}}$ distinct undirected graphs with n labeled vertices.

(d) $\sum_{k=3}^{n} \binom{n}{k}$ finds the number of distinct cycles there are in a complete graph. The sum counts the number of length k cycles, up to n, e.g., if this was a graph with 5 vertices, it would count the 10 distinct 3 length cycles, the 5 distinct 4 length cycles, and the one 5 length cycle.

2 Proofs of the Combinatorial Variety

Note 10 Prove each of the following identities using a combinatorial proof.

(a) For every positive integer $n > 1$,

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k}.$$

(b) For each positive integer m and each positive integer $n > m$,

$$\sum_{a+b+c=m} \binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c} = \binom{3n}{m}.$$

(Notation: the sum on the left is taken over all triples of nonnegative integers (a,b,c) such that $a+b+c=m$.)

Solution:

(a) Both sides of the equation represent the number of ways to pick out an element from a set with at most n elements. On the left hand side, each set size k up to n is considered, with there being k ways to pick out an element from a set of size k and $\binom{n}{k}$ ways to determine that set in the first pace. On the right hand side, the selected element is first considered out of n and then the remaining elements are considered to form the set, which can be up to size $n-1$ now. The summation all the different set sizes up to $n-1$. The right hand side can be read then as, pick one element from n, then $\binom{n-1}{0}$ sets of size 0, OR $\binom{n-1}{1}$ sets of size 1, OR ... and so on until $k = n-1$.

(b) If there are three sets, A, B, and C, each with n elements, to choose m elements from all three of these sets, i.e., choose m elements from $3n$, it's the same as if you choose a elements from set A, b elements from set B, and c elements from set C, where $a+b+c=m$. The sum on the left hand side represents taking into consideration all the different possible ways one can choose a, b, and c such that their sum is m.

3 Strings

Note 10 Show your work/justification for all parts of this problem.

(a) How many different strings of length 5 can be constructed using the characters A, B, C?

(b) How many different strings of length 5 can be constructed using the characters A, B, C that contain at least one of each character?

Solution:

(a) For strings of length 5, in each of the 5 positions, there is a choice from 3 characters: A, B, and C. Therefore, there are $3^5 = 243$ different strings that can be constructed.

(b) From the Inclusion/Exclusion Rule, the number of different strings of length 5 containing at least one of A, B, and C is equal to the total number of strings that can be constructed, minus those with only two characters, plus those with only one character, because we'll subtract each of AAAAA, BBBBBB, and CCCCC double in the previous step. There are $3 \times 2^5 = 96$ two character strings ($\times 3$ for either (A, B), (B, C), and (A, C)), and there are 3 ways to construct only one character strings. Therefore, $243 - 96 + 3 = 150$ is the number of different strings that can be constructed from A, B, and C that contain at least one of each character.

4 Unions and Intersections

Note 11 Given:

- X is a countable, non-empty set. For all $i \in X$, A_i is an uncountable set.
- Y is an uncountable set. For all $i \in Y$, B_i is a countable set.

For each of the following, decide if the expression is "Always countable", "Always uncountable", "Sometimes countable, Sometimes uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" cases, provide two examples – one where the expression is countable, and one where the expression is uncountable.

(a) $X \cap Y$

(b) $X \cup Y$

(c) $\bigcup_{i \in X} A_i$

(d) $\bigcap_{i \in X} A_i$

(e) $\bigcup_{i \in Y} B_i$

(f) $\bigcap_{i \in Y} B_i$

Solution:

(a) This expression is always countable. Any elements shared between X and Y is be some subset of X (Y will be larger), which will then therefore be countable as well, e.g. $\mathbb{N} \cap \mathbb{R} = \mathbb{N}$.

(b) This expression is always uncountable. The union between these two sets will contain an uncountable number of sets from Y.

(c) The union of a countable number of uncountable sets is always uncountable since the expression will necessarily either be the same as any one element in X, i.e, all A_i represent the same uncountable set, or a 'larger' uncountable set.

(d) The intersection of a countable number of uncountable sets is sometimes countable. Consider the case where each A_i represents a different subset of the real numbers, e.g., A_1 numbers between 0 and 1, A_2 numbers between 1 and 2, etc. The intersection between all these sets will be the empty set $\emptyset$ which is countable. It could then also be the case that the intersection is uncountable, if all the countable number of sets A_i were the same uncountable set.

(e) The union of an uncountable number of countable sets is sometimes countable. For example, if each set A_i is the same countable set, e.g., the natural numbers, then the union will be this same countable set. However, it's possible for the union to produce an uncountable set as well. For example, if there are an uncountable number of sets, each set A_i could just contain one real number from the subset of the reals between 0 and 1. The union would then be the numbers in that range.

(f) Using the last example from above, it is possible to construct a countable empty set via the intersection of an uncountable number of A_i, where each A_i simply contains a different number from the subset of the real numbers between 0 and 1. This is always countable, however, since any intersection of countable sets will yield another countable set.

5 Count It!

Note 11

For each of the following collections, determine and briefly explain whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

(a) The integers which divide 8.

(b) The integers which 8 divides.

(c) The functions from $\mathbb{N}$ to $\mathbb{N}$.

(d) The set of strings over the English alphabet. (Note that the strings may be arbitrarily long, but each string has finite length. Also the strings need not be real English words.)

(e) The set of finite-length strings drawn from a countably infinite alphabet, $\mathscr{C}$.

(f) The set of infinite-length strings over the English alphabet.

Solution:

(a) This collection is finite since all numbers that divide 8 must be less than or equal to 8 and greater than or equal to -8.

(b) This collection is countably infinite as it is possible to enumerate the integers which 8 divides, i.e., there is a bijection between this set and the natural numbers. The mapping from the natural numbers to this set is defined by $f(x) = 8x$, where $x \in \mathbb{N}$.

(c) This collection is uncountably infinite, like the reals. Assuming that it is possible to enumerate all functions $f : \mathbb{N} \to \mathbb{N}$ in a list, it is then possible to construct a new function not on this list by changing the ith natural number output of the ith function by some constant, e.g., by adding or subtracting 1. There are many functions that exist that map natural numbers to other natural numbers, so one that only changes one output of another function while leaving all others the same will exist. The constructed function is not on the list because it, by definition, differs from every function on the list. Therefore, it is not possible to enumerate this set, and is uncountably infinite.

(d) The set of strings is countably infinite. It can be enumerated in steps that sets up a bijection between the natural numbers and each string. By listing the strings one step at a time (where each step contains a finite number of strings), it's like taking the union of a countably infinite amount of finite sets, which results in a set of countably infinite size. To do this, list strings by their length, ordered alphabetically, i.e., list all strings of length one in alphabetical order, then strings of length 2, and so on. Each step will have 26^i elements, where i is the length of the string.

(e) This set of strings is countably infinite as well. It can be enumerated by listing all possible strings alphabetically, one step at a time, like before. This sets up a bijection between the natural numbers and the strings. Although, there is a problem with just listing one step at a time by length, because each step has an infinite amount of items (since there are an infinite number of letters). To list in steps with finite numbers of elements (akin to taking a union of countable sets), first list only those strings using the first letter of the alphabet with length one. Then list only those strings using the first two letters of the alphabet with length two, making sure not to repeat strings already mentioned. Then proceed with strings of length 3 that contain the first three letters, and so on.

(f) This set is uncountably infinite, like the reals, because it can be expressed as a subset of the real numbers between 0 and 1. Again, every letter in each string can be mapped to an integer in the natural numbers. Each of these infinite-lengthed numbers can then be appended with a '0.' at the start. The crucial difference with this question is that each string is now of infinite length, making it uncountably infinite, through its parralel with the real numbers which falls under the diagonalization proof.