Discrete Mathematics: Homework #5

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Problem 1: Equivalent Polynomials

This problem is about polynomials with coefficients in $\text{GF}(p)$ for some prime $p \in \mathbb{N}$. We say that two such polynomials f and g are *equivalent* if $f(x) \equiv g(x) \pmod{p}$ for every $x \in \text{GF}(p)$.

A) Show that $f(x) = x^{p-1}$ and $g(x) = 1$ are **not** equivalent polynomials under $\text{GF}(p)$.

B) Use Fermat's Little Theorem to find a polynomial with degree strictly less than 5 that is equivalent to $f(x) = x^5$ over $\text{GF}(5)$; then find a polynomial with degree strictly less than 11 that is equivalent to $g(x) = 4x^{70} + 9x^{11} + 3$ over $\text{GF}(11)$.

C) In $\text{GF}(p)$, prove that whenever $f(x)$ has degree $\geq p$, it is equivalent to some polynomial $\tilde{f}(x)$ with degree $< p$.

Part A
To show $f(x) \not\equiv g(x) \pmod{p}$, consider every $x \in \text{GF}(p)$. When $x = 0$, $f(0) = 0^{p-1} = 0$ and $g(0) = 1$, which are not equivalent under modulo p. Since $f(x)$ and $g(x)$ are not equivalent for all $x \in \text{GF}(p)$, they are not equivalent polynomials under $\text{GF}(p)$.

Part B
Using exponent rules in modular arithmetic in conjuction with Fermat's Little Theorem, $f(x) \equiv x^5 \equiv x^4 \cdot x \equiv x \pmod{5}$, and x is a polynomial with degree strictly less than 5. For the second part, $g(x) \equiv 4x^{70} + 9x^{11} + 3 \equiv 4(x^{10})^7 + 9x^{10}x + 3 \equiv 9x + 7 \pmod{11}$, and $9x + 7$ is a polynomial with degree strictly less than 11.

Part C
If $f(x)$ is a polynomial under $\text{GF}(p)$ with degree $\geq p$, then by recursively applying Fermat's Little Thereom to every term x^k, where $k \geq p$, can be reduced to $x^k \equiv 1 \cdot x^r \pmod{p}$, since $x^k = x^{l(p-1)+r}$ where $l, r \in \mathbb{Z}$. This results in a polynomial $\tilde{f}$ with degree $< p$ under in $\text{GF}(p)$ and is true for any $f(x)$ with degree $\geq p$ over $\text{GF}(p)$.

Problem 2: Secret Sharing

Suppose the Oral Exam questions are created by 2 TAs and 3 Readers. The answers are all encrypted, and we know that:

  A) Two TAs together should be able to access the answers

  B) Three Readers together should be able to access the answers

  C) One TA and one Reader together should also be able to access the answers

  D) One TA by themself or two Readers by themselves should not be able to access the answers.

Design a Secret Sharing scheme to make this work.

Solution
Create a polynomial of degree 6, give each TA four points and give each Reader three points. If both TAs collaborate they can reconstruct the polynomial to find the secret since they have at least 7 points. If all Readers collaborate they can also reconstruct the polynomial to find the secret with 2 points to spare. If one TA and one Reader collaborate they can reconstruct the polynomial with no points to spare. Since one TA only has four points and two Readers only have 6 points (1 point away from being able to reconstruct the polynomial), this Secret Sharing scheme satisfies the condition.

Problem 3: To The Moon!

A secret number s is required to launch a rocket, and Alice distributed the values $(1, p(1)), (2, p(2)), \ldots, (n+1, p(n+1))$ of a degree n polynomial $p(x)$ to a group of \$GME holders $\text{Bob}_1, \ldots, \text{Bob}_{n+1}$. As usual, she chose $p(x)$ such that $p(0) = s$. Bob_1 through Bob_{n+1} now gather to jointly discover the secret. However, Bob_1 is secretly a partner at Melvin Capital and already knows s, and wants to sabotage $\text{Bob}_2, \ldots, \text{Bob}_{n+1}$, making them believe that the secret is in fact some fixed $s' \neq s$. How could he achieve this? In other words, what value should he report (in terms variables known in the problem, such as s', s or $p(1)$) in order to make the others believe that the secret is s'?

Solution
He should report some value for $p(1)$ such that the reconstructed polynomial from $(1, p(1)), (2, p(2)), \ldots, (n+1, p(n+1))$, evaluated at 0, returns a fixed s' that is not the true secret. Reconstructing the polynomial with Lagrange Interpolation, absent the fixed s', the polynomial is

$$p(x) = \sum_{i=1}^{n+1} p(i)\Delta_i(x)$$

But if Bob_1 wants the \$GME holders to reconstruct a polynomial where $p(0) = s'$ instead of s, the first term of the summation must be modified to use $p(1)'$ instead of the true $p(1)$ value

$$p(x)' = p(1)'\Delta_1(x) + \sum_{i=2}^{n+1} p(i)\Delta_i(x)$$

To solve for $p(1)'$ given a fixed $s' \neq s$, a new basis polynomial $\Delta_1(x)'$ is required

$$p(1)' = \frac{s' - \sum_{i=2}^{n+1} p(i)\Delta_i(0)}{\Delta_1(0)'}$$

Problem 4: Lagrange? More like Lamegrange.

A) Let's say we wanted to interpolate a polynomial through a single point, (x_0, y_0). What would be the polynomial that we would get? (This is not a trick question. A degree 0 polynomial is fine.)

B) Call the polynomial from the previous part $f_0(x)$. Now say we wanted to define the polynomial $f_1(x)$ that passes through the points (x_0, y_0) and (x_1, y_1). If we write $f_1(x) = f_0(x) + a_1(x - x_0)$, what value of a_1 causes $f_1(x)$ to pass through the desired points?

C) Now say we want a polynomial $f_2(x)$ that passes through (x_0, y_0), (x_1, y_1), and (x_2, y_2). If we write $f_2(x) = f_1(x) + a_2(x - x_0)(x - x_1)$, what value of a_2 gives us the desired polynomial?

D) Suppose we have a polynomial $f_i(x)$ that passes through the points (x_0, y_0), ..., (x_i, y_i) and we want to find a polynomial $f_{i+1}(x)$ that passes through all those points and also (x_{i+1}, y_{i+1}). If we define $f_{i+1}(x) = f_i(x) + a_{i+1} \prod_{j=0}^{i}(x - x_j)$, what value must a_{i+1} take on?

Part A
If we interpolate a polynomial through a single point (x_0, y_0), we get a 0 degree polynomial, i.e., the point itself for all x: $f_0(x) = y_0$.

Part B
When $x = x_0$, $f_1(x)$ is equal to $f_1(x_0) = f_0(x_0) + a_1(x_0 - x_0) = y_0$. Then for $x = x_1$, $f_1(x_1) = f_0(x_1) + a_1(x_1 - x_0) = y_0 + a_1(x_1 - x_0)$. $f_1(x_1)$ should equal y_1, so

$$y_1 = y_0 + a_1(x_1 - x_0)$$
$$a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - f_0(x_1)}{x_1 - x_0}$$

Part C
For $f_2(x)$ evaluated at $x = x_0$ and $x = x_1$, we know that $f_1(x)$ will output our desired values of y_0 and y_1, and since the rest of the terms in $f_2(x)$ evaluate to zero, we know $f_2(x)$ passes through the first two of our points. For $x = x_2$, $f_2(x)$ is $f_2(x_2) = f_1(x_2) + a_2(x_2 - x_0)(x_2 - x_1)$. Therefore, a_2 must be

$$a_2 = \frac{y_2 - f_1(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

Part D
Like in the previous parts, for values $x_0, x_1, \ldots, x_i$, $f_{i+1}(x)$ will output $y_0, y_1, \ldots, y_i$ since the product $\prod_{j=0}^{i}(x - x_j)$ will end up being zero and $f_{i+1}(x)$ collapses to $f_i(x)$. Like before, consider $f_{i+1}(x_{i+1}) = f_i(x_{i+1}) + a_{i+1} \prod_{j=0}^{i}(x_{i+1} - x_j)$ and that it should equal y_{i+1}. Therefore, a_{i+1} is

$$y_{i+1} = f_i(x_{i+1}) + a_{i+1} \prod_{j=0}^{i}(x_{i+1} - x_j)$$
$$a_{i+1} = \frac{y_{i+1} - f_i(x_{i+1})}{\prod_{j=0}^{i}(x_{i+1} - x_j)}$$

Problem 5: Error-Correcting Codes

A) Recall from class the error-correcting code for erasure errors, which protects against up to k lost packets by sending a total of $n + k$ packets (where n is the number of packets in the original message). Often the number of packets lost is not some fixed number k, but rather a *fraction* of the number of packets sent. Suppose we wish to protect against a fraction α of lost packets (where $0 < \alpha < 1$). At least how many packets do we need to send (as a function of n and α)?

B) Repeat part (A) for the case of general errors.

Part A
For a total amount of packets sent l, if α are lost, the amount recieved is $(1 - \alpha) \cdot l$ packets. To reconstruct the polynomial to find the missing m's, the reciever needs to get at the very least n packets. So $n \geq (1 - \alpha) \cdot l$ packets and l, the number of packets we need to send as a function of n and α, is $\frac{n}{1 - \alpha}$, at least this amount.

Part B
For the case of gneeral errors, when we know that k amount of packages are corrupted, we must send $n + 2k$ to recover the message. If l is again the total amount of packages sent, then we need to protect against $\alpha \cdot l$ errors. So, $l \geq n + 2l \cdot \alpha$ from $n + 2k$ or $l \geq \frac{n}{1 - 2\alpha}$.

Problem 6: Alice and Bob

A) Alice decides that instead of encoding her message as the values of a polynomial, she will encode her message as the coefficients of a degree 2 polynomial $P(x)$. For her message $[m_1, m_2, m_3]$, she creates the polynomial $P(x) = m_1 x^2 + m_2 x + m_3$ and sends the five packets $(0, P(0))$, $(1, P(1))$, $(2, P(2))$, $(3, P(3))$, and $(4, P(4))$ to Bob. However, one of the packet y-values (one of the $P(i)$ terms; the second attribute in the pair) is changed by Eve before it reaches Bob. If Bob receives

$$(0, 1), (1, 3), (2, 0), (3, 1), (4, 0)$$

and knows Alice's encoding scheme and that Eve changed one of the packets, can he recover the original message? If so, find it as well as the x-value of the packet that Eve changed. If he can't, explain why. Work in mod 7. Also, feel free to use a calculator or online systems of equations solver, but make sure it can work under mod 7.

B) Bob gets tired of decoding degree 2 polynomials. He convinces Alice to encode her messages on a degree 1 polynomial. Alice, just to be safe, continues to send 5 points on her polynomial even though it is only degree 1. She makes sure to choose her message so that it can be encoded on a degree 1 polynomial. However, Eve changes two of the packets. Bob receives $(0, 5)$, $(1, 7)$, $(2, x)$, $(3, 5)$, $(4, 0)$. If Alice sent $(0, 5)$, $(1, 7)$, $(2, 9)$, $(3, -2)$, $(4, 0)$, for what values of x will Bob not uniquely be able to determine Alice's message? Assume that Bob knows Eve changed two packets. Work in mod 13. Again, feel free to use a calculator or graphing calculator software.

Hint: Observe that since Bob knows that Eve changed two packets, he's looking for a polynomial that passes through at least 3 of the given points. Think about what must happen in order for Bob to be unable to uniquely identify the original polynomial.

C) Alice wants to send a length n message to Bob. There are two communication channels available to her: Channel X and Channel Y. Only 6 packets can be sent through channel X. Similarly, Channel Y will only deliver 6 packets, but it will also corrupt (change the value) of one of the delivered packets. Using each of the two channels once, what is the largest message length n such that Bob so that he can always reconstruct the message?