Due: Saturday, 3/15, 4:00 PM
Grace period until Saturday, 3/15, 6:00 PM

# Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

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# 1 Counting on Graphs + Symmetry

Note 10

(a) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

**Solution:** The number of colorings, without considering rotations, is 6!. For a particular coloring, there are 16 other colorings that can be created, just by rotating the cube. 16 because you can rotate the cube 4 ways on the horizontal axis, and then another 4 ways on the vertical axis. So, the total number of colorings is $\frac{6!}{16}$.

(b) How many ways are there to color a bracelet with $n$ beads using $n$ colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

**Solution:** Again, without considering rotation, the number of colorings is $n!$. However, since it possible to rotate the entire string over by one bead at a time to generate an equivalent but different coloring, there are $n! \div n = (n-1)!$ colorings.

(c) How many distinct undirected graphs are there with $n$ labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.

**Solution:** For each vertex, there are $n$ possible 'connections' to be made with two possibilities: either there is an edge, or there isn't an edge. For the first vertex, there are $2^n$ permutations, then for the next edge, there are $2^{n-1}$ as the previous edge already determined what connection it has with it, and $2^{n-2}$ and so on. Therefore, there are $2^{n!}$ graphs.

(d) How many distinct cycles are there in a complete graph $K_n$ with $n$ vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. $(v_1, v_2, v_3, v_1)$, $(v_2, v_3, v_1, v_2)$ and $(v_1, v_3, v_2, v_1)$ all count as the same cycle).

**Solution:** $\sum_{k=3}^{n} \binom{n}{k}$ finds the number of distinct cycles there are in a complete graph. The sum counts the number of length $k$ cycles, up to $n$, e.g., if this was a graph with 5 vertices, it would count the 10 distinct 3 length cycles, the 5 distinct 4 length cycles, and the one 5 length cycle.

## 2 Proofs of the Combinatorial Variety

Note 10 Prove each of the following identities using a combinatorial proof.

(a) For every positive integer $n > 1$,

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k}.$$

**Solution:** Consider for each term in the summation on the left hand side that it is possible to express the binomial $\binom{n}{k}$ equivalently as $n \times \binom{n-1}{k}$. Additionally, $k \times \binom{n-1}{k}$ can be expressed as $\binom{n-1}{k-1}$. Reindexing the summation to end at $n-1$ allows for expressing the binomial like the original with $k$ again, producing the right-hand-side expression.

(b) For each positive integer $m$ and each positive integer $n > m$,

$$\sum_{a+b+c=m} \binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c} = \binom{3n}{m}.$$

(Notation: the sum on the left is taken over all triples of nonnegative integers $(a, b, c)$ such that $a + b + c = m$.)

**Solution:** If there are three sets, $A$, $B$, and $C$, each with $n$ elements, to choose $m$ elements from all three of these sets, i.e., choose $m$ elements from $3n$, it's the same as if you choose $a$ elements from set $A$, $b$ elements from set $B$, and $c$ elements from set $C$, where $a + b + c = m$. The sum on the left hand side represents taking into consideration all the different possible ways one can choose $a$, $b$, and $c$ such that their sum is $m$.

## 3 Strings

Note 10 Show your work/justification for all parts of this problem.

(a) How many different strings of length 5 can be constructed using the characters $A, B, C$?

**Solution:** I found that $3 \times \sum_{k=3}^{5} \binom{5}{k} \cdot 2^{5-k}$ represents the number of different strings of length 5 there are that can be constructed with $A, B, C$.

(b) How many different strings of length 5 can be constructed using the characters $A,B,C$ that contain at least one of each character?

**Solution:** For using at least one character, the number of strings is $\binom{5}{3} \times 3^2$. This is because there must at least be an $A$, a $B$, and a $C$ in any of the five possible positions, and then for the remaining two possible positions there are each 3 possible letters to choose from.

# 4 Unions and Intersections

Note 11

Given:

- $X$ is a countable, non-empty set. For all $i \in X$, $A_i$ is an uncountable set.
- $Y$ is an uncountable set. For all $i \in Y$, $B_i$ is a countable set.

For each of the following, decide if the expression is "Always countable", "Always uncountable", "Sometimes countable, Sometimes uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" cases, provide two examples – one where the expression is countable, and one where the expression is uncountable.

(a) $X \cap Y$

**Solution:** This expression is always countable. There are no elements shared between sets $X$ and $Y$, and the intersection will be the empty set $\emptyset$, which is countable.

(b) $X \cup Y$

**Solution:** This expression is always uncountable. The union between these two sets will contain an uncountable number of sets from $Y$.

(c) $\bigcup_{i \in X} A_i$

**Solution:** The union of a countable number of uncountable sets is always uncountable since the expression will necessarily either be the same as any one element in $X$, i.e, all $A_i$ represent the same uncountable set, or a ‘larger’ uncountable set.

(d) $\bigcap_{i \in X} A_i$

**Solution:** The intersection of a countable number of uncountable sets is sometimes countable. Consider the case where each $A_i$ represents a different subset of the real numbers, e.g., $A_1$ numbers between 0 and 1, $A_2$ numbers between 1 and 2, etc. The intersection between all these sets will be the empty set $\emptyset$ which is countable. It could then also be the case that the intersection is uncountable, if all the countable number of sets $A_i$ were the same uncountable set.

(e) $\bigcup_{i \in Y} B_i$

**Solution:** The union of an uncountable number of countable sets is sometimes countable. For example, if each set $A_i$ is the same countable set, e.g., the natural numbers, then the union will be this same countable set. However, it's possible for the union to produce an uncountable set as well. For example, if there are an uncountable number of sets, each set $A_i$ could just contain one real number from the subset of the reals between 0 and 1. The union would then be the numbers in that range.

(f) $\bigcap_{i \in Y} B_i$

**Solution:** Using the last example from above, it is possible to construct a countable empty set via the intersection of an uncountable number of $A_i$, where each $A_i$ simply contains a different number from the subset of the real numbers between 0 and 1. This is always countable, however, since any intersection of countable sets will yield another countable set.

## 5 Count It!

Note 11 For each of the following collections, determine and briefly explain whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

(a) The integers which divide 8.

**Solution:** This collection is finite since all numbers that divide 8 must be less than or equal to 8 and greater than or equal to 1.

(b) The integers which 8 divides.

**Solution:** This collection is countably infinite as it is possible to enumerate the integers which 8 divides, i.e., there is a bijection between this set and the natural numbers. The mapping from the natural numbers to this set is defined by $f(x) = 8x$, where $x \in \mathbb{N}$.

(c) The functions from $\mathbb{N}$ to $\mathbb{N}$.

**Solution:** This collection is uncountably infinite, like the reals. Assuming that it is possible to enumerate all functions $f : \mathbb{N} \to \mathbb{N}$, it is then possible to construct a new function not on this list by adding $+1$ to each function. Therefore, it is not possible to enumerate, and is uncountably infinite.

(d) The set of strings over the English alphabet. (Note that the strings may be arbitrarily long, but each string has finite length. Also the strings need not be real English words.)

**Solution:** Since each string is of finite length, like a natural number, and since it is possible to express each string as a natural number ($A$ as 1, $B$ as 2, etc.), the set of strings over the English alphabet is countably infinite.

(e) The set of finite-length strings drawn from a countably infinite alphabet, $\mathscr{C}$.

**Solution:** With a countably infinite alphabet, it is possible to enumerate each letter in the alphabet, i.e., pair it with a corresponding integer in the natural numbers. Each string in the set of finite-length strings over this alphabet can then be expressed as a finite natural number and the collection is therefore countably infinite.

(f) The set of infinite-length strings over the English alphabet.

**Solution:** This set is uncountably infinite, like the reals, because it can be expressed as a subset of the real numbers between 0 and 1. Again, every letter in each string can be mapped to an integer in the natural numbers. Each of these infinite-lengthed numbers can then be appended with a '0.' at the start. The crucial difference with this question is that each string is now of infinite length, making it uncountably infinite, through its parralel with the real numbers.