Chapter 6

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6.7 Least-Squares Problem

Definition

If A is $m * n$ matrix and b is in R^m, a least-squares problem of $Ax = b$ is an $\hat{x}$ in R^n such that

$$\|b - A\hat{x}\| \leq \|b - Ax\|$$

for all x in R^n .

Theorem 13

The set of least squares solutions of $Ax = b$ coincides with the non-empty set of solutions of the equations: $A^T Ax = A^T b$

Theorem 14

Let A be an $n * n$ matrix. TFAE:
a. the equation $Ax = b$ has a unique least-squares solution for each b in R^n.
b. The columns of A are linearly independent.
c. The Matrix $A^T A$ is invertible.

When these statements are true, the least-squares solution $\hat{x}$ is given by:

$$\hat{x} = (A^T A)^{-1} A^T b$$

Theorem 15

Given an $m * n$ matrix A with linearly indepedent columns, let $A = QR$ be a QR factorization of A as in Theorem 12. Then for each b in R^n, the equation $Ax = b$ has a unique least-squares solution given by:

$$\hat{x} = R^{-1} Q^T b$$

Conclusion

In summary, The least squares solution aims to find the smallest $\|b - Ax\|$.
To find it we use the equation from **Theorem 14** to find $\hat{x}$ which is the least squares solutions.
You can also use QR factorization to find the least squares solution as well. This time, use **Theorem 15** to find the least squares solution. However, the equation from **Theorem 14** still works regardless.