Chapter 7

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7.1 Diagonalization of Symmetric Matrices

Definition: A symmetric matrix is a matrix A such that $A^T = A$
This means:
A will be square The diagonals can be arbitrary but its other entries occur in pairs- on opposite sides of the main diagonal.

Example
Symmetric Matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & 8 \\ 0 & 8 & 7 \end{bmatrix}, \begin{bmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{b} & \mathrm{d} & \mathrm{e} \\ \mathrm{c} & \mathrm{e} & \mathrm{f} \end{bmatrix}$$

Non-Symmetric Matrices:

$$\begin{bmatrix} 1 & -3 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -4 & 0 \\ -6 & 1 & -4 \\ 0 & -6 & 1 \end{bmatrix}$$

Theorem
If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

Theorem
An $n * n$ matrix is orthogonally diagonizable if and only if A is a symmetric matrix.

Theorem The Spectral Theorem for Symmetric Matrices:
An $n * n$ symmetric matrix A has the following properties:
1) A has n real eignvalues, counting multiplicites.
2) The dimension of the eigenspace for each eigenvalue λ equals the multiplcity of λ as a root of the characteristic equation.
3) The eigenspaces are mutually orthogonal.
4) A is orthogonally diagonalizable.

Spectral Decomposition

If $A = PDP^{-1}$, where the columns of P are orthonormal eigenvectors $u_1, ..., u_n$ of A and the corresponding eigenvalues $\lambda_1, ..., \lambda_n$ are in the diagonal matrix D. Then, $P^T = P^{-1}$

The following equation represents A as a spectral decomposition since it breaks up A into pieces determined by the spectrum (eigenvalues).

$$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + ... + \lambda_n u_n u_n^T$$

Conclusion

So basically...
Symmetric matrices are matrices where the pairs on opposite sides are equivalent to each other. While the diagonals can be anything.