Chapter 4

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4.1 Vector Spaces and Subspaces

Definition: Subspace

A vector space is a set of V objects, called vectors. Vector spaces have two defined operations vector addition and vector multiplcation. Axioms are in textbook.
Basically, Communtative and Associatve properties of addition and multiplication must hold.
Additionally, the zero vector must be in the vector space.

Vector Space: $\mathbb{P}^n$

For the set $\mathbb{P}^n$, for $n \geq 0$,

$$p(t) = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n$$

Definition: Vector Space

A subspace of a vector space is a subset H of V that has three properties:
1. The zero vector of V is in H
2. H is closed under vector addition.
3. H is closed under multiplciation of scalars.

Theorem 1

If $v_1, ..., v_p$ are in the vector space V, then $span\{v_1, ..., v_p\}$ is a subspace in V.

4.2 Null Spaces, Column Spaces, and Linear Transformations

Definition: Nullspace

The **Null Space** of a $m * n$ matrix A, writen as $Nul A$, is the set of all solutions that satisfies the homogenous equation $Ax = 0$. In set notation this would be:

$$Nul A = \{x : x \text{ is in } \mathbb{R}^n \text{ and } Ax = 0\}$$

Theorem 2

The null space of a $m \times n$ matrix A is in a subspace of $\mathbb{R}^n$. In other words, the number of components in the nullspace of A will be equal to the number of columns in A.

How to find Nullspace

Given a matrix A :
1. Row Reduce A to echelon form (RREF is unnecessary).
2. Create a vector that holds the general solutions. For example:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 3x_3 \\ x_2 \\ 5x_2 \end{bmatrix}$$

3. Decompose the vector into a linear combination where the weights are the free variables.

$$x_2 \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

4. Null space:

$$Span\{ \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} \}$$

Definition: Column Space

The column space of a $m \times n$ matrix A, written as $ColA$ is the set of linear combinations of the columns of A. if $A = [a_1, ..., a_n]$ then $ColA = span\{a_1, ..., a_n\}$. In other words, the column space of A is just the linearly indepedent columns of A.

Theorem 3

The column space of a $m \times n$ matrix A is a subspace of A.

How to Find Column Space

Given a $m \times n$ matrix A.
1. Row reduce until you can find the pivots.
2. Take every column that is linearly independent to create a subspace. That subspace will be the column space of A.
Note: if Ax=b has a solution for each b(one solution), then the column space of A is A itself.

Definition: Linear Transformation

A linear transformation T from a vector space V into a Vector Space W is a rule that assigns to each x in V a unique vector $T(x)$ in W such that:
1. $T(u + v) = T(u) + T(v)$
2. $T(cu) = cT(u)$

4.3: Linearly Independent Sets; Bases

Theorem 4

An indexed set $\{v_1, ..., v_p\}$ of two or more vectors, with $v_1 \neq 0$, is linearly dependent if and only if some v_j where $(v_j > 1)$ is a linear combination of preceding vectors.

Definition

Let H be a subspace of a vector space V. An indexed set of vectors $B = \{b_1, ..., b_p\}$ in V is a basis for H if:
1. B is a linearly independent set, and
2. The span of B coincides with H such that $H = span\{b_1, ..., b_p\}$

Example

I_2 identity matrix is the standard basis for $\mathbb{R}^2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Each vector is independent, so it forms a basis.

Theorem 5

Spanning Set Theorem
Let $S = \{v_1, ..., v_p\}$ and $H = Span\{v_1, ..., v_p\}$
1. If a vector in S, v_k, is a linear combination of the other Vectors in S, then the set without v_k still spans H.
2. If $H \neq \{0\}$ then some subset of S is a basis for H.

Theorem 6

The pivot columns of A form a basis for $Col A$.

4.4 Coordinate Systems

Theorem 7

Let $B = \{b_1, ..., b_n\}$ be a basis for the vector space V. Then for each x in V, there exists a unique set of scalars $c_1, ..., c_n$ such that

$$x = c_1 b_1 + c_2 b_2 + ... + c_n b_n$$

Definition: Coordinates Relative to the Basis B

Suppose $B = \{b_1, ..., b_n\}$ is a basis for V and x is in V. The coordinates of x relative to the Basis B are the scalars $c_1, ..., c_n$ such that:

$$x = c_1 b_1 + ... + c_n b_n$$

So basically:

$$[x]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Example

Let $b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $x = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and $B = \{b_1, b_2\}$.
Find the coordinate vector $[x]_B$ of x relative to B.

Goal: Given x and B, find $[x]_B$.
Basically, find the scalars $c_1, ..., c_n$ so that $[x]_B = c_1 b_1 + ... + c_n b_n$.

$$[\mathrm{x}]_B = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

This is equal to:

$$[\mathrm{x}]_B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

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Now just just find c_1 and c_2.

$$[\mathrm{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

In this example, the matrix $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ changes the B-coordinates into the standard coordinates of x. This is known as the **change of coordinates** matrix.

Definition: Change of Coordinates Matrix

Let

$$P_B = \{b_1, ..., b_n\}$$

Then the vector equation $[x]_B = c_1 b_1 + ... + c_n b_n$ is equivalent to:

$$x = P_B [x]_B$$

Theorem 8

Let $B = \{b_1, ..., b_n\}$ be a basis for a vector space V. Then the coordinate mapping $x \mapsto [x]_B$ is one-to-one and on-to from V onto $\mathbb{R}^n$.

4.5 The Dimension of a Vector Space

Theorem 9

If a Vector space V has a basis $B = \{b_1, ..., b_n\}$ then any set in V containing more than n vectors is linearly dependent.

Definition: Dimension

In terms of MATH240, $dimV$ is the number of vectors in a basis for V.

Theorem 11

Let H be a subspace of vector space V.

$$dimH \leq dimV$$

Theorem 12

The Basis Theorem
Let V be a p-dimensional vector space where $p \geq_1$. Any linearly independent set of exactly p elements in V is automatically a basis for V.

Conclusion

Just know that dimension is the number of vectors in a basis.

4.6

Theorem 13

If two matrices A and B are row equivalent, then their row spaces will also be equivalent. If B is in echelon form, then the nonzero rows of B will be equal to the row space of B.

Definition: Rank

The **Rank** of A is the dimension of the column space of A.

Theorem 14: The Rank Theorem

$$rankA + dim(NulA) = n$$

in an $m \times n$ matrix.

Conclusion

Remember Rank theorem row space.

4.7 Change of Basis

Theorem 15

Let $B = \{b_1, ..., b_n\}$ and $C = \{c_1, ..., c_n\}$ be bases in vector space V. Then there exists a unique $n \times n$ matrix $\underset{C \leftarrow B}{P}$ such that:

$$[x]_C = \underset{C \leftarrow B}{P}[x]_B$$

The columns in $\underset{C \leftarrow B}{P}$ are the C-coordinate vectors of the vectors in the basis of B. That is:

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} _1C...[b_n]_C \end{bmatrix}$$