MATH241 Chapter 11

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11.1 Cartesian Coordinates in Space

Given a point in 3 dimensional space P, then there are 3 planes that intersect P and are perpendicular to the x, y, and z axis.
So, P can be associated with an ordered triple of numbers (x, y, z). This way of writing P is called the rectangular, or Cartesian, coordinates.

Distance

The distance between two points, P and Q, can be found using the equation:

$$|PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$$

Some other notes:

$$|PQ| = 0 \text{ iff } P = Q$$
$$|PQ| = |QP|$$
$$|PQ| \leq |PR| + |RQ| \text{ for any third point } R$$

11.2 Vectors in Space

Definition: Vector

A vector is an ordered triple (a_1, a_2, a_3) of numbers. The numbers a_1, a_2, and a_3 are called the components of the vector. The vector $\vec{PQ}$ associated with the directed line segement with the inital point $P = (x_0, y_0, z_0)$ and the terminal point $Q = (x_1, y_1, z_2)$ is $(x_1 - x_0, y_1 - y_0, z_1 - z_0)$

Definition: Norm

The length(norm) of a vector $a = (a_1, a_2, a_3)$ is denoted as $\|a\|$ is defined as:

$$\|a\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

A **Unit Vector** is a vector with a norm of 1.
Some special unit vectors:
$i = (1, 0, 0)$
$j = (0, 1, 0)$
$k = (0, 0, 1)$
A vector $a = (a_1, a_2, a_3)$ can be written as:

$$a = a_1 i + a_2 j + a_3 k$$

Vector Operations

Let $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ and c be a scalar.

$$a + b = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$
$$a - b = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$
$$ca = (ca_1, ca_2, ca_3)$$

There are four ways to describe a vector: 1. as (a_1, a_2, a_3), an ordered triple of numbers
2. as (a_1, a_2, a_3), a point in space
3. as a directed line segment with an initial point at (x_0, y_0, z_0) and a terminal point at $(x_0 + a_1, y_0 + a_2, z_0 + a_3)$
4. as $a_1 i + a_2 j + a_3 k$

Parallel Vectors

Two nonzero vectors a and b are parllel iff there is exists a scalar c such that $b = ca$.

Some other notes...

11.3 The Dot Product

Definition: Dot Product

Let $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$ be two vectors. The dot product(or scalar product, or inner product) of a and b is the number $a \cdot b$ is defined as:

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

The dot product satisfies many of the laws that hold for real numbers-

$$a \cdot b = b \cdot a$$
$$c(a \cdot b) = c(a) \cdot b = c(b) \cdot a$$
$$c \cdot (a + b) = a \cdot c + b \cdot c$$

Theorem 11.6

Given the nonzero vectors $a = a_1 i + a_2 j + a_3 k$, and $b = b_1 i + b_2 j + b_3 k$ be nonzero vectors, and let θ be the angle between a and b. Then:

$$a \cdot b = \|a\| \|b\| \cos \theta$$

Notes:

a. nonzero vectors a and b are perpendicular iff $a \cdot b = 0$

b. For any vector a, $a \cdot a = \|a\|^2$

c. $\cos\theta = \frac{a\cdot b}{\|a\|\|b\|}$ (an extension of $a \cdot b = \|a\|\|b\|\cos\theta$)
d. a and b are perpendicular iff $a \cdot b = 0$

something something direction angles...

Definition: Projections

Let a be a nonzero vector. The projection of a vector b onto a is the vector $pr_a b$ defined by:

$$pr_a b = (\frac{a\cdot b}{\|a\|^2})a$$

also to find the :

$$\|pr_a b\| = \frac{|a\cdot b|}{\|a\|}$$

11.4 The Cross Product and Triple Products

Definition: Cross Product

The cross product (or vector product) of $a \times b$ of two vectors $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$ is defined by:

$$a \times b = (a_2 b_3 - a_3 b_2)i - (a_1 b_3 - a_3 b_1)j + (a_1 b_2 - a_2 b_1)k$$

This comes from:

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Notes:
a. $a \times b = -(b \times a)$ -no commutative property.
b. $a \times a = 0$
c. $a \times (b + c) = (a \times b) + (a \times c)$
d. $\|a \times b\| = \|a\|\|b\|\sin\theta$
e. a and b are parallel iff $a \times b = 0$

triple product...
$|a \cdot (b \times c)|$ = the volume of the parallelpiped with sides a, b, and c.

11.5 Lines in Space

vector L and line l are parallel if L is parallel to the vector $\overrightarrow{P_0P}$ where P_0 and P join two distinct points on l.