Maximum Lyapunov Exponents for Different Systems

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Abstract: The maximum Lyapunov exponent of a physical system characterizes the rate at which trajectories that begin with infinitesimally close initial conditions diverge from each other. The maximum Lyapunov exponent was calculated for a linearly damped driven pendulum and a double pendulum varying the driving constant (γ) and initial angle ($\theta_{1,0}$) respectively. It was found that the maximum Lyapunov Exponent for the linearly damped pendulum varied in a significantly more erratic manner than the double pendulum. Additionally, the linearly damped pendulum's maximum Lyapunov exponent decreased for extremely high values of gamma ($\gamma \sim 10$).

1 Introduction

Chaotic systems appear abundantly in nature. With their applications including weather, fluid dynamics, and several mechanical systems [1]. The study of these systems and specifically the characterization of their chaos is thus of great interest. The purpose of our study is the quantification of chaos via the maximum Lyapunov exponent. The local Lyapunov exponent for a system can be found at any given time for two infinitesimally different trajectories and quantifies the rate of separation of those two trajectories at that time. For any given system, the number of Lyapunov exponents is equal to the number of variables describing said system. However, the maximum Lyapunov exponent can be calculated by taking the average of the Lyapunov exponent at each time step over of period of several times steps. As described by Chen [2], the equation for the average Lyapunov exponent, λ, over n time steps (each of length Δt) is shown below in equation (1).

$$\lambda = \frac{1}{n\Delta t}\sum_{i=1}^{n} \ln \frac{d(t_i)}{d_0} \tag{1}$$

(where the two trajectories are initially separated by d_0 and $d(t_i)$ is the separation of the two trajectories after i time steps)

It was shown by Chen that this average will always converge to the maximum Lyapunov exponent of the system for large values of n. This is because, after a large number of iterations, n, the maximum Lyapunov exponent will have the most influence on our average and thus the value which it converges to will be our maximum [2].

For our study, the systems for which the maximum Lyapunov exponent was varied for different system parameters were the linearly damped driven pendulum (LDDP), and the double pendulum

(DP), for which the equations of motion are shown below in equation (2) (checked with [3]) and system of equations (3) respectively. For the LDDP, we varied the driving constant, γ, shown below. For the DP, we varied the initial value of the first angle, $\theta_{1,0}$.

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0^2 sin\phi = \gamma\omega_0^2 cos(\omega t) \tag{2}$$
$$with\ \omega_0 = \frac{g}{L}, 2\beta = \frac{b}{m}, \gamma = \frac{F_0}{mg}$$

$$\begin{aligned} \ddot{\theta}_1 + \alpha_1(\theta_1, \theta_2)\ddot{\theta}_2 &= f_1(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \\ \ddot{\theta}_2 + \alpha_2(\theta_1, \theta_2)\ddot{\theta}_1 &= f_2(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \end{aligned} \tag{3}$$

$$\alpha_1(\theta_1, \theta_2) = \frac{l_2}{l_1}\left(\frac{m_2}{m_1+m_2}\right) cos(\theta_1 - \theta_2) \quad f_1(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = -\frac{l_2}{l_1}\left(\frac{m_2}{m_1+m_2}\right)\dot{\theta}_2^2 sin(\theta_1 - \theta_2) - \frac{g}{l_1} sin(\theta_1)$$
$$\alpha_2(\theta_1, \theta_2) = \frac{l_1}{l_2} cos(\theta_1 - \theta_2) \quad f_2(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = -\frac{l_1}{l_2}\dot{\theta}_1^2 sin(\theta_1 - \theta_2) - \frac{g}{l_2} sin(\theta_2)$$

2 Methods

We numerically calculated max Lyapunov exponents with Python code based on a module written by Thomas Savary [4]. We rewrote the module to be purely functional so that we could use both multithreading (via just-in-time compiling) and multiprocessing to significantly speed up the large amount of calculation, which reduced the runtime of a loop calculating one exponent from ~ 5 minutes to 2 seconds. The core algorithm is a Runge-Kutta 4th-order ODE solver. At each step, it calculates the trajectory of the given system and the deviation vector (the difference between two nearby trajectories). We then randomly select an initial deviation vector and iterate the solver (norming the deviation vector at each step to prevent error) for 10^6 steps and converge at the value of the max Lyapunov exponent of the system. We describe each system with a function describing trajectory and a Jacobian matrix that locally linearizes the ODE. The LDDP is known to be eq. 2 [5] and we derived the DP from the Lagrangian of the system (checked with [3]). (We analytically calculated the Jacobian for each system.)

$$\mathcal{L} = \frac{1}{2}m_1 l_1\dot{\theta}_1 + \frac{1}{2}m_2[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2 cos(\theta_1 - \theta_2)] + m_1gl_1cos(\theta_1) + m_2g(l_1cos(\theta_1) + l_2cos(\theta_2))$$

We then ran the max Lyapunov solver for a range of constants that significantly impact the behavior and energy of the system: the forcing constant γ for the damped driven pendulum and the initial angle of release for the double pendulum. For the damped driven pendulum, we modeled an underdamped scenario with $\omega_0 = 3\pi$, $\omega_d = 2\pi$, $\beta = \frac{3}{4}\pi$, an initial position of $\phi_0 = 1rad$ and no initial velocity. For the double pendulum, we set $m_1 = m_2 = 1kg$, $l_1 = l_2 = 1m$, and no initial velocity. We varied the initial position $\theta_{1_0} = 1rad$ with the second pendulum aligned ($\theta_{2_0} = 0rad$).

3 Results

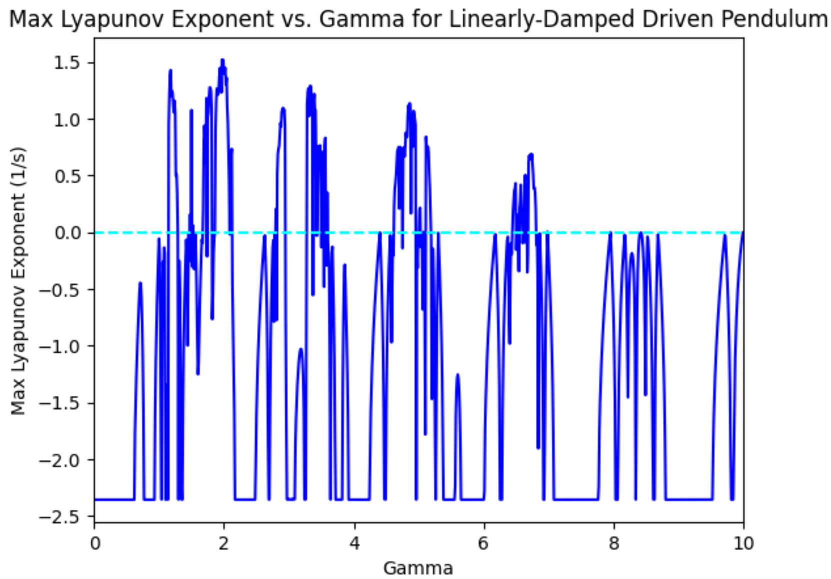

Figure 1: MLE of LDDP vs. γ for $\gamma \in [0, 10]$

As shown above in Figure 1, λ for the LDDP appeared to oscillate between positive and negative values for increasing γ. However, it was also notable that the peaks of the λ vs. γ graph were higher for lower γ and appeared to asymptotically approach $\lambda = 0$ as γ increased above $\gamma \sim 2$. Recall that $\lambda = 0$ represents a state of the system where the separation between infinitesimally close trajectories does not change. One interpretation of the graphs peaks approaching $\lambda = 0$ is that the chaotic motion dies out for **both** extremes (both low and high values of γ). The decrease in cases for high γ (at roughly $\gamma > 7$) can be understood as the driving force of the LDDP completely overtaking the damping, resulting in a constant driven motion. These trends seem to indicate that if either the damping or the driving forces significantly overtakes the other, the LDDP system will approach a non-chaotic state.

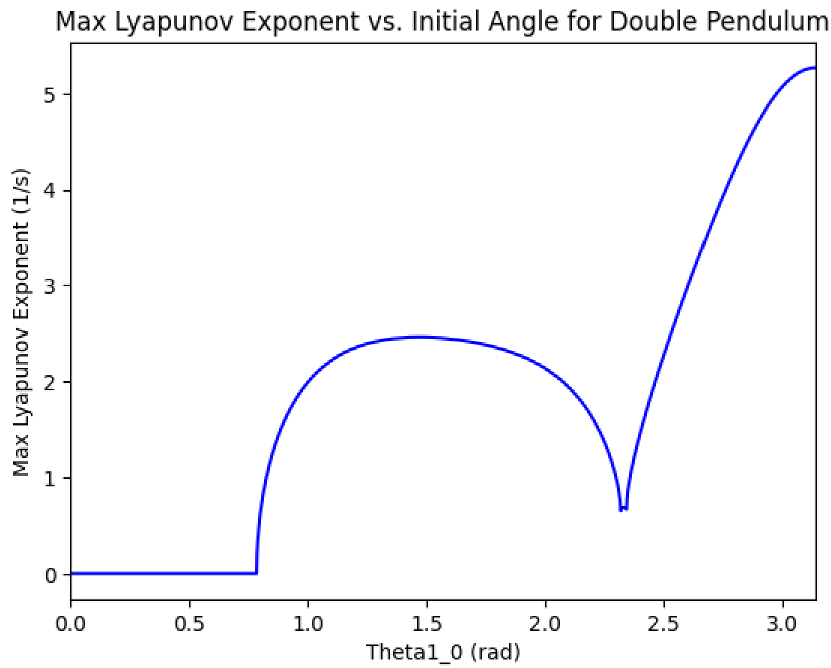

Figure 2: MLE of DP vs. $\theta_{1,0}$ for $\theta_{1,0} \in [0, \pi]$

The behavior of λ for the DP, as shown above in Figure 2, was significantly more ordered than the behavior of the LDDP. For $\theta_{1,0} \in [0, \sim 0.80]$, λ stayed constant at $\lambda = 0$. Thus, in this range of $\theta_{1,0}$, the separation between infinitesimally close trajectories stays constant. Past this range, we see in Figure 2 that λ initially increases rapidly from $\theta_{1,0} \sim 0.8$ to $\theta_{1,0} \sim 1.5$ where it peaks at $\lambda \sim 2.5$. λ then decreases until it reaches $\lambda \sim 0.9$ at $\theta_{1,0} \sim 2.375$. Finally, λ increases rapidly above $\theta_{1,0} \sim 2.375$, hitting it's maximum value at $\lambda \sim 5.5$ when $\theta_{1,0} = \pi$.

4 Conclusion

We found the maximum Lyapunov exponents of the LDDP and the DP while varying their γ and $\theta_{1,0}$ parameters respectively. It was discovered that the relationship between the maximum Lyapunov exponent and γ was highly irregular for the LDDP whereas it's relationship with $\theta_{1,0}$ was significantly more ordered. For both systems, we observed sectors of chaos and sectors of non-chaos when varying our chosen parameters. Interestingly, we observed the the peaks of the maximum Lyapunov exponent vs. γ graph for the LDDP seemed to decrease for both extremely small and extremely large values of γ. Future investigations of interest may include probing different systems to try to ascertain why certain systems may oscillate between states of chaos and non-chaos (as observed for the LDDP) while others observe a more ordered relationship between maximum Lyapunov exponent and initial conditions (as seen in our DP observations).

References

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