Parametric Models: Prior Information, MAP

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Thanks to past instructors
A.Moore tutorials www.cs.cmu.edu/~awm/tutorials

Machine Learning
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Recall: Your first consulting job

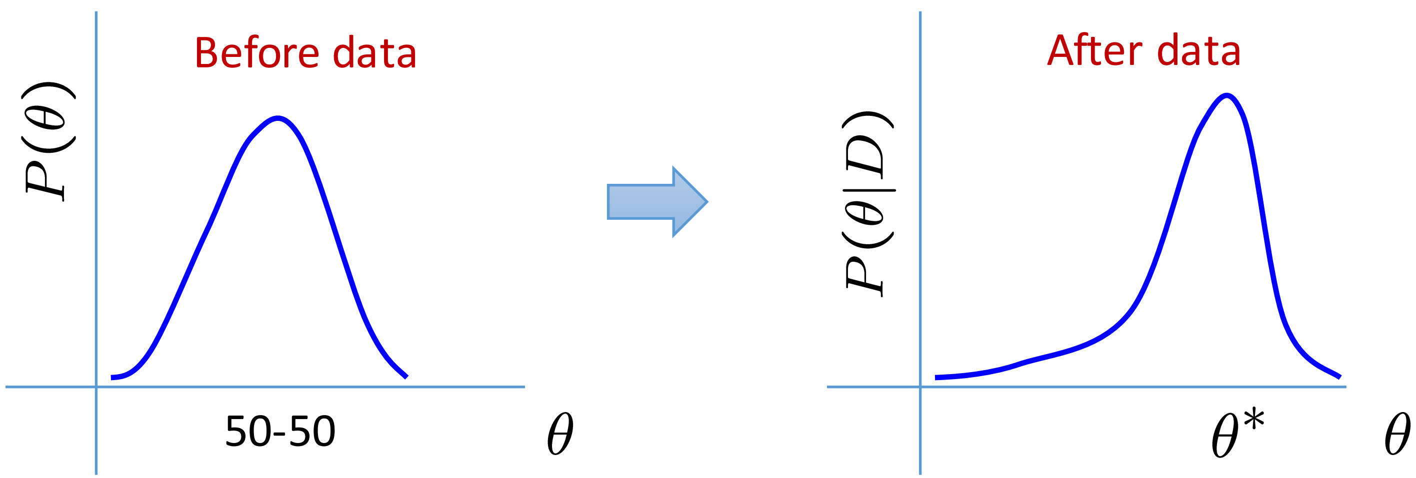
- A billionaire from the suburbs of Seattle asks you a question:
  - He says: I have a coin, if I flip it, what’s the probability it will fall with the head up?
  - You say: Please flip it a few times:

  - You say: The probability is: **3/5** because… frequency of heads in all flips
  - **He says: But can I put money on this estimate?**
  - You say: ummm…. Maybe not.
    - Not enough flips (less than sample complexity)

What about prior knowledge?

- Billionaire says: Wait, I know that the coin is "close" to 50-50. What can you do for me now?
- **You say: I can learn it the Bayesian way…**

- Use the *prior* formation; Estimate a *distribution over possible values of* θ*, given the data*



Bayesian Learning

- Use Bayes rule:

$$P(\theta \mid \mathcal{D}) = \frac{P(\mathcal{D} \mid \theta) P(\theta)}{P(\mathcal{D})}$$

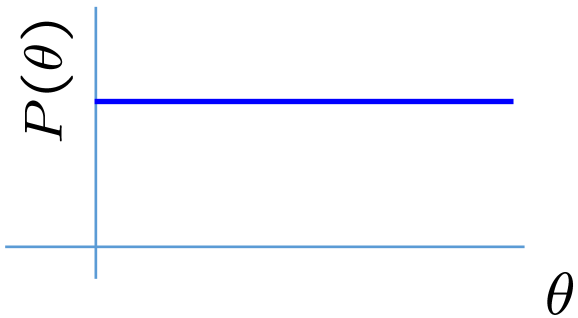
- Or equivalently:

$$\underbrace{P(\theta \mid \mathcal{D})}_{\text{posterior}} \propto \underbrace{P(\mathcal{D} \mid \theta)}_{\text{likelihood}} \underbrace{P(\theta)}_{\text{prior}}$$

Bayes, Thomas (1763) An essay towards solving a problem in the doctrine of chances. *Philosophical Transactions of the Royal Society of London*, **53:370-418**

Prior distribution

- From where do we get the prior?
  - Represents expert knowledge (philosophical approach)
  - Simple posterior form (engineer's approach)
- Uninformative priors:
  - Uniform distribution



- General prior: computational issues for online learning

Conjugate Prior

- Consider a *family* of probability distributions characterized by some parameter θ (possibly a single number, possibly a tuple).
- A prior is a *conjugate prior* if:
  - If it is a member of this family;
  - and if all possible posterior distributions are also members of this family.

- $P(\theta)$ and $P(\theta|D)$ have the same form as a function of θ.

- Closed-form representation of posterior!

Conjugate Prior

- P(θ) and P(θ|D) have the same form as a function of theta

Eg. 1 Coin flip problem

Likelihood given Bernoulli model:

$$P(\mathcal{D} \mid \theta) = \theta^{\alpha_H}(1-\theta)^{\alpha_T}$$

If prior is Beta distribution,

$$P(\theta) = \frac{\theta^{\beta_H - 1}(1-\theta)^{\beta_T - 1}}{B(\beta_H, \beta_T)} \sim Beta(\beta_H, \beta_T)$$

Then posterior is Beta distribution

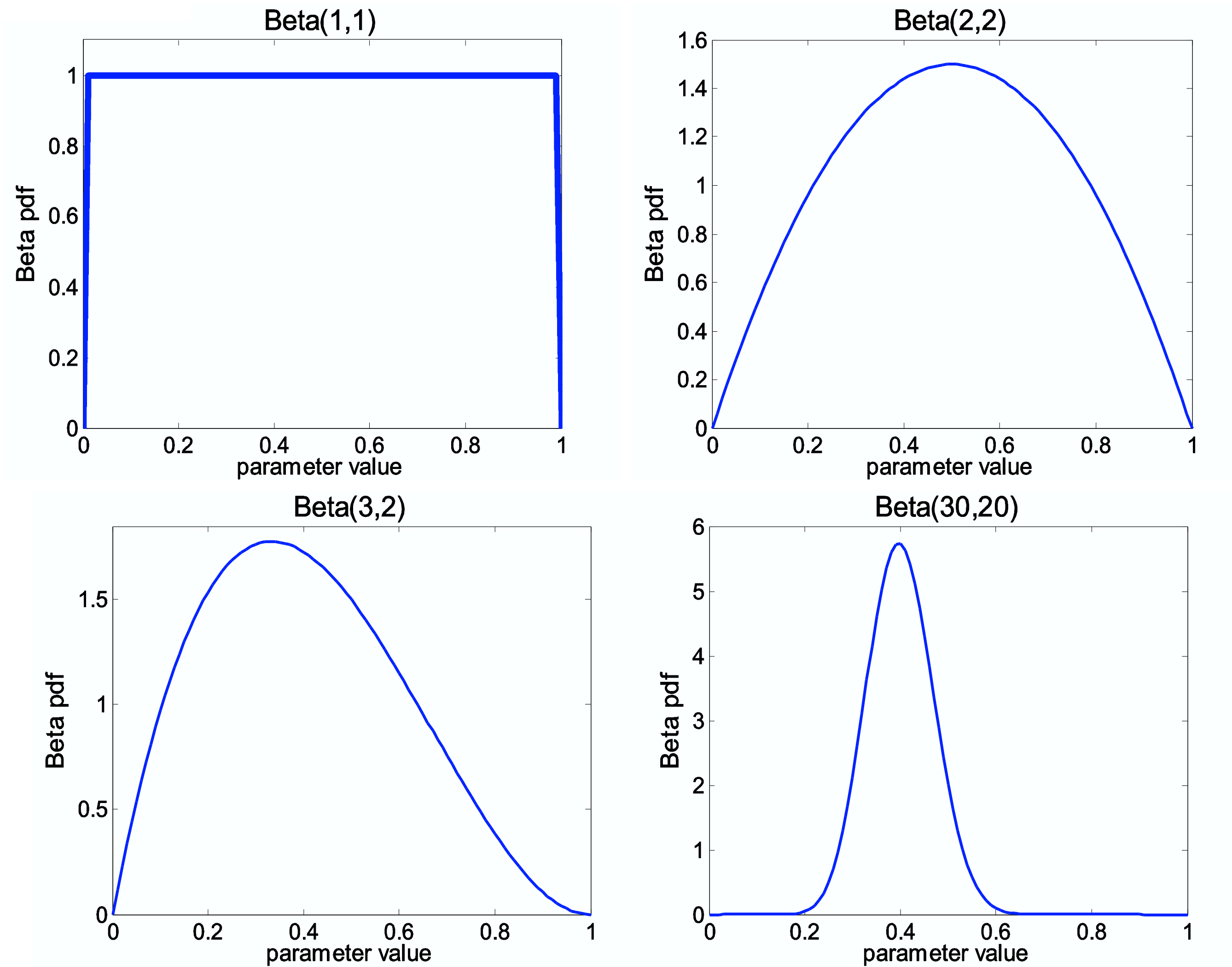
$$P(\theta|D) \sim Beta(\beta_H + \alpha_H, \beta_T + \alpha_T)$$

For Binomial, conjugate prior is Beta distribution.

Beta distribution

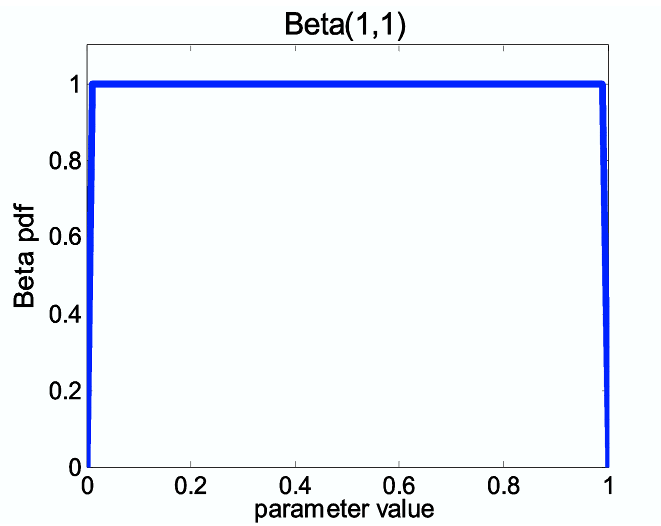
$Beta(\beta_H, \beta_T)$

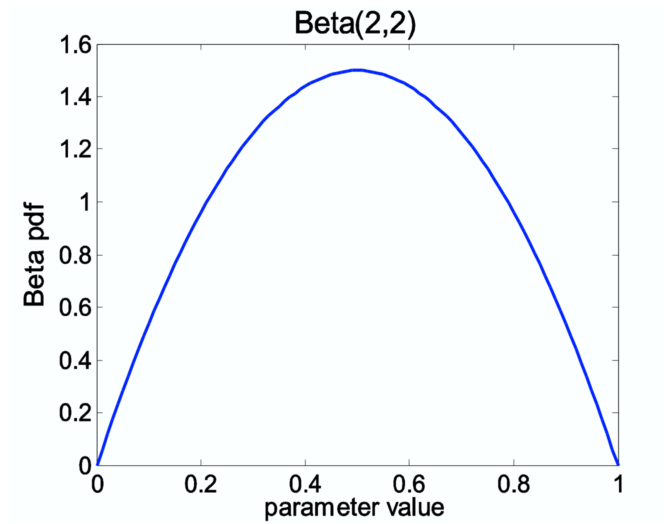
More concentrated as values of β_H, β_T increase



I think the coin is fair – it is "close" to 50-50

$$P(\theta) = \frac{\theta^{\beta_H - 1}(1-\theta)^{\beta_T - 1}}{B(\beta_H, \beta_T)} \sim Beta(\beta_H, \beta_T)$$





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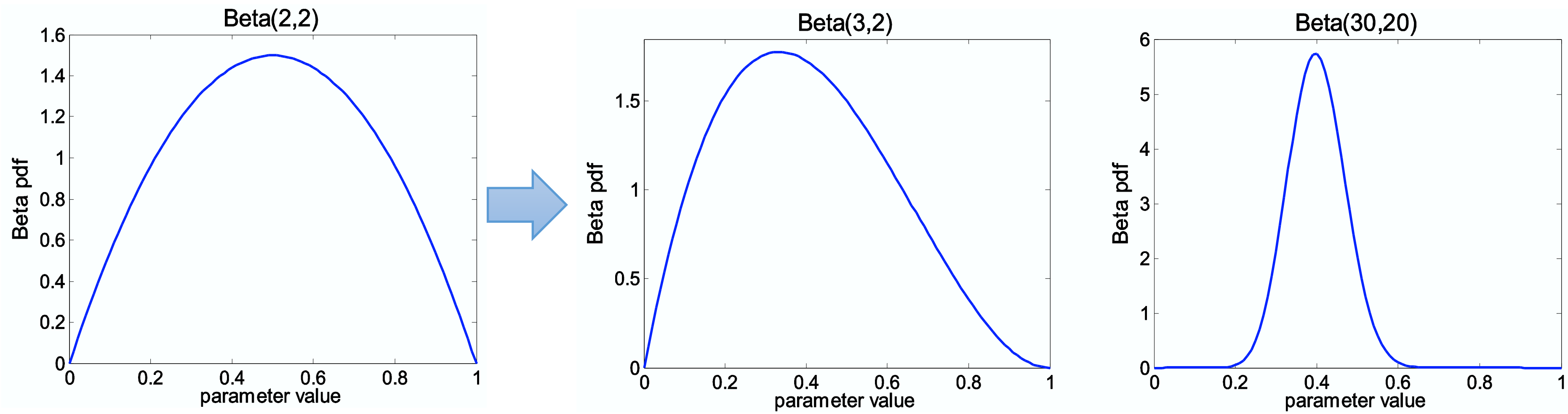
Beta conjugate prior

$$P(\theta) \sim Beta(\beta_H, \beta_T) \qquad P(\theta|D) \sim Beta(\beta_H + \alpha_H, \beta_T + \alpha_T)$$



As $n = \alpha_H + \alpha_T$ increases

As we get more samples, effect of prior is "washed out"

Conjugate Prior

- P(θ) and P(θ|D) have the same form

Eg. 2 Dice roll problem (6 outcomes instead of 2)

Likelihood is ~ Multinomial($\theta = \{\theta_1, \theta_2, \ldots, \theta_k\}$)

$$P(\mathcal{D} \mid \theta) = \theta_1^{\alpha_1} \theta_2^{\alpha_2} \ldots \theta_k^{\alpha_k}$$

If prior is Dirichlet distribution,

$$P(\theta) = \frac{\prod_{i=1}^{k} \theta_i^{\beta_i - 1}}{B(\beta_1, \ldots, \beta_k)} \sim \text{Dirichlet}(\beta_1, \ldots, \beta_k)$$

Then posterior is Dirichlet distribution

$$P(\theta|D) \sim \text{Dirichlet}(\beta_1 + \alpha_1, \ldots, \beta_k + \alpha_k)$$

For Multinomial, conjugate prior is Dirichlet distribution.

Posterior Distribution

- The approach seen so far is what is known as a **Bayesian** approach
- Prior information encoded as a **distribution** over possible values of parameter
- Using the Bayes rule, you get an updated **posterior** distribution over parameters, which you provide with flourish to the Billionaire
- But the billionaire is not impressed
  - Distribution? I just asked for one number: is it 3/5, 1/2, what is it?
  - How do we go from a distribution over parameters, to a single estimate of the true parameters?

Maximum A Posteriori Estimation

Choose θ that maximizes a posterior probability

$$\widehat{\theta}_{MAP} = \arg\max_{\theta} \; P(\theta \mid D)$$
$$= \arg\max_{\theta} \; P(D \mid \theta) P(\theta)$$

$$P(\theta) = \frac{\theta^{\beta_H - 1}(1-\theta)^{\beta_T - 1}}{B(\beta_H, \beta_T)} \sim Beta(\beta_H, \beta_T)$$

MAP estimate of probability of head:

$$P(\theta|D) \sim Beta(\beta_H + \alpha_H, \beta_T + \alpha_T)$$

$$\widehat{\theta}_{MAP} = \frac{\alpha_H + \beta_H - 1}{\alpha_H + \beta_H + \alpha_T + \beta_T - 2}$$

Mode of Beta distribution

MLE vs. MAP

- Maximum Likelihood estimation (MLE)

  Choose value that maximizes the probability of observed data

$$\widehat{\theta}_{MLE} = \arg\max_{\theta} P(D|\theta)$$

- Maximum *a posteriori* (MAP) estimation

  Choose value that is most probable given observed data and prior belief

$$\widehat{\theta}_{MAP} = \arg\max_{\theta} P(\theta|D)$$
$$= \arg\max_{\theta} P(D|\theta)P(\theta)$$

When is MAP same as MLE?

MLE vs. MAP

$$\widehat{\theta}_{MLE} = \frac{\alpha_H}{\alpha_H + \alpha_T}$$

What if we toss the coin too few times?

- You say: Probability next toss is a head = 0
- Billionaire says: You're fired! ...with prob 1 ☺

$$\widehat{\theta}_{MAP} = \frac{\alpha_H + \beta_H - 1}{\alpha_H + \beta_H + \alpha_T + \beta_T - 2}$$

- Beta prior equivalent to extra coin flips
- As $n \longrightarrow$ infty, prior is "forgotten"
- **But, for small sample size, prior is important!**

MLE vs MAP

You are no good when sample is small

You give a different answer for different priors

MAP for Gaussian mean and variance

- Conjugate priors
  - Mean: Gaussian prior
  - Variance: Wishart Distribution

- Prior for mean:

$$P(\mu \mid \eta, \lambda) = \frac{1}{\lambda\sqrt{2\pi}} e^{\frac{-(\mu-\eta)^2}{2\lambda^2}}$$ $= N(\eta, \lambda^2)$

MAP for Gaussian Mean

$$\widehat{\mu}_{MLE} \;=\; \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\widehat{\mu}_{MAP} \;=\; \frac{\frac{1}{\sigma^2}\sum_{i=1}^{n} x_i + \frac{\eta}{\lambda^2}}{\frac{n}{\sigma^2} + \frac{1}{\lambda^2}}$$

MAP of Gaussian variance - Later

Prior Information

- In the Bayesian approach, the prior information is encoded through a prior distribution over the parameters
- Seems onerous: the distribution typically seems to be obtained from convenience (conjugate distribution)
- What other ways can we encode our prior knowledge about the parameters?
- A non-Bayesian approach is via constraints: later

Autonomous Robot Navigation

- Mobile robots have sensors and motors
- How do they move?
  - Need to know *where* they are
  - Combine apriori knowledge with data
  - Not learning, but computing ~MAP

- Apriori: own motion, and sensing

Discussion

- MAP can be seen as “superior” to MLE
  - Use of priors
  - Good estimates from few data

- Robustness tradeoff
  - What if the prior is wrong?

Summary

- Conditional probabilities
- Bayes Rule
- Priors, conjugate prior
- MAP - maximum aposteriori estimate
- MLE and MAP
- Example: Bayesian update for robot localization estimate