方　差

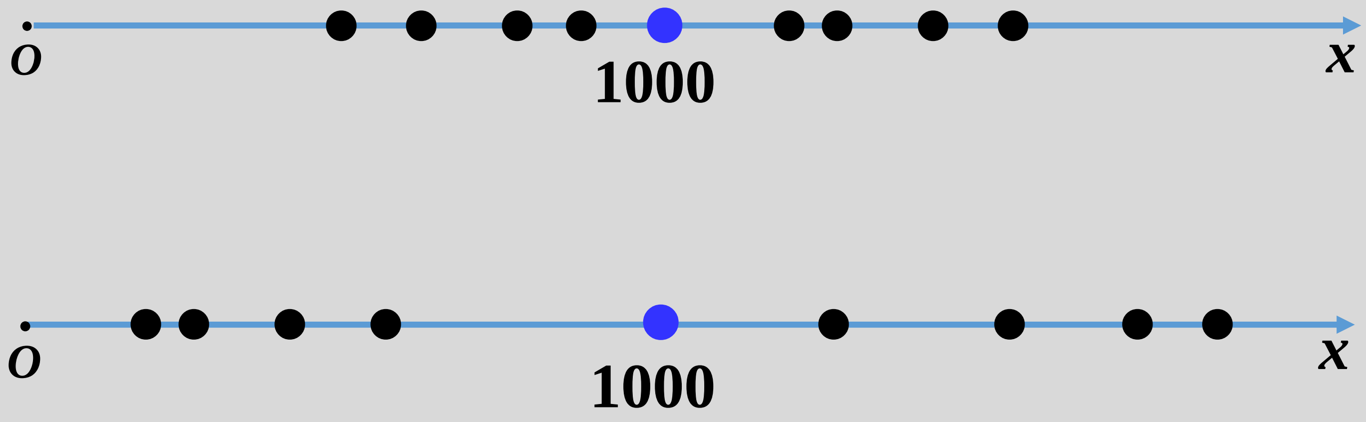
方差的定义

几种常见随机变量的方差

方差的性质

方差是一个常用来体现随机变量取值分散程度的量

有两批灯泡, 其平均寿命都是 $E(X) = 1000$ 小时.



$$|X - E(X)| \Longrightarrow E\left(|X - E(X)|\right) \Longrightarrow E\left([X - E(X)]^2\right)$$

方差的定义

设X是随机变量，若$E([X-E(X)]^2)$存在，则称$E([X-E(X)]^2)$为X的方差，即

$$D(X) = E([X-E(X)]^2)$$

且称$\sqrt{D(X)}$为X的标准差或均方差，记作σ_X, 即 $\sigma_X = \sqrt{D(X)}$

$D(X) = E([X - E(X)]^2)$

- 若X为离散型随机变量，其分布律为$P(X = x_k) = p_k, k = 1, 2, \ldots,$

则 $$D(X) = E([X - E(X)]^2) = \sum_{k \geq 1} [x_k - E(X)]^2 p_k.$$

- 若X为连续型随机变量，其概率密度为$f(x)$, 则

$$D(X) = E([X - E(X)]^2) = \int_{-\infty}^{\infty} [x - E(X)]^2 f(x)\,\mathrm{d}x$$

$D(X) = E([X - E(X)]^2)$

- 由定义式可得 $D(X) = E(X^2) - [E(X)]^2$

证明: $D(X) = E([X - E(X)]^2)$

$= E\{X^2 - 2XE(X) + [E(X)]^2\}$

常数

$= E(X^2) - 2E(X)E(X) + [E(X)]^2$

$= E(X^2) - [E(X)]^2$

几种常见随机变量的方差

1. 两点分布　已知随机变量 X 的分布律为

$$X \sim \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$$

则 $E(X) = p$ 且 $E(X^2) = p$

故 $D(X) = E(X^2) - [E(X)]^2 = p(1-p)$

$$D(X) = E(X^2) - [E(X)]^2$$

2. 泊松分布 设 $X \sim P(\lambda)$，则 $P(X=k) = \frac{\lambda^k}{k!}e^{-\lambda}$ $k = 0,1,\ldots$ 且 $E(X) = \lambda$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k}{k!} e^{-\lambda} + \boxed{\sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda}}$$

$$= \lambda^2 e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} + \lambda = \lambda^2 + \lambda \qquad E(X) = \lambda$$

故 $D(X) = E(X^2) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

$$D(X) = E(X^2) - [E(X)]^2$$

3. 均匀分布

设 $X \sim U(a,b)$, 则 $f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & otherwise \end{cases}$ 且 $E(X) = \frac{a+b}{2}$

故 $D(X) = E(X^2) - [E(X)]^2 = \int_a^b \frac{x^2}{b-a} \mathrm{d}x - (\frac{a+b}{2})^2 = \frac{(b-a)^2}{12}$

$$D(X) = E(X^2) - [E(X)]^2$$

4. 指数分布

设 $X \sim \mathrm{E}(\lambda)$，则 $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & otherwise \end{cases}$ 且 $E(X) = \dfrac{1}{\lambda}$

$$D(X) = E\left(X^2\right) - [E(X)]^2 = \int_0^{\infty} x^2 \lambda e^{-\lambda x}\, \mathrm{d}x - \frac{1}{\lambda^2} = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$D(X) = E([X - E(X)]^2)$

5. 正态分布 $X \sim N(\mu, \sigma^2)$, 则 $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R}$ 且 $E(X) = \mu$

$$D(X) = E([X - E(X)]^2) = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \mathrm{d}x$$

$$t = \frac{x-\mu}{\sigma}$$

$$= \int_{-\infty}^{\infty} (\sigma t)^2 \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{t^2}{2}} \sigma \mathrm{d}t$$

$$= \sigma^2 \boxed{\int_{-\infty}^{\infty} t^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \mathrm{d}t}$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \mathrm{d}t = \sigma^2$$

方差的性质

1. 设 C 是常数，则有 $D(C) = 0$

2. 设C 是常数, 随机变量X 的方差存在, 则CX 的方差存在, 且

$$D(CX) = C^2 D(X)$$

特别地，
$$D(-X) = D(X)$$

方差的性质

3. 设$n \geq 2, X_1, X_2, \dots X_n$ 是相互独立的随机变量，方差均存在，则

$X_1 \pm \cdots \pm X_n$ 的方差存在，且

$$D(X_1 \pm \cdots \pm X_n) = D(X_1) + \cdots + D(X_n).$$

特别地，设随机变量X, Y 相互独立且方差存在, 则 $X \pm Y$ 的方差也存在, 且

$$D(X \pm Y) = D(X) + D(Y).$$

$$D(X) = E([X - E(X)]^2)$$

证： $D(X+Y) = E\left(\{(X+Y) - E[(X+Y)]\}^2\right)$

$= E(\{[X - E(X)] + [Y - E(Y)]\}^2)$

$= E([X - E(X)]^2 + [Y - E(Y)]^2 + 2[X - E(X)][Y - E(Y)])$

$= E([X - E(X)]^2) + E([Y - E(Y)])^2 + 2E([X - E(X)][Y - E(Y)])$

$= D(X) + D(Y) + 2E([X - E(X)][Y - E(Y)])$

$$D(X+Y)=D(X)+D(Y)+2E([X-E(X)][Y-E(Y)])$$

因为X, Y 相互独立，故$X-E(X)$与$Y-E(Y)$也相互独立，所以

常数

$$E([X-E(X)][Y-E(Y)])=E[X-E(X)]\,E[Y-E(Y)]$$

$$=[E(X)-E(X)][E(Y)-E(Y)]\ \ =0$$

由此可得： $D(X+Y)=D(X)+D(Y).$

方差的性质

4. 对任意常数C, 若随机变量X方差存在，则$D(X) \le E\left((X-C)^2\right)$,

且等号成立的充分必要条件是 $E(X)=C$

证： $D(X)-E\left((X-C)^2\right)= E([X-E(X)]^2) - E\left((X-C)^2\right)$

$= -([E(X)]^2 - 2CE(X) + C^2) = -[E(X)-C]^2 \le 0$

且等号成立的充分必要条件是 $E(X)=C$

方差的性质

5. 若 $D(X)=0$ 则 $P(X=E(X))=1$

回忆期望的一个性质：若 $E(|X|)=0$，则 $P(X=0)=1$.

证：由 $D(X)=E([X-E(X)]^2)=0$ 可知

$$P(X-E(X)=0)=1.$$

即 $P(X=E(X))=1$.

$D(aX) = a^2 D(X) \qquad D(C) = 0$

例1. 设 $X \sim \begin{pmatrix} -3 & 0 & 2 & 3 \\ \frac{1}{3} & \frac{1}{12} & \frac{1}{4} & \frac{1}{3} \end{pmatrix}$ 求$D(2X^2 - 5)$

解: $D(2X^2 - 5) = 4D(X^2) + D(5) = 4\left(E(X^4) - [E(X^2)]^2\right) + 0$

$$E(X^2) = (-3)^2 \times \frac{1}{3} + 0^2 \times \frac{1}{12} + 2^2 \times \frac{1}{4} + 3^2 \times \frac{1}{3} = 7$$

$$E(X^4) = (-3)^4 \times \frac{1}{3} + 0^4 \times \frac{1}{12} + 2^4 \times \frac{1}{4} + 3^4 \times \frac{1}{3} = 58$$

故 $D(2X^2 - 5) = 4\left(E(X^4) - [E(X^2)]^2\right) = 36$

若 $X \sim N(\mu_1, {\sigma_1}^2)$, $Y \sim N(\mu_2, {\sigma_2}^2)$, 且 X, Y独立,

则 $aX + bY \sim N(a\mu_1 + b\mu_2,\ a^2{\sigma_1}^2 + b^2{\sigma_2}^2)$

例2　设活塞的直径$X \sim N(22.40, 0.03^2)$, 气缸的直径$Y \sim N(22.50, 0.04^2)$,

(上述直径均以cm计), X, Y相互独立. 任取一个活塞，任取一个气缸,

求活塞能装入气缸的概率.

解: 要求 $P(X < Y)$, 即$P(X - Y < 0)$, 因为

$X \sim N(22.40, 0.03^2)$, $Y \sim N(22.50, 0.04^2)$, 且X, Y相互独立,故

$$X - Y \sim N(22.40 - 22.50,\ 0.03^2 + 0.04^2)$$

$$X - Y \sim N(-0.10, 0.05^2)$$

故 $P(X - Y < 0) = P\left\{\frac{(X-Y)+0.10}{0.05} < \frac{0.10}{0.05}\right\}$

$$= P\left\{\frac{(X-Y)+0.10}{0.05} < 2\right\} = \Phi(2) = 0.9772$$

$$D\left(\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} D(X_k) \text{（独立）}$$

例3. 计算二项分布的方差

解：设 $X \sim B(n, p)$，在n重伯努利试验中，令

$$X_k = \begin{cases} 1, & \text{若事件}A\text{在第}k\text{次试验中发生} \\ 0, & \text{若事件}A\text{在第}k\text{次试验中不发生} \end{cases}$$

则$X_1, \dots, X_n$相互独立，$E(X_k) = p$，$D(X_k) = p(1-p)$，$k = 1, \dots, n$

$$\text{且 } X = \sum_{k=1}^{n} X_k \quad \text{故 } D(X) = \sum_{k=1}^{n} D(X_k) = n\,p\,(1 - p)$$

$$E\left(\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} E(X_k)$$

例4. 设随机变量$X_1, \dots, X_n$独立同分布于正态分布$N(\mu, \sigma^2)$，

$$令\ \bar{X} = \frac{1}{n}\sum_{k=1}^{n} X_k,\ 求\ E(\bar{X}), D(\bar{X}), E(\bar{X}^2).$$

解：由条件可知 $E(X_k) = \mu,\quad D(X_k) = \sigma^2,\quad k = 1, \dots, n.$

$$故\quad E(\bar{X}) = E\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right) = \frac{1}{n}\sum_{k=1}^{n} E(X_k) = \mu$$

$$D\left(\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} D(X_k)$$ （独立）

随机变量 $X_1, \dots, X_n$ 独立同分布于正态分布 $N(\mu, \sigma^2)$ ，

$$E(X_k) = \mu, \quad D(X_k) = \sigma^2, \quad k = 1, \dots, n.$$

$X_1, \dots, X_n$ 相互独立

$$D(\bar{X}) = D\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right) = \frac{1}{n^2}\sum_{k=1}^{n} D(X_k) = \frac{\sigma^2}{n}$$

$$E(\bar{X}^2) = \mathrm{D}(\bar{X}) + [E(\bar{X})]^2 = \frac{\sigma^2}{n} + \mu^2$$

期望与方差的计算

- 设$X_1, X_2, \dots, X_n$独立同分布于 $f(x) = \begin{cases} \frac{1}{\theta}, & 0 \le x \le \theta \\ 0, & else \end{cases}$, $\theta > 0$

求 $E(\max\{X_1, X_2, \dots, X_n\})$； $D(\max\{X_1, X_2, \dots, X_n\})$.

分析： 令 $M = \max\{X_1, X_2, \dots, X_n\}$

$$E(M) = \int_{-\infty}^{\infty} x f_M(x)\, dx \longleftarrow f_M(x) = F'_M(x) \longleftarrow F_M(x) = [F(x)]^n$$

Step 4　Step 3　Step 2　Step 1

期望与方差的计算

- 设$X_1, X_2, \dots, X_n$ 独立同分布于 $f(x) = \begin{cases} \frac{1}{\theta}, & 0 \le x \le \theta \\ 0, & else \end{cases}$, $\theta > 0$

求 $E(\max\{X_1, X_2, \dots, X_n\})$; $D(\max\{X_1, X_2, \dots, X_n\})$.

解: Step 1. 求$F(x)$ $X_1, X_2, \dots, X_n \sim f(x) = \begin{cases} \frac{1}{\theta}, & 0 \le x \le \theta \\ 0, & else \end{cases}$

$$\Rightarrow F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0, & x < 0 \\ \int_0^x \frac{1}{\theta} dt, & 0 \le x \le \theta \\ 1, & x \ge \theta \end{cases} = \begin{cases} 0, & x < 0 \\ \frac{x}{\theta}, & 0 \le x \le \theta \\ 1, & x \ge \theta \end{cases}$$

$$F(x)=\begin{cases}0, & x<0\\ \dfrac{x}{\theta}, & 0\le x\le\theta\\ 1, & x\ge\theta\end{cases}$$

令 $M=\max\{X_1,X_2,\dots,X_n\}$

Step 2. 求$F_M(x)$

$$F_M(x)=[F(x)]^n=\frac{x^n}{\theta^n},\qquad 0\le x\le\theta$$

Step 3. 求$f_M(x)$

$$f_M(x)=F'_M(x)=\frac{nx^{n-1}}{\theta^n},\qquad 0\le x\le\theta$$

$$f_M(x) = \frac{nx^{n-1}}{\theta^n}, \qquad 0 \le x \le \theta$$

Step 4. 求$E(M)$和$D(M)$

$$E(M) = E(\max\{X_1, X_2, \dots, X_n\}) = \int_{-\infty}^{\infty} x f_M(x)\,dx = \int_0^{\theta} \frac{nx^n}{\theta^n}dx = \frac{n}{n+1}\theta$$

$$E(M^2) = \int_{-\infty}^{\infty} x^2 f_M(x)\,dx = \int_0^{\theta} \frac{nx^{n+1}}{\theta^n}dx = \frac{n}{n+2}\theta^2$$

$$D(M) = E(M^2) - [E(M)]^2 = \frac{n}{(n+2)(n+1)^2}\theta^2$$

方差的计算方法

定义

$$D(X) = E([X - E(X)]^2)$$

公式

$$D(X) = E(X^2) - [E(X)]^2$$

性质

$$D(X \pm Y) = D(X) + D(Y)\text{（独立）}$$