正态总体的抽样分布

- 一个正态总体下的基本定理
- 两个正态总体下的基本定理

一个正态总体下的基本定理

设$X_1, X_2, \dots, X_n$ 是来自正态总体$X \sim N(\mu, \sigma^2)$的样本，

$$\bar{X} = \frac{1}{n}\sum_{k=1}^{n} X_k, \quad S^2 = \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \bar{X})^2$$

则有

- $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$;
- $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2(\mathrm{n} - 1)$;
- $\bar{X}$与S^2相互独立；
- $\dfrac{\bar{X} - \mu}{s/\sqrt{n}} \sim t(n-1)$.

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

证明： $X_1, X_2, \ldots, X_n$ 是来自正态总体$X \sim N(\mu, \sigma^2)$的样本，故

$$\bar{X} = \frac{1}{n}\sum_{k=1}^{n} X_k \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

标准化后即得 $$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$T\sim t(n) \Longleftrightarrow T=\frac{X}{\sqrt{Y/n}} \quad (X\sim N(0,1)\ \ Y\sim\chi^2(n)\ \ X\text{与}Y\text{独立})$$

- $$\frac{\bar{X}-\mu}{s/\sqrt{n}}\sim t(n-1).$$

证明：

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\sim N(0,1)$$

$$\frac{(n-1)S^2}{\sigma^2}\sim\chi^2(\mathrm{n}-1)$$

$\bar{X}$与S^2相互独立

$$\Longrightarrow \frac{\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{(n-1)S^2}{\sigma^2}/(n-1)}}\sim t(n-1)$$

$$\Longrightarrow \frac{\bar{X}-\mu}{s/\sqrt{n}}\sim t(n-1).$$

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

例1 设$X_1, X_2, \dots, X_n$ 是来自正态总体$X \sim N(\mu, 4)$的样本，当样本容量为多大时，

$$P(|\bar{X}-\mu| \le 0.1) = 0.95.$$

解：$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1) \Longrightarrow 0.95 = P(|\bar{X}-\mu| \le 0.1) = P\left(\left|\frac{\bar{X}-\mu}{2/\sqrt{n}}\right| \le \frac{0.1}{2/\sqrt{n}}\right)$

$\Longrightarrow 0.95 = \Phi(0.05\sqrt{n}) - \Phi(-0.05\sqrt{n}) = 2\Phi(0.05\sqrt{n}) - 1$

$\Longrightarrow \Phi(0.05\sqrt{n}) = 0.975$ 查表得 $\Phi(1.96) = 0.975$

$\Longrightarrow 0.05\sqrt{n} = 1.96, n \approx 1537$

$$\frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t(n-1).$$

例2 设$X_1, X_2, \dots, X_{10}$ 是来自正态总体$X \sim N(3, \sigma^2)$的样本，$s^2 = 4$,

求 $P(2.1253 \leq \bar{X} \leq 3.8747)$

解：由 $\frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t(n-1)$ 得

$$P(2.1253 \leq \bar{X} \leq 3.8747) = P\left(\frac{2.1253-3}{2/\sqrt{10}} \leq \frac{\bar{X}-3}{2/\sqrt{10}} \leq \frac{2.1253-3}{2/\sqrt{10}}\right)$$

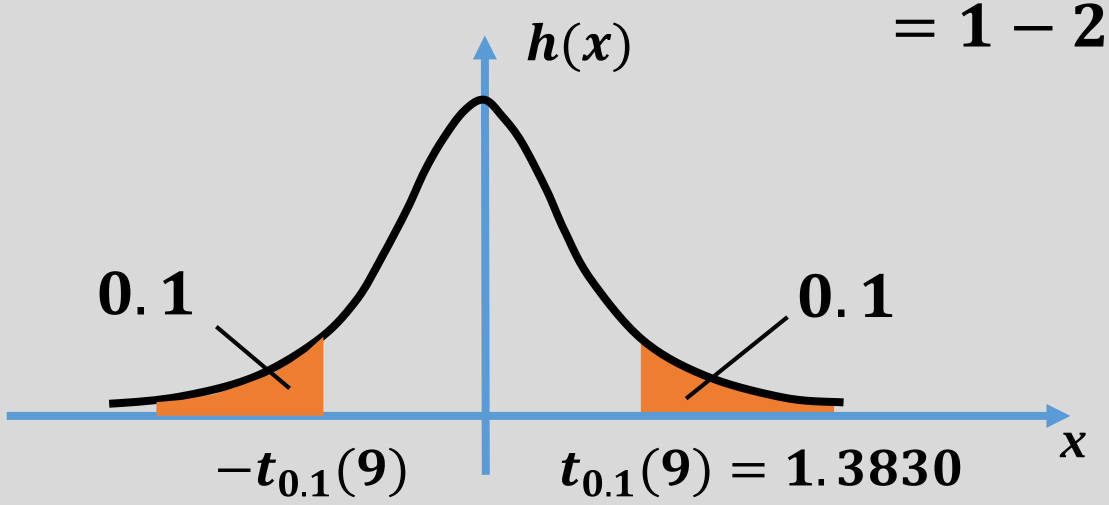
$$= P\left(-1.3830 \leq \frac{\bar{X}-3}{2/\sqrt{10}} \leq 1.3830\right)$$

$$P(2.1253 < \bar{X} \le 3.8747) = P\left(-1.3830 < \frac{\bar{X}-3}{2/\sqrt{10}} \le 1.3830\right)$$

查表得 $t_{0.1}(9) = 1.3830$ 故

$$P(2.1253 \le \bar{X} \le 3.8747) = P\left(-1.3830 \le \frac{\bar{X}-3}{2/\sqrt{10}} \le 1.3830\right)$$

$$= 1 - 2 \times 0.1 = 0.8$$



$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(\mathrm{n}-1)$$

例3 设$X_1, X_2, \dots, X_{10}$是来自正态总体$X \sim N(\mu, 4)$的样本，求 $P\left(S^2 > 2.622\right)$.

解：由 $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$ 得

$$P\left(S^2 > 2.622\right) = P\left(\frac{9S^2}{4} > \frac{9 \times 2.622}{4}\right) = P\left(\frac{9S^2}{4} > 5.899\right)$$

查表得 $\chi^2{}_{0.75}(9) = 5.899$

故 $P\left(S^2 > 2.622\right) = P\left(\frac{9S^2}{4} > 5.899\right) \approx 0.75$

两个正态总体下的基本定理

设$X_1, X_2, \dots, X_m$ 是来自正态总体$X \sim N(\mu_1, \sigma^2)$的样本，$Y_1, Y_2, \dots, Y_n$ 是来自正态总体$X \sim N(\mu_2, \sigma^2)$的样本，且两样本相互独立，记

$$\bar{X} = \frac{1}{m}\sum_{k=1}^{m} X_k, \qquad S_1^2 = \frac{1}{m-1}\sum_{k=1}^{m}(X_k - \bar{X})^2$$

$$\bar{Y} = \frac{1}{n}\sum_{k=1}^{n} Y_k, \qquad S_2^2 = \frac{1}{n-1}\sum_{k=1}^{n}(Y_k - \bar{Y})^2$$

则 $T = \dfrac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{S_w\sqrt{\dfrac{1}{m} + \dfrac{1}{n}}} \sim t(m+n-2)$ 其中 $S_w^2 = \dfrac{(m-1)S_1^2 + (n-1)S_2^2}{m+n-2}$

$$T=\frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{S_w\sqrt{\frac{1}{m}+\frac{1}{n}}}\sim t(m+n-2) \qquad S_w^2=\frac{(m-1)S_1^2+(n-1)S_2^2}{m+n-2}$$

证： $X_1,X_2,\dots,X_m\sim N(\mu_1,\sigma^2) \Longrightarrow \bar{X}\sim N(\mu_1,\frac{\sigma^2}{m})$

$Y_1,Y_2,\dots,Y_n\sim N(\mu_2,\sigma^2) \Longrightarrow \bar{Y}\sim N(\mu_2,\frac{\sigma^2}{n})$

两样本相互独立

$$\Longrightarrow \bar{X}-\bar{Y}\sim N(\mu_1-\mu_2,\frac{\sigma^2}{m}+\frac{\sigma^2}{n})$$

$$\Longrightarrow U=\frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sigma\sqrt{\frac{1}{m}+\frac{1}{n}}}\sim N(0,1)$$

$$T=\frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{S_w\sqrt{\frac{1}{m}+\frac{1}{n}}}\sim t(m+n-2) \qquad S_w^2=\frac{(m-1)S_1^2+(n-1)S_2^2}{m+n-2}$$

$$X_1,X_2,\ldots,X_m\sim N(\mu_1,\sigma^2) \Longrightarrow \frac{(m-1)S_1^2}{\sigma^2}\sim\chi^2(m-1)$$

$$Y_1,Y_2,\ldots,Y_n\sim N(\mu_2,\sigma^2) \Longrightarrow \frac{(n-1)S_2^2}{\sigma^2}\sim\chi^2(n-1)$$

两样本相互独立

$$\Longrightarrow V=\frac{(m-1)S_1^2}{\sigma^2}+\frac{(n-1)S_2^2}{\sigma^2}\sim\chi^2(m+n-2)$$

$$T=\frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{S_w\sqrt{\frac{1}{m}+\frac{1}{n}}}\sim t(m+n-2) \qquad S_w^2=\frac{(m-1)S_1^2+(n-1)S_2^2}{m+n-2}$$

$$U=\frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sigma\sqrt{\frac{1}{m}+\frac{1}{n}}}\sim N(0,1)$$

$$V=\frac{(m-1)S_1^2}{\sigma^2}+\frac{(n-1)S_2^2}{\sigma^2}\sim\chi^2(m+n-2)$$

$\bar{X}$与S^2相互独立 $\Longrightarrow$ U,V相互独立

$$\Longrightarrow \quad \frac{U}{\sqrt{\frac{V}{m+n-2}}}=\frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{S_w\sqrt{\frac{1}{m}+\frac{1}{n}}}\sim t(m+n-2)$$

小　结

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