Ch5 定积分　习题课例子参考解答

例1 $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\sqrt{1+\frac{i}{n}}$

$= \int_0^1\sqrt{1+x}\,dx = \int_0^1(1+x)^{\frac{1}{2}}d(1+x) = \frac{2}{3}(1+x)^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}(2\sqrt{2}-1)$

例2 $\lim\limits_{n\to\infty}\frac{1}{n^2}\sum\limits_{i=1}^{n}\sqrt{in}$

$= \lim\limits_{n\to\infty}\frac{1}{n^2}\sum\limits_{i=1}^{n}\sqrt{n^2\cdot\frac{i}{n}} = \lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\sqrt{\frac{i}{n}} = \int_0^1\sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}$

补1 $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{e^{\frac{k}{n}}}{n+ne^{\frac{2k}{n}}}$

$= \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{e^{\frac{k}{n}}}{1+e^{\frac{2k}{n}}}\cdot\frac{1}{n} = \int_0^1\frac{e^x}{1+e^{2x}}dx$

$= \int_0^1\frac{de^x}{1+(e^x)^2} = \arctan e^x\Big|_0^1 = \arctan e - \frac{\pi}{4}$

例3 $\lim\limits_{n\to\infty}\frac{1}{n}\sqrt[n]{(n+1)(n+2)\cdots(2n)}$

$= \lim\limits_{n\to\infty}\frac{1}{n}\cdot n\sqrt[n]{(1+\frac{1}{n})(1+\frac{2}{n})\cdots(1+\frac{n}{n})}$

$= \lim\limits_{n\to\infty}e^{\frac{1}{n}\sum\limits_{k=1}^{n}\ln(1+\frac{k}{n})} = e^{\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n}\ln(1+\frac{k}{n})} = e^{\int_0^1\ln(1+x)dx}$

$= e^{\int_0^1\ln(1+x)d(1+x)}$

$= \exp\left\{(1+x)\ln(1+x)\Big|_0^1 - \int_0^1dx\right\}$

$= e^{2\ln 2-1} = \frac{4}{e}$

补2 $\lim\limits_{n\to\infty}\ln\frac{\sqrt[n]{n!}}{n}$

$= \lim\limits_{n\to\infty}\ln\sqrt[n]{\frac{1}{n}\cdot\frac{2}{n}\cdots\frac{n}{n}} = \lim\limits_{n\to\infty}\frac{1}{n}\ln(\frac{1}{n}\cdot\frac{2}{n}\cdots\frac{n}{n})$

$= \lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n}\ln\frac{k}{n} = \int_0^1\ln x\,dx = x\ln x\Big|_{0^+}^1 - \int_0^1dx$

$= -1$

例4 $\int_{-1}^{1}(\sin x+\sqrt{1-x^2})\,dx$

$=\int_{-1}^{1}\sin x\,dx+\int_{-1}^{1}\sqrt{1-x^2}\,dx \quad = 0+2\int_{0}^{1}\sqrt{1-x^2}\,dx = 2\cdot\frac{1}{4}\pi\cdot1^2=\frac{\pi}{2}$

例5 $\int_{-1}^{1}\frac{2x^2+x\cos x}{1+\sqrt{1-x^2}}\,dx$

$=\int_{-1}^{1}\frac{2x^2}{1+\sqrt{1-x^2}}\,dx+\int_{-1}^{1}\frac{x\cos x}{1+\sqrt{1-x^2}}\,dx$

$=2\int_{0}^{1}\frac{2x^2}{1+\sqrt{1-x^2}}\,dx+0$

$=4\int_{0}^{1}\frac{x^2(1-\sqrt{1-x^2})}{1-(1-x^2)}\,dx \quad =4\int_{0}^{1}(1-\sqrt{1-x^2})\,dx$

$=4\int_{0}^{1}dx-4\int_{0}^{1}\sqrt{1-x^2}\,dx \quad =4-4\cdot\frac{1}{4}\pi\cdot1^2 \quad =4-\pi$

补3 $\int_{-1}^{2}x^3|x|\,dx$

$=\int_{-1}^{1}x^3|x|\,dx+\int_{1}^{2}x^3|x|\,dx$

$=0+\int_{1}^{2}x^4\,dx \quad =\frac{1}{5}x^5\Big|_{1}^{2} \quad =\frac{31}{5}$

例6 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\sqrt{1-\cos^2x}\,dx$

$=2\int_{0}^{\frac{\pi}{2}}\cos x\sqrt{\sin^2x}\,dx \quad =2\int_{0}^{\frac{\pi}{2}}\cos x\sin x\,dx \quad =2\int_{0}^{\frac{\pi}{2}}\sin x\,d\sin x$

$=2\cdot\frac{1}{2}\sin^2x\Big|_{0}^{\frac{\pi}{2}} \quad =1$

例7 $\int_{0}^{\pi}\cos^n x\,dx$

$=\int_{0}^{\frac{\pi}{2}}\cos^n x\,dx+\int_{\frac{\pi}{2}}^{\pi}\cos^n x\,dx$

又 $\int_{\frac{\pi}{2}}^{\pi}\cos^n x\,dx \overset{\pi-x=t}{=\!=\!=} -\int_{+\frac{\pi}{2}}^{0}\cos^n(\pi-t)\,dt = \int_{0}^{\frac{\pi}{2}}\cos^n(\pi-t)\,dt$

$=\int_{0}^{\frac{\pi}{2}}(-1)^n\cos^n t\,dt$

$\therefore \int_{0}^{\pi}\cos^n x\,dx=\int_{0}^{\frac{\pi}{2}}[1+(-1)^n]\cos^n x\,dx$

$=\begin{cases}0 & n\text{为奇数}\\ 2\int_{0}^{\frac{\pi}{2}}\cos^n x\,dx & n\text{为偶数}\end{cases}$

例8 $\int_0^{2\pi} \sin^4 x\,dx$

$= \int_{-\pi}^{\pi} \sin^4 x\,dx$ （2π是$\sin^4 x$的周期）

$= 2\int_0^{\pi} \sin^4 x\,dx = 4\int_0^{\frac{\pi}{2}} \sin^4 x\,dx = 4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3}{4}\pi$

补4 $\int_0^{n\pi} \sqrt{1-\sin 2x}\,dx$

$= n\int_0^{\pi} \sqrt{(\sin x-\cos x)^2}\,dx$

$= n\int_0^{\pi} |\sin x-\cos x|\,dx$

$= \sqrt{2}\,n\int_0^{\pi} |\sin(x-\frac{\pi}{4})|\,dx$

$\xlongequal{x-\frac{\pi}{4}=t} \sqrt{2}\,n\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} |\sin t|\,dt$

$= \sqrt{2}\,n\int_{-\frac{\pi}{4}}^{0} (-\sin t)\,dt + \sqrt{2}\,n\int_0^{\frac{3\pi}{4}} \sin t\,dt$

$= 2\sqrt{2}\,n$

或 $\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} |\sin t|\,dt$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin t|\,dt$

$= 2\int_{-\frac{\pi}{2}}^{0} (-\sin t)\,dt$

$= 2(\cos t\Big|_{-\frac{\pi}{2}}^{0})$

$= 2$

例9 $\int_0^{\frac{1}{2}} \frac{1-2x}{\sqrt{1-x^2}}\,dx$

$= \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} + \int_0^{\frac{1}{2}} \frac{-2x}{\sqrt{1-x^2}}\,dx$

$= \arcsin x\Big|_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \frac{d(1-x^2)}{\sqrt{1-x^2}}$

$= \frac{\pi}{6} + 2\sqrt{1-x^2}\Big|_0^{\frac{1}{2}}$

$= \frac{\pi}{6} + \sqrt{3} - 2$

补5 $\int_0^{\frac{\pi}{2}} \frac{\sin x\cos x}{a^2\sin^2 x+b^2\cos^2 x}dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{d\sin^2 x}{a^2\sin^2 x+b^2\cos^2 x} \xlongequal{\sin^2 x=t} \frac{1}{2}\int_0^1 \frac{dt}{a^2 t+b^2(1-t)}$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{(a^2-b^2)t+b^2} = \begin{cases} \frac{1}{2}a^2 & a=b \\ \frac{\ln a^2-\ln b^2}{2(a^2-b^2)} & a\neq b \end{cases}$$

例10 $\int_{-\pi}^{\pi} \sin^4(\frac{x}{2})\,dx$

$$\xlongequal{\frac{x}{2}=t} 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 t\,dt = 4\int_0^{\frac{\pi}{2}} \sin^4 t\,dt = 4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3}{4}\pi$$

例11 $\int_0^1 \sqrt{2x-x^2}\,dx$

$$= \int_0^1 \sqrt{1-(x-1)^2}\,dx \xlongequal{x-1=t} \int_{-1}^0 \sqrt{1-t^2}\,dt = \frac{1}{4}\pi\cdot 1^2 = \frac{\pi}{4}$$

补6 $\int_0^{2a} x\sqrt{2ax-x^2}\,dx$

$$= \int_0^{2a} x\sqrt{a^2-\underbrace{\cancel{(a-x)}^2}_{x-a}}\,dx$$

$$\xlongequal{\cancel{a-x=}\ x-a=t} \int_{-a}^{a} (a+t)\sqrt{a^2-t^2}\,dt$$

$$= a\int_{-a}^{a}\sqrt{a^2-t^2}\,dt + \int_{-a}^{a} t\sqrt{a^2-t^2}\,dt$$

$$= a\cdot\frac{1}{2}\cdot\pi a^2 + 0 = \frac{\pi}{2}a^3$$

例12 $\int_0^1 (1-x)^{50}x\,dx$

$$\xlongequal{1-x=t} -\int_1^0 t^{50}(1-t)\,dt = \int_0^1 (t^{50}-t^{51})\,dt$$

$$= \left[\frac{1}{51}t^{51}-\frac{1}{52}t^{52}\right]_0^1 = \frac{1}{51\times 52} = \frac{1}{2652}$$

例13 $\int_{-2}^{2} \max\{x, x^2\}\, dx$

$$\because \quad \max\{x, x^2\} = \begin{cases} x^2 & -2 \le x \le 0 \\ x & 0 < x < 1 \\ x^2 & 2 \ge x \ge 1 \end{cases}$$

$$\therefore \text{原式} = \int_{-2}^{0} x^2 dx + \int_0^1 x\, dx + \int_1^2 x^2 dx$$

$$= \frac{1}{3}x^3 \Big|_{-2}^{0} + \frac{1}{2}x^2 \Big|_0^1 + \frac{1}{3}x^3 \Big|_1^2 = \frac{11}{2}$$

例14 $\int_0^{\pi} \sqrt{1-\sin x}\, dx$

$$= \int_0^{\pi} \sqrt{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}\, dx = \int_0^{\pi} \sqrt{(\cos\frac{x}{2} - \sin\frac{x}{2})^2}\, dx$$

$$= \int_0^{\pi} \left|\cos\frac{x}{2} - \sin\frac{x}{2}\right| dx$$

$$\overset{\frac{x}{2}=t}{=\!=\!=} 2\int_0^{\frac{\pi}{2}} |\cos t - \sin t|\, dt$$

$$= 2\sqrt{2} \int_0^{\frac{\pi}{2}} \left|\sin(t - \frac{\pi}{4})\right| dt$$

$$\overset{t-\frac{\pi}{4}=u}{=\!=\!=} 2\sqrt{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |\sin u|\, du = 4\sqrt{2}\int_0^{\frac{\pi}{4}} \sin u\, du = -4\sqrt{2}\cos u \Big|_0^{\frac{\pi}{4}}$$

$$= 4(\sqrt{2}-1)$$

例15 $f(x) = \begin{cases} kx, & 0 \le x \le \frac{l}{2} \\ c & \frac{l}{2} < x \le l \end{cases}$ 求 $\int_0^x f(t)dt$

$$\int_0^x f(t)dt = \begin{cases} \int_0^x kt\,dt = \frac{1}{2}kx^2 & 0 \le x \le \frac{l}{2} \\ \int_0^{l/2} kt\,dt + \int_{l/2}^{x} c\,dt = \frac{1}{8}kl^2 + cx - \frac{1}{2}cl & \frac{l}{2} < x \le l \end{cases}$$

例16 $\int_0^1 x|x-\alpha|\, dx \quad (-\infty < \alpha < +\infty)$

$$= \begin{cases} \int_0^1 x(x-\alpha)dx & \alpha \le 0 \\ \int_0^{\alpha} x(\alpha - x)dx + \int_{\alpha}^{1} x(x-\alpha)dx & 0 < \alpha < 1 \\ \int_0^1 x(\alpha - x)dx & \alpha \ge 1 \end{cases}$$

例 17 $\displaystyle\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$

①
$$\int_0^{\frac{\pi}{4}} \ln\left(1+\frac{\sin x}{\cos x}\right)dx$$
$$= \int_0^{\frac{\pi}{4}} \ln(\sin x+\cos x)\,dx - \int_0^{\frac{\pi}{4}} \ln\cos x\,dx$$
$$= \int_0^{\frac{\pi}{4}} \ln \sqrt{2}\cos\left(x-\frac{\pi}{4}\right)dx - \int_0^{\frac{\pi}{4}} \ln\cos x\,dx$$
$$= \int_0^{\frac{\pi}{4}} \ln\sqrt{2}\,dx + \int_0^{\frac{\pi}{4}} \ln\cos\left(\frac{\pi}{4}-x\right)dx - \int_0^{\frac{\pi}{4}} \ln\cos x\,dx$$
$$= \frac{\pi}{4}\ln\sqrt{2} + \int_{\frac{\pi}{4}}^{0} \ln\cos t\; d(-t) - \int_0^{\frac{\pi}{4}} \ln\cos x\,dx \quad \left(令\ \frac{\pi}{4}-x=t\right)$$
$$= \frac{\pi}{8}\ln 2 + \int_0^{\frac{\pi}{4}} \ln\cos t\,dt - \int_0^{\frac{\pi}{4}} \ln\cos x\,dx$$
$$= \frac{\pi}{8}\ln 2$$

②
$$\xlongequal[令\ \frac{\pi}{4}-x=t]{} \int_{\frac{\pi}{4}}^{0} \ln\left[1+\tan\left(\frac{\pi}{4}-t\right)\right]d(-t)$$
$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan t}{1+\tan t}\right)dt$$
$$= \int_0^{\frac{\pi}{4}} [\ln 2 - \ln(1+\tan t)]\,dt$$
$$= \frac{\pi}{4}\ln 2 - \int_0^{\frac{\pi}{4}} \ln(1+\tan t)\,dt$$
$$\therefore\ 原式 = \frac{1}{2}\cdot\frac{\pi}{4}\ln 2 = \frac{\pi}{8}\ln 2$$

例18 $\int_0^{\frac{\pi}{2}} \ln\sin x\,dx$

由 $\int_0^{\frac{\pi}{2}} \ln\sin x\,dx = \int_0^{\frac{\pi}{2}} \ln\cos x\,dx$

有

$$\text{原式} = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\ln\sin x + \ln\cos x)\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin x\cos x\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\frac{\sin 2x}{2}\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin 2x\,dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln 2\,dx$$

$$\overset{2x=t}{=\!=} \frac{1}{4}\int_0^{\pi} \ln\sin t\,dt - \frac{\pi}{4}\ln 2$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin x\,dx - \frac{\pi}{4}\ln 2$$

$$\therefore \int_0^{\frac{\pi}{2}} \ln\sin x\,dx = -\frac{\pi}{2}\ln 2$$

例19 $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 x}{1+e^{-x}}\,dx$ 利用 $\int_{-a}^{a} f(x)dx = \int_0^a [f(x)+f(-x)]\,dx$.

$$= \int_0^{\frac{\pi}{4}} \left(\frac{\sin^2 x}{1+e^{-x}} + \frac{\sin^2 x}{1+e^{x}}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \sin^2 x\,dx = \int_0^{\frac{\pi}{4}} \frac{1-\cos 2x}{2}\,dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4}$$

补7 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x\sin^4 x}{1+e^x}\,dx$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{e^x\sin^4 x}{1+e^x} + \frac{e^{-x}\sin^4 x}{1+e^{-x}}\right]dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^4 x\,dx$$

$$= \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3}{16}\pi$$

补8 $\int_0^{\frac{\pi}{2}} \frac{f(\sin t)}{f(\sin t)+f(\cos t)}dt$

因 $\int_0^{\frac{\pi}{2}} \frac{f(\sin t)}{f(\sin t)+f(\cos t)}dt = \int_0^{\frac{\pi}{2}} \frac{f(\cos t)}{f(\cos t)+f(\sin t)}dt$

故 原式 $= \frac{1}{2}\int_0^{\frac{\pi}{2}}\left[\frac{f(\sin t)}{f(\sin t)+f(\cos t)} + \frac{f(\cos t)}{f(\cos t)+f(\sin t)}\right]dt$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} dt = \frac{\pi}{4}$

例20 $\frac{d}{dx}\int_a^b \sin x^2\,dx = 0$

$\frac{d}{db}\int_a^b \sin x^2\,dx = \sin b^2$

$\frac{d}{da}\int_a^b \sin x^2\,dx = -\sin a^2$

例21 $\left(\int_{x^2}^{x^3}\frac{dt}{\sqrt{1+t^4}}\right)'$

$= \frac{1}{\sqrt{1+x^{12}}}(x^3)' - \frac{1}{\sqrt{1+x^8}}(x^2)'$

$= \frac{3x^2}{\sqrt{1+x^{12}}} - \frac{2x}{\sqrt{1+x^8}}$

例22 $F(x) = \int_a^b f(x+y)dy$, 求 $F'(x)$.

$F(x) \xlongequal{令\ x+y=t} \int_{x+a}^{x+b} f(t)dt$

$F'(x) = f(x+b) - f(x+a)$

补9 $\varphi(x) = \int_0^1 f(x^2+t)\,dt$，求 $\varphi'(x)$

$\because \varphi(x) \xlongequal{x^2+t=u} \int_{x^2}^{x^2+1} f(u)\,du$

$\therefore \varphi'(x) = f(x^2+1)\cdot 2x - f(x^2)\cdot 2x$

例23 $F(x) = \int_a^b f(xt)\,dt$，求 $F'(x)$

$F(x) \xlongequal{xt=u} \frac{1}{x}\int_{xa}^{xb} f(u)\,du$

$$F'(x) = \frac{[bf(bx) - af(ax)]x - \int_{xa}^{bx} f(u)\,du}{x^2}$$

补10 $F(x) = \int_0^x t f(x^2-t^2)\,dt$，求 $F'(x)$.

$\cancel{F(x)} \xlongequal{x^2-t^2=u} -\frac{1}{2}\int_{x^2}^{0} f(u)\,du = \frac{1}{2}\int_0^{x^2} f(u)\,du$

$F(x) = -\frac{1}{2}\int_0^x f(x^2-t^2)\,d(x^2-t^2)$

$\therefore F'(x) = \frac{1}{2}f(x^2)\cdot 2x = x f(x^2)$

例24 $f(x) = \int_0^x \frac{\sin t}{\pi - t}\,dt$，求 $\int_0^\pi f(x)\,dx$

解: $\int_0^\pi f(x)\,dx = x f(x)\Big|_0^\pi - \int_0^\pi x f'(x)\,dx$

$= \pi f(\pi) - \int_0^\pi x \frac{\sin x}{\pi - x}\,dx$

（这样解答，出现两处困难：一是 $f(\pi)$ 不易求，二是 $\int_0^\pi x\frac{\sin x}{\pi-x}dx$ 中 x 与 $\pi-x$ 未消去.）

① $= \pi f(\pi) - \int_0^\pi \frac{(x-\pi)+\pi}{\pi-x}\sin x\,dx$

$= \pi f(\pi) + \int_0^\pi \sin x\,dx - \pi\int_0^\pi \frac{\sin x}{\pi-x}\,dx$

$= \int_0^\pi \sin x\,dx = 2$

$\int_0^\pi f(x)\,dx \overset{②}{=\!=} \int_0^\pi f(x)\,d(x-\pi)$

$= (x-\pi)f(x)\Big|_0^\pi - \int_0^\pi (x-\pi)\frac{\sin x}{\pi-x}\,dx$

$= \int_0^\pi \sin x\,dx = 2$

补11 $f(x)=\int_1^x \frac{1}{\sqrt{1+t^4}}dx$，求 $\int_0^1 x^2 f(x)dx$

$$\int_0^1 x^2 f(x)dx = \int_0^1 f(x)\,d\left(\tfrac{1}{3}x^3\right)$$
$$= \tfrac{1}{3}x^3 f(x)\Big|_0^1 - \tfrac{1}{3}\int_0^1 x^3\cdot\frac{1}{\sqrt{1+x^4}}dx$$
$$= -\frac{1}{12}\int_0^1 \frac{d(1+x^4)}{\sqrt{1+x^4}}$$
$$= -\frac{1}{6}\sqrt{1+x^4}\Big|_0^1 = \frac{1}{6}(1-\sqrt{2})$$

例25 证明 $\int_1^a f\left(x^2+\frac{a^2}{x^2}\right)\frac{dx}{x} = \int_1^a f\left(x+\frac{a^2}{x}\right)\frac{dx}{x}$ $(a>0)$

证：左 $= \int_1^a f\left(x^2+\frac{a^2}{x^2}\right)\frac{x}{x^2}dx$
$$= \frac{1}{2}\int_1^a f\left(x^2+\frac{a^2}{x^2}\right)\frac{d(x^2)}{x^2}$$
$$\overset{x^2=t}{=\!=} \frac{1}{2}\int_1^{a^2} f\left(t+\frac{a^2}{t}\right)\frac{dt}{t}$$
$$= \frac{1}{2}\left[\int_1^a f\left(t+\frac{a^2}{t}\right)\frac{dt}{t} + \int_a^{a^2} f\left(t+\frac{a^2}{t}\right)\frac{dt}{t}\right]$$
$$\overset{\frac{a^2}{t}=u}{=\!=} \frac{1}{2}\left[\int_1^a f\left(t+\frac{a^2}{t}\right)\frac{dt}{t} + \int_a^1 f\left(\frac{a^2}{u}+u\right)\frac{u}{a^2}\cdot\left(-\frac{a^2}{u^2}\right)du\right]$$
$$= \frac{1}{2}\left[\int_1^a f\left(x+\frac{a^2}{x}\right)\frac{dx}{x} + \int_1^a f\left(x+\frac{a^2}{x}\right)\frac{dx}{x}\right]$$
$$= \int_1^a f\left(x+\frac{a^2}{x}\right)\frac{dx}{x} = 右$$

例26 证明: $\int_0^1 \ln f(x+t)\,dt = \int_0^x \ln\frac{f(1+t)}{f(t)}\,dt + \int_0^1 \ln f(t)\,dt$

$$\int_0^1 \ln f(x+t)\,dt \ \underline{\underline{x+t=u}}\ \int_x^{1+x} \ln f(u)\,du$$

$$= \int_x^0 \ln f(u)\,du + \int_0^1 \ln f(u)\,du + \int_1^{1+x} \ln f(u)\,du$$

对 $\int_1^{1+x} \ln f(u)\,du \ \underline{\underline{u-1=t}}\ \int_0^x \ln f(1+t)\,dt$

$\therefore$ 左 $=\int_0^1 \ln f(x+t)\,dt = \int_0^x \ln f(1+t)\,dt - \int_0^x \ln f(t)\,dt + \int_0^1 \ln f(t)\,dt$

$$= \int_0^x \ln\frac{f(1+t)}{f(t)}\,dt + \int_0^1 \ln f(t)\,dt = \text{右}$$

注: 本题也可以用微分学中的方法.(求导)

例27 $f(x) = x - \int_0^\pi f(x)\cos x\,dx$, 求 $f(x)$

解: 令 $a = \int_0^\pi f(x)\cos x\,dx$, 由已知有 $f(x)\cos x = x\cos x - a\cos x$

$$\therefore a = \int_0^\pi f(x)\cos x\,dx = \int_0^\pi (x\cos x - a\cos x)\,dx$$

$$\therefore a = \int_0^\pi (x-a)\cos x\,dx = \int_0^\pi (x-a)\,d\sin x$$

$$= (x-a)\sin x\Big|_0^\pi - \int_0^\pi \sin x\,dx$$

$$= -2$$

$$\therefore f(x) = x+2$$

例28. $f(x) = x^2 - x\int_0^2 f(x)\,dx + 2\int_0^1 f(x)\,dx$, 求 $f(x)$

令 $\int_0^2 f(x)\,dx = a$, $\int_0^1 f(x)\,dx = b$, 则 $f(x) = x^2 - ax + 2b$

$$\therefore a = \int_0^2 (x^2 - ax + 2b)\,dx = \left[\frac{1}{3}x^3 - \frac{a}{2}x^2 + 2bx\right]_0^2 = \frac{8}{3} - 2a + 4b$$

$$b = \int_0^1 (x^2 - ax + 2b)\,dx = \left[\frac{1}{3}x^3 - \frac{a}{2}x^2 + 2bx\right]_0^1 = \frac{1}{3} - \frac{a}{2} + 2b$$

$$\Rightarrow \begin{cases} a = 4/3 \\ b = 1/3 \end{cases}$$

$$\therefore f(x) = x^2 - \frac{4}{3}x + \frac{2}{3}$$

例 29　$f(0)=1$, $f(2)=3$, $f'(2)=5$, 求 $\int_0^1 x f''(2x)\,dx$

解：$\int_0^1 x f''(2x)\,dx = \frac{1}{2}\int_0^1 x f''(2x)\,d(2x)$

$= \frac{1}{2}\int_0^1 x\,d f'(2x)$

$= \frac{1}{2} x f'(2x)\Big|_0^1 - \frac{1}{2}\int_0^1 f'(2x)\,dx$

$= \frac{1}{2} f'(2) - \frac{1}{4}\int_0^1 f'(2x)\,d(2x)$

$= \frac{1}{2} f'(2) - \frac{1}{4} f(2x)\Big|_0^1$

$= \frac{1}{2} f'(2) - \frac{1}{4}\left[f(2) - f(0)\right]$

$= \frac{5}{2} - \frac{1}{4}(3-1) \quad = 2$