第2章习题课中例子的参考解答

例1 $f'(x_0)$存在，求 $\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x+(\Delta x)^2)-f(x_0)}{\Delta x}$

解：原式 $=\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x+(\Delta x)^2)-f(x_0)}{\Delta x+(\Delta x)^2}\cdot\dfrac{\Delta x+(\Delta x)^2}{\Delta x}$

$$=f'(x_0)\lim_{\Delta x\to 0}\frac{\Delta x+(\Delta x)^2}{\Delta x}=f'(x_0)\cdot 1=f'(x_0)$$

例2 $f(1)=0$, $f'(1)=2$, 求 $\lim\limits_{x\to 0}\dfrac{f(\sin^2x+\cos x)}{x\tan x}$

解：原式 $=\lim\limits_{x\to 0}\dfrac{f(1+\sin^2x+\cos x-1)-f(1)}{x^2}$

$$=\lim_{x\to 0}\frac{f(1+\sin^2x+\cos x-1)-f(1)}{\sin^2x+\cos x-1}\cdot\frac{\sin^2x+\cos x-1}{x^2}$$

$$=f'(1)\lim_{x\to 0}\frac{\sin^2x+\cos x-1}{x^2}$$

$$=f'(1)\left(1-\frac{1}{2}\right)$$

$$=1$$

（或 $\lim\limits_{x\to 0}\dfrac{\sin^2x+\cos x-1}{x^2}$

$=\lim\limits_{x\to 0}\dfrac{\cos x-\cos^2x}{x^2}$

$=\lim\limits_{x\to 0}\dfrac{\cos x(1-\cos x)}{x^2}$

$=\dfrac{1}{2}$ ）

补1 $f'(0)$存在，$f(0)=0$，求 $\lim\limits_{x\to 0}\dfrac{f(1-\cos x)}{\tan x^2}$

解 $\lim\limits_{x\to 0}\dfrac{f(1-\cos x)}{\tan x^2}=\lim\limits_{x\to 0}\dfrac{f(1-\cos x)}{x^2}$

$$=\lim_{x\to 0}\frac{f(1-\cos x)-f(0)}{1-\cos x}\cdot\frac{1-\cos x}{x^2}$$

$$=f'(0)\cdot\frac{1}{2}$$

$$=\frac{1}{2}f'(0)$$

例3 $f'(a)$存在，$f(a)>0$，求 $\lim\limits_{n\to\infty}\left(\dfrac{f(a)}{f(a+\frac{1}{n})}\right)^n$

解：属于 1^∞ 型.

$$原式 = \lim_{n\to\infty}\left[1+\frac{f(a)-f(a+\frac{1}{n})}{f(a+\frac{1}{n})}\right]^{\frac{f(a+\frac{1}{n})}{f(a)-f(a+\frac{1}{n})}\cdot\frac{f(a)-f(a+\frac{1}{n})}{f(a+\frac{1}{n})}\cdot n}$$

$$= e^{-\lim\limits_{n\to\infty}\frac{f(a+\frac{1}{n})-f(a)}{\frac{1}{n}}\cdot\frac{1}{f(a+\frac{1}{n})}}$$

$$= e^{-\frac{f'(a)}{f(a)}}$$

例4 $\lim\limits_{x\to0}\dfrac{\arctan(1+x)-\frac{\pi}{4}}{x}$

$$= \lim_{x\to0}\frac{\arctan(1+x)-\arctan 1}{x}$$

$$= (\arctan x)'\Big|_{x=1}$$

$$= \frac{1}{1+x^2}\Big|_{x=1}$$

$$= \frac{1}{2}$$

补2 $\lim\limits_{x\to0}\dfrac{\sin(\frac{\pi}{4}-x)-\frac{1}{\sqrt2}}{x}$

$$= -\lim_{x\to0}\frac{\sin(\frac{\pi}{4}-x)-\sin\frac{\pi}{4}}{-x}$$

$$= -(\sin x)'\Big|_{x=\frac{\pi}{4}}$$

$$= \cos\frac{\pi}{4}$$

$$= \frac{\sqrt2}{2}$$

例5 $f(x)=x(x+1)\cdots(x+n)$，求 $f'(0)$

$$f'(0)=\lim_{x\to0}\frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to0}(x+1)(x+2)\cdots(x+n)$$

$$= n!$$

注：$f(x)=x\,p(x)$ $\quad p(x)=(x+1)(x+2)\cdots(x+n)$

$f'(x)=p(x)+x\,p'(x)$

$f'(0)=p(0)=n!$

例6 $f(x)=x+(x-1)\arcsin\frac{x}{x+1}$，求 $f'(1)$.

$$f'(1)=\lim_{x\to1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1}\frac{x+(x-1)\arcsin\frac{x}{x+1}-1}{x-1}$$

$$= 1+\lim_{x\to1}\arcsin\frac{x}{x+1}$$

$$= 1+\frac{\pi}{6}$$

补3 $f(x)=x^2+(x-1)\arctan\dfrac{2x-1}{x^3+x^2-1}$，求 $f'(1)$

解：
$$f'(1)=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$$
$$=\lim_{x\to 1}\frac{x^2+(x-1)\arctan\frac{2x-1}{x^3+x^2-1}-1}{x-1}$$
$$=\lim_{x\to 1}\left(x+1+\arctan\frac{2x-1}{x^3+x^2-1}\right)$$
$$=2+\arctan\frac{1}{1}=2+\frac{\pi}{4}$$

例7 $f(x)=\begin{cases}\dfrac{x}{1-e^{\frac{1}{x}}}, & x\neq 0\\ 0, & x=0\end{cases}$ 求 $f'(0)$

解
$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{1}{1-e^{\frac{1}{x}}}=\begin{cases}\lim\limits_{x\to 0^+}\dfrac{1}{1-e^{\frac{1}{x}}}=0\\ \lim\limits_{x\to 0^-}\dfrac{1}{1-e^{\frac{1}{x}}}=1\end{cases}$$

（$x\to 0$ 时 $e^{\frac{1}{x}}$ 要分 $x\to 0^+$，$x\to 0^-$ 来求）

故 $f'(0)$ 不存在

例8 $f(x)=\begin{cases}g(x)\cos\frac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$ $g(0)=g'(0)=0$，求 $f'(0)$

解：
$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{g(x)\cos\frac{1}{x}}{x}$$
$$=\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}\cdot\cos\frac{1}{x}$$

因 $g'(0)=0$，故 $\lim\limits_{x\to 0}\dfrac{g(x)-g(0)}{x-0}=g'(0)=0$，即 $\dfrac{g(x)-g(0)}{x-0}$ 在 $x\to 0$ 时是无穷小量，

而 $\cos\frac{1}{x}$ 有界，所以 $f'(0)=0$．

例 9. $f(x+y) = f(x)g(y) + f(y)g(x)$

$f'(0)=1,\ g'(0)=0,\ f(0)=0,\ g(0)=1.$

证 $\forall x\in R$, $f'(x) = g(x)$.

证:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)g(h)+f(h)g(x)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)[g(h)-1]+f(h)g(x)}{h}$$

$$= \lim_{h\to 0} f(x)\cdot\frac{g(h)-g(0)}{h} + \lim_{h\to 0} g(x)\cdot\frac{f(h)-f(0)}{h}$$

$$= f(x)g'(0) + g(x)f'(0)$$

$$= g(x)$$

例 10 (1) $y = \ln\frac{\sqrt{x^2+1}}{\sqrt[3]{x+2}}$

解 $y = \frac{1}{2}\ln(x^2+1) - \frac{1}{3}\ln(x+2)$

$$y' = \frac{1}{2}\cdot\frac{1}{1+x^2}\cdot 2x - \frac{1}{3}\cdot\frac{1}{x+2}$$

$$= \frac{x}{1+x^2} - \frac{1}{3(x+2)}$$

2) $y = \sqrt{x\sqrt{x\sqrt{x}}}$

解 $y = x^{\frac{7}{8}}$

$$y' = \frac{7}{8}x^{-\frac{1}{8}}$$

3) $y = x^{a^a} + a^{x^a} + a^{a^x}$

$$y' = \cancel{x}a^a x^{a^a-1} + a^{x^a}\ln a\cdot(x^a)' + a^{a^x}\ln a\,(a^x)'$$

4) $y = \left(\frac{a}{b}\right)^x\left(\frac{b}{x}\right)^a\left(\frac{x}{a}\right)^b$

$$\ln y = x\ln\frac{a}{b} + a(\ln b - \ln|x|) + b(\ln|x| + \ln a)$$

$$\frac{1}{y}y' = \ln\frac{a}{b} \cancel{\ } \frac{a}{x} + \frac{b}{x}$$

补4 $y = x^{x^a} + x^{a^x} + a^{x^x}$

$= y_1 + y_2 + y_3$

$\ln y_1 = x^a \ln x \Rightarrow \frac{1}{y_1}\cdot y_1' = (x^a)'\ln x + x^a\cdot(\ln x)'$

$= a x^{a-1}\ln x + x^{a-1}$

$y_1' = x^{x^a}\left(a x^{a-1}\ln x + x^{a-1}\right)$

$\ln y_2 = a^x \ln x \Rightarrow \frac{1}{y_2}\cdot y_2' = (a^x)'\ln x + a^x\cdot(\ln x)'$

$= a^x \ln a\cdot\ln x + \frac{a^x}{x}$

$y_3' = a^{x^x}\ln a\,(x^x)'$

$= a^{x^x}\ln a\,(e^{x\ln x})'$

$= a^{x^x}\ln a\cdot x^x\,(x\ln x)'$

$= a^{x^x}\ln a\cdot x^x\cdot(\ln x + 1)$

例11. $f(x) = \begin{cases} x^\alpha \sin\frac{1}{x}, & x>0 \\ 0, & x\le 0 \end{cases}$，求 $f'(x)$

解: $x>0$ 时. $f'(x) = \left(x^\alpha \sin\frac{1}{x}\right)' = \alpha x^{\alpha-1}\sin\frac{1}{x} + x^\alpha\cdot\cos\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)$

$= \alpha x^{\alpha-1}\sin\frac{1}{x} - x^{\alpha-2}\cos\frac{1}{x}$

$x<0$ 时. $f'(x) = 0$

$x=0$ 时. $f_+'(0) = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} x^{\alpha-1}\sin\frac{1}{x} = \begin{cases} 0 & \alpha>1 \\ 不存在 & \alpha \end{cases}$

$f_-'(0) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = 0$

例12 $f(x)=\begin{cases} e^{x^2} & x\le 1 \\ ax+b & x>1 \end{cases}$ $f'(1)$存在，求a,b.

解 $f(x)$在$x=1$处连续.

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(ax+b) = a+b$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} e^{x^2} = e \ \cancel{= f(1)}$$

$$\Rightarrow \cancel{a+b=e} \qquad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1) \Rightarrow a+b=e$$

由$f'(1)$存在，有

$$f'_+(1) = \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+}\frac{ax+b-e}{x-1} = \lim_{x\to 1^+}\frac{ax-a}{x-1} = a$$

$$f'_-(1) = \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-}\frac{e^{x^2}-e}{x-1} = \lim_{x\to 1^-}\frac{e(e^{x^2-1}-1)}{x-1}$$

$$= e\lim_{x\to 1^-}\frac{x^2-1}{x-1} = 2e$$

$$\Rightarrow f'_+(1) = f'_-(1) \Rightarrow a=2e \qquad \therefore b=-e$$

例13 $f'(x)$存在，$y=f(e^x)e^{f(x)}$，求$\frac{dy}{dx}$.

解 $$\frac{dy}{dx} = f'(e^x)e^x e^{f(x)} + f(e^x)e^{f(x)}f'(x)$$

例14. $f''(x)$存在，$y=\sin[f(x^2)]$，求$\frac{d^2y}{dx^2}$.

解 $$\frac{dy}{dx} = \cos[f(x^2)]\, f'(x^2)\cdot 2x$$

$$\frac{d^2y}{dx^2} = -\sin[f(x^2)]\,[f'(x^2)\cdot 2x]^2 + \cos[f(x^2)]\, f''(x^2)(2x)^2 + \cos[f(x^2)]\, f'(x^2)\cdot 2$$

例15 (1) $x^y=y^x$，求$\frac{dy}{dx}$

解 $y\ln x = x\ln y$

两边对x求导，$y'\ln x + y\cdot\frac{1}{x} = \ln y + x\cdot\frac{1}{y}y'$

$$\frac{dy}{dx} = \frac{xy\ln y - y^2}{xy\ln x - x^2}$$

(2) $\arctan\frac{y}{x} = \ln\sqrt{x^2+y^2}$

解 方程两边求微分

$$\frac{1}{1+\frac{y^2}{x^2}} d\left(\frac{y}{x}\right) = \frac{1}{2}\frac{1}{x^2+y^2} d(x^2+y^2)$$

$$\frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{x\,dy - y\,dx}{x^2} = \frac{1}{2}\frac{1}{x^2+y^2}(2x\,dx + 2y\,dy)$$

$$\frac{1}{x^2+y^2}(x\,dy - y\,dx) = \frac{1}{x^2+y^2}(x\,dx + y\,dy)$$

$$x\,dy - y\,dx = x\,dx + y\,dy$$

$$(x-y)dy = (x+y)dx$$

$$dy = \frac{x+y}{x-y}dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}.$$

例16 $ye^y = e^{x+1}$，求 $\left.\frac{dy}{dx}\right|_{y=1}$，$\left.\frac{d^2y}{dx^2}\right|_{y=1}$.

解: 方程两边对 x 求导

$$y'e^y + ye^y \cdot y' = e^{x+1} \qquad ①$$

①式两边继续对 x 求导，

$$y''e^y + y'e^y \cdot y' + e^y y' + y\,e^y(y')^2 + ye^y y'' = e^{x+1} \qquad ②$$

$y=1$ 时，$x=0$，代入①式有

$$\cancel{y'(1)} \quad \left.\frac{dy}{dx}\right|_{\substack{y=1\\x=0}} = \frac{1}{2}$$

代入②式，有 $\left.\frac{d^2y}{dx^2}\right|_{y=1} = \cancel{a}\,\frac{1}{8}$.

例17. $y^2e^x + x\sin y = 0$，求 dy.

解 $e^x d(y^2) + y^2 d(e^x) + \sin y\,dx + x\,d\sin y = 0$

$$e^x 2y\,dy + y^2e^x dx + \sin y\,dx + x\cos y\,dy = 0$$

$$dy = -\frac{y^2e^x + \sin y}{2e^x y + x\cos y}dx$$

例18. 已知 $\frac{dx}{dy}=\frac{1}{y'}$.

证明：(1) $\frac{d^2x}{dy^2}=-\frac{y''}{(y')^3}$

(2) $\frac{d^3x}{dy^3}=\frac{3(y'')^2-y'y'''}{(y')^5}$.

证：(1)
$$\frac{d^2x}{dy^2}=\frac{d}{dy}\left(\frac{dx}{dy}\right)=\frac{d}{dy}\left(\frac{1}{y'}\right)$$
$$=\frac{d}{dx}\left(\frac{1}{y'}\right)\cdot\frac{dx}{dy}$$
$$=-\frac{y''}{(y')^2}\cdot\frac{1}{y'}=-\frac{y''}{(y')^3}$$

(2)
$$\frac{d^3x}{dy^3}=\frac{d}{dy}\left(\frac{d^2x}{dy^2}\right)=-\frac{d}{dy}\left(\frac{y''}{(y')^3}\right)$$
$$=-\frac{d}{dx}\left(\frac{y''}{(y')^3}\right)\cdot\frac{dx}{dy}$$
$$=-\frac{y'''(y')^3-3y''(y')^2y''}{(y')^6}\cdot\frac{1}{y'}$$
$$=\frac{3(y'')^2-y'y'''}{(y')^5}$$

例19. $\begin{cases}x=3t^3+2\\ y=e^{3t}+2\end{cases}$，求 $\frac{d^2y}{dx^2}$.

解：
$$\frac{dy}{dx}=\frac{y'_t}{x'_t}=\frac{3e^{3t}}{9t^2}=\frac{e^{3t}}{3t^2}$$
$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{e^{3t}}{3t^2}\right)=\frac{d}{dt}\left(\frac{e^{3t}}{3t^2}\right)\Big/\frac{dx}{dt}$$
$$=\frac{3e^{3t}\cdot3t^2-e^{3t}\cdot6t}{9t^4}\cdot\frac{1}{9t^2}$$
$$=\frac{e^{3t}(3t-2)}{27t^5}$$

例20 $\begin{cases} x = \arctan t \\ 2y - ty^2 + e^t = 5 \end{cases}$，求 $\dfrac{dy}{dx}$.

解 方程组两边分别对t求导，有

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{1+t^2} \\ 2\dfrac{dy}{dt} - y^2 - 2ty\dfrac{dy}{dt} + e^t = 0 \end{cases}$$

$$\therefore \begin{cases} \dfrac{dx}{dt} = \dfrac{1}{1+t^2} \\ \dfrac{dy}{dt} = \dfrac{y^2 - e^t}{2(1-ty)} \end{cases}$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(y^2 - e^t)(1+t^2)}{2(1-ty)}.$$

补5 $\begin{cases} xe^t + t\cos x = \pi \\ y = \sin t + \cos^3 t \end{cases}$ 求 $\dfrac{dy}{dx}\Big|_{x=0}$.

解

$$\begin{cases} \dfrac{dx}{dt}e^t + xe^t + \cos x + t(-\sin x)\dfrac{dx}{dt} = 0 \\ \dfrac{dy}{dt} = \cos t + 3\cos^2(t)(-\sin t) \end{cases}$$

$x=0$时，$t=\pi$. 代入上式，得

$$\frac{dy}{dx}\Big|_{x=0} = \frac{-1}{-\frac{1}{e^\pi}} = e^\pi.$$