离散型随机变量函数的分布

已知二维随机变量(X,Y)的联合分布，$g(x,y)$为定义在平面上$\mathbb{R}^2$的实值函数，

求随机变量$Z=g(X,Y)$ 的概率分布.

已知二维离散型随机变量(X,Y)的联合分布律

$\Downarrow$

确定随机变量$Z=g(X,Y)$ 的取值范围

$\Downarrow$

求随机变量Z 取每个可能值的概率

离散型随机变量函数的分布

例1. 设X与Y是相互独立的离散型随机变量，分布律分别为

$$P(X=i),i=0,1,2,\ldots,\quad P(Y=j),j=0,1,2,\ldots$$

求 $Z=X+Y$的分布律.

解： $Z=X+Y$的可能取值为 $0,1,2,\ldots$

$$P(Z=k)=P(X+Y=k)=P\left(\sum_{i=0}^{k}\{X=i,Y=k-i\}\right)$$

$$=\sum_{i=0}^{k}P(X=i,Y=k-i)=\sum_{i=0}^{k}P(X=i)P(Y=k-i),$$

$$k=0,1,2,\ldots$$

$$P\{X+Y=k\}=\sum_{i=0}^{k} P\{X=i\}\, P\{Y=k-i\}$$

例2. 设 X 与 Y 是相互独立的随机变量，它们分别服从参数为λ_1, λ_2的泊松分布，则 $Z=X+Y$服从参数为$\lambda_1+\lambda_2$的泊松分布.

解：

$$X \sim P(\lambda_1) \Longrightarrow P(X=i)=\frac{{\lambda_1}^i}{i!}e^{-\lambda_1}, \qquad i=0,1,2,\dots,$$

$$Y \sim P(\lambda_2) \Longrightarrow P(Y=j)=\frac{{\lambda_2}^j}{j!}e^{-\lambda_2}, \qquad j=0,1,2,\dots,$$

$$\Longrightarrow P(Z=k)=\sum_{i=0}^{k} P(X=i)P(Y=k-i)=\sum_{i=0}^{k}\frac{{\lambda_1}^i}{i!}e^{-\lambda_1}\cdot\frac{{\lambda_2}^{k-i}}{(k-i)!}e^{-\lambda_2}$$

$$P\{X+Y=k\}=\sum_{i=0}^{k} P\{X=i\}\,P\{Y=k-i\}$$

$$P(Z=k)=\sum_{i=0}^{k}\frac{\lambda_1{}^{i}}{i!}e^{-\lambda_1}\cdot\frac{\lambda_2{}^{k-i}}{(k-i)!}e^{-\lambda_2}$$

$$=\frac{e^{-(\lambda_1+\lambda_2)}}{k!}\sum_{i=0}^{k}\frac{k!}{i!\,(k-i)!}\cdot\lambda_1{}^{i}\cdot\lambda_2{}^{k-i}$$

$$=\frac{(\lambda_1+\lambda_2)^k}{k!}\cdot e^{-(\lambda_1+\lambda_2)}\quad k=0,1,2,\dots,$$

$$\Longrightarrow Z\sim P(\lambda_1+\lambda_2)$$

$X\sim P(\lambda_1)$, $Y\sim P(\lambda_2)$且X与 Y相互独立 $\Longrightarrow$ $X+Y\sim P(\lambda_1+\lambda_2)$

$P\{X+Y=k\}=\sum_{i=0}^{k} P\{X=i\} P\{Y=k-i\}$

例3. 设 X 与 Y 是相互独立的随机变量，它们分别服从二项分布

$B(n_1,p)$和 $B(n_2,p)$, 求 $Z=X+Y$ 的概率分布.

解： $X\sim B(n_1,p) \Longrightarrow P(X=i)=C_{n_1}^{i}p^{i}(1-p)^{n_1-i}, i=0,1,2,\ldots,n_1$

$Y\sim B(n_2,p) \Longrightarrow P(Y=j)=C_{n_2}^{j}p^{j}(1-p)^{n_2-j}, j=0,1,2,\ldots,n_2$

$Z=X+Y$ 的可能取值为 $0,1,\ldots,n_1+n_2$

$$P(Z=k)=\sum_{\{(i,j):i+j=k,i,j\in\{0\}\cup\mathbb{Z}^{+}\}} P(X=i,Y=j)$$

$$C_{n_1+n_2}^{k} = \sum_{\{(i,j):i+j=k,i,j\in\{0\}\cup\mathbb{Z}^+\}} C_{n_1}^{i} C_{n_2}^{j}$$

$$P(Z=k) = \sum_{\{(i,j):i+j=k,i,j\in\{0\}\cup\mathbb{Z}^+\}} P(X=i)P(Y=j)$$

$$= \sum_{\{(i,j):i+j=k,i,j\in\{0\}\cup\mathbb{Z}^+\}} C_{n_1}^{i} p^{i}(1-p)^{n_1-i} C_{n_2}^{j} p^{j}(1-p)^{n_2-j}$$

$$= p^{k}(1-p)^{n_1+n_2-k} \sum_{\{(i,j):i+j=k,i,j\in\{0\}\cup\mathbb{Z}^+\}} C_{n_1}^{i} C_{n_2}^{j}$$

$$= C_{n_1+n_2}^{k} p^{k}(1-p)^{n_1+n_2-k} \Longrightarrow Z\sim B(n_1+n_2,p)$$

$X\sim B(n_1,p)$, $Y\sim B(n_2,p)$ 且X与 Y相互独立 $\Longrightarrow X+Y\sim B(n_1+n_2,p)$

小　　结

已知二维离散型随机变量(X,Y)的联合分布分布律

$\Downarrow$

确定随机变量$Z=g(X,Y)$ 的取值范围

$\Downarrow$

求随机变量Z 取每个可能值的概率

$X \sim P(\lambda_1), Y \sim P(\lambda_2)$且$X$与 Y相互独立 $\Longrightarrow$ $X+Y \sim P(\lambda_1+\lambda_2)$

$X \sim B(n_1,p), Y \sim B(n_2,p)$ 且X与 Y相互独立 $\Longrightarrow$ $X+Y \sim B(n_1+n_2,p)$

连续型随机变量函数的分布

分布函数微分法

$$Z = aX + bY$$

$$Z = \frac{X}{Y}$$

$$M = \max\{X, Y\}$$
$$N = \min\{X, Y\}$$

积分转化法

连续型随机变量和的分布

已知二维连续型随机变量(X,Y)的联合概率密度，求 $Z=X+Y$的概率密度

方法一：　分布函数微分法

推　广：　求$Z=aX+bY$的概率密度, 其中$a\neq 0, b\neq 0$

$Z=X+Y$ 的分布

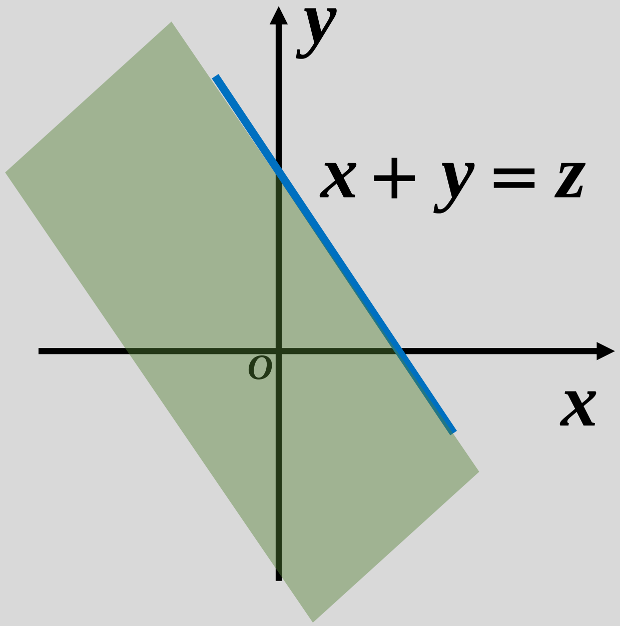
例1 设二维随机变量(X, Y)的联合概率密度为 $f(x, y)$, 求 $Z = X + Y$ 的概率密度$f_Z(Z)$.

解： 先求 $Z = X + Y$ 的分布函数

$$F_Z(z) = P(Z \le z) = P(X + Y \le z)$$

$$= \iint\limits_{x+y\le z} f(x,y)dxdy$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z-y} f(x,y)dx\right)dy$$



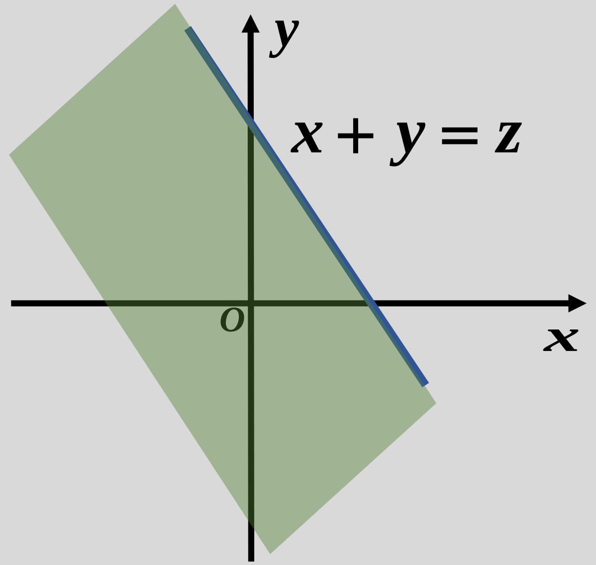
$$F(z)=\int_{-\infty}^{z} f(u)du$$

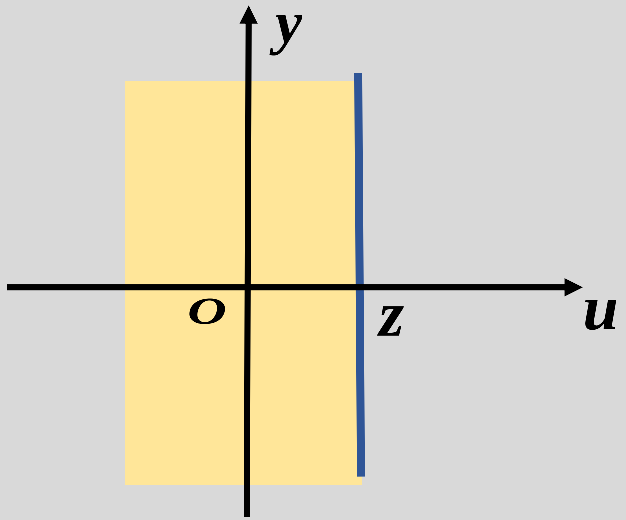
$$F_Z(z)=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{z-y} f(x,y)dx\right)dy$$

$$\xlongequal{x=u-y}\int_{-\infty}^{\infty}\boxed{\left(\int_{-\infty}^{z} f(u-y,y)du\right)}dy$$

$$=\int_{-\infty}^{z}\boxed{\left(\int_{-\infty}^{\infty} f(u-y,y)dy\right)}du$$

$$\Longrightarrow f_Z(z)=\int_{-\infty}^{\infty} f(z-y,y)dy$$





$Z=X+Y$ 的分布

$$f_Z(z)=\int_{-\infty}^{\infty}f(z-y,y)dy$$

由于X 与Y 对称

$$f_Z(z)=\int_{-\infty}^{\infty}f(x,z-x)dx$$

当X, Y 独立时

$$f_Z(z)=\int_{-\infty}^{\infty}f_X(z-y)f_Y(y)dy$$

$$f_Z(z)=\int_{-\infty}^{\infty}f_X(x)f_Y(z-x)dx$$

$$f_Z(z)=\int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx$$

例2 设两个独立的随机变量X 与Y 都服从标准正态分布，求$Z = X + Y$ 的概率密度.

解:

$$f_X(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \qquad f_Y(y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$$

$$f_Z(z)=\int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx=\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{-\frac{x^2}{2}}e^{-\frac{(z-x)^2}{2}}dx$$

$$=\frac{1}{2\pi}e^{-\frac{z^2}{4}}\int_{-\infty}^{\infty}e^{-\left(x-\frac{z}{2}\right)^2}dx \xlongequal{t=x-\frac{z}{2}} \frac{1}{2\pi}e^{-\frac{z^2}{4}}\int_{-\infty}^{\infty}e^{-t^2}dt$$

$$=\frac{1}{2\sqrt{\pi}}e^{-\frac{z^2}{4}} \Longrightarrow Z\sim N(0,2)$$

两个相互独立的正态随机变量之和仍然服从正态分布

$$\left.\begin{array}{l} X \sim N(\mu_1, \sigma_1^2) \\ Y \sim N(\mu_2, \sigma_2^2) \\ X\text{ 与 }Y\text{ 独立} \end{array}\right] \Longrightarrow X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

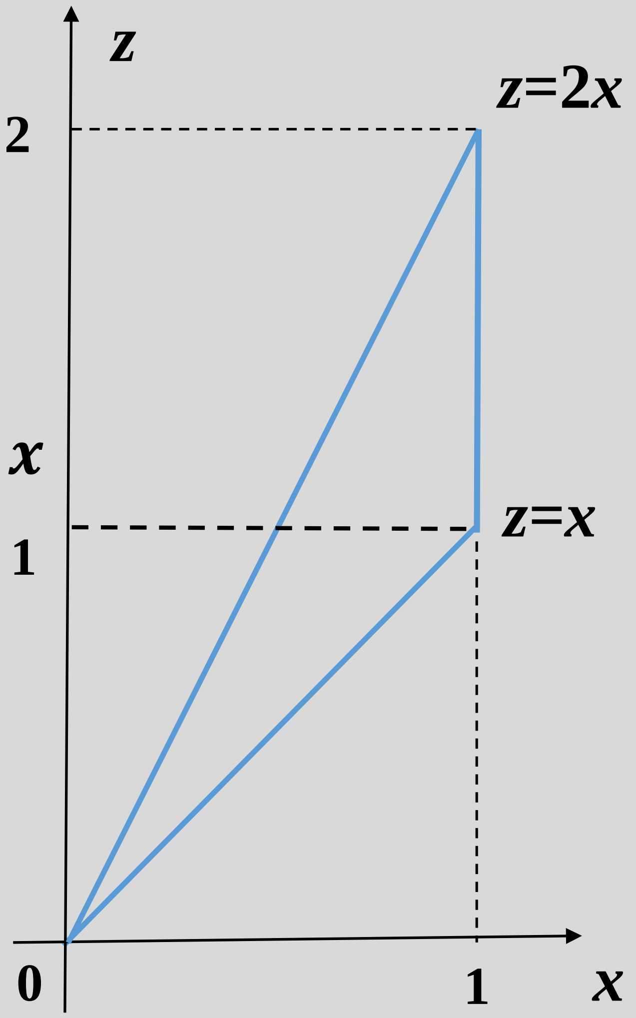
$$f_Z(z)=\int_{-\infty}^{\infty}f(x,z-x)dx$$

例3. 设 $(X,Y)\sim f(x,y)=\begin{cases}24y(1-x), & 0\le x\le 1, 0\le y\le x\\ 0, & otherwise\end{cases}$

求 $Z=X+Y$ 的概率密度函数。

解：$f(x,z-x)=\begin{cases}24(z-x)(1-x), & 0\le x\le 1, 0\le z-x\le x\\ 0, & otherwise\end{cases}$

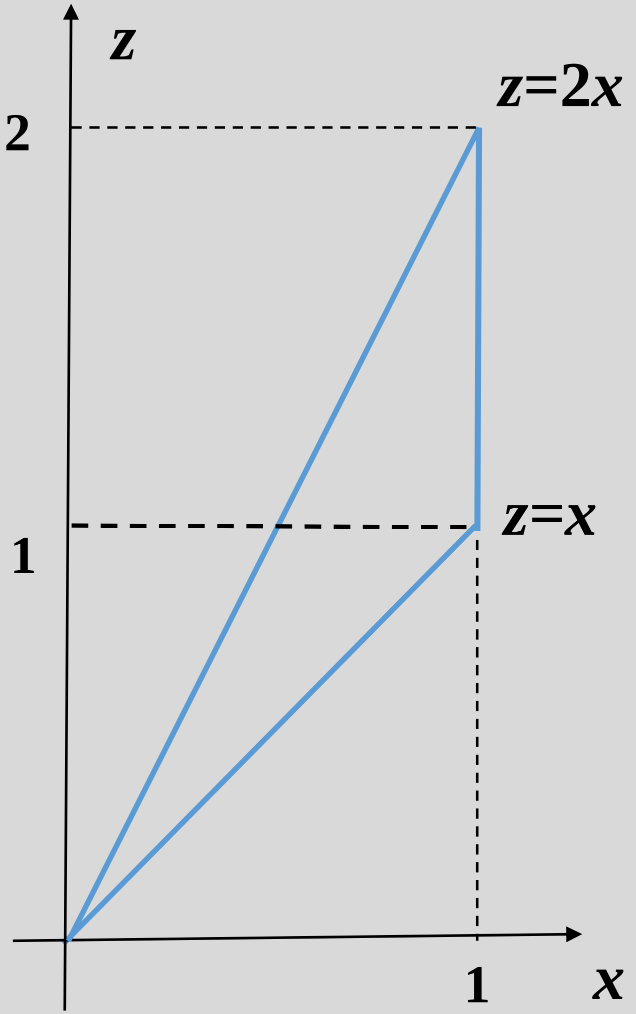
$$D_{xz}:\begin{cases}0\le x\le 1\\ 0\le z-x\le x\end{cases}\Longleftrightarrow\begin{cases}0\le z\le 1\\ \frac{z}{2}\le x\le z\end{cases}\cup\begin{cases}1\le z\le 2\\ \frac{z}{2}\le x\le 1\end{cases}$$



$$f_Z(z)=\int_{-\infty}^{\infty} f(x,z-x)dx$$

$$f_Z(z)=\begin{cases}\int_{\frac{z}{2}}^{z} 24(z-x)(1-x)dx\,, & 0\le z\le 1\\ \int_{\frac{z}{2}}^{1} 24(z-x)(1-x)dx\,, & 1\le z\le 2\\ 0, & \text{其它}\end{cases}$$

$$=\begin{cases}3z^2-2z^3, & 0\le z\le 1\\ -4+12z-9z^2+2z^3, & 1\le z\le 2\\ 0, & otherwise\end{cases}$$



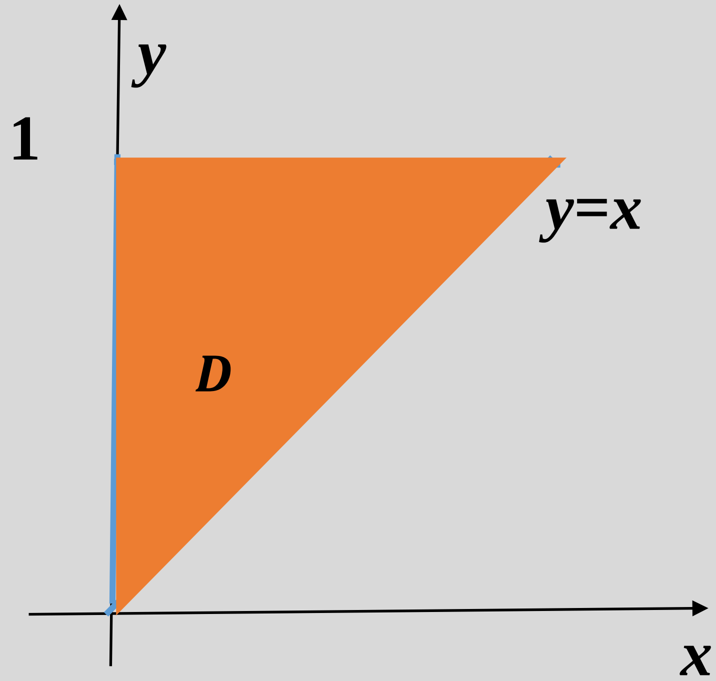
$Z = aX + bY$ 的分布

例4. 设二维随机变量 (X,Y) 在区域 $D=\{(x,y)|0 \le x \le y, 0 \le y \le 1\}$

内服从均匀分布，求 $Z=6X+2Y$ 的概率密度函数。

解： $D=\{(x,y)|0 \le x \le y, 0 \le y \le 1\}$

$$f(x,y) = \begin{cases} 2, & (x,y) \in D \\ 0, & (x,y) \notin D \end{cases}$$



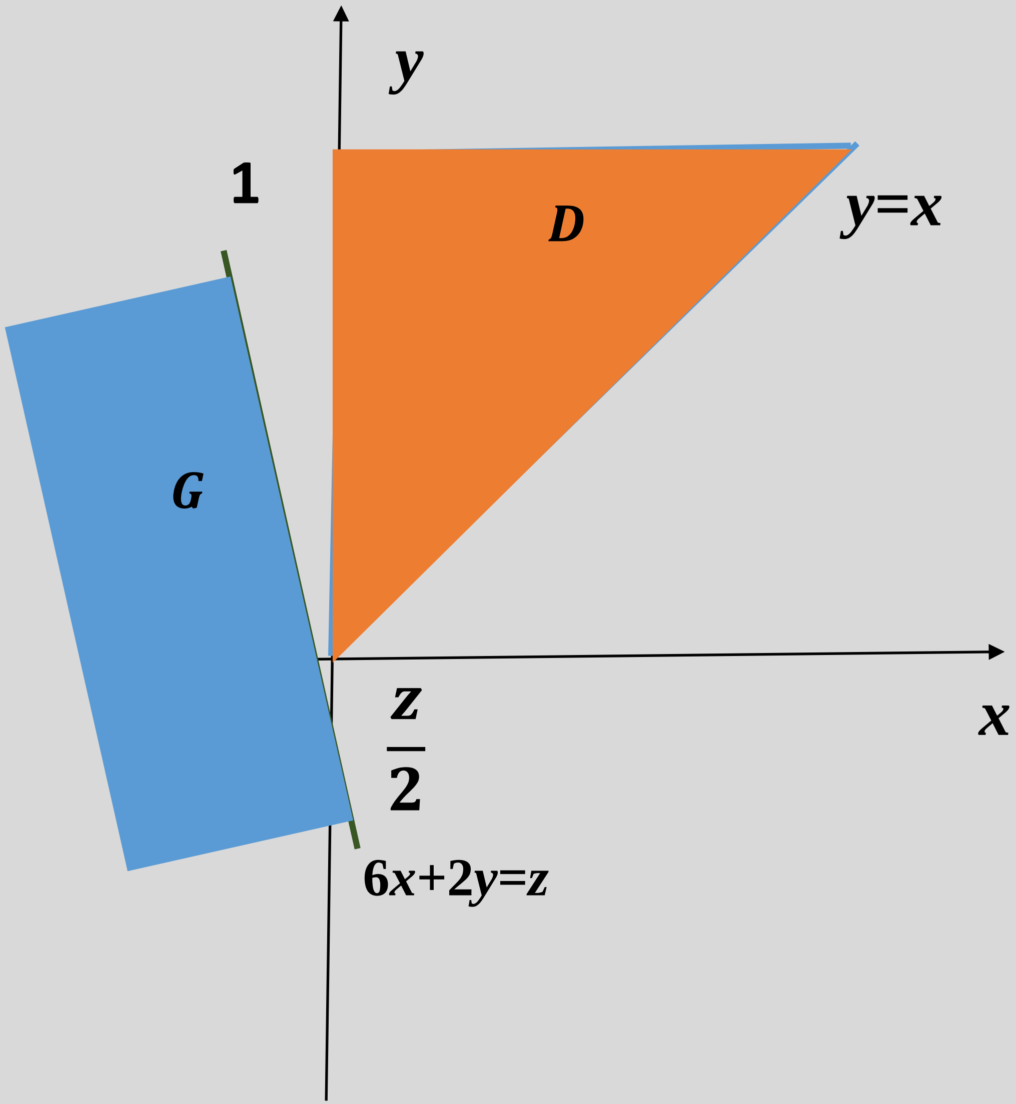
$$f(x,y)=\begin{cases}2, & (x,y)\in D\\ 0, & (x,y)\notin D\end{cases}$$

$D=\{(x,y)|0\le x\le y, 0\le y\le 1\}$

$$F_Z(z)=P(Z\le z)=P(6X+2Y\le z)$$

$$=\iint_{\{6X+2Y\le z\}} f(x,y)dxdy$$

1. $\frac{z}{2}<0, \qquad F_Z(z)=0$

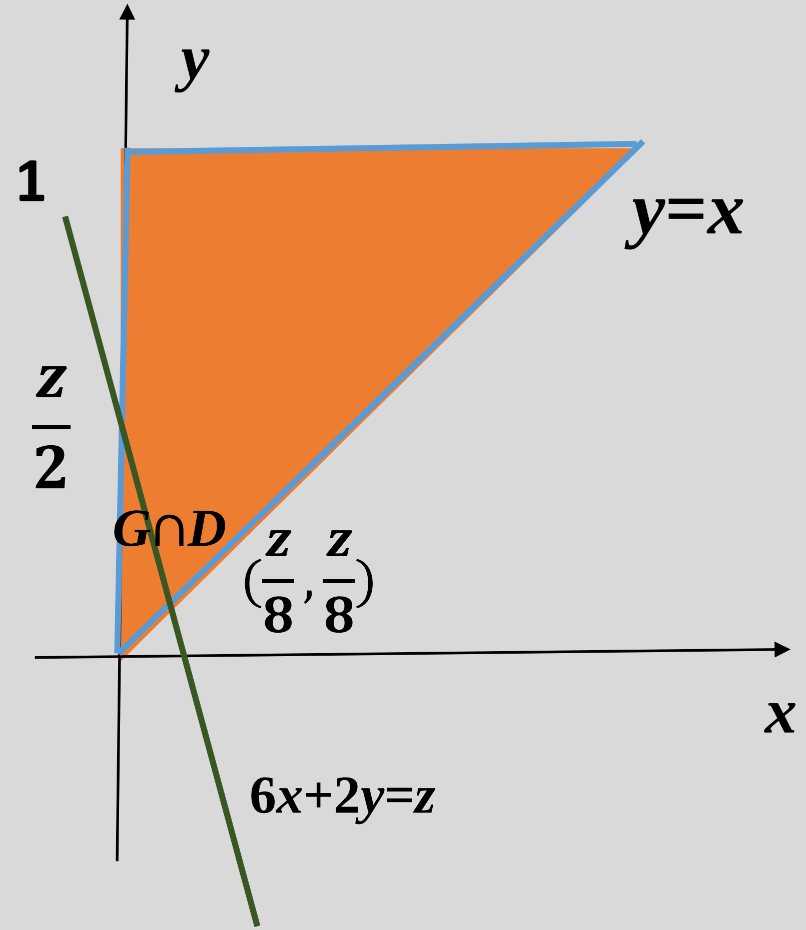


$$f(x,y)=\begin{cases}2, & (x,y)\in D\\ 0, & (x,y)\notin D\end{cases}$$

2. $0\le \frac{z}{2}<1,$

$$F_Z(z)=\iint\limits_{\{6X+2Y\le z\}} f(x,y)\,dxdy$$

$$=\int_0^{\frac{z}{8}}\left(\int_x^{\frac{-6x+z}{2}}2dy\right)dx=\frac{z^2}{16}$$

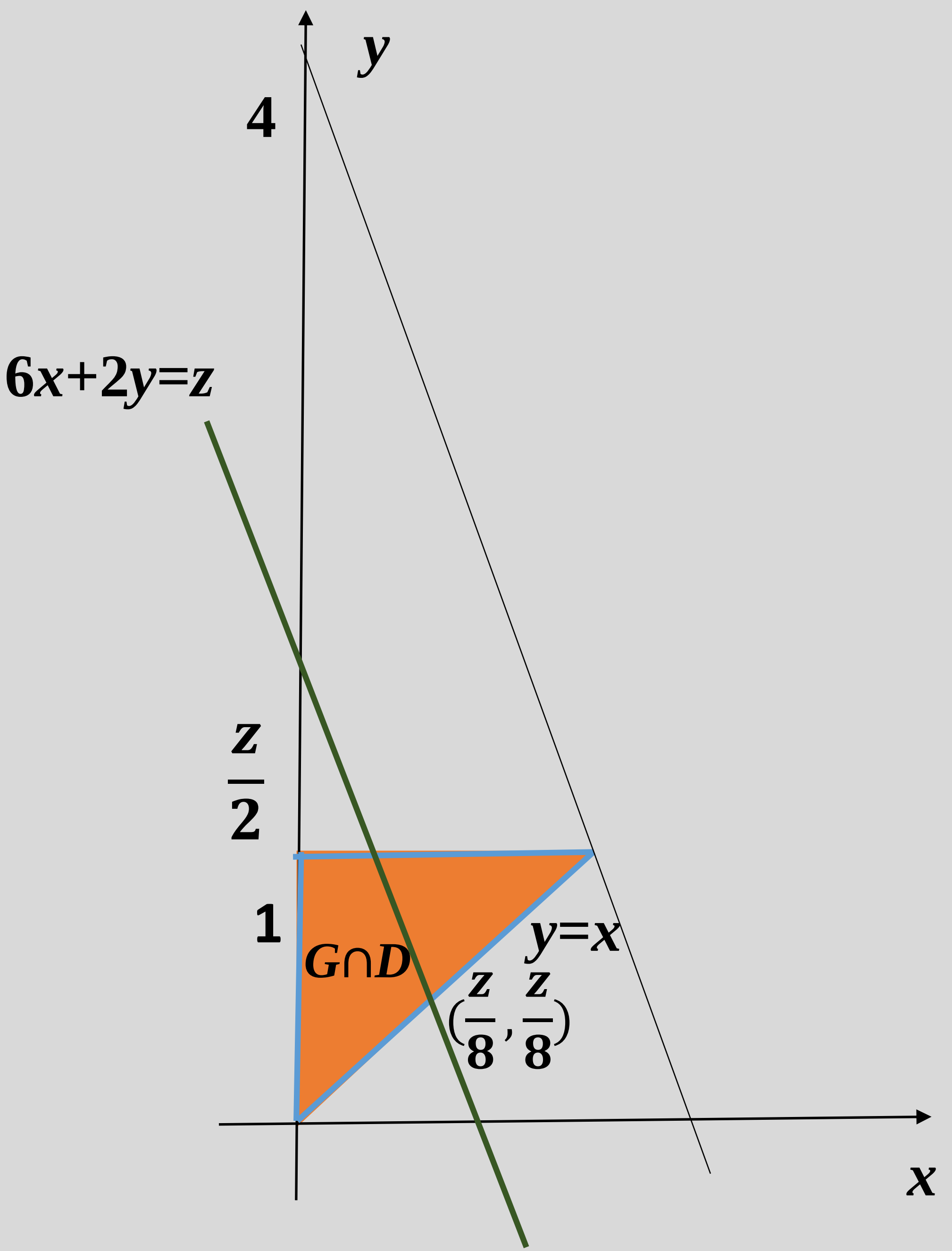


$$3.\ 1 \le \frac{z}{2} < 4, \qquad f(x,y) = \begin{cases} 2, & (x,y) \in D \\ 0, & (x,y) \notin D \end{cases}$$

$$F_Z(z) = \iint\limits_{G \cap D} 2dxdy$$

$$= 2\left(\frac{1}{2} - \int_{\frac{z}{8}}^{1}\left(\int_{\frac{-2y+z}{6}}^{y} dx\right)dy\right) = -\frac{z^2}{48} + \frac{z}{3} - \frac{1}{3}$$

$$4.\ \frac{z}{2} \ge 4, \qquad F_Z(z) = 1$$



$F'(z) = f(z)$

$$F_Z(z) = \begin{cases} 0, & z < 0 \\ \dfrac{z^2}{16}, & 0 \le z < 2 \\ -\dfrac{z^2}{48} + \dfrac{z}{3} - \dfrac{1}{3}, & 2 \le z < 8 \\ 1, & z \ge 8 \end{cases}$$

$$f_Z(z) = \begin{cases} \dfrac{z}{8}, & 0 \le z < 2 \\ -\dfrac{z}{24} + \dfrac{1}{3}, & 2 \le z < 8 \\ 0, & \text{其它} \end{cases}$$

小　　结

已知二维连续型随机变量(X,Y)的联合概率密度$f(x,y)$，$Z=X+Y$的概率密度为

$$f_Z(z)=\int_{-\infty}^{\infty}f(z-y,y)dy \quad \text{或} \quad f_Z(z)=\int_{-\infty}^{\infty}f(x,z-x)dx$$

当X,Y 独立时

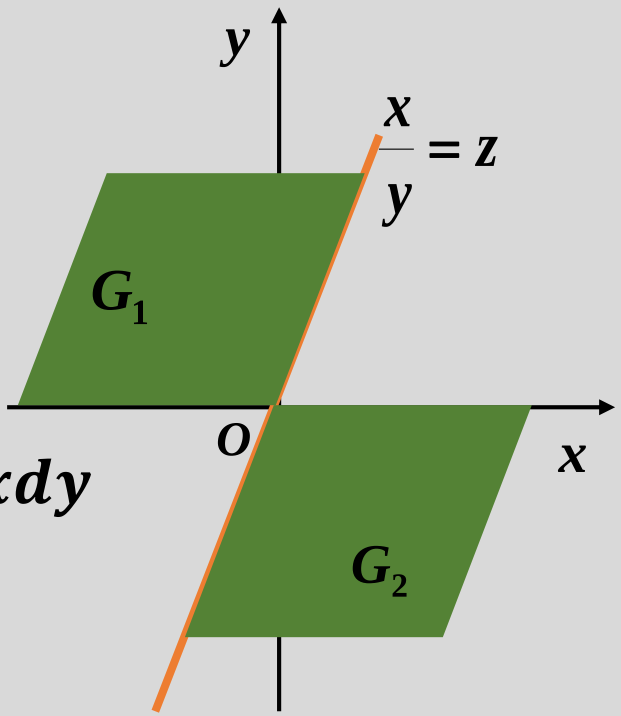
$$f_Z(z)=\int_{-\infty}^{\infty}f_X(z-y)f_Y(y)dy \quad \text{或} \quad f_Z(z)=\int_{-\infty}^{\infty}f_X(x)f_Y(z-x)dx$$

推　广：　求$Z=aX+bY$的概率密度, 其中$a\neq 0, b\neq 0$

$Z = \dfrac{X}{Y}$ 的分布

例1 设二维随机变量(X, Y)的联合概率密度为$f(x, y)$, 试证明 $Z = \dfrac{X}{Y}$ 是连续型随机变量，具有概率密度

$$f_Z(z) = \int_{-\infty}^{\infty} |y| f(zy, y) dy$$



解： $F_Z(z) = P(Z \le z) = P\left(\dfrac{X}{Y} \le z\right) = \displaystyle\iint_{\left\{\frac{x}{y} \le z\right\}} f(x, y) dxdy$

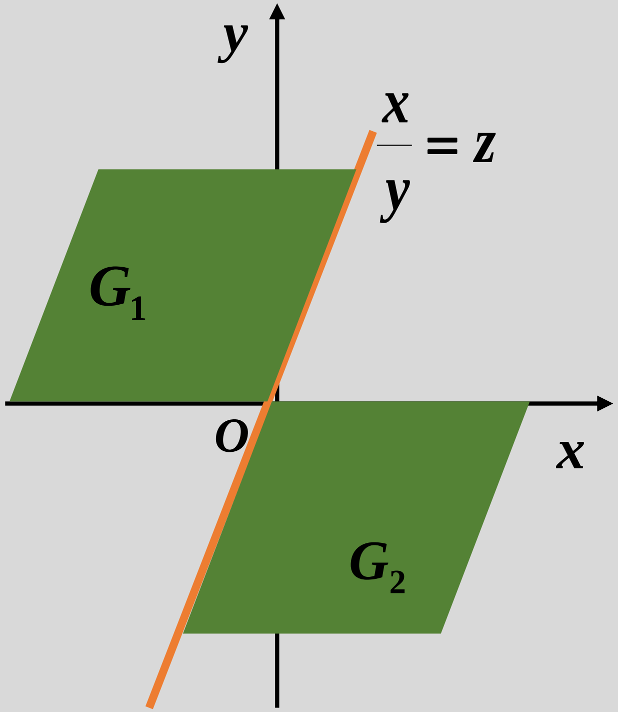
$$= \iint_{G_1} f(x, y) dxdy + \iint_{G_2} f(x, y) dxdy$$

$$F(z)=\int_{-\infty}^{z} f(u)du$$

$$\iint\limits_{G_1} f(x,y)dxdy=\int_0^{\infty}\left(\int_{-\infty}^{yz} f(x,y)dx\right)dy$$

$$\underline{\underline{u=\frac{x}{y}}}\quad \int_0^{\infty}\left(\int_{-\infty}^{z} yf(yu,y)du\right)dy$$

$$=\int_{-\infty}^{z}\left(\int_0^{\infty} yf(yu,y)dy\right)du$$



同理可得 $\iint\limits_{G_2} f(x,y)dxdy=\int_{-\infty}^{z}\left(\int_{-\infty}^{0} -yf(yu,y)dy\right)du$

$Z=\frac{X}{Y}$ 的分布

$$F_Z(z)=\iint_{G_1} f(x,y)dxdy+\iint_{G_2} f(x,y)dxdy$$

$$=\int_{-\infty}^{z}\left(\int_0^{\infty} yf(yu,y)dy\right)du+\int_{-\infty}^{z}\left(\int_{-\infty}^{0} -yf(yu,y)dy\right)du$$

$$=\int_{-\infty}^{z}\left(\int_0^{\infty} |y|f(yu,y)dy+\int_{-\infty}^{0} |y|f(yu,y)dy\right)du$$

$$=\int_{-\infty}^{z}\left(\int_{-\infty}^{\infty} |y|f(yu,y)dy\right)du \Longrightarrow f_Z(z)=\int_{-\infty}^{\infty} |y|f(yz,y)dy$$

M=max{X, Y}, N=min{X, Y}，X与Y独立

例2. 设X和Y是两个相互独立的随机变量，其分布函数分别为 $F_X(x), F_Y(y)$,

令 M=max{X, Y}, N=min{X, Y}, 证明

1. M的分布函数 $F_{\max}(z)=F_X(z)F_Y(z)$

2. N的分布函数 $F_{\min}(z)=1-[1-F_X(z)][1-F_Y(z)]$

证明： $F_{\max}(z)=P(M\le z)=P(\max\{X,Y\}\le z)=P(X\le z, Y\le z)$

$$=P(X\le z)P(Y\le z)=F_X(z)F_Y(z)$$

M=max{X, Y}, N=min{X, Y}, X与Y独立

$$F_{\min}(z) = P(N \le z) = P(\min\{X, Y\} \le z)$$

$$= 1 - P(\min\{X, Y\} > z)$$

$$= 1 - P(X > z, Y > z)$$

$$= 1 - P(X > z)P(Y > z)$$

$$= 1 - [1 - F_X(z)][1 - F_Y(z)]$$

M=max$\{X_1,\dots,X_n\}$, N=min$\{X_1,\dots,X_n\}$的分布

设 $X_1,\dots,X_n$ 是相互独立的随机变量，X_i 的分布函数为 $F_{X_i}(x), i=1,\dots,n.$

令 M=max$\{X_1,\dots,X_n\}$, N=min$\{X_1,\dots,X_n\}$, 则

1. M的分布函数　　$F_{\max}(z)=F_{X_1}(z)\dots F_{X_n}(z)$

2. N的分布函数　　$F_{\min}(z)=1-\left[1-F_{X_1}(z)\right]\dots\left[1-F_{X_n}(z)\right]$

若 $X_1,\dots,X_n$ 独立同分布，不妨设分布函数为$F(x)$, 则

$$F_{\max}(z)=[F(z)]^n \qquad F_{\min}(z)=1-[1-F(z)]^n$$

随机变量函数的分布

例3. 设系统L由两个相互独立的子系统L_1, L_2联接而成，连接的方式分别为(1)串联 (2)并联 (3)备用(当系统L_1损坏时，系统L_2开始工作)，设L_1, L_2的寿命分别为X, Y, 已知它们的概率密度分别为

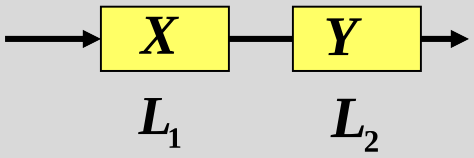
$$f_X(x) = \begin{cases} \alpha e^{-\alpha x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \quad f_Y(y) = \begin{cases} \beta e^{-\beta y}, & y > 0 \\ 0, & y \leq 0 \end{cases}$$

其中$\alpha > 0, \beta > 0$且$\alpha \neq \beta$.

试分别就以上三种联接方式求出L的寿命Z的概率密度.

$$F_{\min}(z) = 1 - [1 - F_X(z)][1 - F_Y(z)]$$

解: (1) 串联　系统L 的寿命 $Z=\min(X, Y)$



$$f_X(x) = \begin{cases} \alpha e^{-\alpha x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$$F_X(x) = \begin{cases} \int_0^x \alpha e^{-\alpha t} dt, & x > 0 \\ 0, & x \leq 0 \end{cases} = \begin{cases} 1 - e^{-\alpha x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

同理可得　$F_Y(y) = \begin{cases} 1 - e^{-\beta y}, & y > 0 \\ 0, & y \leq 0 \end{cases}$

$$F_{\min}(z) = 1 - [1 - F_X(z)][1 - F_Y(z)]$$

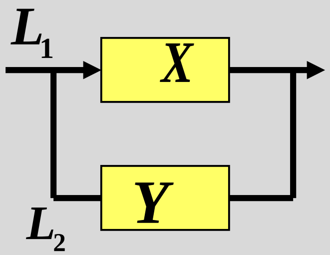
$$F_X(x) = \begin{cases} 1 - e^{-\alpha x}, & x > 0 \\ 0, & x \leq 0 \end{cases} \qquad F_Y(y) = \begin{cases} 1 - e^{-\beta y}, & y > 0 \\ 0, & y \leq 0 \end{cases}$$

$$F_{\min}(z) = 1 - [1 - F_X(z)][1 - F_Y(z)] = \begin{cases} 1 - e^{-(\alpha+\beta)z}, & z > 0 \\ 0, & z \leq 0 \end{cases}$$

$$\Longrightarrow \quad f_{\min}(z) = \begin{cases} (\alpha + \beta)e^{-(\alpha+\beta)z}, & z > 0 \\ 0, & z \leq 0 \end{cases}$$

$F_{\max}(z) = F_X(z)F_Y(z)$

(2) 并联 L 的寿命 $Z=\max(X, Y)$



$$F_X(x) = \begin{cases} 1 - e^{-\alpha x}, & x > 0 \\ 0, & x \le 0 \end{cases} \qquad F_Y(y) = \begin{cases} 1 - e^{-\beta y}, & y > 0 \\ 0, & y \le 0 \end{cases}$$

$$\Longrightarrow F_{\max}(z) = \begin{cases} (1 - e^{-\alpha z})(1 - e^{-\beta z}), & z > 0 \\ 0, & z \le 0 \end{cases}$$

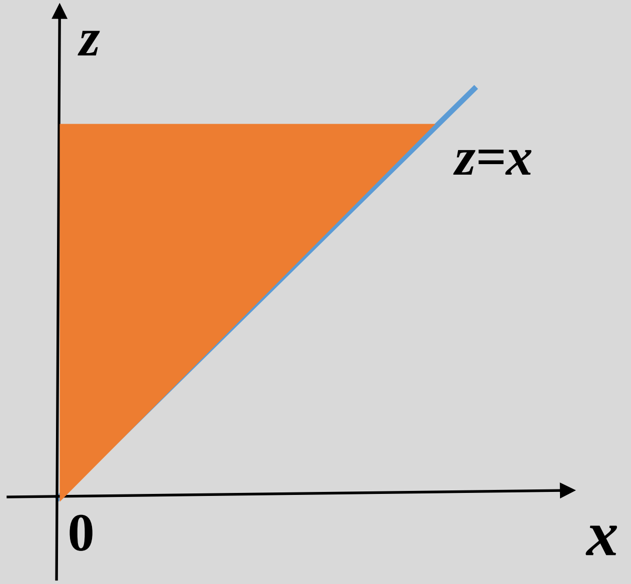
$$\Longrightarrow f_{\max}(z) = \begin{cases} \alpha e^{-\alpha z} + \beta e^{-\beta z} - (\alpha + \beta) e^{-(\alpha+\beta) z}, & z > 0 \\ 0, & z \le 0 \end{cases}$$

$$f_Z(z)=\int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx$$

(3) 备用(当系统L_1损坏时，系统L_2开始工作)

$$D_{xz}:\begin{cases}x\ge 0\\ z>x\end{cases} \Longleftrightarrow \begin{cases}z>0\\ 0\le x<z\end{cases}$$

L 的寿命 $Z=X+Y$



$$f_X(x)=\begin{cases}\alpha e^{-\alpha x}, & x>0\\ 0, & x\le 0\end{cases}$$

$$f_Y(z-x)=\begin{cases}\beta e^{-\beta(z-x)}, & z>x\\ 0, & z\le x\end{cases}$$

$$f_Z(z)=\begin{cases}\int_0^z \alpha e^{-\alpha x}\beta e^{-\beta(z-x)}\,dx, & z>0\\ 0, & z\le 0\end{cases}=\begin{cases}\dfrac{\alpha\beta}{\beta-\alpha}[e^{-\alpha z}-\beta e^{-\beta z}], & z>0\\ 0, & z\le 0\end{cases}$$

积 分 转 化 法

设随机变量(X, Y)的联合概率密度为 $f(x, y)$, $g(x,y)$ 是(分块连续的) 实值函数,

$Z = g(X,Y)$. 如果对任何有界连续函数$h(z)$, 成立

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h[g(x,y)]f(x,y)dxdy = \int_{\alpha}^{\beta} h(z)p(z)dz, \quad -\infty \le \alpha < \beta \le \infty,$$

则 $Z = g(X,Y)$ 的概率密度为 $f_Z(z) = \begin{cases} p(z), & \alpha < z < \beta \\ 0, & otherwise \end{cases}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h[g(x,y)]f(x,y)dxdy = \int_{\alpha}^{\beta} h(z)p(z)dz$$

例4. 设随机变量 $(X,Y)\sim f(x,y) = \begin{cases} 3x, & 0 \le x \le 1, 0 \le y \le x \\ 0, & otherwise \end{cases}$

求 $Z = X - Y$ 的概率密度函数。

解：用积分转化法 此时 $g(x,y) = x - y$,对任何有界连续函数$h(z)$,

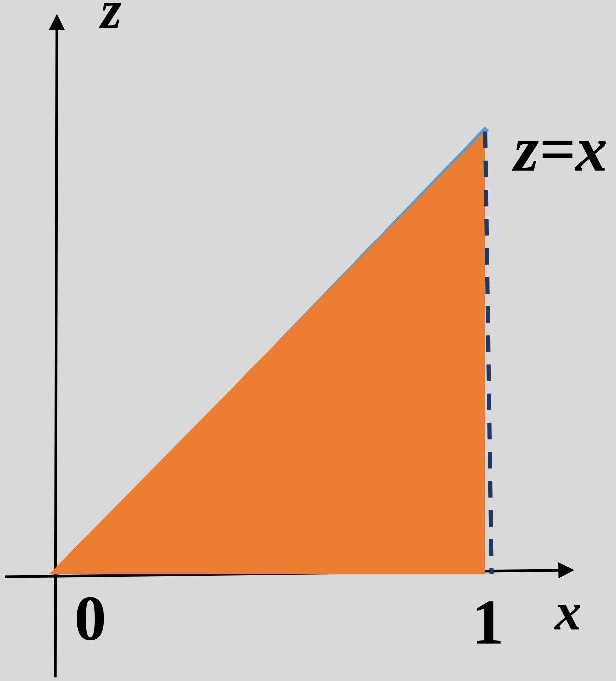
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h[g(x,y)]f(x,y)dxdy = \int_{0}^{1}\left(\int_{0}^{x} h(x-y)3x\,dy\right)dx$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h[g(x,y)]f(x,y)dxdy = \int_{\alpha}^{\beta} h(z)p(z)dz$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h[g(x,y)]f(x,y)dxdy = \int_{0}^{1}\left(\int_{0}^{x} h(x-\ y)3x\,dy\right)dx$$

$z = x - \ y$

$dy = -dz$



$$= \int_{0}^{1}\left(\int_{x}^{0} -h(z)3x\,dz\right)dx$$

$$= \int_{0}^{1}\left(\int_{0}^{x} h(z)3x\,dz\right)dx$$

交换积分次序

$$= \int_{0}^{1}\left(\int_{z}^{1} h(z)3x\,dx\right)dz$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h[g(x,y)]f(x,y)dxdy = \int_{\alpha}^{\beta} h(z)p(z)dz$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h[g(x,y)]f(x,y)dxdy = \int_{0}^{1}(\int_{z}^{1} h(z)3x\,dx)dz$$

$$= \int_{0}^{1} h(z)\left(\int_{z}^{1} 3x\,dx\right)dz = \int_{\underset{\alpha}{0}}^{\overset{\beta}{1}} h(z)\underbrace{\boxed{\left(\frac{3}{2}(1-z^2)\right)}}_{p(z)}dz$$

故 $Z = X - Y$ 的密度函数为 $f_Z(z) = \begin{cases} \frac{3}{2}(1-z^2), & 0 \le z \le 1 \\ 0, & otherwise \end{cases}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h[g(x,y)]f(x,y)dxdy = \int_{\alpha}^{\beta} h(z)p(z)dz$$

例5. 设X和Y是相互独立且服从同一正态分布$N(0,\sigma^2)$的随机变量，

求 $Z=\sqrt{X^2+Y^2}$ 的概率密度函数。

解：用积分转化法　此时　$g(x,y)=\sqrt{x^2+y^2}, f(x,y)=\dfrac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}$

对任何有界连续函数$h(z)$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h[g(x,y)]f(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\left(\sqrt{x^2+y^2}\right)\frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}dx\,dy$$

$$x=\rho\cos\theta \quad y=\rho\sin\theta \quad dxdy=\rho d\rho d\theta \quad 0\le\rho<\infty, 0\le\theta<2\pi$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h[g(x,y)]f(x,y)dxdy = \int_{\alpha}^{\beta} h(z)p(z)dz$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h[g(x,y)]f(x,y)dxdy = \int_0^{\infty}\left(\int_0^{2\pi} h(\rho)\frac{1}{2\pi\sigma^2}e^{-\frac{\rho^2}{2\sigma^2}}\rho \mathrm{d}\theta\right)d\rho$$

$$= \int_0^{\infty} h(\rho)\frac{\rho}{\sigma^2}e^{-\frac{\rho^2}{2\sigma^2}}d\rho$$

$$\Longrightarrow \quad f_Z(z) = \begin{cases} \dfrac{z}{\sigma^2}e^{-\frac{z^2}{2\sigma^2}}, & z \geq 0 \\ 0, & otherwise \end{cases}$$

$\Longrightarrow$ Z服从参数为σ的瑞利分布

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$

练习： 设X和Y相互独立， 具有共同的概率密度

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & else \end{cases}$$

求 $Z=X+Y$ 的密度函数与分布函数。

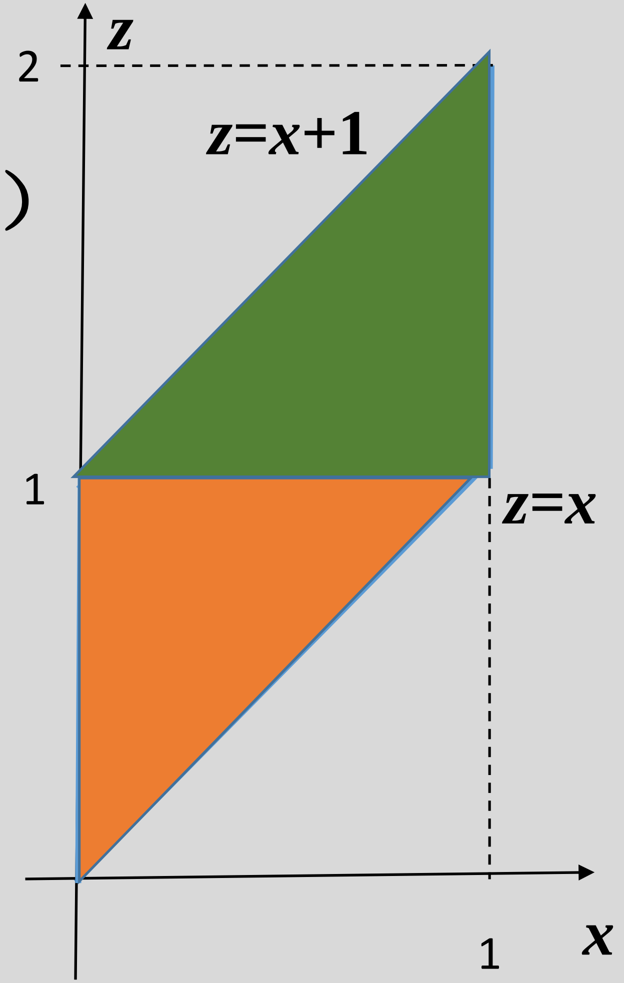
解：法一（先求密度再求分布）

$$D_{xy}: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1 \end{cases} \xrightarrow{x+y=z} D_{xz}: \begin{cases} 0 \le x \le 1 \\ 0 \le z-x \le 1 \end{cases}$$

$$f_Z(z)=\int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx$$

$$D_{xz}:\begin{cases}0\le x\le 1\\ x\le z\le x+1\end{cases}\quad（x\text{-型}）\qquad f_X(x)f_Y(z-x)=\begin{cases}1, & (x,y)\in D_{xz}\\ 0, & otherwise\end{cases}$$

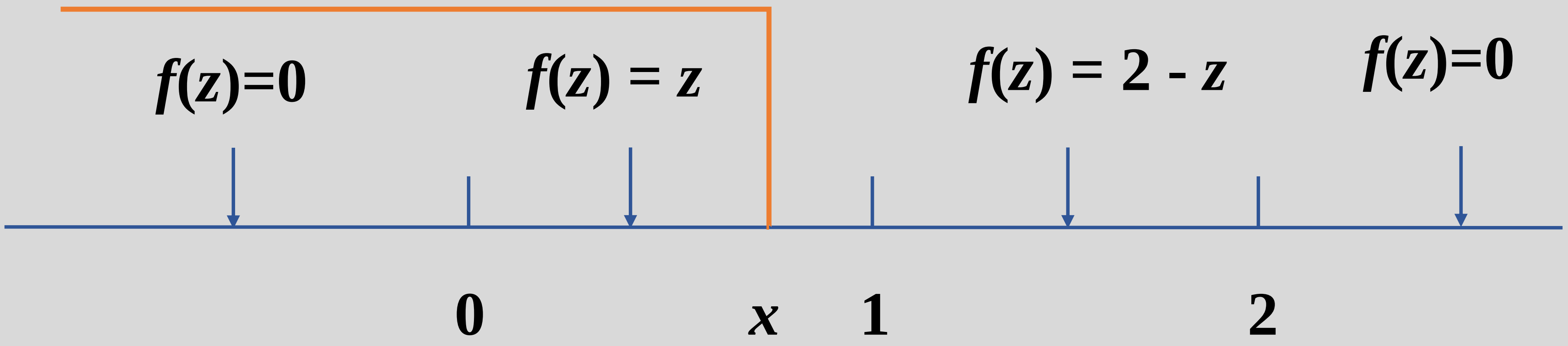
$$\Longleftrightarrow\quad D_{xz}:\begin{cases}0\le z\le 1\\ 0\le x\le z\end{cases}\cup\begin{cases}1\le z\le 2\\ z-1\le x\le 1\end{cases}\quad（z\text{-型}）$$



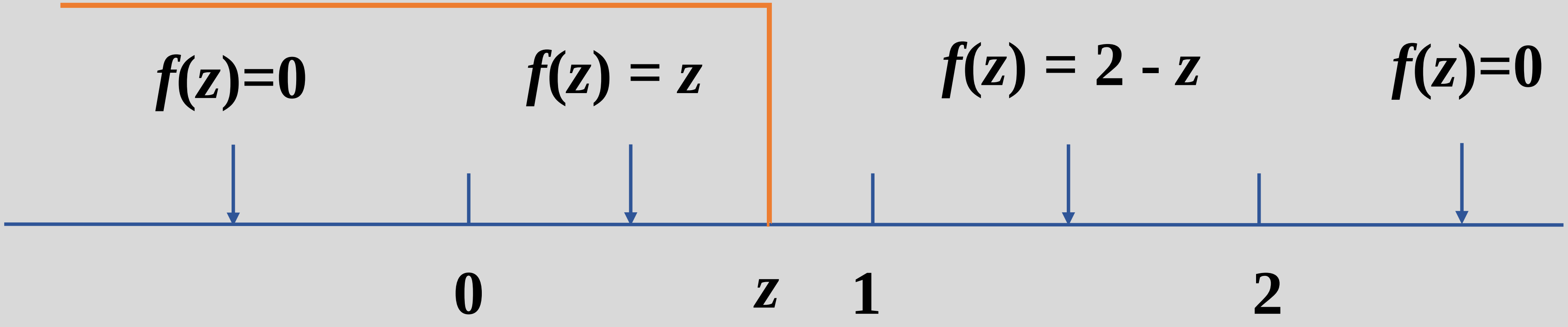
故 $$f_Z(z)=\begin{cases}\int_0^z 1\,dx, & 0\le z\le 1\\ \int_{z-1}^1 1\,dx, & 1\le z\le 2\\ 0, & else\end{cases}$$

$$F_Z(z) = \int_{-\infty}^{z} f_Z(t)dt$$

即 $$f_Z(z) = \begin{cases} z, & 0 \le z \le 1 \\ 2 - z, & 1 \le z \le 2 \\ 0, & else \end{cases}$$



$$F_Z(z) = \int_{-\infty}^{z} f_Z(t)dt$$



$$F_Z(z) = \begin{cases} 0, & z < 0 \\ \displaystyle\int_0^z t\,dt\,, & 0 \le z < 1 \\ \displaystyle\int_0^1 t\,dt + \int_1^z (2-t)\,dt\,, & 1 \le z < 2 \\ 1, & z \ge 2 \end{cases}$$

$$F_Z(z) = \int_{-\infty}^{z} f_Z(t)dt$$

即 $$F_Z(z) = \begin{cases} 0, & z < 0 \\ \dfrac{z^2}{2}, & 0 \le z < 1 \\ -\dfrac{z^2}{2} + 2z - 1, & 1 \le z < 2 \\ 1, & z \ge 2 \end{cases}$$

$$P\{(X,Y)\in G\}=\iint_G f(x,y)dxdy$$

练习．设X 和Y 相互独立， 具有共同的概率密度

$$f(x)=\begin{cases}1, & 0\le x\le 1\\ 0, & else\end{cases}$$

求 $Z=X+Y$ 的密度函数与分布函数。

解：法二（先求分布再求密度）

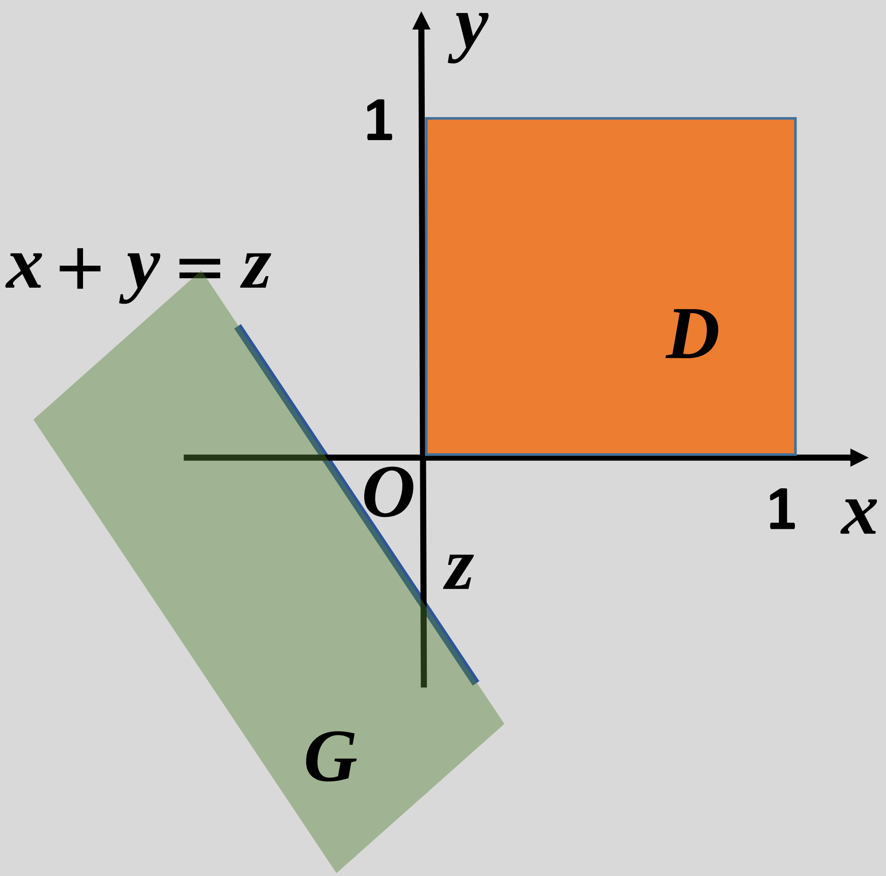
$$f(x,y)=\begin{cases}1, & 0\le x\le 1, 0\le y\le 1\\ 0, & else\end{cases}$$

$$f(x,y)=\begin{cases}1, & 0\le x\le 1, 0\le y\le 1\\ 0, & else\end{cases}$$

$$F_Z(z)=P(Z\le z)=P(X+Y\le z)$$

$$=\iint\limits_{x+y\le z} f(x,y)\,dxdy$$

1. $z<0$ 时，$F_Z(z)=0$

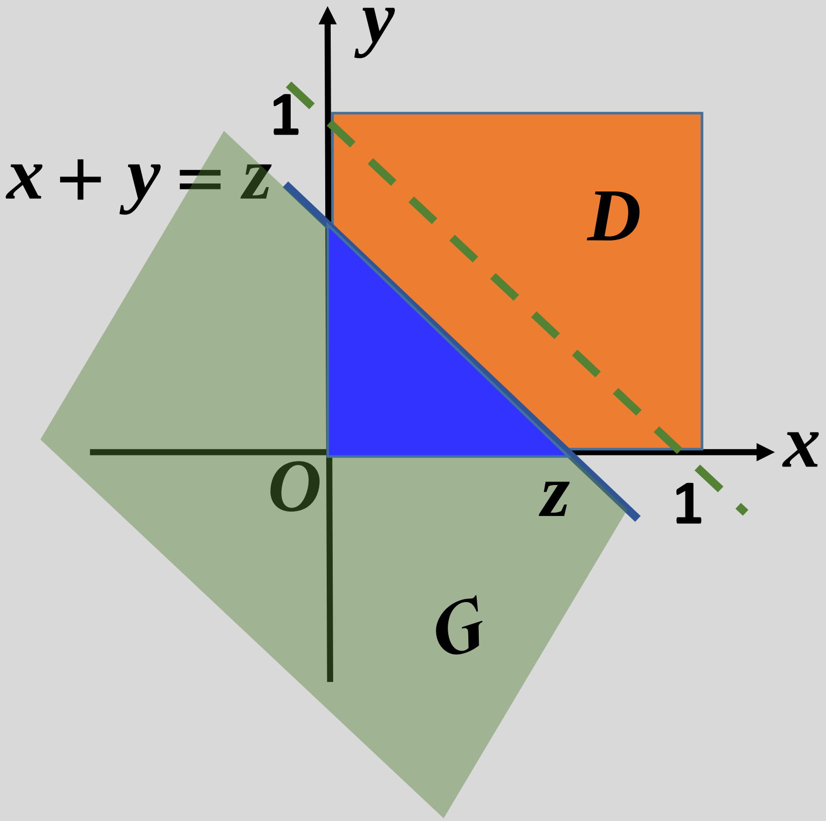


$$f(x,y)=\begin{cases}1, & 0\le x\le 1, 0\le y\le 1\\ 0, & else\end{cases}$$

$$F_Z(z)=P(X+Y\le z)=\iint\limits_{x+y\le z} f(x,y)dxdy$$

2. $0\le z<1$ 时，

$$F_Z(z)=\int_0^z\left(\int_0^{z-x}1\,dy\right)dx=\frac{z^2}{2}$$



$$f(x,y)=\begin{cases}1, & 0\le x\le 1, 0\le y\le 1\\ 0, & else\end{cases}$$

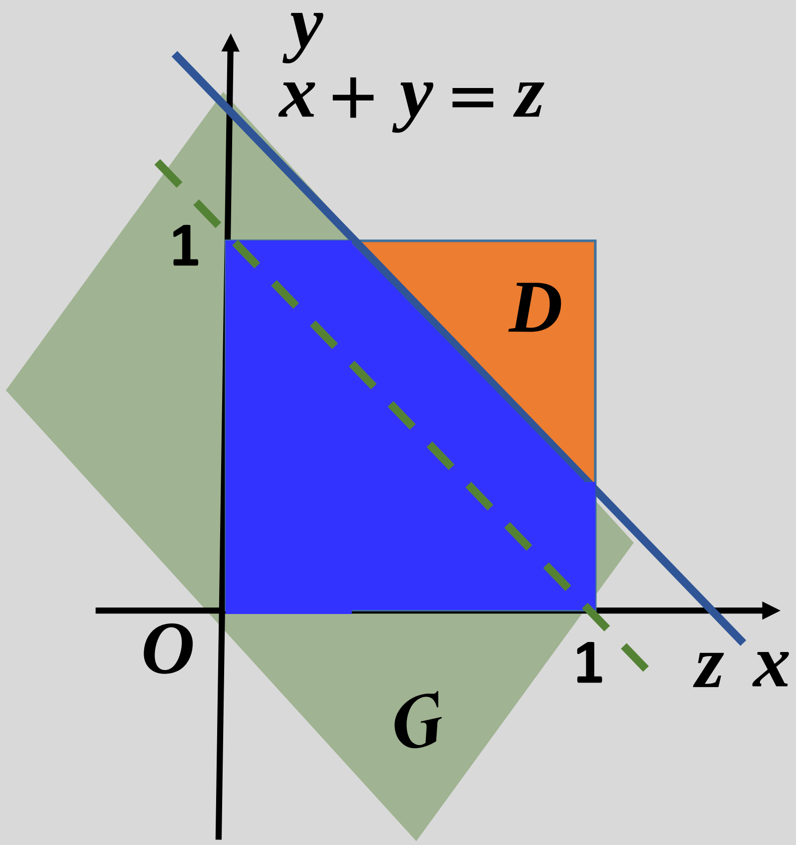
$$F_Z(z)=P(X+Y\le z)=\iint\limits_{x+y\le z} f(x,y)dxdy$$

3. $1\le z<2$ 时，

$$F_Z(z)=1-\int_{z-1}^{1}(\int_{z-x}^{1}1\,dy)dx$$

$$=1-\frac{(2-z)^2}{2}$$

4. $z\ge 2$ 时， $F_Z(z)=1$



$$F_Z(z) = \begin{cases} 0, & z < 0 \\ \dfrac{z^2}{2}, & 0 \le z < 1 \\ 1 - \dfrac{(2-z)^2}{2}, & 1 \le z < 2 \\ 1, & z \ge 2 \end{cases}$$

$$f_Z(z) = \begin{cases} z, & 0 \le z < 1 \\ 2 - z, & 1 \le z < 2 \\ 0, & \text{其它} \end{cases}$$

连续型随机变量函数的分布

主要方法

分布函数微分法

$Z=X+Y$ 型的第三种方法：公式法

积分转化法

一维随机变量 ⟹ 二维随机变量 ⟹ n维随机变量