例1. 设$X_1, X_2, \dots, X_n$ 独立同分布于$N(\mu, \sigma^2)$, 记

$$\bar{X} = \frac{1}{n}\sum_{k=1}^{n} X_k\,, \quad S^2 = \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \bar{X})^2\,, \quad \bar{Y} = \frac{1}{n}\sum_{k=1}^{n}(X_k - \mu)^2\,,$$

求 $E\left(S^2\right)$; $D(\bar{Y})$.

例2. 设随机变量$X_1, \dots X_n$独立同分布于正态分布$N\left(\mu, \sigma^2\right)$, 若

$$\bar{X} = \frac{1}{n}\sum_{k=1}^{n} X_k\,, \quad Z_1 = X_1 - \bar{X}, \qquad Z_2 = X_2 + \bar{X},$$

求Z_1, Z_2的相关系数.

例3. 设X, Y独立同分布于标准正态分布$N(0,1)$，求 $E(\max(X,Y))$.

例4. 设$X_1, X_2, \dots, X_n$ 独立同分布于 $f(x) = \begin{cases} \frac{1}{2\theta}, & -\theta \le x \le \theta \\ 0, & else \end{cases}$, $\theta > 0$

求 $E(\max\{|X_1|, |X_2|, \dots, |X_n|\})$; $D(\max\{|X_1|, |X_2|, \dots, |X_n|\})$.

例5. 设$X_1, X_2, \dots, X_n$ 独立同分布于 $f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x-\mu}{\theta}}, & x \ge \mu \\ 0, & x < \mu \end{cases}$, $\theta > 0$

求 $E(\min\{X_1, X_2, \dots, X_n\})$.

例1. 设$X_1, X_2, \dots, X_n$ 独立同分布于$N(\mu, \sigma^2)$, 记

$$\bar{X} = \frac{1}{n}\sum_{k=1}^{n} X_k\,,\quad S^2 = \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \bar{X})^2\,,\quad \bar{Y} = \frac{1}{n}\sum_{k=1}^{n}(X_k - \mu)^2\,,$$

求 $E(S^2)$; $D(\bar{Y})$.

解： $$E(S^2) = \frac{1}{n-1}\sum_{k=1}^{n} E[(X_k - \bar{X})^2]$$

$$= \frac{1}{n-1}\sum_{k=1}^{n}\{D(X_k - \bar{X}) + [E(X_k - \bar{X})]^2\}$$

$X_1, X_2, \ldots, X_n$ 独立同分布于$N(\mu, \sigma^2)$

$$D(X_k - \bar{X}) = D(X_k) + D(\bar{X}) - 2cov(X_k, \bar{X})$$

$$D(X_k) = \sigma^2 \qquad D(\bar{X}) = \frac{\sigma^2}{n}$$

$$Cov(X_k, \bar{X}) = Cov(X_k, \frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n}\sum_{i=1}^{n} Cov(X_k, X_i)$$

由于$X_1, X_2, \ldots, X_n$ 相互独立，故 $Cov(X_k, X_i) = 0\ (k \neq i)$

$$Cov(X_k, \bar{X}) = \frac{1}{n}Cov(X_k, X_k) = \frac{1}{n}D(X_k) = \frac{\sigma^2}{n}$$

$$D(X_k) = \sigma^2 \quad D(\bar{X}) = \frac{\sigma^2}{n} \quad Cov(X_k, \bar{X}) = \frac{\sigma^2}{n}$$

于是得

$$D(X_k - \bar{X}) = D(X_k) + D(\bar{X}) - 2cov(X_k, \bar{X}) = \frac{(n-1)\sigma^2}{n}$$

$$E(S^2) = \frac{1}{n-1}\sum_{k=1}^{n}\{D(X_k - \bar{X}) + [E(X_k - \bar{X})]^2\} = \sigma^2$$

$$X_1,\dots X_n\ i.i.d.\sim N\left(\mu,\sigma^2\right),\overline{Y}=\frac{1}{n}\sum_{k=1}^{n}(X_k-\mu)^2$$

- $X_1,\dots X_n\ i.i.d.\sim N\left(\mu,\sigma^2\right)$

  记 $Y_k=\dfrac{X_k-\mu}{\sigma},k=1,\dots,n$

  $\longrightarrow Y_1,\dots Y_n\ i.i.d.\sim N(0,1)$

$$\longrightarrow D\left({Y_k}^2\right)=E\left({Y_k}^4\right)-\left[E\left({Y_k}^2\right)\right]^2=3-1=2$$

$$D(\overline{Y})=\frac{\sigma^4}{n^2}\sum\nolimits_{k=1}^{n}D\left[\left(\frac{X_k-\mu}{\sigma}\right)^2\right]=\frac{\sigma^4}{n^2}\sum\nolimits_{k=1}^{n}D\left({Y_k}^2\right)=\frac{2\sigma^4}{n}$$

$X_1, \ldots X_n$独立 $\longrightarrow$ $\operatorname{cov}(X_i, X_j) = 0 \; for \; i \neq j$

例2. 设随机变量$X_1, \ldots X_n$独立同分布于正态分布$N(\mu, \sigma^2)$, 若

$$\bar{X} = \frac{1}{n}\sum_{k=1}^{n} X_k, \quad Z_1 = X_1 - \bar{X}, \qquad Z_2 = X_2 + \bar{X},$$

求Z_1, Z_2的相关系数.

提示：

- $X_1, \ldots X_n$同分布 $\longrightarrow$ $\operatorname{cov}(X_1, \bar{X}) = \operatorname{cov}(X_2, \bar{X}) = \dfrac{\sigma^2}{n}$

$$\operatorname{cov}(Z_1, Z_2) = \operatorname{cov}(X_1 - \bar{X}, X_2 + \bar{X}) = -\operatorname{cov}(\bar{X}, \bar{X}) = -D(\bar{X}) = -\frac{\sigma^2}{n}$$

$$\operatorname{cov}(Z_1, Z_2) = -\frac{\sigma^2}{n}$$

$$D(Z_1) = D(X_1 - \bar{X}) = D(X_1) + D(\bar{X}) - 2\operatorname{cov}(X_1, \bar{X}) = \frac{(n-1)\sigma^2}{n}$$

$$D(Z_2) = D(X_2 + \bar{X}) = D(X_2) + D(\bar{X}) + 2\operatorname{cov}(X_2, \bar{X}) = \frac{(n+3)\sigma^2}{n}$$

$$\rho_{Z_1 Z_2} = \frac{\operatorname{cov}(Z_1, Z_2)}{\sqrt{D(Z_1)}\sqrt{D(Z_2)}} = -\frac{1}{\sqrt{(n-1)(n+3)}}$$

$$f_X(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \qquad f_Y(y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$$

例3. 设X, Y独立同分布于标准正态分布$N(0,1)$，求 $E(\max(X,Y))$.

解法一： $$E(\max(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\max(x,y)f_X(x)f_Y(y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\max(x,y)\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dxdy$$

$$E(\max(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \max(x,y)\,\frac{1}{2\pi}\,e^{-\frac{x^2+y^2}{2}}\,dxdy$$

$$E(\max(X,Y)) = \underbrace{\iint\limits_{\{x \geq y\}} x\,\frac{1}{2\pi}\,e^{-\frac{x^2+y^2}{2}}\,dxdy}_{I_1} + \underbrace{\iint\limits_{\{x < y\}} y\,\frac{1}{2\pi}\,e^{-\frac{x^2+y^2}{2}}\,dxdy}_{I_2}$$

$$I_1 = \underset{\{x \geq y\}}{\iint} x \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dxdy$$

则 $$I_1 = \int_{-\infty}^{\infty} \int_{y}^{\infty} x \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dxdy = \int_{-\infty}^{\infty} \frac{e^{-\frac{y^2}{2}}}{2\pi} \left(\int_{y}^{\infty} xe^{-\frac{x^2}{2}} dx \right) dy$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{y^2}{2}}}{2\pi} \underline{\left(\int_{y}^{\infty} e^{-\frac{x^2}{2}} d\frac{x^2}{2} \right) dy} = \int_{-\infty}^{\infty} \frac{e^{-\frac{y^2}{2}}}{2\pi} e^{-\frac{y^2}{2}} dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-y^2} dy = \frac{1}{2\sqrt{\pi}}$$

$$I_2=\iint\limits_{\{x<y\}} y\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dxdy$$

同理 $$I_2=\int_{-\infty}^{\infty}\int_{x}^{\infty} y\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dydx \;=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}\left(\int_{x}^{\infty}ye^{-\frac{y^2}{2}}dy\right)dx$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}e^{-\frac{x^2}{2}}dx \;=\frac{1}{2\sqrt{\pi}}$$

故 $I_1=I_2=\dfrac{1}{2\sqrt{\pi}}$ 则 $E(\max(X,Y)\,)=I_1+I_2\;=\dfrac{1}{\sqrt{\pi}}$

$$E(X \pm Y) = E(X) \pm E(Y) \quad D(X \pm Y) = D(X) + D(Y)\ (\text{独立})$$

例3. 设X, Y独立同分布于标准正态分布$N(0,1)$，求 $E(\max(X,Y))$.

解法二：

$$\max(X,Y) = \frac{1}{2}(X + Y + |X - Y|)$$

$$E(\max(X,Y)) = \frac{1}{2}E(X+Y) + \frac{1}{2}E(|X-Y|)$$

$$X, Y \overset{i.i.d.}{\sim} N(0,1) \Longrightarrow X + Y \sim N(0,2) \quad X - Y \sim N(0,2)$$

$$\max(X,Y) = \frac{1}{2}(X+Y+|X-Y|)$$

$$Z = X - Y \sim N(0,2)$$

$$E(|Z|) = \int_{-\infty}^{\infty} |x| \frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{4}} dx \quad = 2\int_{0}^{\infty} x \frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{4}} dx \quad = \frac{1}{\sqrt{\pi}}$$

$$E(\max(X,Y)\,) = \frac{1}{\sqrt{\pi}}$$

期望与方差的计算

例4 设$X_1, X_2, \dots, X_n$ 独立同分布于 $f(x) = \begin{cases} \dfrac{1}{2\theta}, & -\theta \le x \le \theta, \\ 0, & else \end{cases}$ $\quad \theta > 0$

求 $E(\max\{|X_1|, |X_2|, \dots, |X_n|\})$; $D(\max\{|X_1|, |X_2|, \dots, |X_n|\})$.

分析： 令 $M = \max\{|X_1|, |X_2|, \dots, |X_n|\}$

$$E(M) = \int_{-\infty}^{\infty} x f_M(x)\, dx \longleftarrow f_M(x) = F'_M(x) \longleftarrow F_M(x) = [F_{|X_i|}(x)]^n$$

Step 4 Step 3 Step 2 Step 1

期望与方差的计算

例4 设$X_1, X_2, \dots, X_n$ 独立同分布于 $f(x) = \begin{cases} \dfrac{1}{2\theta}, & -\theta \le x \le \theta \\ 0, & else \end{cases}, \quad \theta > 0$

求 $E(\max\{|X_1|, |X_2|, \dots, |X_n|\})$; $D(\max\{|X_1|, |X_2|, \dots, |X_n|\})$.

解: Step 1. 求$F_{|X_i|}(x)$ $\quad X_i \sim f(x) = \begin{cases} \dfrac{1}{2\theta}, & -\theta \le x \le \theta \\ 0, & else \end{cases}, \quad i = 1, \dots, n$

$$\Longrightarrow F_{|X_i|}(x) = P(|X_i| \le x) = \begin{cases} 0, & x < 0 \\ \dfrac{x}{\theta}, & 0 \le x \le \theta, \\ 1, & x \ge \theta \end{cases} \quad i = 1, \dots, n$$

$$F_{|X_i|}(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x}{\theta}, & 0 \le x \le \theta, \\ 1, & x \ge \theta \end{cases} \qquad i = 1, \dots, n$$

令 $M = \max\{|X_1|, |X_2|, \dots, |X_n|\}$

Step 2. 求$F_M(x)$

$$F_M(x) = [F_{|X_i|}(x)]^n = \frac{x^n}{\theta^n}, \qquad 0 \le x \le \theta$$

Step 3. 求$f_M(x)$

$$f_M(x) = F'_M(x) = \frac{nx^{n-1}}{\theta^n}, \qquad 0 \le x \le \theta$$

$$f_M(x) = \frac{nx^{n-1}}{\theta^n}, \qquad 0 \le x \le \theta$$

Step 4. 求$E(M)$和$D(M)$

$$E(M) = E(\max\{|X_1|, |X_2|, \dots, |X_n|\}) = \int_{-\infty}^{\infty} x f_M(x)\, dx = \frac{n}{n+1}\theta$$

$$E(M^2) = \int_{-\infty}^{\infty} x^2 f_M(x)\, dx = \frac{n}{n+2}\theta^2$$

$$D(M) = E(M^2) - [E(M)]^2 = \frac{n}{(n+2)(n+1)^2}\theta^2$$

期望与方差的计算

例5. 设$X_1, X_2, \dots, X_n$ 独立同分布于 $f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x-\mu}{\theta}}, & x \geq \mu, \\ 0, & x < \mu \end{cases}$ $\theta > 0$

求 $E(\min\{X_1, X_2, \dots, X_n\})$.

分析： 令 $N = \min\{X_1, X_2, \dots, X_n\}$

$$E(N) = \int_{-\infty}^{\infty} x \underline{f_N(x)}\, dx \longleftarrow f_N(x) = \underline{F_N'(x)} \longleftarrow F_N(x) = 1 - [1 - \underline{F_{X_i}(x)}]^n$$

Step 4 Step 3 Step 2 Step 1

$$X_1, X_2, \dots, X_n \text{ 独立同分布于} f(x) = \begin{cases} \dfrac{1}{\theta} e^{-\frac{x-\mu}{\theta}}, & x \ge \mu \\ 0, & x < \mu \end{cases}, \qquad \theta > 0$$

解：Step 1. $F_{X_i}(x)$

$$X_i \sim f_{X_i}(x) = \begin{cases} \dfrac{1}{\theta} e^{-\frac{x-\mu}{\theta}}, & x \ge \mu \\ 0, & x < \mu \end{cases}, \qquad i = 1, \dots, n$$

$$\Longrightarrow \quad X_i \sim F_{X_i}(x) = \begin{cases} 0, & x < \mu \\ \displaystyle\int_{\mu}^{x} \frac{1}{\theta} e^{-\frac{t-\mu}{\theta}} dt, & x \ge \mu \end{cases}, \qquad i = 1, \dots, n$$

$$= \begin{cases} 0, & x < \mu \\ 1 - e^{-\frac{x-\mu}{\theta}}, & x \ge \mu \end{cases}, \qquad i = 1, \dots, n$$

$$X_i \sim F_{X_i}(x) = \begin{cases} 0, & x < \mu \\ 1 - e^{-\frac{x-\mu}{\theta}}, & x \geq \mu \end{cases}, \qquad i = 1, \dots, n$$

解:Step 2. 求$F_N(x)$

$$F_N(x) = 1 - [1 - F_{X_i}(x)]^n = 1 - e^{-\frac{n(x-\mu)}{\theta}}, \qquad x \geq \mu$$

Step 3. 求 $f_N(x)$

$$f_N(x) = F'_N(x) = \frac{n}{\theta} e^{-\frac{n(x-\mu)}{\theta}}, \qquad x \geq \mu$$

$$N=\min\{X_1,X_2,\dots,X_n\}\sim f_N(x)=\frac{n}{\theta}e^{-\frac{n(x-\mu)}{\theta}},\qquad x\geq\mu$$

Step 4. 求 $E(N)$

$$E(N)=E(\min\{X_1,X_2,\dots,X_n\})=\int_{-\infty}^{\infty}xf_N(x)\,dx=\int_{\mu}^{\infty}x\frac{n}{\theta}e^{-\frac{n(x-\mu)}{\theta}}dx$$

$$\overset{t=\frac{n(x-\mu)}{\theta}}{=\!=}\int_0^{\infty}\left(t+\frac{n}{\theta}\mu\right)e^{-t}\frac{\theta}{n}dt=\int_0^{\infty}te^{-t}\frac{\theta}{n}dt+\int_0^{\infty}\mu e^{-t}dt=\mu+\frac{\theta}{n}$$