Ch4 不定积分 习题课 例子参考解答

例1 (1) $\int x\sqrt[3]{1-3x}\,dx$

$= -\frac{1}{3}\int[(1-3x)-1]\sqrt[3]{1-3x}\,dx$

$= \frac{1}{9}\int[(1-3x)-1](1-3x)^{\frac{1}{3}}d(1-3x)$

$= \frac{1}{9}\int[(1-3x)^{\frac{4}{3}}-(1-3x)^{\frac{1}{3}}]d(1-3x)$

$= \frac{1}{21}(1-3x)^{\frac{7}{3}}-\frac{1}{12}(1-3x)^{\frac{4}{3}}+C$

(2) $\int x^3\sqrt[3]{1+x^2}\,dx$

$= \frac{1}{2}\int x^2(1+x^2)^{\frac{1}{3}}d(1+x^2)$

$= \frac{1}{2}\int[(1+x^2)^{\frac{4}{3}}-(1+x^2)^{\frac{1}{3}}]d(1+x^2)$

$= \frac{3}{14}(1+x^2)^{\frac{7}{3}}-\frac{3}{8}(1+x^2)^{\frac{4}{3}}+C$

(3) $\int\frac{x^5}{\sqrt[3]{1+x^3}}dx$

$= \frac{1}{3}\int\frac{x^3\,d(1+x^3)}{(1+x^3)^{\frac{1}{3}}} = \frac{1}{3}\int\frac{(x^3+1)-1}{(1+x^3)^{\frac{1}{3}}}d(1+x^3)$

$= \frac{1}{3}\int[(1+x^3)^{\frac{2}{3}}-(1+x^3)^{-\frac{1}{3}}]d(1+x^3)$

$= \frac{1}{5}(1+x^3)^{\frac{5}{3}}-\frac{1}{2}(1+x^3)^{\frac{2}{3}}+C$

(4) $\int\frac{1}{1+e^x}dx$

$= \int\frac{(1+e^x)-e^x}{1+e^x}dx = \int(1-\frac{e^x}{1+e^x})dx = x-\ln(1+e^x)+C$

(5) $\int \frac{e^x}{e^x+2+2e^{-x}}dx$

$= \int \frac{e^x}{e^{-x}(e^{2x}+2e^x+2)}dx = \int \frac{(e^x+1)-1}{(e^x+1)^2+1}d(e^x+1)$

$= \int \frac{e^x+1}{(e^x+1)^2+1}d(e^x+1) - \int \frac{d(e^x+1)}{(e^x+1)^2+1}$

$= \frac{1}{2}\ln(e^{2x}+2e^x+2) - \arctan(e^x+1) + C$

(6) $\int \frac{x^4+1}{x^6+1}dx$

$= \int \frac{x^4+1}{(x^2+1)(x^4-x^2+1)}dx = \int \frac{(x^4-x^2+1)+x^2}{(x^2+1)(x^4-x^2+1)}dx = \int \frac{dx}{1+x^2} + \int \frac{x^2}{x^6+1}dx$

$= \int \frac{dx}{1+x^2} + \frac{1}{3}\int \frac{d(x^3)}{1+(x^3)^2}$

$= \arctan x + \frac{1}{3}\arctan(x^3) + C$

例2 (1) $\int \frac{dx}{x\sqrt{1+x^4}}$

$\overset{①}{=} \int \frac{dx}{x^3\sqrt{1+\frac{1}{x^4}}} = -\frac{1}{2}\int \frac{d(\frac{1}{x^2})}{\sqrt{1+\frac{1}{x^4}}} \underset{(公式)}{=} -\frac{1}{2}\ln\left(\frac{1}{x^2}+\sqrt{1+\frac{1}{x^4}}\right) + C$

$= \int \frac{x^3}{x^4\sqrt{1+x^4}}dx \overset{\sqrt{1+x^4}=t}{=\!=\!=} \frac{1}{4}\int \frac{d(t^2-1)}{t(t^2-1)} = \frac{1}{2}\int \frac{dt}{t^2-1} = \frac{1}{4}\ln\left|\frac{t-1}{t+1}\right| + C$

$= \frac{1}{4}\ln\frac{\sqrt{1+x^4}-1}{\sqrt{1+x^4}+1} + C$

(2) $\int \frac{dx}{\sqrt{x}\sqrt{1+\sqrt{x}}}$

$= 2\int \frac{d\sqrt{x}}{\sqrt{1+\sqrt{x}}} = 2\int \frac{d(1+\sqrt{x})}{(1+\sqrt{x})^{\frac{1}{2}}} = 4\sqrt{1+\sqrt{x}} + C$

(3) $\int \frac{dx}{\cos^4 x}$

$= \int \sec^2 x \cdot \sec^2 x \, dx = \int (1+\tan^2 x) \, d\tan x = \tan x + \frac{1}{3}\tan^3 x + C$

(4) $\int \frac{\sin 2x}{1+\sin^4 x} dx$

$= \int \frac{2\sin x \cos x}{1+\sin^4 x} dx = \int \frac{d(\sin^2 x)}{1+\sin^4 x} = \arctan(\sin^2 x) + C$

(5) $\int \frac{\sin 2x}{\sqrt{4-\cos^4 x}} dx = \int \frac{2\sin x \cos x}{\sqrt{4-\cos^4 x}} = -\int \frac{d(\cos^2 x)}{\sqrt{4-\cos^4 x}}$

$= - \arcsin \frac{\cos^2 x}{2} + C$

补1 (1) $\int \frac{\arctan\sqrt{x}}{\sqrt{x}\,(1+x)} dx$

$= 2\int \frac{\arctan\sqrt{x}}{1+x} d\sqrt{x} = 2\int \arctan\sqrt{x} \, d \arctan\sqrt{x} = (\arctan\sqrt{x})^2 + C$

(2) $\int e^{5+\sin^2 2x} \sin 4x \, dx$

$= \frac{1}{2}\int e^{5+\sin^2 2x} \, d\sin^2 2x = \frac{1}{2}\int e^{5+\sin^2 2x} d(5+\sin^2 2x) = \frac{1}{2} e^{5+\sin^2 2x} + C$

(6) $\int \frac{\cos 2x}{1+\sin x \cos x} dx$

$= \frac{1}{2}\int \frac{d\sin 2x}{1+\sin x \cos x} = \int \frac{d \sin x \cos x}{1+\sin x \cos x} = \int \frac{d(1+\sin x \cos x)}{1+\sin x \cos x} = \ln|1+\sin x \cos x| + C$

(7) $\int \frac{\tan x}{\ln \cos x} dx$

$= \int \frac{\sin x}{\cos x \, (\ln \cos x)} dx = -\int \frac{d \ln \cos x}{\ln \cos x} = -\ln|\ln \cos x| + C$

(8) $\int \frac{\ln \tan x}{\cos x \sin x} dx$

$= \int \frac{\ln \tan x}{\cos^2 x \tan x} dx = \int \frac{\ln \tan x}{\tan x} d\tan x = \int \ln \tan x \, d \ln \tan x$

$= \frac{1}{2}(\ln \tan x)^2 + C$

(9) $\int \frac{x+1}{x(1+xe^x)} dx$

$= \int \frac{(x+1)e^x}{xe^x(1+xe^x)} dx = \int \frac{d(xe^x)}{xe^x(1+xe^x)} = \int \left(\frac{1}{xe^x} - \frac{1}{1+xe^x}\right) d(xe^x)$

$= \ln(xe^x) - \ln(1+xe^x) + C$

$= x + \ln x - \ln(1+xe^x) + C$

(10) $\int \frac{1+\ln x}{2+(x\ln x)^2} dx$

$= \int \frac{d(x\ln x)}{2+(x\ln x)^2} = \frac{1}{\sqrt{2}} \arctan \frac{x\ln x}{\sqrt{2}} + C$

(11) $\int \frac{1-\ln x}{(x-\ln x)^2} dx$

$= \int \frac{1-\ln x}{x^2(1-\frac{\ln x}{x})^2} dx = \int \frac{d(\frac{\ln x}{x})}{(1-\frac{\ln x}{x})^2} = -\int \frac{d(1-\frac{\ln x}{x})}{(1-\frac{\ln x}{x})^2} = \frac{1}{1-\frac{\ln x}{x}} + C$

$= \frac{x}{x-\ln x} + C$

(12) $\int \frac{\sqrt{\ln(x+\sqrt{1+x^2})}}{\sqrt{1+x^2}} dx$

$= \int \sqrt{\ln(x+\sqrt{1+x^2})}\, d\ln(x+\sqrt{1+x^2})$

$= \frac{2}{3}(\ln x+\sqrt{1+x^2})^{\frac{3}{2}} + C.$

例3. (1) $\displaystyle\int \frac{\sqrt{1-x}}{\sqrt{x+1}-\sqrt{1-x}}\,dx$

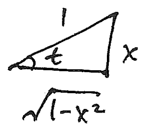
① $\displaystyle\int \frac{1}{\sqrt{\frac{1+x}{1-x}}-1}\,dx \xlongequal[x=\frac{t^2-1}{t^2+1}]{\sqrt{\frac{1+x}{1-x}}=t} \int \frac{1}{t-1}\cdot\frac{4t}{(t^2+1)^2}\,dt$

②
$$= \int \frac{\sqrt{1-x}\,(\sqrt{1+x}+\sqrt{1-x})}{2x}\,dx = \frac{1}{2}\int \frac{\sqrt{1-x^2}+1-x}{x}\,dx$$

$$= \frac{1}{2}\int \frac{\sqrt{1-x^2}}{x}\,dx + \frac{1}{2}\int\left(\frac{1}{x}-1\right)dx$$

$$\xlongequal{x=\sin t} \frac{1}{2}\int \frac{\cos t}{\sin t}\cos t\,dt + \frac{1}{2}\ln|x| - \frac{1}{2}x$$

$$= \frac{1}{2}\int \frac{1-\sin^2 t}{\sin t}\,dt + \frac{1}{2}\ln|x| - \frac{1}{2}x$$



$\frac{1}{2}\ln|\csc t-\cot t| + \frac{1}{2}\cos t$

$$= -\frac{1}{2}\ln|\csc t+\cot t| + \frac{1}{2}\ln|x| - \frac{1}{2}x + C$$

$\frac{1}{2}\ln\left|\frac{1}{x}-\frac{\sqrt{1-x^2}}{x}\right| + \frac{1}{2}\sqrt{1-x^2}$

$$= \cancel{-\frac{1}{2}\ln\left|\frac{1}{x}+\frac{\sqrt{1-x^2}}{x}\right| + \frac{1}{2}\ln|x| - \frac{1}{2}x + C}$$

$$= \cancel{-\frac{1}{2}\ln(1+\sqrt{1-x^2}) + \ln|x| - \frac{1}{2}x + C}$$

$$= \frac{1}{2}\ln(1-\sqrt{1-x^2}) + \frac{1}{2}\sqrt{1-x^2} - \frac{1}{2}x + C$$

(2) $\displaystyle\int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx$

$$= -\frac{1}{2}\int \frac{\arcsin x}{\sqrt{1-x^2}}\,d(1-x^2)$$

$$= -\int \arcsin x\,d\sqrt{1-x^2}$$

$$= -\sqrt{1-x^2}\arcsin x + \int \sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= -\sqrt{1-x^2}\arcsin x + x + C$$

注: 原式 $= \int x\arcsin x\,d\arcsin x$

$= \frac{1}{2}\int x\,d(\arcsin x)^2 = \frac{1}{2}x(\arcsin x)^2 - \frac{1}{2}\int(\arcsin x)^2dx$ 不行.

⑥

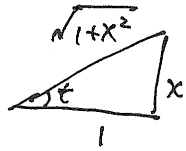
(3) $$\int \frac{e^{\arctan x}}{(1+x^2)^{3/2}}\,dx$$

$$\xrightarrow[x=\tan t]{\text{令}\arctan x = t} \int \frac{e^t}{\sec^3 t}\sec^2 t\,dt = \int e^t \cos t\,dt = \int e^t\,d\sin t$$

$$= e^t \sin t - \int \sin t\, e^t\,dt = e^t \sin t + \int e^t\,d\cos t$$

$$= e^t \sin t + e^t \cos t - \int \cos t\, e^t\,dt$$

$$\therefore \int e^t \cos t\,dt = \frac{1}{2}e^t(\sin t + \cos t) + C$$



$$\therefore \text{原式} = \frac{1}{2}e^{\arctan x}\left(\frac{x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}}\right) + C$$

$$= \frac{(1+x)\,e^{\arctan x}}{2\sqrt{1+x^2}} + C$$

补2 (1) $$\int \frac{x\arctan x}{\sqrt{1+x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{\arctan x\,d(1+x^2)}{\sqrt{1+x^2}} = \int \arctan x\,d\sqrt{1+x^2}$$

$$= \sqrt{1+x^2}\arctan x - \int \sqrt{1+x^2}\,\frac{1}{1+x^2}\,dx$$

$$= \sqrt{1+x^2}\arctan x - \int \frac{dx}{\sqrt{1+x^2}} \qquad \left(\text{利用} \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x+\sqrt{x^2\pm a^2}| + C\right)$$

$$= \sqrt{1+x^2}\arctan x - \ln|x+\sqrt{x^2+1}| + C$$

(2) $$\int \frac{xe^x}{\sqrt{e^x-1}}\,dx$$

$$\xrightarrow[x=\ln(1+u^2)]{u=\sqrt{e^x-1}} \int \frac{(1+u^2)\ln(1+u^2)}{u}\cdot\frac{2u}{1+u^2}\,du = 2\int \ln(1+u^2)\,du$$

$$= 2u\ln(1+u^2) - 4\int \frac{u^2+1-1}{1+u^2}\,du$$

$$= 2u\ln(1+u^2) - 4u + 4\arctan u + C$$

$$= 2x\sqrt{e^x-1} - 4\sqrt{e^x-1} + 4\arctan\sqrt{e^x-1} + C$$

例4 (1) $\int \frac{x\ln x}{(1+x^2)^{3/2}}\,dx$

$= \frac{1}{2}\int \frac{\ln x}{(1+x^2)^{3/2}}\,d(1+x^2) = -\int \ln x\,d(1+x^2)^{-\frac{1}{2}}$

$= -\frac{\ln x}{\sqrt{1+x^2}} + \int \frac{1}{\sqrt{1+x^2}}\cdot\frac{1}{x}\,dx$

$= -\frac{\ln x}{\sqrt{1+x^2}} + \frac{1}{2}\int \frac{1+\cancel{x^2-x^2}}{x^2\sqrt{1+x^2}}\,d(x^2) \xlongequal[x^2=t^2-1]{\sqrt{1+x^2}=t} \frac{1}{2}\int \frac{2t\,dt}{t(t^2-1)} = \int \frac{dt}{t^2-1}$

$= \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C$

$= \frac{1}{2}\ln\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + C$

$= -\frac{\ln x}{\sqrt{1+x^2}} + \cancel{\frac{1}{2}\int \frac{1}{x^2}\,d(x^2) - \frac{1}{2}\int \frac{d(x^2)}{\sqrt{1+x^2}}}$?

$= -\frac{\ln x}{\sqrt{1+x^2}} + \cancel{\ln x - \sqrt{1+x^2} + C}\ \frac{1}{2}\ln\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + C$

(2) $\int e^{\sin x}\sin 2x\,dx$

$= \int e^{\sin x}\,2\sin x\,d\sin x \xlongequal{\sin x = t} 2\int te^t\,dt$

$= 2\int t\,de^t = 2te^t - 2\int e^t\,dt = 2te^t - 2e^t + C$

$= +2\sin x\,e^{\sin x} - 2e^{\sin x} + C$

(3) $\int \frac{\ln x - 1}{(\ln x)^2}\,dx$

$= \int \frac{1}{\ln x}\,dx - \int \frac{dx}{(\ln x)^2}$

$= \frac{x}{\ln x} - \int x\cdot\left(-\frac{1}{\ln^2 x}\right)\cdot\frac{1}{x}\,dx - \int \frac{dx}{(\ln x)^2}$

$= \frac{x}{\ln x} + C$

4, $\int \frac{x e^x}{\sqrt{e^x-2}} dx$

$= \int \frac{x\, d(e^x-2)}{\sqrt{e^x-2}} = 2\int x\, d\sqrt{e^x-2}$

$= 2x\sqrt{e^x-2} - 2\int \sqrt{e^x-2}\, dx$

$\xlongequal[x=\ln(2+t^2)]{\sqrt{e^x-2}=t} 2x\sqrt{e^x-2} - 2\int t \cdot \frac{2t}{2+t^2} dt$

$= 2x\sqrt{e^x-2} - 4\int \frac{t^2+2-2}{2+t^2} dt$

$= 2x\sqrt{e^x-2} - 4t + \frac{8}{\sqrt{2}} \arctan\frac{t}{\sqrt{2}} + C$

$= 2x\sqrt{e^x-2} - 4\sqrt{e^x-2} + 4\sqrt{2} \arctan\frac{\sqrt{e^x-2}}{2} + C$

(5) $\int \frac{x e^{\arctan x}}{(1+x^2)^2} dx$

$\xlongequal[x=\tan t]{\arctan x=t} \int \frac{\tan t \cdot e^t}{\sec^4 t} \cdot \sec^2 t\, dt = \int e^t \sin t \cos t\, dt$

$= \frac{1}{2}\int \sin 2t\, de^t$

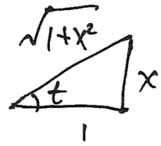
$= \frac{1}{2} e^t \sin 2t - \frac{1}{2}\int e^t \cos 2t \cdot 2\, dt$

$= \frac{1}{2} e^t \sin 2t - \int \cos 2t\, de^t$

$= \frac{1}{2} e^t \sin 2t - \cancel{\int} e^t \cos 2t - \int e^t \sin 2t \cdot 2\, dt$

$\therefore \frac{5}{2}\int \sin 2t\, e^t\, dt = \frac{1}{2} e^t \sin 2t - e^t \cos 2t + C_1$

$\frac{1}{2}\int \sin 2t\, e^t\, dt = \frac{1}{10} e^t \sin 2t - \frac{1}{5} e^t \cos 2t + C$



原式 $= \frac{1}{5} e^{\arctan x} \cdot \frac{x}{1+x^2} - \frac{1}{5} e^{\arctan x} \left(\frac{1-x^2}{1+x^2}\right) + C$

$= \frac{1}{5} e^{\arctan x} \cdot \frac{x^2+x-1}{1+x^2} + C$

(6) $\int \sin(\ln x)dx$

$\xlongequal[x=e^t]{\ln x=t} \int \sin t\, e^t dt = \int \sin t\, de^t = e^t\sin t - \int e^t\cos t\, dt$

$= e^t\sin t - \int \cos t\, de^t$

$= e^t\sin t - e^t\cos t + \int e^t \sin t\, dt$

$\therefore \int \sin t\, e^t dt = \frac{1}{2}e^t(\sin t - \cos t) + C$

即原式 $= \frac{1}{2}x\,[\sin(\ln x) - \cos(\ln x)] + C$

例5 $f(x)$ 有原函数 e^{-x^2}，求 $\int x f'(x)dx$

解 $\int x f'(x)dx = \int x\, df(x)$

$= x f(x) - \int f(x)\,dx$

$= x(e^{-x^2})' - e^{-x^2} + C$

$= -2x^2e^{-x^2} - e^{-x^2} + C$

例6 (1) $\int f'(x)f''(x)\,dx$

$= \int f'(x)\, d\, f'(x) = [f'(x)]^2 - \int f'(x) f''(x)\,dx$

$\therefore$ 原式 $= \frac{1}{2}[f'(x)]^2 + C$

(2) $\int x\left(\frac{\sin x}{x}\right)'' dx$

$= \int x\, d\left(\frac{\sin x}{x}\right)' = x\left(\frac{\sin x}{x}\right)' - \int \left(\frac{\sin x}{x}\right)' dx$

$= x\left(\frac{\sin x}{x}\right)' - \frac{\sin x}{x} + C$

$= \frac{x\cos x - 2\sin x}{x} + C$

例7 (1) $I_n = \int \tan^n x\,dx$

$$= \int \tan^{n-2}x\,(\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2}x\, d\tan x - \int \tan^{n-2}x\,dx$$

$$= \tan^{n-1}x - \int \tan x\cdot(n-2)\tan^{n-3}x\cdot\sec^2 x\,dx - I_{n-2}$$

$$= \tan^{n-1}x - (n-2)\int\tan^{n-2}x\,(\tan^2 x+1)\,dx - I_{n-2}$$

$$= \tan^{n-1}x - (n-2)I_n - (n-1)I_{n-2}$$

$$\therefore (n-1)I_n = \tan^{n-1}x - (n-1)I_{n-2}$$

(2) $I_n = \int \frac{1}{\sin^n x}\,dx$

$$= \int\csc^n x\,dx = -\int\csc^{n-2}x\,d\cot x$$

$$= -\csc^{n-2}x\cot x + \int \cot x\cdot(n-2)\csc^{n-3}x\,(-\csc x\cot x)\,dx$$

$$= -\csc^{n-2}x\cot x - (n-2)\int\csc^{n-2}x\,(\csc^2 x-1)\,dx$$

$$= -\csc^{n-2}x\cot x - (n-2)I_n + (n-2)I_{n-2}$$

$$\therefore (n-1)I_n = -\csc^{n-2}x\cot x + (n-2)I_{n-2}$$

例8 (1) $\int\frac{x^9-8}{x^{10}+8x}\,dx$

$$=\int\frac{x^9+8-16}{x(x^9+8)}\,dx = \int\frac{dx}{x} - 16\int\frac{dx}{x(x^9+8)}$$

$$=\int\frac{dx}{x} - 16\int\frac{x^8\,dx}{x^9(x^9+8)}$$

$$=\int\frac{dx}{x} - \frac{16}{9}\int\frac{d(x^9)}{x^9(x^9+8)} = \int\frac{dx}{x} - \frac{2}{9}\int\left(\frac{1}{x^9} - \frac{1}{x^9+8}\right)d(x^9)$$

$$= \ln|x| - 2\ln|x| + \frac{2}{9}\ln|x^9+8| + C$$

$$= -\ln|x| + \frac{2}{9}\ln|x^9+8| + C$$

(2) $\int \frac{x^2-1}{x^4+1}dx$ (3) $\int \frac{x^2+1}{x^4+1}dx$ (4) $\int \frac{1}{x^4+1}dx$

(参见课件)

(5) $\int \frac{1}{\sin^4 x+\cos^4 x}dx$

$$= \int \frac{1}{(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x}dx = \int \frac{1}{1-\frac{1}{2}\sin^2 2x}dx$$

$$= \int \frac{1}{1-\frac{1}{4}(1-\cos 4x)}dx = \int \frac{4}{3+\cos 4x}dx$$

$$\overset{t=\tan 2x}{=\!=\!=} \int \frac{4}{3+\frac{1-t^2}{1+t^2}} \cdot \frac{1}{2(1+t^2)}dt = \int \frac{1}{t^2+2}dt$$

$$= \frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}} + C$$

$$= \frac{\sqrt{2}}{2}\arctan\frac{\tan 2x}{\sqrt{2}} + C$$

(6) $\int \frac{\sin x}{\sin x+\cos x}dx$

令 ~~I~~ $\sin x = a(\sin x+\cos x) + b(\sin x+\cos x)'$

$= a(\sin x+\cos x) + b(\cos x-\sin x)$

$\therefore \begin{cases} 1 = a-b \\ 0 = a+b \end{cases} \Rightarrow a=\frac{1}{2},\ b=-\frac{1}{2}$

$$\therefore \int \frac{\sin x}{\sin x+\cos x}dx = \int \frac{\frac{1}{2}(\sin x+\cos x)-\frac{1}{2}(\sin x+\cos x)'}{\sin x+\cos x}dx$$

$$= \frac{1}{2}x - \frac{1}{2}\ln|\sin x+\cos x| + C$$

例19 $\left(\frac{\sin x}{x}\right)' = f(x)$， 求 $\int x f'(2x)\,dx$

解: $\int x f'(2x)\,dx = \frac{1}{2}\int x f'(2x)\,d(2x) = \frac{1}{2}\int x\,d f(2x)$

$= \frac{1}{2} x f(2x) - \frac{1}{2}\int f(2x)\,dx$

$\cancel{= \frac{1}{2} x f'(2x) - \frac{1}{4} f(2x) + C}$

又 $f(x) = \left(\frac{\sin x}{x}\right)' = \frac{x\cos x - \sin x}{x^2}$

$\therefore \int x f'(2x)\,dx = \frac{1}{2}x\cdot\frac{2x\cos 2x - \sin 2x}{4x^2} - \frac{1}{4}\cdot\frac{\sin 2x}{2x} + C$

$= \frac{x\cos 2x - \sin 2x}{4x} + C$

例10 $f'(\sin^2 x) = \cos 2x + \tan^2 x$， $0 < x < \frac{\pi}{2}$， 求 $f(x)$

解: 令 $\sin^2 x = t$，则 $\cos 2x = 1 - 2\sin^2 x = 1 - 2t$

$\tan^2 x = \frac{\sin^2 x}{1-\sin^2 x} = \frac{t}{1-t}$

得 $f'(t) = 1 - 2t + \frac{t}{1-t} = -2t - \frac{1}{t-1}$

$f(t) = \int\left(-2t - \frac{1}{t-1}\right)dt = -t^2 - \ln|t-1| + C = -t^2 - \ln(1-t) + C$ $\quad 0<t<1$

$\cancel{= -\frac{2}{3}t^3 - \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C}$

$f(x) = -x^2 - \ln(1-x) + C$ $\quad 0<x<1$

$\cancel{f(x) = -\frac{2}{3}x^3 - \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C}$

$\cancel{\therefore f(x) = -\frac{2}{3}\sin^6 x - \ln\frac{1-\sin^2 x}{1+\sin^2 x} + C}$

$\cancel{= -\frac{2}{3}\sin^6 x - 2\ln\cos x + \ln(1+\sin^2 x) + C}$

例11 $\int f'(\sqrt{x})\,dx = x(e^{\sqrt{x}} + 1) + C$， 求 $f(x)$

①令 $\xlongequal{\sqrt{x} = t}$ $\int f'(t)\,2t\,dt = t^2(e^t + 1) + C$

$\therefore 2\int t\,df(t) = 2t f(t) - 2\int f(t)\,dt = t^2(e^t+1) + C$

上式两边求导，有 $f'(t) = e^t + 1 + \frac{1}{2}te^t$

$f(t) = e^t + t + \frac{1}{2}\int t\,de^t = e^t + t + \frac{1}{2}te^t - \frac{1}{2}e^t + C_1$

$\therefore f(x) = \frac{1}{2}e^{\sqrt{x}} + \sqrt{x} + \frac{1}{2}\sqrt{x}e^{\sqrt{x}} + C_1$

② 对 $\int f'(\sqrt{x})\,dx = x(e^{\sqrt{x}}+1)+C$

两边对x求导，得

$$f'(\sqrt{x}) = e^{\sqrt{x}}+1 + x\,e^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}$$
$$= e^{\sqrt{x}}+1 + \frac{1}{2}\sqrt{x}\,e^{\sqrt{x}}$$

令 $\sqrt{x}=t$

$$f'(t) = e^{t}+1+\frac{1}{2}t\,e^{t}$$

余下同上.

例12. $\int \max(1,|x|)\,dx$

解: $\max(1,|x|) = \begin{cases} 1, & |x|<1 \\ |x| & |x|\geqslant 1 \end{cases} = \begin{cases} -x & x\leqslant -1 \\ 1 & -1<x<1 \\ x & x\geqslant 1 \end{cases}$

$$\int \max(1,|x|)\,dx = \begin{cases} -\frac{1}{2}x^2+C_1 & x\leqslant -1 \\ x+C_2 & -1<x<1 \\ \frac{1}{2}x^2+C_3 & x\geqslant 1 \end{cases}$$

又原函数连续，有

$$\begin{cases} -\frac{1}{2}+C_1 = -1+C_2 \\ 1+C_2 = \frac{1}{2}+C_3 \end{cases} \Longrightarrow \frac{1}{2}+C_1 = C_2 = -\frac{1}{2}+C_3 = C$$

$$\therefore \int \max(1,|x|)\,dx = \begin{cases} -\frac{1}{2}x^2-\frac{1}{2}+C \\ x+C \\ \frac{1}{2}x^2+\frac{1}{2}+C \end{cases}.$$