HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR ELLIPTIC PROBLEM

1. 2-D problem

1.1. Model problem.

$$\begin{cases} -\nabla\cdot\nabla u(x) = f(x) & x\in\Omega \\ u = g & on \quad \partial\Omega \end{cases} \tag{1.1}$$

1.2. HDG scheme. Some notations:

(1) Denote by $\mathcal{T}_h$ a collection of disjoint regular elements K that partition Ω and set $\partial\mathcal{T}_h := \{\partial K : K\in\mathcal{T}_h\}$.
(2) For an element K of the collection $\mathcal{T}_h$, $F=\partial K\cap\partial\Omega$ is the boundary face if the $d-1$ Lebesgue measure of F is nonzero. For two elements K^+ and K^- of the collection $\mathcal{T}_h$, $F=\partial K^+\cap\partial K^-$ is the interior face between K^+ and K^- if the $d-1$ Lebesgue measure of F is nonzero. Denote by $\mathcal{E}_h^o$ and $\mathcal{E}_h^\partial$ the set of interior and boundary faces, respectively and set $\mathcal{E}_h=\mathcal{E}_h^o\cup\mathcal{E}_h^\partial$.
(3) Let $\mathbf{n}^+$ and $\mathbf{n}^-$ be the outward unit normal vectors on two neighboring elements K^+ and K^-, respectively. We use $(\mathbf{G}^\pm,\mathbf{v}^\pm,q^\pm)$ to denote the trace of $(\mathbf{G},\mathbf{v},q)$ on F from the interior of $K^\pm$, where $\mathbf{G}$, $\mathbf{v}$ and q are second-order tensorial, vectorial and scalar functions, respectively.
(4) For $F\in\mathcal{E}_h^o$, we set

$$[[\mathbf{G}\mathbf{n}]] = \mathbf{G}^+\mathbf{n}^+ + \mathbf{G}^-\mathbf{n}^-, \quad [[\mathbf{v}\odot\mathbf{n}]] = \mathbf{v}^+\odot\mathbf{n}^+ + \mathbf{v}^-\odot\mathbf{n}^-$$
$$[[q\mathbf{n}]] = q^+\mathbf{n}^+ + q^-\mathbf{n}^-$$

(5) Let $\mathcal{P}_k(D)$ denote the space of polynomial of degree at most k on a domain D and let $L^2(D)$ be the space of square integrable functions on D. We set $\mathbf{P}_k(D)=[\mathcal{P}_k(D)]^d$, $\mathbf{L}^2(D)=[L^2(D)]^d$.

We introduce the following discontinuous approximation spaces

$$\begin{aligned} \boldsymbol{\mathcal{V}}_h^k &:= \{\boldsymbol{v}\in(\mathbf{L}^2(\mathcal{T}_h))^d : \boldsymbol{v}|_K\in(\mathcal{P}^k(K))^d \ \ \forall K\in\mathcal{T}_h\}, \\ \mathcal{W}_h^k &:= \{w\in L^2(\mathcal{T}_h) : w|_K\in\mathcal{P}^k(K) \ \ \forall K\in\mathcal{T}_h\}. \end{aligned} \tag{1.2}$$

In addition, we introduce a finite element approximation spce for the approximate trace of the solution

$$\mathcal{M}_h := \{\mu\in L^2(\mathcal{E}_h) : \mu|_F\in\mathcal{P}^k(F) \ \ \forall F\in\mathcal{E}_h\} \tag{1.3}$$

For the sake of the definition of the HDG scheme, we rewrite (1.1) into a first order system

$$\begin{aligned} \mathbf{q}-\nabla u &= 0 \quad \forall x\in\Omega \\ -\nabla\cdot\mathbf{q} &= f(x) \quad \forall x\in\Omega \\ u &= g \quad on \quad \partial\Omega \end{aligned} \tag{1.4}$$

Local solver: find $\mathbf{q}_h\in\mathbf{V}_h$, $u_h\in\mathbf{W}_h$, $\hat{u}_h\in\mathbf{M}_h$, s.t.

$$\begin{aligned} (\mathbf{q}_h,\mathbf{v}_h)+(u_h,\nabla\cdot\mathbf{v}_h) &= \langle\hat{u}_h,\mathbf{v}_h\cdot\mathbf{n}\rangle \\ (\mathbf{q}_h,\nabla w_h)-\langle\hat{\mathbf{q}}_h\cdot\mathbf{n},w_h\rangle &= (f,w_h) \end{aligned} \tag{1.5}$$

for all $\mathbf{v}_h\in\mathbf{V}_h$ and $w_h\in\mathbf{W}_h$.

We choose the numerical flux $-\hat{\mathbf{q}}_h$ as

$$\hat{\mathbf{q}}_h = \mathbf{q}_h - \tau(u_h - \hat{u}_h)\mathbf{n}, \quad on \quad \mathcal{E}_h. \tag{1.6}$$

Then equations (1.5) can be rewritten as

$$\begin{aligned}(\mathbf{q}_h, \mathbf{v}_h) + (u_h, \nabla\cdot\mathbf{v}_h) &= \langle \hat{u}_h, \mathbf{v}_h\cdot\mathbf{n}\rangle \\ -(\nabla\cdot\mathbf{q}_h, w_h) + \tau\langle u_h, w_h\rangle &= (f, w_h) + \tau\langle \hat{u}_h, w_h\rangle\end{aligned} \tag{1.7}$$

Continuity condition of the numerical flux

$$\left\langle \hat{\mathbf{q}}_h\cdot\mathbf{n}, \mu\right\rangle_{\partial\mathcal{T}_h} = 0 \tag{1.8}$$

i.e.

$$\left\langle (\mathbf{q}_h - \tau(u_h - \hat{u}_h)\mathbf{n})\cdot\mathbf{n}, \mu\right\rangle_{\partial\mathcal{T}_h} = 0 \tag{1.9}$$

for all $\mu \in \mathbf{M}_h(0)$.

Denote by $\Phi_i\big|_{i=1}^{d_q}$, $\phi_i\big|_{i=1}^{d_u}$ and $m_i\big|_{i=1}^{d}$ are basis of spaces $\mathbf{V}_h(K)$, $\mathbf{W}_h(K)$ and $\mathbf{M}_h(e)$. Then we expand $\mathbf{q}_h$, u_h and $\hat{u}_h$ as follows

$$\mathbf{q}_h\big|_K = \sum_{j=1}^{d_q} q_K^j\Phi_j, \quad u_h\big|_K = \sum_{j=1}^{d_u} u_K^j\phi_j, \quad \hat{u}_h\big|_e = \sum_{j=1}^{d} \hat{u}_e^j m_j.$$

where Φ_j has the form

$$\Phi_1 = \begin{pmatrix}\phi_1\\0\end{pmatrix}, \quad \Phi_2 = \begin{pmatrix}\phi_2\\0\end{pmatrix}, \cdots \Phi_{du} = \begin{pmatrix}\phi_{du}\\0\end{pmatrix} \quad \Phi_{du+1} = \begin{pmatrix}0\\\phi_1\end{pmatrix}, \cdots \Phi_{dq} = \begin{pmatrix}0\\\phi_{du}\end{pmatrix} \tag{1.10}$$

Substitute these expressions into (1.7) and set $v_h = \Phi_i$ and $w_h = \phi_i$ we have

$$\sum_{j=1}^{d_q} q_K^j(\Phi_i, \Phi_j)_K + \sum_{j=1}^{d_u} u_K^j(\nabla\cdot\Phi_i, \phi_j)_K - \sum_{e\in\partial K}\sum_{j=1}^{d_m} \hat{u}_e^j\langle\Phi_i\cdot\mathbf{n}, m_j\rangle_e = 0$$
$$i = 1, 2, \cdots d_q$$

$$-\sum_{j=1}^{d_q} q_K^j(\phi_i, \nabla\cdot\Phi_j)_K + \tau\sum_{j=1}^{d_u} u_K^j\langle\phi_i, \phi_j\rangle_{\partial K} - \tau\sum_{e\in\partial K}\sum_{j=1}^{d_m} \hat{u}_e^j\langle\phi_i, m_j\rangle_e = (f, \phi_i)_K$$
$$i = 1, 2, \cdots d_u$$

$$\sum_{j=1}^{d_q} q_K^j(m_i, \Phi_j\cdot\mathbf{n})_K - \tau\sum_{j=1}^{d_u} u_K^j\langle m_i, \phi_j\rangle_{\partial K} + \tau\sum_{e\in\partial K}\sum_{j=1}^{d_m} \hat{u}_e^j\langle m_i, m_j\rangle_e$$
$$i = 1, 2, \cdots d$$

Denoted by

$$\begin{array}{lll}\mathbb{M} = \left[(\phi_i, \phi_j)_{\mathcal{T}_h}\right]_{N\times N}, & \mathbb{C}_x = \left[(\partial_x\phi_i, \phi_j)_{\mathcal{T}_h}\right]_{N\times N}, & \mathbb{C}_y = \left[(\partial_y\phi_i, \phi_j)_{\mathcal{T}_h}\right]_{N\times N} \\ \tilde{\mathbb{C}}_x = \left[\partial_x\phi_i, \phi_j)_{\mathcal{T}_h}\right]_{N\times N}, & \tilde{\mathbb{C}}_y = \left[\partial_y\phi_i, \phi_j)_{\mathcal{T}_h}\right]_{N\times N}, & \mathbb{D} = \left[\langle\phi_i, \phi_j\rangle_{\partial\mathcal{T}_h}\right]_{N\times N}, \\ \mathbb{T} = \left[\langle\mu_i, \phi_j\rangle_{\partial\mathcal{T}_h}\right]_{M\times N} & \mathbb{T}_x = \left[\langle\mu_i, \phi_j\mathbf{n}_x\rangle_{\partial\mathcal{T}_h}\right]_{M\times N} & \mathbb{T}_y = \left[\langle\mu_i, \phi_j\mathbf{n}_y\rangle_{\partial\mathcal{T}_h}\right]_{M\times N} \\ \tilde{\mathbb{T}}_x = \left[\langle\mu_i, \phi_j\mathbf{n}_x\rangle_{\partial\mathcal{T}_h}\right]_{M\times N} & \tilde{\mathbb{T}}_y = \left[\langle\mu_i, \phi_j\mathbf{n}_y\rangle_{\partial\mathcal{T}_h}\right]_{M\times N} & \mathbb{N} = \left[\langle\mu_i, \mu_j\rangle_{\partial\mathcal{T}_h}\right]_{M\times M}\end{array} \tag{1.11}$$

then the formulation equivalent to the following linear system

$$\begin{pmatrix}\mathbb{K}_{11} & \mathbb{K}_{12} & \mathbb{K}_{13}\\ \mathbb{K}_{21} & \mathbb{K}_{22} & \mathbb{K}_{23}\end{pmatrix}\begin{pmatrix}\mathbb{Q}\\ \mathbb{U}\\ \mathbf{\Lambda}\end{pmatrix} = \begin{pmatrix}\mathbf{0}\\ \mathbb{F}\end{pmatrix} \tag{1.12}$$

and the contributions of each element

$$\mathbb{K}_{31}\mathbb{Q} + \mathbb{K}_{32}\mathbb{U} + \mathbb{K}_{33}\mathbf{\Lambda} \tag{1.13}$$

where

$$\mathbb{K}_{11} = \begin{pmatrix} \mathbb{M} & 0 \\ 0 & \mathbb{M} \end{pmatrix}, \qquad \mathbb{K}_{12} = \begin{pmatrix} \mathbb{C}_x \\ \mathbb{C}_y \end{pmatrix}, \qquad \mathbb{K}_{13} = \begin{pmatrix} -\mathbb{T}_x^T \\ -\mathbb{T}_y^T \end{pmatrix} \tag{1.14}$$

$$\mathbb{K}_{21} = \begin{pmatrix} -\tilde{\mathbb{C}}_x^T & -\tilde{\mathbb{C}}_y^T \end{pmatrix}, \qquad \mathbb{K}_{22} = \tau\mathbb{D}, \qquad \mathbb{K}_{23} = -\tau\mathbb{T}^T \tag{1.15}$$

$$\mathbb{K}_{31} = \begin{pmatrix} \tilde{\mathbb{T}}_x & \tilde{\mathbb{T}}_y \end{pmatrix}, \qquad \mathbb{K}_{32} = -\tau\mathbb{T}, \qquad \mathbb{K}_{33} = \tau\mathbb{N} \tag{1.16}$$

Hence

$$\mathbb{Q} = -\mathbb{K}_{11}^{-1}(\mathbb{K}_{12}\mathbb{U} + \mathbb{K}_{13}\mathbf{\Lambda}) \tag{1.17}$$

Substituting into the second equation we have

$$(\mathbb{K}_{22} - \mathbb{K}_{21}\mathbb{K}_{11}^{-1}\mathbb{K}_{12})\mathbb{U} + (\mathbb{K}_{23} - \mathbb{K}_{21}\mathbb{K}_{11}^{-1}\mathbb{K}_{13})\mathbf{\Lambda} = \mathbb{F} \tag{1.18}$$

Denote by

$$\mathbb{A} = \mathbb{K}_{22} - \mathbb{K}_{21}\mathbb{K}_{11}^{-1}\mathbb{K}_{12}, \qquad \mathbb{B} = \mathbb{K}_{23} - \mathbb{K}_{21}\mathbb{K}_{11}^{-1}\mathbb{K}_{13} \tag{1.19}$$

The (1.13) is

$$-\mathbb{K}_{31}\mathbb{K}_{11}^{-1}(\mathbb{K}_{12}\mathbb{U} + \mathbb{K}_{13}\mathbf{\Lambda}) + \mathbb{K}_{\mathbf{32}}\mathbb{U} + \mathbb{K}_{\mathbf{33}}\mathbf{\Lambda} \tag{1.20}$$

i.e.

$$\left(\mathbb{K}_{33} + \begin{pmatrix} \mathbb{K}_{31} & \mathbb{K}_{32} \end{pmatrix} \begin{pmatrix} \mathbb{K}_{11}^{-1}(\mathbb{K}_{12}\mathbb{A}^{-1}\mathbb{B} - \mathbb{K}_{13}) \\ -\mathbb{A}^{-1}\mathbb{B} \end{pmatrix}\right)\mathbf{\Lambda} + \begin{pmatrix} \mathbb{K}_{31} & \mathbb{K}_{32} \end{pmatrix} \begin{pmatrix} -\mathbb{K}_{11}^{-1}\mathbb{K}_{12}\mathbb{A}^{-1}\mathbb{F} \\ \mathbb{A}^{-1}\mathbb{F} \end{pmatrix} \tag{1.21}$$

Denote by

$$\mathbb{X} = \left(\mathbb{K}_{33} + \begin{pmatrix} \mathbb{K}_{31} & \mathbb{K}_{32} \end{pmatrix} \begin{pmatrix} \mathbb{K}_{11}^{-1}(\mathbb{K}_{12}\mathbb{A}^{-1}\mathbb{B} - \mathbb{K}_{13}) \\ -\mathbb{A}^{-1}\mathbb{B} \end{pmatrix}\right)\mathbf{\Lambda}, \quad \mathbf{Y} = -\begin{pmatrix} \mathbb{K}_{31} & \mathbb{K}_{32} \end{pmatrix} \begin{pmatrix} -\mathbb{K}_{11}^{-1}\mathbb{K}_{12}\mathbb{A}^{-1}\mathbb{F} \\ \mathbb{A}^{-1}\mathbb{F} \end{pmatrix}$$

1.3. **system.** According to (1.8), resolved with each element's boundary we have

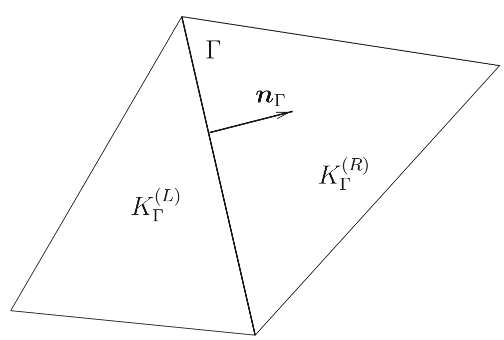
$$\sum_{K\in\mathcal{T}_h} \left\langle \hat{\mathbf{q}}_h \cdot \mathbf{n}, \mu \right\rangle_{\partial K} = 0 \tag{1.22}$$

set $\mu = m_\Gamma^i$, where m_Γ^i is basis function which is defined in boundary Γ, has character

$$m_\Gamma^i \mid_{\Gamma'} \equiv 0, \quad \Gamma' \neq \Gamma.$$

(1.22) can be briefed as follows

$$\left\langle \hat{\mathbf{q}}_h \cdot \mathbf{n}, m_\Gamma^i \right\rangle_{\Gamma^{(L)}+\Gamma^{(R)}} = 0 \qquad \forall\Gamma \in \partial\mathcal{T}_h \quad i = 1, 2, \cdots \mathcal{N}_{bp} \tag{1.23}$$



According to (1.12), the $(\mathbb{Q}^T, \mathbb{U}^T)^T$ can be expressed with $\mathbf{\Lambda}$, so it lead to a linear system of equations, Denote by

$$\mathbb{K}\hat{\mathbf{U}} = \mathbf{L}$$

where $\hat{\mathbf{U}}$ is the unknowns consist of $\mathbf{\Lambda}$ from every element.Suppose $\mathbf{\Lambda}_i$ is one of the $\mathbf{\Lambda}$ derived from i-th element, We have $\hat{\mathbf{U}} = [\mathbf{\Lambda}_1^T, \mathbf{\Lambda}_2^T, \cdots \mathbf{\Lambda}_{\mathcal{N}}^T]^T$.

1.4. **assemble $\mathbb{K}$.** On elements K, We known (1.21) origin from

$$\left\langle \hat{\mathbf{q}}_h \cdot \mathbf{n}, m_\Gamma^i \right\rangle_{\partial K}, \qquad \Gamma \in \partial K, \quad i = 1, 2, \cdots \mathscr{N}_{bp}$$

i.e.

$$\sum_{\Gamma \in \partial K} \left\langle \hat{\mathbf{q}}_h \cdot \mathbf{n}, m_\Gamma^i \right\rangle_\Gamma := \mathbb{X}\mathbf{\Lambda} - \mathbf{Y}, \qquad \Gamma \in \partial K, \quad i = 1, 2, \cdots \mathscr{N}_{bp}$$

We can circled with each element, lead to relevant algorithm

for $e = 0 : \mathscr{N} - 1$
    for $i = 0 : \mathscr{N}_b - 1$
        for $k = 0 : \mathscr{N}_{bp} - 1$
            $I = \tau(e, i, k)$
            for $j = 0 : \mathscr{N}_b - 1$
                for $h = 0 : \mathscr{N}_{bp} - 1$
                    $J = \tau(e, j, h)$
                    $\mathbb{K}[I][J] += \mathbb{X}[k + i\mathscr{N}_{bp}][h + j\mathscr{N}_{bp}]$
                end
            end
        end
    end
end

Where $\mathscr{N}$ is the numbers of all elements, $\mathscr{N}_b$ is the numbers of boundarys of each element ,as triangle element $\mathscr{N}_b = 3$ and rectangle element $\mathscr{N}_b = 4$, $\mathscr{N}_{bp}$ is the numbers of points in boundary of element. $\mathbb{K}$ is left coefficient matrix of (1.23). τ is the mapping from local boundary index to global boundary index.

1.5. **assemble $\mathbf{L}$.** Similarly with assemble $\mathbb{K}$, We can circled with each element, lead to relevant algorithm

for $e = 0 : \mathscr{N} - 1$
    for $i = 0 : \mathscr{N}_b - 1$
        for $k = 0 : \mathscr{N}_{bp} - 1$
            $I = \tau(e, i, k)$
            $\mathbf{L}[I] += \mathbf{Y}[k + i\mathscr{N}_{bp}]$
        end
    end
end