MIXED TRIANGULAR SPECTRAL ELEMENT METHOD FOR ELLIPTIC PROBLEM

1. 2-D problem

1.1. Model problem and mixed element formulation.

$$
\begin{cases} -\nabla \cdot (\beta \nabla u(\boldsymbol{x})) + \gamma u = f(\boldsymbol{x}) & \boldsymbol{x} \in \Omega \\ u = g & on \quad \partial\Omega \end{cases} \tag{1.1}
$$

where $\beta(x, y)$ is a given function. For the sake of the definition of the mixed spectral elment scheme, we rewrite (1.1) into a first order system

$$
\begin{aligned} \beta^{-1}\boldsymbol{q} - \nabla u = 0 \quad & \forall \boldsymbol{x} \in \Omega \\ -\nabla \cdot \boldsymbol{q} + \gamma u = f(\boldsymbol{x}) \quad & \forall \boldsymbol{x} \in \Omega \\ u = g \quad & on \quad \partial\Omega \end{aligned} \tag{1.2}
$$

The variational formulation is difined as: Find $(\boldsymbol{q}, u) \in (H^1(\Omega))^2 \times H^1(\Omega)$ s.t.

$$
\begin{aligned} (\beta^{-1}\boldsymbol{q},\ \boldsymbol{v})_\Omega &= (\nabla u,\ \boldsymbol{v})_\Omega \\ (\boldsymbol{q},\ \nabla w)_\Omega + (\gamma u,\ w)_\Omega &= (f,\ w)_\Omega \end{aligned} \tag{1.3}
$$

Define

$$
\boldsymbol{W}_N = \{v \in H^1(\Omega), v|_{\Omega_e} \in \mathcal{P}_N(\Omega_e)\}, \quad \boldsymbol{V}_N = (\boldsymbol{W}_N)^2 \tag{1.4}
$$

then the mixed spectral element formulation is: find $\boldsymbol{q}_N \in \boldsymbol{V}_N$, $u_N \in \boldsymbol{W}_N$, s.t.

$$
\begin{aligned} (\beta^{-1}\boldsymbol{q}_N,\ \boldsymbol{v}_N)_\Omega &= (\nabla u_N,\ \boldsymbol{v}_N)_\Omega \\ (\boldsymbol{q}_N,\ \nabla w_N)_\Omega + (\gamma u_N,\ w_N)_\Omega &= (f,\ w_N)_\Omega \end{aligned} \tag{1.5}
$$

for all $\boldsymbol{v}_N \in \boldsymbol{V}_N$ and $w_N \in \boldsymbol{W}_N$.

1.2. Assembling linear system.

Denote by $\Phi_i\big|_{i=1}^{d_q}$, $\phi_i\big|_{i=1}^{d_u}$ are basis of spaces $\boldsymbol{V}_N$, $\mathcal{W}_N$. Then $\boldsymbol{q}_N$, u_N have expressions

$$
\boldsymbol{q}_N = \sum q_j \Phi_j, \quad u_N = \sum u_j \varphi_j.
$$

where Φ_j has the form

$$
\Phi_1^{(m)} = \begin{pmatrix} \varphi_1^{(m)} \\ 0 \end{pmatrix}, \cdots, \Phi_N^{(m)} = \begin{pmatrix} \varphi_N^{(m)} \\ 0 \end{pmatrix}, \quad \Phi_{N+1}^{(m)} = \begin{pmatrix} 0 \\ \varphi_1^{(m)} \end{pmatrix}, \quad \Phi_{2N}^{(m)} = \begin{pmatrix} 0 \\ \varphi_N^{(m)} \end{pmatrix}. \tag{1.6}
$$

Substitute these expressions into (1.5) and set $v_N = \Phi_i$ and $w_N = \phi_i$ we have

$$
\begin{aligned} \sum q_j \left(\beta^{-1}\Phi_j, \Phi_i\right)_\Omega &= \sum u_j \left(\nabla\varphi_j, \Phi_i\right)_\Omega \\ \sum q_j (\Phi_j, \nabla\varphi_i)_\Omega + \sum u_j \left(\gamma\varphi_j, \varphi_i\right)_\Omega &= (f, \varphi_i)_\Omega \end{aligned}
$$

Denoted by

$$
\mathbb{M} = \left[(\beta^{-1}\varphi_i, \varphi_j)_\Omega\right], \quad \mathbb{N} = \left[(\gamma\varphi_i, \varphi_j)_\Omega\right], \quad \mathbf{F} = \left[(f, \varphi_i)_\Omega\right]
$$
$$
\mathbb{C}_x = \left[(\varphi_i, \partial_x\varphi_j)_\Omega\right], \quad \mathbb{C}_y = \left[(\varphi_i, \partial_y\varphi_j)_\Omega\right]
$$
$$
\mathbf{Q}_x = [q_1, q_2, \cdots, q_N]^T, \quad \mathbf{Q}_y = [q_{N+1}, q_{N+2}, \cdots, q_{2N}]^T, \quad \mathbf{U} = [u_1, u_2, \cdots, u_N]^T
$$

then the formulation (1.5) equivalent to the following linear systems

$$\begin{aligned} \mathbb{M}\mathbf{Q}_x &= \mathbb{C}_x\mathbf{U} \\ \mathbb{M}\mathbf{Q}_y &= \mathbb{C}_y\mathbf{U} \\ \mathbb{C}_x^T\mathbf{Q}_x + \mathbb{C}_y^T\mathbf{Q}_y + \mathbb{N} &= \mathbf{F} \end{aligned} \tag{1.7}$$

or

$$\begin{pmatrix} -\mathbb{M} & \mathbf{0} & \mathbb{C}_x \\ \mathbf{0} & -\mathbb{M} & \mathbb{C}_y \\ \mathbb{C}_x^T & \mathbb{C}_y^T & \mathbb{N} \end{pmatrix} \begin{pmatrix} \mathbf{Q}_x \\ \mathbf{Q}_y \\ \mathbf{U} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{F} \end{pmatrix}$$

or the following form

$$\Big(\mathbb{C}_x^T\mathbb{M}^{-1}\mathbb{C}_x + \mathbb{C}_y^T\mathbb{M}^{-1}\mathbb{C}_y + \mathbb{N}\Big)\mathbf{U} = \mathbf{F}$$

2. Numerical Test:

Let $\gamma(x) = 1.0,\ \beta = e^{x+y},\ \Omega = [0,1]^2$, and choose exact smooth solution

$$U(x) = \cos(\pi r^2), \quad r = \sqrt{(x^2 + y^2)} \tag{2.8}$$

Then

$$f = \Big(2\pi(x + y + 2)\sin(\pi r^2) + 4\pi^2 r^2 \cos(\pi r^2)\Big)e^{x+y} + \cos(\pi r^2) \tag{2.9}$$

and the other one exact nonsmooth solution

$$U(x) = r^5, \quad r = \sqrt{(x + y)} \tag{2.10}$$

Then

$$f = -\Big(5r^3 + \frac{15}{2}r\Big)e^{x+y} + r^5 \tag{2.11}$$