

Risk-Aware Sparse Predictive Control

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Model Predictive Control: Receding Horizon Scheme

- MPC seems like playing chess and planning moves ahead



- Shift the motivation idea from chess principle to “control theory”

- Finite rolling horizon control (RHC)
- Implement the first action only
- Real-time (Online) optimization

♣ MPC = repeated open-loop control

[Rawlings & Mayne & Diehl (2017)]; picture from wiki

Uncertain Discrete Linear Time-invariant (LTI) System

Consider an uncertain discrete LTI system

$$x_{t+1} = A(\delta)x_t + B(\delta)u_t + Ew_t, \quad x_0 \neq 0$$

- state $x_t \in \mathbb{R}^n$, control input $u_t \in \mathbb{R}^m$, disturbances $w_t \in \mathcal{W} \subseteq \mathbb{R}^{n_w}$
- matrices $A(\delta) \in \mathbb{R}^{n \times n}$, $B(\delta) \in \mathbb{R}^{n \times m}$ w.r.t. model uncertainty $\delta \in \Delta$

– Mild Assumptions

- The pair $(A(\delta), B(\delta))$ is stabilizable for any $\delta \in \Delta \subseteq \mathbb{R}^{n_\delta}$.
- The uncertainty sets Δ and \mathcal{W} are bounded.
- The uncertainties δ and w_t are i.i.d., randomly extracting from the probability \mathbb{P}_δ on Δ (resp. \mathbb{P}_w on \mathcal{W}).

Predictive System: Modeling and Constraints

Prediction System Model

Given the state x_t observed at time t , the predicted state is modeled as

$$x_{j+1|t} = A(\delta)x_{j|t} + B(\delta)u_{j|t} + Ew_{j|t}, \quad x_{0|t} \doteq x_t, \quad j = 0, 1, \dots, N-1$$

- subscript $\bullet_{j|t}$ is predictive instants at sampling time t

e.g., $x_{j|t}$ denotes the j th step forward prediction of the state at time t .

– “Soft State + Hard Input” Constraints

$$\mathbb{P} \left\{ \theta \in \Theta : Cx_{j+1|t} \leq c, \quad j = 0, 1, \dots, N-1 \right\} \geq 1 - \epsilon, \quad Du_{j|t} \leq d$$

- matrix C w.r.t. vector c is with appropriate size (resp. D w.r.t. d)
- a desired level of accuracy or risk $\epsilon \in (0, 1)$
- random variable $\theta \doteq (\delta, \bar{w})$, $\theta \in \Theta \doteq \Delta \times \mathcal{W}^N$, $\Theta \stackrel{\text{i.i.d.}}{\sim} \mathbb{P} \doteq \mathbb{P}_\delta \times \mathbb{P}_w^N$

Sparse Predictive Control: Minimum Control Effort

- MOTIVATION: Minimum pieces for a checkmate in chess.

Goal of Predictive Control

Find a “risk-aware” control sequence that steers the state from the initial state $x_{0|j}$ towards a prescribed terminal set causing a joint soft constraint

$$\mathbb{P} \{h(\bar{u}, \theta) \leq 0\} \doteq \mathbb{P} \left\{ C_f x_{N|t} \leq c_f, C x_{j+1|t} \leq c, j = 0, 1, \dots, N-1 \right\} \geq 1 - \epsilon,$$

QUESTION: Minimum control effort in control systems.

- Sparsity-promoting for predictive control (SPC)*

$$J(\bar{u}) = \sum_{j=0}^{N-1} \|u_{j|t}\|_1 = \|\bar{u}\|_1, \xrightarrow{\text{RHC}} u_t \doteq u_{0|t} = \underbrace{\begin{bmatrix} I_m & 0_{m \times (N-1)} \end{bmatrix}}_F \overbrace{\begin{bmatrix} u_{0|t} \\ \vdots \\ u_{N-1|t} \end{bmatrix}}^{\bar{u}}$$

[Nagahara & Østergaard & Quevedo (2016)]

Risk-Aware Sparse Predictive Control

Chance-Constrained Sparse Optimization Problem (CCSP)

Solving a risk-aware sparse predictive control for uncertain discrete LTI system amounts to a chance-constrained sparse optimization problem

$$\min_{\bar{x}, \bar{u}} \quad \|\bar{u}\|_1$$

$$\text{s.t.} \quad x_{j+1|t} = A(\delta)x_{j|t} + B(\delta)u_{j|t} + Ew_{j|t},$$

$$x_{0|t} = x_t, \quad j = 0, 1, \dots, N-1,$$

$$\mathbb{P}\{C_f x_{N|t} \leq c_f, \quad Cx_{j+1|t} \leq c, \quad j = 0, 1, \dots, N-1\} \geq 1 - \epsilon,$$

$$Du_{j|t} \leq d, \quad j = 0, 1, \dots, N-1,$$

- Pros: Problem is well-defined at hand.
- Cons: Risk-aware (i.e., CCSP) solution \bar{u}_ϵ^* is hard to calculate exactly.
- **Oracle**: Data-driven sampling/scenario approach.

[Tempo & Calafiore & Dabbene (2013); Campi & Garatti (2018)]

Data-Driven Sparse Predictive Control

★ Data-driven sampling: generate scenarios $\{\theta^{(1)}, \theta^{(2)}, \dots, \theta^{(K)}\} \stackrel{\text{i.i.d.}}{\sim} \mathbb{P}$

Random Convex Program (RCP)

Using data-driven sampling, a risk-aware SPC reduces to data-driven SPC, which is a random convex program (resp. CCSP)

$$\begin{aligned} \min_{\bar{x}, \bar{u}} \quad & \|\bar{u}\|_1 \\ \text{s.t.} \quad & x_{j+1|t}^{(i)} = A(\delta^{(i)})x_{j|t}^{(i)} + B(\delta^{(i)})u_{j|t} + Ew_{j|t}^{(i)}, \\ & x_{0|t}^{(i)} = x_t, \quad j = 0, 1, \dots, N-1, i = 1, 2, \dots, K, \\ & C_f x_{N|t}^{(i)} \leq c_f, \quad C x_{j|t}^{(i)} \leq c, \quad j = 0, 1, \dots, N-1, i = 1, 2, \dots, K, \\ & D u_{j|t} \leq d, \quad j = 0, 1, \dots, N-1, \end{aligned}$$



Is data-driven solution \bar{u}_K^* a good approximation for solution \bar{u}_ϵ^* ?

How about sample complexity K ?

Main Result

Theorem (Probabilistic Robustness Guarantee)

Given a convex uncertain function $h(\bar{u}, \theta)$. Let $\Theta^K \doteq \{\theta^{(1)}, \theta^{(2)}, \dots, \theta^{(K)}\}$ be a multi-sample of θ collected from probability \mathbb{P} , where K satisfies

$$\sum_{i=0}^{mN-1} \binom{K}{i} \epsilon^i (1 - \epsilon)^{K-i} \leq \beta$$

for specified risk $\epsilon \in (0, 1)$ and confidence $\beta \in (0, 1)$. For RCP, we have

$$\begin{aligned} \bar{u}_K^* = \arg \min_{\bar{u}} \quad & \|\bar{u}\|_1 \\ \text{s.t.} \quad & h(\bar{u}, \theta^{(i)}) \leq 0, \quad i = 1, 2, \dots, K, \\ & Du_{j|t} \leq d, \quad j = 0, 1, \dots, N-1. \end{aligned}$$

Then, with confidence $1 - \beta$, a probabilistic robustness guarantee

$$\mathbb{P}^K \{h(\bar{u}_K^*, \theta) \leq 0\} \geq 1 - \epsilon$$

holds true for obtained optimal input \bar{u}_K^* .

Finite Sample Complexity

- Sample complexity K can be defined as [Alamo & Tempo & Luque (2010)]

$$K \geq \frac{mN - 1 + \ln(1/\beta) + \sqrt{2(mN - 1) \ln(1/\beta)}}{\epsilon} \doteq \mathcal{K}(mN, \epsilon, \beta).$$

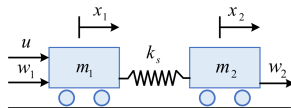
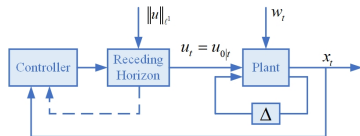
Corollary (Finite Receding Horizon Risk-Aware SPC)

Let $K \geq \mathcal{K}(mN, \epsilon, \beta)$ and \bar{u}_K^* be optimal control for RCP. If $u_{j|t}^*$ is applied to the uncertain LTI dynamics with a finite prediction horizon N , then, w.p. $1 - \beta$, the risk-aware SPC is achieved for $j = 0, 1, \dots, N - 1$.

♣ Role of Probability \mathbb{P} , confidence β , and risk ϵ :

- Probability \mathbb{P} is distribution-free.
- Confidence β should be as small as possible.
- As scenarios $K \rightarrow \infty$, the risk $\epsilon \rightarrow 0$.

Numerical Experiments

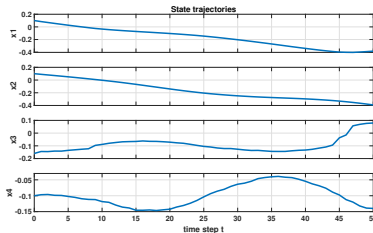
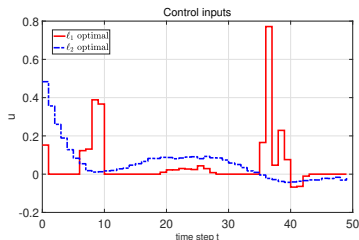


Consider a discretized uncertain two-mass-spring system modeled as

$$A(\delta) = \begin{bmatrix} 1 & 0 & t_s & 0 \\ 0 & 1 & 0 & t_s \\ -\frac{k_s t_s}{m_1} & \frac{k_s t_s}{m_1} & 1 & 0 \\ \frac{k_s t_s}{m_2} & -\frac{k_s t_s}{m_2} & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{t_s}{m_1} \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{t_s}{m_1} & 0 \\ 0 & \frac{t_s}{m_2} \end{bmatrix}, \quad x_t = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad w_t = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

- masses: $m_1 = m_2 = 1$, $t_s = 0.1$. [Kothare et.al (1996)]
- disturbances: $w \sim \mathcal{N}(0, \Sigma_w)$ with covariance $\Sigma_w = \text{diag}(0.02^2, 0.02^2)$.
- parametric uncertainty: $k_s \sim \text{Unif}([0.5, 2.0])$.
- initial state $x_0^\top = [0.15 \quad 0.15 \quad -0.15 \quad -0.1]$, $N = 6$, and $t = 50$.
- risk $\epsilon = 0.05$, confidence $\beta = 10^{-6}$, and scenarios $K = 695$.

Numerical Experiments



- Constraints: $|x_3| \leq 0.15$, $|x_4| \leq 0.15$, and $|u| \leq 1$.
- SPC promotes more zero inputs than quadratic MPC.
- State trajectories are near to the prescribed terminal set.
- Data-driven SPC enjoys good robustness and sparsity.

Conclusions

So far...

"A data-driven sampling approach for risk-aware sparse predictive control for uncertain discrete LTI systems."

The Take Home Messages

- Risk-aware solution \bar{u}_ϵ^* is approximated by data-driven solution \bar{u}_K^* with a high probability.
- Provide probabilistic robustness and sample complexity guarantees.

Outlook

- Recursive feasibility and stability for risk-aware SPC.
- Model-free framework for sparse predictive control.

Thank You for Your Attention !

Suggestions & Comments are Welcome !