Data-Driven Sparse Feedback Control via System Level Synthesis

D2 Zhicheng Zhang

IPS, Fujisaki Lab

Sep 27, 2022

Control System

System Model

Consider an uncertain discrete-time linear time invariant system

$$x(t+1) = Ax(t) + Bu(t) + w(t), \; t = 0, 1, \cdots, T-1,$$

- Here $x \in \mathbb{X} \subseteq \mathbb{R}^n$, $u \in \mathbb{U} \subseteq \mathbb{R}^m$, and $w \in \mathbb{W} \subseteq \mathbb{R}^n$
  - state and input constraints

    e,g., polytopic, $\mathbb{X} = \{Z_x x \leq 1\}$, $\mathbb{U} = \{Z_u u \leq 1\}$
  - e.g., noise $w \sim \mathrm{N}(0, \sigma_w^2 I)$, $\mathbb{E}[w(t)] = 0, \mathbb{E}[w(t)^\top w(t)] = \sigma_w^2 I$
- True system dynamics (A, B) are often unknown
  - Mild assumption: the pair (A, B) is controllable
  - While it can be observed from a finite number of empirical data.
  - Indirect (Identify-then-control) Vs. Direct (Data-driven control)

Control Objective

Sparse Optimal Control

Consider the control objective is sparse control, which is essentially a *sparse optimization* by solving a relaxed ℓ_1 norm program

$$\begin{aligned} \min_{u} \quad & \mathcal{J}(u) = c(u) + \alpha \|u\|_1 \\ \text{s.t.} \quad & x(t+1) = Ax(t) + Bu(t) + w(t) \end{aligned}$$

- $c(u)$ is a convex performance function,
  - e.g., quadratic function, LQR ($x^\top Qx + u^\top Ru$) or LQG
- A rich stories for related *model-based* and *open-loop* problems
  - optimality (dynamic program), tractable algorithms
- Ques: Is it possible to solve *model-free* and *closed-loop* setting ?
  - i.e., pair (A, B) is unknown, and set $u(t) = Fx(t)$

Model Free Dynamics: Sufficient PE Data

- We seek a data-based representation of the model-free dynamics (open or closed-loop) enabled by persistently exciting input.

Persistence of excitation (PE) Data

The signal $u : [0, T-1] \to \mathbb{R}^m$, that is,

$$u(0), u(1), \cdots, u(T-1)$$

is *persistently exciting* of order L if the Hankel matrix associated with

$$\mathscr{H}_L(u_{[0,T-1]}) = U_{0,L,T-L+1} = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-L) \\ u(1) & u(2) & \cdots & u(T-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ u(L-1) & u(L) & \cdots & u(T-1) \end{bmatrix}$$

has full (row) rank mL.

- PE requires sufficiently long input sequences: $T \geq (m+1)L - 1$

Willems' et al Fundamental Lemma

- A PE condition for a controllable system generating data that are sufficiently independent over time

Willems' et al Fundamental Lemma (Rank Condition)

For a controllable nominal system (noise-free, $w(t) = 0, \forall t \geq 0$)

$$x(t+1) = Ax(t) + Bu(t),$$

if the control action $u_{[0,T-1]}$ is PE of order $n + L$, then rank condition

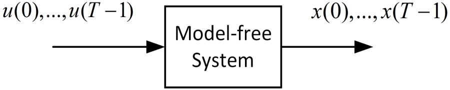
$$\text{rank} \begin{bmatrix} \mathscr{H}_L(u_{[0,T-1]}) \\ \mathscr{H}_1(x_{[0,T-1]}) \end{bmatrix} = n + Lm \quad \text{holds.}$$

$$\mathscr{H}_L(u_{[0,T-1]}) = U_{0,L,T-L+1} = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-L) \\ u(1) & u(2) & \cdots & u(T-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ u(L-1) & u(L) & \cdots & u(T-1) \end{bmatrix}$$

$$\mathscr{H}_1(x_{[0,T-1]}) = X_{0,T-L+1} = [x(0) \ x(1) \ \cdots \ x(T-L)]$$

J.C. Willems, P. Rapisarda, I. Markovsky, B.L. De Moor. A note on persistency of excitation. Syst & Cont. Lett, 2005

Deep implication for Control



Lemma (Necessary and Sufficient Condition)

(i) if $u_{[0,T-1]}$ is PE of order $n+L$, then any L-long input/state trajectory of system can be expressed as

$$\begin{bmatrix} u_{[0,L-1]} \\ x_{[0,L-1]} \end{bmatrix} = \begin{bmatrix} U_{0,L,T-L+1} \\ X_{0,L,T-L+1} \end{bmatrix} g = \begin{bmatrix} \mathscr{H}_L(u_{[0,T-1]}) \\ \mathscr{H}_L(x_{[0,T-1]}) \end{bmatrix} g, \quad g \in \mathbb{R}^{T-L+1}$$

(ii) Any linear combination of the columns of the matrix of data, i.e.,

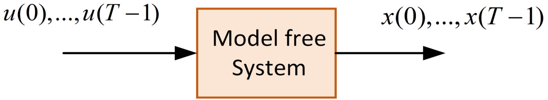
$$\begin{bmatrix} \mathscr{H}_L(u_{[0,T-1]}) \\ \mathscr{H}_L(x_{[0,T-1]}) \end{bmatrix} g,$$

is a L-long input-state trajectory of the system.

- This Lemma result can be applied to input-output trajectory

I. Markovsky and P. Rapisarda. Data-driven simulation and control, Int. J. Contr., 2008.

Data-based System Representation



$$\begin{bmatrix} U_0 \\ X_0 \end{bmatrix} = \begin{bmatrix} u(0) & \cdots & u(T-L) \\ \vdots & \ddots & \vdots \\ u(L-1) & \cdots & u(T-1) \\ x(0) & \cdots & x(T-L) \\ \vdots & \ddots & \vdots \\ x(L-1) & \cdots & x(T-1) \end{bmatrix}$$

$$\begin{bmatrix} u(t) \\ \vdots \\ u(t+L-1) \\ x(t) \\ \vdots \\ x(t+L-1) \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} g(t)$$

- A finite collection input-state data can construct system dynamics
- Both for open-loop and closed-loop system can be written as data-based system representation

System Level Synthesis Framework

SLS framework

- Principle: it shifts (structured, distributed) controller synthesis task from a controller design to "*the entire closed-loop system responses*"
- Implementation: employ a state feedback $u = Fx$ on dynamics

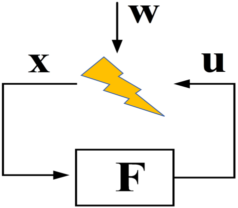
$$x(t+1) = (A+BF)x(t) + w(t), \quad t = 0, 1, \cdots, T-1$$

- Mappings: use maps $\Psi_x : w \to x$ and $\Psi_u : w \to u$ to describe the evolution of state and input as follows

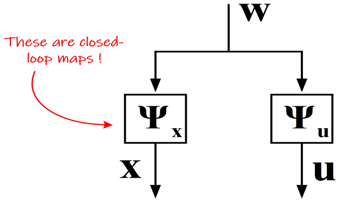
$$\begin{bmatrix} x(k) \\ u(k) \end{bmatrix} = \sum_{t=1}^{k} \begin{bmatrix} \Psi_x(k-t+1) \\ \Psi_u(k-t+1) \end{bmatrix} w(t-1).$$

- Learning from noisy data: closed-loop responses $\{\Psi_x(k), \Psi_u(k)\}$ maps from the external disturbance $\{w(0), \cdots, w(T-1)\}$ to the state $x(k)$ and control action $u(k)$ at time instant k, respectively.

SLS Parameterization



$$\mathbf{x} = \mathcal{Z}(\mathcal{A}\mathbf{x} + \mathcal{B}\mathbf{u}) + \mathbf{w}$$
$$\mathbf{u} = \mathbf{F}\mathbf{x}$$



SLS parameterization

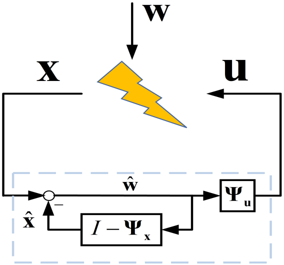
$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Psi}_x \\ \boldsymbol{\Psi}_u \end{bmatrix} \mathbf{w}$$

- Collect time series $t = 0, 1, \cdots, T-1$ and define $\mathbf{w} = w_{[-1,T-2]}$

$$\mathbf{x} = x_{[0,T-1]} = (I - \mathcal{Z}(\mathcal{A} + \mathcal{B}\mathbf{F}))^{-1} w_{[-1,T-2]} = \boldsymbol{\Psi}_x \mathbf{w},$$
$$\mathbf{u} = u_{[0,T-1]} = \mathbf{F}(I - \mathcal{Z}(\mathcal{A} + \mathcal{B}\mathbf{F}))^{-1} w_{[-1,T-2]} = \boldsymbol{\Psi}_u \mathbf{w}.$$

  - noiseless: $w(-1) = x_0$ and $w(t) = 0, \forall t \geq 0$
    noise: $w(-1) = x_0$ and for $t \geq 0$, $w(t)$ is process noise

SLS Controller Implementation



$$\mathbf{F} = \mathbf{\Psi}_{\mathrm{u}} \mathbf{\Psi}_{\mathrm{x}}^{-1}$$

SLS controller Implementation

$$\mathbf{u} = \mathbf{\Psi}_{\mathrm{u}} \hat{\mathbf{w}}$$
$$\hat{\mathbf{x}} = (I - \mathbf{\Psi}_{\mathrm{x}}) \hat{\mathbf{w}}$$
$$\hat{\mathbf{w}} = \mathbf{x} - \hat{\mathbf{x}}$$

$$\mathbf{\Psi} = \begin{bmatrix} \Psi^{0,0} & & & \\ \Psi^{1,1} & \Psi^{1,0} & & \\ \vdots & \ddots & \ddots & \\ \Psi^{T-1,T-1} & \cdots & \Psi^{T-1,1} & \Psi^{T-1,0} \end{bmatrix} \in \mathcal{L}^{T,p\times q}$$

Proposition 1 (Noiseless) [Anderson & Doyle & Low & Matni, 19]

1. For $\mathbf{\Psi}_x \in \mathcal{L}^{T,n\times n}$, $\mathbf{\Psi}_u \in \mathcal{L}^{T,m\times n}$, the affine subspace is defined by

$$[I - Z\mathcal{A} \quad -Z\mathcal{B}] \begin{bmatrix} \mathbf{\Psi}_x \\ \mathbf{\Psi}_u \end{bmatrix} = I, \quad \text{(achievability constraint)}$$

parameterizes all possible system responses from $\mathbf{w}$ to $(\mathbf{x}, \mathbf{u})$.

2. the controller $\mathbf{F} = \mathbf{\Psi}_u \mathbf{\Psi}_x^{-1}$ achieves the desired response.

Sparsity promoting LQR

- LQR with ℓ^1 norm regularization (Vector form)

$$\min_{\mathbf{x},\mathbf{u}} \quad \left\| \begin{bmatrix} \mathcal{Q}^{\frac{1}{2}} & 0 \\ 0 & \mathcal{R}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right\|_F^2 + \alpha \|\mathbf{u}\|_1$$

$$\text{s.t.} \quad \mathbf{x} = \mathcal{Z}\mathcal{A}\mathbf{x} + \mathcal{Z}\mathcal{B}\mathbf{u} + \mathbf{w}, \quad \mathbf{x} \in \mathbb{X}, \quad \mathbf{u} \in \mathbb{U}, \quad \forall \mathbf{w} \in \mathbb{W}$$

Sparse LQR via SLS

Observe that $\mathbf{x} = \mathbf{\Psi}_x \mathbf{w} = \mathbf{\Psi}_x(:,0)x_0$ and $\mathbf{u} = \mathbf{\Psi}_u \mathbf{w} = \mathbf{\Psi}_u(:,0)x_0$, where $\mathbf{\Psi}(:,0)$ denotes the first block column of the matrix $\mathbf{\Psi} \in \mathcal{L}^{T,n\times n}$, then

$$\min_{\mathbf{\Psi}_x,\mathbf{\Psi}_u} \quad \left\| \begin{bmatrix} \mathcal{Q}^{\frac{1}{2}} & 0 \\ 0 & \mathcal{R}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbf{\Psi}_x(:,0) \\ \mathbf{\Psi}_u(:,0) \end{bmatrix} x_0 \right\|_F^2 + \alpha \|\mathbf{\Psi}_u(:,0)x_0\|_1$$

$$\text{s.t.} \quad [I - Z\mathcal{A} \quad -Z\mathcal{B}] \begin{bmatrix} \mathbf{\Psi}_x(:,0) \\ \mathbf{\Psi}_u(:,0) \end{bmatrix} = I(:,0)$$

$$\mathbf{\Psi}_x(:,0)x_0 \in \mathbb{X}, \ \mathbf{\Psi}_u(:,0)x_0 \in \mathbb{U}, \ \forall \mathbf{w} \in \mathbb{W}$$

Data-Driven SLS Representation

Theorem (Equivalence)

Assume that nominal system satisfies Willems' fundamental Lemma, then the feasible solutions to achievability constraint defined over a time horizon $t = 0, 1, \cdots, T-1$ can be equivalent to a SLS formulation, that is,

$$\left\{ \begin{bmatrix} \mathscr{H}_L(x_{[0,T-1]}) \\ \mathscr{H}_L(u_{[0,T-1]}) \end{bmatrix} \mathcal{G},\ \forall \mathcal{G} \in G(x) \right\}$$
$$= \left\{ [I - Z\mathcal{A} \quad - Z\mathcal{B}] \begin{bmatrix} \boldsymbol{\Psi}_x(:,0) \\ \boldsymbol{\Psi}_u(:,0) \end{bmatrix} = I(:,0) \right\}$$

for all $\mathcal{G} \in G(x) := \{\mathcal{G} : \mathscr{H}_1(x_{[0,T-1]})\mathcal{G} = \boldsymbol{\Psi}_x(0,0) = I\}$.

- Data-driven sparse LQR problem via SLS

$$\min_{\mathcal{G} \in G(x)} \quad \left\| \begin{bmatrix} \mathcal{Q}^{\frac{1}{2}} & 0 \\ 0 & \mathcal{R}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathscr{H}_L(x_{[0,T-1]}) \\ \mathscr{H}_L(u_{[0,T-1]}) \end{bmatrix} \mathcal{G} x_0 \right\|_F^2 + \alpha \| \mathscr{H}_L(u_{[0,T-1]}) \mathcal{G} x_0 \|_1$$
$$\text{s.t.} \quad \mathscr{H}_L(x_{[0,T-1]})\mathcal{G}x_0 \in \mathbb{X},\ \mathscr{H}_L(u_{[0,T-1]})\mathcal{G}x_0 \in \mathbb{U}$$

Robust System Level Synthesis

Proposition 2 (Robustness) [Anderson & Doyle & Low & Matni, 19]

1. For $\mathbf{\Psi}_x \in \mathcal{L}^{T,n\times n}, \mathbf{\Psi}_u \in \mathcal{L}^{T,m\times n}$, the affine subspace is defined by
$$[I - Z\mathcal{A} \quad -Z\mathcal{B}] \begin{bmatrix} \mathbf{\Psi}_x \\ \mathbf{\Psi}_u \end{bmatrix} = I + \mathbf{\Delta}, \quad (\text{achievability constraint})$$
parameterizes all possible system responses from $\mathbf{w}$ to $(\mathbf{x}, \mathbf{u})$.
2. the controller $\mathbf{F} = \mathbf{\Psi}_u \mathbf{\Psi}_x^{-1}$ achieves system response
$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Psi}_x \\ \mathbf{\Psi}_u \end{bmatrix} \underbrace{(I + \mathbf{\Delta})^{-1}\mathbf{w}}_{\hat{\mathbf{w}}(\mathbf{\Delta})}.$$

- Uncertainty term $\hat{\mathbf{w}}(\mathbf{\Delta}) := (I + \mathbf{\Delta})^{-1}\mathbf{w}$ contains model parametric uncertainty and external disturbances.
- State and control actions can learn from noise data since we have $\mathbf{x} = \mathbf{\Psi}_x \hat{\mathbf{w}}(\mathbf{\Delta})$ and $\mathbf{u} = \mathbf{\Psi}_u \hat{\mathbf{w}}(\mathbf{\Delta})$.

Sparse Robust LQR

We here focus on a sparse robust LQR problem, that is,

$$\begin{aligned}\min_{x,u} \max_{w,\Delta} \ & \mathcal{J}(u) \\ \text{s.t.} \quad & x(t+1) = Ax(t) + Bu(t) + w(t)\end{aligned}$$

This leads to a robust LQR with SLS as follows

$$\begin{aligned}\min_{\boldsymbol{\Psi}_x,\boldsymbol{\Psi}_u} \max_{\boldsymbol{\Delta}} \quad & \left\| \begin{bmatrix} \mathcal{Q}^{\frac{1}{2}} & 0 \\ 0 & \mathcal{R}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Psi}_x \\ \boldsymbol{\Psi}_u \end{bmatrix} \hat{\mathbf{w}}(\boldsymbol{\Delta}) \right\|_F^2 + \alpha \|\boldsymbol{\Psi}_u \hat{\mathbf{w}}(\boldsymbol{\Delta})\|_1 \\ \text{s.t.} \quad & [I - \mathcal{Z}\mathcal{A} \quad -\mathcal{Z}\mathcal{B}] \begin{bmatrix} \boldsymbol{\Psi}_x \\ \boldsymbol{\Psi}_u \end{bmatrix} = I + \boldsymbol{\Delta}, \\ \Leftrightarrow \quad & \begin{bmatrix} I - \mathcal{Z}\hat{\mathcal{A}} & -\mathcal{Z}\hat{\mathcal{B}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Psi}_x \\ \boldsymbol{\Psi}_u \end{bmatrix} = I,\end{aligned}$$

where $\hat{\mathcal{A}}, \hat{\mathcal{B}}$ are estimated dynamics.

- e.g., Model uncertainty: $\delta_A = \hat{\mathcal{A}} - \mathcal{A}$, and $\delta_B = \hat{\mathcal{B}} - \mathcal{B}$.

Data-driven Robust SLS

Data-based achievability constraint

$$
\begin{bmatrix} I - \mathcal{Z}\mathcal{A} & -\mathcal{Z}\mathcal{B} \end{bmatrix} \begin{bmatrix} \mathscr{H}_L(x_{[0,T-1]}) \\ \mathscr{H}_L(u_{[0,T-1]}) \end{bmatrix} = \begin{bmatrix} \mathscr{H}_1(x_{[0,T-1]}) \\ 0 \end{bmatrix} + \mathcal{Z}\mathscr{H}_L(w_{[-1,T-2]})
$$

$$
\Rightarrow \begin{bmatrix} I - \mathcal{Z}\mathcal{A} & -\mathcal{Z}\mathcal{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Psi}_x(:,0) \\ \boldsymbol{\Psi}_u(:,0) \end{bmatrix} = I(:,0) + \boldsymbol{\Delta}(:,0)
$$

for all $g \in G(x) := \{g : \mathscr{H}_1(x_{[0,T-1]})g = I\}$, $\boldsymbol{\Psi}_x(:,0) = \mathscr{H}_L(x_{[0,T-1]})g$, $\boldsymbol{\Psi}_u(:,0) = \mathscr{H}_L(u_{[0,T-1]})g$, and $\boldsymbol{\Delta}(:,0) = \mathcal{Z}\mathscr{H}_L(w_{[-1,T-2]})g$.

Theorem (Data-based robust SLS)

The data-based system response $\{\boldsymbol{\Psi}_x, \boldsymbol{\Psi}_u\}$ and $\boldsymbol{\Delta}$ can be defined as

$$
\begin{aligned}
\boldsymbol{\Psi}_x &= \mathcal{Z}_L(I_L \otimes \mathscr{H}_L(x_{[0,T-1]}))\mathcal{G} = \bar{\mathscr{H}}_L(x)\mathcal{G}, \;\; \mathcal{Z}_L = [I \;\; \mathcal{Z} \;\; \cdots \;\; \mathcal{Z}^{L-1}] \\
\boldsymbol{\Psi}_u &= \mathcal{Z}_L(I_L \otimes \mathscr{H}_L(u_{[0,T-1]}))\mathcal{G} = \bar{\mathscr{H}}_L(u)\mathcal{G}, \\
\boldsymbol{\Delta} &= \mathcal{Z}_L(I_L \otimes \mathcal{Z}\mathscr{H}_L(w_{[-1,T-2]}))\mathcal{G} = \bar{\mathscr{H}}_L(w)\mathcal{G},
\end{aligned}
$$

and satisfy achievability constraint in Proposition 2.

Data-Driven Robust Sparse LQR

- Data-based sparse LQR via Robust SLS

$$
\begin{aligned}
\min_{\mathcal{G}\in G(x)} \max_{\bar{\mathscr{H}}_L(w)} \quad & \left\| \begin{bmatrix} \mathcal{Q}^{\frac{1}{2}} & 0 \\ 0 & \mathcal{R}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \bar{\mathscr{H}}_L(x) \\ \bar{\mathscr{H}}_L(u) \end{bmatrix} \mathcal{G}(I + \bar{\mathscr{H}}_L(w)\mathcal{G})^{-1} \right\|_F^2 \\
& + \alpha \| \bar{\mathscr{H}}_L(u)\mathcal{G}(I + \bar{\mathscr{H}}_L(w)\mathcal{G})^{-1} \|_1 \\
\text{s.t.} \quad & \text{achievability constraint}
\end{aligned}
$$

- The data-based objective function seems like not easy.
- Translate it into a qusi-convex by evaluating its upper bound
- Structure assumption for uncertainty is necessary.
- Relax worst case uncertainty using probabilistic method

The Take Home Message

- Instead of reasoning about the feedback gain $\mathbf{F}$ it directly concerns about $\mathbf{\Psi}$, this benefits us to implement the sparse controller.