Risk-Aware Sparse Predictive Control

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Model Predictive Control: Receding Horizon Scheme

– MPC seems like playing chess and planning moves ahead

– Shift the motivation idea from chess principle to "control theory"

- Finite rolling horizon control (RHC)
- Take the first action only
- Real-time (Online) optimization

♣ MPC = repeated open-loop control [Rawlings & Mayne & Diehl (2017)]; picture from wiki

Control System Description

Uncertain Discrete Linear Time-invariant (LTI) System

Consider an uncertain discrete LTI system

$$x_{t+1} = A(\delta)x_t + B(\delta)u_t + Ew_t, \quad x_t \neq 0$$

- state $x_t \in \mathbb{R}^n$, control input $u_t \in \mathbb{R}^m$, disturbances $w_t \in \mathcal{W} \subseteq \mathbb{R}^{n_w}$
- matrices $A(\delta) \in \mathbb{R}^{n\times n}$, $B(\delta) \in \mathbb{R}^{n\times m}$ w.r.t. model uncertainty $\delta \in \Delta$

– **Mild Assumptions**

- The pair $(A(\delta), B(\delta))$ is stabilizable for any $\delta \in \Delta \subseteq \mathbb{R}^{n_\delta}$.
- The sets Δ and $\mathcal{W}$ are bounded.
- The uncertainties δ and w_t are i.i.d., randomly extracting from the probability $\mathbb{P}_\delta$ on Δ (resp. $\mathbb{P}_w$ on $\mathcal{W}$).

Predictive System: Modeling and Constraints

Prediction System Model

Given the state x_t observed at time t, the predicted state is modeled as

$$x_{j+1|t} = A(\delta)x_{j|t} + B(\delta)u_{j|t} + Ew_{j|t}, \quad x_{0|t} = x_t, \quad j = 0, 1, \cdots, N-1$$

- subscript $\bullet_{j|t}$ is predictive instants for state x (resp. u, w).

e.g., $x_{j|t}$ denotes the jth step forward prediction of the state at time t.

– **"Soft State + Hard Input" Constraints**

$$\boxed{\mathbb{P}\left\{\theta \in \Theta : Cx_{j+1|t} \leq c,\ j = 0, 1, \ldots, N-1\right\} \geq 1 - \epsilon}, \quad \boxed{Du_{j|t} \leq d}$$

- matrix C w.r.t. vector c is with appropriate size (resp. D w.r.t. d)
- a desired level of accuracy or risk $\epsilon \in (0, 1)$
- random variable $\theta \doteq (\delta, \bar{w}),\ \theta \in \Theta \doteq \Delta \times \mathcal{W}^N,\ \Theta \overset{\text{i.i.d.}}{\sim} \mathbb{P} \doteq \mathbb{P}_\delta \times \mathbb{P}_w^N$

Sparse Predictive Control: Minimum Control Effort

– MOTIVATION: Minimum pieces for a checkmate in chess.

Goal of Predictive Control

Find a "risk-aware" control sequence that steers the state from the initial state $x_{0|j}$ towards a prescribed terminal set causing a joint soft constraint

$$\mathbb{P}\left\{h(\bar{u},\theta) \leq 0\right\} \doteq \mathbb{P}\left\{C_f x_{N|t} \leq c_f,\ C x_{j+1|t} \leq c,\ j = 0, 1, \ldots, N-1\right\} \geq 1-\epsilon,$$

QUESTION: Minimum control effort in control system.

- *Sparsity-promoting for predictive control* (SPC)

$$J(\bar{u}) = \sum_{j=0}^{N-1} \|u_{j|t}\|_1 = \|\bar{u}\|_1, \overset{\text{RHC}}{\Longrightarrow} u_t \doteq \boxed{u_{0|t}} = \underbrace{\begin{bmatrix} I_m & 0_{m\times(N-1)} \end{bmatrix}}_{F} \overbrace{\begin{bmatrix} u_{0|t} \\ \vdots \\ u_{N-1|t} \end{bmatrix}}^{\bar{u}}$$

[Nagahara & Østergaard & Quevedo (2016)]

Risk-Aware Sparse Predictive Control

Chance-Constrained Sparse Optimization Problem (CCSP)

Solving a risk-aware sparse predictive control for uncertain discrete LTI system amounts to a chance-constrained sparse optimization problem

$$
\begin{aligned}
\min_{\bar{x},\bar{u}} \quad & \|\bar{u}\|_1 \\
\text{s.t.} \quad & x_{j+1|t} = A(\delta)x_{j|t} + B(\delta)u_{j|t} + Ew_{j|t}, \\
& x_{0|t} = x_t, \quad j = 0, 1, \ldots, N-1, \\
& \mathbb{P}\{C_f x_{N|t} \le c_f,\ Cx_{j+1|t} \le c,\ j = 0, 1, \ldots, N-1\} \ge 1 - \epsilon, \\
& Du_{j|t} \le d, \quad j = 0, 1, \ldots, N-1,
\end{aligned}
$$

- Pros: Problem is well-defined at hand.
- Cons: Risk-aware (i.e., CCSP) solution $\bar{u}_\epsilon^*$ is hard to calculate exactly.
- Oracle: Data-driven sampling/scenario approach.

[Tempo & Calafiore & Dabbene (2013); Campi & Garatti (2018)]

Data-Driven Sparse Predictive Control

★ Data-driven sampling: generate scenarios $\{\theta^{(1)}, \theta^{(2)}, \dots, \theta^{(K)}\} \overset{\text{i.i.d.}}{\sim} \mathbb{P}$

Random Convex Program (RCP)

Using data-driven sampling, a risk-aware SPC reduces to data-driven SPC, which is a random convex program (resp. CCSP)

$$
\begin{aligned}
\min_{\bar{x},\bar{u}} \quad & \|\bar{u}\|_1 \\
\text{s.t.} \quad & x^{(i)}_{j+1|t} = A(\delta^{(i)})x^{(i)}_{j|t} + B(\delta^{(i)})u_{j|t} + Ew^{(i)}_{j|t}, \\
& x^{(i)}_{0|t} = x_t, \quad j = 0, 1, \dots, N-1,\ i = 1, 2, \dots, K, \\
& C_f x^{(i)}_{N|t} \le c_f, \quad Cx^{(i)}_{j|t} \le c, \quad j = 0, 1, \dots, N-1,\ i = 1, 2, \dots, K, \\
& Du_{j|t} \le d, \quad j = 0, 1, \dots, N-1,
\end{aligned}
$$

- Q: Is data-driven solution $\bar{u}^*_K$ a good approximation for solution $\bar{u}^*_\epsilon$?
- Q: How about sample complexity K ?

Main Result

Theorem (Probabilistic Robustness Guarantee)

Given a convex uncertain function $h(\bar{u}, \theta)$. Let $\Theta^K \doteq \{\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(K)}\}$ be a multi-sample of θ collected from probability $\mathbb{P}$, where K satisfies

$$\sum_{i=0}^{mN-1} \binom{K}{i} \epsilon^i (1-\epsilon)^{K-i} \leq \beta$$

for specified risk $\epsilon \in (0,1)$ and confidence $\beta \in (0,1)$. For RCP, we have

$$\begin{aligned} \bar{u}_K^* = \arg\min_{\bar{u}} \quad & \|\bar{u}\|_1 \\ \text{s.t.} \quad & h(\bar{u}, \theta^{(i)}) \leq 0, \quad i = 1, 2, \ldots, K, \\ & Du_{j|t} \leq d, \quad j = 0, 1, \ldots, N-1. \end{aligned}$$

Then, with confidence $1-\beta$, a probabilistic robustness guarantee

$$\mathbb{P}^K\{h(\bar{u}_K^*, \theta) \leq 0\} \geq 1 - \epsilon$$

holds true for obtained optimal input $\bar{u}_K^*$.

Finite Sample Complexity

- Sample complexity K can be defined as [Alamo & Tempo & Luque (2010)]

$$K \geq \frac{mN - 1 + \ln(1/\beta) + \sqrt{2(mN-1)\ln(1/\beta)}}{\epsilon} \doteq \mathcal{K}(mN, \epsilon, \beta).$$

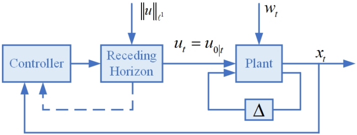
Corollary (Finite Receding Horizon Risk-Aware SPC)

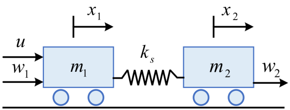
Let $K \geq \mathcal{K}(mN, \epsilon, \beta)$ and $\bar{u}_K^*$ be optimal control for RCP. If $u_{j|t}^*$ is applied to the uncertain LTI dynamics with a finite prediction horizon N, then, w.p. $1-\beta$, the risk-aware SPC is achieved for $j = 0, 1, \ldots, N-1$.

♣ Role of Probability $\mathbb{P}$, confidence β, and risk ϵ:

- Probability $\mathbb{P}$ is distribution-free.
- For practical method, confidence β should take small.
- As scenarios K tends to infinity, the risk ϵ tends to zero.

Numerical Experiments



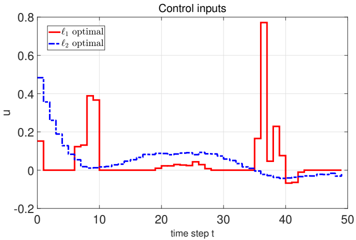


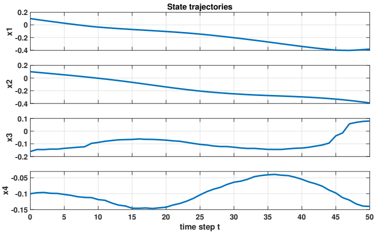
Consider a discretized uncertain two-mass-spring system modeled as

$$A(\delta)=\begin{bmatrix} 1 & 0 & t_s & 0 \\ 0 & 1 & 0 & t_s \\ -\frac{k_s t_s}{m_1} & \frac{k_s t_s}{m_1} & 1 & 0 \\ \frac{k_s t_s}{m_2} & -\frac{k_s t_s}{m_2} & 0 & 1 \end{bmatrix},\ B=\begin{bmatrix} 0 \\ 0 \\ \frac{t_s}{m_1} \\ 0 \end{bmatrix},\ E=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{t_s}{m_1} & 0 \\ 0 & \frac{t_s}{m_2} \end{bmatrix},\ x_t=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},\ w_t=\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

- masses: $m_1 = m_2 = 1,\ t_s = 0.1.$ [Kothare et.al (1996)]
- disturbances: $w \sim \mathrm{N}(0, \Sigma_w)$ with covariance $\Sigma_w = \mathrm{diag}(0.02^2, 0.02^2)$.
- parametric uncertainty: $k_s \sim \mathrm{Unif}([0.5, 2.0])$.
- initial state $x_0^\top = \begin{bmatrix} 0.15 & 0.15 & -0.15 & -0.1 \end{bmatrix}$, $N = 6$, and $t = 50$.
- risk $\epsilon = 0.05$, confidence $\beta = 10^{-6}$, and scenarios $K = 695$.

Numerical Experiments





- Constraints: $|x_3| \leq 0.15$, $|x_4| \leq 0.15$, and $|u| \leq 1$.
- SPC promotes more zero inputs than quadratic MPC.
- State trajectories are near to the prescribed terminal set.
- Data-driven SPC enjoys a good robustness.

Conclusions

So far: A data-driven sampling approach for risk-aware sparse predictive control for uncertain discrete LTI system.

The Take Home Messages

- Risk-aware solution $\bar{u}_\epsilon^*$ is approximated by data-driven solution $\bar{u}_K^*$.
- Provide probabilistic robustness and sample complexity guarantees.

Outlook

- Recursive feasibility and stability for risk-aware SPC.
- Model-free framework for sparse predictive control.

Thank You for Your Attention !

Suggestions & Comments are Welcome !