SST Data-Embedding Sparsity-Promoting Dynamic Mode Decomposition

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Abstract

In this poster, we exploit a sparsity-promoting dynamic mode decomposition approach to explore the equation-free weather system (possibly nonlinear and high-dimensional), where the system dynamics can directly be collected from the historical time series of monthly SST data. By virtue of spectral analysis of data-driven modeling, we discuss the dynamic modes and the related eigenvalues.

Observed Historical Data Collection from SST

Motivated by Navarra et al. (2021), we concern on the ensemble sea surface temperature (SST) data which provided by the Nino-3. The data are available from January 1891 to December 2019, where the monthly SST data measurements are constructed by $\mathbf{10 \times 60 = 600}$ spatial grids points. Specifically, the data consist of $N = \mathbf{1548}$ temporal snapshots for a total of $\mathbf{928,800}$ points. For the sake of convenience, we first pack the monthly data matrix as a vector-field, and then repeat the similar procedure for every three months' data (i.e., stack the spatial measurements from a quarter of a year into a vector). Hence, the dimension of spatial points saved per time snapshot becomes $n = \mathbf{1800}$, and the number of snapshots reduces to $N = \mathbf{515}$, which provides an efficient platform to compute standard dynamic mode decomposition using a *tall-skinny* matrix ($n \gg N$), i.e., $X := \begin{bmatrix} x_0 & x_1 & \cdots & x_N \end{bmatrix}$, where $x_i \in \mathbb{R}^{1800\times 1}$.

DMD Algorithm

(i) Generate two data matrices from observed SST full data X as follows

$$X_0 := \begin{bmatrix} x_0 & x_1 & \cdots & x_{N-1} \end{bmatrix} \in \mathbb{R}^{n\times N}, \quad X_1 := \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix} \in \mathbb{R}^{n\times N}$$

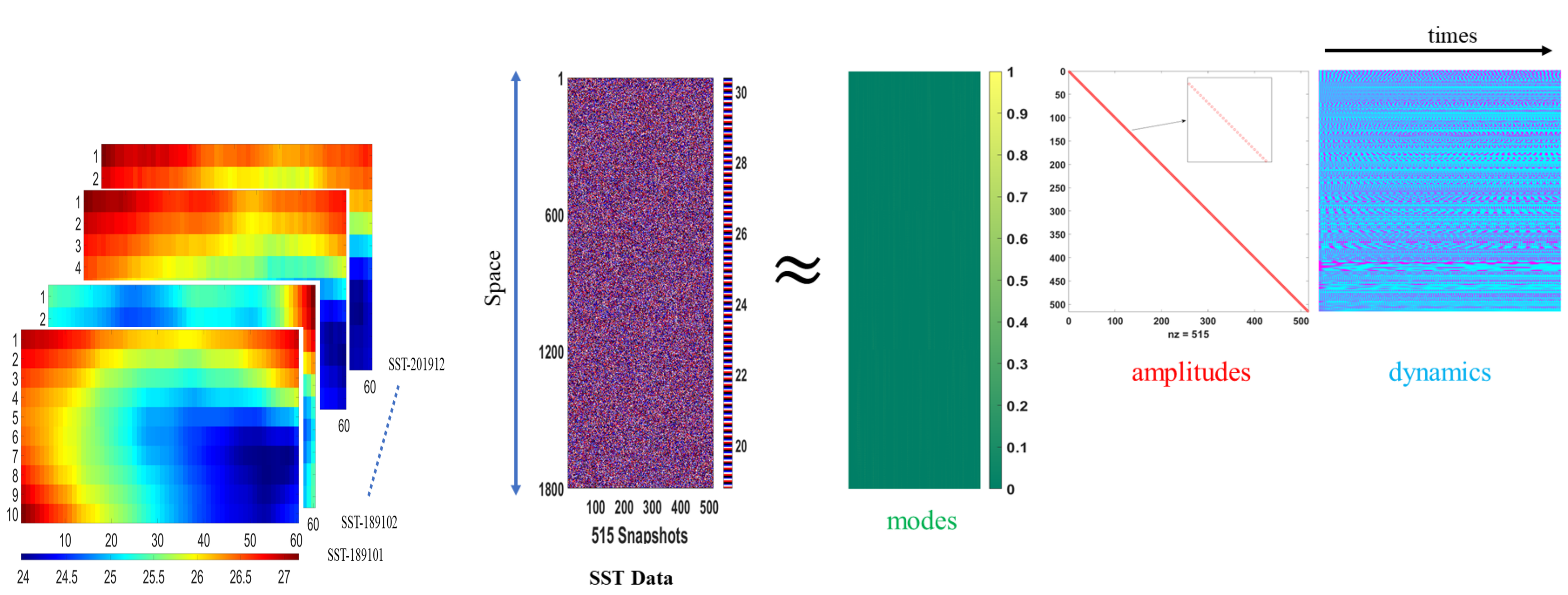
(ii) Simulate a discrete-time LTI system as

$$x_{t+1} = Ax_t, \ t = 0, 1, \dots, N-1 \ \Rightarrow \ X_1 = AX_0, \ A \in \mathbb{R}^{n\times n} \quad (1)$$

(iii) Perform SVD/DMD algorithms (Kutz et al., 2016), and then approximate experimental snapshots using a linear combination of DMD modes. The solution of data-driven system is $x_t = \sum_{i=1}^{r} \phi_i \lambda_i^t b_i$ for all t. We have

$$\underbrace{\begin{bmatrix} x_0 & x_1 & \cdots & x_{N-1} \end{bmatrix}}_{X_0} \approx \underbrace{\begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_r \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_r \end{bmatrix}}_{D_b} \underbrace{\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_r & \cdots & \lambda_r^{N-1} \end{bmatrix}}_{V_\lambda \in \mathbb{C}^{r\times N}}.$$

where $D_b = \mathbf{diag}(b_i)$ represents the amplitudes of the modes Φ, and the Vandermode matrix $V_\lambda \in \mathbb{C}^{r\times N}$ captures the temporal evolution of the dynamic modes. An intuitive graphical sketch is explained as follows:



Methodology: Sparsity-Promoting Dynamic Mode Decomposition

Inspired by the seminal work (Jovanović et al., 2014), we here consider sparsity promoting DMD (spDMD) for SST data, which not only provides a useful tool in quantitative analysis of high-dimensional data-embedding model, but also in the identification of the dominant dynamic modes to realize model reduction.

Sparsity-Promoting DMD Approach

? Why we here introduce spDMD for SST data-embedding modeling?

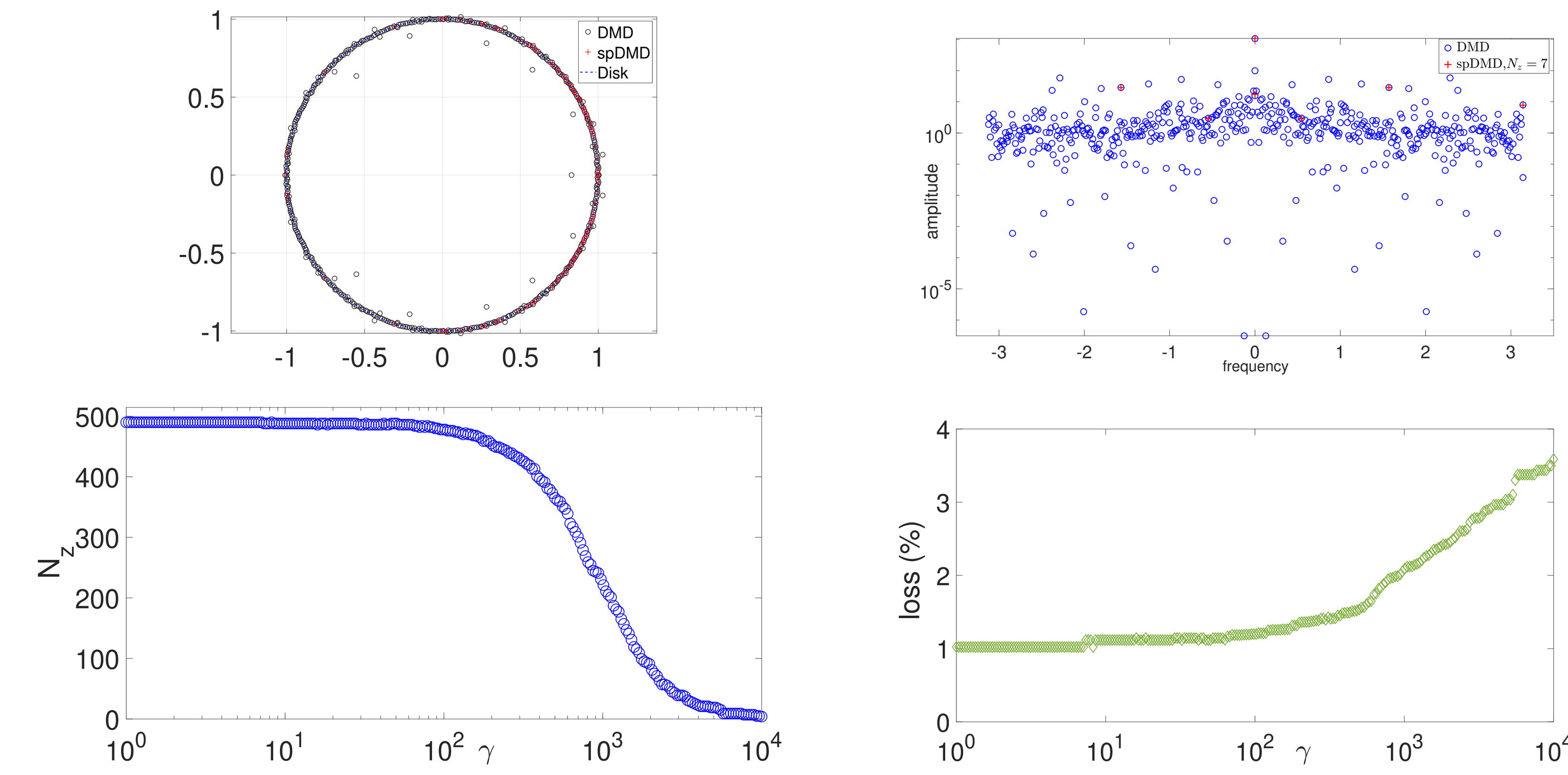
♣ It directly *finds dominant amplitudes* and then easily *reduces model* !

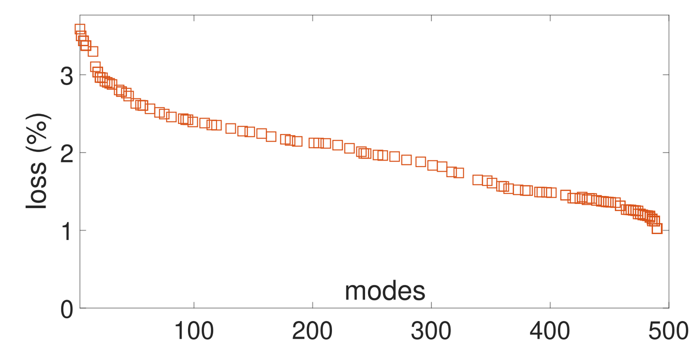
Goal: Minimize a regularized least-squares deviation (between the SST data matrix of snapshots and a linear combination of DMD modes) with an added ℓ_1 norm penalty of the vector of DMD amplitudes, i.e.,

$$J(\mathbf{b}) \triangleq \min_{\mathbf{b}} \|X_0 - \Phi D_b V_\lambda\|_F^2 + \gamma \|\mathbf{b}\|_1 \quad (2)$$

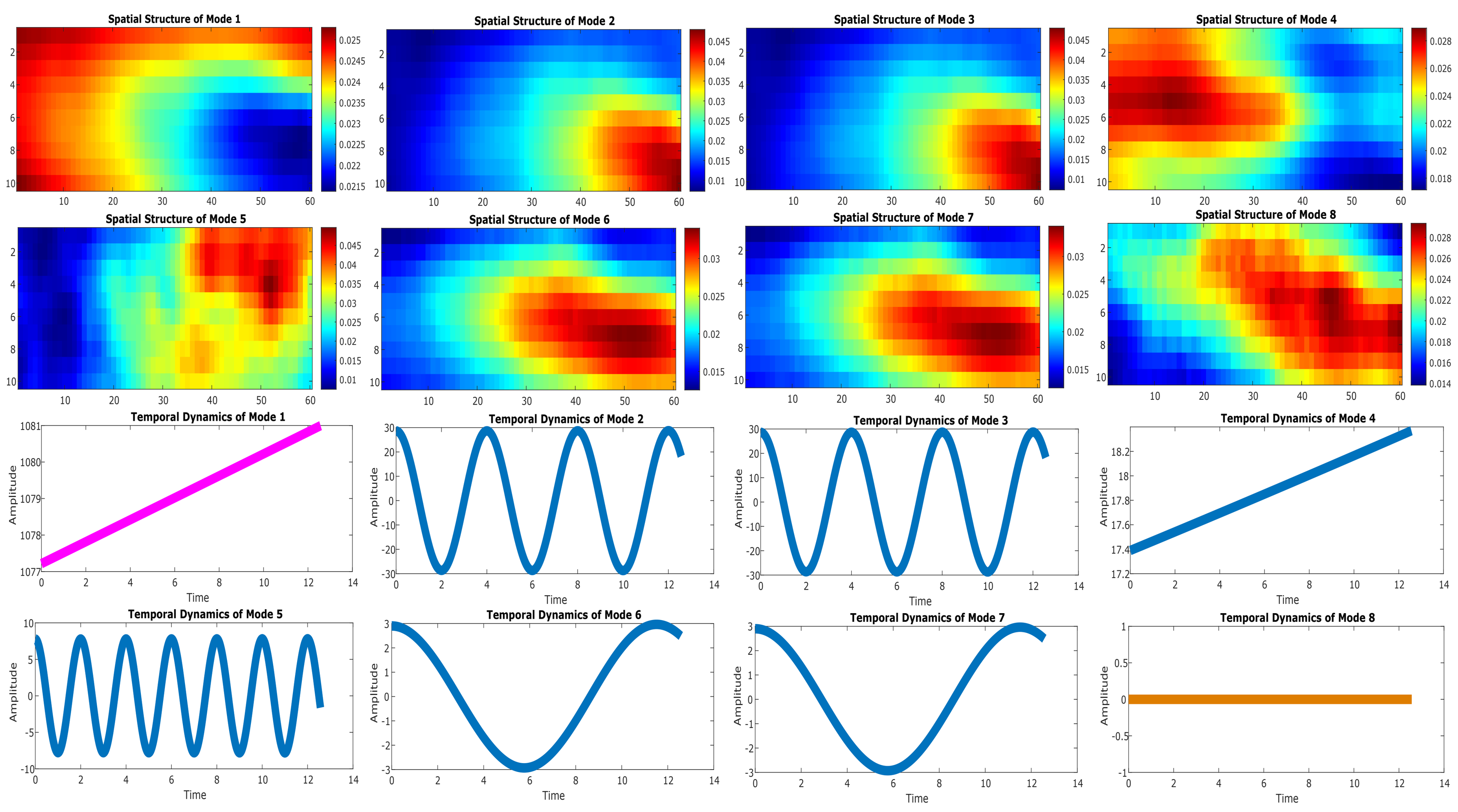
- ℓ_1 norm $\|\mathbf{b}\|_1 = \sum_{i=1}^{r} |b_i|$ is penalized on the amplitudes of the modes
- Weight $\gamma \geq \mathbf{0}$ makes the trade-off between least-square error and sparsity
- Convex optimization & computational tractable, e.g., `cvx`, `YALMIP`, `ADMM`

Numerical Benchmark Results





Weight γ	Cardinality N_z	Cost $J(b)$	Loss $\frac{J(b)}{\|X_0\|}$%
1	490	184036	1.0224
871	245	374125	1.9872
3822	25	670052	2.9174
7719	7	952274	3.4362



▷ Eigenvalues for SST Data in DT/CT cases with different weights γ

▷ spDMD displays performance loss and spatio-temporal modes

Model-Order Reduction: A Low-dimensional Dynamical System

When optimal solution $\mathbf{b}_\gamma^*$ is obtained from spDMD problem (2), the sparsity $\mathbf{card}(\mathbf{b}_\gamma^*) = m$ paves the way to model reduction of original high-dimensional system $x_{k+1} = T(x_k)$ related to (1). Let $V_j := \varphi_j(x_0)v_j$ and λ_j^k as a variable $z_j[k]$, then a low-dimensional system from first reordered m KMs is as

$$y_k = x_k \approx \sum_{j=1}^{m} \lambda_j^k \varphi_j(x_0) v_j = \sum_{j=1}^{m} z_j[k] V_j := VZ^m[k],$$

◇ $Z^m = [z_1, \dots, z_m]^\top \in \mathbb{C}^m$, $V = [V_1, \dots, V_m] \in \mathbb{C}^{n\times m}$, where λ_j is Koopman eigenvalue (KE), φ_j is eigenfunction (KEF), v_j is mode (KM)

Ref. Susuki & Mezic (2012) $\Rightarrow \ VZ^M[k+1] = T(VZ^M[k+1])$

$$(\text{Multiply } V^\dagger) \ \Rightarrow \ Z^m[k+1] = (V^\dagger V)^{-1} V^\dagger T(VZ^m[k+1]) := \mathcal{T}_m(Z^m[k]) \quad (3)$$

Conclusion & Discussion

- Compare the effects for eigenvalues and modes by DMD and spDMD
- spDMD can identify dominant modes for better model reduction in SST
- Relations between spDMD, KEs and KEFs, KMs (Mauroy et al., 2020)

References

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