Koopman Forms of Nonlinear Systems with Inputs

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Introduction

Related Literature for CT System

- In CT input-linear or control-affine nonlinear systems, applying the chain rule results in not entirely linear Koopman models (Kaiser et al., 2020).
- Using state-dependent observables yields a constant state-transition matrix and a state-varying input matrix, allowing interpretation as LPV models (Mohammadpour & Scherer, 2012).
- If the input matrix is within the span of observables, the Koopman model can be bilinear (Goswami & Paley, 2017; Huang et al., 2018).
- However, systematically obtaining Koopman forms for general nonlinear control remains an open question.
- Klus et al. (2020) derive Koopman generators for discrete inputs, resulting in a switched linear system.
- Alternatively, an extended state-input space approach (Kaiser et al., 2020) creates an autonomous lifted model, which is challenging for control. Thus, current methods offer limited control possibilities for nonlinear systems with inputs.

Related Literature for DT System

In DT, deriving Koopman models for linear input and control-affine nonlinear systems is challenging due to the **absence of a chain rule** for the difference operator.

DT models are widely used for system identification and embedded control purposes.

- Various methods exist for handling DT nonlinear systems with inputs:
  - Williams et al. (2016) propose identifying an autonomous LPV Koopman model where the input serves as the scheduling variable.
  - Proctor et al. (2018) utilize a dictionary of state, input, and mixed-dependent observables, restricting the output space of the Koopman operator to state-dependent observables.
- This approach is common in identification-related studies (e.g., Bonnert and Konigorski, 2020; Liu et al., 2018) and investigations of system theoretic properties (Yeung et al., 2018).
- Alternatively, some works (e.g., Kaiser et al., 2020; Korda and Mezić, 2020) employ the spectral properties of the Koopman operator to lift autonomous dynamics using eigenfunction coordinates..
- However, the challenge of deriving Koopman models that directly incorporate inputs remains unresolved.

Contributions

This paper studies the analytic derivation of Koopman forms of general nonlinear systems with inputs both in CT and DT.
The contributions are

- C1 A systematic *factorization* method is devised to obtain, for a wide class of nonlinear systems, an *exact* CT Koopman model suitable for control purposes.
- C2 A method to *analytically* compute an *exact* Koopman form of a wide class of DT nonlinear systems with inputs is developed.
- C3 Providing an interpretation of the resulting Koopman forms as LPV models both in continuous and discrete time.
- C4 A 2-norm based magnitude bound of the state response error between the derived exact Koopman model and its LTI approximations is devised to give characterization of the expected model uncertainty.

Embedding of CT systems

Consider an autonomous time-invariant nonlinear system

$$\dot{x}_t = f_{\mathrm{c}}(x_t), \quad x_0 \in \mathbb{X} \subseteq \mathbb{R}^{n_x}, \tag{1}$$

- $f_{\mathrm{c}} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ is Lipschitz continuous.
- $x_t := x(t)$ is the state variable.
- The solution of (1) at time t starting from x_0 is described as

$$x_t = F_{\mathrm{c}}(t, x_0) = x_0 + \int_0^t f_{\mathrm{c}}(x_\tau)\mathrm{d}\tau. \tag{2}$$

- The set $\mathbb{X}$ is considered to be compact and forward invariant under $F_{\mathrm{c}}(t, \cdot)$, i.e., $F_{\mathrm{c}}(t, \mathbb{X}) \subseteq \mathbb{X}$, $\forall t \geq 0$ (assum. a weak stability)[1].
- The family of Koopman operators $\{\mathcal{K}^t\}_{t\geq 0}$ associated with $F_{\mathrm{c}}(t, \cdot)$ for a given $\mathcal{F} \subseteq \mathcal{C}^1$ Banach function space is defined as

$$\mathcal{K}^t\phi(x_0) = \phi \circ F_{\mathrm{c}}(t, x_0), \quad \forall \phi \in \mathcal{F} \tag{3}$$

- $\phi : \mathbb{X} \to \mathbb{R}$ denotes scalar observable functions in $\mathcal{F}$.

[1] The flow F_c is uniformly Lipschitz continuous w.r.t. t, the Koopman semigroup $\{\mathcal{K}^t\}_{t\geq 0}$ is strongly continuous on $\mathcal{F}$.

- Infinitesimal generator $\mathcal{L} : \mathcal{D}_{\mathcal{L}} \to \mathcal{F}$ of Koopman semigroup

$$\mathcal{L}\phi(x_0) = \lim_{t\to 0} \frac{\mathcal{K}^t\phi(x_0) - \phi(x_0)}{t}, \quad \forall\phi \in \mathcal{D}_{\mathcal{L}} \tag{4}$$

exists[2], where the domain $\mathcal{D}_{\mathcal{L}} \subseteq \mathcal{F}$ is a "dense set" in $\mathcal{F}$.

- Under certain assumption, the generator $\mathcal{L}$ can be *linear*.
- Let $z_\phi(t) = \mathcal{K}^t\phi(x_0)$, then it is the solution of the equation

$$\dot{z}_\phi(t) = \mathcal{L}z_\phi(t), \quad z_\phi(0) = \phi(x_0), \quad \Longrightarrow \quad \mathcal{K}^t = e^{\mathcal{L}t}. \tag{6}$$

[2]the limit exists in the strong sense $\lim_{t\to 0} \|\mathcal{L}\phi(x_0) - \frac{\mathcal{K}^t\phi(x_0)-\phi(x_0)}{t}\| = 0$.

In CT, the infinitesimal generator $\mathcal{L}$ is important, as it is used to describe the dynamics in the lifted space of observables. Under some assumptions, the generator $\mathcal{L}$ is *linear*.

Applying $\mathcal{L}$ to $\phi(x_0)$ gives PDE

$$\dot{\phi} = \frac{\partial \phi}{\partial x} f_c = \mathcal{L}\phi, \quad \phi = \phi_{\mathsf{inf}} \tag{7}$$

which is a linear but infinite-dim. form of the underlying system.

■ Consider that there exists a finite-dimensional Koopman invariant subspace $\mathcal{F}_{\mathrm{n_f}} \subseteq \mathcal{D}_{\mathcal{L}}$, that is, $\mathcal{L} : \mathcal{F}_{\mathrm{n_f}} \to \mathcal{F}_{\mathrm{n_f}}$.

■ If $\mathcal{F}_{n_f}$ is invariant under Koopman generator $\mathcal{L}$, then, due to the linearity of $\mathcal{L}$, $\mathcal{L}\phi$ is a linear combination of the elements of $\mathcal{F}_{\mathrm{n_f}}$.

♦ Choose the basis $\Phi^\top = [\phi_1 \ \cdots \ \phi_{\mathrm{n_f}}]$ of $\mathcal{F}_{\mathrm{n_f}}$, we have ODE

$$\dot{\phi}_j = \mathcal{L}\phi_j = \sum_{i=1}^{\mathrm{n_f}} L_{i,j}\phi_i \tag{8}$$

- L is a matrix representation of the Koopman operator
- the jth column of L has the coordinates of $\mathcal{L}\phi_j$ in the basis Φ.

♦ Introduce $A = L^\top \in \mathbb{R}^{\mathrm{n_f}\times \mathrm{n_f}}$, the lifted dynamics of (1) is as follows

$$\dot{\Phi}(x_t) = A\Phi(t), \tag{9}$$

which further results in the following relation (via eq.(7))

$$\dot{\Phi}(x_t) = \frac{\partial \Phi}{\partial x}(x_t) f_{\mathrm{c}}(x_t) = A\Phi(x_t), \quad \frac{\partial \Phi}{\partial x} \text{is the Jacobian of } \Phi. \tag{10}$$

Hence, a Koopman lifting for eq.(1) is to find a set of observables Φ:

$$\frac{\partial \Phi}{\partial x} f_{\mathrm{c}} \in \mathrm{span}\{\Phi\}. \tag{11}$$

To recover the original state, assume that $\Phi^\dagger(\Phi(x_t)) = x_t$ exists.

♦ Koopman representation of (1) gives an explicit LTI dynamics

$$z_t := \Phi(x_t) \quad \Longrightarrow \quad z_t = Az_t, \quad z_0 = \Phi(x_0). \tag{12}$$

Consider DT autonomous nonlinear systems

$$x_{k+1} = f(x_k) \quad x_0 \in \mathbb{X}, \quad f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x} \tag{26}$$

The Koopman operator acting on observable space $\mathcal{F}$ with $\phi : \mathbb{X} \to \mathbb{R}$

$$\mathcal{K}\phi = \phi \circ f, \quad \forall \phi \in \mathcal{F} \tag{27}$$

$$\implies \quad \mathcal{K}\phi(x_k) = \phi \circ f(x_{k+1}) = \phi(x_{k+1}) \tag{28}$$

Assump: there exists a finite-dim Koopman invariant space $\mathcal{F}_{\mathrm{n_f}} \subseteq \mathcal{F}$

$$\mathcal{K}\phi_j = \sum_{i=1}^{\mathrm{n_f}} K_{i,j}\phi_j \tag{29}$$

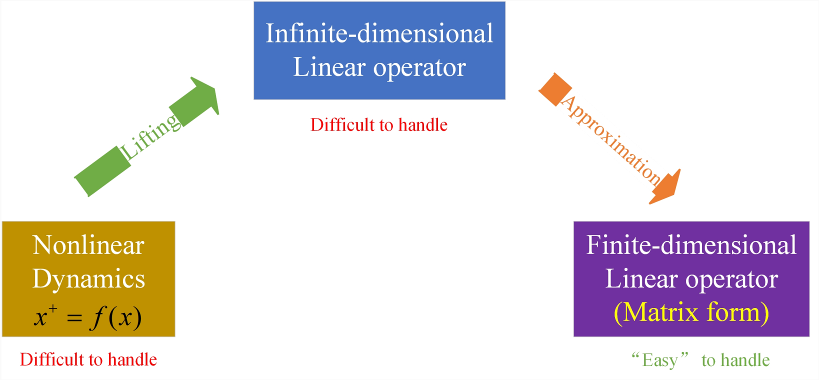
A finite-dim representation of (26) in the lifted space is

$$\Phi(x_{k+1}) = A\Phi(x_k), \quad A = K^\top, \quad \Phi^\top = [\phi_1, \cdots, \phi_{\mathrm{n_f}}] \tag{30}$$

$$\implies \quad \Phi \circ f(x_k) = A\Phi(x_k) \tag{31}$$

$$\implies \quad \Phi \circ f \in \mathrm{span}\{\Phi\},\ \Phi^\dagger(\phi(x_k)) = x_k,\ x = \mathrm{id}(x),\ \mathrm{id} \in \mathrm{span}\{\Phi\} \tag{32}$$

$$\implies \quad z_k = \Phi(x_k), \quad z_{k+1} = Az_k, \quad z_0 = \Phi(x_0) \tag{33}$$


Infinite-dimensional
Linear operator
Difficult to handle
Lifting
Approximation
Nonlinear
Dynamics
$x^+ = f(x)$
Difficult to handle
Finite-dimensional
Linear operator
(Matrix form)
"Easy" to handle

Embedding of CT Systems

Consider the CT nonlinear time-invariant control system

$$\dot{x}_t = f_{\mathrm{d}}(x_t, u_t), \quad x_0 \in \mathbb{X},\ u_t \in \mathbb{U} \subseteq \mathbb{R}^{n_u}, \tag{13}$$

- Lipschitiz continuous for $f_{\mathrm{d}} : \mathbb{R}^{n_x} \times \mathbb{U} \to \mathbb{R}^{n_x}$.
- Here u_t is the constant input such that satisfies the commute.[3]
- Assume that $\mathbb{U}$ is given such that $\mathbb{X}$ is compact (i.e., closed & bounded) and forward invariant set under the induced the flow.
- Construct the (only) state-dependent observables.

Decompose (13) into the sum between the contributions of the *autonomous* and *input-related* dynamics [Surana CDC'16]

$$f_{\mathrm{d}}(x_t, u_t) = \underbrace{f_{\mathrm{d}}(x_t, 0)}_{f_c(x_t)} + \underbrace{f_{\mathrm{d}}(x_t, u_t) - f_{\mathrm{d}}(x_t, 0)}_{g_c(x_t, u_t)}, \tag{14}$$

where $g_c(x_t, 0) = 0$.

[3] Ref.) Mauroy & Mezic & Susuki, Springer'21, Chap. 1.5, pp.28: In the non-autonomous case (the control u is not constant), the Koopman operator $\mathcal{K}_{u(\cdot)}$ and generator $\mathcal{L}_{u(\cdot)}$ do not commute.

Theorem 1 (Exact Finite-Dimensional Lifted Form for CT systems)

Given a nonlinear CT system in the general form $\dot{x}_t = f_\mathrm{d}(x_t, u_t)$ in (13), where f_d is of the form (14), with the observables $\Phi : \mathbb{X} \to \mathbb{R}^{n_\mathrm{f}}$ in $\mathcal{C}^1$ such that (11) holds for $f_c(x_t)$, then there exists an exact finite-dimensional lifted form

$$\dot{\Phi}(x_t) = A\Phi(x_t) + \mathcal{B}(x_t, u_t), \tag{15a}$$

with $A \in \mathbb{R}^{n_\mathrm{f} \times n_\mathrm{f}}$ and $\mathcal{B} : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^{n_\mathrm{f}}$ defined as

$$\mathcal{B}(x_t, u_t) = \frac{\partial \Phi}{\partial x}(x_t) g_c(x_t, u_t). \tag{15b}$$

Proof: Lift dynamics $\dot{\Phi}(x_t) = \frac{\partial \Phi}{\partial x}(x_t) f_c(x_t) = A\Phi(x_t)$ and use chain rule

$$\dot{\Phi}(x_t) = \frac{\partial \Phi}{\partial x}(x_t) f_\mathrm{d}(x_t, u_t) = \underbrace{\frac{\partial \Phi}{\partial x}(x_t) f_c(x_t)}_{A\Phi(x_t)} + \underbrace{\frac{\partial \Phi}{\partial x}(x_t) g_c(x_t, u_t)}_{\mathcal{B}(x_t, u_t)} \tag{16}$$

Embedding of CT Systems

Theorem 1 (Exact Finite-Dimensional Lifted Form for CT systems)

Given a nonlinear CT system in the general form $\dot{x}_t = f_{\mathrm{d}}(x_t, u_t)$ in (13), where f_{d} is of the form (14), with the observables $\Phi : \mathbb{X} \to \mathbb{R}^{n_{\mathrm{f}}}$ in $\mathcal{C}^1$ such that (11) holds for $f_c(x_t)$, then there exists an exact finite-dimensional lifted form

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with $A \in \mathbb{R}^{n_{\mathrm{f}} \times n_{\mathrm{f}}}$ and $\mathcal{B} : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^{n_{\mathrm{f}}}$ defined as

$$\mathcal{B}(x_t, u_t) = \frac{\partial \Phi}{\partial x}(x_t) g_c(x_t, u_t). \tag{15b}$$

- ♣ Eq. (15a) represents an exact lifted form of (13), consistent with the lifting in (16) under Φ.
- ♣ Limitation: nonlinear function $\mathcal{B}(x_t, u_t)$ is NOT good for control
- ♣ Orcale: recast nonlinear term $\mathcal{B}(x_t, u_t)$ into a "LPV" form

$$\mathcal{B}(x_t, u_t) = B(x_t, u_t)u_t \quad (\Longleftarrow \text{ Lemma 1})$$

Lemma 1 (Factorization)

Let $\mathcal{B} : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^{n_f}$ be continuously differentiable in u_t, continuous in x_t and satisfying $\mathcal{B}(x_t, 0) = 0$, and let $\mathbb{U}$ be <u>convex set</u> containing the origin. Then,

$$B(x_t, u_t) = \int_0^1 \frac{\partial \mathcal{B}}{\partial u}(x_t, \lambda u_t)\mathrm{d}\lambda, \tag{18}$$

provides a factorization of $\mathcal{B}$ such that $\mathcal{B}(x_t, u_t) = B(x_t, u_t)u_t$ for any $(x_t, u_t) \in (\mathbb{X}, \mathbb{U})$.

Proof: Let $\lambda \in \mathbb{R}, p, q \in \mathbb{U} \subseteq \mathbb{R}^{n_u}$. For arbitrary input function u can be recast as <u>convex form</u> $u(\lambda) = p + \lambda(q - p)$ with $u \in [p, q]$, $\lambda \in [0, 1]$.

$$\zeta_i(1) - \zeta_i(0) = \int_0^1 \zeta'(\lambda)\mathrm{d}\lambda, \quad \zeta_i = \mathcal{B}_i(x, u(\lambda)), \quad \zeta_i' = \frac{\partial \zeta_i}{\partial \lambda} \tag{A.1}$$

$$\overset{i\text{th row of } \mathcal{B}}{\Longleftrightarrow} \mathcal{B}_i(x, q) - \mathcal{B}_i(x, p) = \int_0^1 \frac{\partial \mathcal{B}_i}{\partial u}(x, u(\lambda))(q - p)\mathrm{d}\lambda \tag{A.2}$$

$$\Longrightarrow \mathcal{B}(x, q) - \mathcal{B}(x, p) = \Big(\int_0^1 \frac{\partial \mathcal{B}}{\partial u}(x, u(\lambda))\mathrm{d}\lambda\Big)(q - p) \tag{A.3}$$

Embedding of CT Systems

Lemma 1 (Factorization)

Let $\mathcal{B} : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^{n_\mathrm{f}}$ be continuously differentiable in u_t, continuous in x_t and satisfying $\mathcal{B}(x_t, 0) = 0$, and let $\mathbb{U}$ be convex set containing the origin. Then,

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provides a factorization of $\mathcal{B}$ such that $\mathcal{B}(x_t, u_t) = B(x_t, u_t)u_t$ for any $(x_t, u_t) \in (\mathbb{X}, \mathbb{U})$.

A general lifted representation for CT nonlinear system (13) or (15a)

$$\dot{\Phi}(x_t) = A\Phi(x_t) + B(x_t, u_t)u_t. \tag{19}$$

LPV form of the Koopman model of (13), (15a), (19)

$$\dot{z}_t = Az_t + B_z(p_t)u_t, \quad z_t = \Phi(x_t), \quad z_0 = \Phi(x_0) \tag{20}$$

- A scheduling map, $p_t = \mu(z_t, u_t)$ such that $B_z \circ \mu = B$ (select μ)
- B_z is a predefined function class, e.g., affine, polynomial, rational

CT systems with control-affine or linear input case

Consider control-affine (or linear input $g_c(x_t) = b \in \mathbb{R}^{n_x \times n_u}$) form of (13)

$$\dot{x}_t = f_c(x_t) + g_c(x_t)u_t, \quad g_c : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x \times n_u}, \quad u_t \in \mathbb{U} \subseteq \mathbb{R}^{n_u}. \tag{21}$$

Taking $B(x_t)u_t = \mathcal{B}(x_t, u_t)$, lifted representation is

$$\Phi(x_t) = A\Phi(x_t) + B(x_t)u_t, \quad B(x_t) = \frac{\partial \Phi}{\partial x}(x_t)g_x(x_t), \tag{22,23}$$

LPV setup: Scheduling map $p_t = \mu(z_t)$ and Jacobian $B(x_t) = \frac{\partial \Phi}{\partial x}(x_t)b$.

If $\frac{\partial \Phi}{\partial x} g_{c_i} \in \operatorname{span}\{\Phi\}$, $B_i \in \mathbb{R}^{n_\mathrm{f} \times n_\mathrm{f}}$ meets $\frac{\partial \Phi}{\partial x} g_{c_i} = B_i \Phi$, a bilinear form is

$$\dot{\Phi}(x_t) = A\Phi(x_t) + \sum_{i=1}^{n_u} B_i \Phi(x_t) u_{t,i}, \tag{24}$$

LPV form of control-affine system

$$\dot{z}_t = Az_t + \sum_{j=1}^{n_\mathrm{f}} z_{t,j}\tilde{B}_j u, \quad z_t = \Phi(x_t), \quad z_0 = \Phi(x_0). \tag{25}$$

- $z_{j,t}$ is the jth element of z_t
- $\tilde{B}_j = [B_{1,j} \;\cdots\; B_{n_u,j}]$, where $B_{i,j} \in \mathbb{R}^{n_\mathrm{f}}$ is the jth column of B_i

Embedding of DT Systems

DT: Koopman representation under inputs

Consider a DT nonlinear time-invariant control system

$$x_{k+1} = f_{\mathrm{d}}(x_k, u_k), \quad x_0 \in \mathbb{U},\ u_k \in \mathbb{U} \subseteq \mathbb{R}^{n_u}. \tag{34}$$

▶ Decompose $f_{\mathrm{d}}(x_k, u_k)$ into "**autonomous**" + "**input-driven**" forms

$$f_{\mathrm{d}}(x_k, u_k) = f(x_k) + g(x_k, u_k). \tag{35}$$

Lift the autonomous dynamics with zero input (i.e., $u_k = 0$) as

$$x_{k+1} = f_{\mathrm{d}}(x_k, 0) = f(x_k), \quad g(x_k, 0) = 0. \tag{36}$$

Apply the observables Φ to lifted autonomous part

$$\Phi(x_{k+1}) = A\Phi(x_k) = \Phi(f(x_k)). \tag{37}$$

Consider a full dynamics (34) applying the lifting Φ:

$$\Phi(x_{k+1}) = \boxed{\Phi\Big(f(x_k) + g(x_k, u_k)\Big).} \tag{38}$$

- Directly separate the "autonomous" and "input-driven" dynamics as in CT case ? $\not\Leftarrow$ Lacks of "chain rule" under the shift operator

$\Rightarrow$ **Fundamental theorem of calculus** (FTC) i.e., $\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$

Embedding of DT systems

Nonlinear DT system

$$x_{k+1} = f_{\mathrm{d}}(x_k, u_k) \tag{34}$$

$$= f_{\mathrm{d}}(x_k, u_k) = f(x_k) + g(x_k, u_k) \tag{35}$$

Theorem 2 (Exact finite-dimensional lifting for DT system)

Given a nonlinear DT system in the general form (34), where f_{d} is written as (35), together with observables $\Phi : \mathbb{X} \to \mathbb{R}^{n_f}$ in $\mathcal{C}^1$ with $\mathbb{X}$ convex (e.g., convex forward invariant set), such that $\Phi(f(\cdot)) \in \mathrm{span}\{\Phi\}$, then there exists an exact finite-dimensional lifting:

$$\Phi(x_{k+1}) = A\Phi(x_k) + \mathcal{B}(x_k, u_k) \tag{39a}$$

with $A \in \mathbb{R}^{n_f \times n_f}$ and

$$\mathcal{B}(x_k, u_k) = \left(\int_0^1 \frac{\partial \Phi}{\partial x} \big(f(x_k) + \lambda g(x_k, u_k)\big) \mathrm{d}\lambda \right) g(x_k, u_k). \tag{39b}$$

LPV form of Koopman Lifting for DT systems

As $\mathbb{X}$ is convex, take $\mathbb{X} \ni x(\lambda) = p + \lambda(q - p)$ with $p, q \in \mathbb{X}$, $\lambda \in [0, 1]$.

As ith component $\phi_i \in \mathcal{C}^1$, $i \in \{1, \cdots, \mathrm{n_f}\}$, define the $\mathcal{C}^1$ function $h_i : \mathbb{R} \to \mathbb{R}$ as $h_i(\lambda) = \phi_i \circ x(\lambda)$. Using FTC, i.e., $\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$,

$$h_i(1) - h_i(0) = \int_0^1 h_i'(\lambda)\mathrm{d}\lambda, \quad h_i' = \frac{\partial h_i}{\partial \lambda} \tag{40}$$

$$\xRightarrow[\text{chain-rule}]{\text{use } \phi_i} \quad \phi_i(q) - \phi_i(p) = \int_0^1 \frac{\partial \phi_i}{\partial x}(x(\lambda))\frac{\partial x}{\partial \lambda}\mathrm{d}\lambda \tag{41}$$

$$\begin{aligned} \Longrightarrow \quad \phi_i(q) - \phi_i(p) &= \Big(\int_0^1 \frac{\partial \phi_i}{\partial x}(x(\lambda))\mathrm{d}\lambda\Big)(q - p) \qquad (42) \\ &= \Big(\int_0^1 \frac{\partial \phi_i}{\partial x}(p + \lambda(q - p))\mathrm{d}\lambda\Big)(q - p) \end{aligned}$$

Choose $q_{k+1} = f(x_k) + g(x_k, u_k) = x_{k+1}$ and $p_{k+1} = f(x_k)$, we have

$$x_{k+1}(\lambda) = p_{k+1} + \lambda(q_{k+1} - p_{k+1}) = f(x_k) + \lambda\, g(x_k, u_k) \tag{43}$$

Substitute (43) into (42) at $k + 1$ instant, we derive

$$\phi_i(x_{k+1}) = \phi_i(f(x_k)) + \Big(\int_0^1 \frac{\partial \phi_i}{\partial x}(f(x_k) + \lambda g(x_k, u_k))\mathrm{d}\lambda\Big)g(x_k, u_k) \tag{44}$$

LPV form of Koopman Lifting for DT system

Similar to CT case, we factorize $\mathcal{B}$ in (39a) to get

$$\Phi(x_{k+1}) = A\Phi(x_k) + B(x_k, u_k)u_k. \tag{45}$$

As $g(x_k, 0) = 0$ by construction in (35), Lemma 1 gives

$$B(x_k, u_k) = \int_0^1 \frac{\partial \mathcal{B}}{\partial u}(x_k, \lambda u_k)\mathrm{d}\lambda. \tag{46}$$

LPV form of the Koopman representation of DT systems (34)

The Lifted form (45) can be written as LPV form

$$z_{k+1} = Az_k + B_z(p_k)u_k, \quad z_k = \Phi(x_k) \tag{47}$$

- A scheduling map $p_k = \mu(z_k, u_k)$ satisfies $B_z \circ \mu = B$

DT systems with control-affine or linear inputs

- For a control-affine DT nonlinear system

$$x_{k+1} = f(x_k) + g(x_k)u_k. \tag{48}$$

By Theorem 2, we obtain a lifted form (45) with a simpler term (39b)

$$\mathcal{B}(x_k, u_k) = \underbrace{\left(\int_0^1 \frac{\partial \Phi}{\partial x}(f(x_k) + \lambda g(x_k)u_k)\mathrm{d}\lambda\right) g(x_k)}_{B(x_k,u_k)} u_k. \tag{49}$$

LPV form of Koopman representation follows as in (47).

- For a DT nonlinear system with linear input

$$x_{k+1} = f(x_k) + bu_k. \tag{50}$$

By Theorem 2, we have

$$\mathcal{B}(x_k, u_k) = \underbrace{\left(\int_0^1 \frac{\partial \Phi}{\partial x}(f(x_k) + \lambda bu_k)\mathrm{d}\lambda\right) b}_{B(x_k,u_k)} u_k. \tag{51}$$

with an LPV form as in (47).

Approximation Error of LTI Koopman Forms

Characteristic of approximation error

The *approximate* LTI Koopman model is given by

$$\hat{z}_{k+1} = A\hat{z}_k + \hat{B}u_k. \tag{53}$$

- $\hat{z}_k$ is the associated state vector
- A meets embedding conditions $\Phi \circ f(x_k) = A\Phi(x_k)$
- $\hat{B}$ is obtained via either an approx. of B_z or data-driven EDMDc

DT Error Dynamics

The error dynamics (i.e., $e_k = z_k - \hat{z}_k$) between the "**exact LPV Koopman form**" (47) and the "**LTI Koopman form**" (53) are

$$e_k = Ae_{k-1} + (B_{k-1} - \hat{B})u_{k-1}, \quad e_0 = 0. \tag{54}$$

- z_k is the state of exact Koopman form (47) $z_{k+1} = Az_k + B_z(p_k)u_k,\ z_k = \Phi(x_k)$
- Initial condition $z_0 = \hat{z}_0$ such that zero error initialization $e_0 = 0$.
- $B_k = B_z(p_k) = B_z(\mu(\Phi(x_k), u_k))$

■ Characterize the expected magnitude of e_k as k involves.

Error Bound

Theorem 3 (Error Bounds)

Consider the *exact* Koopman embedding (47) of a general nonlinear DT system (34) and the approximate LTI Koopman form (53). Under any initial condition $z_0 = \Phi(x_0) = \hat{z}_0$ and input map $u : \mathbb{Z}_0^+ \to \mathbb{R}^{n_u}$ with bounded $\|u\|_{\ell_\infty}$, the state evolution error e_k between these representation given by (54) satisfies[4]

(i) If $\rho(A) \leq 1$, $\|e_k\|_2$ is finite for any $k \in \mathbb{Z}_0^+$ and $\lim_{t\to\infty} \|e_k\|_2$ exists;

(ii) If $\bar{\sigma}(A) \leq 1$, (i) is satisfied and furthermore

$$\|e_k\|_2 \leq \frac{\beta}{1-\bar{\sigma}(A)} \|u\|_{\ell_\infty}, \ \beta = \max_{x\in\mathbb{X}, u\in\mathbb{U}} \|B_z(\mu(\Phi(x), u)) - \hat{B}\|_{2,2} \quad (55)$$

[4]Notation: $\rho(A) = \max_{r\in\lambda(A)} |r|$ denotes the spectral radius of a matrix $A \in \mathbb{R}^{n\times n}$ with eigenvalues $\lambda(A)$ and $\bar{\sigma}(P)$ is the maximum singular value of $P \in \mathbb{R}^{m\times n}$.

Numerical Examples

Continuous-time case

Consider a CT nonlinear system (e.g., $\mu = -0.05,\ \lambda = -1$)

$$\dot{x} \overset{(61)}{=} \begin{bmatrix} \mu x_1 - x_1 \\ \lambda(x_2 - x_1^2) - x_2 \end{bmatrix} + \begin{bmatrix} x_1 e^{u_1} \\ u_1 u_2 + x_2 e^{u_2} \end{bmatrix} \underset{\text{separa.}}{\overset{(62)}{\Longleftrightarrow}} \underbrace{\begin{bmatrix} \mu x_1 \\ \lambda(x_2 - x_1^2) \end{bmatrix}}_{f_c(x)} + \underbrace{\begin{bmatrix} x_1 e^{u_1} - x_1 \\ u_1 u_2 + x_2 e^{u_2} - x_2 \end{bmatrix}}_{g_c(x,u)}$$

- Lift CT system 2D to 3D by choosing the observables:

$$\underbrace{\Phi(x)}_{z} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \end{bmatrix},\ \dot{\Phi}(x) = \underbrace{\begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & -\mu \\ 0 & 0 & 2\mu \end{bmatrix}}_{A = K^\top} \Phi(x) + \underbrace{\begin{bmatrix} x_1 e^{u_1} - x_1 \\ u_1 u_2 + x_2 e^{u_2} - x_2 \\ 2x_1^2 e^{u_1} - 2x_1^2 \end{bmatrix}}_{\mathcal{B}(x,u) := \frac{\partial \Phi}{\partial x}(x_t) g_c(x_t, u_t)\ 15(\text{b})} \tag{63}$$

$$B(x_t, u_t) \overset{(18)}{=} \begin{bmatrix} \frac{x_1}{u_1} e^{u_1} - \frac{x_1}{u_1} & 0 \\ \frac{1}{2} u_2 & \frac{1}{2} u_1 + \frac{x_2}{u_2} e^{u_2} - \frac{x_2}{u_2} \\ 2\frac{x_1^2}{u_1} e^{u_1} - 2\frac{x_1^2}{u_1} & 0 \end{bmatrix},\ \underbrace{B(x,0)}_{\frac{\partial \mathcal{B}}{\partial x}\big|_{(x,0)}} \overset{u_1=0}{\underset{u_2=0}{=}} \begin{bmatrix} 0 & x_1 \\ x_2 & 0 \\ 0 & 2x_1^2 \end{bmatrix} \tag{64}$$

Lifted form

The lifted form of CT system (61):

$$\dot{\Phi}(x) = A\Phi(x) + B(x,u)u, \tag{65a}$$

$$x = C\Phi(x) \tag{65b}$$

LPV form: a scheduling $p = \mu(z,u) = [z^\top \ u^\top]^\top$ such that $B_z \circ \mu = B$,

$$(61) \implies \quad \dot{z} = Az + B_z(p)u, \quad z = \Phi(x) \tag{66a}$$

$$x = Cz, \quad C = [I_2 \ 0_{2\times 1}] \tag{66b}$$

Parameters setup

- Runge-Kutta 4 to solve ODE (61) and (66) with sampling $T_s = 10^{-4}$
- initial guesses $x_0 = [1\ 1]^\top$, $z_0 = \Phi(x_0) = [1\ 1\ 1]^\top$ and white noise input signal $u_i(t_k) \sim \mathcal{N}(0, 0.1)$
- quality of approximated model $\epsilon_i = x_i(t_k) - z_i(t_k)$, $i = 1, 2$, where the difference between the ith entry of the state evolution, i.e.,

$$\epsilon_i = [\epsilon_i(t_0) \ \cdots \ \epsilon_i(t_N)]$$

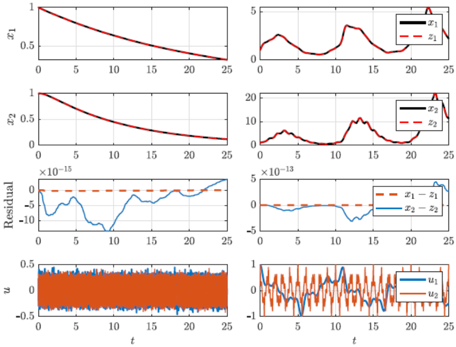

Fig. 1. Continuous-time example: state response of the original nonlinear system (61) given in black and its exact Koopman representation (66) given in red under white noise (left panel) and multisine (right panel) excitation u. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 1
Characterization of the state-evolution error between the original nonlinear system and the Koopman forms in the considered simulation examples.

Input	x_i	Continuous time		Discrete time		Discrete time with constant $\hat{B}$	
		$\|\epsilon_i\|_{\ell_2}$	$\|\epsilon_i\|_{\ell_\infty}$	$\|\epsilon_i\|_{\ell_2}$	$\|\epsilon_i\|_{\ell_\infty}$	$\|\epsilon_i\|_{\ell_2}$	$\|\epsilon_i\|_{\ell_\infty}$
Multisine	$i = 1$	$3.12 \cdot 10^{-13}$	$1.77 \cdot 10^{-15}$	0	0	$1.60 \cdot 10^{-14}$	$8.88 \cdot 10^{-16}$
	$i = 2$	$7.71 \cdot 10^{-11}$	$4.51 \cdot 10^{-13}$	$3.50 \cdot 10^{-14}$	$7.10 \cdot 10^{-15}$	190.82	18.06
White noise	$i = 1$	$6.96 \cdot 10^{-14}$	$2.22 \cdot 10^{-16}$	0	0	$7.31 \cdot 10^{-15}$	$8.88 \cdot 10^{-16}$
	$i = 2$	$2.96 \cdot 10^{-12}$	$1.34 \cdot 10^{-14}$	$1.23 \cdot 10^{-14}$	$3.55 \cdot 10^{-15}$	58.12	9.17

- the error measures $\|\epsilon_i\|_{\ell_2}$ are below 10^{-10} (resp. $\|\epsilon_i\|_{\ell_\infty} < 10^{-12}$)

DT case

Consider a DT system, the $x_{k,i}$ is the value of the ith state at time k

$$x_{k+1} = \underbrace{\begin{bmatrix} a_1 x_{k,1} \\ a_2 x_{k,2} - a_3 x_{k,1}^2 \end{bmatrix}}_{f(x_k)} + \underbrace{\begin{bmatrix} 1 \\ x_{k,1}^2 \end{bmatrix}}_{g(x_k)} u_k, \quad a_1 = a_2 = 0.7,\ a_3 = 0.5 \quad (67)$$

- Lift DT system from $2\mathrm{D}$ to $3\mathrm{D}$ based on observables $\{\phi_1, \phi_2, \phi_3\}$

$$\underbrace{\Phi(x_k)}_{z_k} = \begin{bmatrix} x_{k,1} \\ x_{k,2} \\ x_{k,1}^2 \end{bmatrix}, \ \Phi(x_{k+1}) = \underbrace{\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & -a_3 \\ 0 & 0 & a_1^2 \end{bmatrix}}_{A} \Phi(x_k) + \underbrace{\begin{bmatrix} 1 \\ x_{k,1}^2 \\ 2a_1 x_{k,1} + u_k \end{bmatrix}}_{B(x_k, u_k)} \quad (68a)$$

LPV form: a scheduling $p_k = \mu(z_k, u_k) = [z_k^\top \ u_k^\top]^\top$ such that $B_z \circ \mu = B$,

$$z_{k+1} = Az_k + B_z(p_k)u_k, \quad z_k = Cz_k, \quad C = [I_2 \ 0_{2\times1}] \quad (69)$$

- Take initial conditions as $x_0 = [1\ 1]^\top$, $z_0 = [1\ 1\ 1]^\top$.

Table 1
Characterization of the state-evolution error between the original nonlinear system and the Koopman forms in the considered simulation examples.

Input	x_i	Continuous time		Discrete time		Discrete time with constant $\hat{B}$	
		$\|\epsilon_i\|_{\ell_2}$	$\|\epsilon_i\|_{\ell_\infty}$	$\|\epsilon_i\|_{\ell_2}$	$\|\epsilon_i\|_{\ell_\infty}$	$\|\epsilon_i\|_{\ell_2}$	$\|\epsilon_i\|_{\ell_\infty}$
Multisine	$i = 1$	$3.12 \cdot 10^{-13}$	$1.77 \cdot 10^{-15}$	0	0	$1.60 \cdot 10^{-14}$	$8.88 \cdot 10^{-16}$
	$i = 2$	$7.71 \cdot 10^{-11}$	$4.51 \cdot 10^{-13}$	$3.50 \cdot 10^{-14}$	$7.10 \cdot 10^{-15}$	190.82	18.06
White noise	$i = 1$	$6.96 \cdot 10^{-14}$	$2.22 \cdot 10^{-16}$	0	0	$7.31 \cdot 10^{-15}$	$8.88 \cdot 10^{-16}$
	$i = 2$	$2.96 \cdot 10^{-12}$	$1.34 \cdot 10^{-14}$	$1.23 \cdot 10^{-14}$	$3.55 \cdot 10^{-15}$	58.12	9.17

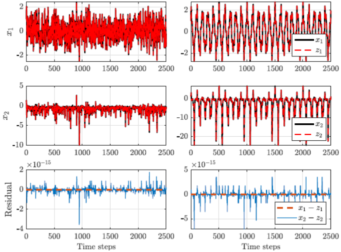

Fig. 2. Discrete-time example: state response of the original nonlinear system (67) given in black and its exact Koopman representation (69) given in red under white noise (left panel) and multisine (right panel) excitation u. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

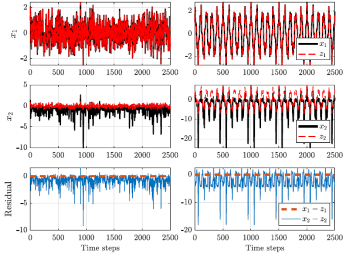

Fig. 3. Comparison to the approx. LTI form: state response of the nonlinear system (67) given in black and its approx. LTI Koopman model (70) given in red under white noise (left panel) and multisine (right panel) excitation u. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

- White noise $u_k \sim \mathcal{N}(0, 0.5)$ and multi-sine signal with sum of 6 sinusoids (frequency range $[0.1, 1]$ Hz for u_1 and $[1, 10]$ Hz for u_2)

Approximation by LTI Koopman form

Numerical result: the accuracy of the resulting model can drastically decrease if only an LTI approximation is used.

For DT case (67), with a constant input matrix $\hat{B}$, the Koopman form is

$$z_{k+1} = Az_k + \hat{B}u_k, \quad x_k = [I_2 \ 0_{2\times 1}]z_k \tag{70}$$

Find $\hat{B}$ is to minimize the average 2-norm between B and $\hat{B}$

1. take grid point in $(\mathbb{X}, \mathbb{U})$ w.r.t. experiment trajectory (67)
2. formulate matrices
   $Z = [\Phi(x_0) \ \cdots \ \Phi(x_{N-1})]$, $Z^+ = [\Phi(x_1) \ \cdots \ \Phi(x_N)]$, $U = [u_0 \ \cdots \ u_{N-1}]$
3. with A derived analytically in (68), a numerical input matrix $\hat{B}$ is

$$\hat{B} = (Z^+ - AZ)U^\dagger \tag{71}$$

- This is a least-squares method used in EDMD [Korda & Mezić, Automatica'18]
  - Compute A and B

$$\begin{bmatrix} A & B \end{bmatrix} = VG^\dagger, \quad V = \begin{bmatrix} Z \\ U \end{bmatrix} \begin{bmatrix} Z \\ U \end{bmatrix}^\top, G = Z^+ \begin{bmatrix} Z \\ U \end{bmatrix}^\top \tag{72}$$

If increase dictionary, EDMD may create spurious eigenvalues that can procedure worse results in simulations [Brunton et.al, SIAM Rew'22]

¶ Lifting + Thikonov regularization $\Gamma = \alpha I$, the estimator is

$$[A \quad B] = Z^{+} Y^{\top} (Y Y^{\top} + \alpha I)^{-1}, \quad Y = [Z^{\top} \; U^{\top}]^{\top} \tag{73}$$

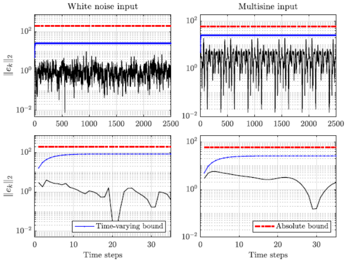

Fig. 4. Evolution of the state-response error $\|e_k\|_2$, the time-varying error bound (60) (blue), and the absolute error bound (55) (red) for the approximate LTI Koopman model under white noise (left panel) and multisine inputs (right panel). Zoomed in view of the initial time steps is provided in the second row. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

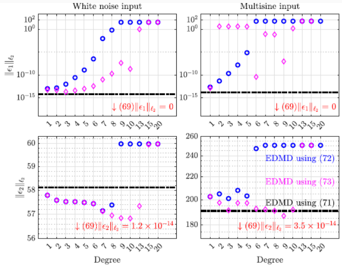

Fig. 5. ℓ_2 error of simulated state trajectories of the LTI Koopman model obtained with observables of a full dictionary of monomials with increasing degrees (blue circles), the model obtained using regularized EDMD (magenta diamonds), the model obtained via (71) (black line), and the exact Koopman representation (69) (red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

- More observables do not improve the second state's approximation and harm the first state's. While Thikonov regularization (a.k.a. ridge regres) can improve.

Conclusion

Summary

The take home messages of this paper are as follows:

- ★ It derives a Koopman representation of both **CT** and **DT** general nonlinear systems with **inputs**
- ★ The lifted forms are **LPV** models, enabling LPV tools for analysis and control of nonlinear systems
- ★ Give an **exact representation** of the original dynamics, unlike purely LTI Koopman models
- ★ An **error bound** is derived to describe the approximation capability of LTI Koopman models.

Outlook: Koopman Forms for Uncertian Nonlinear System

Consider an uncertain nonlinear system

$$x^+ = f(x,u,d) = f_1(x) + g_1(x)u + g_2(x)d, \quad y = h(x) \tag{B.1-1}$$

$$\text{or} \quad x^+ = f_\delta(x,u) = f(x,\delta) + g(x,\delta)u, \quad y = h(x) \tag{B.1-2}$$

where x is the state, u is input, and d is the external disturbance, and $\delta \in \Delta$ is the parametric uncertainty.

♣ "**autonomous**" + "**input-driven**" + "**disturbance-driven**" dynamics

Example: Nonlinear Disturbance Observer Based Control (NDOBC)

- Lost of real-world and industrial applications

Questions & Comments ?

Thank you for your listening!

Appendix

- Notations: $\rho(A) = \max_{r\in\lambda(A)} |r|$ denotes the spectral radius of a matrix $A \in \mathbb{R}^{n\times n}$ with eigenvalues $\lambda(A)$ and $\bar{\sigma}(P)$ is the maxi. singular value of $P \in \mathbb{R}^{m\times n}$. $\|v\|_2$ is the Euclidean norm of a vector $v \in \mathbb{R}^n$ and $\|P\|_{2,2}$ is the induced $2,2$ matrix norm $\|P\|_{2,2} = \sup_{v\in\mathbb{R}^n\setminus 0} \frac{\|P\|_2}{\|v\|_{\ell_2}} = \bar{\sigma}(P)$. For a DT signal $v : \mathbb{Z}_0^+ \to \mathbb{R}^n$, $\|v\|_2 = \sqrt{\sum_{k=0}^{\infty} \|v_k\|_2^2}$, $\|v\|_{\ell_\infty} = \max_{k\in\mathbb{Z}_0^+} \|v\|_2$