Data-Driven Sparse Feedback with Schur α-Stability

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LTI System and Stability

Discrete-Time Linear Time Invariant (LTI) System

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \in \mathbb{N}_{\geq 0},$$

where

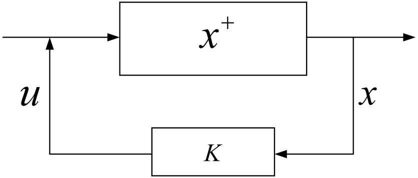
- state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^n$.
- the pair (A, B) is reachable.

Static State Feedback

Find a static state feedback controller

$$u = Kx, \quad K \in \mathbb{R}^{m \times n}$$

that stabilizes closed-loop $x^+ = (A + BK)x$.



LMI: Schur α-Stability

Definition 1 (Schur α-Stability)

Given a LTI system, seek a state feedback gain matrix $K \in \mathbb{R}^{m \times n}$ such that it holds

$$1 - \alpha \triangleq \max_{1 \leq i \leq n} |\lambda_i(A + BK)| < 1.$$

$(\Longleftrightarrow)$

A gain K exists iff introduces new matrices $Q \in \mathbb{R}^{n \times n}$ and $G \in \mathbb{R}^{m \times n}$ which satisfies

$$\begin{bmatrix} Q & Q^\top A^\top + G^\top B^\top \\ AQ + BG & (1 - \alpha)^2 Q \end{bmatrix} \succ 0, \quad Q = Q^\top \succ 0.$$

Then, it admits a Schur α-stabilizing gain

$$K = GQ^{-1}.$$

- Asymptotic stability: $A + BK/(1 - \alpha)$ when $\alpha = 0$.

Temporal Vs. Spatial:

How does one infer Input Sparsity from Structural Sparsity?

Recap of Model-Based Sparse Feedback Control

Problem (Sparse State Feedback Control) For a discrete LTI plant, finding a state feedback $u = Kx$, then sparse feedback control u with input sparsity satisfies the next conditions

(i) the discrete time closed-loop (feedback) system

$$x(t+1) = A_{cl}x(t), \ \ A_{cl} \doteq (A+BK) \in \mathbb{R}^{n \times n},$$

is *Schur stable* if for all eigenvalues have $|\lambda_i(A_{cl})| < 1$;

(ii) gain matrix $K \in \mathbb{R}^{m \times n}$ enjoys "row-sparsity" that penalizes $\ell_{1,\infty}$ matrix norm [Tropp, SP'06]

$$\|K\|_{1,\infty} = \sum_{i=1}^{m} \max_{1 \le j \le n} |k_{i,j}|$$

by promoting *the number of zero-valued rows*

Revisit Condition (i)

- (i) $\Rightarrow$ (i') Let α[1], then A_{cl} is Schur α-stable *iff*

$$\exists Q \succ 0, \quad A_{cl} P A_{cl}^\top - (1-\alpha)^2 Q \prec 0, \quad 1-\alpha \doteq \max_{1\le i\le n} |\lambda_i(A_{cl})| < 1$$

- (i') $\Leftrightarrow$ (i'') introduce new variables G, Q meeting $G = KQ$ and the recovered the expression $A_{cl} = A + BK$, the LMI is as follows (see **Definition 1**)

$$\begin{bmatrix} Q & QA^\top + G^\top B^\top \\ AQ + BG & (1-\alpha)^2 Q \end{bmatrix} \succ 0, \quad Q \succ 0.$$

- (ii) $\Rightarrow$ Finding K equals to solve V, thus K and V have to share the same row-sparsity.

[1] $1-\alpha$ is the spectral radius of the closed-loop matrix A_{cl}

Motivation: Communication, Control, Calculation

- For $u = Kx$, introduce new variables G, Q which fits $K = GQ^{-1}$.

$$\underbrace{\begin{bmatrix} * \\ 0 \\ \vdots \\ * \\ 0 \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} * & * & \cdots & * \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{K} \times \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \cdot \\ x_n \end{bmatrix}}_{x}, \implies K = \underbrace{\begin{bmatrix} * & * & \cdots & * \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{G} \times \underbrace{\begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \\ * & * & \cdots & * \end{bmatrix}}_{Q^{-1}}$$

Asserts:

1. matrix $\|K\|_{1,\infty}$ shares the same **Zero-Rows Structure** with matrix $\|G\|_{1,\infty}$.
2. **Structured Row Sparsity** on state feedback gain induces **Control Input Sparsity.**
3. the "**active**" **inputs** w.r.t. the indices of **non-zero rows** of gain matrix K.

Model-Based Sparse Feedback Control

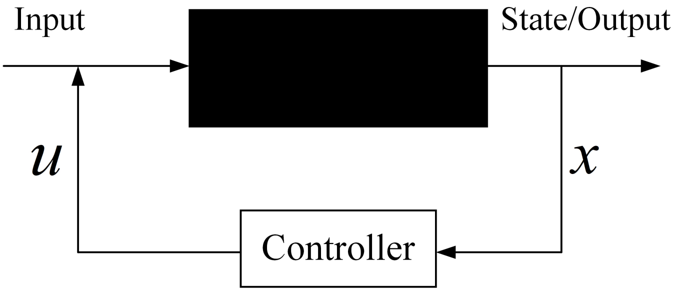
Given the knowledge of model, i.e., (A, B) is known.

Problem 1 (Model-Based Sparse Feedback Control) [B. Polyak, Khlebnikov & Shcherbakov, ECC'13]

For model-based LTI systems, find the solutions G, Q by using a variable $G = KQ$ s.t. LMIs

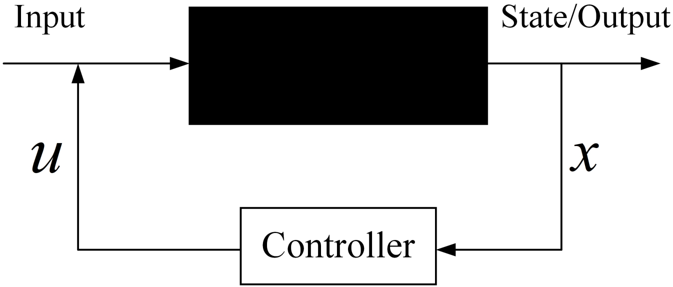
$$\min_{G, Q=Q^\top} \|G\|_{1,\infty} \quad \text{s.t.} \quad \begin{bmatrix} Q & QA^\top + G^\top B^\top \\ AQ + BG & (1-\alpha)^2 Q \end{bmatrix} \succeq 0, \quad Q \succ 0,$$

A Black-Box System?



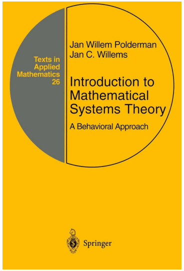
- Question: How should I design a controller?
  - Collect Data → Identify System/Model → Design Controller
    - “Why not Skip SysID if our focus is only control?”

A Black-Box System?



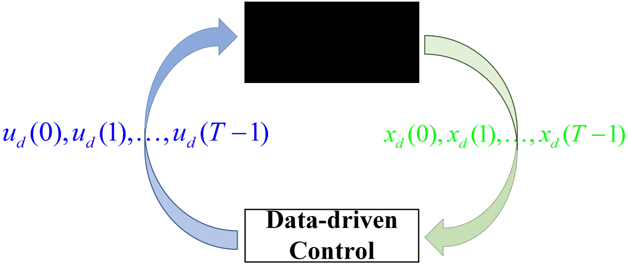
- Question: How should I design a controller?
  - Collect Data → ~~(Identify System/Model)~~ → Design Controller
    - "Why not Try Direct data-driven control?"

Let's try to lift model-based system using experimental data directly ...



Pictures from Google...

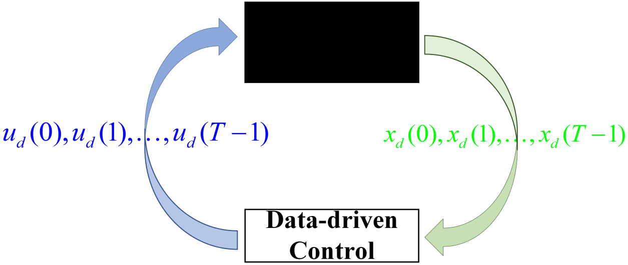
Data-Enabled System Representation



- Harvest Input-State data trajectoires [“d” is a single experimental data samples]

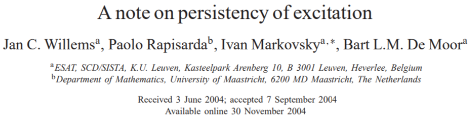
$$U_{0,T} = \begin{bmatrix} u_d(0) & u_d(1) & \cdots & u_d(T-1) \end{bmatrix},$$
$$X_{0,T} = \begin{bmatrix} x_d(0) & x_d(1) & \cdots & x_d(T-1) \end{bmatrix},$$
$$X_{1,T} = \begin{bmatrix} x_d(1) & x_d(2) & \cdots & x_d(T) \end{bmatrix}.$$

Data-Enabled System Representation



Assumption 1 (Full Rank Condition)

$$\operatorname{rank}\begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} = m + n.$$



A note on persistency of excitation

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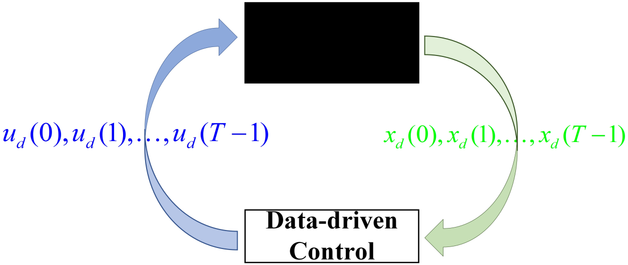
- T implies "rich" if input $U_{0,T}$ meets "persistently exiciting" (PE).

Data-Enabled System Representation



Data-Enabled Open-Loop

$$\begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} = X_{1,T}$$

Data-Based Closed-Loop [De Persis & Tesi, IEEE TAC'19]

$$A + BK = [B \;\; A] \begin{bmatrix} K \\ I_n \end{bmatrix} \Longleftrightarrow [B \;\; A] \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} G_K = X_{1,T} \, G_K$$

Direct Data-Based Stabilization

- A new variable transformation

$$V = G_K Q,$$

and Schur's Complement reduces it to the data-based LMI

$$\begin{bmatrix} X_{0,T} V & V^\top X_{1,T}^\top \\ X_{1,T} V & X_{0,T} V \end{bmatrix} \succ 0$$

with

$$\begin{bmatrix} G \\ Q \end{bmatrix} = \begin{bmatrix} K \\ I_n \end{bmatrix} Q = \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} \underbrace{G_K Q}_{:=V}$$

- The solution to LMI returns V, the feedback gain is derived via

$$\begin{bmatrix} K \\ Q \end{bmatrix} = \begin{bmatrix} U_{0,T} G_K \\ X_{0,T} V \end{bmatrix} \implies \boxed{K = U_{0,T} V (X_{0,T} V)^{-1}}$$

Direct Data-Based Schur α-Stabilization

Lemma Given a degree of stability α, any matrix V satisfying

$$\begin{bmatrix} X_{0,T}V & V^\top X_{1,T}^\top \\ X_{1,T}V & (1-\alpha)^2 X_{0,T}V \end{bmatrix} \succ 0$$

is such that

$$K = U_{0,T}V(X_{0,T}V)^{-1}$$

is a Schur α-stabilizing state feedback gain for LTI system x^+.

Main Result

Theorem 1 (Data-Based Sparse Feedback Control Synthesis)

Given data samples $U_{0,T}$, $X_{0,T}$, $X_{1,T}$, then data-based Problem 1 with decision variable V is as

$$\begin{aligned} \min_{V} \quad & \|U_{0,T}V\|_{1,\infty} \\ \text{s.t.} \quad & \begin{bmatrix} X_{0,T}V & V^\top X_{1,T}^\top \\ X_{1,T}V & (1-\alpha)^2 X_{0,T}V \end{bmatrix} \succ 0, \quad X_{0,T}V = V^\top X_{0,T}^\top. \end{aligned} \tag{1}$$

- Direct data-driven method: use data samples $X_{0,T}$, $U_{0,T}$ instead of model (A, B)
- Computational tractable: convex optimization with SDP using `CVX` or `YALMIP`.

Example

Data-based stabilization of the **He**licopter model with 8-order, 4-inputs (HE3) [Leibfitz, Compleib'06]

- PE Input and data samples takes as follows

```
load('HE3'.mat)
T = 46, n = 8, m = 4;
u = rand(m,T-1),  x0 = rand(n,1),  u0 = rand(m,1);
```

- Give a degree of stability `alpha=0.07`
- Solve Theorem 1, we use `cvx` in MATLAB

```
cvx_begin sdp
    variables V(T,n)  Q(n,n) symmetric
    minimize  sum(max(abs(U*V), [], 2))
    subject to
    [Q, X1*V;  V'*X1', (1-alpha)^2*Q] > 0
    X0*V == Q,  Q>0;
cvx_end
K_dsp= (U*V)*inv(Q);
```

Numerical Benchmark

Q =

1.0e+05 *

9.0445	-2.2118	-0.2702	3.0435	-0.4438	-0.4747	0.5791	0.2015
-2.2118	3.7322	0.1332	-2.5459	0.0487	0.1525	-0.1943	0.0570
-0.2702	0.1332	0.0130	-0.2214	0.0078	-0.0018	-0.0209	0.0031
3.0435	-2.5459	-0.2214	4.8525	0.0190	0.3263	0.2906	-0.2049
-0.4438	0.0487	0.0078	0.0190	0.0365	0.0225	-0.0250	-0.0255
-0.4747	0.1525	-0.0018	0.3263	0.0225	0.2189	-0.0169	-0.0365
0.5791	-0.1943	-0.0209	0.2906	-0.0250	-0.0169	0.0398	0.0065
0.2015	0.0570	0.0031	-0.2049	-0.0255	-0.0365	0.0065	0.0258

Numerical Benchmark

```
K_dsp =

   -0.0000    0.0000    0.0001    0.0000    0.0000    0.0000    0.0001    0.0000
   -1.6284   -0.0115   34.2895   -0.8888   -3.0393   -0.1886   48.4596  -14.0172
    0.3893   -0.0398   -3.3797   -0.7727  -11.4249   -0.5043   -6.2644  -19.1246
    0.0000   -0.0000   -0.0000   -0.0000   -0.0000   -0.0000   -0.0000   -0.0000

>> G=U*V

G =

    0.0002    0.0005   -0.0001   -0.0012    0.0003    0.0002    0.0008    0.0000
  266.7809 -266.7764  266.7776  266.7780  257.9931  266.7767  121.4764 -266.7773
  -69.8611   69.8638   69.8645   69.8640  -69.8650  -69.8649  -69.8654  -69.8646
   -0.0016   -0.0004    0.0000   -0.0002   -0.0001   -0.0003   -0.0003   -0.0001
```

Numerical Benchmark

```
eig_vals =

   0.7507 + 0.1566i
   0.7507 - 0.1566i
   0.8508 + 0.0867i
   0.8508 - 0.0867i
   0.9204 + 0.0422i
   0.9204 - 0.0422i
   0.8583 + 0.0000i
   0.8963 + 0.0000i
```

Conclusions

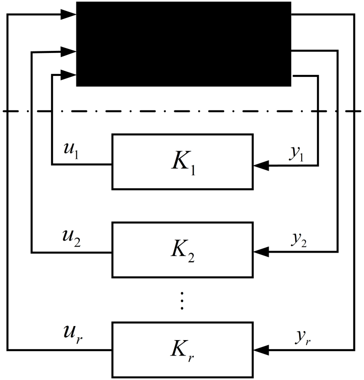
The Take-Away Message

Based on PE input data, learning sparse feedback control with Schur α-stability from a single experimental input-state data samples for a black-box LTI system.

More Specifically...

1. Infer a "direct" data-driven sparse control with **input sparsity** from **structured sparsity**
2. Reformulate **model-based** **sparse feedback control** as **data-based** representation
3. State-feedback Schur α-stabilization

Future Work: Decentralized Sparse Output/State Control



Given a model-free plant, and each pair (A, B_j), $j = 1, \ldots, r$ is *unknown*, i.e.,

$$x^+ = Ax + \sum_{j=1}^{r} B_j u_j, \quad y_i = C_i x, \quad u_i = K_i y_i, \quad i = 1, \ldots, r,$$

Thank you for your attention

Suggestions & Comments are Welcome!

ご清聴ありがとう
ございました