Linear Quadratic Tracking Control with Sparsity-Promoting Regularization

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Abstract

In this poster, we show a novel linear quadratic (LQ) tracking control with sparsity regularization using $\mathcal{L}^0$ norm. Sparsity regularization leads to sparse control, which has a significant length of time over which the control is exactly zero. Since the $\mathcal{L}^0$ cost is non-convex and discontinuous, we introduce $\mathcal{L}^1$ relaxation to make the optimization numerically tractable. The numerical solution is obtained with the aid of time-discretization to derive a discrete-time optimal control problem with ℓ^1 regularization. We also show the robustness for the perturbed tracking system. Some examples illustrate the effectiveness of the proposed control.

Sparse Optimization

1. Problem formulation. Consider a master-slave system tracking problem

$$\begin{cases} \dot{z}_{\mathrm{m}}(t) = Az_{\mathrm{m}}(t), \quad t \geq 0, \quad z_{\mathrm{m}}(0) = \xi_{\mathrm{m}} \in \mathbb{R}^n \\ \dot{z}_{\mathrm{s}}(t) = Az_{\mathrm{s}}(t) + Bu(t), \quad t \geq 0, \quad z_{\mathrm{s}}(0) = \xi_{\mathrm{s}} \in \mathbb{R}^n \end{cases} \tag{1}$$

where $z_s(t), z_m(t) \in \mathbb{R}^n$ are the state variables and $u(t) \in \mathbb{R}$ is a single control input.
The tracking goal is to seek a control $u(t)$ while achieving

$$\lim_{t\to\infty} \|z_s(t) - z_m(t)\| \doteq \lim_{t\to\infty} \|x(t)\| = 0 \tag{2}$$

Then, we have the following tracking error system for final time $T > 0$:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = \xi = \xi_s - \xi_m, \quad x(T) = 0, \quad |u(t)| \leq 1 \tag{3}$$

Standard Linear quadratic (LQ) performance cost is

$$J_{LQ}(u) = \frac{1}{2}\int_0^T \left\{ x(t)^\top Qx(t) + ru(t)^2 \right\} dt, \quad Q^\top = Q > 0, \quad r > 0 \tag{4}$$

2. LQ Hands-off (Sparse) Control. We wish to find a feasible control $\{u(t) : 0 \leq t \leq T\}$ that *achieves tracking as well as minimizes the control efforts.* This urges us to consider the *sparsity method* to promote the zero inputs such that taking more zero values in control actuator, that is, minimal control efforts. To this end, we propose the *LQ hands-off (sparse) control problem* via *sparse optimization*

$$\text{minimize } J_0(u) \quad \text{subject to} \quad (3) \tag{P_0}$$

where $J_0(u) = J_{LQ}(u) + \lambda\|u\|_0$, $\|u\|_0$ is the $\mathcal{L}^0$ norm and $\lambda > 0$ is the weight [1].
Note that this problem is *non-smooth, non-convex, and discontinuous* since the cost is related to $\mathcal{L}^0$ penalty. Here, we tackle this thorny problem directly by means of *non-smooth maximum principle* [2], and derive the related optimality conditions using $\mathcal{L}^0$ norm.

3. Necessary Conditions. For the necessary conditions, we define the *Hamiltonian function* by

$$H^\eta(x, p, u) \triangleq p^\top(Ax + Bu) - \eta\left(\frac{1}{2}(x^\top Qx + ru^2) + \lambda|u|^0\right), \tag{5}$$

where $p \in \mathbb{R}^n$, $\eta \in \{0, 1\}$, $|u|^0 = 1$ if $u \neq 0$ and 0 otherwise. The necessary conditions can be directly derived from [2. Th 22.26].

Numerical Computation

In this part, we introduce the numerical method to make the LQ hands-off control problem computationally tractable. The solvable approach is a three-step process:

1. Convex Relaxation. First, we adopt $\mathcal{L}^1$ norm to relax the (P_0), then the optimization problem is

$$\text{minimize } J_1(u) \quad \text{subject to} \quad (3) \tag{P_1}$$

where $J_1(u) = J_{LQ}(u) + \lambda\|u\|_1$ and $\|u\|_1$ is the $\mathcal{L}^1$ norm of control signal.

2. Time-discretization. Second, we apply time-discretization for (P_1). Let $m \in \mathbb{N}$ be the number of partitions over time interval $[0, T]$

$$[0, T] = [0, h) \cup [h, 2h) \cup \cdots \cup [(m-1)h, mh], \tag{6}$$

where $h = T/m$ is the discretization step. Also, we assume that the control is zero-order hold. Namely,

$$u(t) = u_d^k, \ t \in [kh, (k+1)h), \ k = 0, 1, \ldots, m-1.$$

The discontinuous tracking error system in (3) is discretized as

$$\begin{aligned} &x_d^{k+1} = A_d x_d^k + B_d u_d^k, \ x_d^0 = \xi, \ x_d^m = 0, \\ &|u_d^k| \leq 1, \ k = 0, 1, \ldots, m-1, \end{aligned} \tag{7}$$

where $x_d^k = x(kh)$ for $k = 0, 1, \ldots, m-1$, and

$$A_d \triangleq e^{Ah}, \quad B_d \triangleq \int_0^h e^{At} B dt. \tag{8}$$

Also, the cost J_{LQ} is discretized as

$$J_{LQ}^d = \frac{1}{2}\sum_{k=0}^{m-1} \begin{bmatrix} x_d^k \\ u_d^k \end{bmatrix}^\top \begin{bmatrix} Q_d & S_d \\ S_d^\top & R_d \end{bmatrix} \begin{bmatrix} x_d^k \\ u_d^k \end{bmatrix} \tag{9}$$

Finally, the relaxed $\mathcal{L}^1$ norm of control signal is

$$\|u\|_1 = \sum_{k=0}^{m-1} \int_{kh}^{(k+1)h} |u(t)|\, dt = \sum_{k=0}^{m-1} |u_d^k| = h\|u_d\|_{\ell^1}$$

where $\|u_d\|_{\ell^1}$ is the ℓ^1 norm of $u_d = [u_d^0, \cdots, u_d^{m-1}]^\top$.

3. Finite-dimensional Optimization. The original problem (P_0) can be reduced as (P_1), and can be further discritized as the following problem

$$\text{minimize } J_{LQ}^d + \lambda h\|u_d\|_{\ell^1} \quad \text{subject to} \quad (7) \tag{P_2}$$

The problem (P_2) is a convex optimization problem for finite-dimensional vector $u_d \in \mathbb{R}^m$, which is efficiently solved by toolboxes `CVX` with MATLAB.

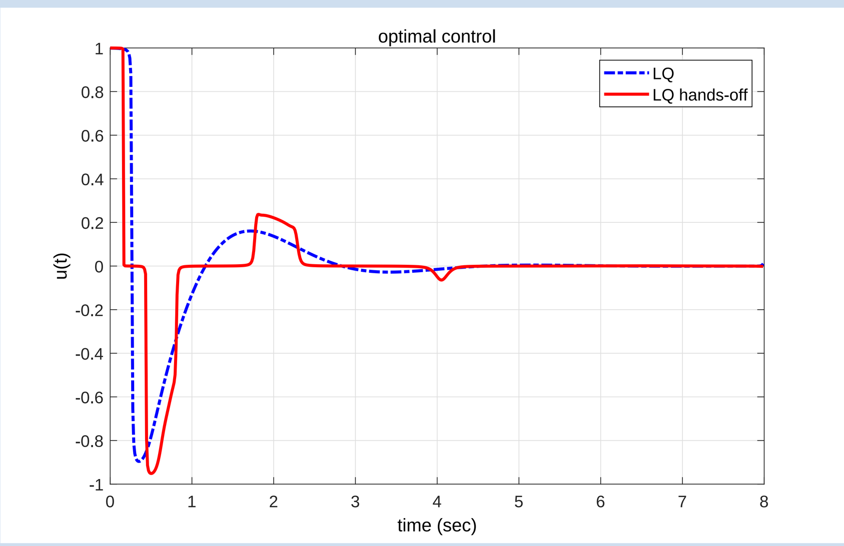

Fig 1. LQ control and LQ hands-off control.

Robustness & Simulations

We assume the master-slave systems share the same matrix A, and we discuss the robustness of the tracking error system (3) since a gap Δ between A matrices. The perturbed linear plant model is expressed by

$$\dot{x}(t; \Delta) = (A + \Delta)x(t; \Delta) + Bu(t). \tag{10}$$

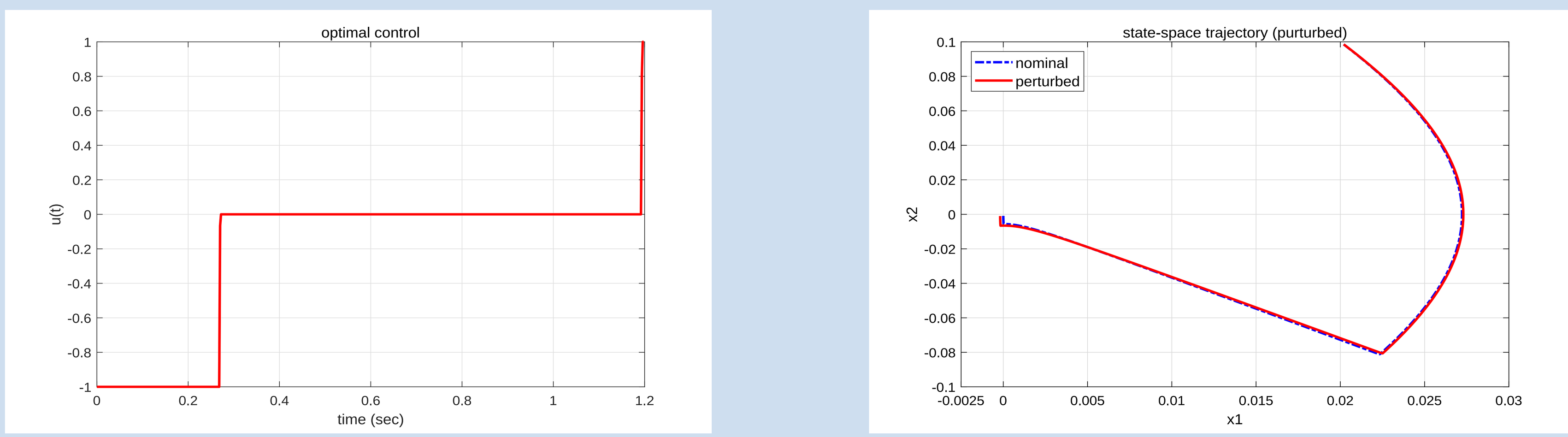

Fig 2. LQ sparse inputs (**Left**) and Phase portraits (**Right**) of the perturbed system with gap Δ.

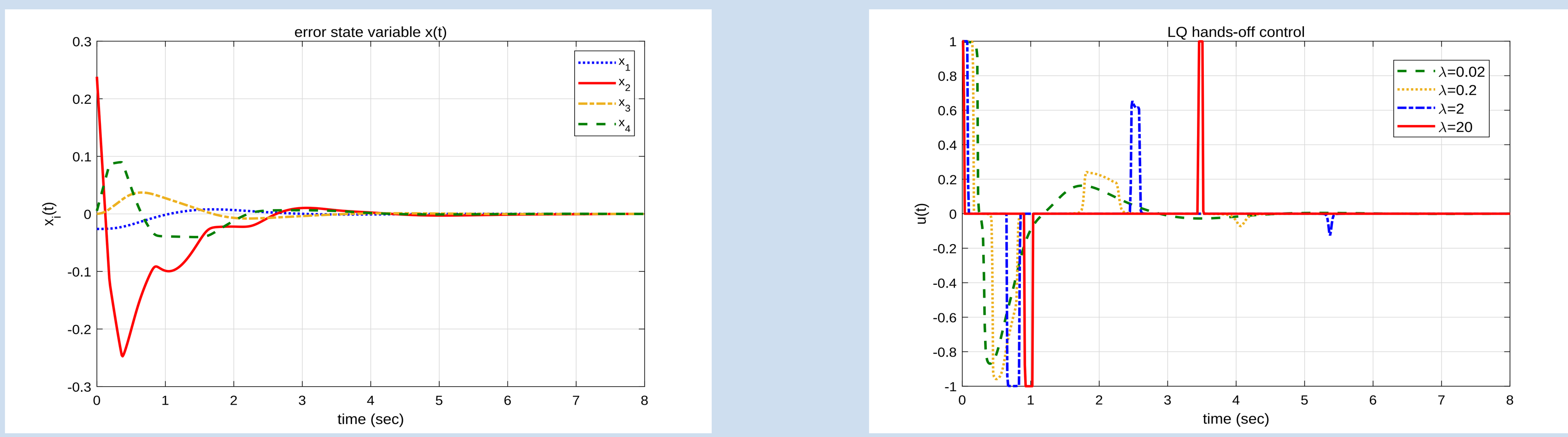

Fig 3. Tracking error states (**Left**) and LQ sparse control with different weights λ (**Right**) for system (3).

Conclusion

- Necessary conditions for LQ hands-off control
- LQ hands-off control may not be continuous
- Worst-case approach for robustness

Reference

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[2] F. Clarke, *Functional Analysis, Calculus of Variations and Optimal Control.* Springer, 2013.