To Minimize Actuator Switchings and Hold Discrete-Valued Control Signals
Part I

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Outline & Goal

- Sparsity-Promoting Method
  - Sparsity of a vector is measured by its ℓ^0 norm.
- Discrete-Valued Control (or Quantized Control) Signal
  - The control contains quantized (i.e., piecewise constant) signals.
  - System whose input takes only values on a discrete-valued set.
  - Control Actuator changes from one value in a finite alphabet to another may leads to many switchings (i.e, control variation).
- Sparse Optimization for Controller (Signals & Switchings)
  - We propose a new control optimization problem, which aims to

Minimize the Actuator Switchings
As Well As
Hold Discrete-Valued Control Signals !

ℓ^0 norm and its sparsity

- For a vector $x \in \mathbb{R}^n$ is sparse, if it contains many zero elements, or has ℓ^0 *"norm"* [Rish & Grabarnik, 2014]

$$\|x\|_0 = \#(\mathrm{supp}(x)) = \text{the number of nonzero elements in } x$$

  where $\mathrm{supp}(x) \triangleq \left\{i \in \{1, 2, \cdots, n\} : x_i \neq 0\right\}$ denotes the support set of x.

- Strictly speaking, the ℓ^0 *norm* is not a norm since it fails to obey the *absolutely homogeneous* property of the norm, that is, for any nonzero scalar β such that $|\beta| \neq 1$, and

$$\|\beta x\|_0 = \|x\|_0 \neq |\beta| \|x\|_0, \quad \forall x \neq 0,$$

  and hence we sometimes call it as ℓ^0 *pseudo-norm* or *cardinality*.

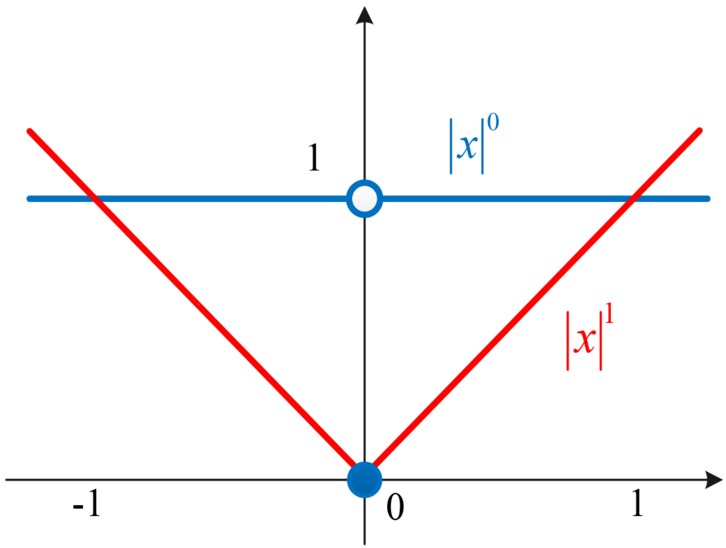
- However, for convenience's sake, we still use "ℓ^0 norm" in this presentation.

Relation between ℓ^0 norm and ℓ^1 norm

ℓ^p norm: $\|x\|_p \triangleq \left(\sum_{i=1}^{n} |x|^p \right)^{\frac{1}{p}}, \ p \in [1, \infty]$

ℓ^0 norm: $\|x\|_0 = \sum_{i=1}^{n} |x|^0, \quad |x|^0 \triangleq \begin{cases} 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$ nonconvex

ℓ^1 norm: $\|x\|_1 = \sum_{i=1}^{n} |x|,$ sum of absolute values $|x|$, convex



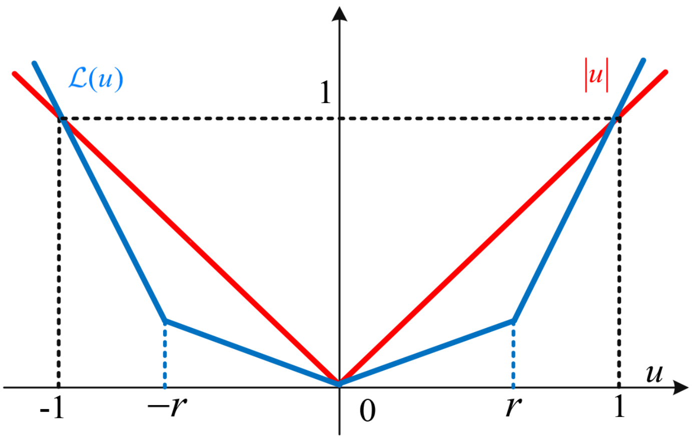
Discrete-Valued Control

- L^1 optimal control (or *Minimum fuel control*)[Nagahara et al, 2016 TAC]

$$\min_{u\in\mathcal{U}} \int_0^T |u(t)|dt \ \to u(t) \text{ takes } \{-1,0,1\} \ \to \textit{Bang-off-Bang}$$

- Let us consider:

$$\min_{u\in\mathcal{U}} \int_0^T \mathcal{L}(u(t))dt \ \to u(t) \text{ takes } \{\pm 1, \pm r, 0\} \ \to \textit{Discrete-Valued}$$



Problem formulation

Nonlinear system [Ikeda & Nagahra, 2017 TAC]

$$\dot{x}(t) = f(x(t)) + y(x(t))u(t), \quad t \in [0, T],\ x(0) = \xi \in \mathbb{R}^n.$$

Feasible control

Fix $T > 0$. Find a *discrete-valued control* $u(t)$ that brings the initial state $x(0)$ to $x(T) = 0$ and satisfies $U_{\min} \le u(t) \le U_{\max}$, $\forall t \in [0, T]$.

Discrete-valued control

Find a feasible control to achieve discrete-valued signals as follows

$$u(t) \in \mathbb{U} \triangleq \{U_1, U_2, \ldots, U_N\},$$

where $U_{\min} = U_1 < U_2 < \cdots < U_N = U_{\max}$.

Example: Discrete-Valued Signals

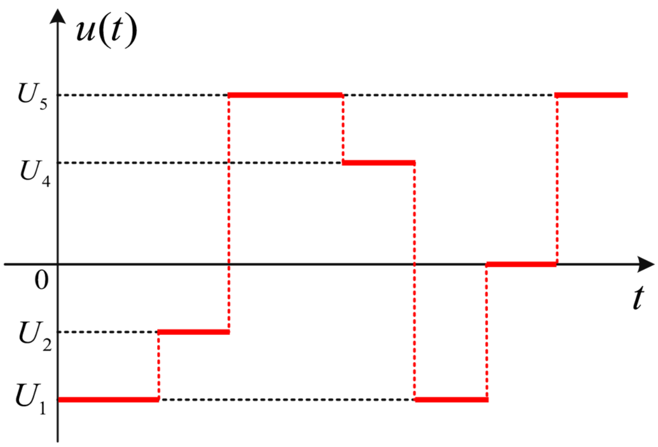
- Let the control $u(t)$ takes $N = 5$ real numbers, then a finite alphabet (or discrete set) $\mathbb{U}$ is given by

$$\mathbb{U} = \{U_1, U_2, U_3, U_4, U_5\},$$

and also satisfying

$$U_1 < U_2 < U_3 < U_4 < U_5.$$

The sketch of discrete-valued control signals is shown as



Construct Discrete-Valued Control Signals

- Sparse discrete-valued control signals:

$$\sum_{i=1}^{N} \gamma_i \|u - U_i\|_0 = \sum_{i=1}^{N} \gamma_i \int_0^T \chi_0(u(t) - U_i)dt.$$

where $\chi_0(\cdot)$ is the kernel function defined by

$$\chi_0(\alpha) = \begin{cases} 1, & \text{if } \alpha \neq 0, \\ 0, & \text{if } \alpha = 0. \end{cases}$$

  - If $u(t) = U_i$ for some intervals $\mathcal{I} \in [0, T]$, then $u(t) - U_i = 0$ over these intervals $\mathcal{I}$, and hence it leads to *sparse*.
  - Extend the real value U_i to a finite alphabet $\{U_1, \cdots, U_N\} \triangleq \mathbb{U}$, then it will generate a series of *piecewise cosntant control signals*.
  - Choose the weights $\gamma_1, \cdots, \gamma_N$ according to the importance of values $\{U_1, \cdots, U_N\}$.
  - This form is L^0 cost, and hence it is difficult to directly solve.
  - Technique: `Mixed-Integer Programming`, L^1 `Relaxation` etc.

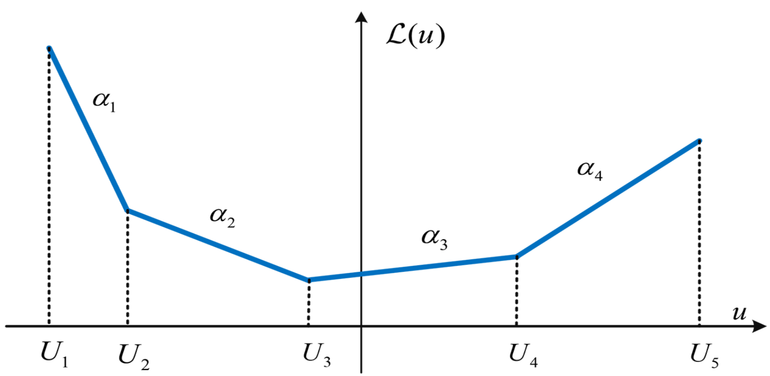
Sum of Absolute Values

- Construct $\mathcal{L}(u)$ by *sum-of-absolute-values* (SOAV) approach:

$$\mathcal{L}(u) := \sum_{i=1}^{N} \gamma_i |u - U_i|, \ \gamma_i > 0, \ \gamma_1 + \cdots + \gamma_N = 1.$$

α_k is the slope of the line between the U_k and U_{k+1},

$$\alpha_k = \sum_{i=1}^{k} \gamma_i - \sum_{i=k+1}^{N} \gamma_i, \quad k = 1, 2, \cdots, N-1.$$



Control Variation (Actuator Switchings)

- The frequency at actuator changes [Kishida et al, 2019 CDC]

$$\bigvee_0^T u = \sum_{l=0}^{q-1} |u(t_{l+1}) - u(t_l)|$$

- Control variations for continuous form [Royden & Fitzpatrick, 2010]

$$\bigvee_0^T u = \int_0^T |\dot{u}(t)|dt, \quad |\dot{u}(t)| \leq \delta,$$

  - The Derivative of Control Signal.
  - The magnitude of the derivative (i.e. the number of control switching) should be bounded.
  - Also known as Minimum Attention Control ! [Brockeet, 1997 CDC]

Discrete-Valued Control with Minimal Switchings

Sparse Discrete-valued Control with Sparse Switchings

Plant: $\dot{x}(t) = f(x(t)) + \sum_{j=1}^{m} y_j(x(t))u_j(t)$, $x(t) \in \mathbb{R}^n$, $u_j(t) \in \mathbb{R}$.

Assumptions: f, y_j, f', y_j' are continuous in x.

Boundary conditions: $x(0) = \xi$, $x(T) = 0$, $\forall t \in [0, T]$, state
$u(0) = \theta$, $u(T) = 0$, $\forall t \in [0, T]$, control

Constraints: $u_j \in \mathbb{U}$, $U_{\min} \le u_j \le U_{\max}$, $|u_j| \le 1$, $|\dot{u}_j| \le \delta$

Cost function: $$J_0(u) = \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{m} \gamma_i \|u_j - U_i\|_0}_{\text{sparse quantized signals}} + \underbrace{\sum_{j=1}^{m} \lambda_j \|\dot{u}_j\|_0}_{\text{sparse switchings}}$$

- Minimize the switchings and hold discrete-valued signals.
- Cost function is *discontinuous, nonconvex & nonsmmoth*.

Convex Relaxation: L^1 relaxation

SOAV Optimal Control with Minimal Actuator Switchings

$$\min \; J_1(u) = \sum_{j=1}^{m} \int_0^T \Bigg(\underbrace{\sum_{i=1}^{N} \gamma_i |u_j(t) - U_i|}_{\mathcal{L}(u_j(t))} + \underbrace{\lambda_j |\dot{u}_j(t)|}_{\mathcal{Q}(\dot{u}_j(t))} \Bigg) dt$$

$$\begin{aligned} s.t. \quad & \dot{x}(t) = f(x(t)) + \textstyle\sum_{j=1}^{m} y_j(x(t)) u_j(t), \; t \in [0, T] \\ & x(0) = \xi, \; x(T) = 0, \; u(0) = \theta, \; u(T) = 0, \\ & U_{\min} \le u_j(t) \le U_{\max}, \; |u_j(t)| \le 1, \; |\dot{u}_j(t)| \le \delta. \end{aligned}$$

- The integral terms penalize the cost of discrete-valued controlling and the frequency of the control variation.
- The control problem is *L^1 norm Convex Optimization* problem.
- Time discretization approach, $\rightarrow$ *Model Predictive Controller ?*

Conclusion

- Minimize Actuator Switchings and Hold Discrete-Valued Control Signals
  - ℓ^1-norm relaxation is one of the most important techniques for sparsity methods as approximation of ℓ^0-norm optimization.
  - Discrete-valued control via *SOAV Optimization*
  - Control variation based on the *Derivative of Control Input*
- Future Work
  - Augmented System Formulation (Optimality Condition) e.g. Maximum principle [Clarke, 2013]
  - Value function (The Continuity Property)
  - Numerical Computation
    - An example to Model Predictive Control
    - An example to Self-Triggered Control

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