Performance Bounds for the Scenario Approach and an Extension to a Class of Non-Convex Programs

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Paper Introduction

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Goal

This paper focuses on

- Consider the scenario convex program (SCP) for (relaxed) robust convex program (RCP) and chance-constrained program (CCP).
- Consider the tail probability of the worst-case violation.
- Build a probabilistic bridge from the optimal value of SCP ($J^\star_N$) to the optimal value of RCP ($J^\star_{\mathrm{RCP}}$) and CCP ($J^\star_{\mathrm{CCP}}$).

Problem Statement

Technical Statements

- Let $\mathbb{X} \in \mathbb{R}^n$ be a compact convex set and $c \in \mathbb{R}^n$ a constant vector.
- Let $(\mathcal{D}, \mathcal{B}(\mathcal{D}), \mathbb{P})$ be probability space, where $\mathcal{D}$ is a metric space with the respective Borel $\sigma-$algebra $\mathcal{B}(\mathcal{D})$.
- A measurable function $f : \mathbb{X} \times \mathcal{D} \mapsto \mathbb{R}$, which is convex in the first argument and bounded in the second argument.

Let us consider the following optimization problems:

Robust Convex Program (RCP)

$$\begin{aligned} \min_{x \in \mathbb{X}} \quad & c^\top x \\ \text{s.t.} \quad & f(x, d) \leq 0 \quad \forall d \in \mathcal{D} \end{aligned}$$

where the optimal value of the RCP is denoted by $J^\star_{\mathrm{RCP}}$.

Relaxed (Perturbed) RCP_γ

Consider the Relaxed (Perturbed) RCP for $\gamma > 0$

$$\begin{aligned} \min_{x \in \mathbb{X}} \quad & c^\top x \\ \text{s.t.} \quad & f(x, d) \leq \gamma, \quad \forall d \in \mathcal{D} \end{aligned}$$

with the optimal value $J^\star_{\mathrm{RCP}_\gamma}$.

Chance-Constrained Program (CCP_ε)

$$\begin{aligned} \min_{x \in \mathbb{X}} \quad & c^\top x \\ \text{s.t.} \quad & \mathbb{P}\big[f(x, d) \leq 0\big] \geq 1 - \varepsilon \end{aligned}$$

with the optimal value $J^\star_{\mathrm{CCP}_\varepsilon}$.

- x is an ε-probability robust design, if it holds that $\mathbb{P}\big[f(x, d) > 0\big] \leq \varepsilon$.

Standard Scenario Convex Program [Campi & Garatti]

- Assume probability measure $\mathbb{P}$ on $\mathcal{D}$
- Sample $(d_i)_{i=1}^N$ independent and identically distributed (i.i.d.) from $\mathbb{P}$
- Formulate the scenario convex program

Scenario Convex Program (SCP)

$$\begin{aligned} \min_{x \in \mathbb{X}} \quad & c^\top x \\ \text{s.t.} \quad & f(x, d_i) \leq 0, \quad \forall i \in \{1, \cdots, N\} \end{aligned}$$

where the optimal solution and value of SCP are denoted by $x_N^\star$ and $J_N^\star$.

SP: Value Bound

Theorem 2.2 (CCP$_\varepsilon$ Feasibility)

Let $\beta \in [0,1]$ and $N \geq N(\varepsilon;\beta)$ where

$$N \geq N(\varepsilon;\beta) := \min\left\{ N \in \mathbb{N} \mid \sum_{i=0}^{n-1} \binom{N}{i} \varepsilon^i (1-\varepsilon)^{N-i} \leq \beta \right\}$$

Then, the optimizer of SCP is a feasible solution of CCP$_\varepsilon$ with probability $\mathbb{P}^N$ (N-fold product probability measure) at least $1-\beta$.

- Alternatively, it states that $\mathbb{P}^N\left[x_N^\star \models \mathrm{CCP}_\varepsilon\right] \geq 1-\beta$, where $\models$ refers to the feasibility satisfaction, i.e., $x_N^\star \models \mathrm{CCP}_\varepsilon$ means that $x_N^\star$ is a feasible solution for the program CCP$_\varepsilon$.
- w.p. at least $1-\beta$, if SCP is feasible it holds that $J_N^\star \geq J^\star_{\mathrm{CCP}_\varepsilon}$
- It is sufficient to select a sample size $N \geq \frac{2}{\varepsilon}\left(n - 1 + \mathrm{In}\ \frac{1}{\beta}\right) \geq N(\varepsilon;\beta)$

Uniform Level-set Bound (ULB) [Kanamori & Takeda]

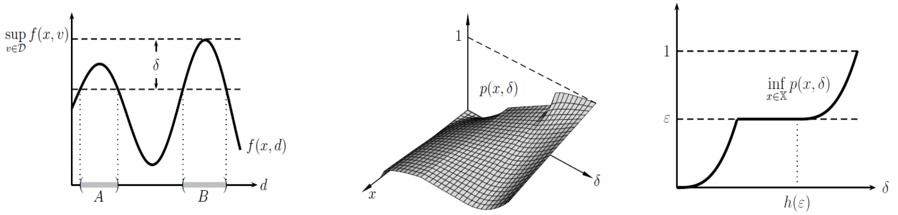
Definition 3.1

The *tail probability of the worst-case violation* $p : \mathbb{X} \times \mathbb{R}_+ \mapsto [0,1]$ is the function defined as

$$p(x,\delta) := \mathbb{P}\left[\sup_{v \in \mathcal{D}} f(x,v) - \delta < f(x,d)\right]$$

We call $h : [0,1] \mapsto \mathbb{R}_+$ a *uniform level-set bound* (ULB) of p if

$$h(\varepsilon) \geq \sup\left\{\delta \in \mathbb{R}_+ \mid \inf_{x \in \mathbb{X}} p(x,\delta) \leq \varepsilon\right\}, \quad \forall\, \varepsilon \in [0,1].$$



(a) $p(x,\delta) = \mathbb{P}[A \cup B]$

(b) Tail probability of the worst-case violation

(c) Uniform level set bound

Lemma 3.2

Let $h : [0,1] \mapsto \mathbb{R}_+$ be a ULB. Then

$$x \models \mathrm{CCP}_\varepsilon \Rightarrow x \models \mathrm{RCP}_{h(\varepsilon)}$$

that is, the feasible set of CCP_ε with constraint violation level ε is a subset of the feasible set of the relaxed program $\mathrm{RCP}_{h(\varepsilon)}$ with $\gamma := h(\varepsilon)$.

Proposition 3.8

Assume that the mapping $\mathcal{D} \ni d \mapsto f(x,d) \in \mathbb{R}$ is Lipschitz continuous with constant L_d uniformly in $x \in \mathbb{X}$. Suppose there exists a strictly increasing function $g : \mathbb{R}_+ \mapsto [0,1]$ such that

$$\mathbb{P}\Big[\mathbb{B}_r(d)\Big] \geq g(r), \quad \forall d \in \mathcal{D}$$

where $\mathbb{B}_r(d) \subset \mathcal{D}$ is an open ball centered at d with radius r. Then, $h(\varepsilon) := L_d g^{-1}(\varepsilon)$ is a ULB in the sense of Definition 3.1, where g^{-1} is the inverse function of g.

Bad news: Curse of dimensionality !

Key Assumption and Lemma

Assumption 3.3 (Slater Point)

There exists an $x_0 \in \mathbb{X}$ such that $\sup_{d \in \mathcal{D}} f(x,d) < 0$. Define the constant:

$$L_{SP} := \frac{\min_{x \in \mathbb{X}} c^\top x - c^\top x_0}{\sup_{d \in \mathcal{D}} f(x,d)}$$

Lemma 3.4

Consider the relaxed program RCP_γ and its optimal value $J^\star_{\mathrm{RCP}_\gamma}$. Under Assumption 3.3, the mapping $\mathbb{R}_+ \ni \gamma \mapsto J^\star_{\mathrm{RCP}_\gamma} \mathbb{R}$ is Lipschitz continuous with constant bounded by L_{SP} i.e., for all $\gamma_2 \geq \gamma_1 \geq 0$ we have

$$0 \leq J^\star_{\mathrm{RCP}_{\gamma_1}} - J^\star_{\mathrm{RCP}_{\gamma_2}} \leq L_{SP}(\gamma_2 - \gamma_1)$$

- *Proof:* Under the strong duality condition the mapping $\gamma \mapsto \mathrm{RCP}_\gamma$, which is Lipschitz continuous with the constant $\|\lambda^\star\|_1$ where $\lambda^\star$ is a dual optimizer of the RCP. In [33] it implies that $\|\lambda^\star\|_1 \leq L_{SP}$.
- Note that Lemma 3.4 also can be applied to the program SCP.

Main Results

Theorem 3.6 (RCP Confidence Interval)

Consider the programs RCP and SCP with the associated optimal values $J^\star_{\mathrm{RCP}}$ and $J^\star_N$, respectively. Suppose Assumption 3.3 holds and with constant L_{SP}. Given a ULB h and $\varepsilon, \beta \in [0,1]$ for all $N \geq N(\varepsilon, \beta)$, we have

$$\mathbb{P}^N\left[J^\star_{\mathrm{RCP}} - J^\star_N \in [0, I(\varepsilon)]\right] \geq 1 - \beta$$

where $I(\varepsilon) := \min\left\{L_{SP}h(\varepsilon), \max_{x\in\mathbb{X}} c^\top x - \min_{x\in\mathbb{X}} c^\top x\right\}$

- *Proof:* $\mathbb{P}^N\left[x^\star_N \models \mathrm{CCP}_\varepsilon\right] \geq 1 - \beta \Rightarrow^{Le3.2} \mathbb{P}^N\left[x^\star_N \models \mathrm{RCP}_{h(\varepsilon)}\right] \geq 1 - \beta$ $\Rightarrow \mathbb{P}^N\left[J^*_{\mathrm{RCP}_{h(\varepsilon)}} \leq J^*_N\right] \geq 1 - \beta \Rightarrow^{Le3.4} J^\star_{\mathrm{RCP}} \leq J^\star_{\mathrm{RCP}_{h(\varepsilon)}} + L_{SP}h(\varepsilon)$ Then, $\mathbb{P}^N\left[J^\star_{\mathrm{RCP}} \leq J^*_N + L_{SP}h(\varepsilon)\right] \geq 1 - \beta$. Meanwhile, SCP is a restricted version of RCP, then it has $J^\star_N \leq J^\star_{\mathrm{RCP}}$.
- Explicit bound: Under Pro 3.8, for any $N \geq N(g(\frac{\varepsilon}{L_{SP}L_d}), \beta)$, we have

$$\mathbb{P}^N\left[J^\star_{\mathrm{RCP}} - J^\star_N \in [0, \varepsilon]\right] \geq 1 - \beta$$

Main Results

Theorem 3.7 (CCP$_\varepsilon$ Confidence Interval)

Consider the programs CCP$_\varepsilon$ and SCP with the associated optimal values $J^\star_{\mathrm{CCP}_\varepsilon}$ and $J^\star_N$, respectively. Let h be a ULB and $\lambda^\star_N$ be the dual optimizer of SCP. Given $\beta \in [0,1]$ for all $N \geq N(\varepsilon, \beta)$, we have

$$\text{A Priori Assessment:} \quad \mathbb{P}^N\left[J^\star_{\mathrm{CCP}_\varepsilon} - J^\star_N \in [-I(\varepsilon), 0]\right] \geq 1 - \beta$$

$$\text{A Posteriori Assessment:} \quad \mathbb{P}^N\left[J^\star_{\mathrm{CCP}_\varepsilon} - J^\star_N \in [-I_N(\varepsilon), 0]\right] \geq 1 - \beta$$

where the *a priori* interval $I(\varepsilon)$ is defined in the former, and the *a posteriori* interval is $I_N(\varepsilon) := \min\left\{\|\lambda^\star_N\|_1 h(\varepsilon), \max_{x\in\mathbb{X}} c^\top x - \min_{x\in\mathbb{X}} c^\top x\right\}$.

- *Proof:* $\mathbb{P}^N\left[J^\star_{\mathrm{RCP}_{h(\varepsilon)}} \leq J^\star_{\mathrm{CCP}_\varepsilon} \leq J^\star_N\right] \geq 1 - \beta$. Lemma 3.4 ensures $J^\star_N \leq J^\star_{\mathrm{RCP}} \leq J^\star_{\mathrm{RCP}_{h(\varepsilon)}} + L_{SP} h(\varepsilon)$. The optimal value of scenario relaxed RCP$_{h(\varepsilon)}$ is denoted by $J^\star_{N,h(\varepsilon)}$. $J^\star_{N,h(\varepsilon)} \leq J^\star_{\mathrm{RCP}_{h(\varepsilon)}}$ w.p. 1. Approximate the L_{SP} as $\|\lambda^\star_N\|_1$, and apply Lemma 3.4 to SCP,

$$J^\star_N - \|\lambda^\star_N\|_1 h(\varepsilon) \leq J^\star_{N,h(\varepsilon)} \leq J^\star_{\mathrm{RCP}_{h(\varepsilon)}}$$

Feasibility of RCP via SCP

An Example to RCP & SCP

$$
\begin{aligned}
&\min_{x\in\mathbb{X}:=[-1,1]} && -x \\
\text{RCP:}\quad & \text{s.t.} && x-d\le 0,\quad \forall d\in\mathcal{D} \\
\text{SCP:}\quad & \text{s.t.} && x-d_i\le 0,\quad \forall i\in\{1,\cdots,N\}
\end{aligned}
$$

- The feasible set of RCP is $[-1,0]$ with the optimizer $x^\star = 0$.
- The optimizer of SCP is $x_N^\star = \min_{i\le N} d_i$.
- If probability $\mathbb{P}$ does not have atoms (point measure), we have $\mathbb{P}^N[\min_{i\le N} d_i > 0] = 1$. Thus, one can deduce that
$$\mathbb{P}^N[x_N^\star \models \text{RCP}] = 0,\quad \forall \mathbb{P}\in\mathcal{P},\quad \forall N\in\mathbb{N}$$
where $\mathcal{P}$ is the family of all non-atomic measure on $(\mathcal{D},\mathcal{B}(\mathcal{D}))$.
- If the set $\arg\max_{d\in\mathcal{D}} f(x,d)$ has measure zero for any $x\models\text{RCP}$, then SCP almost surely (a.s.) returns infeasible solutions to RCP, as worst-case scenarios are a.s. neglected.

Measurability of SCP Optimizer

Two-stage SCP

Let $\phi : \mathbb{R}^n \mapsto \mathbb{R}$ is a strictly convex function. Then, consider the program

$$\begin{aligned} \min_{x \in \mathbb{X}} \quad & \phi(x) \\ \text{s.t.} \quad & f(x, d_i) \leq 0, \quad \forall i \in \{1, \cdots, N\} \\ & c^\top x \leq J_N^\star \end{aligned}$$

with the optimal solution (optimizer) $\tilde{x}_N^\star$.

- $f(x, d)$ with respect to the x is lower semi-continuous.
- Clearly, $\tilde{x}_N^\star$ is an optimizer of the program SCP.
- Proposition 3.10 (Measurability of the Optimizer)
- Proposition 3.11 (Measurability of the Feasible Set)
- Constraint function $f(x, d)$: measurability w.r.t. the uncertainty and lower semi-continuity w.r.t. the decision variables.

Example: Quadratic Constraints via Infinite Hyperplanes

Model Setup

- Take decision variable $x = [x_1, x_2]^\top \in \mathbb{X} : [0,1]^2 \subset \mathbb{R}^2$
- Define linear objective function $c := [-1, -1]^\top$
- Choose constraint function $f(x, d) := x_1 cos(d) + x_2 sin(d) - 1$
- $\mathbb{P}$ is the uniform probability measure on uncertainty $d \in \mathcal{D} := [0, 2\pi]$

Many infinite hyperplane constraints are rewritten as quadratic constraint

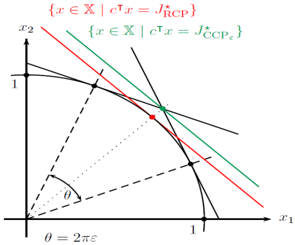
$$\max_{d \in [0,2\pi]} x_1 cos(d) + x_2 sin(d) - 1 \leq \gamma \iff x_1^2 + x_2^2 \leq (\gamma + 1)^2,\ \gamma \geq 0$$

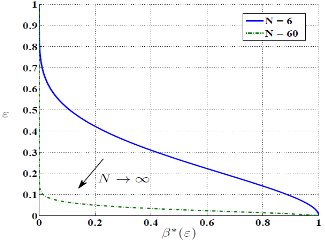
And the analytical solutions of optimal values for programs

$$J^\star_{\mathrm{RCP}_\gamma} = \max\{-\sqrt{2}(\gamma+1)\}, \quad J^\star_{\mathrm{CCP}_\varepsilon} = \max\{-\sqrt{2}/cos(\pi\varepsilon), -2\}$$

Fixed N scenarios for SCP with the optimal value $J^\star_N$, Given $\varepsilon, \beta \in [0,1]$,

$$\beta^*(\varepsilon) = \sum_{i=0}^{n-1} \binom{N}{i} \varepsilon^i (1-\varepsilon)^{N-i} = (1-\varepsilon)^N + N\varepsilon(1-\varepsilon)^{N-1},\ (i.e., n = 2)$$





- In this case $x_0 = [0, 0]^\top$ is Slater point, then $L_{SP} = \frac{-2-0}{-1} = 2$.
- The mapping $d \mapsto f(x, d)$ has Lipschitz constant $L_d = \sqrt{2}$ over $\mathbb{X}$.
- From Pro. 3.8, it derives that $g(r) = \frac{r}{\pi}$ and the ULB $h(\varepsilon) := \sqrt{2}\pi\varepsilon$.
- Then the confidence interval $I(\varepsilon) := \max\{2\sqrt{2}\pi\varepsilon, 2\}$, this arrives at $J^\star_{\mathrm{RCP}} - J^\star_N \in [0, I(\varepsilon)]$ (resp. $J^\star_{\mathrm{CCP}} - J^\star_N \in [-I(\varepsilon), 0]$) w.p. $1 - \beta^*(\varepsilon)$.

To validate this result, Solving program SCP for M different experiments:

- For each $k \in \{1, 2, \cdots, M\}$, drawing N scenarios $(d_i(k))_{k=1}^N \subset [0, 2\pi]$ w.r.t. uniform probability distribution $\mathbb{P}$, and then solve the SCP.
- Let $J^\star_N(k)$ be the optimal value of the kth experiment.

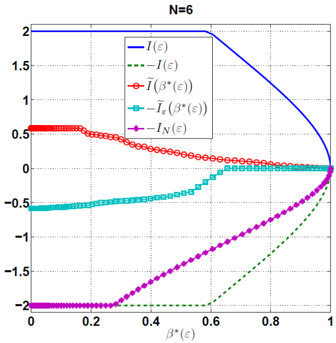
- Given $\beta \in [0,1]$, the empirical confidence interval of RCP can be represented by the minimal $\tilde{I}(\beta)$ so that the interval $[0, \tilde{I}(\beta)]$ contains $J_N^\star(m) - J_{\mathrm{RCP}}^\star$ for at least m experiments where $\frac{m}{M} \geq 1 - \beta$, i.e.,

$$\tilde{I}(\beta) := \min\Big\{\tilde{I} \in \mathbb{R}_+ \mid \exists A \subset \{1, \cdots, M\} : |A| \geq (1-\beta)M \\ \text{and } J_{\mathrm{RCP}}^\star - J_N^\star(k) \in [0, \tilde{I}] \quad \forall k \in A\Big\}$$

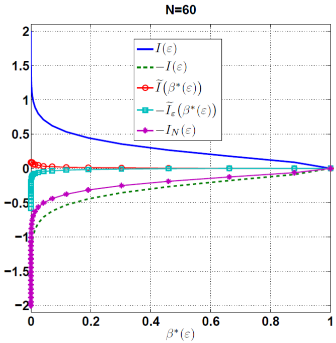
- Regarding CCP_ε, notice that the empirical confidence interval $\tilde{I}_\varepsilon(\beta)$ depends on both parameters ε and β since the analytical optimal values $J_{\mathrm{CCP}}^\star$ depends on ε as well. Then we can define

$$\tilde{I}_\varepsilon(\beta) := \min\Big\{\tilde{I} \in \mathbb{R}_+ \mid \exists A \subset \{1, \cdots, M\} : |A| \geq (1-\beta)M \\ \text{and } J_{\mathrm{CCP}_\varepsilon}^\star - J_N^\star(k) \in [-\tilde{I}, 0] \quad \forall k \in A\Big\}$$

- The sets $\tilde{I}(\beta)$ and $\tilde{I}_\varepsilon(\beta)$ are in close relation with sample quantiles.
- Set the number of experiments as $M = 2000$, scenarios $N = 6, 60$.
- The confidence interval $[0, I(\varepsilon)]$ (resp. $[-I(\varepsilon), 0]$) contains the empirical confidence interval $[0, \tilde{I}(\beta^*(\varepsilon))]$ (resp. $[-\tilde{I}_\varepsilon(\beta^*(\varepsilon)), 0]$)



(a) Simulations for $N = 6$



(b) Simulations for $N = 60$

- Choose one of the experiments and depict the corresponding *a posteriori* confidence interval $I_N(\varepsilon)$ versus $\beta^*(\varepsilon)$.
- Figs shows that a posteriori confidence interval $I_N(\varepsilon)$ (violet) proposes a tighter bound than the a priori confidence interval $I(\varepsilon)$ (green).
- We conjecture that in general the dual optimizer of SCP may happen to be a better approximation in comparison with the constant L_{SP}.

Conclusion

- Propose probabilistic performance bounds for both RCP and CCP_{ε} via SCP.
- Based on the tail probability of the worst-case constraint violation of the SCP solution and perturbation theory of convex optimization to give bounds.
- Given confidence bounds for the objective performance of RCPs and CCPs based on SCP.

Outlook in this work

- Study the derivation of ULBs, depending on the uncertainty set and the constraint functions.
- Investigate the relation between the constant L_{SP} and the dual optimizers of the program SCP.