Koopman Analysis of Mean-Field Dynamics in Weather Prediction Data

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Summary

- ❖ **Spatial mean-field** provides a coarse, spatially averaged view of the field response, offering a rough understanding of weather conditions.
- ❖ Koopman analysis of **deviation dynamics** between a single trajectory and its spatial mean-trajectory to quantify changes in the original field response.
- ❖ **Theoretical residual error bounds** for the deviation dynamics to assess the Koopman latent space's effectiveness as a data-driven surrogate model.
- ❖ Numerical analysis for mean sea level pressure (MSLP) weather data by **dynamic mode decomposition** (**DMD**) methods (e.g., **sparsity**-based selection).
- ❖ Utility: The error bound can be utilized to detect weather anomalies, such as heavy rainfall conditions.

Dynamic Mode Decompositions

❑Consider a large-scale nonlinear weather dynamics

$$\boldsymbol{x}_{k+1} = \boldsymbol{T}(\boldsymbol{x}_k), \quad \boldsymbol{x}_k \in \mathcal{X} \subseteq \mathbb{R}^p,\ k \in \mathbb{Z}_{\geq 0}$$
$$y_k = f(\boldsymbol{x}_k), \quad f \in \mathcal{F} \qquad (1)$$

❑***Spatial (averaged) mean flow*** (center across the spatial grids)

$$\mathrm{m}_k = \frac{1}{p}\sum_{i=1}^{p}[\boldsymbol{x}_k]_i = \frac{1}{p}\sum_{i=1}^{p}[\boldsymbol{T}^k(\boldsymbol{x}_0)]_i \qquad (2)$$

❑ **Data Collection:** From observed dataset $\{\boldsymbol{y}_k\}_{k=1}^{N}$ to build a LTI system

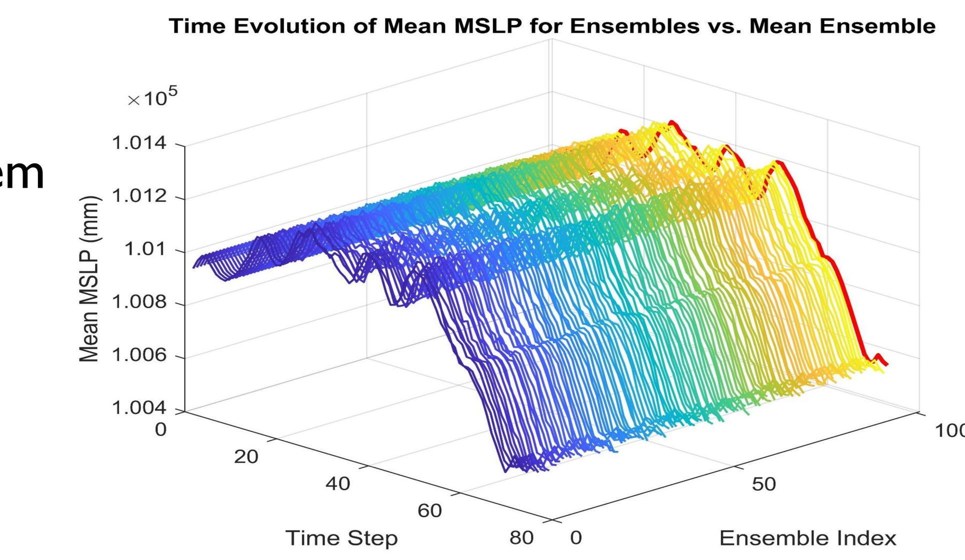
$$\boldsymbol{y}_{k+1} = \boldsymbol{A}\boldsymbol{y}_k\,. \qquad (\text{i.e.}, \boldsymbol{Y}^+ = \boldsymbol{A}\boldsymbol{Y})$$

❑ **Dynamic Mode Decomposition (DMD):**

$$\boldsymbol{y}_k = \sum_{j=1}^{N} \phi_j \lambda_j^t b_j \xleftarrow[\text{Truncated SVD}]{\text{Approximation}} \boldsymbol{y}_k = \sum_{j=1}^{r} \phi_j \lambda_j^t b_j$$

❑Given M initial states $x_0^1, x_0^2, \ldots, x_0^M$, we have the next ensemble-based data matrix:

$$\begin{bmatrix} \boldsymbol{y}_0^1 & \boldsymbol{y}_1^1 & \boldsymbol{y}_2^1 & \cdots & \boldsymbol{y}_{N-1}^1 \\ \boldsymbol{y}_0^2 & \boldsymbol{y}_1^2 & \boldsymbol{y}_2^2 & \cdots & \boldsymbol{y}_{N-1}^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \boldsymbol{y}_0^M & \boldsymbol{y}_1^M & \boldsymbol{y}_2^M & \cdots & \boldsymbol{y}_{N-1}^M \end{bmatrix}$$



➢ **Least Square Error Optimization**

$$\underbrace{[\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{N-1}]}_{\boldsymbol{Y}\in\mathbb{R}^{n\times N}}^{\text{Original data}} \approx \underbrace{\begin{bmatrix} | & \cdots & | \\ \phi_1 & \cdots & \phi_r \\ | & \cdots & | \end{bmatrix}}_{\Phi}^{\text{Spatial Modes}} \overbrace{\underbrace{\begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_r \end{bmatrix}}_{\text{Amplitudes}}}^{\mathrm{D_b}} \underbrace{\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_r & \cdots & \lambda_r^{N-1} \end{bmatrix}}_{\Xi}^{\text{Temporal Modes}}$$

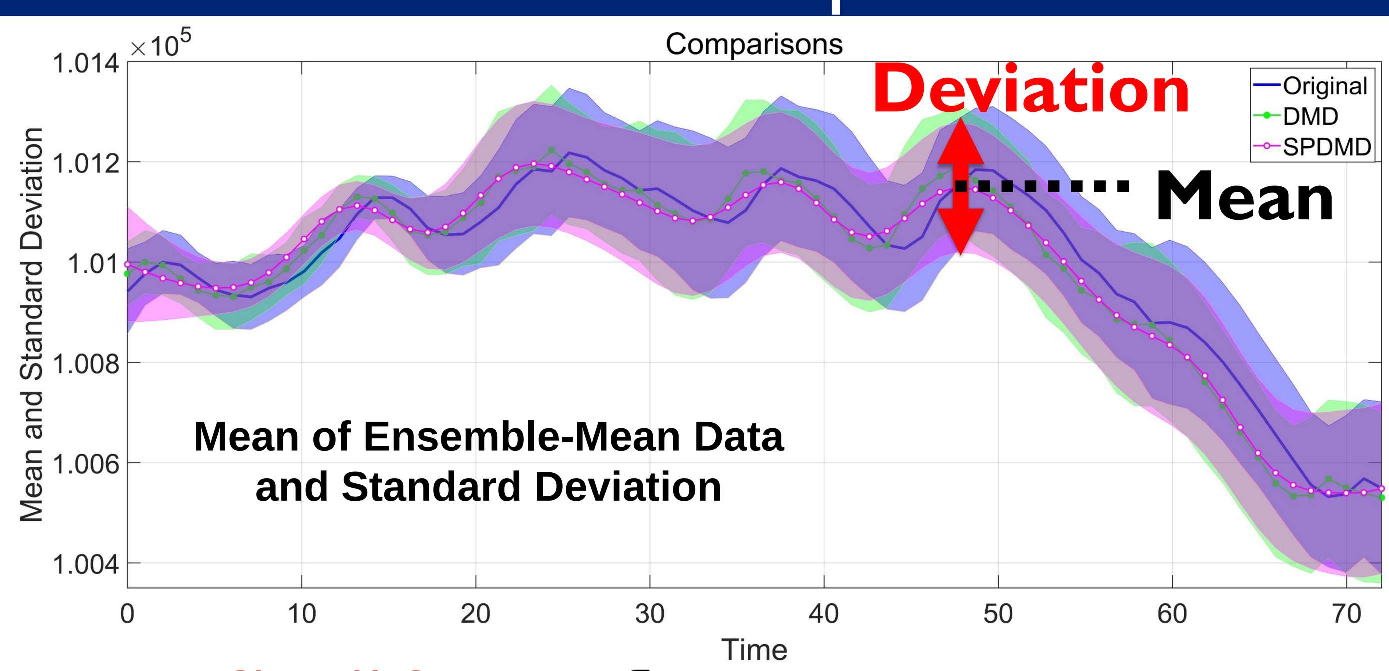
Penalize an ℓ_1 norm on DMD "amplitudes"

➢ **(Regularized) Least Square Error Optimization**

Sparsity-Promoting DMD

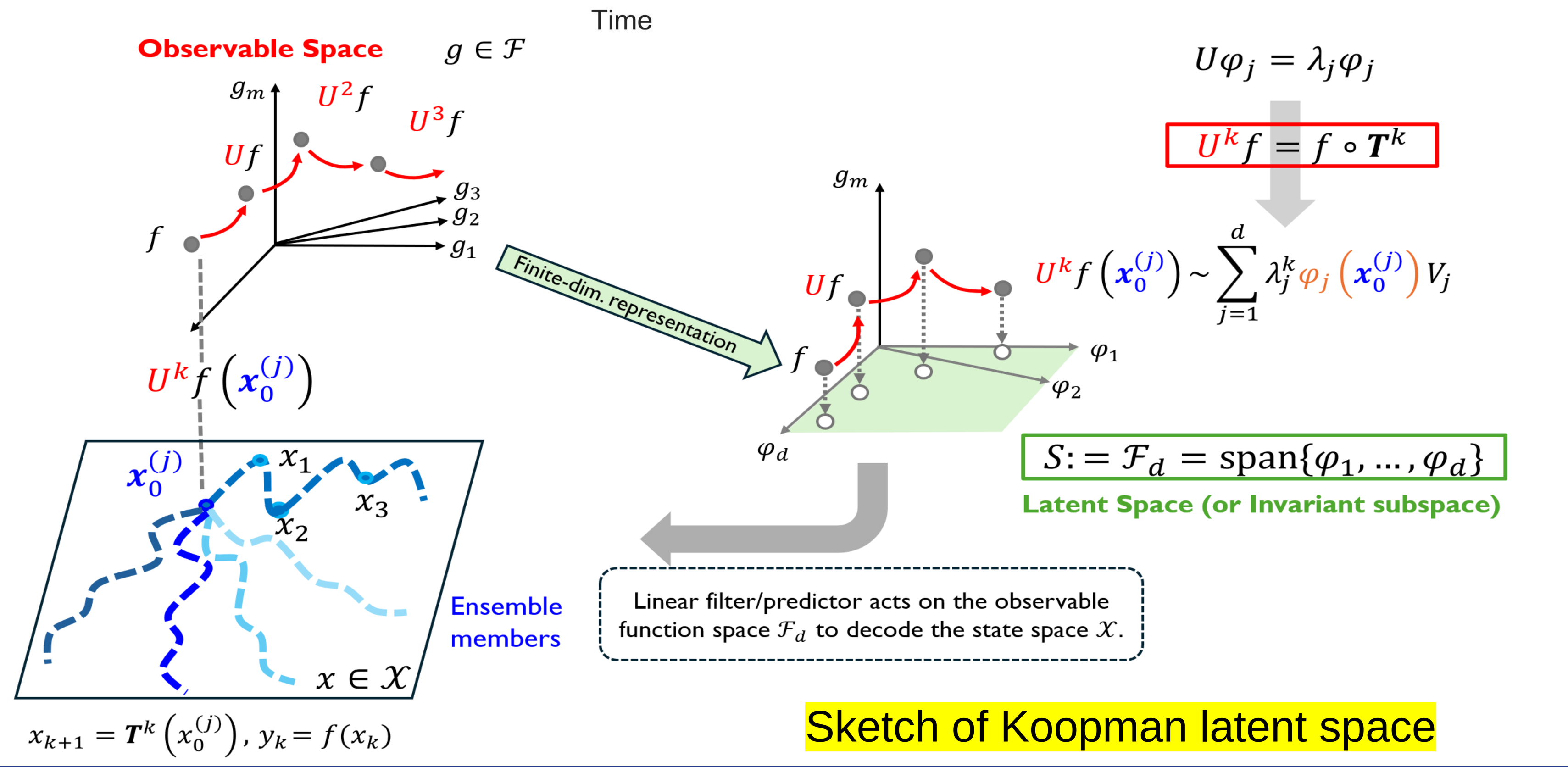
$$J_r(\boldsymbol{b}) = \min_{\boldsymbol{b}} ||\boldsymbol{Y} - \boldsymbol{\Phi}\mathrm{D_b}\Xi||_2 + \gamma||\boldsymbol{b}||_1$$

Motivation: Spatial Mean Field



❑**Observation:** DMD methods can provide a surrogate model with a good deviation using a small number of modes.

❑**Question: Why?** What factors impact the deviations? (KMS or KEFs)?



Sketch of Koopman latent space

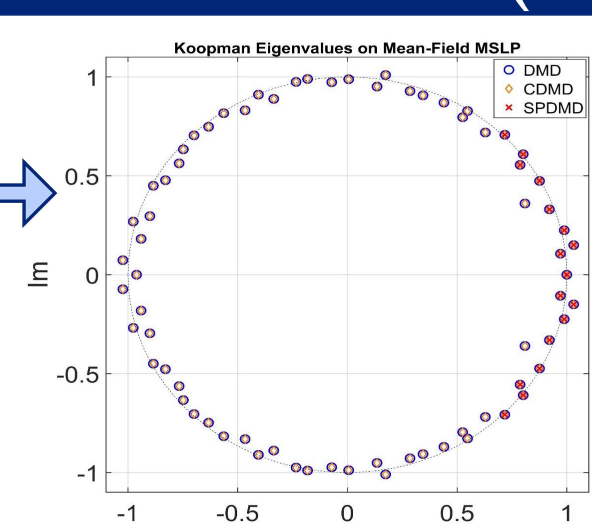
Mean Sea Level Pressure (MSLP) Weather Data

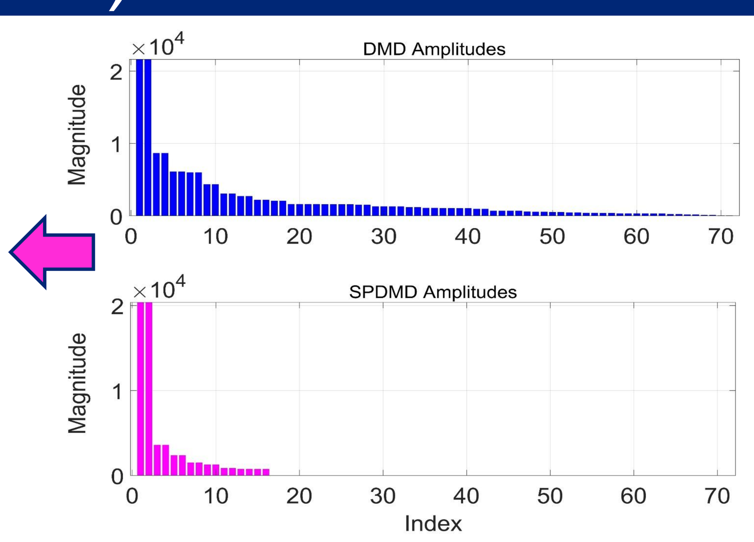
MSLP Weather data:

- 127--132 E, 30--34 N
- Snapshots =73
- Ensembles M = 100

$\boldsymbol{Y} \in \mathbb{R}^{20860\times 73}$

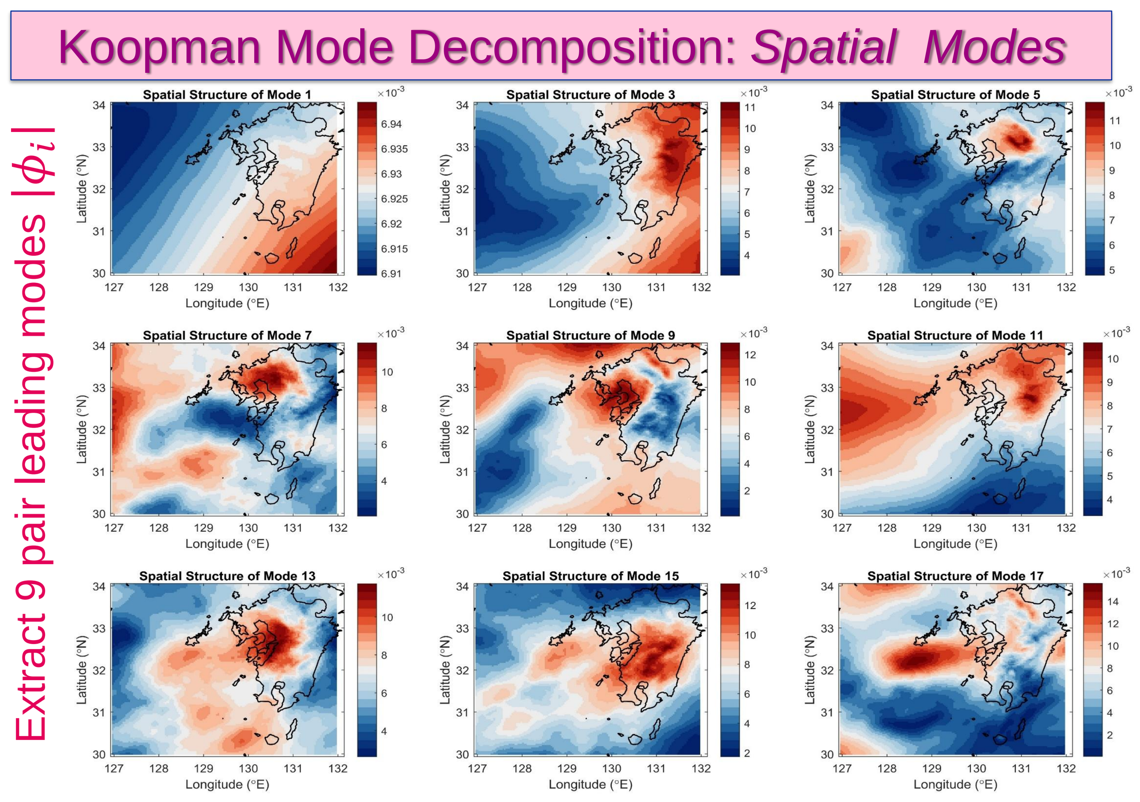
SPDMD + SVD

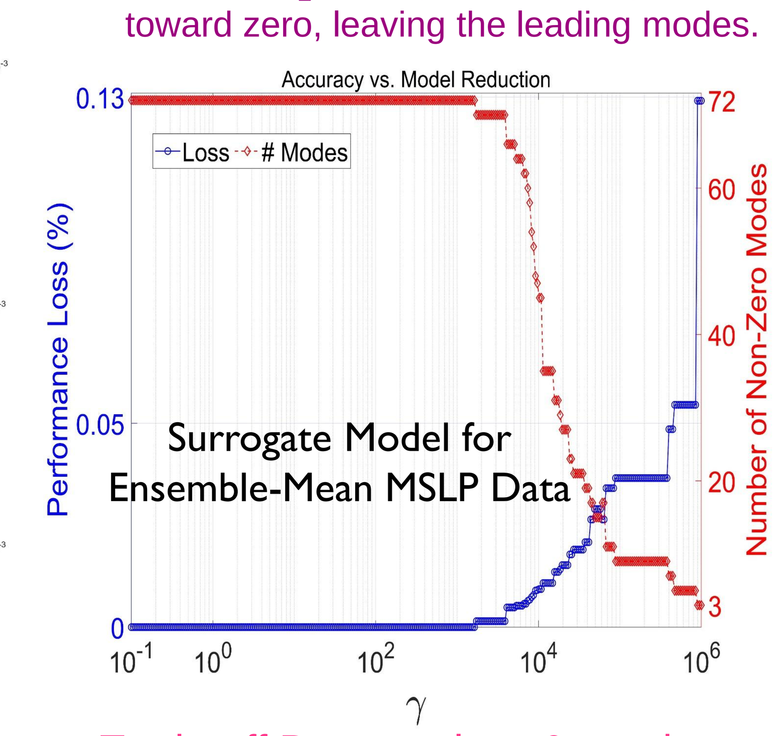




SPDMD: ℓ_1 norm penalizes amplitudes toward zero, leaving the leading modes.

Koopman Mode Decomposition: *Spatial Modes*





Trade-off Between loss & modes

❖SPDMD selects dominant Koopman modes to span the Koopman latent space, controlling weight γ to reduce residual errors of deviations and enhance surrogate model effectiveness.

Error Bound Guarantees on Koopman Latent Space (Single vs. Spatial Mean Trajectory)

Problem 1 (Single Trajectory vs. Spatial Mean Trajectory): *Given a discrete-time dynamical system (1) and its spatial mean flow (2), find an explicit formula for quantifying their deviation using Koopman Mode Decomposition (KMD) with Koopman eigenvalues (KE), eigenfunctions (KEF), and modes (KM).*

Assumption (Koopman Invariant Subspace): *Let the observables $f \in \mathcal{L}^2(\mathcal{X}, \mu)$ (the Hilbert space). Assume there exists a finite-dim. Koopman-invariant subspace of dimension d, spanned by the leading Koopman eigenfunctions $\{\varphi_1, \ldots, \varphi_d\}$, such that any observable f can be expressed as a linear combination of these eigenfunctions.*

Proposition 1 (KMD Viewpoint of Problem 1): *For Problem 1, KMD provides an explicit formula to quantify the deviation from the mean, $[\boldsymbol{x}_k]_l - m_k$, in the single trajectory, which can be reformulated as the deviation of Koopman modes*

$$U^k(f_l - \bar{f})(\boldsymbol{x}_0) = \sum_{i=1}^{\infty} \lambda_i^k \varphi_i(x_0)(V_l - \bar{V}_j)$$
$$= \sum_{i=1}^{d} \lambda_i^k \varphi_i(x_0)(V_l - \bar{V}_j) + [\boldsymbol{r}_k]_l$$

where the scalar $V_l := \boldsymbol{V}_{i,l}$ is the l-th component of the i-th Koopman mode $\boldsymbol{V}_i$, the mean (average) observable is defined as $\bar{f} = \frac{1}{p}\sum_{i=1}^{p} f_i(\boldsymbol{x}) = \frac{1}{p}\int_A f(x)d\mu(A)$ associated with the mean Koopman modes $\bar{V}_j = \frac{1}{p}\sum_{i=1}^{p} \boldsymbol{V}_{j,i} = \langle \bar{f}, \varphi_j \rangle$.

Theorem 1 (Error Bound): *For Problem 1, let $\mathcal{F}_d = \text{span}\{\varphi_1, \ldots, \varphi_d\}$ be the subspace (i.e., candidate of the Koopman latent space) of dominant Koopman eigenfunctions with (non-resonant) eigenvalues $\lambda_j \in \mathbb{C}$, ordered as*

$$1 \geq |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_d| \geq |\lambda_{d+1}| \geq \cdots$$

and $\mathcal{P}_d$ as the orthogonal projection onto $\mathcal{F}_d$. For a linear bounded Koopman operator U^k, the evolution of derivation dynamics $e_{l,k} := [\boldsymbol{e}_k]_l = U^k(f_l - \bar{f})(\boldsymbol{x})$ is decomposed as

$$e_{l,k} = U_d^k(f_l - \bar{f})(\boldsymbol{x}) + [\boldsymbol{r}_k]_l = \mathcal{P}_d e_{l,k} + \boldsymbol{r}_{l,k},$$

in which the residual error

$$\boldsymbol{r}_{l,k} = \mathcal{P}_d e_{l,k} + (I - \mathcal{P}_d)e_{l,k}$$

satisfies

$$\|\boldsymbol{r}_{l,k}\| \leq C|\lambda_{d+1}|^k \|f_l - \bar{f}\|$$

with $sup_j\|\varphi_j\|_\infty \leq C < \infty$ if $\|\boldsymbol{r}_{l,k}\| \ll 1$ for all l, then $\mathcal{F}_d$ captures well the deviation dynamics of $\boldsymbol{y}_k$, enabling an effective data-driven surrogate model by Koopman latent space $\mathcal{F}_d$.

Remarks & Conclusion

- ✓ **Error Bound:** Implies a finite d-dimensional Koopman-invariant subspace enabling a surrogate model that quantifies deviations between single and spatial mean dynamics via leading d-modes.
- ✓ **A Priori Knowledge:** If system (1) has a globally asymptotic fixed point (or regular attractor) with $|\lambda_i| \leq 1$ and bounded KEFs φ_i (with constant C), then error bound can be estimated a priori.
- ✓ **Utility:** When the error bound is small ($||r_{l,k}|| \ll 1$), the finite d-dimensional invariant subspace effectively captures the original dynamics in a low-dimensional linear form.
- ✓ **Sparsity-Selection:** SPDMD can induce the good Koopman latent space when deviation dynamics are well captured using a small leading modes and errors are small (e.g., mean-field MSLP data).

References

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