Linear Quadratic Tracking Control with Sparsity-Promoting Regularization

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Linear Quadratic (LQ) Tracking

Master System: $\dot{z}_m(t) = Az_m(t),\ t \geq 0,\ z_m(0) = \xi_m \in \mathbb{R}^n$
Slave System: $\dot{z}_s(t) = Az_s(t) + Bu(t),\ t \geq 0,\ z_s(0) = \xi_s \in \mathbb{R}^n$

Tracking Goal:

$$\lim_{t\to\infty} \|z_s(t) - z_m(t)\| \doteq \lim_{t\to\infty} \|x(t)\| = 0$$

Tracking Error System:

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = \xi_s - \xi_m = \xi \in \mathbb{R}^n,\ t \geq 0 \qquad (1)$$

Performance Index (LQ Cost)

$$J_{LQ} = \frac{1}{2}\int_0^T \left\{ x(t)^\top Qx(t) + ru(t)^2 \right\} dt,\ Q = Q^T \geq 0,\ r > 0$$

$\mathcal{Q}$: Is it possible to find a *feasible control* $\{u(t) : 0 \leq t \leq T\}$ that

achieves tracking as well as minimizes the control effort ?

Recall Control Signals

▷ Minimum energy $\Leftrightarrow$ $\mathcal{L}^2$ norm of control signal $\int_0^T u(t)^2 dt$

▷ Minimum fuel $\Leftrightarrow$ $\mathcal{L}^1$ norm of control signal $\int_0^T |u(t)| dt$

▷ ? $\Leftrightarrow$ $\mathcal{L}^0$ norm of control signal $\int_0^T |u(t)|^0 dt$

where $|u|^0 = 1$ if $u \neq 0$ and 0 otherwise

$\Longrightarrow$ $\mathcal{L}^0$ norm of control signal is related to the *sparsity*.
It can induce *more zero elements* compared to its dimension

$\mathcal{A}$: *Sparsity-promoting method* is a powerful technique !

- Compressed Sensing
- Maximum Hands-off Control

Sparse Optimization: LQ Hands-off Control

A novel LQ hands-off control problem via sparse optimization

$$\min \quad \underbrace{\overbrace{\frac{1}{2}\int_0^T \left\{ x(t)^\top Q x(t) + r u(t)^2 \right\} dt}^{J_{LQ}\text{: LQ cost}} + \overbrace{\lambda \int_0^T |u(t)|^0 dt}^{\text{weighted } \mathcal{L}^0 \text{norm}}}_{J_0\text{: LQ hands-off control cost}}$$

$$\begin{aligned} \text{s.t.} \quad & \dot{x}(t) = Ax(t) + Bu(t) \\ & x(0) = \xi, \quad x(T) = 0 \\ & |u(t)| \leq 1, \quad \forall t \in [0, T] \end{aligned} \qquad (P_0)$$

Pros: Minimize the control inputs and achieve tracking

Cons: (P_0) is *non-convex, non-smooth and discontinuous*

Necessary Conditions: Non-smooth Maximum Principle

Lemma: Let (x^*, u^*) be a local minimizer for (P_0). Then there exist the $\{p(t) \in \mathbb{R}^n : t \in [0, T]\}$ and Hamiltonian function

$$H^\eta(x, p, u) \triangleq p^\top(Ax + Bu) - \eta(\frac{1}{2}Ru^2 - \lambda|u|^0) \qquad (2)$$

with a scalar $\eta \in \{0, 1\}$ satisfying the following properties:

1. the non-triviality condition for a.e. $t \in [0, T]$:

$$(\eta, p(t)) \neq 0$$

2. the adjoint equation for a.e. $t \in [0, T]$:

$$\dot{p}(t) = -A^\top p(t) + \eta Q x^*(t)$$

3. the maximum condition for a.e. $t \in [0, T]$:

$$u^*(t) = \arg \max_{u \in [-1,1]} p(t)^\top Bu - \eta(\frac{1}{2}ru^2 - \lambda|u|^0)$$

4. the constancy of the Hamiltonian for a.e. $t \in [0, T]$:

$$H^\eta(x^*(t), p(t), u^*(t)) = h \in \mathbb{R}$$

Optimal Solution

Theorem: The optimal control $u^*(t)$ (if it exists) satisfies

1. If $\eta = 1$, then

$$u^*(t) = \mathrm{sat}\big(\mathbf{H}_\theta(r^{-1}B^\top p(t))\big), \tag{3}$$

where $\theta = \sqrt{2\lambda/r}$, $\mathrm{sat}(\cdot)$ is the saturation function defined by

$$\mathrm{sat}(v) \triangleq \begin{cases} -1, & \text{if } v < -1 \\ v, & \text{if } -1 \le v \le 1 \\ 1, & \text{if } v > 1. \end{cases}$$

and $\mathbf{H}_\theta(\cdot)$ is the *hard-thresholding function* defined by

$$\mathbf{H}_\theta(w) \triangleq \begin{cases} 0, & \text{if } -\theta < w < \theta \\ w, & \text{if } w < -\theta \text{ or } \theta < w, \end{cases}$$

and $\mathbf{H}_\theta(w) \in \{0, w\}$, if $w = \pm\theta$.

Numerical Computation

Convex relaxation:

$$\min\ J_{LQ} + \lambda \|u\|_1 \quad \text{s.t. } (P_0) \text{ constraints} \tag{P_1}$$

Time-discretization:

$(P_1) \rightarrow$ Finite-dimensional Optimization Problem (P_2)

$$\begin{aligned} \min \quad & \frac{1}{2} \sum_{k=0}^{m-1} \begin{bmatrix} x_d^k \\ u_d^k \end{bmatrix}^\top \begin{bmatrix} Q_d & S_d \\ S_d^\top & R_d \end{bmatrix} \begin{bmatrix} x_d^k \\ u_d^k \end{bmatrix} + \lambda \frac{T}{m} \sum_{k=0}^{m-1} |u_d^k| \\ \text{s.t.} \quad & x_d^{k+1} = A_d x_d^k + B_d u_d^k \\ & x_d^0 = \xi, \quad x_d^m = 0 \\ & |u_d^k| \leq 1, \quad k = 0, 1, \ldots, m-1 \end{aligned} \tag{P_2}$$

- Method: $\mathtt{L}^1$ `relaxation`, `Time-discretization`, `CVX`

Robustness Analysis: Uncertainties in the initial states

We assume that the initial states are perturbed as

$$z_{\rm m}(0) = z_{\rm m} + \delta_{\rm m}, \quad z_{\rm s}(0) = z_{\rm s} + \delta_{\rm s}, \tag{4}$$

where $\delta_{\rm m}$ and $\delta_{\rm s}$ are uncertain vectors in $\mathbb{R}^n$. Define

$$\delta \triangleq \delta_{\rm s} - \delta_{\rm m}. \tag{5}$$

Then, the initial state $x(0)$ in (1) is described as

$$x(0) = \xi + \delta. \tag{6}$$

Lemma: Let $u^*(t)$ be the LQ hands-off control that solves (P_1) with initial state $x(0) = \xi$. Let $x(t;\delta)$ denote the state variable of (1) with the optimal control u^* from the perturbed initial state in (6). Then we have

$$\|x(T;\delta)\|_{\ell^2} \leq \|e^{AT}\| \|\delta\|_{\ell^2}, \tag{7}$$

where $\|e^{AT}\|$ is the maximum singular value of e^{AT}, and $\|\delta\|_{\ell^2}$ is the ℓ^2 norm of vector δ defined by $\|\delta\|_{\ell^2} \triangleq \sqrt{\delta^\top \delta}$.

Robustness Analysis: Gap between the A matrices

We consider the case when there is a gap between the A matrices in master-slave system. We denote the gap matrix by Δ, that is, we consider the following state-space equation instead of (1):

$$\dot{x}(t;\Delta) = (A+\Delta)x(t;\Delta) + Bu(t). \tag{8}$$

Lemma: Let $u^*(t)$ be the LQ hands-off control that solves (P_1) for the ideal plant (1). Then we have

$$\begin{aligned} \|x(T;\Delta)\|_{\ell^2} &\leq \alpha(\Delta)(\|\xi\|_{\ell^2} + \|B\|\|u^*\|_1) \\ &\leq \alpha(\Delta)(\|\xi\|_{\ell^2} + \|B\|\|u^*\|_0) \end{aligned} \tag{9}$$

where

$$\alpha(\Delta) \triangleq e^{\min\{\|A\|,\|A+\Delta\|\}T}(e^{\|\Delta\|T} - 1). \tag{10}$$

Simulations

Consider a inverted pendulum-cart system without perturbations, which can be written as a linearized model

$$\begin{cases} \ddot{\epsilon} = \dfrac{(M_c + m_p)g}{M_c l_p}\epsilon - \dfrac{1}{M_c l_p}u, \\ \ddot{q} = -\dfrac{m_p g}{M_c}\epsilon + \dfrac{1}{M_c}u, \end{cases} \tag{11}$$

where parameters are: $M_c = 2.4$, $m_p = 0.23$, $l_p = 0.36$, $g = 9.81$, $Q = 5I$, $R = 1$, and $x(0) = [-\pi/120, \pi/12, 0, 0]^\top$.

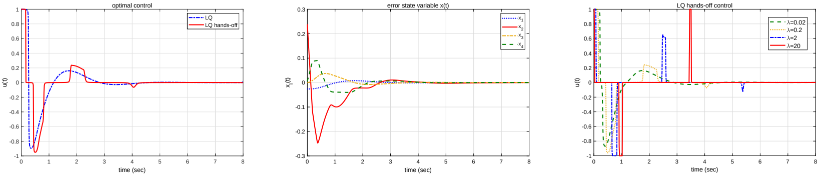

Fig. LQ and LQ hands-off control (left), tracking error states (middle), and different weights for LQ hands-off control (right)

Simulations

Consider a second-order linearized inverted pendulum system with a small disturbance

$$A = \begin{bmatrix} 0 & 1 \\ \frac{m_p l_p g}{L_p} & \frac{-b}{L_p} \end{bmatrix}, \ \Delta = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The parameters are $m_p = 3$, $l_p = 1.5$, $g = 9.81$, $L_p = m_p l_p^2/2$, $b = 0.06$, $\lambda = 1$, $Q = 3I$, $R = 1$ and $x(0) = [0.02, 0.1]^\top$.

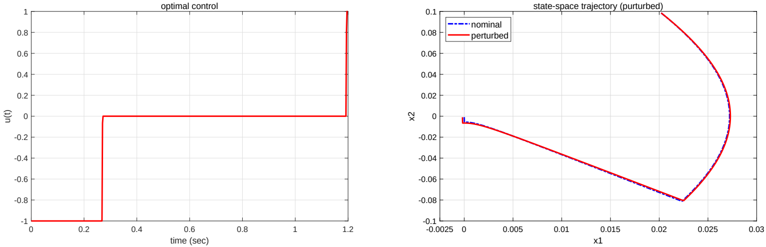

Fig. LQ hands-off control (Left) and Phase portraits (Right) of the perturbed system with gap Δ.

Conclusion

- Necessary conditions for LQ hands-off control
- LQ hands-off control may not be continuous
- Future work
  - $\mathcal{L}^p$ norm approximation, Majorization-mininization algorithms
  - Improve Robustness: Probabilistic Methods, Scenario approach

THANK YOU FOR YOUR ATTENTION !