Sparsity-Promoting Dynamic Mode Decomposition Applied to Sea Surface Temperature Fields

Zhicheng Zhang*

(Joint work with Yoshihiko Susuki*, & Atsushi Okazaki†)

*Department of Electrical Engineering, Kyoto University

†Institute of Advanced Academic Research, Chiba University

2025 SICE Festival with Annual Conference (SICE FS'25)

September 11, 2025, Chiang Mai, Thailand

Purpose and Outline of Presentation

Goal: Koopman/Dynamic mode decomposition (K/DMD) applied to climate systems.

- **Ques 1**: Why should we capture the ***dominant*** Koopman modes embedded in sea surface temperature (SST) data field?
  (i.e., Why use KMD? Why explore SST?)
- **Ques 2**: Why we focus on Sparsity-Promoting technique?

1. Background on long-term SST and its real-world (physical information) (e.g., monthly, seasonal, annual cycle of SST data)
2. Review of Sparsity-Promoting Dynamic Mode Decomposition (SPDMD)
3. Discussion on extracted dominant Koopman modes (including ***accuracy & model reduction***, ***reconstruction***, etc…)
4. Summary

State-of-the-Art Reviews on Climate Systems

nature communications

Article

https://doi.org/10.1038/s41467-024-48033-6

Revealing trends and persistent cycles of non-autonomous systems with autonomous operator-theoretic techniques

Received: 13 August 2023

Accepted: 16 April 2024

Gary Froyland [1], Dimitrios Giannakis [2,3], Edoardo Luna [4] & Joanna Slawinska [2]

Geophysical Research Letters

RESEARCH LETTER

10.1029/2023GL102743

Identification of the Madden–Julian Oscillation With Data-Driven Koopman Spectral Analysis

Benjamin R. Lintner[1,2], Dimitrios Giannakis[3], Max Pike[1], and Joanna Slawinska[3]

Key Points:

- An operator-theoretic approach,

SCIENTIFIC REPORTS

natureresearch

Check for updates

Exponentially decaying modes

2024 NAVARRA ET AL.

Variability of SST through Koopman Modes

ANTONIO NAVARRA,[a,b] JOE TRIBBIA,[c] STEFAN KLUS,[d] AND PAULA LORENZO-SÁNCHEZ[a,e]

2021 NAVARRA ET AL.

Estimation of Koopman Transfer Operators for the Equatorial Pacific SST

ANTONIO NAVARRA,[a,b] JOE TRIBBIA,[c] AND STEFAN KLUS[d]

SCIENTIFIC REPORTS

natureresearch

ARTICLE

Check for updates

https://doi.org/10.1038/s41467-021-26357-x OPEN

Spectral analysis of climate dynamics with operator-theoretic approaches

Gary Froyland[1], Dimitrios Giannakis[2,3], Benjamin R. Lintner[4], Maxwell Pike[4] & Joanna Slawinska[5,6,7]

...tistical ENSO

...operator-

...s for nonlinear

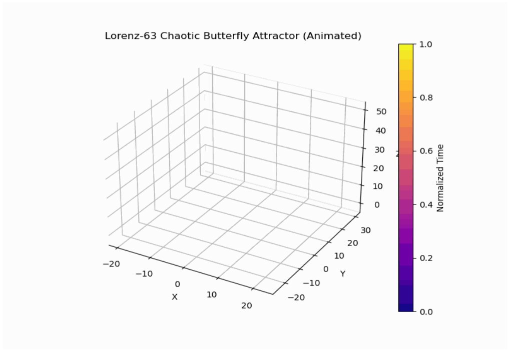
...iannakis[1*]

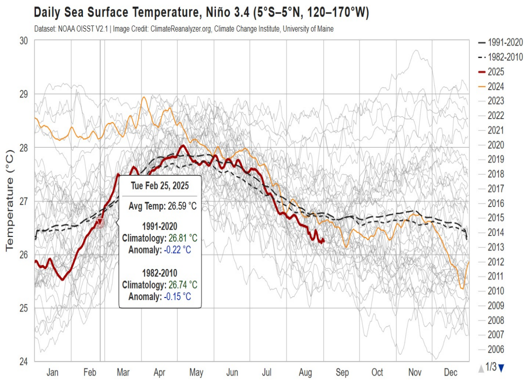
Recently, (Koopman) operator theoretic methods are being increasingly popular and practical in climate science and weather forecasting!

So…Why?

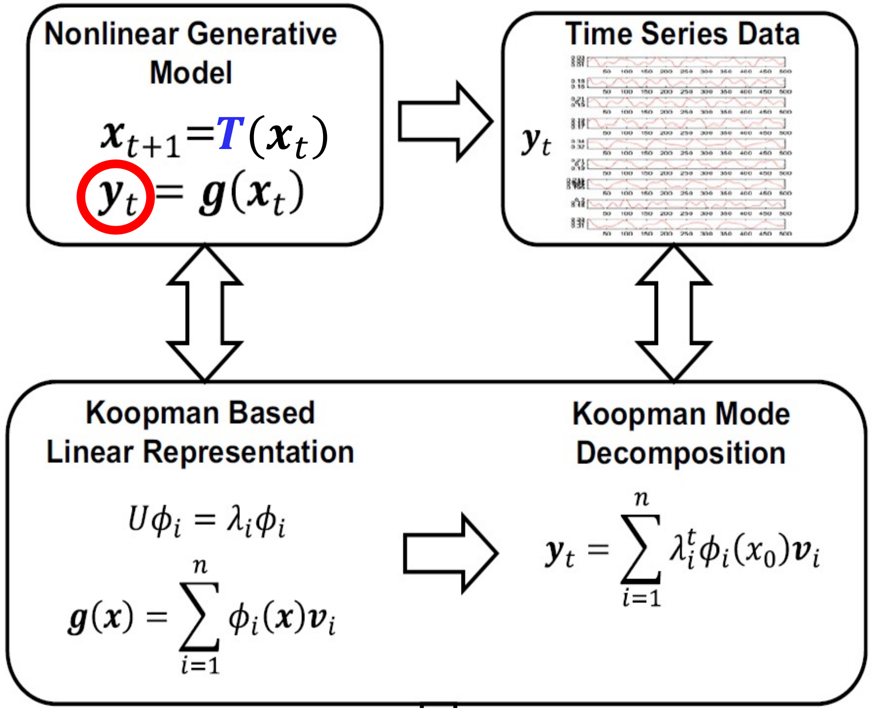
• • • • • •

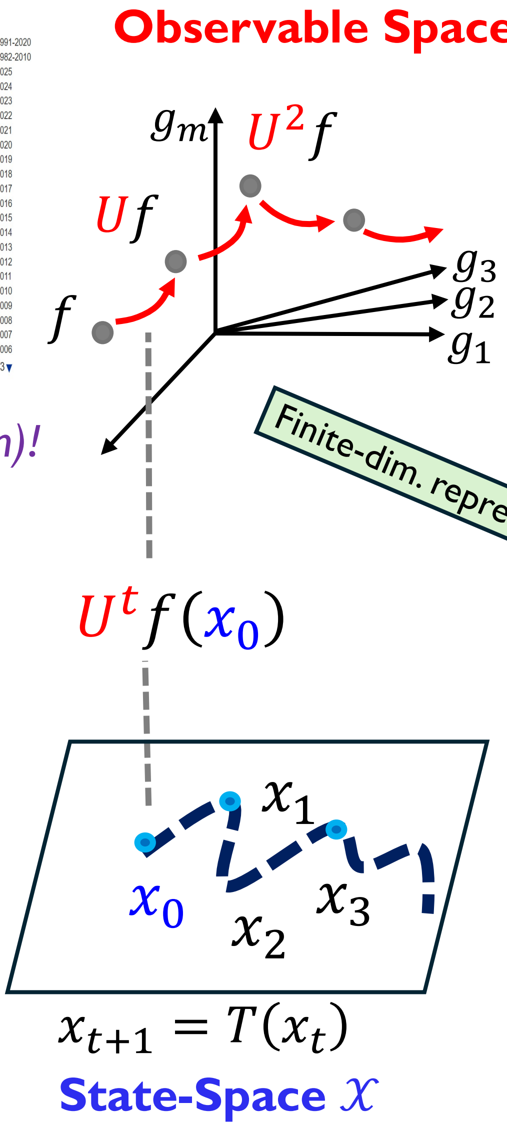
Why Koopman Operator for Time-Series Modeling Works?

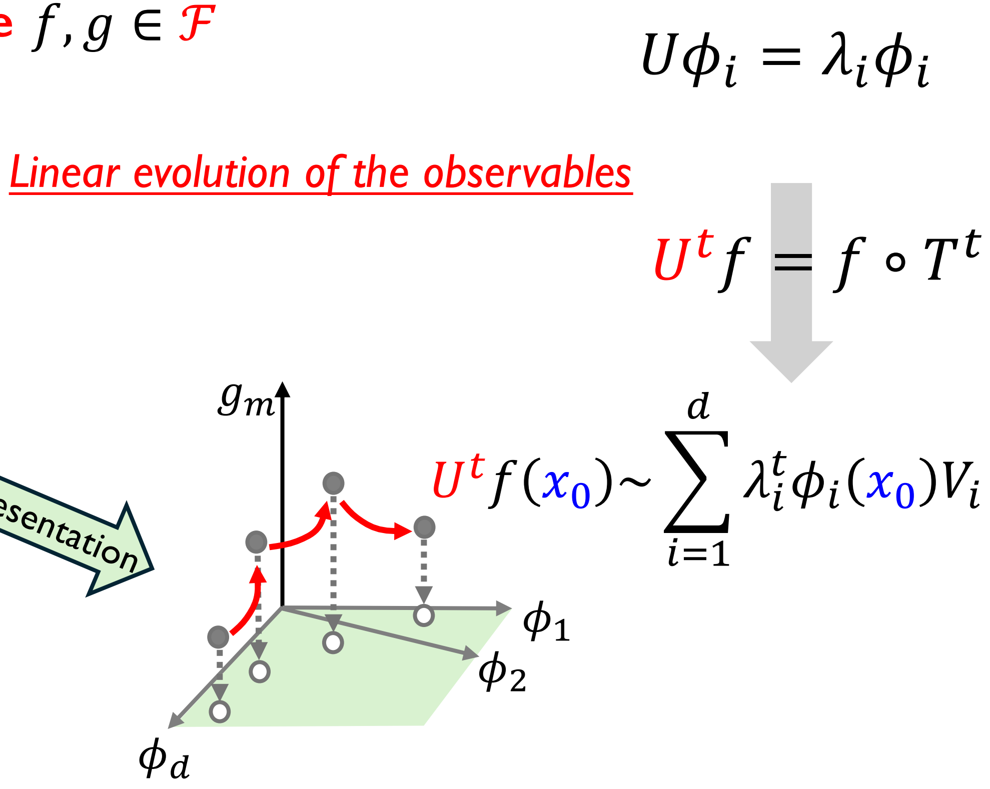




Climate systems go beyond toy model (Chaotic system)!



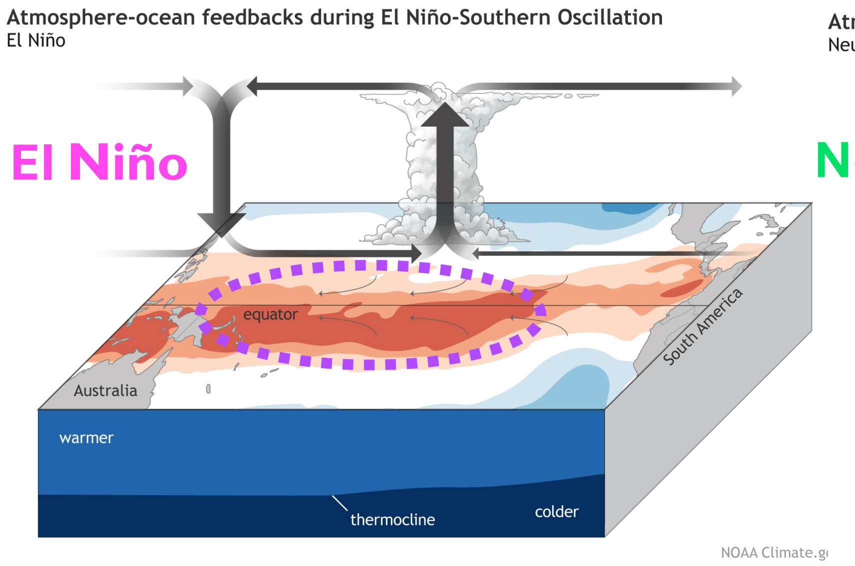


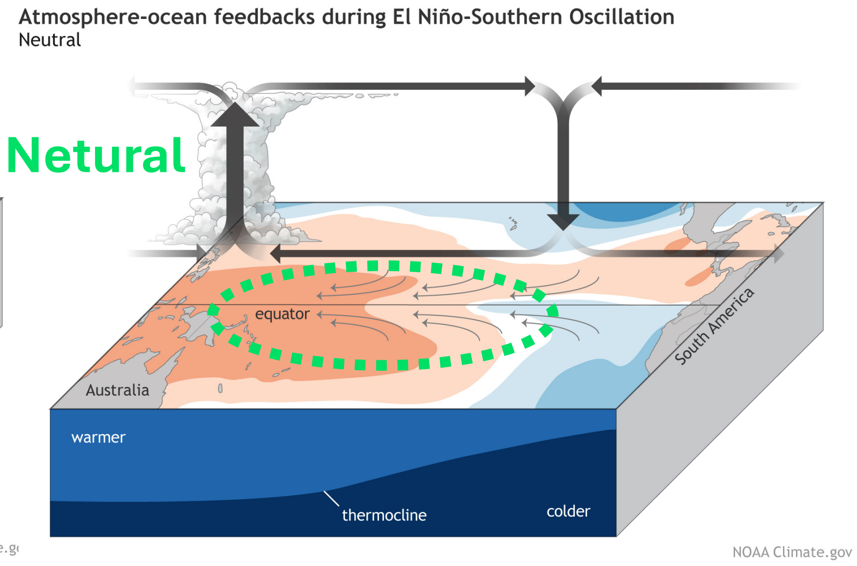


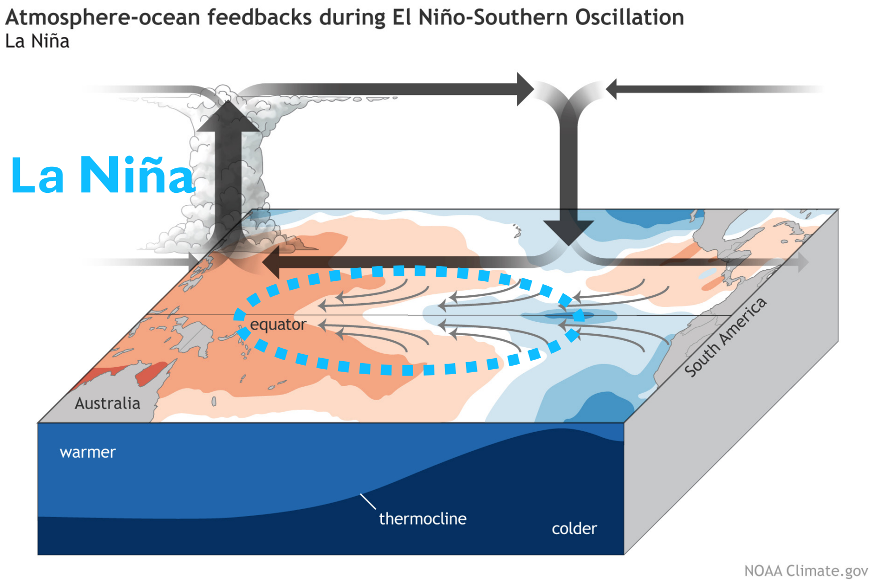
- The (flow) dynamics acts on **finite-dim. state-space**, while the system is ***nonlinear.***
- Koopman operator *lifts a nonlinear system into a* ***linear representation*** acting on the **space of observables**, but it is **infinite-dimensional.**

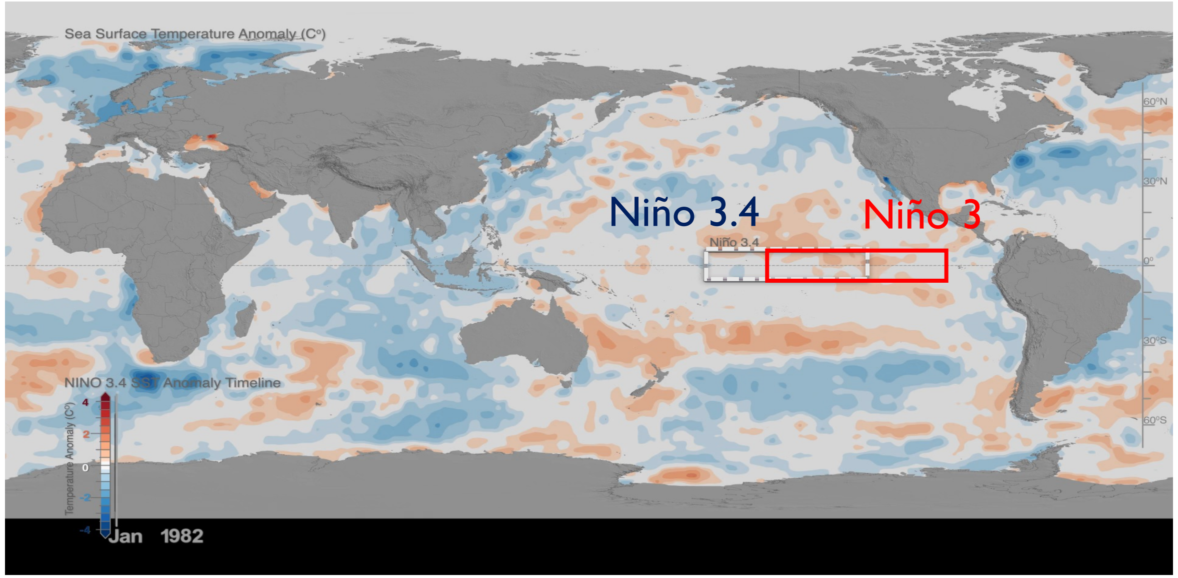
Refs.) Koopman. *PNAS'1931*;Mezic, *Nonlinear Dyna' 2005*; Mauroy, Mezic & Susuki, Springer'*2020*

Background: Why is (Niño-3) SST Data worth Exploring?









Why SST data?

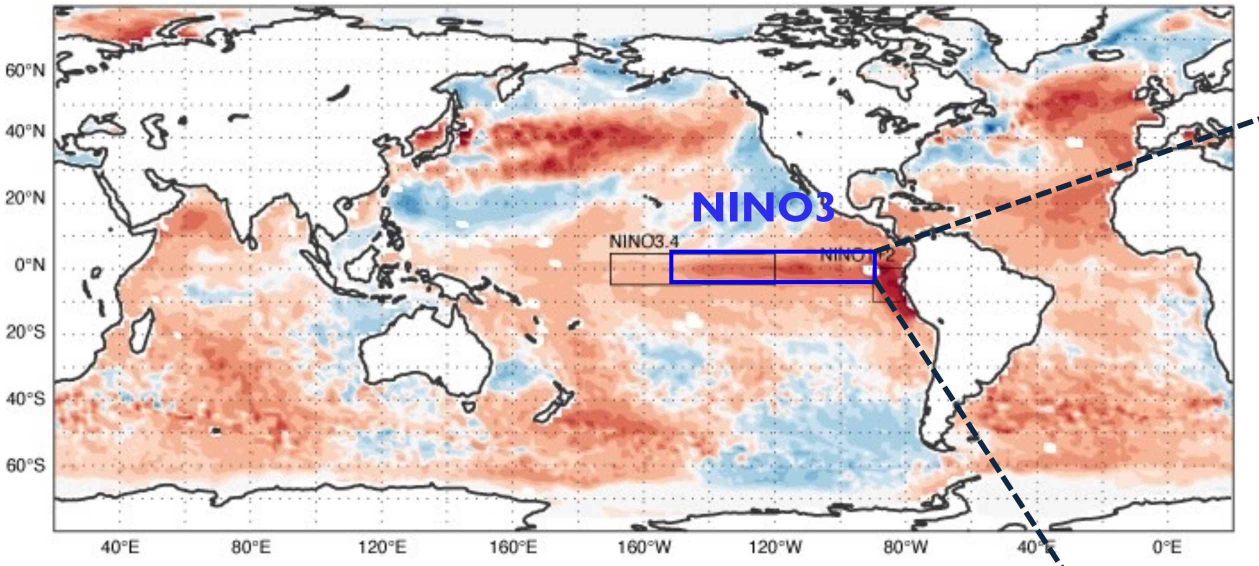
- ❑ Nino-3 tracks SST anomalies in the eastern tropical Pacific (5°N–5°S, 150°W–90°W), is a vital dataset for understanding ***global climate patterns***, such as El Nino-Southern Oscillation (ENSO).

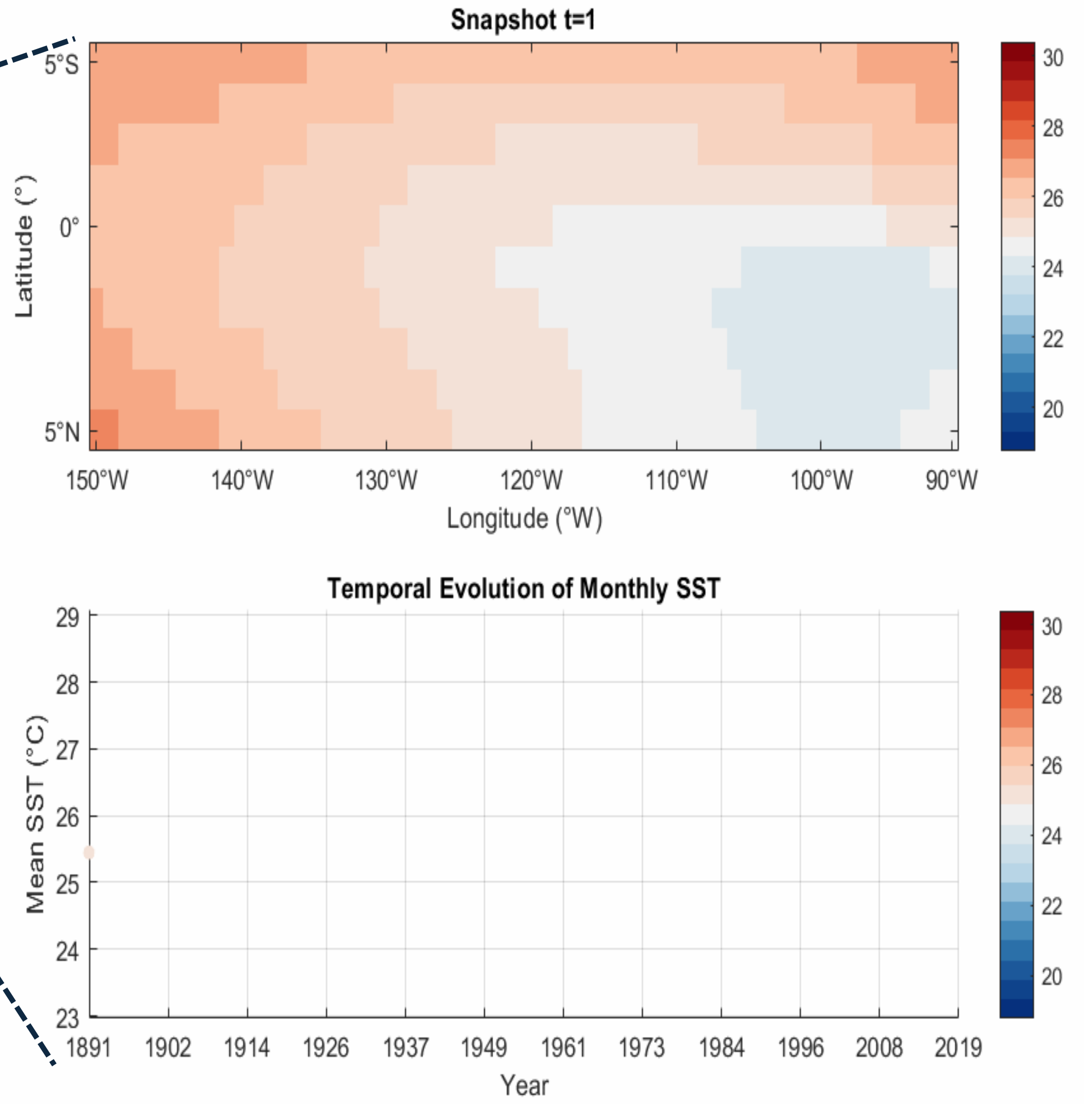
Purpose:

- ❑ Use Koopman Mode Decomposition (KMD) capture ***dominant climate patterns*** by using a few modes!

Ref.) Pictures comes from *NOAA Climate*, https://www.climate.gov/news-features/blogs/enso/
Animation takes from *NASA Scientific Visualization Studio*, https://svs.gsfc.nasa.gov/4695#section_credits

Niño-3 Index Sea Surface Temperature (SST) Data





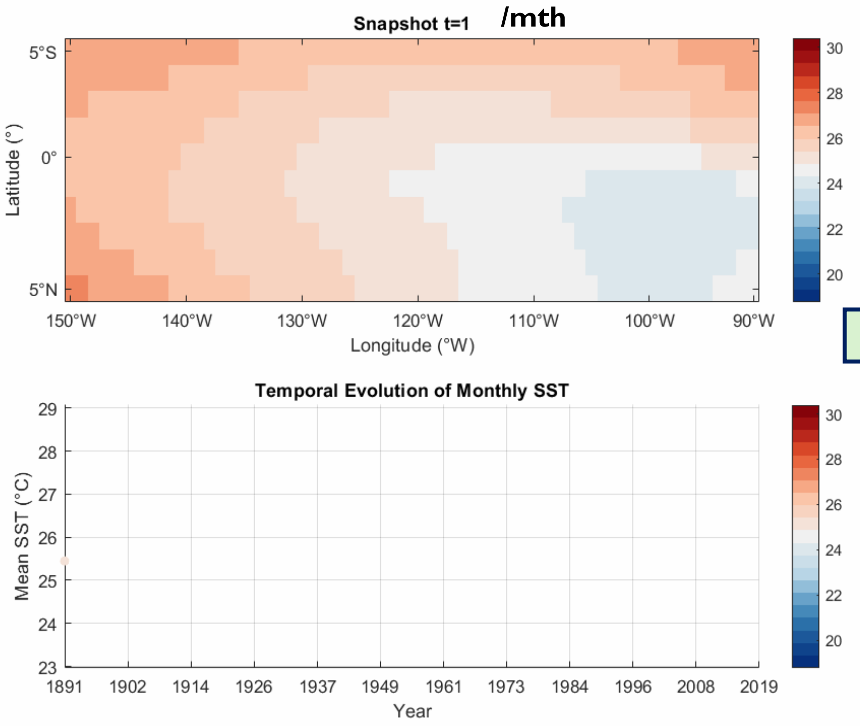
Niño 3 index Observed SST Data

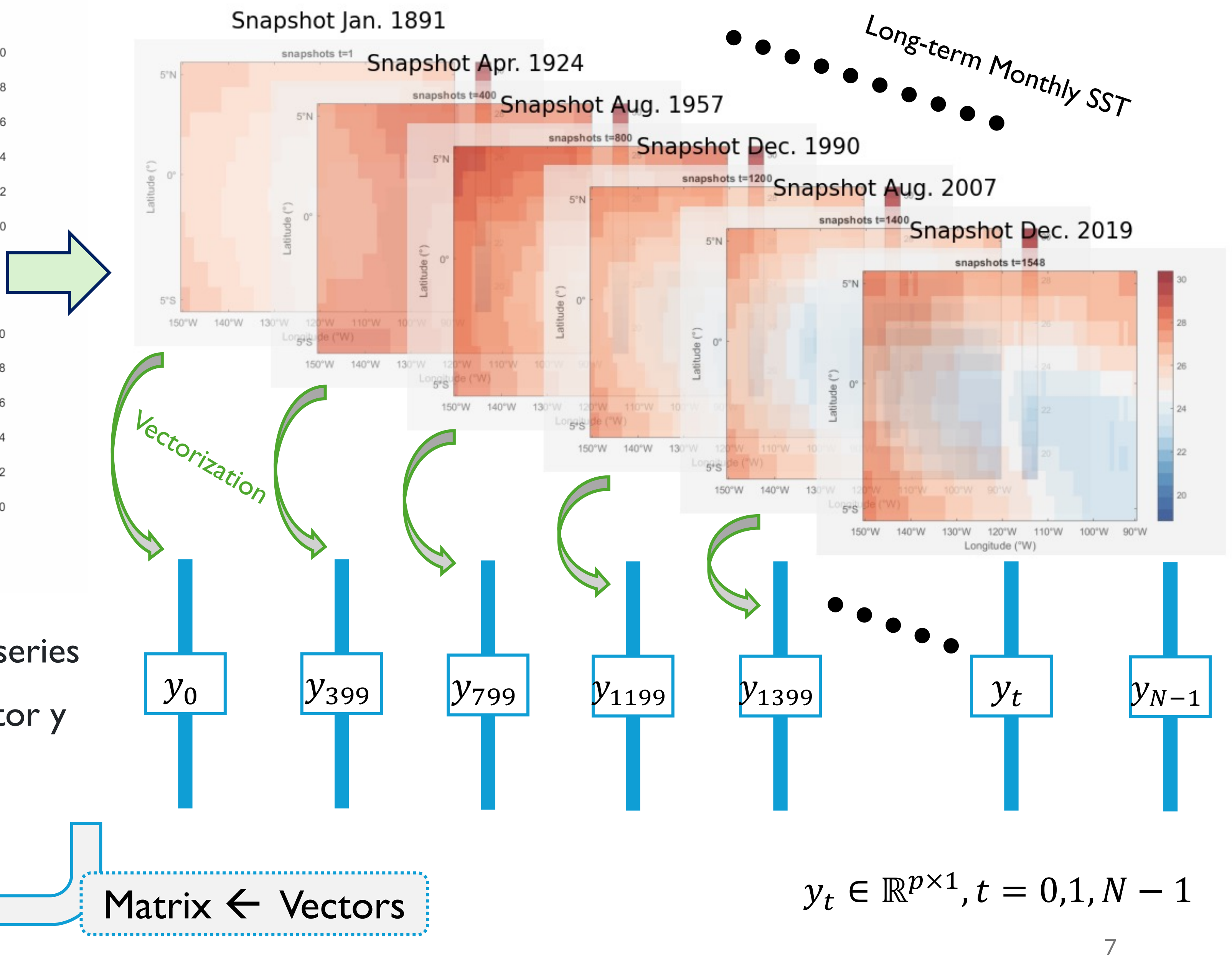
<u>Nino-3 index (Original) Data</u>:

- Area Region:
  Latitude: **5N-5S**, Longitude: **90W-150W**
- Total Grids (Lat. x Longt.):
  10 x 60 = 600 girds
- Month/Year Range:
  Jan. 1891 to **Dec. 2019** (129 yrs)
- Snapshots: t =1548/mths
- Climate modes: ENSO events, etc…

Time-Series Data Collection: Data-Driven Climate Modeling

Niño 3 index (Monthly) SST Data Field





Data Processing:

- Extract the snapshot matrices from time-series
- Stack the data matrix into a high-dim. vector y
- Span the vectors as a new big matrix **Y**.

$$\mathbf{Y} := \left\{\begin{matrix} | & | & & | \\ y_0, & y_1, & \ldots, & y_{N-1} \\ | & | & & | \end{matrix}\right\}$$

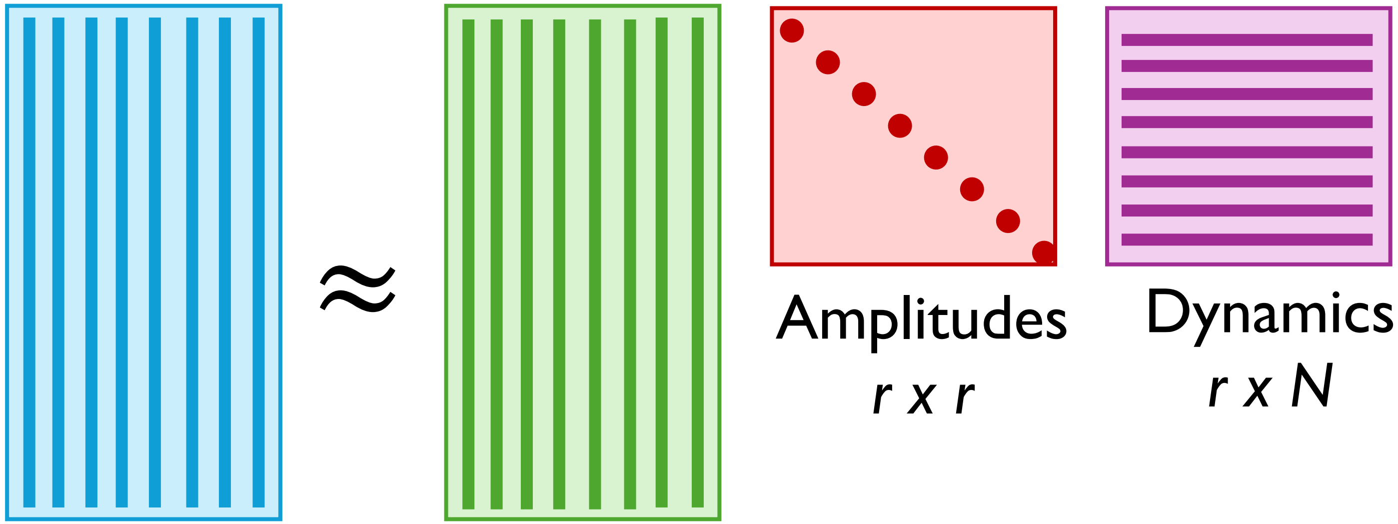
$$y_t \in \mathbb{R}^{p\times 1}, t = 0,1, N-1$$

Ref.) Data Source-*NOAA Physical Sciences Laboratory*, https://psl.noaa.gov/gcos_wgsp/Timeseries/Nino3

Method: Dynamic Mode Decomposition (DMD)

Refs.) Rowley et al, Spectral analysis of nonlinear flows, *J. Fluid Mech.'2009*
Schmid, Dynamic mode decomposition of numerical and experimental data, *J. Fluid Mech.'2010*

Data p x N Modes p x r



Amplitudes r x r

Dynamics r x N

$$\mathbf{Y} \approx \boldsymbol{\Phi} \quad \mathrm{Diag}(\mathbf{b}) \quad \boldsymbol{\Xi}$$

Singular Value Decomposition (SVD)

$$y_t \approx \sum_{j=1}^{N} \phi_j \lambda_j^t b_j \quad \xRightarrow[\mathbf{Y} = \mathbf{U\Sigma V}^*]{\text{SVD}} \quad y_t \approx \sum_{j=1}^{r} \phi_j \lambda_j^t b_j$$

Linear Time Invariant (LTI):

$$\mathbf{Y}^+ = \mathbf{AY} \quad \Longleftarrow \quad y_{t+1} = \mathbf{A} y_t$$

$$\boxed{J(\mathbf{b}) = \min_{\mathbf{b}} \|\mathbf{Y} - \boldsymbol{\Phi}\ \mathrm{Diag}(\mathbf{b})\boldsymbol{\Xi}\|_F^2}$$

➢ DMD via Least Square Error Optimization

$$\underbrace{\begin{bmatrix} | & \cdots & | \\ y_0 & \cdots & y_{N-1} \\ | & \cdots & | \end{bmatrix}}_{\mathbf{Y}\in\mathbb{R}^{p\times N}} \approx \underbrace{\begin{bmatrix} | & \cdots & | \\ \phi_1 & \cdots & \phi_r \\ | & \cdots & | \end{bmatrix}}_{\boldsymbol{\Phi}} \overbrace{\begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_r \end{bmatrix}}^{\mathrm{Diag}(\mathbf{b})} \underbrace{\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_r & \cdots & \lambda_r^{N-1} \end{bmatrix}}_{\boldsymbol{\Xi}}$$

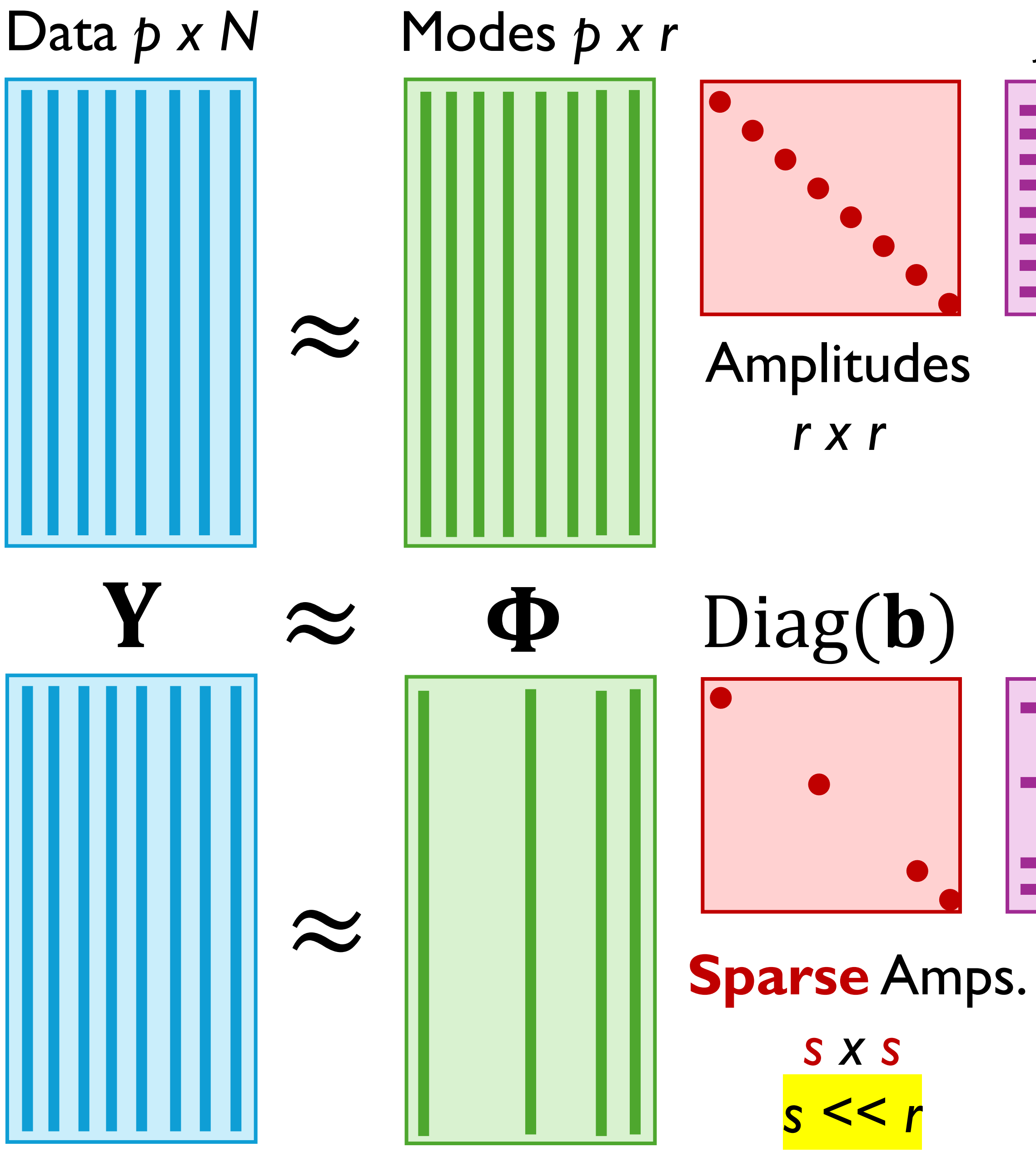
Connection between DMD & KMD:

❖ DMD is a ***numerical*** counterparts of Koopman mode decomposition (KMD)
❖ Koopman operator: *linear evolutions of observables* (directly measured from data)
❖ DMD modes → Koopman modes

Refs.) Tu et al, On dynamic mode decomposition: theory and applications, *J. Compt. Dyna.'2014*; Mezic, *Nonlinear Dyna' 2005*; Mauroy, Mezic & Susuki, Springer'2020

Method: Sparsity-Promoting Dynamic Mode Decomposition (SPDMD)

Ref.) Kutz et al, Dynamic mode decomposition: Data-driven modeling of complex systems, *SIAM'2016*
Jovanovic, Schimd & Nichols, Sparsity-promoting dynamic mode decomposition, *Physics. Fluids'2014*



$$J(\mathbf{b}) = \min_{\mathbf{b}} \|\mathbf{Y} - \boldsymbol{\Phi}\ \mathrm{Diag}(\mathbf{b})\boldsymbol{\Xi}\|_F^2$$

- *Promote sparsity on **"Amplitudes"!***
- *Take L1 norm regularization!*
  (Convex program, ADMM Algorithm)

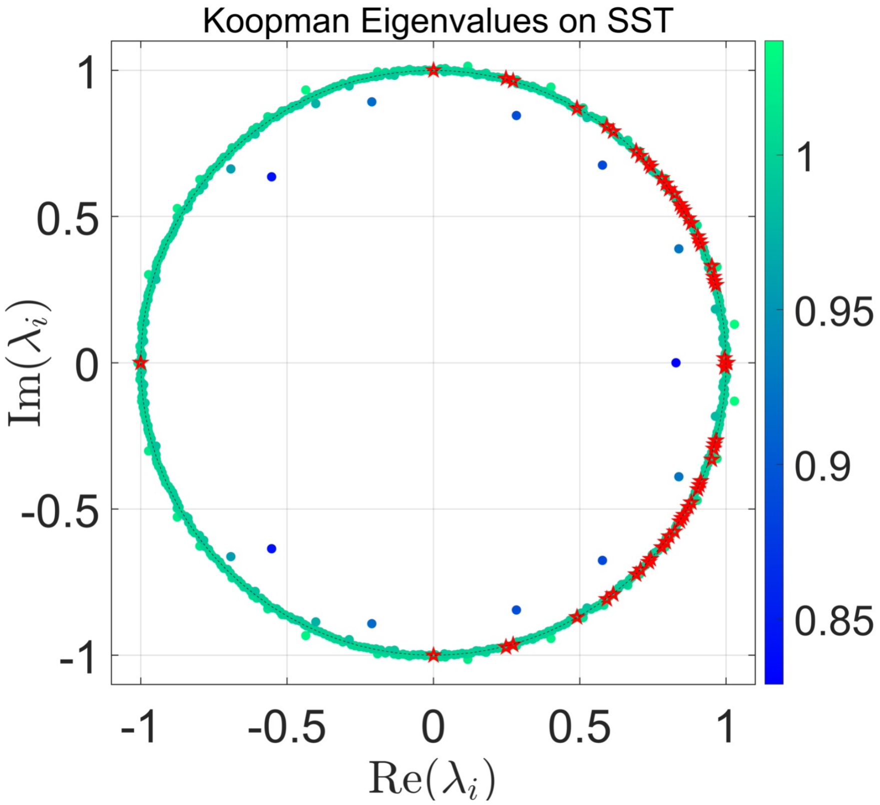
$$J_\gamma(\mathbf{b}) = \min_{\mathbf{b}} \|\mathbf{Y} - \boldsymbol{\Phi}\ \mathrm{Diag}(\mathbf{b})\boldsymbol{\Xi}\|_F^2 + \gamma \sum_{i=1}^{r} |b_i|$$

- Dominant modes determined by order of the *magnitude of amplitudes* $|b_i|, i = 1, \ldots, r$
- Associated with leading pairs (λ_i, ϕ_i)

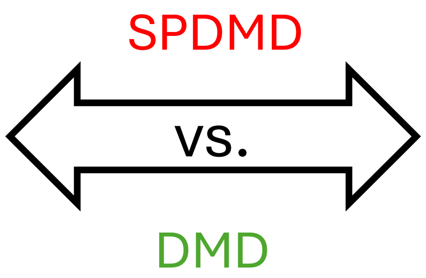
Numerical Benchmark : Seasonal Cycle of SST Data Experiment

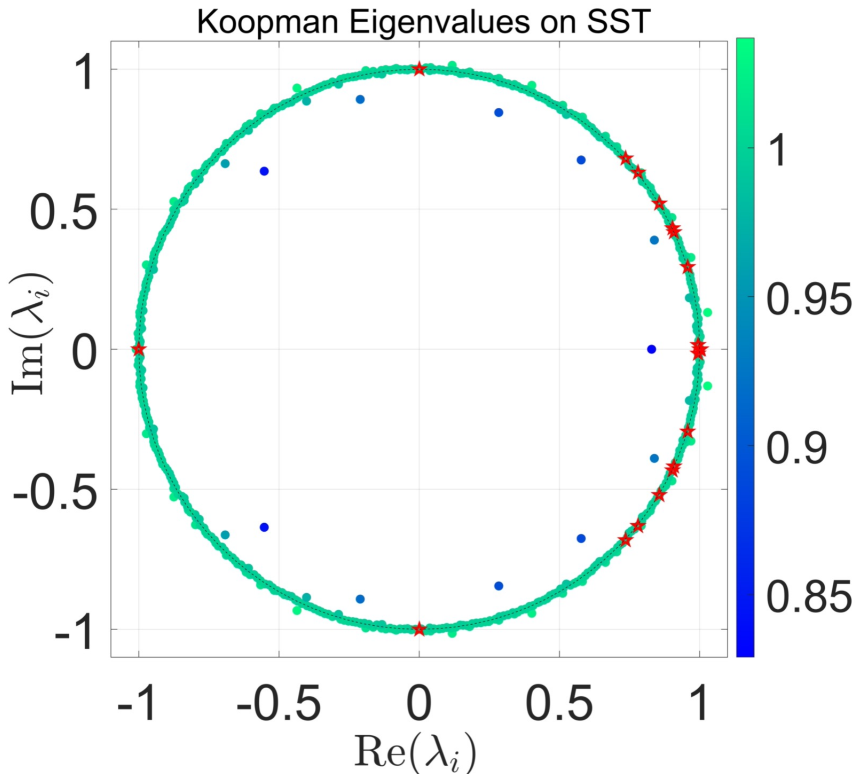
- ❑ Seasonal cycle SST data (3-mth period)
- ❑ $y_t \in \mathbb{R}^{1800}, p = 1800, N = 515$
- ❑ **Tall** and **Skinny** matix: $\mathbf{Y} \in \mathbb{R}^{1800\times 515}$ estimates $\boldsymbol{A}$
- ❑ One column denotes 3-mth (medium-resolution)

- ❑ DMD (●), SPDMD (★)
- ❑ DMD and SPDMD estimates the Koopman SST eigenvalues distributed around unit cycle!
- ❑ Seasonal SST data enjoys ***steady state!***



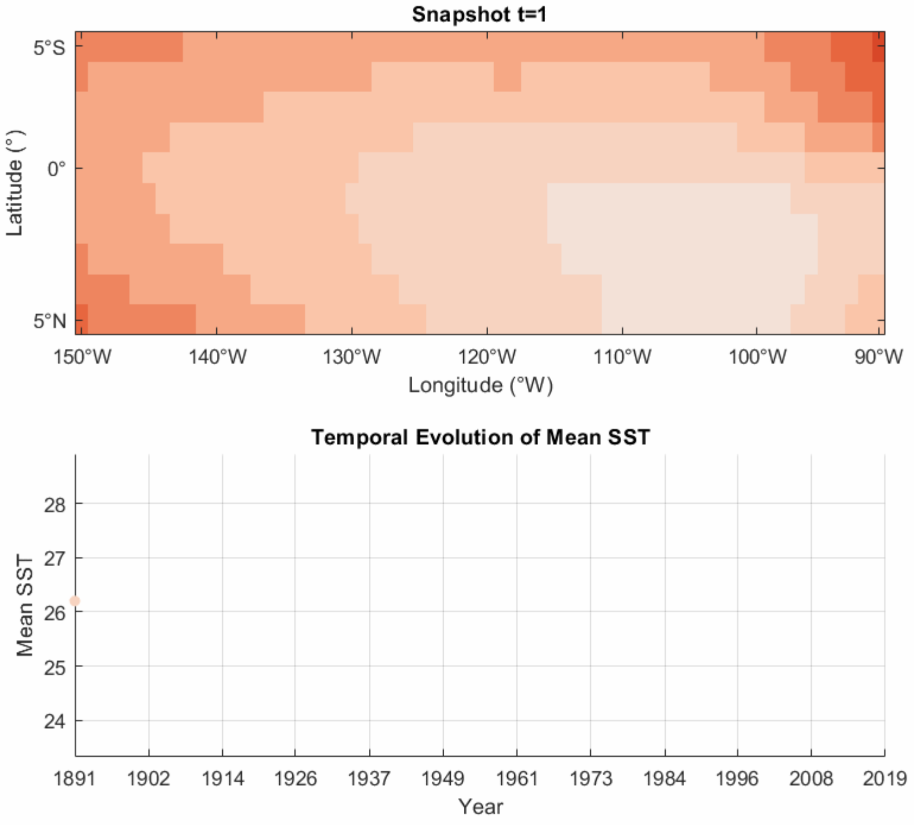
Captures 59 modes of 515



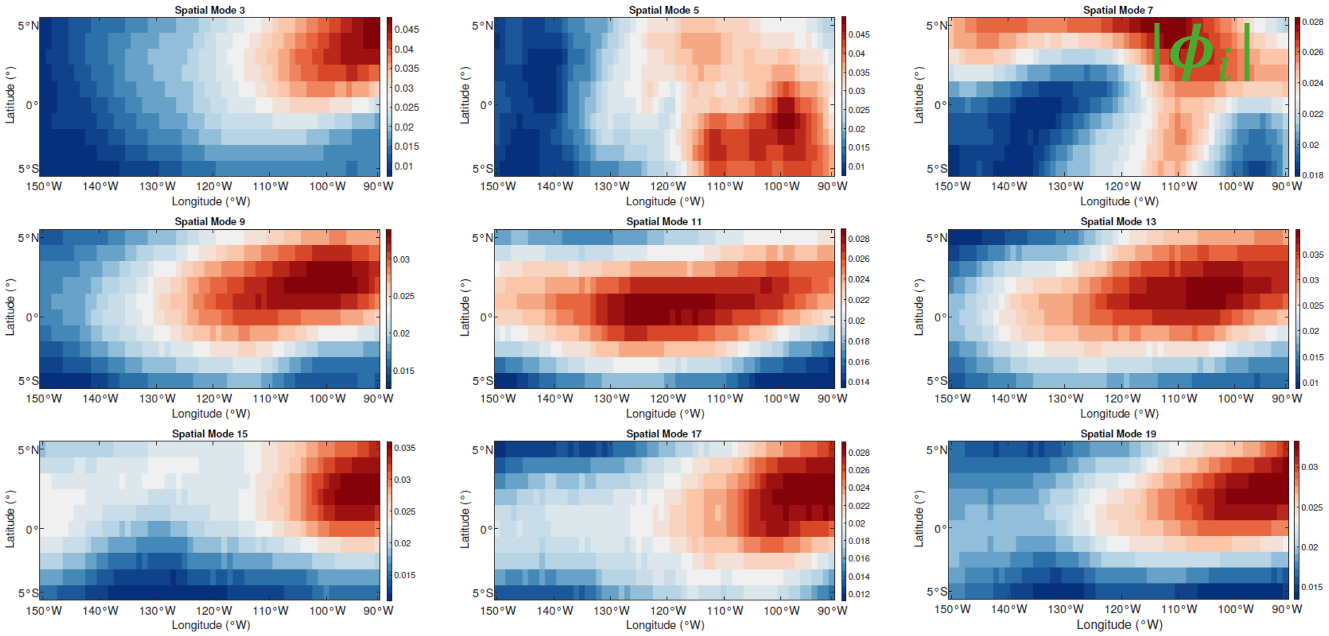


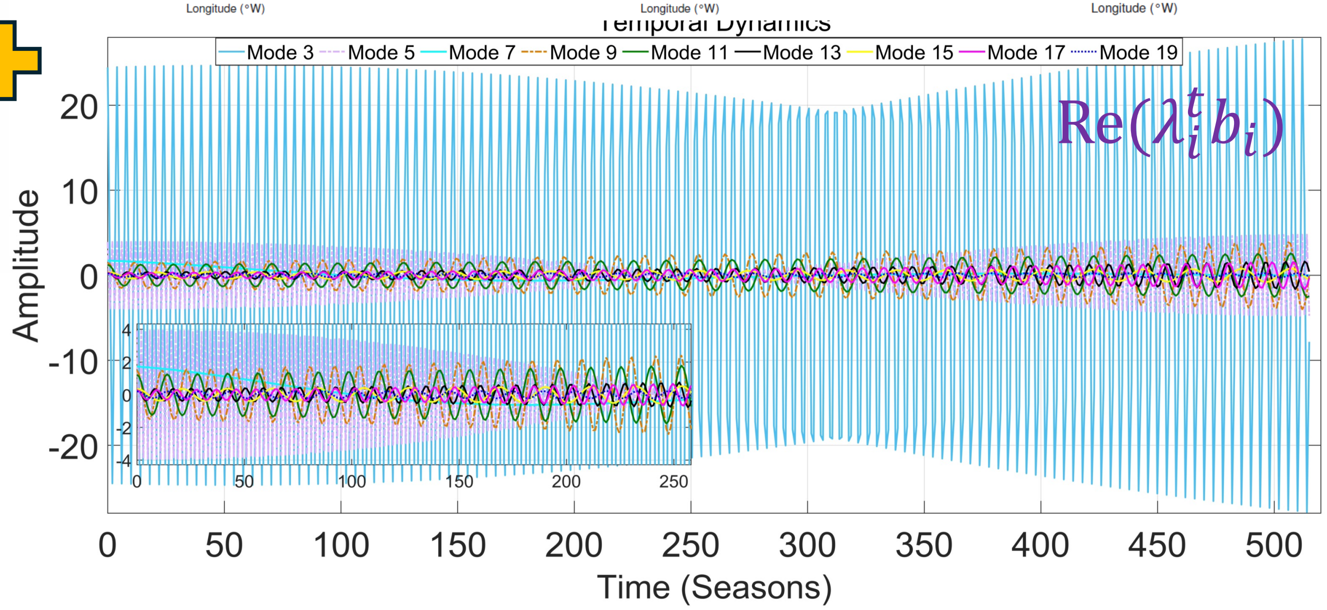
Captures 19 modes of 515

Extraction: Dominant Koopman (Spatial & Temporal) Modes



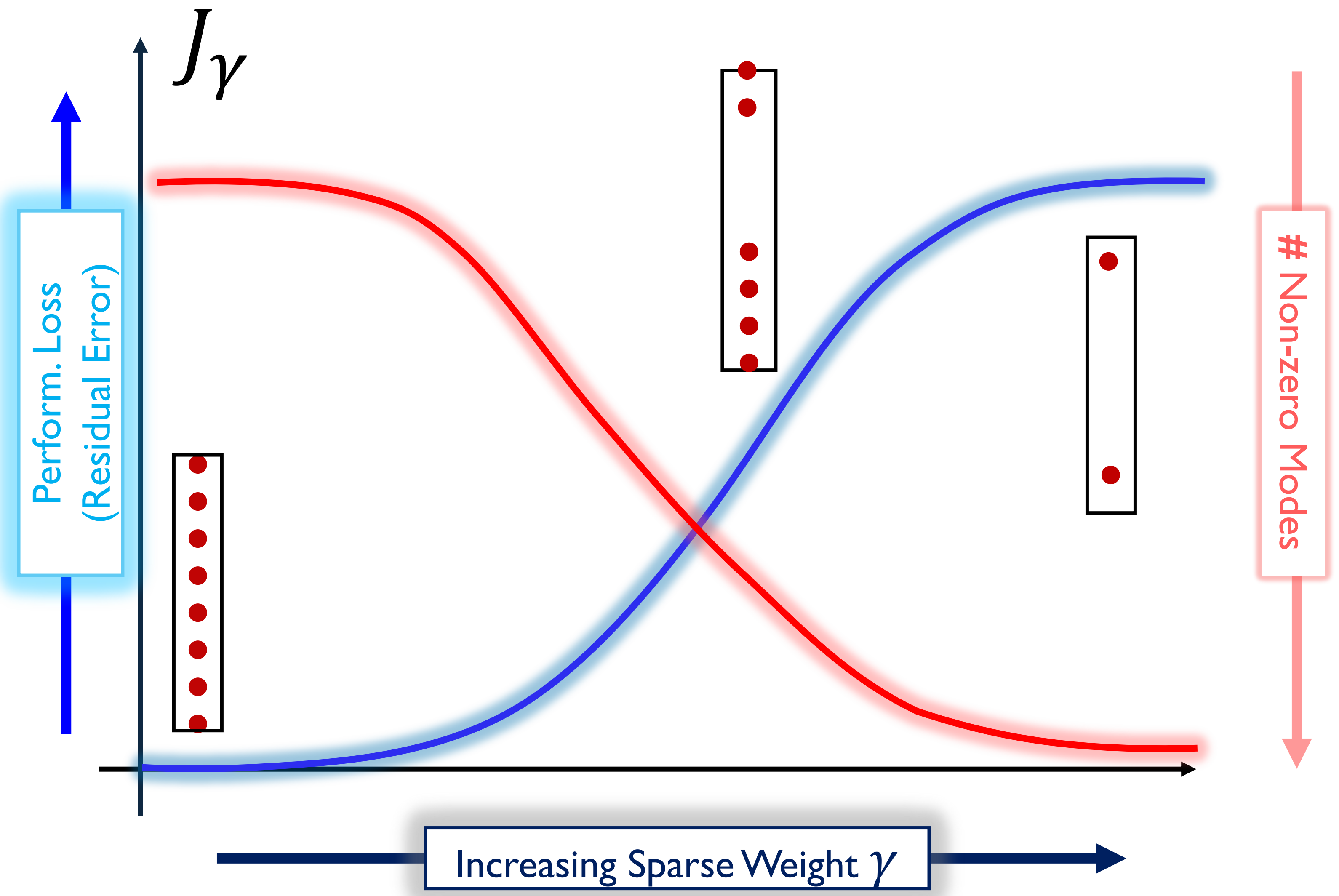
Seasonal SST Data with 516 snapshots





- SPDMD extracts leading **spatial** & **temporal** climate modes.
- It helps us understand the variability of climate dynamics, e.g., **ENSO** (El Niño, La Niña events)

Trade-off: Accuracy vs. Model Reduction



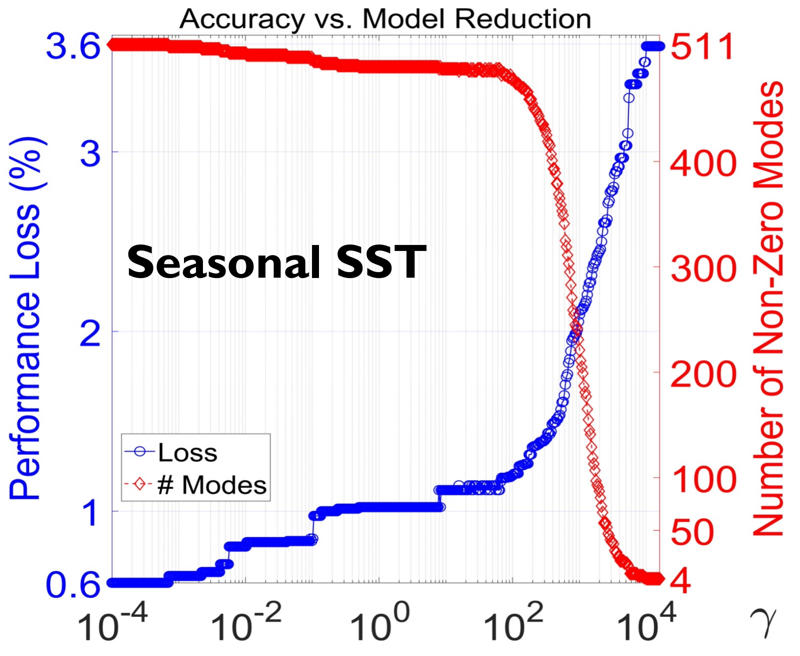


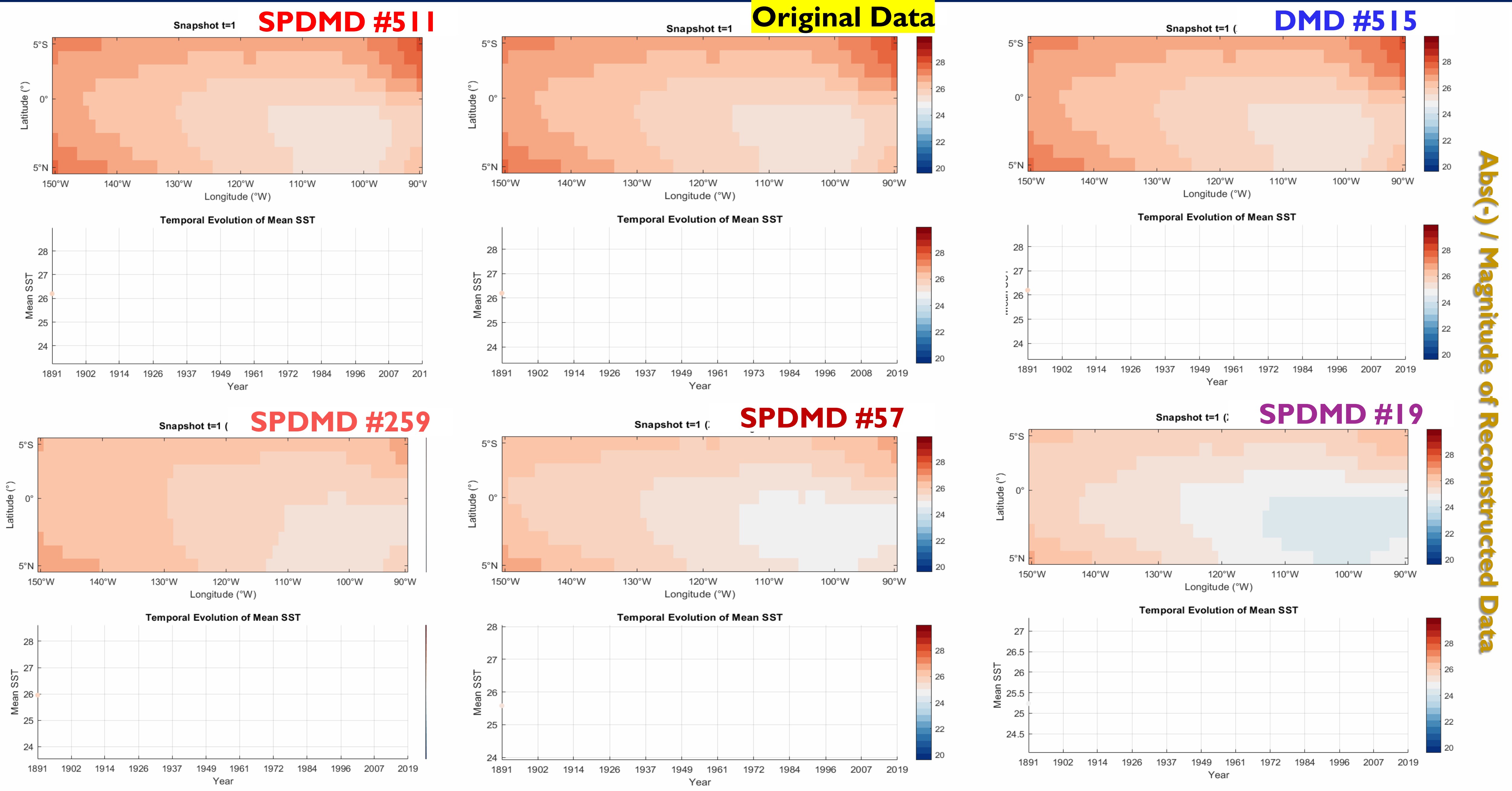
Table 2: Model reduction with the number of modes, the cost, and the performance loss in seasonal cycle of SST.

Weight γ	Modes $\#i$	Cost $\mathcal{J}(\mathbf{b}^\star)$	Loss $\Pi\%$
0.0001	511	2.2308×10^4	0.6010
0.0105	501	1.1602×10^5	0.8277
0.5503	490	1.8414×10^5	1.0224
8.5573	488	2.2331×10^5	1.1172
100.24	478	2.3250×10^5	1.1983
691.05	303	3.4864×10^5	1.8255
1714.3	107	4.7381×10^5	2.3815
4948.2	19	7.7276×10^5	3.0350
5757.2	9	8.6025×10^5	3.3761
16000	4	1.6447×10^6	3.5880

Ques:

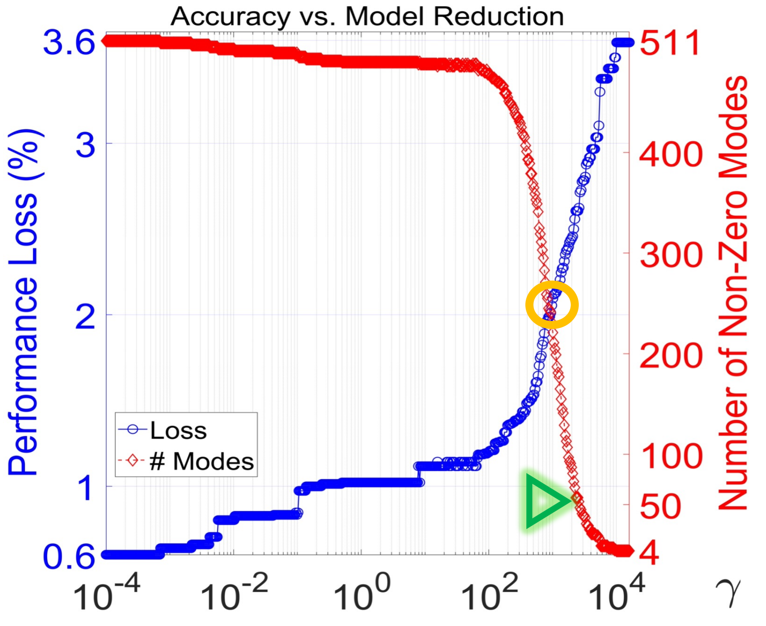
- Does the SPDMD method perform effectively?
- How does the number of modes (few/many) impact the accuracy of the climate model?

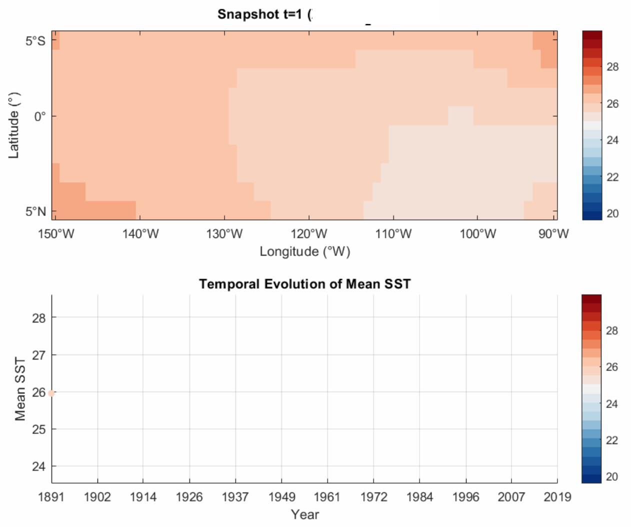
SPDMD Reproducibility: Reconstruction, Validation of Seasonal SST



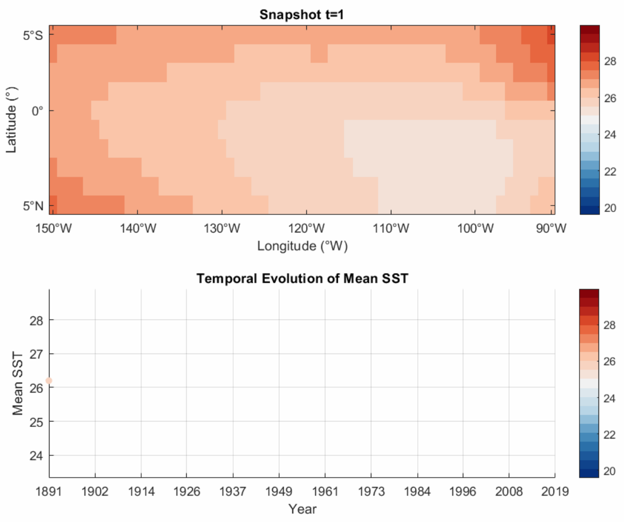
The Take-Home Messages:

- ✓ Koopman analysis for SST data by using SPDMD method to identify the dominant (spatial & temporal) modes.
- ✓ The captured Koopman modes (growth/decay) reveal climate patterns, suggesting the climate changes/variability…
- ✓ By reconstruction, KMD with SPDMD is effective method to make trade-off between accuracy and model reduction!

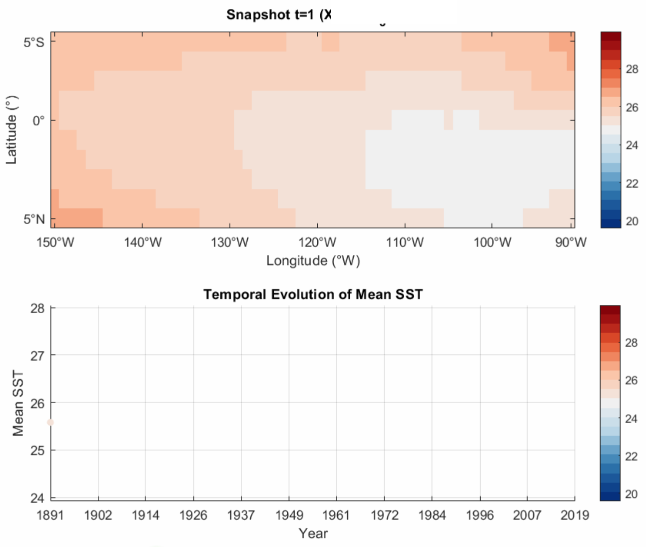




#259 Modes



Original SST (#515 Modes)



#57 Modes

arXiv:2507.05711
SPDMD applied to SST

Thank you!

ขอบคุณครับ/ค่ะ

Suggestions and Comments are Welcome!

zhang.zhicheng.2c@kyoto-u.ac.jp

MOONSHOT
RESEARCH & DEVELOPMENT PROGRAM

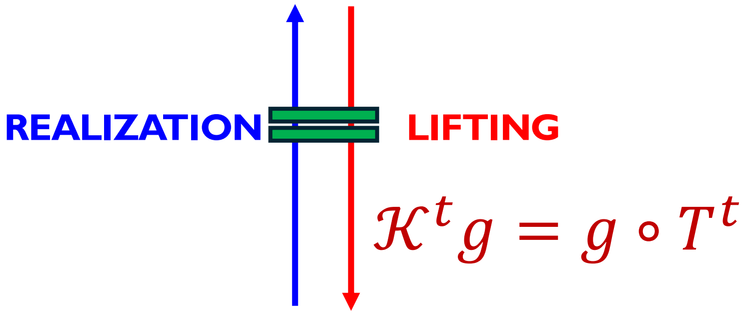
京都大学
KYOTO UNIVERSITY

Backup Slides

Koopman Operator: A (Bounded) Linear Operator

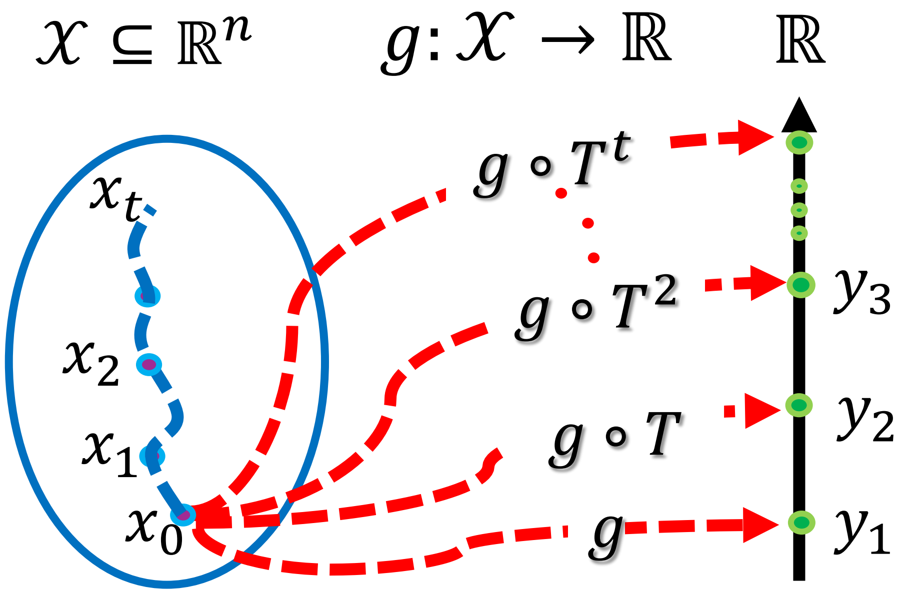
Koopman Framework

$$x_{t+1} = T(x_t)$$



$$g_{t+1} = \mathcal{K}^t g_t = g \circ T = g(x_{t+1})$$

- Lifts the **nonlinear evolution of state** into a ***linear evolution of functional observable***
- Linearity: $\mathcal{K}^t(\alpha_1 g_1 + \alpha_2 g_2) = \alpha_1 \mathcal{K}^t g_1 + \alpha_2 \mathcal{K}^t g_2$
  $\forall \alpha_1, \alpha_2 \in \mathbb{R},\ g_1, g_2 \in \mathcal{O}$



State space $\mathcal{X}$ (***Finite dim. Nonlinear*** **dynamics**)

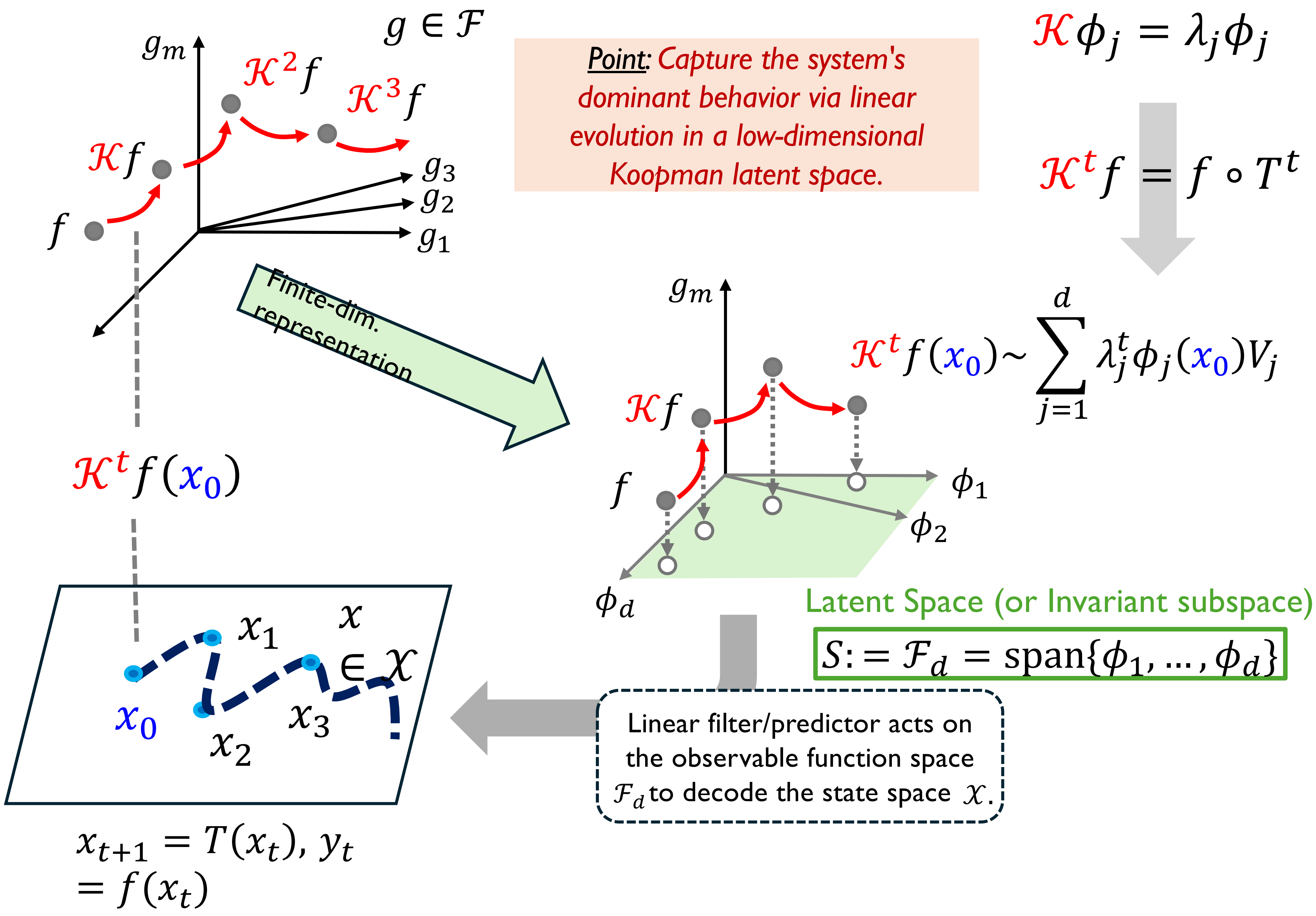
LIFTING

Observable space $\mathcal{F}$ (***Infinite dim.*** **linear evolution**)

$$\mathcal{K}^t g_t = g \circ T^t$$

- **Linear operator** → (**Spectral Analysis**)

Refs.) Koopman, PNAS'1931; Koopman & von Neumann. PNAS'1932; Mezic, Nonlinear Dynamics'05


$g \in \mathcal{F}$
g_m
g_1 g_2 g_3
f $\mathcal{K}f$ $\mathcal{K}^2 f$ $\mathcal{K}^3 f$
Point: Capture the system's dominant behavior via linear evolution in a low-dimensional Koopman latent space.
$\mathcal{K}\phi_j = \lambda_j \phi_j$
$\mathcal{K}^t f = f \circ T^t$
Finite-dim. representation
$\mathcal{K}^t f(x_0)$
$\mathcal{K}^t f(x_0) \sim \sum_{j=1}^{d} \lambda_j^t \phi_j(x_0) V_j$
ϕ_1 ϕ_2 ϕ_d
Latent Space (or Invariant subspace)
$S := \mathcal{F}_d = \mathrm{span}\{\phi_1, \ldots, \phi_d\}$
$x \in \mathcal{X}$
x_0 x_1 x_2 x_3
Linear filter/predictor acts on the observable function space $\mathcal{F}_d$ to decode the state space $\mathcal{X}$.
$x_{t+1} = T(x_t),\ y_t = f(x_t)$