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Sparse Robust Control Design via Scenario Program

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Sparse optimal control penalizes L^0 *norm* on the control signal to perform sparse optimization, where we minimize the length of the time duration on which the control input is exactly zero. This *sparse control*, also called *maximum hands-off control* [1], not only guarantees optimal performance, but also reduces the control effort. In general, the precise L^0 optimization problem for sparse control is discontinuous and non-convex, and hence it causes heavy computational burden. A universal approach is convex relaxation that uses L^1 norm to approximate L^0 optimal solution. In [1], it is claimed that the L^1 optimal control with *"bang-off-bang" property* is equivalent to L^0 optimal control based on the *normality* assumption for continuous systems. On the other hand, for discrete-time systems, an equivalence between ℓ^0 and ℓ^1 optimal control can be established on the *restricted isometry property* (RIP) [2]. Although other contributions in the sparse control setup already exist [1–3], in this work, we take a *probabilistic robust* perspective to perform sparse optimization for uncertain systems.

Let us consider a discrete-time uncertain system

$$x(t+1) = A(q)x(t) + b(q)u(t), \quad t = 0, 1, \cdots, M-1, \quad x(0) = \xi, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable and $u(t) \in \mathbb{R}$ is the scalar input. The coefficients $A(q) \in \mathbb{R}^{n\times n}$ and $b(q) \in \mathbb{R}^n$ are functions of uncertain parameters $q \in \mathbb{Q}$, where arbitrary dependence is allowed.

To deal with such an uncertain system, one can consider designing robust control to provide a performance guarantee for *all* admissible value of uncertain parameters $q \in \mathbb{Q}$. Precisely, the goal is to find a sequence input $\{u(t)\}_{t=0}^{M-1}$ that brings the terminal state $x(M; q)$ near the target $\bar{x}$ (or origin, $\bar{x} = 0$) with a small metric $\gamma \geq 0$. Note that the state at terminal time M can be written as

$$x(M; q) = A(q)^M \xi + \sum_{t=0}^{M-1} A(q)^{M-1-t} b(q)u(t) \doteq A(q)^M \xi + \mathcal{R}(q)u,$$

where the reachability matrix $\mathcal{R}(q)$ and the control input vector u are as follows:

$$\mathcal{R}(q) = [A(q)^{M-1}b(q) \ A(q)^{M-2}b(q) \ \cdots \ A(q)b(q) \ b(q)], \quad u = [u(0) \ u(1) \ \cdots \ u(M-2) \ u(M-1)]^\top.$$

We assume that the pair $(A(q), b(q))$ is robustly reachable, that is, rank $\mathcal{R}(q) = n$ for all $q \in \mathbb{Q}$. Then we can define the desired constraint

$$f(u, q) \doteq \|x(M; q) - \bar{x}\|_2 - \gamma = \|A(q)^M \xi + \mathcal{R}(q)u - \bar{x}\|_2 - \gamma \leq 0 \quad \forall q \in \mathbb{Q}, \tag{2}$$

where $f(u, q) : \mathbb{U} \times \mathbb{Q} \to [-\infty, \infty]$ is a scalar-valued function that describes the performance guarantee.

Motivated by the above, we consider a *worst-case sparse robust design*

$$\begin{aligned} \min_{u \in \mathbb{U} \subseteq \mathbb{R}^M} \quad & \|u\|_0 \\ \text{s.t.} \quad & \|A(q)^M \xi + \mathcal{R}(q)u - \bar{x}\|_2 - \gamma \leq 0 \quad \forall q \in \mathbb{Q}, \end{aligned} \tag{3}$$

where $\|u\|_0 = \sum_{t=0}^{M-1} |u(t)|^0$ is the ℓ^0 *norm* of input vector u, and $|u(t)|^0 = 1$ if $u(t) \neq 0$ and 0 otherwise.

Note that Problem (3) not only includes ℓ^0 non-convex optimization problem but also contains worst-case program, and hence it is difficult to solve. In order to make Problem (3) computationally tractable, in what follows, we address the two obstacles by means of relaxation method in which *convex relaxation* for ℓ^0 cost $\|u\|_0$ and *probabilistic relaxation* for constraint $f(u, q)$ for all $q \in \mathbb{Q}$.

As mentioned in [1–3], ℓ^1 relaxation is a covex approximation for ℓ^0 penalty. For the worst-case constraint like (2), a probabilistic relaxation approach for (2) is proposed [4], where the uncertainty parameters q are regarded as random variables which follow a probability measure $\mathbb{P}$ over the uncertainty set $\mathbb{Q}$.

We introduce the following definition of risk.

Definition 1 (Risk) [4] *Given a design variable u, the probability of violation (risk) is defined as*

$$V(u) = \mathbb{P}\{q \in \mathbb{Q} : \ f(u,q) > 0\}.$$

Hence, $V(u)$ is the probability with which the constraint is not satisfied by the design variable u. We say that u is an *ϵ-probability robust design* if it holds that $V(u) \leq \epsilon$.

This probabilistic method suggests us to relax (2) as the *probabilistic robust design*

$$\mathbb{P}\{\|A(q)^M\xi + \mathcal{R}(q)u - \bar{x}\|_2 > \gamma\} = \mathbb{P}\{q \in \mathbb{Q} : \ f(u,q) > 0\} \leq \epsilon. \tag{4}$$

The probabilistic form of constraint in optimization is referred to as *chance-constrained program* (CCP).

Recently, a *scenario program* (SP) [4] is presented, where only a finite number N of randomly sampled scenarios of an uncertain variable q are considered. In our context, it means that we consider a set of randomly selected constraints

$$\|A(q^{(i)})^M\xi + \mathcal{R}(q^{(i)})u - \bar{x}\|_2 - \gamma \leq 0, \quad \text{i.e.,} \quad f(u, q^{(i)}) \leq 0, \quad i = 1, \cdots, N. \tag{5}$$

This randomized approach would approximate the feasible solution for the CCP with high confidence. Besides, SP is practical whenever it is possible to generate samples from the uncertainty, since it does not require any further knowledge on the underlying probability distribution.

Motivated by these seminal works, in this study, we consider the *sparse robust control design via scenario program* for dealing with the uncertain discrete-time system (1). With the help of convex and probabilistic relaxation techniques, the original Problem (3) becomes solvable. The centerpiece of this study is the *ℓ^1-relaxation scenario robust design* of the form

$$\begin{aligned} \min_{u \in \mathbb{U} \subseteq \mathbb{R}^M} \quad & \|u\|_1 \\ \text{s.t.} \quad & \|A(q^{(i)})^M\xi + \mathcal{R}(q^{(i)})u - \bar{x}\|_2 - \gamma \leq 0, \quad i = 1, \cdots, N, \end{aligned} \tag{6}$$

where $\|u\|_1 = \sum_{t=0}^{M-1} |u(t)|$ is the ℓ^1 norm of control input vector u.

We can interpret the above relaxed ℓ^1 norm of control input design as a good approximation for ℓ^0 *norm* form in Problem (3) since it also induces the sparse solution.

Then we have the following result.

Theorem 1 *Assume that the pair $(A(q), b(q))$ is robustly reachable and $M > n$. Let u_N^* be the feasible and unique solution to the ℓ^1-relaxation scenario robust problem* (6). *Then for given $\epsilon \in (0,1)$ and $N \geq M$, it holds that*

$$\mathbb{P}^N\{V(u_N^*) > \epsilon\} \leq \beta, \tag{7}$$

where

$$\beta = \sum_{k=0}^{M-1} \binom{N}{k} \epsilon^k (1-\epsilon)^{N-k}. \tag{8}$$

That is, with probability (N-fold) at least $1-\beta$, the solution u_N^* is the ϵ-probability robust design.

This result is tight and (7) holds with equality if the convex program (6) is fully-supported [5].

References

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