Sparse Robust Control via Scenario Program

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What is sparsity ?

[1] quasi-norm (since it lacks the required homogeneity).

What is sparsity ?

- A vector $x \in \mathbb{R}^n$ is sparse if it contains many 0's, or has small ℓ^0 "norm"[1]

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where $\mathrm{supp}(x) \doteq \{i \in \{1, \cdots, N\} : x_i \neq 0\}$.

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- Applications: signal processing, machine learning, statistics, etc.

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- Applications: signal processing, machine learning, statistics, etc.
- In control system, it is referred to as sparse control. (ℓ^0 norm optimal control or maximum hands-off control).
- Difficulties: nonconvex problem (ℓ^0 cost); computational burden (if n is very large, like 1 million).

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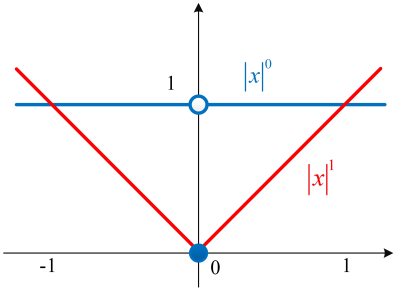
ℓ^0 norm and ℓ^1 norm

- ℓ^0 norm: $\|x\|_0 = \sum_{i=1}^{n} |x|^0, \quad |x|^0 \triangleq \begin{cases} 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$ nonconvex



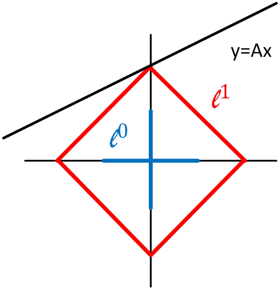
- ℓ^1 norm: $\|x\|_1 = \sum_{i=1}^{n} |x|$, sum of absolute values $|x|$, convex

Picture is taken from Nagahara, *Now Pub*, 2020.

ℓ^1 Optimization

- ℓ^1 norm optimization (or LASSO)[2]:

$$\min_{x \in \mathbb{R}^n} \|x\|_1 \quad \text{s.t.} \quad y = Ax$$



- ℓ^1 and ℓ^0 contours: a diamond and a plus, resp.

[2] Off-the-shelf packages: `CVX` or `YALMIP` in `MATLAB`.

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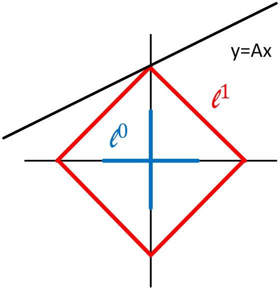

- ℓ^1 and ℓ^0 contours: a diamond and a plus, resp.
- ℓ^1 norm is a *good approximation* for ℓ^0 norm.

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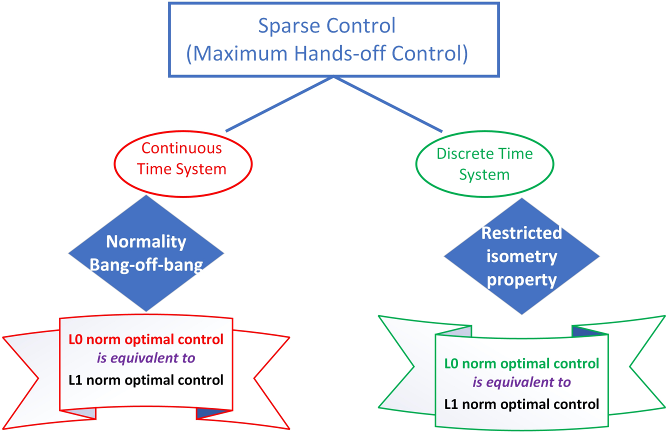
Existing Results



- Sparse control aims to maximize the length of the time duration on which the control input is exactly zero.
- Continuous/discrete time systems does not include any perturbations, or it only studies worst-case.

How about sparse control for *general* uncertain systems ?

Or, is it possible to adopt *sparse control* for uncertain systems as well as *enjoy a high probabilistic robustness* ?

Questions ?

How about sparse control for *general* uncertain systems ?

Or, is it possible to adopt *sparse control* for uncertain systems as well as *enjoy a high probabilistic robustness* ?

That is, exploring a probabilistic robust design rather than a worst case robust ...

This is our focus !

Problem Formulation

Consider an uncertain discrete-time linear system

$$x(t+1) = A(q)x(t) + b(q)u(t), \quad x(0) = \xi,\ t = 0, 1, \cdots, M-1$$

- $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}$ is the scalar input.
- Uncertain coefficients: $A(q) \in \mathbb{R}^{n\times n}$ and $b(q) \in \mathbb{R}^n$, $q \in \mathbb{Q}$.
- Terminal state constraint:

$$f(u, q) \doteq \|x(M; q) - \bar{x}\|_2 - \gamma \leq 0, \quad \forall q \in \mathbb{Q},$$

where $x(M; q) = A(q)^M \xi + \mathcal{R}_M(q)u$, $\bar{x}$ is target, γ is a gap,

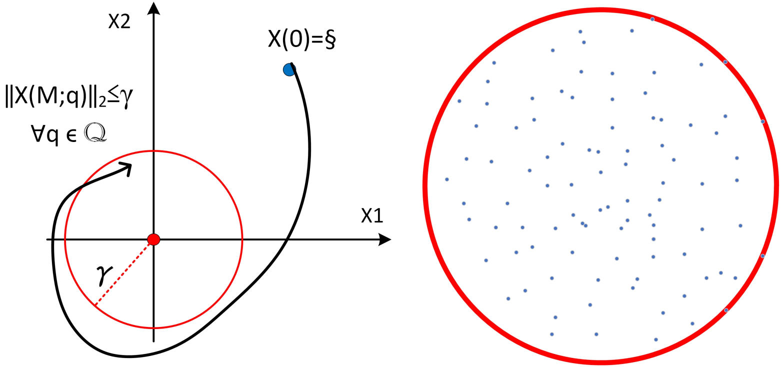
$$\mathcal{R}_M(q) = [A(q)^{M-1}b(q)\ \cdots\ A(q)b(q)\ b(q)],$$

$$u = [u(0)\ u(1)\ \cdots\ u(M-1)]^\top.$$

- Assume the pair $(A(q), b(q))$ is robustly reachable.

(i.e., $\text{rank}(\mathcal{R}_M(q)) = n$ for all $q \in \mathbb{Q}$).

Revisit Worst-case



- Find a feasible control such that drives the initial state near the origin at terminal time M with a small gap γ.
- For *all* uncertainty $q \in \mathbb{Q}$, terminal state falls into a disk centered at origin with radius γ. (worst-case !)

Worst-case Sparse Robust Control Design

$$
\begin{aligned}
&\min_{u \in \mathbb{R}^M} \ \|u\|_0 \\
&\text{s.t. } f(u,q) \leq 0, \quad \forall q \in \mathbb{Q}.
\end{aligned}
$$

- Computationally intractable (NP hard !)
- Enforce satisfaction of *all* constraints.
- Worst-case robust constraint (overly conservative).

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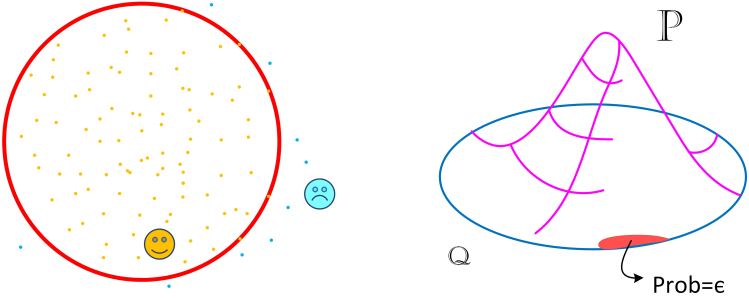
- Computationally intractable (NP hard !)
- Enforce satisfaction of *all* constraints.
- Worst-case robust constraint (overly conservative).
- Relaxation techniques:

  convex (ℓ^1) relaxation for ℓ^0 cost (i.e., $\|u\|_0 \Rightarrow \|u\|_1$);

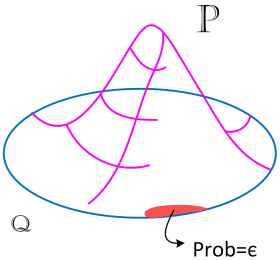
  probabilistic relaxation for constraint $f(u,q)$, $\forall q \in \mathbb{Q}$.



- This relaxation wants to seek a solution that violates at most *a small fraction of constraints*.

- See the uncertainty parameter $q \in \mathbb{Q}$ as a random element with a probability $\mathbb{P}$.
- Does not limit on *all* uncertainty set $\mathbb{Q}$.
  Admissible for a violation under a risk level $\epsilon \in (0, 1)$.

Violation of Probability

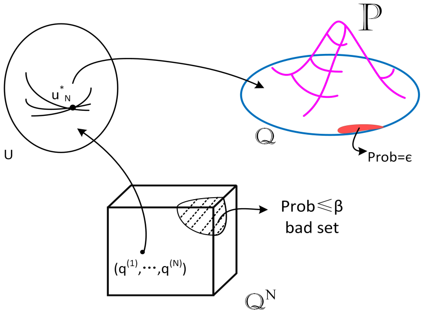


- Violation of Probability (Risk):

$$
\begin{aligned}
V(u) &= \mathbb{P}\{q \in \mathbb{Q} : f(u, q) > 0\} \\
&= \mathbb{P}\{q \in \mathbb{Q} : \|x(M; q)\|_2 > \gamma\}.
\end{aligned}
$$

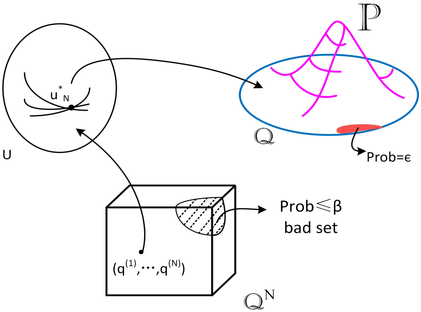
- This is called probabilistic (or chance) constrained.
- ϵ-probabilistic robust design: $V(u) \leq \epsilon$.

Scenario Program



- Generate a finite number of N samples $\{q^{(i)}\}_{i=1}^{N}$ i.i.d. according to the probability $\mathbb{P}$ from the uncertainty set $\mathbb{Q}$.
- Scenario counterpart: $f(u, q^{(i)}) \leq 0, \quad i = 1, \cdots, N.$

Scenario Program



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- Scenario counterpart: $f(u, q^{(i)}) \leq 0, \quad i = 1, \cdots, N.$
- A finite-sample guarantee: $\mathbb{P}^N\{V(u_N^*) > \epsilon\} \leq \beta.$

Sparse Scenario Robust Design

$$\begin{aligned} \min_{u \in \mathbb{R}^M} \quad & \|u\|_1 \\ \text{s.t.} \quad & f(u, q^{(i)}) \leq 0, \quad i = 1, \cdots, N. \end{aligned}$$

- It is a random convex program.
- Relaxed ℓ^1 cost and relaxed scenario counterpart.
- It satisfies with high probability for "unvisible" scenarios.
- Computationally tractable.
- Optimal solution u_N^* exists uniquely (random variable).
- Support set $\mathcal{S}$: its removal does not change solution.

Theorem (Main Result)

Assume that the pair $(A(q), b(q))$ is robustly reachable and $M > n$. Given a robust level $\epsilon \in (0,1)$ and a confidence parameter $\beta \in (0,1)$, choose the scenarios $N > M$ such that

$$\sum_{i=0}^{M-1} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i} \leq \beta.$$

Let u_N^* be the feasible and unique solution to the sparse (ℓ^1 relaxation) scenario robust problem. Then it holds that

$$\mathbb{P}^N\{V(u_N^*) > \epsilon\} \leq \beta.$$

In other words, with probability (N-fold product probability measure) at least $1-\beta$, the solution u_N^* is the ϵ-level probability robust design.

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- Equality holds if $|\mathcal{S}| = M$.
- Sample complexity: $N \geq \frac{2}{\epsilon}\left(\ln \frac{1}{\beta} + M - 1\right)$.

Numerical Example (Aircraft)

- Uncertain discrete-time plant

$$x(t+1) = \begin{bmatrix} a_{q11} & -1+a_{q12} & a_{q13} & a_{q14} \\ 1+a_{q21} & a_{q22} & a_{q23} & a_{q24} \\ a_{q31} & 1+a_{q32} & a_{q33} & a_{q34} \\ a_{q41} & a_{q42} & 1+a_{q43} & a_{q44} \end{bmatrix} + \begin{bmatrix} 2+b_{q1} \\ b_{q2} \\ b_{q3} \\ b_{q4} \end{bmatrix} u(t).$$

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- State constraints: $\xi = [0.1\ 0.1\ 0.1\ 0.1]^\top$, $\|x(20)\|_2 \leq \gamma \doteq 0.02$.

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- Control constraint: $|u(t)| \leq 1$, $\forall t = 0, 1, \cdots, 19$.

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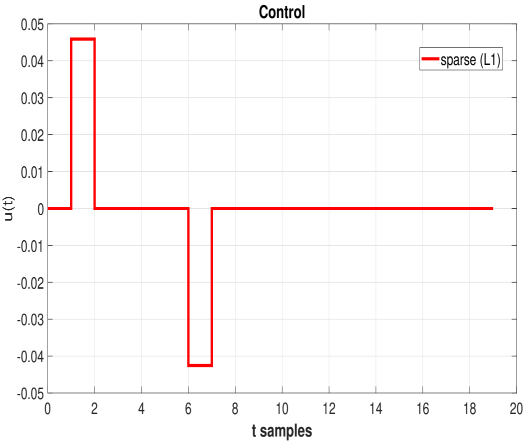
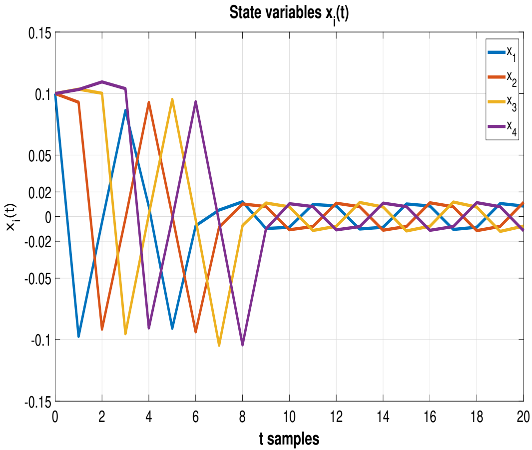
- Control constraint: $|u(t)| \leq 1$, $\forall t = 0, 1, \cdots, 19$.
- Required finite scenarios $N \geq 888$, taking $N = 900$.

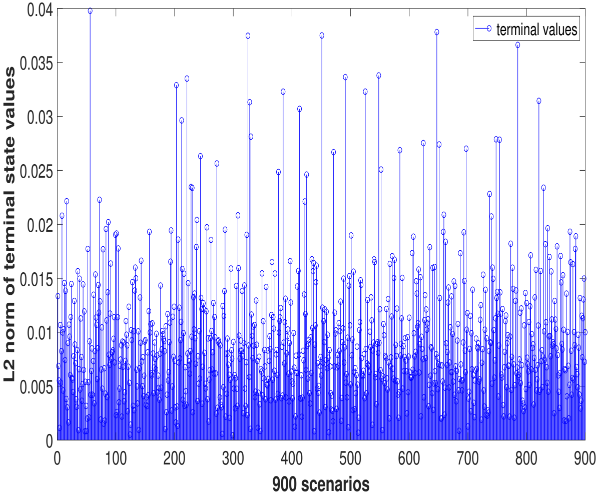


- Optimal control solution $\|u^*_{900}\|_1 = 0.0884$.
- The designed ℓ^1 norm optimal control is quite sparse.

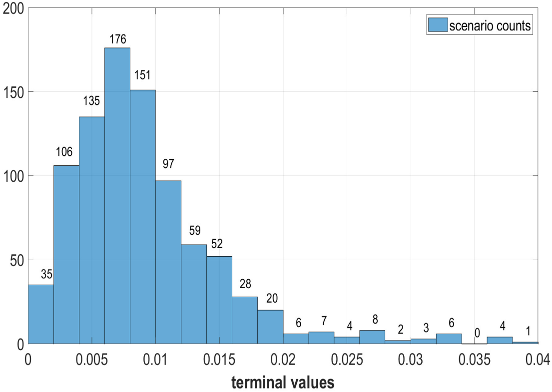


- The terminal state of four states are close to the origin, and they are fluctuated under a prescribed gap value $\gamma = 0.02$.

Terminal State



Validation Test



- Validation test: $\mathbb{P}^{900}\{\|x(20; q^{(i)})\|_2 > 0.02\} \leq 5\%$
- Violation counts are $41 (< 45 = 900 \times 5\%)$.
- The designed sparse control for uncertain discrete time system is with a high probabilistic robustness (95%).

Conclusion

- Take *probabilistic robust* perspective to study sparse control for *general* uncertain discrete-time systems.
- Provide *a finite-sample guarantee* for the designed sparse control with a high probabilistic robustness.
- By means of relaxation techniques, i.e.,
  convex (or ℓ^1) relaxation for ℓ^0 cost;
  probabilistic relaxation for constraint $f(u, q)$,
  the sparse scenario program becomes tractable.

Thank You Very Much !

Q & A