Sparse Feedback Control Realization Using a Dynamic Linear Compensator

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Abstract: In this note, we explore the discrete time sparse feedback control realization for a constrained sparse optimal control problem, where the feedback controller enjoys sparse inputs and a dynamic linear compensator. We then investigate the resulting augmented dynamics to ensure the closed-loop stability. Finally, numerical example demonstrates the effectiveness of proposed control approach.

Keywords: Sparse optimal control, feedback stabilization, dynamic linear compensator, closed-loop solution.

1. INTRODUCTION

Minimizing control efforts is a critical issue in control community, the goal is to *maximize the time horizon over which the control value is exactly zero*, and the resulting control signal is often sparse and low consumption. Hence, we refer to such control as "sparse control", "hands-off control", or "ℓ_1 optimal control", etc [1].

In this note, we develop sparse optimal control strategy *from open-loop control to closed-loop control realization*. In fact, the related works for sparse optimal controller is almost devoted to open-loop scenarios. We thus intend to seek an explicit and analytic closed-loop solution of sparse control problem. Different from a real-time feedback algorithm (e.g., model predictive control) approach to sparse control [1], we here focus on a *sparse feedback control* realization using a *dynamic compensator*, where the proposed dynamic state feedback controller is *linear* and the explicit compensator derivation leverages a technique built on relatively optimal control [2, 3].

2. PROBLEM STATEMENT

Let us consider a discrete linear time invariant (LTI) plant of the form

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the controlled output, and A, B, C, and D are real constant matrices of appropriate sizes. We assume that the pair (A, B) is reachable. In addition, we impose an output constraint

$$-\mu \le y(t) \le \mu, \tag{2}$$

for all $\|x_0\|_1 \le 1$, where $\mu \in \mathbb{R}^p$ dominates the magnitude of the output.

The compensator $\mathcal{K}$ which we want to design is a dynamic state feedback

$$\begin{aligned} z(t+1) &= Fz(t) + Gx(t), \qquad z(0) = 0, \\ u(t) &= Hz(t) + Kx(t), \end{aligned} \tag{3}$$

where $z(t)$ is the state of the compensator and F, G, H, and K are real matrices of appropriate sizes. Note that we set the initial state $z(0)$ of the compensator as *zero*.

Then, the closed-loop augmented system composed of the plant (1) and the compensator (3) can be described as

$$\begin{aligned} \psi(t+1) &= (\mathcal{A} + \mathcal{B}\mathcal{K})\psi(t), \qquad \psi(0) = \psi_0 \\ y(t) &= (\mathcal{C} + \mathcal{D}\mathcal{K})\psi(t) \end{aligned} \tag{4}$$

where

$$\psi(t) \doteq \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \quad \psi_0 \doteq \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$$

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} K & H \\ G & F \end{bmatrix},$$

$$\mathcal{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} D & 0 \end{bmatrix}.$$

The objective of this note is to give a compensator of the form (3) which generates a desirable input sequence $\{u(0), u(1), \cdots, u(N-1)\}$ such that it drives the plant state $x(t)$ from a nonzero initial state $x(0)$ to the origin in a finite time steps N (i.e., the terminal state $x(N) = 0$) with a minimum/sparse control effort.

3. MAIN RESULTS

To address the compensator design in a compact way, we introduce a stable matrix $P \in \mathbb{R}^{N\times N}$, which is an N-Jordan block associated with 0 eigenvalue, defined by

$$P = \begin{bmatrix} 0 & 0 \\ I_{N-1} & 0 \end{bmatrix}, \tag{5}$$

where $I_{N-1} \in \mathbb{R}^{(N-1)\times(N-1)}$ is the identity matrix.

In the following, we show how to implement sparse feedback control, which can be divided into two steps. The first one is to address a constrained sparse input matrix *optimization*. The second one is to achieve a feedback *realization* using a dynamic linear compensator.

We first investigate the following ℓ_1 optimal control problem by sparse optimization.

† Zhicheng Zhang is the presenter of this paper.

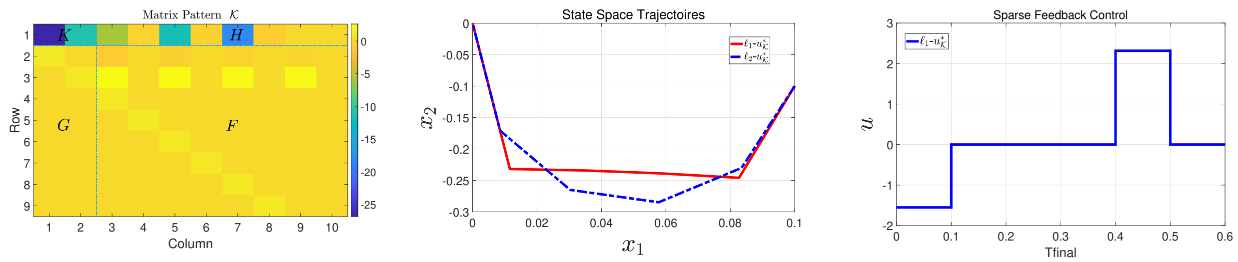

Fig. 1 The compensator pattern (left), the state-space trajectories (middle), and the sparse feedback control input (right).

Problem 1 (Sparse Input Matrix Optimization) Find the matrices $X \in \mathbb{R}^{n\times nN}$ and $U \in \mathbb{R}^{m\times nN}$ such that the obtained U is sparse, which amounts to solve a constrained ℓ_1 norm input matrix optimization

$$
\begin{aligned}
\min_{X,U} \quad & \|U\|_1 \\
\text{s.t.} \quad & AX + BU = X(P\otimes I_n), \\
& I_n = X(e_1 \otimes I_n), \\
& \operatorname{abs}(CX + DU) \le \mu(\mathbf{1}_n \otimes \mathbf{1}_N)^\top,
\end{aligned}
$$

where $\operatorname{abs}(\cdot)$ returns the absolute value of each element in a matrix, $e_i \in \mathbb{R}^N$ denotes the vector with a 1 in the ith element and 0's elsewhere, and $\mathbf{1}_n = \begin{bmatrix}1 \cdots 1\end{bmatrix} \in \mathbb{R}^n$.

The above control problem is an open-loop sparse optimal control strategy stemmed from [1], which provides an open-loop ℓ_1 optimal control solution u^*. We here extend the idea to simultaneously satisfy for all initial state $x_0 \in \{e_1, e_2, \cdots, e_n\}$. Clearly, Problem 1 is a convex optimization, and hence is computationally tractable by using the off-the-shelf packages, such as CVX, or YALMIP in MATLAB.

Once the optimal solution (X, U) of Problem 1 is attained, we then proceed sparse feedback control synthesis. Namely, we study the feedback realization problem.

Problem 2 (Realization) Based on the solution of Problem 1, solve a linear equation

$$
\begin{bmatrix} K & H \\ G & F \end{bmatrix}\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \tag{6}
$$

with respect to (F, G, H, K) and determine the compensator matrices, where

$$
Z = \begin{bmatrix} 0 & I_{n(N-1)} \end{bmatrix}, \qquad V = Z(P \otimes I_n).
$$

In fact, the matrices of (6) gives a compensator (3) which generates a desirable input sequence, which is stated in the following theorem.

Theorem 1 (Realization of Sparse Feedback Control) Suppose that Problem 1 has the minimizer (X, U). Then the equation (6) has the unique solution (F, G, H, K) and the corresponding compensator (3) generates the input sequence $u(t) = U(e_{t+1} \otimes x_0)$, $t = 0, 1, \ldots, N-1$, which drives the plant state $x(t)$ from $x(0) = x_0$ to $x(N) = 0$ under the output constraint (2). Furthermore, the closed-loop system (4) is internally stable.

Proof: We first describe X and U as

$$
X = \begin{bmatrix} X_0 & X_1 & \cdots & X_{N-1} \end{bmatrix},
$$

$$
U = \begin{bmatrix} U_0 & U_1 & \cdots & U_{N-1} \end{bmatrix},
$$

where $X_t \in \mathbb{R}^{n\times n}$, $U_t \in \mathbb{R}^{m\times n}$, and $t = 0, 1, \ldots, N-1$. Since the second constraint of Problem 1, we see $X_0 = I_n$. With this fact and the definition Z, we have that

$$
\det \Psi \ne 0, \qquad \Psi = \begin{bmatrix} X \\ Z \end{bmatrix}.
$$

Thus we see that (6) has the unique (F, G, H, K). Also, the second constraint of Problem 1 with (6) says that

$$
(\mathcal{A} + \mathcal{B}\mathcal{K})\Psi = \Psi(P \otimes I_n),
$$

which implies that the closed-loop system is internally stable and $x(N) = 0$ is achieved for any initial state $\psi(0)$ of the system. Moreover, since

$$
\begin{aligned}
& X(e_{t+1} \otimes x_0) = X_t x_0, \quad U(e_{t+1} \otimes x_0) = U_t x_0, \\
& (P \otimes I_n)e_{t+1} \otimes x_0 = e_{t+2} \otimes x_0,
\end{aligned}
$$

we see that the sequences $x(t) = X(e_{t+1} \otimes x_0)$, $u(t) = U(e_{t+1} \otimes x_0)$ indeed satisfy the LTI dynamics (1), which proves the theorem for realizing sparse feedback control. ■

Based on the proposed Theorem 1, we can directly give a corollary that establishes the connection between the open-loop sparse optimal control solution and the closed-loop sparse optimal control solution.

Corollary 1 (Equivalence) Suppose that Problems 1 has the minimizer (X, U). Let $u_{\mathcal{K}}^*$ be optimal sparse feedback (closed-loop) control solution using a dynamic linear compensator $\mathcal{K}$ (3) and u^* be open-loop ℓ_1 optimal control solution u^* of a discrete LTI system (1), respectively. Then, it holds that

$$
u^* = u_{\mathcal{K}}^* = Hz + Kx^*, \tag{7}
$$

for any initial condition x_0 such that $\|x_0\|_1 \le 1$.

4. SIMULATIONS

Let us consider a continuous second-order system

$$
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \qquad x(0) = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}
$$

and discretize it with sampling time $0.2\ s$ to be the plant (1). We then solve Problems 1 (Optimization) and 2 (Realization) to implement the sparse feedback control. By taking time steps $N = 5$, the simulations are shown in

Fig. 1. It is seen that the pattern $\mathcal{K}$ reveals the real values of four matrices (F, G, H, K). The related state-space trajectories with two different ℓ_1 (red solid) and ℓ_2 (blue dash) optimal feedback controllers display that its always bring the initial state to the origin. Meanwhile, the obtained feedback control input is sparse as desired.

5. CONCLUSION

In this report, we have proposed a sparse feedback control realization from open-loop solution to closed-loop solution. By means of implementing a dynamic linear compensator, the stability, optimality, and sparsity of the sparse feedback controller are ensured.

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