Sparse Feedback Control Realization

Using a Dynamic Linear Compensator

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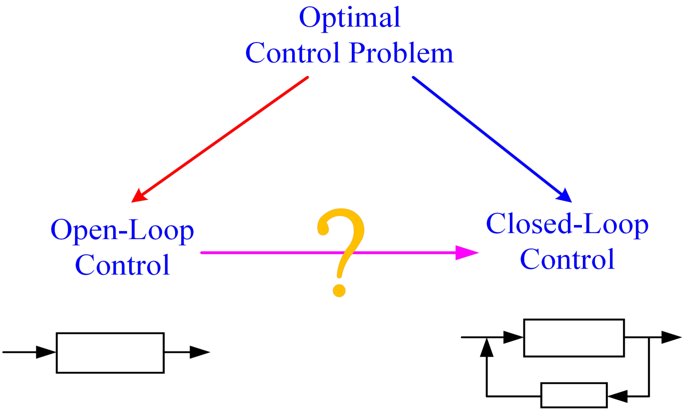
Background: Optimal Control



Q: *Can we infer closed-loop controller from its open-loop solutions* ?

Major challenge: *Closed-Loop Sparse Optimal Control Inputs*

Problem Statement

LTI Dynamics

Consider a discrete linear time invariant (LTI) system

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0$$
$$y(t) = Cx(t) + Du(t), \quad t = 0, 1, \cdots, N-1.$$

where

- $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, and $y \in \mathbb{R}^p$ is the output
  - state and input constraints

    e.g., output constraint: $\mathcal{Y} = \{y : -\mu \leq y(t) \leq \mu, \ \mu \in \mathbb{R}^p\}$
- Model-based system matrices A, B, C, D are known
  - Mild assumption: the pair (A, B) is reachable
- Goal: Finding a sparse control sequence $\{u(t)\}_{t=0}^{N-1}$ such that it brings the state $x(t)$ from initial value x_0 to the origin under finite N steps

Open-Loop **S**parse **O**ptimal **C**ontrol

Sparse Control (or Maximum Hands-off Control, ℓ_1 Optimal Control)

A sparse optimal control problem aims to *maximize the time interval over which the control input is exactly zero*, i.e., *control effort minimization*, which is equivalent to solving a constrained ℓ_1 norm optimization problem

$$
\begin{aligned}
\text{(SOC)} \quad \min_{u} \quad & \mathcal{J}(u) = \sum_{i=1}^{m} \sum_{t=0}^{T-1} |u_i(t)| = \|u\|_1 \\
\text{s.t.} \quad & x(t+1) = Ax(t) + Bu(t) \\
& x(0) = x_0, \quad x(N) = 0, \\
& -\mu \leq Cx(t) + Du(t) \leq \mu, \quad \forall t = 0, 1, \cdots, N-1
\end{aligned}
$$

- Open-loop control: simplicity for setup, construction and design
- Convex optimization and computationally tractable (`CVX`,`YALMIP`)
- Pros & Cons: Preference theory, but Fails To real-world Applications

Why Sparse Feedback Control ?

Drawbacks of Open-Loop: Poor Reliability & Flexibility & Accuracy

" $\cdots$, *open-loop control is something like riding a bicycle with your eyes closed, which is very fragile against disturbance*, $\cdots$ " 🙈

Question: How to obtain *closed-loop solution* of Problem (SOC) ?

- Real-Time Algorithms/Iterations (Implicit Feedback)
  - e.g., model predictive control (MPC) [Nagahara et al., EURASIP J. ASP, 76, 2016]
  - e.g., self-triggered control [Nagahara et al., IEEE TAC, 61(3), 2016, Sec. 4]
  - e.g., dynamic programming [Lewis et al., Optimal Control, 3rd ed, 2012, Sec. 6]

👉 *lots of recent momentum* with contributions by Maciejowski, Quevedo, Nagahara, Rao, Bakolas, Kenji, Kishida, Oishi, ...

- Demerit: "*Online optimization*" $\Longrightarrow$ heavy computation !

Seeking Closed-Loop Sparse Solution

- Classic State Feedback Gain Design
  - e.g., static state feedback gain: $u(t) = Kx(t)$
  - e.g. linear quadratic regulator (LQR): $\mathcal{J}(u) = x^\top Qx + u^\top Ru$

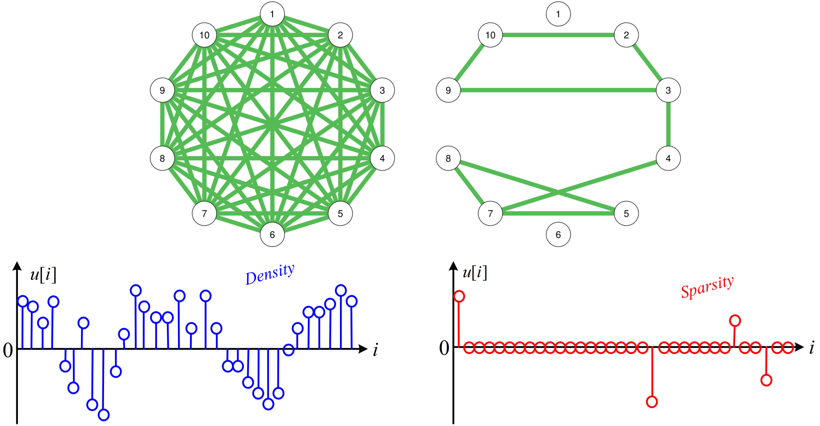
    [Lin & Fardad & Jovanovic, IEEE TAC, 58(9), 2013]

A rich stories for sparse feedback control design, however, ...

- Fact 1: Sparse feedback control majors in *structured sparsity*
- Fact 2: Determining a feedback gain matrix K is not a simple task ...
  - e.g., taking $\mathcal{J}(u) = \|u\|_1, \quad \mathcal{J}(u) = \lambda_1\|u\|_1 + \lambda_2\|u\|_2^2$, ... $\Rightarrow \quad K$?

🌵 Dilemma: look for sparse feedback solution with an *explicit feedback* gain and enjoy *input sparsity*, instead of structured sparsity

Sparsity: Spatial vs. Temporal



- Structured sparsity (Spatial): reduce the no. of communication links
- Input sparsity (Temporal): promote the no. of zero input values
  - Notice: $\mathcal{J}(K) = \|K\|_1, \quad u = Kx \quad$ vs. $\quad \mathcal{J}(u) = \|u\|_1$

Dynamic Linear Compensator

How Should I Design a Feedback Controller ?

Open-loop solutions at hand

$\Longrightarrow$ **Closed-loop** solutions

Oracle: Dynamic state feedback control [Blanchini & Pellegrino, IEEE TAC, 48(12), 2003]

Dynamic Linear Compensator

The compensator $\mathcal{K}$ which we want to design is a dynamic state feedback

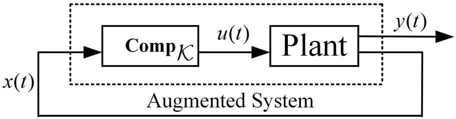
$$
\begin{aligned}
z(t+1) &= Fz(t) + Gx(t), \quad z(0) = 0, \\
u(t) &= Hz(t) + Kx(t)
\end{aligned}
$$

where F, G, H and Z are real matrices with appropriate sizes.

- *Feedback control realization from open-loop solutions*

Infer sparse feedback control using dynamic linear compensator !

Augmented Closed-Loop System



Augmented Closed-Loop System

We introduce an augmented closed-loop system composed of the LTI plant and compensator $\mathcal{K}$, expressed by as follows

$$\begin{aligned} \psi(t+1) &= (\mathcal{A} + \mathcal{B}\mathcal{K})\psi(t), \quad \psi(0) = \psi_0 \\ y(t) &= (\mathcal{C} + \mathcal{D}\mathcal{K})\psi(t), \quad \mathcal{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \; \mathcal{D} = \begin{bmatrix} D & 0 \end{bmatrix}, \end{aligned}$$

$$\psi(t) \doteq \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \; \psi_0 \doteq \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \; \mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \; \mathcal{B} = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, \; \mathcal{K} = \begin{bmatrix} K & H \\ G & F \end{bmatrix}.$$

- Ensure internally stability & determine feedback matrix $\mathcal{K}$

Compact Form

We introduce a nilpotent matrix $P \in \mathbb{R}^{N \times N}$, which is an N-Jordan block associated with 0 eigenvalue, defined by

$$P = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ I_{N-1} & 0 \end{bmatrix}$$

To implement sparse feedback control, we perform two steps:

- First: *sparse optimization*.
- Second: *feedback realization*

First Step: Sparse Optimization

Problem 1 (Sparse Optimization)

Find the matrices $X \in \mathbb{R}^{n \times nN}$ and $U \in \mathbb{R}^{m \times nN}$ such that the obtained U is sparse, which amounts to solve an ℓ_1 norm (sparse) matrix optimization

$$\begin{aligned} \min_{X,U} \quad & \|U\|_1 = \textstyle\sum_{i=1}^{m} \sum_{j=1}^{nN} |u_{ij}| \,, \\ \text{s.t.} \quad & AX + BU = X(P \otimes I_n) \,, \\ & I_n = X(e_1 \otimes I_n) \,, \\ & \mathrm{abs}(CX + DU) \leq \mu(\mathbf{1}_n \otimes \mathbf{1}_N)^\top, \end{aligned}$$

where $e_i \in \mathbb{R}^N$ is the vector with a 1 in the ith element and 0's elsewhere.

- Extend to a more general initial scenario: $x_0 \in \{e_1, e_2, \cdots, e_n\}$
- Generate n possible trajectories: vectors (x,u) $\Longrightarrow$ matrices (X,U)
- Convex program: optimal solution (X, U) is available.

Second Step: Feedback Realization

Once the optimal solution (X, U) of Problem 1 is attained, we then implement sparse feedback control realization.

Problem 2 (Feedback Realization)

Based on the solution (X, U) of Problem 1, solve a linear equation

$$\begin{bmatrix} K & H \\ G & F \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix}$$

w.r.t. (F, G, H, K) and determine the compensator $\mathcal{K}$, where

$$Z = \begin{bmatrix} 0_{n(N-1)\times n} & I_{n(N-1)} \end{bmatrix}, \quad V = Z(P \otimes I_n).$$

$$\mathcal{K} = \boxed{\begin{bmatrix} K & H \\ G & F \end{bmatrix}} = \boxed{\begin{bmatrix} U \\ V \end{bmatrix}} \boxed{\begin{bmatrix} X \\ Z \end{bmatrix}^{-1}} \quad ? \Leftarrow \quad \left[\begin{array}{c:ccc} I_n & X_1 & \cdots & X_{N-1} \\ \hdashline 0_{n(N-1)\times n} & & I_{n(N-1)} & \end{array}\right]$$

Main Result (Realization of Sparse Feedback Control)

Theorem (Realization of Sparse Feedback Control)

Suppose that Problem 1 has the minimizer (X, U).
Then Problem 2 determines the unique solution (F, G, H, K), resulting in the compensator $\mathcal{K}$ generates the input sequence $u(t) = U(e_{t+1} \otimes x_0)$, $t = 0, 1, \ldots, N-1$, which drives the system state $x(t)$ from $x(0) = x_0$ to $x(N) = 0$ under the output constraint $y \in \mathcal{Y}$.
Furthermore, the closed-loop augmented system is internally stable.

Corollary (Equivalence)

Suppose that $u^*_{\mathcal{K}}$ be sparse optimal feedback control solution by using a linear dynamic compensator $\mathcal{K}$, and u^* be the optimal open-loop solution of Problem 1, respectively. Then, for $x_0 \in \{e_1, \cdots, e_n\}$, we have the result

$$u^* = u^*_{\mathcal{K}} = Hz + Kx^*.$$

- Remark: Sparse feedback controller $u^*_{\mathcal{K}}$ is an N step deadbeat control

Numerical Example

Let us consider a continuous second-order system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad x(0) = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}.$$

and discretize it with ZoH sampling 0.1 s to be the discrete plant.

- Take time steps $N = 5$;
- Solve Problems 1 (Optimization) and 2 (Realization)

```
X =

    1.0000         0    0.8707    0.0435    0.6158    0.0308    0.3670    0.0183    0.1219    0.0061
         0    1.0000   -2.5884   -0.1293   -2.5142   -0.1256   -2.4651   -0.1232   -2.4406   -0.1219
```

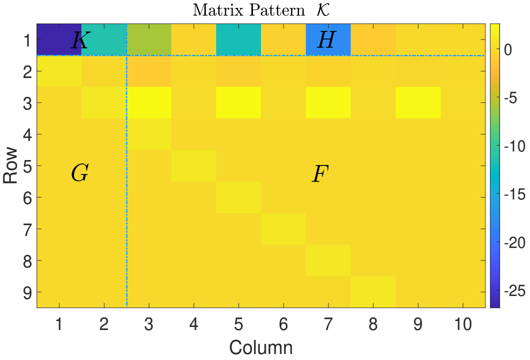
```
U =

  -26.8413  -11.3243   -0.0000   -0.0000    0.0000    0.0000    0.0000    0.0000   24.3659    1.2173
```

- Sparse feedback control realization

Numerical Results

By computing, we obtain the pattern of compensator $\mathcal{K}$

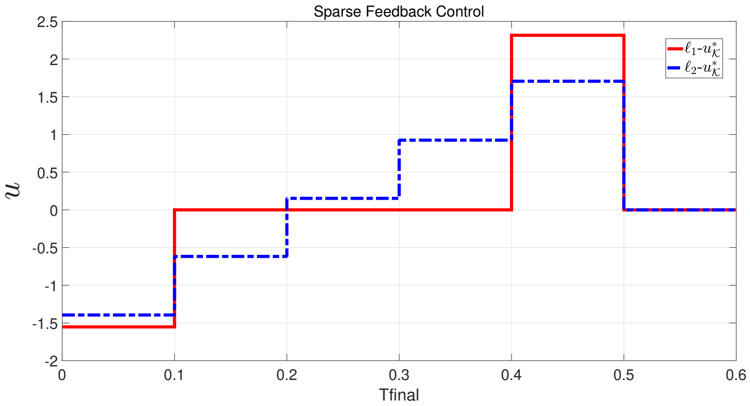


- Colorbar reveals the real values of the elements of matrices

e.g. $K = \begin{bmatrix} -26.8413 & -11.3243 \end{bmatrix}$

$H = \begin{bmatrix} -5.9419 & -0.2968 & -11.9433 & -0.5967 & -18.0642 & -0.9025 & 0.0000 & 0.0000 \end{bmatrix}$

Feedback Control Inputs: ℓ_1 vs. ℓ_2 optimal controller

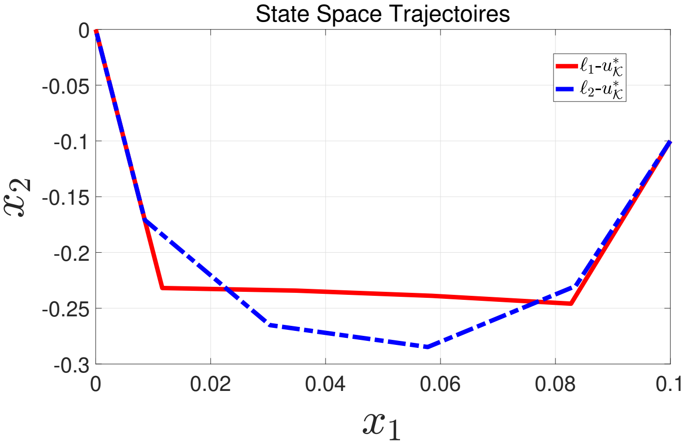


- Sparse feedback control (i.e., closed-loop ℓ_1 optimal control, red)

$$u^*_{\mathcal{K}} = Hz + Kx^* = \begin{bmatrix} -1.5517 & -0.0000 & -0.0000 & -0.0000 & 2.3149 & 0.0000 & -0.0000 \end{bmatrix}$$

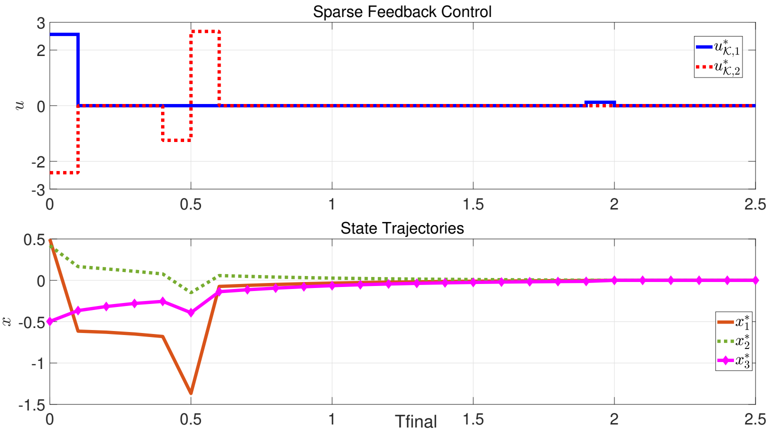
- Sparse feedback $\|U\|_1$ vs. Mini. energy feedback $\|U\|_2$

State-Space Trajectories



- Designed (sparse) optimal feedback controller (ℓ_1-$u^*_{\mathcal{K}}$ or ℓ_2-$u^*_{\mathcal{K}}$) drives the system state $x(t)$ from initial state $x_0^\top = [0.1 \ \ -0.1]$ to the origin.

Multiple Inputs Case



- Proposed sparse feedback control is successful for multi-inputs 

Simple, offline, low-cost, economic, and practical tool !

Conclusion

The Take-Home Message

We develop sparse optimal control from open-loop control to closed-loop control realization, which makes real-world applications be possible !

- Dynamic linear compensator is explicit, which paves the way towards *open-loop sparse control to closed-loop sparse control*.
- Provide sparsity, optimality, and stability for designed feedback control
- Beyond sparse cost: Useful for any "convex" optimal control index
  - e.g., $\mathcal{J}(u) = c(x, u)$, where $c(\cdot)$ is a convex cost function.

Future Work

- *Direct data-driven* sparse feedback control
- *Continuous-time* sparse feedback control

Thank you for your attention !

Suggestions & Comments are Welcome !

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Appendix: Proof Sketch

We first describe X and U as

$$X = \begin{bmatrix} X_0 & X_1 & \cdots & X_{N-1} \end{bmatrix},$$
$$U = \begin{bmatrix} U_0 & U_1 & \cdots & U_{N-1} \end{bmatrix},$$

where $X_t \in \mathbb{R}^{n\times n}$, $U_t \in \mathbb{R}^{m\times n}$, and $t = 0, 1, \ldots, N-1$.
Since the second constraint of Problem 1, we see that $X_0 = I_n$.
With this fact and the definition Z, we have that

$$\det \Psi \neq 0, \quad \Psi = \begin{bmatrix} X \\ Z \end{bmatrix} = \left[\begin{array}{c:c} I_n & X_1 \ \cdots \ X_{N-1} \\ \hdashline 0_{n(N-1)\times n} & I_{n(N-1)} \end{array}\right].$$

Thus we see that linear equation in Problem 2 has the unique (F, G, H, K). Also, the second constraint of Problem 1 claims that

$$(\mathcal{A} + \mathcal{B}\mathcal{K})\Psi = \left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} K & H \\ G & F \end{bmatrix} \right) \begin{bmatrix} X \\ Z \end{bmatrix} = \Psi(P \otimes I_n)$$

which implies that the closed-loop system is internally stable.

Proof ('Cont)

Moreover, since

$$\begin{aligned} X(e_{t+1} \otimes x_0) &= X_t x_0, \\ U(e_{t+1} \otimes x_0) &= U_t x_0, \\ (P \otimes I_n)(e_{t+1} \otimes x_0) &= e_{t+2} \otimes x_0, \end{aligned}$$

we see that the sequences

$$x(t) = X(e_{t+1} \otimes x_0), \quad u(t) = U(e_{t+1} \otimes x_0)$$

indeed satisfy discrete time system.
Similar to the above discussion, it is easy to verify the output (or input and state) constraints of Problem 1, which establishes the theorem.