Modeling, Robustness, and Stability for Sparse Optimal Control of Dynamical Systems

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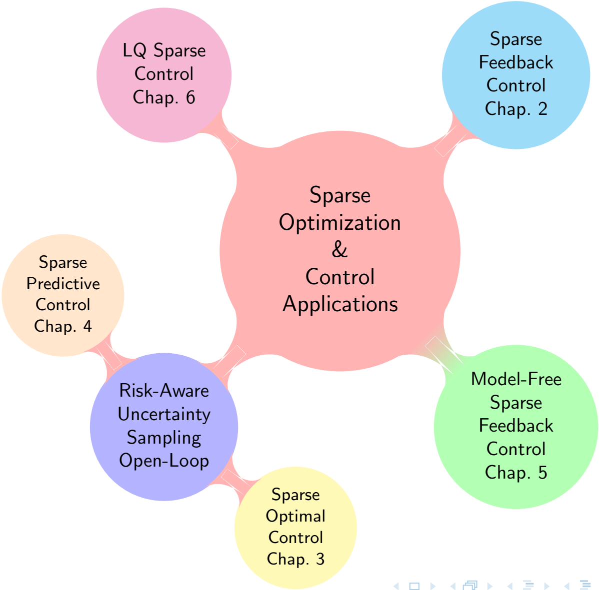
OSAKA UNIVERSITY

Overview of Thesis's Chapters



Comparisons & Contributions

Sparse Optimal Control of Dynamical Systems

- Open-loop vs. Closed-Loop Models
  - ★ **Stability of Sparse Feedback Systems** **(Chap. 2)**

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♣ *Modeling, Robustness, and Stability* are of Interests !

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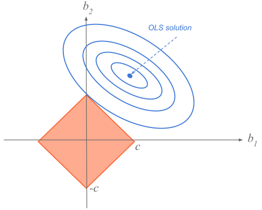
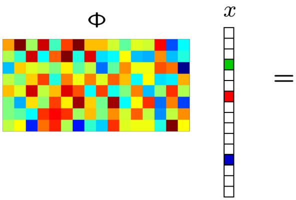
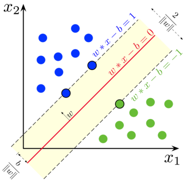
Sparse Modeling

Lots of momentums for sparsity ... pictures from wiki







Lasso in statistics, compressed sensing in images/signals, SVM in ML

- Parsimony: reduce the dimensions/sizes/components of variables

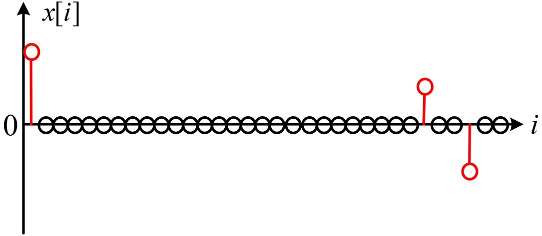
Sparse Optimization

Sparse Optimization Problem

Find sparsest solution that minimizes

$$\begin{aligned} \min_{x} \quad & \|x\|_0 \\ \text{s.t.} \quad & x \in \mathcal{X} \subseteq \mathbb{R}^n \end{aligned}$$

- $\|x\|_0 \triangleq \#\{i : x_i \neq 0\}$ counts the number of non-zero elements



Sparse solution is $\|x\|_0 = 3$ with $x \in \mathbb{R}^{33}$.

Sparse Optimization

Sparse Optimization Problem

Find sparsest solution that minimizes

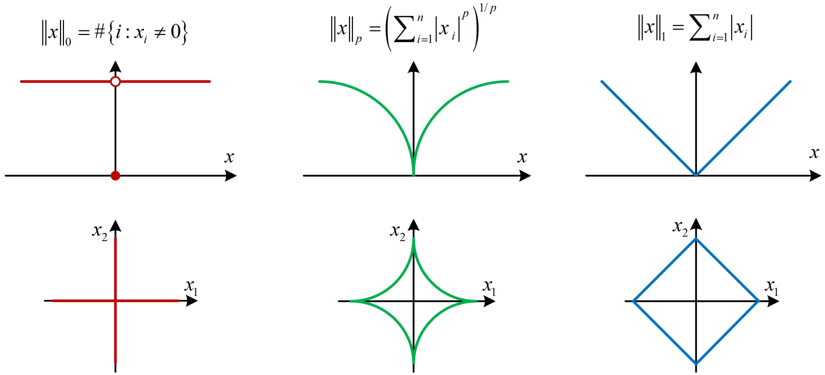
$$\begin{aligned} \min_{x} \quad & \|x\|_0 \\ \text{s.t.} \quad & x \in \mathcal{X} \subseteq \mathbb{R}^n \end{aligned}$$

- $\|x\|_0 \triangleq \#\{i : x_i \neq 0\}$ counts the number of non-zero elements

$$\|x\|_0 \leq s, \quad \Rightarrow \quad \binom{n}{s} \text{ possible combinations,} \quad \Leftarrow \quad n \text{ is large } (\times)$$

NP hard: best algorithm takes at least *exponential time* in problem size !

Sparsity: Exactness vs. Relaxation



Exact ℓ_0 “quasi-norm” [1], approximated ℓ_p “norm”, relaxed ℓ_1 norm

- **Convex Relaxation**: non-convex ℓ_0 “norm” $\Longrightarrow$ convex ℓ_1 norm

[1] Does not satisfy absolute homogeneity

"Shift the principle from compressed sensing to sparse optimal control using sparse optimization"

A Discrete Linear Time Invariant (LTI) Dynamics

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\
y(t) &= Cx(t) + Du(t), \quad t = 0, 1, \cdots, T-1.
\end{aligned}
$$

- State $x(t) \in \mathbb{R}^n$, Control input $u(t) \in \mathbb{R}^m$, Output $y(t) \in \mathbb{R}^p$.

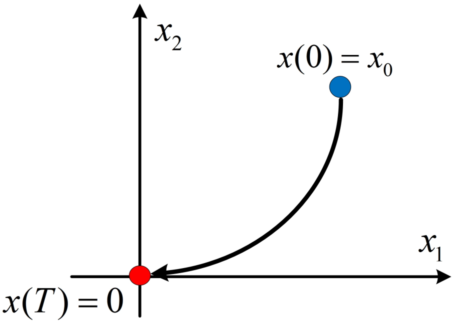
$$
\begin{aligned}
x(1) &= A{\color{blue}x_0} + Bu(0), \\
x(2) &= A(Ax_0 + Bu(0)) + Bu(1), \\
&\vdots
\end{aligned}
$$

$$
{\color{red}x(T)} = A^T x_0 + \underbrace{\begin{bmatrix} A^{T-1}B & \cdots & AB & B \end{bmatrix}}_{\Phi_T \in \mathbb{R}^{n \times mT}} \underbrace{\begin{bmatrix} u(0) \\ \vdots \\ u(T-1) \end{bmatrix}}_{u \in \mathbb{R}^{mT}}
$$

Sparse Control Problem

Control Objective

Finding a control sequence $\{u(t)\}_{t=0}^{T-1}$ with input sparsity that drives a discrete reachable LTI system state *from an initial state $x(0)$ to the origin $x(T) = 0$ within a finite time horizon $T > 0$.*



Open-loop Sparse Optimal Control

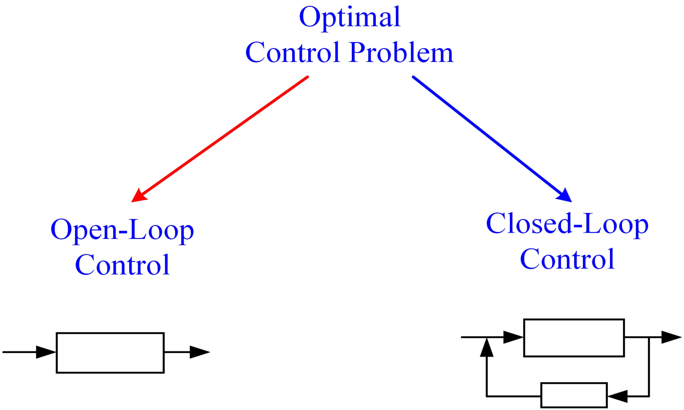
Discrete-Time Open-loop Sparse Optimal Control Problem

$$
\begin{aligned}
\min_{x,u} \quad & \|u\|_1 \triangleq \sum_{i=1}^{m} \sum_{t=0}^{T-1} |u_i(t)| \\
\text{s.t.} \quad & x(t+1) = Ax(t) + Bu(t), \\
& x(0) = x_0, \quad x(T) = 0, \\
& -\mu \leq Cx(t) + Du(t) \leq \mu, \quad \forall t = 0, 1, \cdots, T-1
\end{aligned}
$$

★ Open-loop optimal solution (x^*, u^*) is available (`CVX`, `YALMIP`).

■ Lack stability for system performance (without feedback response) !

What is Open-loop vs. Closed-loop Control ?



- Open-loop control: leveraging input-state/output, initial state x_0
- Closed-loop control: finding feedback gain $u = Kx$ (or $u = Ky$)

Why Studies (Sparse) Feedback Control ?

> " $\cdots$, *open-loop control is something like riding a bicycle with your eyes closed, which is very fragile against disturbance,* $\cdots$ "

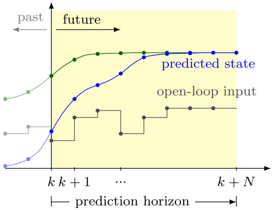
- Open-loop Control : Poor reliability & flexibility & accuracy !

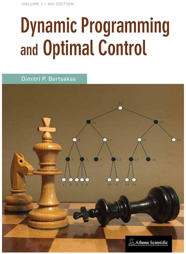
How to Realize Feedback Response

Question:

"How to obtain closed-loop solution for (sparse) optimal control ? "

Answer: Real-Time Algorithms/Iterations (Implicit Feedback) [pictures from Google]





Model Predictive Control, Dynamic Programming, Self-triggered Control, ...

- "*Online Optimization*" $\Longrightarrow$ heavy computation !

Seeking Sparse Feedback Solution

So far, a rich stories for sparse feedback control design, however, ...

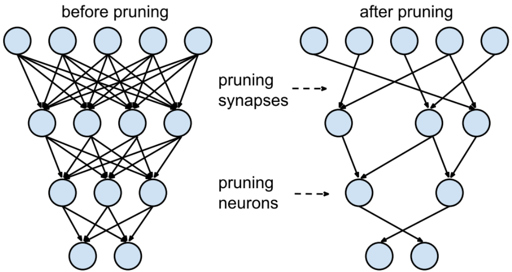
- Sparse feedback control majors in *structured sparsity*
  - e.g., static state feedback gain: $u(t) = Kx(t)$
- Determining a feedback gain matrix K is not a simple task ...
  - e.g., taking $\mathcal{J}(u) = \|u\|_1, \quad \mathcal{J}(u) = \lambda_1\|u\|_1 + \lambda_2\|u\|_2^2$, ... $\Rightarrow \quad K$?

Fall into Dilemma:

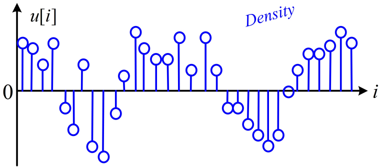
Goal

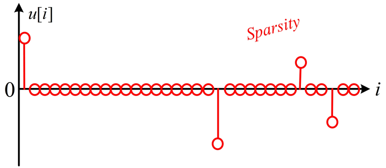
Look for sparse feedback solution with an *explicit feedback* gain as well as enjoy *input sparsity*, instead of structured sparsity

Sparsity: Spatial vs. Temporal

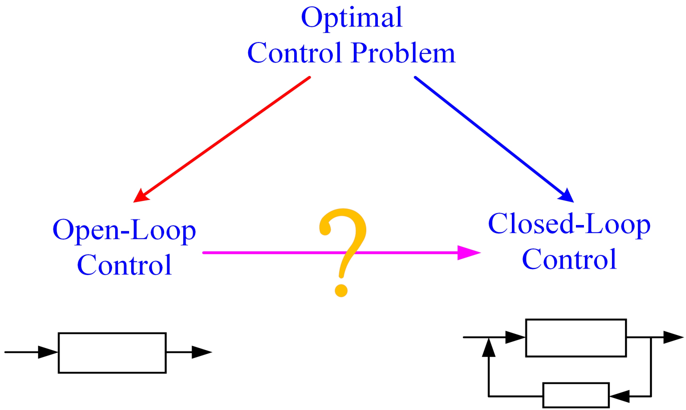


Spatial (Structured): Dense/Sparse (Neural) Networks



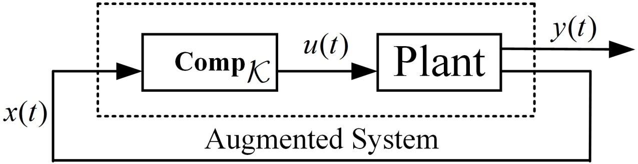


Temporal (Signal): Dense/Sparse Signals



Q: Can we infer sparse feedback controller from its open-loop solutions ?

ORACLE: Relatively Optimal Control Technique



Dynamic Linear Compensator [Blanchini & Pellegrino, IEEE TAC, 48(12), 2003]

The compensator $\mathcal{K}$ which we want to design is a dynamic state feedback

$$
\begin{aligned}
z(t+1) &= Fz(t) + Gx(t), \quad z(0) = 0, \\
u(t) &= Hz(t) + Kx(t)
\end{aligned}
$$

where F, G, H and Z are real matrices with appropriate sizes.

■ Determine $\mathcal{K} = \begin{bmatrix} K & H \\ G & F \end{bmatrix}$ $\Leftrightarrow$ Ensure internally stability

First Step: Sparse Optimization

Problem 1 (Constrained Sparse Matrix Optimization) [Chap. 2]

Find the matrices $X \in \mathbb{R}^{n \times nT}$ and $U \in \mathbb{R}^{m \times nT}$ such that U is sparse, i.e.,

$$\begin{aligned}
\min_{X,U} \quad & \|U\|_1 = \textstyle\sum_{i=1}^{m} \sum_{j=1}^{nT} |u_{ij}| \,, \\
\text{s.t.} \quad & AX + BU = X(P \otimes I_n) \,, \quad P = \begin{bmatrix} 0 & 0 \\ I_{T-1} & 0 \end{bmatrix} \\
& I_n = X(e_1 \otimes I_n) \,, \\
& \mathrm{abs}(CX + DU) \leq \mu(\mathbf{1}_n \otimes \mathbf{1}_T)^\top ,
\end{aligned}$$

where $e_i \in \mathbb{R}^N$ is the vector with a 1 in the ith element and 0's elsewhere.

- A general initial scenario: $x_0 \in \{e_1, e_2, \cdots, e_n\}$
- Generate n possible trajectories: vectors (x,u) $\Longrightarrow$ matrices (X,U)
- Convex program: optimal solution (X, U) is available.

Second Step: Feedback Realization

Problem 2 (Sparse Feedback Realization)

Based on the solution (X, U), solve a linear equation

$$\begin{bmatrix} K & H \\ G & F \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix}$$

w.r.t. (F, G, H, K) and determine the compensator $\mathcal{K}$, where

$$Z = \begin{bmatrix} 0_{n(T-1)\times n} & I_{n(T-1)} \end{bmatrix}, \quad V = Z(P \otimes I_n).$$

$$\mathcal{K} = \begin{bmatrix} K & H \\ G & F \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix}^{-1} \Leftarrow \begin{bmatrix} X \\ Z \end{bmatrix} \left[\begin{array}{c:ccc} I_n & X_1 & \cdots & X_{T-1} \\ \hdashline 0_{n(T-1)\times n} & & I_{n(T-1)} & \end{array}\right]$$

Realization of Sparse Feedback Control

Theorem (Realization of Sparse Feedback Control)

Suppose that Problem 1 has the minimizer (X, U).
Problem 2 gives the unique solution (F, G, H, K), resulting in the control $\mathcal{K}$ generates the inputs $u(t) = U(e_{t+1} \otimes x_0)$, $t = 0, 1, \ldots, T-1$, which drives the system state $x(t)$ from $x(0) = x_0$ to $x(T) = 0$ under output constraint $y \in \mathcal{Y}$. Then, closed-loop augmented system is internally stable.

Corollary (Equivalence)

For $x_0 \in \{e_1, \cdots, e_n\}$, we have $u^* = u^*_{\mathcal{K}} = Hz + Kx^*$.

Proposition (Deadbeat Control)

The realized sparse feedback controller $u^*_{\mathcal{K}}$ is a T-step deadbeat control.

Numerical Example

Let us consider a continuous second-order system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad x(0) = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}.$$

and discretize it with ZoH sampling 0.1 s to be the discrete plant.

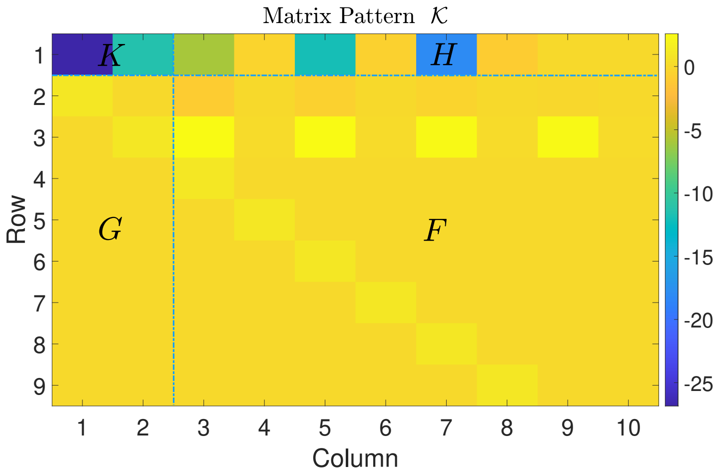
- Take time steps $T = 5$;
- Solve Problems 1 (Optimization) and 2 (Realization)

```
X =

    1.0000         0    0.8707    0.0435    0.6158    0.0308    0.3670    0.0183    0.1219    0.0061
         0    1.0000   -2.5884   -0.1293   -2.5142   -0.1256   -2.4651   -0.1232   -2.4406   -0.1219
```

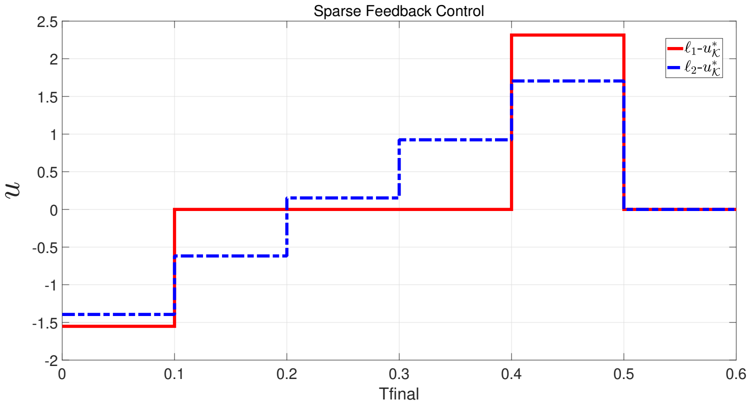
```
U =

  -26.8413  -11.3243   -0.0000   -0.0000    0.0000    0.0000    0.0000    0.0000   24.3659    1.2173
```

- Sparse feedback control realization

By computing, we obtain the pattern of compensator $\mathcal{K}$



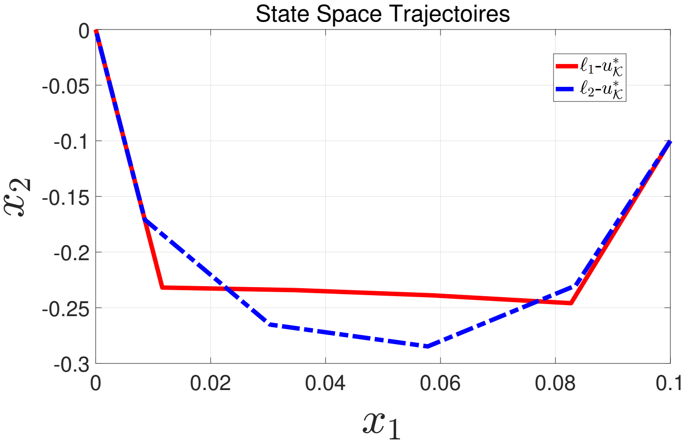
Feedback Control Inputs



- Sparse feedback controller

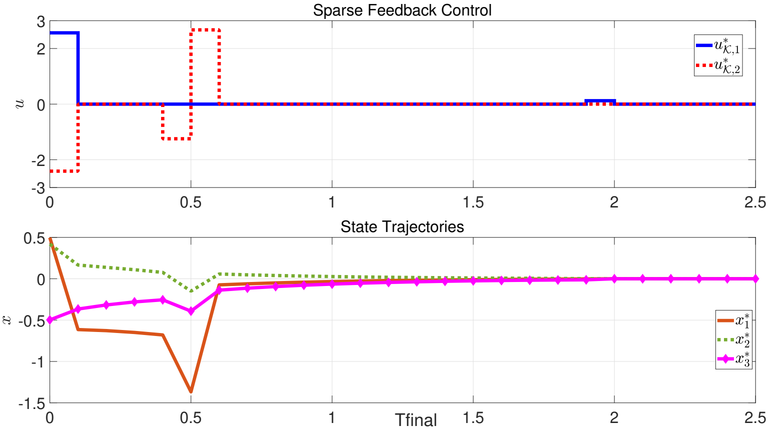
$$u_{\mathcal{K}}^* = Hz + Kx^* = \begin{bmatrix} -1.5517 & -0.0000 & -0.0000 & -0.0000 & 2.3149 & 0.0000 & -0.0000 \end{bmatrix}$$

State-Space Trajectories



- (Sparse) feedback controller (ℓ_1-$u^*_{\mathcal{K}}$ or ℓ_2-$u^*_{\mathcal{K}}$) drives the system state $x(t)$ from initial state $x_0^\top = [0.1 \;\; -0.1]$ to the origin.

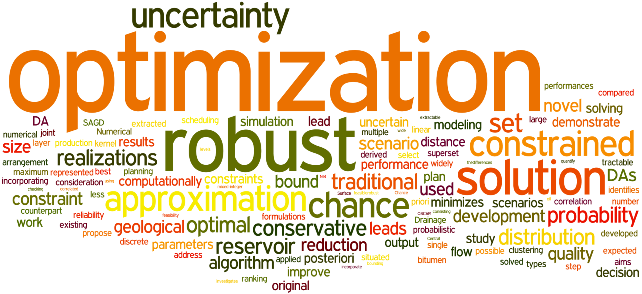
Multiple Inputs Case



- Proposed sparse feedback control is successful for multi-inputs
- Simple, offline, low-cost, economic, and practical tool !

Optimization Model in Uncertainty

"There is nothing certain, but the uncertain..."



Sparse Optimization Meets Uncertainty

Worst-Case Sparse Optimization Problem

$$\begin{aligned} \min_{x \in \mathcal{X}} \quad & \|x\|_0 \\ \text{s.t.} \quad & h(x, \delta) \leq 0, \quad \forall \delta \in \Delta \end{aligned}$$

Chance Constrained Sparse Optimization Problem

$$\begin{aligned} \min_{x \in \mathcal{X}} \quad & \|x\|_0 \\ \text{s.t.} \quad & \mathbb{P}\left\{\delta \in \Delta : h(x, \delta) \leq 0\right\} \geq 1 - \epsilon, \end{aligned}$$

- $h(x, q) : \mathbb{R}^n \times \mathbb{R}^{n_\delta} \to \mathbb{R}$ is *convex* in x, $\forall \delta \in \Delta$, and *bounded* in q, $\forall x$

Chance-Constrained Sparse Convex Program

Chance Constrained Sparse Convex Program

$$\begin{aligned} \min_{x \in \mathcal{X}} \quad & \|x\|_1 \\ \text{s.t.} \quad & \mathbb{P}\left\{\delta \in \Delta : h(x,\delta) \le 0\right\} \ge 1 - \epsilon \end{aligned}$$

here $h(x, q) : \mathbb{R}^n \times \mathbb{R}^{n_\delta} \to \mathbb{R}$ is *convex* in x, $\forall \delta \in \Delta$, and *bounded* in q, $\forall x$

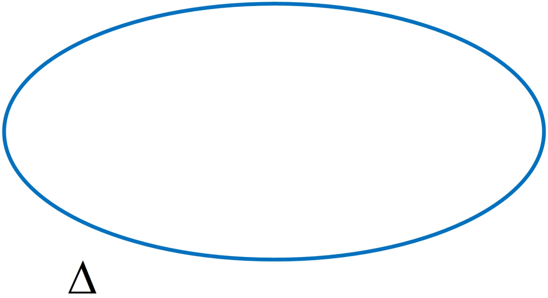
Risk (Violation of Probability)

$$V(x) \triangleq \mathbb{P}\Big\{\delta \in \Delta : h(x,\delta) > 0\Big\} \quad \Rightarrow \quad V(x) \le \epsilon$$

- $\epsilon \in (0,1)$ is a prescribed risk level, e.g., $\epsilon = 5\%$.

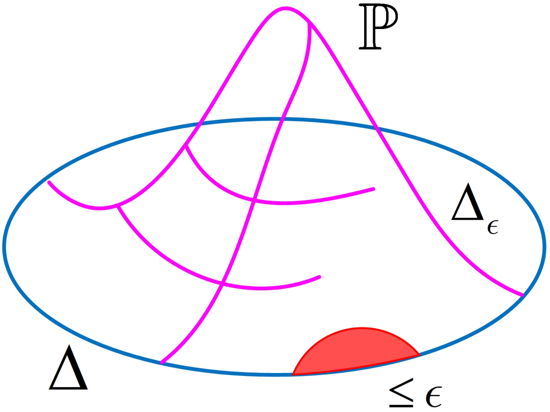
- Well-defined, while it is intractable

Data-Driven Sampling: Lifting Chance Constraints



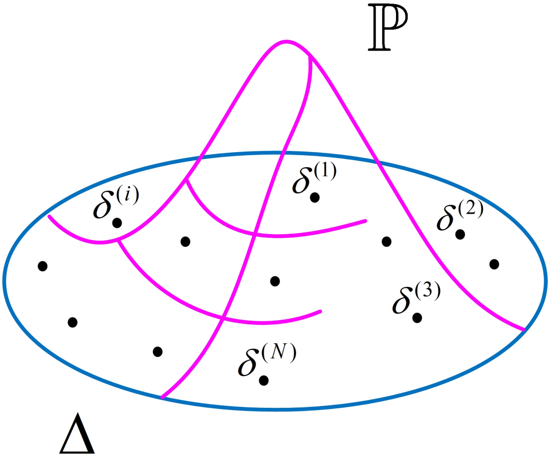
Worst-case uncertainty: $h(x, \delta) \leq 0$, for all $\delta \in \Delta$

Data-Driven Sampling: Lifting Chance Constraints



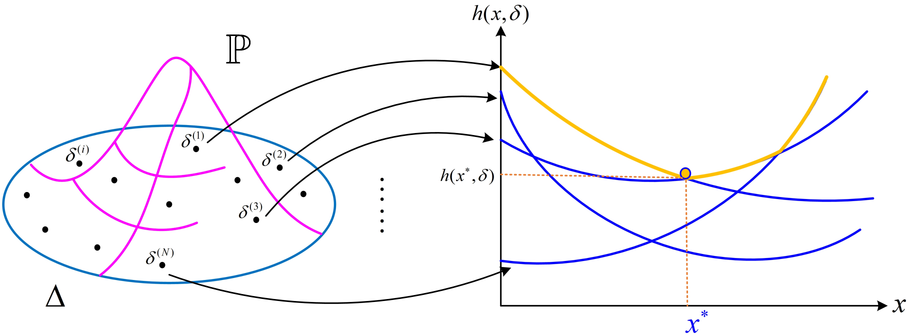
Chance-constrained uncertainty: $\mathbb{P}\{\delta \in \Delta : h(x, \delta) > 0\} \leq \epsilon$

Data-Driven Sampling: Lifting Chance Constraints



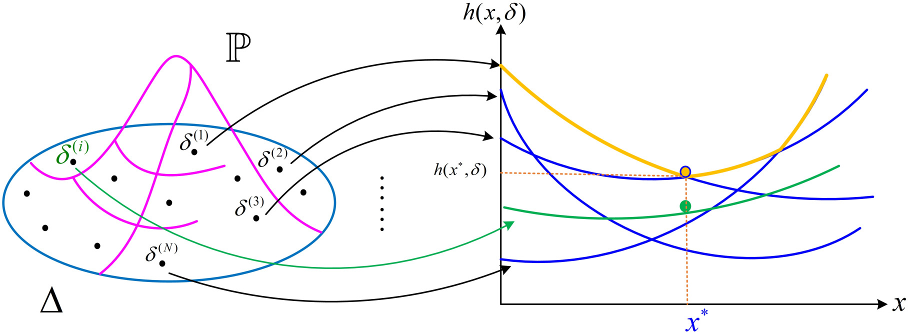
Randomly i.i.d. generate N scenarios from the probability $\mathbb{P}$

Data-Driven Sampling: Lifting Chance Constraints



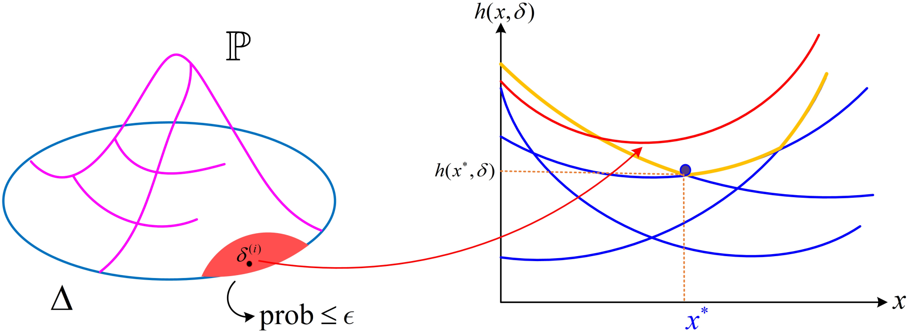
$\{\delta^{(1)}, \cdots, \delta^{(N)}\} \quad \rightsquigarrow \quad \{h(x, \delta^{(1)}), \ldots, h(x, \delta^{(N)})\} \quad \Leftrightarrow \quad \bigcap_{i=1}^{N} h(x, \delta^{(i)})$ i.e., Feasibility

Data-Driven Sampling: Lifting Chance Constraints



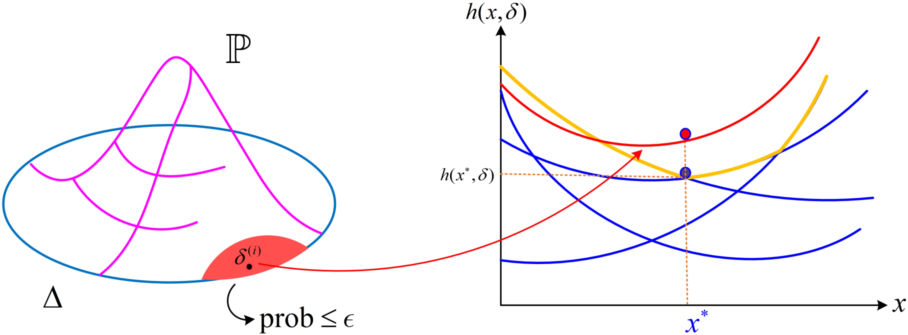
In general, for most of scenarios $\delta^{(i)}$'s $\rightsquigarrow$ $h(x,\delta^{(i)}) \leq h(x^*,\delta)$

Data-Driven Sampling: Lifting Chance Constraints



While, for some of scenarios $\delta^{(i)}$'s $\rightsquigarrow$ $h(x, \delta^{(1)}) > h(x^*, \delta)$, i.e., violation

Data-Driven Sampling: Lifting Chance Constraints



A small violation is allowed, e.g., $\epsilon = 1\%$, $V(x) \leq \epsilon$

Sparse Random Convex Program

Sparse Random Convex Program [Chap. 3]

$$\begin{aligned} \min_{x \in \mathcal{X}} \quad & \|x\|_1 \\ \text{s.t.} \quad & h(x, \delta^{(i)}) \leq 0, \quad i = 1, \dots, N. \end{aligned}$$

- Assumption 1: Optimal solution x_N^* exists and is unique.
- Assumption 2: For every $x \in \mathcal{X}$, $\mathbb{P}\{\delta : h(x, \delta) = 0\} = 0$.

- Probabilistic robustness guarantee [Calafiore & Campi, IEEE TAC'08]

$$\mathbb{P}^N\{V(x_N^*) \leq \epsilon\} \geq 1 - \beta, \quad \beta = \sum_{i=0}^{n-1} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i}$$

- A priori and explicit sample complexity: $N \geq \frac{2}{\epsilon}\left(\ln \frac{1}{\beta} + n\right)$

We further concern on:

- Balance the "*Sparse Performance*" and the associated "*Risk*".
- A *posterioi* probabilistic robustness guarantees

Relaxed Scenario-based Sparse Convex Optimization Problem

$$
\begin{aligned}
\min_{x \in \mathcal{X}, \xi^i \geq 0} \quad & \|x\|_1 + \rho \sum_{i=1}^{N} \xi^i \\
\text{s.t.} \quad & h(x, \delta^{(i)}) \leq \xi^i, \quad i = 1, \ldots, N.
\end{aligned}
$$

Sparse Optimization with Relaxation

Relaxed Scenario-based Sparse Convex Optimization Problem

$$\min_{x\in\mathcal{X},\xi^i\geq 0} \quad \overbrace{\|x\|_1}^{\text{sparse cost}} + \rho \underbrace{\sum_{i=1}^{N} \xi^i}_{\text{regrets}}$$

$$\text{s.t.} \quad h(x,\delta^{(i)}) \leq \xi^i, \quad i = 1, \ldots, N.$$

- $n + N$ decision variables.
- Slack variables $\xi^i \geq 0$, $i = 1, \cdots, N$ are "regrets".
- Weight ρ: balance the sparse cost and the violated constraints
  - $\rho \to 0$ non-regret , $\rho = 1/N$ empirical regret, $\rho \to \infty$ infinite regret

Application: Robust Control

Consider an uncertain discrete LTI system

$$x(t+1) = A(\delta)x(t) + B(\delta)u(t), \quad t = 0, 1, \cdots, T-1,$$

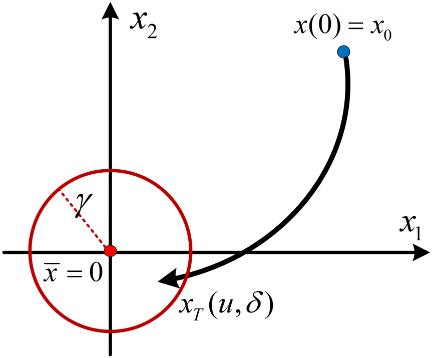
- $A(\delta) \in \mathbb{R}^{n\times n}$, $B(\delta) \in \mathbb{R}^{n\times m}$ depend on the uncertainty $\delta \in \Delta \subseteq \mathbb{R}^{n_\delta}$
- Uncertainties δ are random variables endowed with probability $\mathbb{P}$.

$$x_T(u,\delta) = A(\delta)^T x_0 + \underbrace{\begin{bmatrix} A(\delta)^{T-1}B(\delta) & \dots & A(\delta)B(\delta) & B(\delta) \end{bmatrix}}_{\Phi_T(\delta)\in\mathbb{R}^{n\times mT}} \underbrace{\begin{bmatrix} u_0 \\ \vdots \\ u_{L-1} \end{bmatrix}}_{u\in\mathbb{R}^{mT}}$$

Uncertainty in Control Problem

Robustness in Control Objective

Seek a control sequence $\{u(t)\}_{t=0}^{T-1}$ with input sparsity that drives the system $x(t)$ from the initial state x_0 *near to* the terminal state $x_T(u,\delta)$.



Uncertainty in Control Problem

Robustness in Control Objective

Seek a control sequence $\{u(t)\}_{t=0}^{T-1}$ with input sparsity that drives the system $x(t)$ from the initial state x_0 *near to* the terminal state $x_T(u,\delta)$.

$h(u,\delta)$: measure the target $\bar{x}$ and final state $x_T(u,\delta)$ with radius $\gamma > 0$

$$\begin{aligned} h(u,\delta) &\triangleq \|x_T(u,\delta) - \bar{x}\|_2 - \gamma \\ &= \|A(\delta)^T x_0 + \Phi_T(\delta)u - \bar{x}\|_2 - \gamma \leq 0, \quad \forall \delta \in \Delta \end{aligned}$$

Using probabilistic/chance-constrained framework

$$\mathbb{P}\{\delta \in \Delta : h(u,\delta) \leq 0\} \leq 1 - \epsilon$$

Risak-Aware Sparse Optimal Control

Problem 3 (Risk-aware Sparse Optimal Control)

For an uncertain LTI system, minimizes

$$\begin{aligned} \min_{u} \quad & \|u\|_1 \\ \text{s.t.} \quad & \mathbb{P}\{\delta \in \Delta : h(u,\delta) \leq 0\} \leq 1-\epsilon \end{aligned}$$

Problem 4 (Sparse Random Optimal Control)

$$\begin{aligned} \min_{u} \quad & \|u\|_1 \\ \text{s.t.} \quad & h(u,\delta^{(i)}) \leq 0, \quad \forall\, i = 1,\ldots,N \end{aligned}$$

Theorem (Risk-aware Sparse Optimal Control)

Assume that the pair $(A(\delta), B(\delta))$ is robustly reachable and $mT > N$. Given a robust level $\epsilon \in (0,1)$ and a confidence parameter $\beta \in (0,1)$, choose the scenarios $N > mT$ such that

$$\sum_{i=0}^{mT-1} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i} \leq \beta.$$

Let u_N^* be the feasible and unique solution to program. Then it holds that

$$\mathbb{P}^N\{V(u_N^*) > \epsilon\} \leq \beta.$$

In other words, with probability (N-fold product probability measure) at least $1-\beta$, the solution u_N^* is the ϵ-level probabilistic robust control.

- finite-sample guarantee: $N \geq \frac{2}{\epsilon}\left(\ln \frac{1}{\beta} + mT - 1\right)$
- sparse random solution u_N^* $\rightsquigarrow$ risk-aware sparse solution u_ϵ^*

Relaxed Risk-Aware Sparse Optimal Control Problem

Given a reachable LTI system, and the parameters $x_0, \bar{x}, T, \rho, \gamma$, achieving a trade-off between the sparse control and the risk amounts to minimizing

$$
(\mathrm{RaSOCP}_N^{\rho}): \quad \begin{aligned} \min_{u \in \mathbb{R}^{mL}, \xi^i \geq 0} \quad & \|u\|_1 + \rho \sum_{i=1}^{N} \xi^i \\ \text{s.t.} \quad & h(u, \delta^{(i)}) \leq \xi^i, \quad i = 1, \cdots, N. \end{aligned}
$$

Theorem (Violation of RaSOCP_N^{ρ}) cf. [Garatti & Campi, 2022, Math. Program]

With polynomial $\underline{\epsilon}(\cdot)$ and $\overline{\epsilon}(\cdot)$, given a confidence $\beta \in (0,1)$, the risk assessment for violation $V(u_N^*)$ in problem (RaSOCP_N^{ρ}) is as follows

$$
\mathbb{P}^N\{\underline{\epsilon}(s_N^*) \leq V(u_N^*) \leq \overline{\epsilon}(s_N^*)\} \geq 1 - \beta,
$$

where s_N^* counts the number of the violated constraints $h(u_N^*, \delta^{(i)}) \geq 0$.

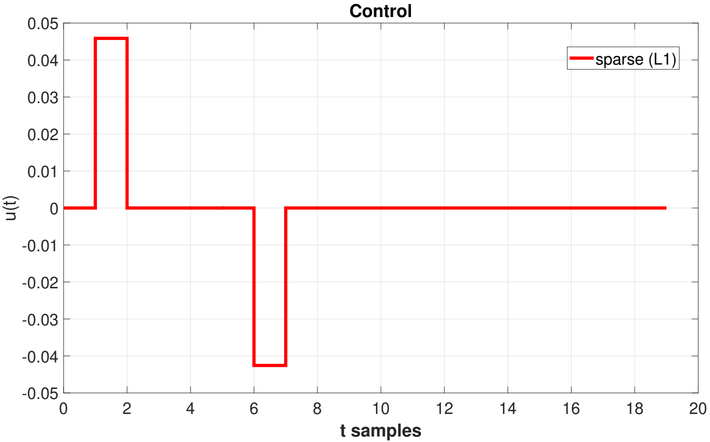
Numerical Example (Risk-Aware Sparse Optimal Control)

- Uncertain discrete-time plant

$$x(t+1) = \begin{bmatrix} a_{\delta_{11}} & -1+a_{\delta_{12}} & a_{\delta_{13}} & a_{\delta_{14}} \\ 1+a_{\delta_{21}} & a_{\delta_{22}} & a_{\delta_{23}} & a_{\delta_{24}} \\ a_{\delta_{31}} & 1+a_{\delta_{32}} & a_{\delta_{33}} & a_{\delta_{34}} \\ a_{\delta_{41}} & a_{\delta_{42}} & 1+a_{\delta_{43}} & a_{\delta_{44}} \end{bmatrix} + \begin{bmatrix} 2+b_{\delta_1} \\ b_{\delta_2} \\ b_{\delta_3} \\ b_{\delta_4} \end{bmatrix} u(t).$$

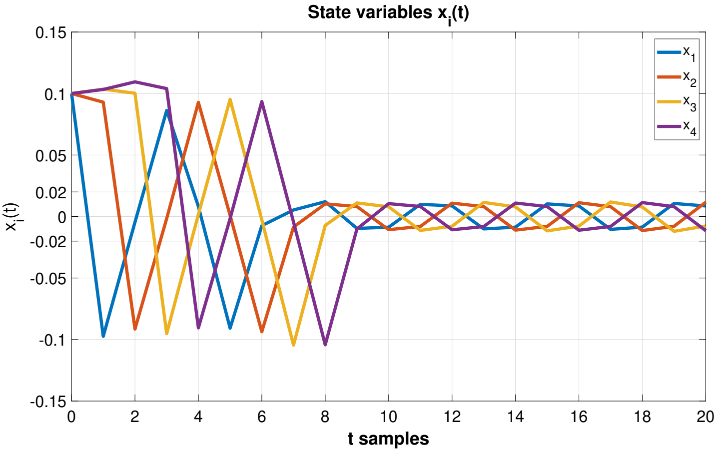
- $a_{\delta_{ij}} \sim U(-0.05, 0.05)$, $b_{\delta_i} \sim U(0, 0.2)$, $i, j = 1, 2, 3, 4$.
- Terminal time $T = 20$, risk $\epsilon = 0.05$, confidence $\beta = 10^{-5}$.
- State constraints: $x_0 = [0.1\ 0.1\ 0.1\ 0.1]^\top$, $\|x(20)\|_2 \leq \gamma \doteq 0.02$.
- Control constraint: $|u(t)| \leq 1$, $\forall t = 0, 1, \cdots, 19$.
- Sample complexity $N \geq 888$, taking $N = 900$.

Control Inputs



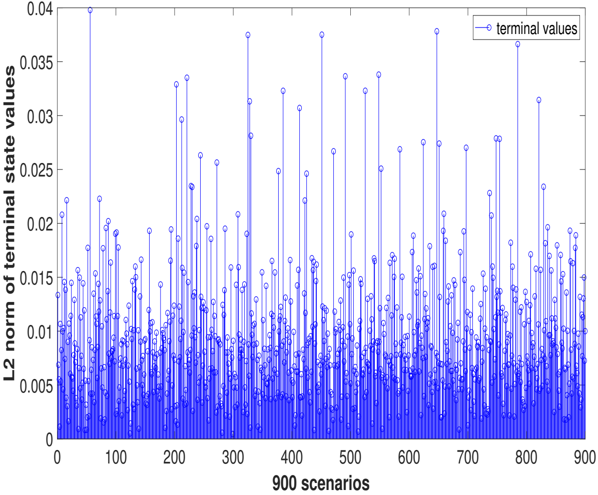
- Optimal value $\|u_{900}^{*}\|_1 = 0.0884$.

State Trajectories

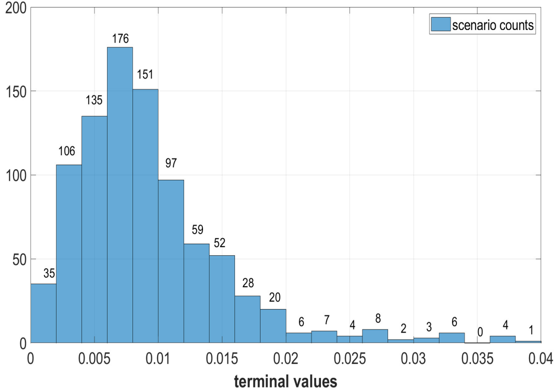


- The terminal state of four states are close to the origin, and they are fluctuated under a prescribed gap value $\gamma = 0.02$.

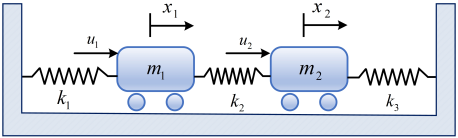
Terminal State



Validation Test



- Validation test: $\mathbb{P}^{900}\{\|x(20;\delta^{(i)})\|_2 > 0.02\} \leq 5\%$
- Violation counts are $41(< 45 = 900 \times 5\%)$.
- The designed sparse control for uncertain discrete time system is with a high probabilistic robustness (95%).



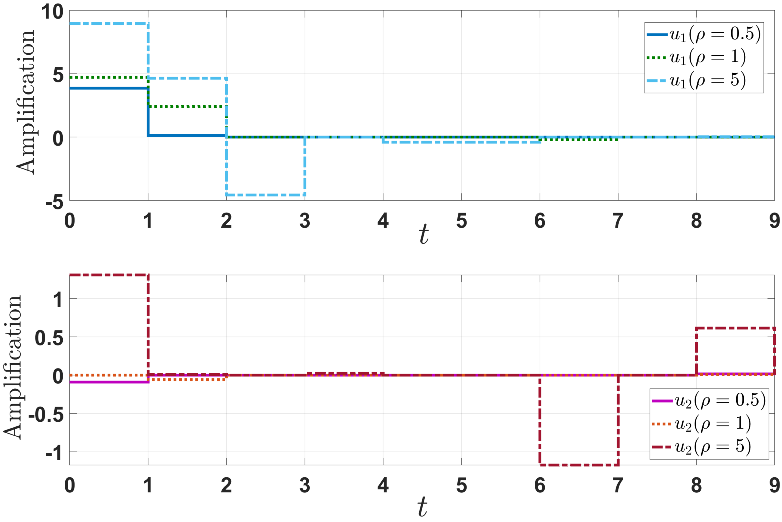
Consider a continuous fourth-order mass-spring system with two inputs

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-(k_1+k_2)}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_1} & 0 & \frac{-(k_2+k_3)}{m_2} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} u(t), \;\; x(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and discretize it with sampling time 0.05 s to be the discrete plant.

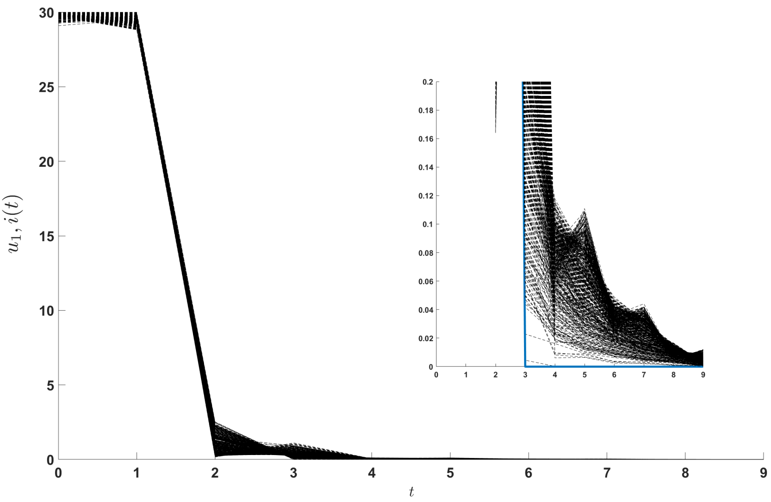
- $m_1 = 1,\ m_2 = 2,\ T = 10,\ \delta = [k_1\ \ k_2\ \ k_3] \overset{i.i.d.}{\sim} \mathcal{U}[0.1, 1]^3, \gamma = 0.5,\ \bar{x} = 0$
- Solve $(\mathrm{RaSOCP}_N^{\rho})$ by using `CVX` and $N = 1000$.

Numerical Benchmarks



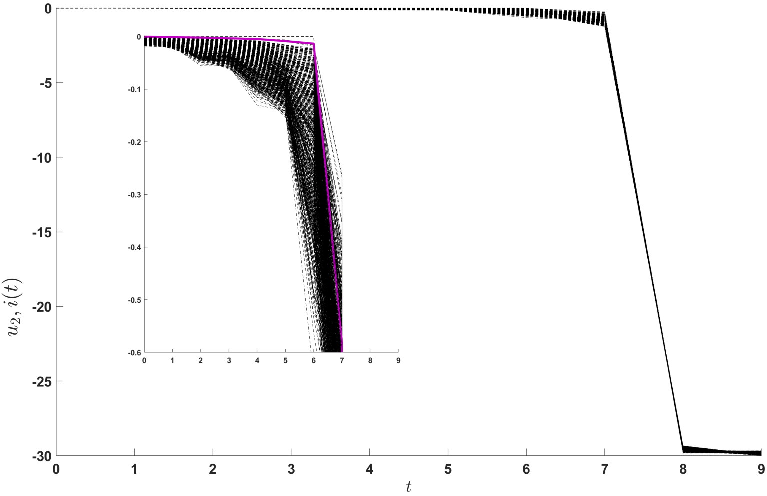
A lower value ρ improves the sparse cost (Non Free Lunch!)

Numerical Benchmarks



Additional samples testing $N_{\mathrm{V}} = 2000$ for $u_{1,i}(t)$, $i = 1, \ldots, 2000$

Numerical Benchmarks



Additional samples testing $N_{\mathrm{V}} = 2000$ for $u_{2,i}(t)$, $i = 1, \dots, 2000$

Conclusions

In this talk, the take-home messages are as

♠ Sparse feedback control synthesis (Chap. 2)

- Infer sparse feedback controller from its open-loop solutions.
- Provide initialization robustness, stability and sparsity

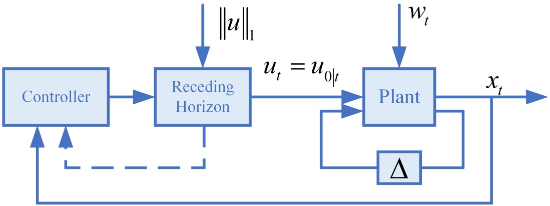
♣ Risk-aware sparse optimal control (Chap. 3)

- Provide probabilistic robustness
- Balance the sparse cost and risk
- Provide priori and posterioi probabilistic guarantees

Conclusions

The rest of this dissertation:

Chap.4

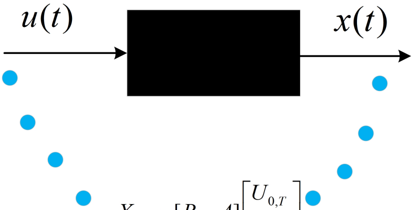


Chapter 4: Risk-aware sparse predictive control

Conclusions

The rest of this dissertation:

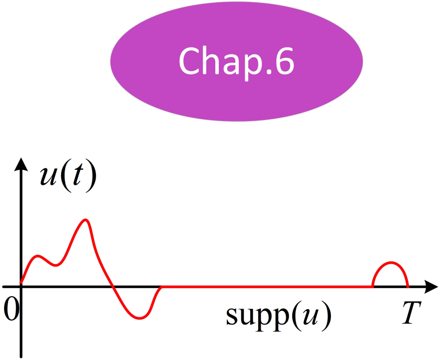
Chap.5



$$X_{1,T} = [B \quad A]\begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix}$$

Chapter 5: Data-driven sparse feedback control

The rest of this dissertation:



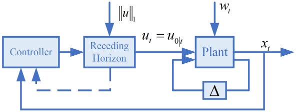
$$\min_{x,u} \int_0^T \left(x^\top(t) Q x(t) + R u^2(t) \right) dt + \lambda \int_0^T |u(t)|^0 \, dt$$

Chapter 6: Linear quadratic (LQ) sparse optimal control

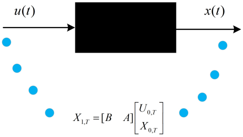
Conclusions

The rest of this dissertation:

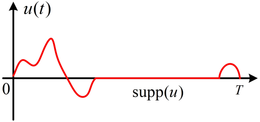
Chap.4



Chap.5



Chap.6



$$\min_{x,u} \int_0^T \left(x^\top(t) Q x(t) + R u^2(t) \right) dt + \lambda \int_0^T |u(t)|^0 \, dt$$

Ch.4:Sparse predictive control; Ch.5:Model-free sparse feedback; Ch.6:LQ sparse control

★ Studies on *sparse optimization with uncertainty* using "probabilistic approach" are increased. The research is still developing.


ご清聴
ありがとうございました


Q&A