Exploring SCALE Weather Data via Koopman Modes

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Abstract: In this note, we consider the Koopman mode decomposition (KMD) for SCALE simulation weather data through a data-driven approach, called sparsity-promoting dynamic mode decomposition (spDMD). A difference with other existed works such as sufficient long-term weather data or steady-steady SST data is that SCALE data is a sequence of short-term snapshots and possibly emerges the transient behavior. This note shows that using spDMD and KMD, we can capture the dominant amplitudes associated with the leading Koopman modes, and it effectively reduces the order of the system model.

Keywords: Koopman operator, sparsity-promoting, transient behavior, weather data application

1 Introduction

Modeling weather system is a non-trivial task due to the inherent complexity and potential unpredictability of real meteorological systems, which may go beyond the evolution of mathematical models, such as classical Lorenz and chaotic systems. However, it is possible to build a purely data-driven climate dynamical systems by measuring historical weather data [1].

Koopman operator [2] is the state-of-the-art technique that lifts the nonlinear systems from a finite-dimensional state space to an infinite-dimensional, but linear, dynamics acting on the space of observable functions. Through spectral analysis, Koopman modes can reveal dominant patterns in weather data, which are useful for predicting future behavior and explaining the variability of weather: see, e.g., [3].

Nonetheless, exploring non-steady state weather data remains a challenging and unresolved problem. This note attempts to exploit the KMD and spDMD to identify the transient behavior of a short-term SCALE data and perform the model-reduction.

2 Data-driven method

2.1 Dynamic Mode Decomposition

We harvest the following time series data matrices

$$\begin{aligned} \mathrm{Y} &:= \begin{bmatrix} y_0 & y_1 & \dots & y_{N-1} \end{bmatrix} \in \mathbb{R}^{m\times N}, \\ \mathrm{Y}' &:= \begin{bmatrix} y_1 & y_2 & \dots & y_N \end{bmatrix} \in \mathbb{R}^{m\times N}. \end{aligned} \tag{1}$$

where y_k represents the observed snapshot matrix which can be reshaped as vectors, and Y' is the shifted data matrix of Y. Therefore, we have the relationship

$$y_{k+1} = Ay_k, \quad \rightarrow \quad \mathrm{Y}' = A\mathrm{Y}, \tag{2}$$

where A is a linear matrix that approximates the Koopman operator, and can be estimated using singular value decomposition (SVD) and dynamic mode decomposition (DMD) algorithms [1, Chap. 1].

As we all known, DMD is a direct data-driven numerical method that minimizes the least-square errors in the linear model (2). It computes the the eigenvalues λ_j and the eigenvector v_j of A, which yields $Av_j = \lambda_j v_j$. The associated DMD modes Φ_j are given by $\Phi_j = \mathrm{Y}v_j$, and the system (2) is reconstructed as

$$y_k \approx \sum_{j=1}^{r} \Phi_j \lambda_j^k b_j, \quad y_0 = \sum_{j=1}^{r} \Phi_j b_j, \tag{3}$$

where r depends on the truncated SVD and b_j are the coefficients (a.k.a., amplitudes), determined by the initial condition. Let the observable be $y_k = f(x_k)$, which implies that DMD (3) makes the connection with a *finite-approximation* of KMD[1]. Besides, the compact form of (3) is characterized by $\mathrm{Y} \approx \Phi D_b V_{\text{and}}$, that is, Y is approximated as

$$\underbrace{\begin{bmatrix} | & \cdots & | \\ \Phi_1 & \cdots & \Phi_r \\ | & \cdots & | \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_r \end{bmatrix}}_{\mathrm{D}_b} \underbrace{\begin{bmatrix} 1 & \lambda_1 & \dots & \lambda_1^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_r & \dots & \lambda_r^{N-1} \end{bmatrix}}_{V_{\text{and}}}.$$

2.2 Sparsity-promoting DMD

In what follows, we introduce sparsity-promoting DMD (spDMD) [4] that minimizes a regularized least-squares deviation (between the data matrices (3) and a linear combination of DMD modes) with an added ℓ_1 penalty of the vector of DMD amplitudes, i.e.,

$$J_\gamma(b) = \|\mathrm{Y} - \Phi D_b V_{\text{and}}\| + \gamma\|b\|_1, \tag{4}$$

where γ is a weight to tune the sparsity level.

3 Application to SCALE weather data

To illustrate the effectiveness of the proposed methods, we consider the real-world weather application via KMD, where the weather data-embedding dynamics is generated from the SCALE simulation data [5].

The data consist of a short-term sequence of 42 snapshots of vortex magnitude $|\omega| = |\nabla \times \mathrm{v}|$ in scalar fields with 40×97 grids, calculated using the finite-difference method. Collecting the each snapshot matrix horizontally yields a tall and short matrix of dimension $\mathrm{Y} \in \mathbb{R}^{3880\times 42}$, that is, the columns are the stacked snapshot $y_k \in \mathbb{R}^{40\times 97}$.

[1]Recall that [2], an observable f can be written as a linear combination of Koopman eigenfunctions, expressed as $f(x_k) = \sum_{j=1}^{\infty} \lambda_j^k \varphi_j(x_0)\mathbf{v}_j$, where λ_j is the Koopman eigenvalue, φ_j is the Koopman eigenfunction, and $\mathbf{v}_j$ is the Koopman modes.

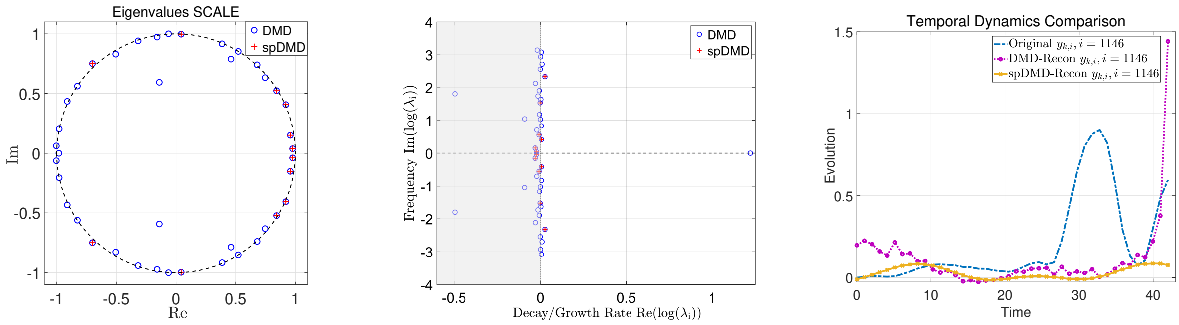

Fig. 1: Eigenvalues of SCALE (left), the decay/growth rate (middle), and the temporal dynamics (right).

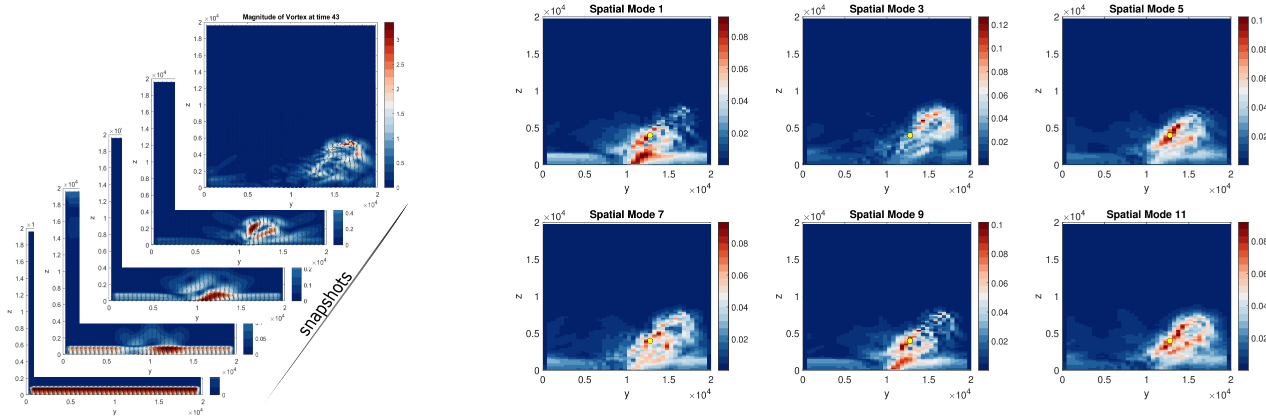

Fig. 2: The snapshots of original SCALE data with full 43 snapshots (left). The dominant Koopman modes (magnitude $|\Phi_j|$) extracted from the SCALE data via spDMD (right), where sparsity takes $\gamma = 18.1629$, and the number of non-zero amplitudes **card**(b)=12 (i.e., 6 pairs), ordered by the absolute value of amplitudes.

Fig. 1 shows the eigenvalues obtained from DMD and spDMD for the SCALE data (left). The dependence of the identified dominant modes along with frequencies versus the associated growth or decay rate is shown in Fig. 1 (middle). In this case, we specify one element $i = 1146$ with the location $[26, 29]$ in the 40×97 grid of spatial modes (yellow point), and the comparisons of temporal dynamics are shown in Fig. 1 (right). Fig. 2 shows the snapshots of the original SCALE full data with 43 snapshots (left), and it also shows the (reduced-ordered) dominant spatial modes with magnitude $|\Phi_j|$ (right), which reveals that spDMD can capture the spatial modes with transient growth pattern. Unlike the SST data [3], SCALE data does not enjoy the steady state. Thus, we focus on analyzing its transient behavior. In the right of Fig. 1, the growth of the vortex around 40 seconds is captured by DMD. However, during the mid-term period (from 25 sec to 40 sec), the transient growth data is not well extracted by both the DMD methods. Since DMDs are based on the point spectrum of the underlying Koopman operator, this might suggests that we require to consider its continuous spectrum.

4 Conclusion

This note explored the KMD for a short-term SCALE weather data and captured those transient modes with large amplitudes by spDMD. Future work will focus on ensemble data and the continuous spectrum using DMD to analyze transient behavior, along with related temperature and pressure data.

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