Formulas for Data-Driven Control:

Stabilization, Optimality and Robustness

D2 Zhicheng Zhang

IPS, Fujisaki Lab
@Paper Introduction
Claudio De Persis and Pietro Tesi, IEEE TAC, Vol.65, No. 3, 2020.

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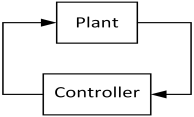
Control System



Mathematical Models

Consider a discrete-time linear (input-output) system

$$x(k+1) = f(x(k), u(k)), \quad y(k) = h(x(k), u(k)),$$
$$\text{e.g.} \quad x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) + Du(k),$$

- The system dynamic are often known (model-based)
- Here $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$
- Time horizon $[0, t-1]$
- Q: How about model-free system ?

Learning From Data: Indirect VS. Direct Approach

When system matrices (A, B) are *unknown*, one can follow 2 approaches:

1. **Indirect**: **System Identification** + **Control** of the identified system
   - B. Recht. A tour of reinforcement learning: The view from continuous control. Ann. Rev. Contr, Robot, and Auto. Sys., 2019.
   - M. Verhaegen and V. Verdult, Filtering and system identification: a least squares approach. Cambridge, U.K.: Cambridge Univ. Press, 2007.
2. **Direct**: **Data-Driven** control design (Skip identification)
   - M.C. Campi, A. Lecchini, and S.M. Savaresi, Virtual reference feedback tuning: a direct method for the design of feedback controllers. Automatica, 2002.
   - C. Novara, L. Fagiano, and M. Milanese, Direct feedback control design for nonlinear systems. Automatica, 2013.
   - Y.S. Wang, N. Matni, and J.C. Doyle, A system level approach to controller synthesis, IEEE TAC, 2019.

Persistence of excitation

- We seek a data-driven representation of the unknown closed-loop dynamics enabled by persistently exciting input.

Persistently Exciting (PE) Data

The signal $u : [0, T-1] \to \mathbb{R}^m$, that is,

$$u(0), u(1), \cdots, u(T-1)$$

is *persistently exciting* of order L if the Hankel matrix associated with

$$U_{0,L,T-L+1} = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-L) \\ u(1) & u(2) & \cdots & u(T-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ u(L-1) & u(L) & \cdots & u(T-1) \end{bmatrix}$$

has full (row) rank mL.

- PE requires sufficiently long input sequences: $T \geq (m+1)L - 1$

PE generation

PE Signal Generation

```
global L m T ud
T = L*(m+1)-1;
aux = zeros (m,T);
aux(:)=0.5;
ud(1:m, 1:T) = rand(m,T)-aux;
% computing Hankel matrix Ud over [0,T-1]
for j=1:T-L+1
    for i=1:L
      Ud((i-1)*m+1:(i-1)*m+m,j)=ud(1:m, j+i-1);
    end
end
```

```
% if rank(Ud)-M*L then the sequence ud(0),...,ud(T-1) is PE of
order L
```

```
if rank(Ud)==m*L
disp('input sequence is PE');
end
```

Willems' et al Fundamental Lemma

- A PE condition for a controllable system generating data that are sufficiently independent over time

Lemma

Let the system

$$x(k+1) = Ax(k) + Bu(k)$$

be *controllable*. Then for any $t \geq 1$,

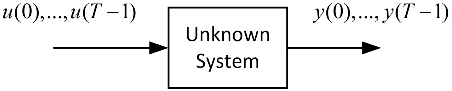
$$u_{[0,T-1]} \text{ PE of order } n+t, \quad \Rightarrow \text{rank} \begin{bmatrix} U_{0,t,T-t+1} \\ X_{0,T-t+1} \end{bmatrix} = n + tm$$

$$U_{0,t,T-t+1} = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-t) \\ u(1) & u(2) & \cdots & u(T-t+1) \\ \vdots & \vdots & \ddots & \vdots \\ u(t-1) & u(t) & \cdots & u(T-1) \end{bmatrix}$$

$$X_{0,T-t+1} = [x(0)\ x(1)\ \cdots\ x(T-t)]$$

J.C. Willems, P. Rapisarda, I. Markovsky, B.L. De Moor. A note on persistency of excitation. Syst & Cont. Lett, 2005

Deep implication for Control



Lemma

(i) if $u_{[0,T-1]}$ is PE of order $n+t$, then any t-long input/output trajectory of system can be expressed as

$$\begin{bmatrix} u_{[0,T-1]} \\ y_{[0,t-1]} \end{bmatrix} = \begin{bmatrix} U_{0,t,T-t+1} \\ Y_{0,t,T-t+1} \end{bmatrix} g, \quad g \in \mathbb{R}^{T-t+1}$$

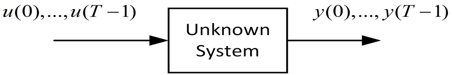
(ii) Any linear combination of the columns of the matrix of data, i.e.,

$$= \begin{bmatrix} U_{0,t,T-t+1} \\ Y_{0,t,T-t+1} \end{bmatrix} g,$$

is a t-long input-output trajectory of the system.

I. Markovsky and P. Rapisarda. Data-driven simulation and control, Int. J. Contr., 2008.

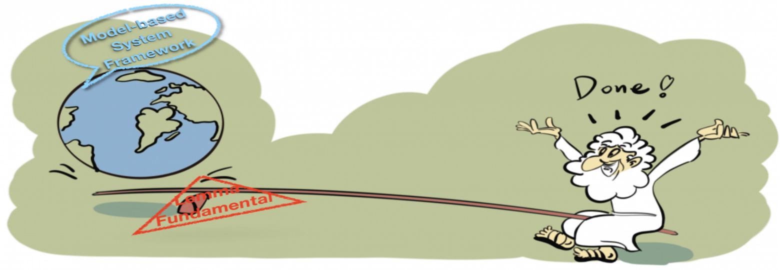
Data-Driven Process

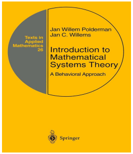


$$\begin{bmatrix} U_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} u(0) & \cdots & u(T-t) \\ \vdots & \ddots & \vdots \\ u(t-1) & \cdots & u(T-1) \\ y(0) & \cdots & y(T-t) \\ \vdots & \ddots & \vdots \\ y(t-1) & \cdots & y(T-1) \end{bmatrix}$$

$$\begin{bmatrix} u(k) \\ \vdots \\ u(k+t-1) \\ y(k) \\ \vdots \\ y(k+t-1) \end{bmatrix} = \begin{bmatrix} U_0 \\ Y_0 \end{bmatrix} g(k)$$

Lift Model-based Framework





Data-Driven System Representation

Theorem 2 (Data-Based Closed-Loop Representation)

Let PE condition holds, system $x(k+1) = Ax(k) + Bu(k)$ in closed-loop with a state-feedback $u = Kx$ has the following equivalent representation

$$x(k+1) = X_{1,T} G_K x(k)$$

where G_K is a $T \times n$ matrix satisfying

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_{0,1,T} \\ X_{0,T} \end{bmatrix} G_K = \begin{bmatrix} U_{0,1,T} G_K \\ X_{0,T} G_K \end{bmatrix}.$$

Then, shift design from K to G_k, namely, $u(k) = U_{0,1,T} G_K x(k)$.

$$A + BK = [B \;\; A] \begin{bmatrix} K \\ I_n \end{bmatrix} \Longleftrightarrow [B \;\; A] \begin{bmatrix} U_{0,1,T} \\ X_{0,T} \end{bmatrix} G_K = X_{1,T} G_K$$

$$\begin{aligned} AX_{0,T} + BU_{0,1,T} &= A[x(0) \;\; x(1) \cdots x(T)] + B[u(0) \;\; u(1) \cdots u(T-1)] \\ &= [x(1) \;\; x(2) \cdots x(T)] := X_{1,T} \end{aligned}$$

Direct Data-Driven Approach: Stability

Data-driven State-feedback Stabilization

Find G_K such that the closed-loop system

$$x(k+1) = X_{1,T} G_K \quad \text{is } \textit{asymptotically stable}.$$

A necessary and sufficient condition is given by the *Lyapunov inequality*

$$P \succ 0, \quad X_{1,T} G_K \cdot P \cdot G_K^\top X_{1,T}^\top - P \prec 0$$

Let $Q := G_K P$. Stability equals to (a LMI with Schur's complement)

$$\begin{bmatrix} X_{0,T} Q & Q^\top X_{1,T}^\top \\ X_{1,T} Q & X_{0,T} Q \end{bmatrix} \succ 0 \quad \textit{with} \quad \begin{bmatrix} K \\ P \end{bmatrix} = \begin{bmatrix} U_{0,1,T} G_K \\ X_{0,T} Q \end{bmatrix}$$

The solution to the LMI returns Q. The state-feedback control gain is

$$K = U_{0,1,T} Q (X_{0,T} Q)^{-1}$$

Direct Data-Driven Stabilization

Theorem 3

Let PE condition holds. Then any matrix Q satisfying

$$\begin{bmatrix} X_{0,T}Q & X_{1,T}Q \\ Q^\top X_{1,T}^\top & X_{0,T}Q \end{bmatrix} \succ 0$$

is such that

$$K = U_{0,1,T}Q(X_{0,T}Q)^{-1}$$

is a stabilizing state-feedback gain for system

$$x(k+1) = Ax(k) + Bu(k)$$

- Converse result: if K is a stabilizing sate-feedback gain for the system, then it can be written as $K = U_{0,1,T}Q(X_{0,T}Q)^{-1}$.
- The result still holds even using data *not obtain from PE data*.

H. van Waarde et al, Data Informativity: A New Perspective on Data-Driven Analysis and Control, IEEE TAC, 2020.

Optimality: Linear Quadratic Regulator (LQR)

Problem (LQR Control Design)

Design control sequence $u(0), u(1), \cdots$ that minimizes

$$J_\infty(x_0, u) = \sum_{k=0}^{\infty} (x^\top(k) Q_x x(k) + u^\top(k) R u(k))$$

for the system $x(k+1) = Ax(k) + Bu(k),\ x(0) = x_0$.
There exists a unique solution given by the controller

$$u = Kx, \quad K := -(R + B^\top PB)^{-1} B^\top PA$$

with P is the unique positive define solution to the DARE

$$A^\top PA - P - A^\top PB(R + B^\top PB)^{-1} B^\top PA + Q = 0.$$

```
% returns the solution K, P to LQR problem:
    dlqr(A, B, Q_x, R)
```

Optimality-LQR reformulation

LQR Reformulation

A reformulation of LQR as an optimization problem

$$
\begin{aligned}
\min_{K,P,X} \quad & \mathrm{trace}(Q_x P) + \mathrm{trace}(X) \\
\text{s.t.} \quad & (A+BK)P(A+BK)^\top - P + I_n \preceq 0 \\
& P \succeq I_n \\
& X - R^{\frac{1}{2}} K P K^\top R^{\frac{1}{2}} \preceq 0
\end{aligned}
$$

E. Feron, V. Balakrishnan, S. Boyd, L. El Ghaoui, Numerical methods for $\mathcal{H}_2$ related problems, in Proc. 1992 ACC.

Data-Driven Solution to LQR

Theorem 4 (Data-driven LQR Representation)

Let PE condistion holds. Then the optimal $\mathcal{H}_2$ state-feedback controller K for the system can be described as the semidefinite program (SDP)

$$\begin{aligned} \min_{Q,X} \quad & \mathrm{trace}(Q_x X_{0,T} Q) + \mathrm{trace}(X) \\ \text{s.t.} \quad & \begin{bmatrix} X & R^{\frac{1}{2}} U_{0,1,T} Q \\ Q^\top U_{0,1,T}^\top R^{\frac{1}{2}} & X_{0,T} Q \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} X_{0,T} Q - I_n & X_{1,T} Q \\ Q^\top X_{1,T}^\top & X_{0,T} Q \end{bmatrix} \succeq 0 \end{aligned}$$

The resulting optimal state-feedback gain is

$$K = U_{0,1,T} Q (X_{0,T} Q)^{-1}$$

which coincides with the DARE based solution

$$K = -(R + B^\top P B)^{-1} B^\top P A$$

Algorithm via CVX in Matlab

Data-Driven LQR

```
cvx_begin sdp
    variable Q(T,n)
    variable X(m,m) symmetric
    minimize (trace(Q_x*X0*Q)+trace(X))
    [X, sqrtm(R)*U0*Q;  Q'*U0'*sqrtm(R)', X0*Q] >= 0
    [X0*Y-eye(n),X1*Q; Q'*X1', X0*Q] >= 0
cvx_end
K = U0*Q*(inv(X0*Q));
```

Robustness: Noisy Measurements

Consider the system, but suppose that one can only measure the signal

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ \zeta(k) &= x(k) + w(k), \quad k = 0, 1, 2, \cdots \end{aligned}$$

where w is an unknown measurement noise.

Experiment

The objective is to design a stabilizing controller for above system

- Consider a PE input $u_{[0,T-1]}$ of order $n + t$ with $t = 1$;
- Apply it to the system and collect the measured (hence, noisy) state response in the $n \times T$ matrix

$$Z_{0,T} := X_{0,T} + W_{0,T} \quad \text{or} \quad Z_{1,T} := X_{1,T} + W_{1,T}$$

where

$$X_{0,T} := [x(0)\ x(1) \cdots x(T-1)], \quad W_{0,T} := [w(0)\ w(1) \cdots w(T-1)]$$

Data-Driven with Noisy Measurements

From Willems' Fundamental Lemma

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U_{0,1,T} \\ Z_{0,T} \end{bmatrix} G_K = \begin{bmatrix} U_{0,1,T} \\ X_{0,T} + W_{0,T} \end{bmatrix} G_K$$

Hence

$$A + BK = [B \ \ A] \begin{bmatrix} U_{0,1,T} \\ Z_{0,T} \end{bmatrix} G_K = (\ Z_{1,T} \ + \ R_{0,T} \ ) G_K$$

$$\begin{aligned} AZ_{0,T} + BU_{0,1,T} &= AX_{0,T} + AW_{0,T} + BU_{0,1,T} \\ &= X_{1,T} + AW_{0,T} \\ &= Z_{1,T} + AW_{0,T} - W_{1,T} \\ &:= Z_{1,T} + R_{0,T} \end{aligned}$$

Robust Stabilization

Theorem 5

Let a single-to-noise ratio (SNR) condition

$$R_{0,T} R_{0,T}^\top \preceq \gamma Z_{1,T} Z_{1,T}^\top$$

holds for some $\gamma > 0$ and $R_{0,T} = AW_{0,T} - W_{1,T}$. Then any matrix Q and scalar $\alpha > 0$ satisfying $\gamma < \alpha^2/(4+2\alpha)$ and

$$\begin{bmatrix} Z_{0,T} Q - \alpha Z_{1,T} Z_{1,T}^\top & Z_{1,T} Q \\ Q^\top & Z_{0,T} \end{bmatrix} \succ 0, \quad \begin{bmatrix} I_T & Q \\ Q^\top & Z_{0,T} Q \end{bmatrix} \succ 0$$

is such that

$$K = U_{0,1,T} Q (X_{0,T} Q)^{-1}$$

is a stabilizing state-feedback gain for system $x(k+1) = Ax(k) + Bu(k)$

- Search for the *feasible solution maximizing* α
- The results can be applied to $x(k+1) = Ax(k) + Bu(k) + d(k)$

Conclusion

Key

A direct data-driven control design for unknown dynamic system under PE (persistently exciting) condition.

- Sate-feedback Stabilization
- Optimality for LQR
- Robustness to noise
- LMI and SDP settings.
- Direct data-based technique is useful for open-loop system, nonlinear system, output feedback, MIMO system (details in paper)

Motivation for future work

- Probabilistic (robustness) guarantee in data-driven control
- Sparsity-promoting for state-feedback gain K