Linear Quadratic Tracking Control with Sparsity-Promoting Regularization

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Linear Quadratic (LQ) Tracking

Master System: $\dot{z}_m(t) = Az_m(t),\ t \geq 0,\ z_m(0) = \xi_m \in \mathbb{R}^n$
Slave System: $\dot{z}_s(t) = Az_s(t) + Bu(t),\ t \geq 0,\ z_s(0) = \xi_s \in \mathbb{R}^n$

Tracking Goal:

$$\lim_{t\to\infty} \|z_s(t) - z_m(t)\| \doteq \lim_{t\to\infty} \|x(t)\| = 0$$

Tracking Error System:

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = \xi_s - \xi_m = \xi \in \mathbb{R}^n,\ t \geq 0$$

Performance Index (LQ Cost)

$$J_{LQ} = \frac{1}{2}\int_0^T \left\{ x(t)^\top Qx(t) + r u(t)^2 \right\} dt,\ Q = Q^T \geq 0,\ r > 0$$

$\mathcal{Q}$: Is it possible to find a *feasible control* $\{u(t) : 0 \leq t \leq T\}$ that

achieves tracking as well as minimizes the control effort ?

Recall Control Signals

▷ Energy $\Leftrightarrow$ $\mathcal{L}^2$ norm of control signal

▷ Fuel $\Leftrightarrow$ $\mathcal{L}^1$ norm of control signal

▷ ? $\Leftrightarrow$ $\mathcal{L}^0$ norm of control signal

$\Longrightarrow$ $\mathcal{L}^0$ norm of control signal is related to *sparsity*, which contains *only a small number of non-zero elements* compared to its dimension

$\mathcal{A}$: *Sparsity-promoting method* is a powerful technique !

- Compressed Sensing
- Maximum Hands-off Control

Sparse Optimization: LQ Hands-off Control

A novel LQ hands-off control problem via sparse optimization

$$\begin{aligned}
\min \quad & \underbrace{\overbrace{\frac{1}{2}\int_0^T \left\{ x(t)^\top Q x(t) + r u(t)^2 \right\} dt}^{J_{LQ}:\ \text{LQ cost}} + \overbrace{\lambda \int_0^T |u(t)|^0 dt}^{\textit{Sparsity}:\ \mathcal{L}^0 \text{norm}}}_{J_0:\ \text{LQ hands-off control cost}} \\
\text{s.t.} \quad & \dot{x}(t) = Ax(t) + Bu(t) \\
& x(0) = \xi, \quad x(T) = 0 \\
& |u(t)| \leq 1, \quad \forall t \in [0, T]
\end{aligned} \qquad (P_0)$$

Pros: *Minimize the control inputs and achieve tracking*

Cons: (P_0) is *non-convex, non-smooth and discontinuous*

Proposed Approach: Convex relaxation

$$\min\ J_{LQ} + \lambda \|u\|_1 \quad \text{s.t. } (P_0) \text{ constraints} \qquad (P_1)$$

- $\mathcal{L}^1$ optimality $\rightsquigarrow$ $\mathcal{L}^0$ solution approxiamtion

Numerical Computation

- Method: L^1 `relaxation`, `Time-discretization`, `CVX`

$(P_0) \rightarrow (P_1) \rightarrow$ Finite-dimensional Optimization Problem (P_2)

$$\begin{aligned} \min \quad & \frac{1}{2}\sum_{k=0}^{m-1} \begin{bmatrix} x_d^k \\ u_d^k \end{bmatrix}^\top \begin{bmatrix} Q_d & S_d \\ S_d^\top & R_d \end{bmatrix} \begin{bmatrix} x_d^k \\ u_d^k \end{bmatrix} + \lambda \frac{T}{m} \|u_d\|_{\ell^1} \\ \text{s.t.} \quad & x_d^{k+1} = A_d x_d^k + B_d u_d^k \\ & x_d^0 = \xi, \quad x_d^m = 0 \\ & |u_d^k| \le 1, \quad k = 0, 1, \ldots, m-1 \end{aligned} \qquad (P_2)$$

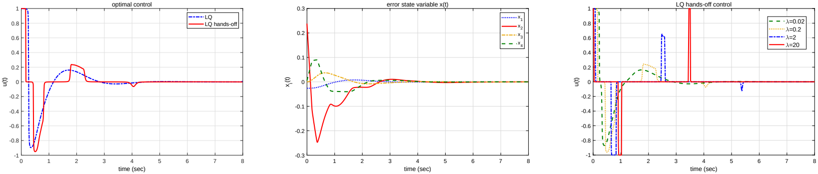

Fig. LQ and LQ hands-off control (left), tracking error states (middle), and different wights for LQ hands-off control (right)

Conclusion

- Necessary conditions for LQ hands-off control
- LQ hands-off control may not be continuous
- Further Details (Paper & Poster)
  - Theoretical analysis
  - Worst-case approach for robustness

THANK YOU FOR YOUR ATTENTION !

Textbook: **Sparsity Methods for Systems and Control**
https://www.nowpublishers.com/article/BookDetails/9781680837247