Data-Driven Sparse Feedback with Schur α-Stability

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Abstract: A data-driven framework for a discrete linear time invariant (LTI) system is employed in conjunction with sparse feedback control synthesis. Instead of a priori knowledge of an actual system model, this note concerns on the black-box control systems by purely exploiting experimental input-state data samples. An $\ell_{1,\infty}$ matrix norm on a feedback gain is penalized to promote a row-sparse structure that maximizes the number of zero-valued rows within the feedback matrix itself, resulting in a preferable sparse actuator setting.

Keywords: Sparse feedback gain, Schur α-stability, data-driven control.

1. INTRODUCTION

In this note, we explore the data-driven sparse feedback control for a *model-free* discrete-time linear time-invariant plant, where the true dynamics of the plant is *unknown* and the control inputs of the feedback response is *sparse*. More concretely, the control objective tends to learn sparse feedback controllers directly from the input-state data trajectories, in contrast to the common approaches where a priori knowledge of the actual system model is given or identified to synthesize a model-based sparse (feedback) controller [1, 2]. Assuming that the input signals for a single or multiple experiments either hold the persistently exciting (PE) condition or go beyond one using data informativity [3-5], the plant can be successfully reconstructed by the input-state data samples, resulting in a data assisted system representation. Then we pursue a sparse stabilizing feedback with input sparsity and a prescribed degree of stability.

This note presents several distinctions compared to existing works: Firstly, the considered system operates under a black-box setup, meaning it lacks a known model but is accessible through input-state data based on a PE condition [5]. Secondly, the input sparsity is achieved by using a *row-sparsity* structured feedback gain matrix [1, 6], accomplished through penalizing an $\ell_{1,\infty}$ norm on itself [7]. This results in a preferable sparse actuator setting, and the derived controller naturally is a data-driven sparse feedback controller enjoying Schur α-stability.

2. MODEL-BASED SPARSE FEEDBACK

Consider a discrete linear time invariant (LTI) plant

$$x(t+1) = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and the control input at time $t \in \mathbb{N}_{\geq 0}$, respectively. Besides, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are system matrices, where the pair (A, B) is assumed to be reachable. Then we introduce a static state feedback

$$u(t) = Kx(t), \tag{2}$$

where $K \in \mathbb{R}^{m\times n}$ is the gain to be determined.

In this note, we explore a *row-sparse* gain K such that the closed-loop system composed of (1) and (2)

$$x(t+1) = (A + BK)x(t) \tag{3}$$

is *Schur α-stable*, i.e., $\max_{1\leq i\leq n} |\lambda_i(A+BK)| < 1-\alpha$. Thus we see that such a gain K exists if and only if there exist a symmetric and positive-definite matrix $Q \in \mathbb{R}^{n\times n}$ (i.e., $Q = Q^\top \succ 0$) and a matrix $G \in \mathbb{R}^{m\times n}$ which satisfy

$$\begin{bmatrix} Q & Q^\top A^\top + G^\top B^\top \\ AQ + BG & (1-\alpha)^2 Q \end{bmatrix} \succ 0, \tag{4}$$

which follows a standard LMI technique. Then, the existence of solutions Q and G admits a Schur α-stabilizing gain

$$K = GQ^{-1}.$$

Deriving the aforementioned statement directly is a simple task, as it parallels the concept of asymptotic stability. In essence, the Schur α-stability of $A + BK$ can be interpreted as Schur stability of $A + BK/(1-\alpha)$ [1].

In this case, K becomes row-sparse if G is row-sparse, and the matrices K and G share an identical row-sparse pattern. Specifically, minimizing $\|G\|_{0,\infty}$ with respect to Q and G subject to (4) gives a design procedure for a row-sparse state feedback gain with Schur α-stability, where $\|G\|_{0,\infty}$ is defined as the number of nonzero elements of the set $\{\max_{1\leq j\leq n} |K_{i,j}|,\ i = 1, 2, \ldots, m\}$. Since $\|G\|_{0,\infty}$ is not a norm and is not convex, we use its convex relaxation $\|G\|_{1,\infty}$ instead of $\|G\|_{0,\infty}$, where $\|K\|_{1,\infty} = \sum_{i=1}^m \max_{1\leq j\leq n} |K_{i,j}|$.

In this way, we can formulate the row-sparse state feedback design with Schur α-stability as

$$\min_{G, Q=Q^\top} \quad \|G\|_{1,\infty} \quad \text{s.t.} \quad \text{LMI (4)}. \tag{5}$$

This model-based framework (5) works well under the condition that the matrices A and B of (1) are given. In what follows, we discard this model-based framework, namely, assume that the knowledge of A and B of (1) is partially or even completely unknown, giving rise to a *model-free* setup. This motivates us to investigate the following data-driven sparse feedback control paradigm.

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3. DATA-DRIVEN SPARSE FEEDBACK

We now harvest a finite input-state measurements over time T from the actual plant (1). They are supposed to be recorded by the matrices $U_{0,T} \in \mathbb{R}^{m\times T}$, $X_{0,T} \in \mathbb{R}^{n\times T}$, and $X_{1,T} \in \mathbb{R}^{n\times T}$ as

$$\begin{aligned} U_{0,T} &= \begin{bmatrix} u_d(0) & u_d(1) & \cdots & u_d(T-1) \end{bmatrix}, \\ X_{0,T} &= \begin{bmatrix} x_d(0) & x_d(1) & \cdots & x_d(T-1) \end{bmatrix}, \\ X_{1,T} &= \begin{bmatrix} x_d(1) & x_d(2) & \cdots & x_d(T) \end{bmatrix}, \end{aligned} \tag{6}$$

where the subscript d in u_d and x_d stands for the data samples collected from the system performing a single experiment in offline. We here assume that the following full rank condition

$$\operatorname{rank} \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} = m + n, \tag{7}$$

holds true, where the length T should be sufficient long, since it requires that the input sequence $U_{0,T}$ is *persistently exciting* (PE) [3-5].

Let the matrices of (6) be the input-state data trajectories collected by the plant (1) during an experiment, they must satisfy

$$X_{1,T} = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix}. \tag{8}$$

Under the condition (7), the system matrices A and B are determined uniquely for given data matrices $U_{0,T}$, $X_{0,T}$, and $X_{1,T}$. Thus the data matrices (6) can be used instead of A and B, which provides a data-driven framework. We now formulate the main result of this note as follows:

Theorem 1: Suppose that the LTI dynamics (1) generates data matrices $U_{0,T}$, $X_{0,T}$, and $X_{1,T}$ of (6) which satisfy the condition (7). Then, with $U_{0,T}$, $X_{0,T}$, and $X_{1,T}$, the problem (5) has the following equivalent data-driven representation as

$$\begin{aligned} \min_{V} \quad & \|U_{0,T}V\|_{1,\infty} \\ \text{s.t.} \quad & \begin{bmatrix} X_{0,T}V & V^\top X_{1,T}^\top \\ X_{1,T}V & (1-\alpha)^2 X_{0,T}V \end{bmatrix} \succ 0, \\ & X_{0,T}V = V^\top X_{0,T}^\top, \end{aligned} \tag{9}$$

where $V \in \mathbb{R}^{T\times n}$. The solution V of the problem gives a Schur α-stabilizing gain

$$K = U_{0,T}V(X_{0,T}V)^{-1}.$$

Proof: Let us introduce a relation between the variable pair (G, Q) of (4) and the variable V of (9) as

$$\begin{bmatrix} G \\ Q \end{bmatrix} = \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} V, \tag{10}$$

that is,

$$G = U_{0,T}V, \qquad Q = X_{0,T}V,$$

which is analogous to [5, Theorem 2]. For any given G and Q, the equation (10) has a solution V under the condition (7). Conversely, if V is given, G and Q are always determined. With $X_{0,T}V = V^\top X_{0,T}^\top$, we have $Q = Q^\top$. Thus by using the equation (10) with the symmetric constraint, we can replace G and Q with V, where we see

$$\begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} G \\ Q \end{bmatrix} = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_{0,T} \\ X_{0,T} \end{bmatrix} V = X_{1,T}V.$$

Substituting the above into (4), the Schur α-stability is effectively provided for a model-free LTI system (1) based on the data-driven representation (9). We therefore complete the proof of Theorem 1. ■

This theorem clarifies that a data-driven sparse feedback design is possible for Schur α-stabilization. The static state feedback gain is purely derived from input-state data based on the persistently exciting condition (7), where the system matrices A and B are not needed for LTI dynamics (1).

4. CONCLUSION

In this note, we established an equivalence relationship between "model-based" sparse feedback control and "model-free" sparse feedback control for an LTI dynamics with Schur α-stability. In order to deal-with the problem setup of a black-box system, we exploited the data-driven technique that generates the experimental input-state data trajectories to reconstruct the LTI system behavior, then the row-sparse feedback gain can be characterized as a data-driven form. Future works are interested in the effects of external noise/disturbances and the sparse input-state data samples for data-driven controller design.

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