Quantum Mechanics Head notes

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Contents

Notes to jog my memory.

1 Heisenberg Matrix Approach

Add notes to talk about some rules on linear algebra generalisation of Quantum Mechanics.

2 Algebraic Approach to Harmonic Oscillator

Add notes to talk about the raising and lowering operators

3 Generalised Angular Momentum

3.1 Preliminaries

Begin with:

$$\left[\hat{J}_x, \hat{J}_y\right] = i\hbar\hat{J}_z$$

Other pairing can be achieved through cyclic permutations.

The raising and lowering operators are:

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$$

They are Hermitian conjugates of each other.

The product is defined by:

$$\hat{J}_\pm\hat{J}_\mp = \hat{J}^2 - \hat{J}_z^2 \pm \hbar\hat{J}_z$$

(The last sign follows the first.)

Thus, the commutator is:

$$\left[\hat{J}_\pm, \hat{J}_\mp\right] = \pm 2\hbar\hat{J}_z$$

(Again, the sign follows the leader.)

The raising/lowering operators commutes with $\hat{J}^2$, but with the components:

$$\left[\hat{J}_z, \hat{J}_\pm\right] = \pm\hbar\hat{J}_\pm$$

(Follows the sign.)

3.2 Eigenstates and Eigenvalues

We begin again by stating that:

$$\hat{J}^2|\phi\rangle = \alpha|\phi\rangle$$
$$\hat{J}_z|\phi\rangle = \beta|\phi\rangle$$

The raising/lowering operators raises or lowers $\hat{J}_z$ in $\hbar$ increments.

However, as it commutes with $\hat{J}^2$, it does nothing.

Since α is fixed and by definition bigger than $\hat{J}_z$ (as it is a component), it follows that there is a termination i.e. bounded upper and lower eigenvalues.

3.2.1 Proof

To show, write a top eigenstate where applying the raising operator would yield 0, and vice versa for the lowest state. The difference in eigenvalues is $n\hbar$. Using the commutator of the raising and lowering operator, this can be easily shown.

From:

$$m_T - m_B = n\hbar$$

Then:

$$\alpha = mT(mT + \hbar)$$

And since:

$$m_T = n\frac{\hbar}{2}$$

Rewriting $j = \frac{n}{2}$:

$$\alpha = j(j+1)\hbar^2$$
$$\beta = m\hbar$$

3.3 Actions of the Operators

Rewrite $|\phi\rangle = |j, m\rangle$ The raising operator gives: $\hat{J}_z \left(\hat{J}_+ |j, m\rangle \right) = (m + 1)\hbar \left(\hat{J}_+ |j, m\rangle \right)$, but applying $\hat{J}_z$ to $|j, m+1\rangle$ yields the same eigenvalue(?), implying they are proportional to each other. With this, we write:

$$\hat{J}_+ |j, m\rangle = C_+ |j, m+1\rangle$$

Rewriting the above as a bra and noticing that the raising/lowering operators are Hermitian adjoint, using their product yields (usual convention takes positive root):

$$C_+ = \hbar\sqrt{j(j+1) - m^2 - m}$$
$$C_- = \hbar\sqrt{j(j+1) - m^2 + m}$$
$$\therefore C_\pm = \hbar\sqrt{j(j+1) - m(m \pm 1)}$$

(Again, m sign follows leading sign)

4 Spin (Spin $\frac{1}{2}$ systems)

The spin states are a carbion copy of the algebraic theory of generalised angular momentum (Griffith). They do not have an equivalent position representation.

4.1 Preliminaries

Spin up and spin down states:

$$
\begin{aligned}
|\frac{1}{2},\frac{1}{2}\rangle &\equiv |\alpha\rangle = |+\rangle \\
|\frac{1}{2},-\frac{1}{2}\rangle &\equiv |\beta\rangle = |-\rangle
\end{aligned}
$$

Again:

$$
\begin{aligned}
&\hat{S}^2|+\rangle = s(s+1)\hbar^2|+\rangle = \frac{3}{4}\hbar^2|+\rangle \\
&\hat{S}_z|+\rangle = s\hbar|+\rangle = \frac{1}{2}|+\rangle \\
&\hat{S}^2|-\rangle = s(s+1)\hbar^2|-\rangle = \frac{3}{4}\hbar^2|-\rangle \\
&\hat{S}_z|-\rangle = s\hbar|-\rangle = -\frac{1}{2}|-\rangle
\end{aligned}
$$

4.2 Raising and Lowering Operators

Same as before:

$$
\begin{aligned}
\hat{S}_\pm|s,m\rangle &= \hbar\sqrt{s(s+1)-m(m\pm 1)}|s,m\pm 1\rangle \\
\therefore \hat{S}_\pm|\frac{1}{2},m\rangle &= \hbar\sqrt{\frac{3}{4}-m(m\pm 1)}|\frac{1}{2},m\pm 1\rangle
\end{aligned}
$$

Thus:

- Raising spin down becomes $\hbar$ times spin up
- Lowering spin up becomes $\hbar$ times spin down

- Anything else is zero (cannot raise above the ceiling or go below the floor)

This might come in handy:

$$\hat{S}_x = \frac{1}{2}\left(\hat{S}_+ + \hat{S}_-\right); \qquad \hat{S}_y = \frac{1}{2i}\left(\hat{S}_+ - \hat{S}_-\right)$$

4.2.1 The arbitrary state

An arbitrary state can be written as:

$$|\chi\rangle = a|+\rangle + b|-\rangle$$

4.2.2 Additional Relations

The $\hat{S}^2$ is a purely numerical operator for any state:

$$\hat{S}^2 \equiv \frac{3}{4}\hbar^2$$

Additionally, for spin-$\frac{1}{2}$ states:

$$\hat{S}_x^2 = \hat{S}_y^2 = \hat{S}_x^2 = \frac{\hbar^2}{4}$$

1. Anticommutators:

$$\left\{\hat{S}_x, \hat{S}_y\right\} = 0$$

4.2.3 Matrix Representation

This section introduces the Pauli matrices. These are formed by performing: $\langle s', m'|\hat{S}|s, m\rangle$ for each respective operators. Each operator, $\hat{S}^2, \hat{S}_x, \hat{S}_y, \hat{S}_z$ can be formed by multiplying their respective matrices with $\hbar/2$:

I =
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The trace of the Pauli matrices are zero, and the determinant is -1.

4.3 Determination of eigenstates and eigenvectors

To solve for $\hat{S}_x|\chi\rangle = a|\chi\rangle$, we write $a = \frac{\hbar}{2}\lambda$ and $\hat{S}_x = \frac{\hbar}{2}\sigma_x$, and let $|\chi\rangle = (u, v)^T$. Thus the matrix approach gives an easy way to determine the eigenstates and eigenvalues of the operators.

Equivalently, this can be performed via the basis states and raising/lowering operators approach.

See page 54 of notes for a table showing the results.

4.4 Spin along arbitrary direction

The operator $\hat{S}_n$ is defined as $\hat{S} \cdot \hat{n}$, where:

$$\hat{n} = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$$

Thus:

$$\hat{S}_n = \frac{\hbar}{2}(\hat{\sigma}_x \cos\phi \sin\theta + \hat{\sigma}_y \sin\phi \sin\theta + \hat{\sigma}_z \cos\theta) = \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\theta} \\ \sin\theta e^{i\theta} & -\cos\theta \end{pmatrix}$$

Then, using trigonometric identities (half angle?):

$$|+\rangle_{\hat{n}=} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

$$|\text{-}\rangle_{\hat{n}=} \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

- Spin up along $+z$ is $P_+ = cos^2\frac{\theta}{2}$
- Spin down is $P_- = sin^2\frac{\theta}{2}$

4.5 Addition of Angular Momentum

In the special case of spin-$\frac{1}{2}$ particles, the addition of angular momentum is simple:

$\hat{\mathbb{S}} = \hat{\mathbb{S}_{\not{1}}} + \hat{\mathbb{S}_{\not{2}}}$

There are four possible states for the 2 particles:

$$|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$$

Then, from above:

$$\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}$$

Eigenvalues are thus: 1,0,0,-1.

$1\backslash 2$	α	β
α	$\alpha_1\ \alpha_2$	$\alpha_1\ \beta_1$
β	$\beta_1\ \alpha_2$	$\beta_1\ \beta_2$

Denote a general state by $|S, M\rangle$.

If $M = \frac{1}{2} + \frac{1}{2} = 1$ must have $S = 1$, as:

$$\hat{S}_z|1,1\rangle = \left(\hat{S}_{1z} + \hat{S}_{2z}\right)\alpha_1\alpha_2 = \frac{\hbar}{2}\alpha_1\alpha_2 + \frac{\hbar}{2}\alpha_1\alpha_2 = \hbar\alpha_1\alpha_2$$

5 Total Angular Momenta

We define:

$$\hat{J} = \hat{L} + \hat{S}$$

While $Y_{lm}\alpha$ and $Y_{lm}\beta$ are eigenstates of $\hat{J}_z$, they are not eigenstates of $\hat{J}^2$.

5.1 Example

Let $J = l+\frac{1}{2}$ and $m_j = l+\frac{1}{2}$ i.e. the state of maximum projection $Y_{ll\alpha} \equiv |j,j\rangle$. Applying the lowering operator $\hat{J}_-|j,m\rangle = \sqrt{j(j+1) - m(m-1)}\hbar|j, m-1\rangle$:

$$\left(\hat{L}_- + \hat{S}_-\right)Y_{ll}\alpha = \left(\hat{L}_- + \hat{S}_-\right)|l,l\rangle|\alpha\rangle = \sqrt{2l}\hbar|l, l-1\rangle|\alpha\rangle + |l,l\rangle\hbar|\beta\rangle$$

Equating the two and using orthogonality conditions:

Introduce a new state $|l - \frac{1}{2}, l - \frac{1}{2}\rangle = a|l, l-1\rangle|\alpha\rangle + b|l, l\rangle|\beta\rangle$ which is orthogonal: