Overall Notes for General Relativity

Kelvin

October 21, 2021

Contents

1 Week 1

1.1 Vector that lives on the sheet

$f : V \to R$

- Parameterization that forces vector to live inside a curved surface
  - Recall from Newtonian mechanics
- a bunch of stuff I have to go through again
  - Covariant and contravariant vectors
    * Derivation
  - Coordinate transformation

2 Week 2

2.1 Tensors

Tensor of type (p,q) is: $T^{i_1,i_2,\ldots,i_p}_{j_1,j_2,\ldots,j_q}$

2.1.1 Tensors have upper and lower indices:

- Upper indices transform contravariantly
- Lower indices transform covariantly

2.1.2 Ranks

- Scalar: 0
- Vectors: 1
- We will not go beyond rank 4 tensors in this course

2.1.3 Contraction

- Summing over a pair of indices results in a tensor of rank = p+q-2
- Proof requires showing that the a set of Jacobian cancels out
- This forms a kronecker delta, which does not change transformation
- QED

2.1.4 Partial Derivatives

- Not a tensor
- Transformation properties of PD
- Rewriting it as a chain rule and product rule
- One term obeys tensor trans. but the other does not

2.1.5 Manifolds

- An n dimensional, smooth (inf. diff.) is a set
- with collection of open subsets ω_α
- a collection of coordinates $X^i_{(\alpha)}$
- Where $\alpha \in N$ and $i = 1, 2, ..., n$
  - every point on the manifold belongs to at least one \${Ω_α}
- EXAMPLE: Surface of Earth is a manifold
- Collection are pages of atlas
- Overlap

2.1.6 Distance

- g_{ij} is a rank 2 tensor known as metric tensor
- Represents distance
- Inverse: g^{jk}
- The line element: \$$\mathrm{ds}^2 = \mathrm{g_{ij} dX^{idXj}}$
  - Where g_{ij} are arbitrary function of coordinates
  - If it is I, then it becomes $ds^2 = dx^2 + dy^2 + dz^2$

Remark:

Dimensions	Independent components
2D	3
3D	6
4D	10
N-D	$\frac{N(N+1)}{2}$

- Other motivation for the metric structure:

To introduce a mapping that links the covariant and the contravariant

2.1.7 Metric Signature

- Counts the number of positive and negative eigenvalues
- Kronecker delta: (/+,/+,+)
- GR: (-,/+,+,+)
- Line elements or metric in GR are called Lorentzian manifolds

2.1.8 Geodesic Equation

- Def: A curve C given by $X^i(\lambda)$ is **geodesic**
- if it satisfies $\ddot(X)^n + Gamma^n_{ms}\dot(X)^m\dot(X)^s = 0$

Where \$$\Gamma^{\mathrm{n}}_{\mathrm{ms}} = \frac{1}{2}\, \mathrm{g}^{\mathrm{nk}}(\partial_{\mathrm{m}}\ \mathrm{g}_{\mathrm{sk}} + \partial_{\mathrm{s}}\ \mathrm{g}_{\mathrm{km}}$ - $\partial_{\mathrm{k}}\ \mathrm{g}_{\mathrm{ms}}\backslash$

- Geodesics are shortest distances between points
- Get metric tensors and you can computed
- **Christoffel Symbol**: Γ^n_{ms}
- Derive from E-L and learn the actual derivations, watch it a few times

2.2 Summary:

3 Week 3

3.0.1 Geodesic Equation in Polar Coordinates

- Solutions are given by straight lines (Exercise is to show this)

3.0.2 The Christoffel symbol can be found in two ways

- Explicit formula
- via Lagrangian approach

3.0.3 Simplified/reduced Lagragian gives the same EL equations

- However one is used over the other
- Because sqrt is motivated from arc

3.0.4 Metric tensor

- $g_{ij}T^iT^j = ||T||^2$
- Can be used to raise or lower indices (see Zee)
- Upper is column, lower is row
  1. $T_i^j = g_{is}T^{sj}$
  2. $T_j^i = g_{js}T^{is}$
  3. $T_{ij} = g_{im}g_{jn}T^{mn}$

3.0.5 Covariant Derivative

- A covariant derivative ∇_a on a mapping
- From tensors of type (p,q) to tensors type (p,q+1)
- Has the following properties:
  1. If f is a smooth function, then $\nabla_a f = \partial_a f$
  2. ∇_a is linear (like Operators)
  3. Leibniz (product) rule
  4. Commutes with contraction
     - Contract first then differentiate or vice versa

3.0.6 Metricity

- Something is metric compatitable if:
- $\nabla_a g_{bc} = 0$
- Else, the non-metricity is $Q_a bc = \nabla_a g_{bc} = 0$

3.0.7 Torsion

- Torsion is defined by:
- $(\nabla_a \nabla_b f - \nabla_b \nabla) a f) = T^c_{ab} \partial_{cf}$
- GR assumes $T^c_{ab} = 0$

3.0.8 Fine $\nabla_a A^i$

- $\partial_a A^i$ is not a rank 2 tensor
- $\partial_a f = \nabla_a A^i$
- Thus $\partial_a A^i$ should be linear in A^i
- Something about showing a C from the equation of this function

3.0.9 Christoffel symbol components

- The C are actually the Christoffel symbol
  - Show that it is symmetric
  - Torsion free
  - Examinable proof of writing out the three equations
    - Adding up and subtracting

3.0.10 Parallel Transport

- In GR, the straightest curves are shortest
- However, in presence of torsion, this is no longer the case