PHAS2441 Session 7:
Second-order Runge Kutta method.

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We're starting from Euler's method for solving differential equations of the form

$$\frac{dx}{dt} = f(x,t) \tag{1}$$

to obtain $x(t)$.

Given starting condition for some t, Euler's method uses a Taylor expansion to solve the equation at a later time $t+h$:

$$x(t+h) = x(t) + h\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1}{2}h^2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \ldots \tag{2}$$

$$= x(t) + hf(x,t) + \mathrm{O}(h^2). \tag{3}$$

In second-order Runge Kutta, we do a Taylor expansion around $t+\frac{1}{2}h$ to obtain an expression for $x(t+h)$:

$$x(t+h) = x(t+\frac{1}{2}h) + \frac{1}{2}h\left(\frac{dx}{dt}\right)_{t+\frac{1}{2}h} + \frac{1}{8}h^2\left(\frac{d^2x}{dt^2}\right)_{t+\frac{1}{2}h} + \mathrm{O}(h^3) \tag{4}$$

We can also calculate $x(t)$ using a Taylor expansion around $t+\frac{1}{2}h$, which gives us

$$x(t) = x(t+\frac{1}{2}h) - \frac{1}{2}h\left(\frac{dx}{dt}\right)_{t+\frac{1}{2}h} + \frac{1}{2}\left(-\frac{1}{2}\right)^2h^2\left(\frac{d^2x}{dt^2}\right)_{t+\frac{1}{2}h} + \mathrm{O}(h^3) \tag{5}$$

(note the sign change in the second term). Subtracting equation 4 from equation 5 and doing some minor rearrangement then gives us

$$x(t+h) = x(t) + h\left(\frac{dx}{dt}\right)_{t+\frac{1}{2}h} + \mathrm{O}(h^3). \tag{6}$$

Conveniently, by doing this we've eliminated the troublesome terms involving $\frac{d^2x}{dt^2}$. Even better, we've now reduced our errors from $\mathrm{O}(h^2)$ in Euler's method to $\mathrm{O}(h^3)$ here.

However, we're now left with the need to calculate

$$\left(\frac{dx}{dt}\right)_{t+\frac{1}{2}h} = f\left(x(t+\frac{1}{2}h), t+\frac{1}{2}h\right). \tag{7}$$

The problem with this is that we don't know $x(t + \frac{1}{2}h)$. We can *approximate* it though, by using equation 3, to give

$$x(t + \frac{1}{2}h) = x(t) + \frac{1}{2}hf(x,t). \tag{8}$$

Substituting this result into equation 6 then gives us the set of equations which are known as the second-order Runge Kutta method:

$$k_1 = hf(x,t), \tag{9}$$

$$k_2 = hf(x + \frac{1}{2}k_1, t + \frac{1}{2}h) = hf\left(\left[x(t) + \frac{1}{2}hf(x,t)\right], t + \frac{1}{2}h\right) \tag{10}$$

$$x(t+h) = x(t) + k_2. \tag{11}$$

The fourth-order Runge-Kutta method (i.e. the one that is accurate to h^4 and has errors of order $\mathrm{O}(h^5)$) follows a similar procedure of taking Taylor expansions around different points and then eliminating unknowns by using different linear combinations, but is considerably more complicated.