解: 显然 $x > 0$

(1) 由题意 $a \leqslant \frac{e^x}{x} - \ln x + x$

记 $g(x) = \frac{e^x}{x} - \ln x + x$

则 $g'(x) = \dfrac{xe^x - e^x}{x^2} - \dfrac{1}{x} + 1 \qquad x < 1$ 时 $g(x) < 0$,$x > 1$ 时 $g(x) > 0$

$$= \frac{e^x(x-1) + x(x-1)}{x^2}$$

$$= \frac{(e^x + x)(x-1)}{x}$$

故 $g(x)$ 在 $x = 1$ 处取得最小值 $g(1) = 1 + e$.

故 $a \leqslant 1 + e$

(2) 不妨设 $0 < x_1 < 1 < x_2$

$x_1 x_2 < 1 \iff x_1 < \frac{1}{x_2} < 1$

由 1 知:$x < 1$ 时 , $f(x)$ 单调递减

则 $x_1 < \frac{1}{x_2} \Leftrightarrow f(\frac{1}{x_2}) < f(x_1) = 0 = f(x_2)$

记 $\phi(x) = f(x) - f(\frac{1}{x}), x \in (0,1]$

$$\phi'(x) = \frac{(e^x + x)(x-1)}{x} - \frac{(e^x + x)(x-1)}{x}(\frac{-1}{x^2})$$

$$= \frac{(e^x + x)(x-1)}{x}(\frac{x^2+1}{x^2})$$

故 $\phi'(x) < 0, \phi(x)$ 单调递减

$\phi(1) = f(1) - f(1) = 0$

故 $x \in (0,1)$ 时 $\phi(x) > \phi(1) = 0$

$\Rightarrow f(x_1) = f(x_2) > f(\frac{1}{x_2})$

$\Rightarrow x_1 < \frac{1}{x_2} \Rightarrow x_1 x_2 < 1$