

(1) 由题意 $f(x) = (x-a)^3, f(4) = (4-a)^3 = 8 \Rightarrow a = 2$

(2) 由题意 $f(x) = (x-a)(x-b)^2$

$$f'(x) = (x-b)^2 + 2(x-a)(x-b) = (x-b)(3x-2a-b)$$

故 $\frac{2a+b}{3}$ 则 a, b 只能为 -3 和 3

若 $a = -3, b = 3$ 时 $\frac{2a+b}{3} = -1$ (舍去)

若 $a = 3, b = -3$ 时 $\frac{2a+b}{3} = 1$ (符合题意)

$$\text{故 } f(x) = (x-3)(x+3)^2, f'(x) = (x+3)(3x-3)$$

$$x < -3, f'(x) > 0; -3 \leq x \leq 1, f'(x) \leq 0; x > 1, f'(x) > 0;$$

$$\text{故在 } x = 1 \text{ 处取得极小值 } f(1) = -2 * 16 = -32$$

(3) 由题意 $f(x) = x(x-b)(x-1) = (x^2-x)(x-b) = x^3 - (b+1)x^2 + bx$

$$f'(x) = 3x^2 - 2(b+1)x + b$$

$$f'(x) = 0 \text{ 时 } 3x^2 - 2(b+1)x + b = 0$$

$$\text{易知 } f(x) \text{ 在 } x_0 = \frac{2(b+1) - \sqrt{4(b+1)^2 - 4 \cdot 3b}}{6} = \frac{b+1 - \sqrt{b^2 - b + 1}}{3} \text{ 处取得极大值}$$

$$x_0^2 = \frac{2b^2 + b + 2 - 2(b+1)\sqrt{b^2 - b + 1}}{9}, x_0^3 = \frac{4b^3 + 3b^2 + 3b - 3b\sqrt{b^2 - b + 1}}{27}$$