设抛物线 $C: y^2 = 2px, A(a,0), B(ab,0)$, 经过 A 的直线交 C 于 M,N 两点, 直线 MB, NB 与 C 的另一交点分别为 P,Q, 求 PQ 与 x 轴的交点.

解: 设 $M(\frac{m^2}{2p}, m), N(\frac{n^2}{2p}, n)$

$k_{MN} = \frac{m-n}{\frac{m^2}{2p}-\frac{n^2}{2p}} = \frac{2p}{m+n}$

$l_{MN}: y = \frac{2p}{m+n}(x-\frac{m^2}{2p}) = \frac{2p}{m+n}x - \frac{mn}{m+n} = \frac{2px-mn}{m+n}$

l 经过 $(a,0)$ 则 $mn = 2pa$

$K_{MD} = \frac{m}{\frac{m^2}{2p}-ab}$

则 $MD: y = \frac{1}{\frac{m}{2p}-\frac{ab}{m}}(x-ab)$

$$\begin{cases} y = \dfrac{1}{\frac{m}{2p}-\frac{ab}{m}}(x-ab), \\ y^2 = 2px \end{cases}$$

消去 x 得:$(\frac{m}{2p}-\frac{ab}{m})y = \frac{y^2}{2p} - ab \Leftrightarrow \frac{y^2}{2p} - aby^2 - (m-\frac{8}{m})y - 8 = 0$

$y_1y_2 = -8$, 故 $A(\frac{16}{m^2}, -\frac{8}{m})$

同理 $B(\frac{16}{n^2}, -\frac{8}{n})$

故 $k_{AB} = \frac{\frac{8}{m}-\frac{8}{n}}{16(\frac{1}{n^2}-\frac{1}{m^2})} = \frac{-1}{2(\frac{1}{m}+\frac{1}{n})} = \frac{-mn}{2(m+n)} = \frac{k}{2}$

$$\begin{aligned} l_{AB} &= \frac{-mn}{2(m+n)}(x-\frac{16}{m^2}) - \frac{8}{m} \\ &= \frac{-mn}{2(m+n)}x + \frac{8n}{m(m+n)} - \frac{8}{m} \\ &= \frac{-mnx-16}{2(m+n)} \\ &= \frac{k}{2}(x-4) \end{aligned}$$

$y=0$ 时 $-\frac{8}{m}\cdot\frac{2(m+n)}{mn} = x - \frac{16}{m^2} \Rightarrow x = \frac{-16}{mn} = 4$

故 α, β 同是第一象限角或第二象限角, 则 $\alpha-\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

此时 $\tan x$ 单调递增

$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} = \frac{k-\frac{k}{2}}{1+\frac{k^2}{2}} = \frac{1}{\frac{2}{k}+k}$

显然 $k<0$ 时,$\tan(\alpha-\beta)<0, \alpha-\beta<0$

$k>0$ 时,$\tan(\alpha-\beta) = \frac{1}{\frac{2}{k}+k} \leqslant \frac{1}{2\sqrt{2}}$

$k=\sqrt{2}$ 时取等号, 此时 $\tan(\alpha-\beta)$ 取得最大值,$\alpha-\beta$ 也取得最大值

此时 m满足$m^2-2\sqrt{2}m-4=0$,取$m=\sqrt{2}+\sqrt{6}$

则$A(8-4\sqrt{3}, 2\sqrt{2}-2\sqrt{6})$

故$k_{AB} = \frac{\sqrt{2}}{2}, AB: y = \frac{\sqrt{2}}{2}(x-8+4\sqrt{3}) + 2\sqrt{2} - 2\sqrt{6} = \frac{\sqrt{2}}{2}x - 2\sqrt{2}$