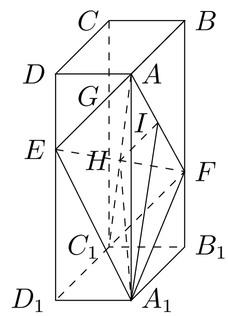
几何法: 复杂不推荐



(1) 证明:

连接 C_1F, 取 G 使 $C_1G = 2CG$, 连接 GB

平面 BCC_1B_1 内, $C_1G = \frac{2}{3}CC_1 = \frac{2}{3}BB_1 = FB$

又 ,$FB \mathbin{/\!/} C_1G$, 则 C_1GBF 为平行四边形

则 $GB \underline{\mathbin{/\!/}} C_1F$

又 $CG = \frac{1}{3}CC_1 = \frac{1}{3}DD_1 = DE, DE \mathbin{/\!/} CG$

则 $ABGE$ 共面且为平行四边形, 则 $GB \underline{\mathbin{/\!/}} AE$

故 $AE \underline{\mathbin{/\!/}} C_1F, AEC_1F$ 共面

(2) 解：

记 EF 中点为 H, 由 (1) 知 AEC_1F 为平行四边形

$C_1F = \sqrt{1+1} = \sqrt{2}, AF = \sqrt{2^2+2^2} = 2\sqrt{2},$

$AC_1 = \sqrt{1^2+2^2+3^2} = \sqrt{14}, AH = HC_1 = \frac{\sqrt{14}}{2}$

由余弦定理:

$\cos AHF = \frac{(\frac{\sqrt{14}}{2})^2+(\frac{EF}{2})^2-(2\sqrt{2})^2}{2\frac{\sqrt{14}}{2}\times\frac{EF}{2}}$

$= -\cos C_1HF = -\frac{(\frac{\sqrt{14}}{2})^2+(\frac{EF}{2})^2-(\sqrt{2})^2}{2\frac{\sqrt{14}}{2}\times\frac{EF}{2}}$

故有: $\dfrac{7}{2}+\dfrac{1}{4}EF^2-8 = -\dfrac{7}{2}-\dfrac{1}{4}EF^2+2$

$$\frac{1}{2}EF^2 = 2+8-7$$

$$EF = \sqrt{6}$$

由 $AE^2+EF^2 = 2+6 = 8 = AF^2$

故 $\angle AEF = \frac{\pi}{2}$, 即 $AE \perp EF$

$\triangle A_1EF$ 中, 易知 $A_1F = A_1E = \sqrt{5}$

故 $A_1H \perp EF$, 作 $HI \mathbin{/\!/} AE$ 交 AF 于 I

则 $\angle IHA_1$ 即为所求二面角的二面角

易知 HI 为 $\triangle FAE$ 中位线,$IH = \frac{1}{2}AE = \frac{\sqrt{2}}{2}$

又 H 为 AC_1 中点, 即立方体几何中心, 故 $A_1H = \frac{1}{2}AC_1 = \frac{\sqrt{14}}{2}$

在平面 AFA_1 中, 通过辅助线易得 $A_1I = \sqrt{5}$

$\triangle IHA_1$ 中,

$$\begin{aligned}\cos\angle IHA_1 &= \frac{(\frac{\sqrt{2}}{2})^2 + (\frac{\sqrt{7}}{2})^2 - (\sqrt{5})^2}{2 \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{14}}{2}} \\ &= \frac{\frac{1}{2} + \frac{7}{2} - 5}{\sqrt{7}} \\ &= -\frac{\sqrt{7}}{7}\end{aligned}$$

故 $\sin IHA = \sqrt{1 - (-\frac{\sqrt{7}}{7})^2} = \frac{\sqrt{42}}{7}$

故二面角 $A - EF - A_1$ 的正弦值为 $\frac{\sqrt{42}}{7}$