(I) $a=-\frac{3}{4}$ 时,$f(x)=-\frac{3}{4}\ln a+\sqrt{1+x}$

$f'(x)=-\frac{3}{4x}+\frac{1}{2\sqrt{1+x}}=\frac{2x-3\sqrt{1+x}}{4x\sqrt{1+x}}$

$f'(x)=0$ 时 $2x=\sqrt{1+x}\Rightarrow 4x^2-9x-9=0$

$\Rightarrow (x-3)(4x+3)=0$

$x_1=3,x_2=-\frac{3}{4}$,$x>3,f'(x)>0,0<x<3,f(x)<0$

故单调递增区间为 $(3,+\infty)$, 单调递减区间为 $(0,3)$

(II) $a\ln x+\sqrt{1+x}-\frac{\sqrt{x}}{2a}\leqslant 0$

注意到带入 $x=1$, 得 $\frac{1}{2a}\geqslant\sqrt{2}\Rightarrow 0<a\leqslant\frac{\sqrt{2}}{4}$

$a>0$ 时, 原式等价于 $\sqrt{x}\cdot\frac{1}{a^2}-2\sqrt{x+1}\cdot\frac{1}{a}-2\ln x\geqslant 0$

记 $t=\frac{1}{a},t\geqslant 2\sqrt{2}$

则 $g(t)=\sqrt{x}\cdot t^2-2\sqrt{x+1}\cdot t-2\ln x=\sqrt{x}(t-\sqrt{1+\frac{1}{x}})^2-2\ln x-\frac{1+x}{\sqrt{x}}$

$g(t)$ 开口向上, 其对称轴 $t=\sqrt{1+\frac{1}{x}}\leqslant 2\sqrt{2}$ 即 $x\geqslant\frac{1}{7}$ 时,$g(t)$ 单调递增

故 $g(t)\geqslant g(2\sqrt{2})=8\sqrt{x}-4\sqrt{2}\sqrt{x+1}-2\ln x$

记 $p(x)=4\sqrt{x}-2\sqrt{2}\sqrt{x+1}-\ln x,x\geqslant\frac{1}{7}$

$$
\begin{aligned}
p'(x)&=\frac{2}{\sqrt{x}}-\frac{\sqrt{2}}{\sqrt{1+x}}-\frac{1}{x}\\
&=\frac{\sqrt{1+x}-\sqrt{2x}}{\sqrt{x(1+x)}}+\frac{\sqrt{x}-1}{x}\\
&=\frac{1-x}{\sqrt{x(x+1)}(\sqrt{2x}+\sqrt{x+1})}+\frac{x-1}{x(\sqrt{x}+1)}\\
&=(x-1)\frac{(\sqrt{2x}+\sqrt{x+1})(\sqrt{x}+1)-\sqrt{x}(\sqrt{x}+1)}{x\sqrt{x+1}(\sqrt{x}+1)(\sqrt{2x}+\sqrt{x+1})}\\
&=\frac{(x-1)(\sqrt{2x^2+2x}+1-\sqrt{x})}{x\sqrt{x+1}(\sqrt{x}+1)(\sqrt{2x}+\sqrt{x+1})}
\end{aligned}
$$

$x>1,p'(x)>0;x<1,p'(x)<0$ 故 $p(x)$ 在 $x=1$ 处取得极小值 $p(1)=0$

故 $x\geqslant\frac{1}{7}$时, $g(t)\geqslant g(2\sqrt{2})=2p(x)\geqslant 0$

$\frac{1}{e^2}\leqslant x<\frac{1}{7}$ 时, 对称轴 $\sqrt{1+\frac{1}{x}}>2\sqrt{2}$

此时 $g(t)\geqslant g(\sqrt{1+\frac{1}{x}})=\sqrt{x}+\frac{1}{\sqrt{x}}-2\sqrt{(1+\frac{1}{x})(1+x)}-2\ln x=\frac{-(x+1+2\sqrt{x}\ln x)}{\sqrt{x}}$

记 $q(x)=x+1+2\sqrt{x}\ln x,\frac{1}{e^2}\leqslant x<\frac{1}{7},q'(x)=1+\frac{\ln x}{\sqrt{x}}+\frac{2}{\sqrt{x}}>0$

故 $q(x)$ 单调递增,$q(x)<q(\frac{1}{7})$

故此时 $g(t)>g(\sqrt{1+\frac{1}{\frac{1}{7}}})=g(2\sqrt{2})\geqslant 0$