

- (1)  $a_2 a_4 = a_5 \Rightarrow a_1^2 q^4 = a_1 q^4 \Rightarrow a_1 = 1$   
 $a_3 - 4a_2 + 4a_1 = 0 \Rightarrow q^2 - 4q + 4 = 0 \Rightarrow q = 2$   
 故  $\{a_n\} = 2^{n-1}$  是 “M-数列”

- (2) ①  $S_n = \frac{1}{\frac{2}{b_n} - \frac{2}{b_{n+1}}} = \frac{b_n b_{n+1}}{2(b_{n+1} - b_n)}$   
 $S_{n+1} = \frac{b_{n+1} b_{n+2}}{2(b_{n+2} - b_{n+1})}$   
 $b_{n+1} = S_{n+1} - S_n$   
 $= b_{n+1} \left[ \frac{b_{n+2}(b_{n+1} - b_n) - b_n(b_{n+2} - b_{n+1})}{2(b_{n+2} - b_{n+1})(b_{n+1} - b_n)} \right]$   
 $2(b_{n+2}b_{n+1} - b_{n+2}b_n - b_{n+1}^2 + b_nb_{n+1})$   
 $= (b_{n+2}b_{n+1} - 2b_nb_{n+2} + b_nb_{n+1})$   
 $b_{n+2}b_{n+1} - 2b_{n+1}^2 + b_nb_{n+1} = 0$   
 $b_{n+1}(b_{n+2} - 2b_{n+1} + b_n) = 0$   
 故  $\{b_n\}$  为等差数列  
 $n = 1$  时,  $1 = 2 - \frac{2}{b_2} \Rightarrow b_2 = 2$   
 故  $\{b_n\} = n$

- ② 设  $\{c_n\} = q^{n-1}$  由题意:  $c_k \leq b_k \leq c_{k+1}$

$$q^{k-1} \leq k \leq q^k$$

$$(k-1) \ln q \leq \ln k \leq k \ln q$$

$$\frac{\ln k}{k} \leq \ln q \leq \frac{\ln k}{k-1}$$

$$\text{设 } f(x) = \frac{\ln x}{x} \text{ 则 } f'(x) = \frac{1 - \ln x}{x^2}$$

$$f'(x) = 0 \text{ 时 } x = e$$

$x < e$  时  $f(x)$  单调递增  $x > e$  时  $f(x)$  单调递减

故  $f(x)$  在  $x = e$  处取得最大值。

$$\text{若 } x \text{ 取整数, } f(2) = \frac{\ln 2}{2} = \ln(\sqrt[6]{8}), f(3) = \frac{\ln 3}{3} = \ln(\sqrt[6]{9})$$

故  $f(x)$  在  $x = 3$  处取得最大值

即只需  $q \geq \sqrt[3]{3}, f(x) \leq \ln q$  恒成立

$$\text{设 } g(x) = \frac{\ln x}{x-1}, f'(x) = \frac{1 - \frac{1}{x} - \ln x}{(x-1)^2}$$

$$\text{记 } h(x) = 1 - \frac{1}{x} - \ln x, h'(x) = \frac{1}{x^2} - \frac{1}{x}$$

$$x > 0 \text{ 时, } x = 1 \text{ 时取得最大值 } h(1) = 0$$

故  $g'(x) \leq 0, g(x)$  单调递减

$k \leq 3$  时  $g(x)$  递减,  $f(x)$  递增

$$\text{只需 } \sqrt[3]{3} \leq q \leq \sqrt{3},$$

便有  $f(x) \leq f(3) \leq \ln q \leq g(3) \leq g(x)$

$k > 3$  后  $g(x)$  递减, 当  $g(x) < \ln \sqrt[3]{3}$  时, 不等式在  $k = 3$  处不成立

而  $g(5) = \ln(\sqrt[4]{5}) = \ln(\sqrt[12]{125})$

$g(6) = \ln(\sqrt[5]{6}) = \ln(\sqrt[15]{216})$

而  $\sqrt[3]{3} = \sqrt[12]{81} = \sqrt[15]{243}$

故  $k$  最大为 5, 即  $m = 5$