Improved algorithms for efficient active learning halfspaces with Massart and Tsybakov noise

Chicheng Zhang and Yinan Li
The University of Arizona

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THE UNIVERSITY OF ARIZONA

Abstract

We give an efficient PAC active halfspace learning algorithm that has improved noise-tolerance and label efficiency under benign noise conditions, given that the unlabeled data distribution satisfies certain structural properties [DKKTZ20]. Specifically:

1. Under Massart noise, it achieves optimal label complexity; such efficient and label-optimal results were previously only known when the unlabeled data distribution is uniform [YZ17].
2. Under two subfamilies of Tsybakov noise, it achieves improved label complexities compared to passive learning algorithms.

Problem: efficient active learning halfspaces with benign noise

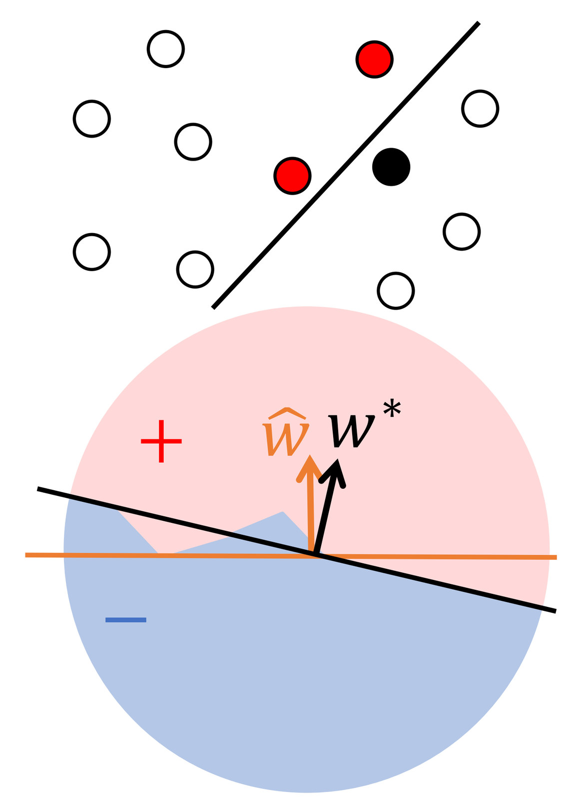
- ($\boxed{x}$, $\boxed{y}$) drawn from a distribution D

  features   **interactive** label queries

- x drawn from a structured distribution [DKKTZ20] (e.g. isotropic log-concave distributions)

- Linear classifiers: $H = \{\mathrm{sign}(w \cdot x): w \in \mathrm{R}^d\}$
- Error $\mathrm{err}(w) = P(y \neq \mathrm{sign}(w \cdot x))$
- Optimal linear classifier $w^* = \mathrm{argmin}_w \mathrm{err}(w)$

- Goal: computationally efficient algorithm that returns a vector $\hat{w}$, such that
  $\mathrm{err}(\hat{w}) - \mathrm{err}(w^*) \leq \epsilon$,
  using a few label queries

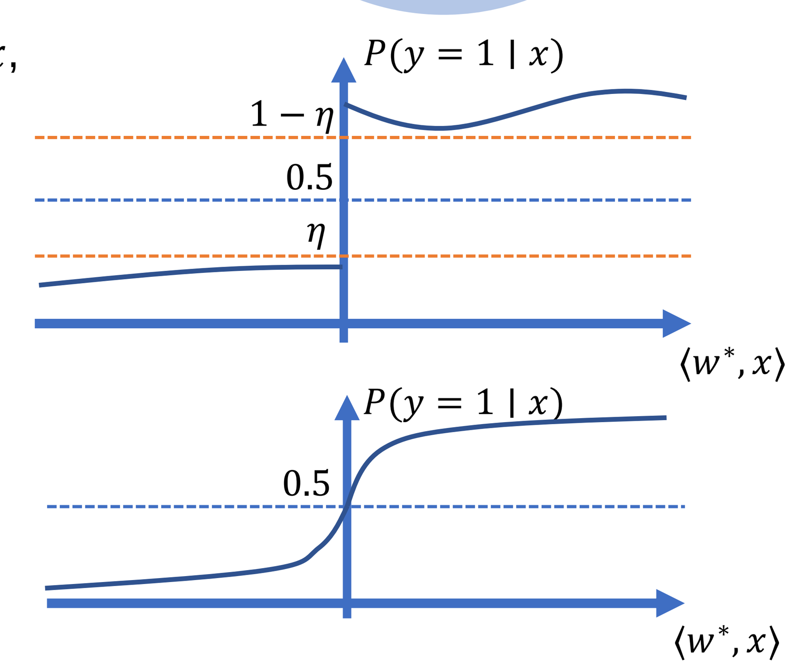


- Main assumption: there exists some halfspace w^* that is Bayes optimal, i.e., for all x,
  $\eta(x) := P_D(y \neq \mathrm{sign}(w^* \cdot x) | x) \leq 1/2$

- η-Massart [MN06]: for all x, $\eta(x) \leq \eta < \frac{1}{2}$

- α-Tsybakov [T04] for $\alpha \in (0,1)$: for all t, $P_D(1/2 - \eta(x) \leq t) \leq O\left(t^{\alpha/(1-\alpha)}\right)$

- α-Geometric Tsybakov [e.g., CN08]: for all x, $\frac{1}{2} - \eta(x) \geq |w^* \cdot x|^{\frac{1-\alpha}{\alpha}}$



Main result: Massart noise

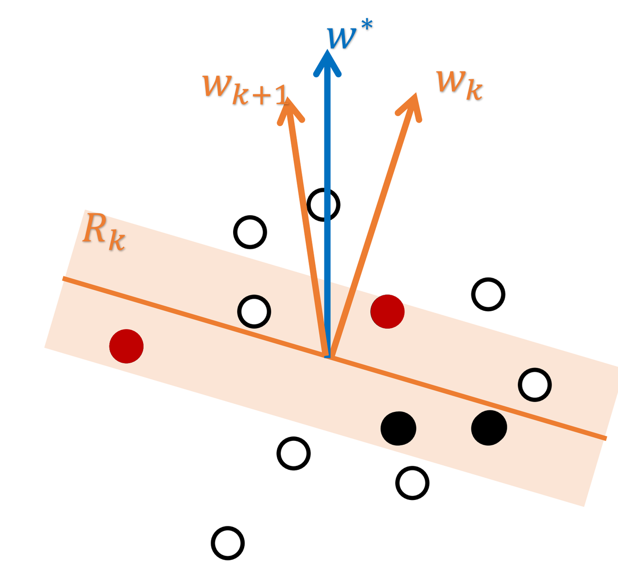
Algorithm	Efficient?	Label complexity in $\tilde{O}$
[BL13]	No	$\frac{d}{(1-2\eta)^2}\mathrm{polylog}(1/\epsilon)$
[ZSA20]	Yes	$\frac{d}{(1-2\eta)^4}\mathrm{polylog}(1/\epsilon)$
This work	Yes	$\frac{d}{(1-2\eta)^2}\mathrm{polylog}(1/\epsilon)$

Main result: Tsybakov noise

Algorithm	Efficient?	Label complexity in $\tilde{O}$
[BL13]	No	$d\left(\frac{1}{\epsilon}\right)^{2-2\alpha}$
[DKKTZ20]	Yes	$\mathrm{poly}(d)\left(\frac{1}{\epsilon}\right)^{O(1/\alpha)}$
This work ($\alpha \in \left(\frac{1}{2}, 1\right]$)	Yes	$d\left(\frac{1}{\epsilon}\right)^{\frac{2-2\alpha}{2\alpha-1}}$
This work (Geometric Tsybakov)	Yes	$d\left(\frac{1}{\epsilon}\right)^{\frac{2-2\alpha}{\alpha}}$

Algorithm skeleton

$w_1 \leftarrow \mathrm{Initialize}()$.   //Acute Initialization
In phases $k = 1,2,..,k_0 = \log(1/\epsilon)$:
    $w_{k+1} \leftarrow \mathrm{Refine}(w_k, 2^{-(k+1)})$.   // Refinement
Return w_{k_0+1}.



Refine: design challenges and related work

A series of prior works combine margin-based sampling with loss minimization techniques to design Refine:

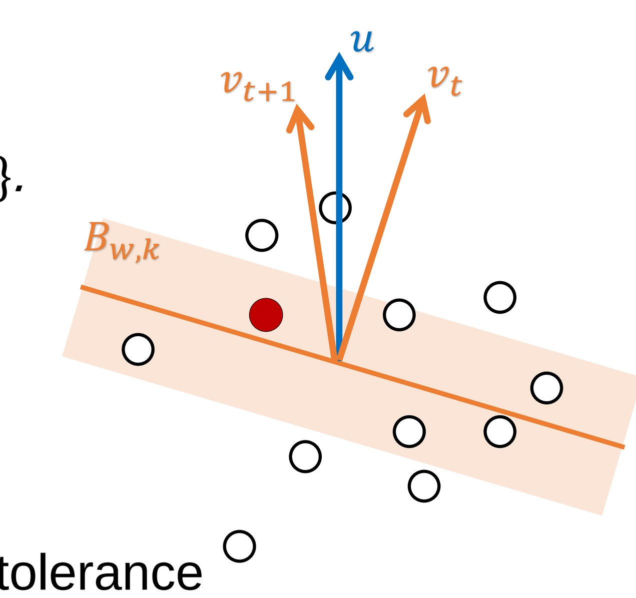
- [BL13]:computationally inefficient (0-1 loss minimization)
- [ABHU15, ABHZ16]: analysis only tolerates $\eta \leq$ small constant, or requires high label complexity
- [ZSA20]: specialized to Massart noise (needs to know η)

Refine: our design

For $t = 1,2,\ldots,T$:

1. **Sample:** $(x_t, y_t) \leftarrow$ example drawn from $D|_{B_t}$, where $B_t = \{x: |v_t \cdot x| \leq b\}$.
2. **Update:** $v_{t+1} \leftarrow v_t - \alpha g_t$, where $g_t = -y_t x_t$

Return average: $v \leftarrow \frac{1}{T}\sum_{t=1}^{T} v_t$



Key difference from [ZSA20]: simpler definition of g_t leads to broader noise tolerance

Algorithmically similar to ``nonconvex optimization'' view [GCB09, DKTZ20], but analysis very different (see next)

Analysis: key ideas

Theorem: If $\theta(v_1, w^*) \leq 2\theta$, then with high probability, $\mathrm{Refine}(v_1, \theta)$ returns a vector v with $\theta(v, w^*) \leq \theta$, if T is of order:

- $\frac{d}{(1-2\eta)^2}$, under η-Massart noise;
- $d\left(\frac{1}{\theta}\right)^{\frac{2-2\alpha}{2\alpha-1}}$, under α-Tsybakov noise with $\alpha \in \left(\frac{1}{2}, 1\right]$;
- $d\left(\frac{1}{\theta}\right)^{\frac{2-2\alpha}{\alpha}}$, under α-Geometric Tsybakov noise.

Key observation: Refine optimizes the following ``proximity function'' in a nonstandard way:

$$\psi_b(v) = \mathrm{E}\left[\left(1 - 2\eta(x)\right)|w^* \cdot x| \mid |v \cdot x| \leq b\right]$$

Idea: rewriting OGD's regret guarantees over g_t's:

$$\frac{1}{T}\sum_{t=1}^{T}\langle -w^*, g_t\rangle \leq \frac{1}{T}\sum_{t=1}^{T}\langle -v_t, g_t\rangle + O\left(\frac{1}{\sqrt{T}}\right)$$

Concentrates to $\frac{1}{T}\sum_{t=1}^{T}\psi_b(v_t)$ Can be made small by tuning b, T

The ``proximity function'' ψ_b

Lemma (simplified): For ``structured'' D, under one of the three noise conditions, $\psi_b(v)$ is lower bounded by an increasing function of $\theta(v, w^*)$.

Consequently, optimizing $\psi_b(v) \Rightarrow$ optimizing $\theta(v, w^*)$