CSC 665: Homework 3

Chicheng Zhang

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Please complete the following set of problems. **You are free to discuss with your classmates on your solutions, but only at a high level; if that is the case, please mention your collaborators.** The exercise is due **on Dec 3, 12:30pm, on Gradescope**. You are free to cite existing theorems from the textbooks and course notes.

Problem 1

In this exercise we will prove a special case of von Neumann's minimax theorem using online learning.

Theorem 1 (von Neumann's minimax theorem)**.** *For any matrix* $M \in [0,1]^{n \times n}$*,*

$$\min_{p \in \Delta^{n-1}} \max_{q \in \Delta^{n-1}} p^\top M q = \max_{q \in \Delta^{n-1}} \min_{p \in \Delta^{n-1}} p^\top M q. \tag{1}$$

1. (Optional) Show that for any function $f(x, y)$ and domains $\mathcal{X}$ and $\mathcal{Y}$, we always have

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y) \geq \min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y),$$

   and use it to conclude that the left hand side is always at least the right hand side in Equation (1).

2. Consider two players R and C (denoting Row and Column respectively) playing a repeated game of T rounds against each other. At time t, R (resp. C) selects a probability distribution of rows $p_t \in \Delta^{n-1}$ (resp. $q_t \in \Delta^{n-1}$). For each player, it is associated with a DTOL game: for R (resp. C), its loss vector at time t is defined as $\ell_{R,t} = M q_t$ (resp. $\ell_{C,t} = (\mathbf{1} - M)^\top p_t$, where $\mathbf{1}$ is the $n \times n$ matrix of all 1's). R and C applies the Hedge algorithm with learning rate $\sqrt{\frac{8 \ln N}{T}}$ on their respective loss vectors.

   (a) Write down the regret guarantees provided by Hedge for both players (your answer should be in terms of M, p_t, q_t's.)

   (b) Define $\bar{p} = \frac{1}{T} \sum_{t=1}^T p_t$ and $\bar{q} = \frac{1}{T} \sum_{t=1}^T q_t$. Show that

$$\max_{q \in \Delta^{n-1}} \bar{p}^\top M q - \min_{p \in \Delta^{n-1}} p^\top M \bar{q} \leq \sqrt{\frac{2 \ln N}{T}}, \tag{2}$$

   and use this to conclude Equation (1).

3. Suppose we have a modified rock-paper-scissor game where the game matrix M is defined as follows:

	R	P	S
R	0.5	0.7	0
P	0.2	0.5	1
S	1	0	0.5

   Write a piece of code that simulates the learning process of both players in item 2, and plot the left hand side of Equation (2) as a function of T, for $T = 10^i$, $i = 1, 2, \ldots, 6$. Use this to experimentally verify the correctness of Equation (2). What are the $\bar{p}$ and $\bar{q}$'s for each T?

Problem 2 (Optional)

Show that in realizable online classification with a finite hypothesis class $\mathcal{H} \subset (\mathcal{X} \to \{0,1\})$, if at time t, one predicts label 1 with probability $\frac{|V_t^+|}{|V_t|}$ (in other words, $\hat{y}_t = \frac{|V_t^+|}{|V_t|}$), the algorithm has a mistake bound of $\ln|\mathcal{H}|$, that is,

$$\sum_{t=1}^{T} |\hat{y}_t - y_t| \leq \ln|\mathcal{H}|.$$

Problem 3 (Optional)

Consider realizable online classification with hypothesis class $\mathrm{Ldim}(\mathcal{H}) = \infty$. If the learner is allowed to randomly predict a label at every timestep, can it achieve a finite mistake bound? Why or why not?

Problem 4 (Optional)

Show that Hedge with learning rate $\eta > 0$ has a regret as follows:

$$\sum_{t=1}^{T} \langle p_t, \ell_t \rangle - \min_{i=1}^{N} \sum_{t=1}^{T} \ell_{t,i} \leq \frac{\ln N}{\eta} + \eta \sum_{t=1}^{T} \sum_{i=1}^{N} p_{t,i} \ell_{t,i}^2.$$

You can use the fact that $e^x \leq 1 + x + x^2$ for $x \leq 1$.

(This bound has many useful applications, for example, adversarial multi-armed bandits, as we will see in the next few lectures.)