Contextual Combinatorial Cascading Bandits

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Multi-Armed Bandits Problem

- System of base arms, ***K***
- Reward of each arm ***i*** are random samples from unknown distributions with unknown means.
    - best arm $\mu^* = max\, \mu_i$ (mean $= \mu_i$)

- In each round ***t*****,** the learning agent selects one arm i_t to play and observe the reward, $R_t(i_t)$
- Regret after playing T rounds: $\text{Regret} = T\mu^* - \mathbb{E}[\sum_{t=1}^{T} R_t(i_t)]$

Goal: Minimize the Cumulative regret

Problem of MAB: dealing with trade-off between exploitation and exploration

Feedbacks

At every time step, a learning agent chooses a subset of ground items (super arm) under certain combinatorial constraints.

1. **Bandit feedback**: only reward of chosen super arm is obtained
2. **Semi-bandit feedback**: observe outcomes of the individual base arms in super arm
3. **Cascading feedback**: can obtain the reward of the super arm and the weights of some base arms in the chosen subset of arm, according to some stopping criterion.

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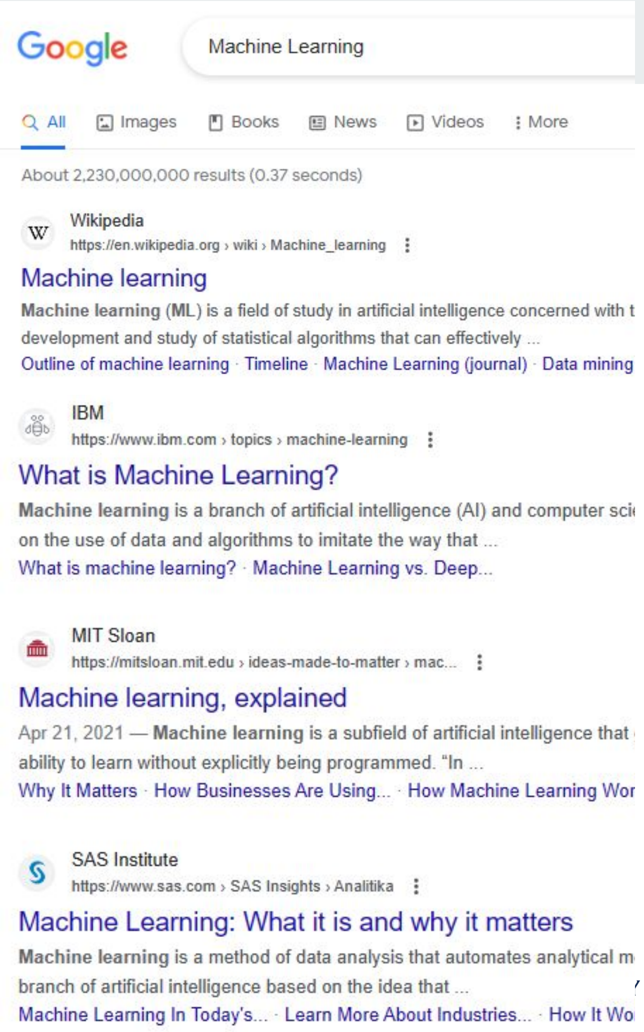
Related Work

Contextual Combinatorial Multi-Armed Bandit
• Involves Semi-bandit feedback and nonlinear reward • May have other combinatorial constraints Ref: Qin et al., 2014

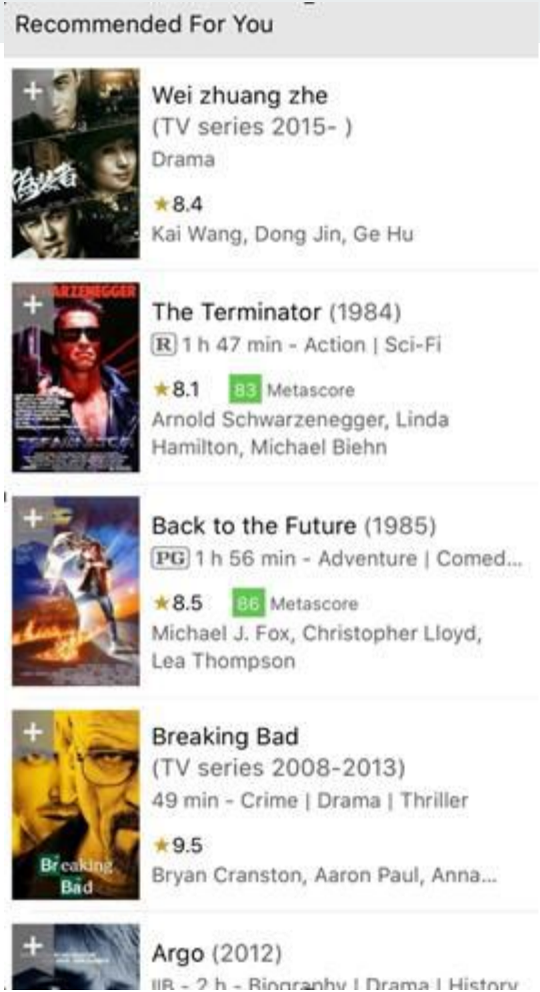
Cascading Bandit
• Feasible action is Sequential list • Stopping at first satisfactory item from the list • Feedback is recorded as Clicks • Disjunctive objective: reward of an action is 1 if there is at least one “good” item in the list Ref: Kveton et al. 2015a



Related Work - Cont.

Position Discount

- **Combes et al. 2015a** considered a **cascading model** with a particular case of **contextual information**,
- User is recommended with **K** items.
- The position discounts introduced makes the recommended set in the decreasing order of UCBs, which is more realistic
- Considered in the list order, so that the agent's reward is discounted depending on the position where the stopping criterion is met.

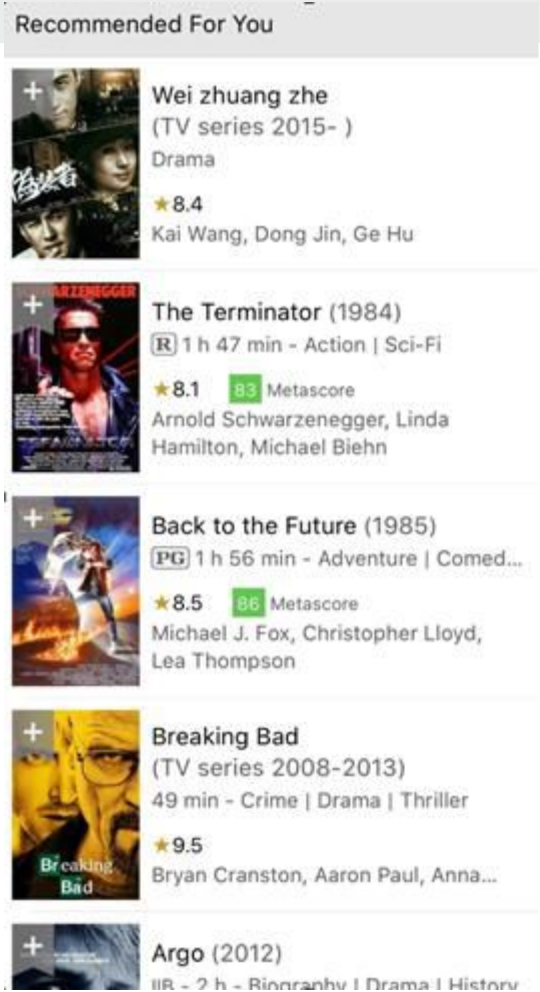


Related Work - Cont.

Combinatorial Cascading Multi-Armed Bandit

- Feasible action is subset of items under combinatorial constraints
- Can involve Semi-bandit feedback
- Conjunctive objective: reward of an action is 1 if all the items are “good” in the list
- Challenges:
    - Exponential number of actions
    - Offline optimization may already be hard

Ref: Kveton et al., 2015c



Author's Contribution

- Formulate the **Contextual Combinatorial Cascading Bandits** problem which handles
  - contextual information
  - cascading feedback
  - position discount
  - general reward function (non-linear)
- Proposed C^3 - Algorithm
- Satisfy Monotonicity and Lipschitz continuity conditions

	context	cascading	Position Discount	General Reward
Combinatorial UCB[1]	No	Yes	No	Yes
Contextual Combinatorial UCB[2]	Yes	No	No	Yes
Comb-Cascade[3]	No	Yes	No	No
C^3-UCB	Yes	Yes	Yes	Yes

Ref:
1: Chen et al. 2016
2: Qin et al., 2014
3: Kveton et al., 2015c

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Learning Protocol

- $E=\{1,\ldots,L\}$: set of base arms
- **Action** $A = (a_1,\ldots,a_k)$: $a_1,\ldots,a_k \in$ **E**; a sequence of base arms (each with length of k)
  - There is a feasible action set $\mathcal{S}$ with length at most K
- At each time $t \geq 1$
  - set of contexts $\{x_{t,a}\}_{a\in E}$ are given (e.g. user/keyword features) and is ≤ 1
  - learning agent recommends a feasible action to user, $A_t = \left(a_1^t, \ldots, a_{|A_t|}^t\right) \in \mathcal{S}$
  - Cascading Feedback Model: The user checks from the first item and stops at O_t-th item.
  - Feedback: observe weights ("quality") of first O_t items in A_t at time t, $w_t(a^t_k)$, $k \leq O_t$.

$$\mathbb{E}[w_t(a)|H_t] = \underbrace{\theta_*^T}_{\text{Fixed but unknown}} x_{t,a} = w_{t,a}$$

Learning Protocol - Cont.

- **Assume the expected reward of an action A is a function $f(A,w_t)$ of expected weight, $w_t=\{w_{t,a}\}_{a\in E} = (\theta_*^{T} x_{t,a})_{a\in E}$ of each base arm.**
- **Satisfy the following assumptions:**
  - **Monotonicity:** expected reward function $\boldsymbol{f(A,w)}$ is a non-decreasing with respect to **w**: for any $w,w' \in [0, 1]^E$, if $w(a) \le w'(a)$, we have $f(A,w) \le f(A,w')$.
  - **Lipschitz continuity:** The expected reward function $\boldsymbol{f(A,w)}$ is B-Lipschitz continuous with respect to $\boldsymbol{w}$ together with position discount parameters $\gamma_k \in [0,1]$, $k \le K$. For any $w,w' \in [0, 1]^E$, we h

$$|f(A,w) - f(A,w')| \le B \sum_{k=1}^{|A|} \gamma_k \, |w(a_k) - w'(a_k)|$$

where $A = (a_1, \ldots, a_{|A|})$.

Learning Protocol - Cont.

- An Oracle $\mathcal{O}_{\mathcal{S}}(\boldsymbol{w})$ is called an α-approximation oracle for some $\alpha \leq 1$, if on given input $\boldsymbol{w}$, the oracle returns an action $\mathbf{A} = \mathcal{O}_{\mathcal{S}}(\boldsymbol{w}) \in \mathcal{S}$ satisfying $\boldsymbol{f(A,w)} \geq \boldsymbol{\alpha}\, \boldsymbol{f(A^*,w)}$ where $\boldsymbol{A} = \mathbf{argmax}_{\mathbf{A}\in\mathcal{S}}\, \boldsymbol{f(A,w)}$
- Regret in $\boldsymbol{T}$ rounds

$$\textbf{Regret} = \underbrace{\sum_{t=1}^{T} f_t^*}_{\text{Best cumulative reward}} - \mathbb{E}\left[\sum_{t=1}^{T} f(A_t, w_t)\right]$$

f_t^* :max expected reward in round t

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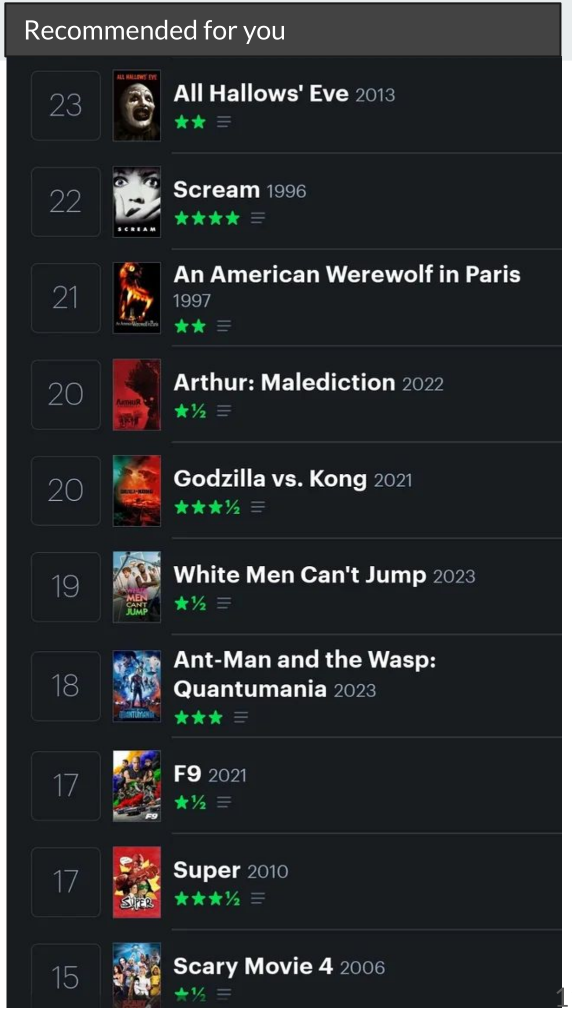
Example - Movie Recommendation

- Each Movie i has a feature vector m_i.
- At time t,
  - A random user comes with feature vector $\mathbf{u}_t$
  - Use $\boldsymbol{x}_{i,t} = \mathbf{g}(\mathbf{m}_i, \mathbf{u}_t)$, a function of $\mathbf{m}_i$ and $\mathbf{u}_t$, as context
  - The learning agent recommends a list of movies $\boldsymbol{A}_t$.
  - The user checks from the first movie and stops at the attractive one.
  - The learning agent received reward γ_k if the user stops at position k.

    $$1 = \gamma_k \geq \ldots \geq \gamma_k \geq 0$$



Algorithm

- To estimate the expected reward, we get an estimate $\boldsymbol{\theta}_*$ using ridge regression problem on context vector $\boldsymbol{x}$ and observed rewards $\boldsymbol{w}$.
- We get an l^2 **-regularized least-squares estimation** of $\boldsymbol{\theta}_*$ with **regularization parameter, λ > 0**

$$\hat{\theta}_t = \left(X_t^T X_t + \lambda I\right)^{-1} X_t^T Y_t$$

Where $X_t \in \mathbb{R}^{(\sum_{s=1}^{t} O_s).d}$: matrix with rows as $\gamma_k \, x_{s,a_k^s}^T$ and Y_t: column vector with elements as $\gamma_k \, w_s(a_k^s), k \in [O_s], s \in [t]$

- Let V_t is a symmetric positive definite matrix, $\in R^{d \times d}$

$$V_t = X_t^T X_t + \lambda I = \lambda I + \sum_{s=1}^{t} \sum_{k=1}^{O_s} \gamma_k^2 \, x_{s,a_k^s} \, x_{s,a_k^s}^T$$

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Algorithm

- **Lemma 1**: $\beta_t(\delta) = R\sqrt{ln(\frac{det(V_t)}{\lambda^d \delta^2})} + \sqrt{\lambda}$ **. For any δ>0, with probability at least 1-δ, for all t>0, we have $||\hat{\theta}_t - \theta_* ||_{V_t} \leq B_t(\delta)$**

Proof: By theorem 2 in (Abbasi-Yadkori et al., 2011), we can obtain a good estimate of difference between θ_* and $\hat{\theta}_t$.

Interpretation: With high probability, the estimate $\hat{\theta}$ lies in the ellipsoid centered at θ_* with confidence radius $\beta_t(\delta)$ under V_t norm.

- The upper confidence bound (UCB) of the expected weight for each base arms:

$$U_t(a) = min\left\{\hat{\theta}_{t-1}^T x_{t,a} + \beta_{t-1}(\delta)\left||x_{t,a}\right||_{V_{t-1}^{-1}}, 1\right\}$$

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- **Lemma 2**: When $||\boldsymbol{\hat{\theta}}_t - \boldsymbol{\theta}_* ||_{Vt} \leq \mathbf{B}_t\ (\boldsymbol{\delta})$ holds for time t-1, we have

$$0 \leq U_t(a) - w_{t,a} \leq 2\beta_{t-1}(\delta)||x_{t,a}||_{V_{t-1}^{-1}}$$

Proof: Note $w_{t,a} = (\theta_*^T x_{t,a})$

By Holder's inequality

$$\begin{aligned}
|\hat{\theta}_{t-1}^T x_{t,a} - \theta_*^T x_{t,a}| &= |[V_{t-1}^{1/2}(\hat{\theta}_{t-1}^T - \theta_*]^T (V_{t-1}^{-1/2} x_{t,a})| \\
&\leq ||V_{t-1}^{1/2}(\hat{\theta}_{t-1} - \theta_*||_2 ||V_{t-1}^{-1/2} x_{t,a}||_2 \\
&= ||\hat{\theta}_{t-1} - \theta_*||_{V_{t-1}} ||x_{t,a}||_{V_{t-1}^{-1}} \\
&\leq \beta_{t-1}(\delta)||x_{t,a}||_{V_{t-1}^{-1}}
\end{aligned}$$

Because $1 - \theta_*^T x_{t,a} \geq 0$ and

$$\begin{aligned}
0 &\leq (\hat{\theta}_{t-1}^T x_{t,a} + \beta_{t-1}(\delta)||x_{t,a}||_{V_{t-1}^{-1}}) - \theta_*^T x_{t,a} \\
&\leq 2\beta_{t-1}(\delta)||x_{t,a}||_{V_{t-1}^{-1}}
\end{aligned}$$

Claimed result is obtained…

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Algorithm

1. Learning agent computer (UCBs), $\boldsymbol{U_t} \in [0,1]^E$ for expected weights of all base arms in E.

2. Uses $\boldsymbol{U_t}$ to select an action $\boldsymbol{A}=(a^t_1,\ldots,a^t_{|A_t|})$

3. User selects from the first base arm in $\boldsymbol{A_t}$ and stops at $\boldsymbol{O_t}$-th base arm,

Agent observes $\boldsymbol{w_t}(a^t_k)$, $k \le O_t$

4. **Learning agent updates $\boldsymbol{V_t}$, X_t Y_t to get newer estimates $\hat{\boldsymbol{\theta}}_t$ and $\boldsymbol{\theta}_*$ and new confidence radius, $\boldsymbol{\beta_t}(\delta)$**

Algorithm 1 C^3-UCB

```
Parameters:
    {γ_k ∈ [0,1]}_{k≤K}; δ = 1/√n; λ ≥ C_γ = Σ_{k=1}^K γ_k^2
Initialization:
    θ̂_0 = 0, β_0(δ) = 1, V_0 = λI, X_0 = ∅, Y_0 = ∅
for all t = 1, 2, ..., n do
    Obtain context x_{t,a} for all a ∈ E                                    (1)
    ∀a ∈ E, compute
      U_t(a) = min{θ̂_{t-1}^T x_{t,a} + β_{t-1}(δ)||x_{t,a}||_{V_{t-1}^{-1}}, 1}
    //Choose action A_t using UCBs U_t                                      (2)
    A_t = (a_1^t, ..., a_{|A_t|}^t) ← O_S(U_t)
    Play A_t and observe O_t, w_t(a_k^t), k ∈ [O_t]                          (3)
    //Update statistics                                                     (4)
    V_t ← V_{t-1} + Σ_{k=1}^{O_t} γ_k^2 x_{t,a_k^t} x_{t,a_k^t}^T
    X_t ← [X_{t-1}; γ_1 x_{t,a_1^t}^T; ...; γ_{O_t} x_{t,a_{O_t}^t}^T]
    Y_t ← [Y_{t-1}; γ_1 w_t(a_1^t); ...; γ_{O_t} w_t(a_{O_t}^t)]
    θ̂_t ← (X_t^T X_t + λI)^{-1} X_t^T Y_t
    β_t(δ) ← R√(ln(det(V_t)/(λ^d δ^2))) + √λ
end for t
```

Result

Theorem: Suppose the expected reward function ***f(A,w)*** is a function of expected weights and satisfies the requirements of monotonicity and B-Lipschitz continuity. Then the α-regret of **C^3-UCB** algorithm, satisfies

$$Regret^{\alpha}(n) = O(\frac{dB}{p^*}\sqrt{TKln(T)})$$

Proof:

$$R^{\alpha}(n) \leq \frac{2\sqrt{2}B}{p^*}\sqrt{nKdln(1+C_{\gamma}n/(\lambda d))}$$
$$.(R\sqrt{ln[(1+C_{\gamma}n/(\lambda d))^d n]}+\sqrt{\lambda})+\alpha\sqrt{n}$$
$$= O(\frac{dBR}{p^*}\sqrt{nKln(C_{\gamma}n)})$$

$\mathbf{p_{t,A}}$ be the probability of full observation of A.

$\mathbf{p^* = \min_{1 \leq t \leq T} \min_{A \in S} p_{t,A}}$: Minimal probability that action has all base arms observed all time.

d**:** dimension of latent and feature vectors;

K**:** largest length of the sequence

n = T (number of rounds)

R is the sub-Gaussian constant

$C\gamma \leq K$ (sum of discounts $\leq$ number of positions)

$$C_{\gamma} = \sum_{k=1}^{K} \gamma_k^2 \leq K$$

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Experiment 1

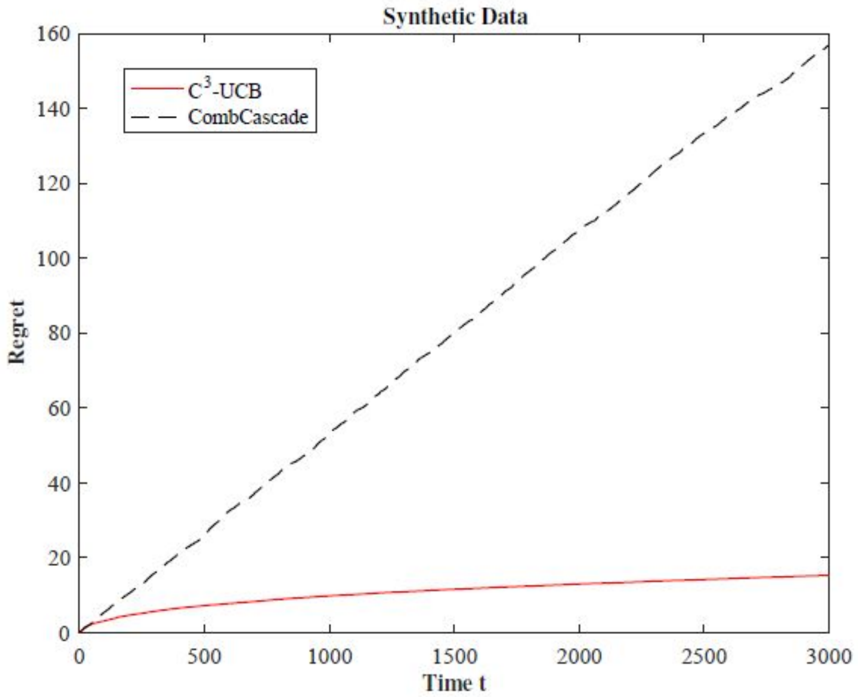
<u>Synthetic Data:</u> compare C^3-UCB to CombCascade (Combinatorial Cascading) on synthetic problems.

Setup:

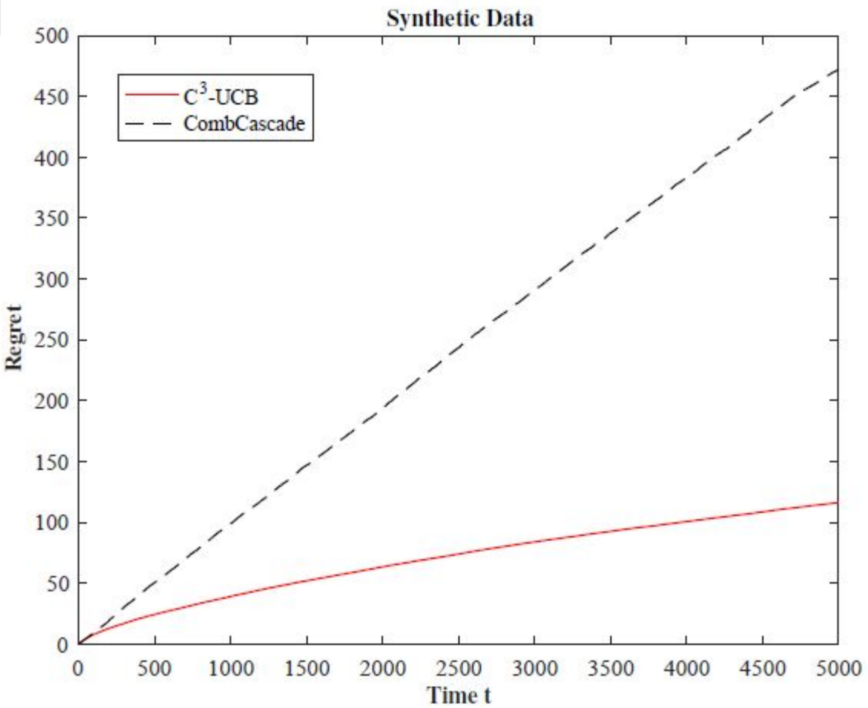
1. The problem is a contextual cascading bandit with L = 200 items and K = 4, d = 20 where at each time t.
2. The agent recommends K items to the user.
3. At first, we randomly choose a $\boldsymbol{\theta}$ in R^{d-1} with $\|\boldsymbol{\theta}\|_2 = \mathbf{1}$ and let $\boldsymbol{\theta}_* = (\boldsymbol{\theta}/2, 1/2)$.
4. Then at each time t, we randomly assign $\mathbf{x}'_{t,a}$ in $\mathbf{R}^{d-1}$ with $\|\mathbf{x}'_{t,a}\|_2 = \mathbf{1}$ to arm **a** and use $\mathbf{x}_{t,a} = (\mathbf{x}'_{t,a}, \mathbf{1})$ to be the contextual information for arm a.

This processing will guarantee the inner product $\theta_*^\top x_{t,a} = \frac{1}{2}(\theta^\top x'_{t,a} + 1) \in [0, 1]$

Next we generate the weight for arm a at time t by a random sample from the Bernoulli distribution with mean $\boldsymbol{\theta}_*^{\mathrm{T}} \boldsymbol{x}_{t,a}$

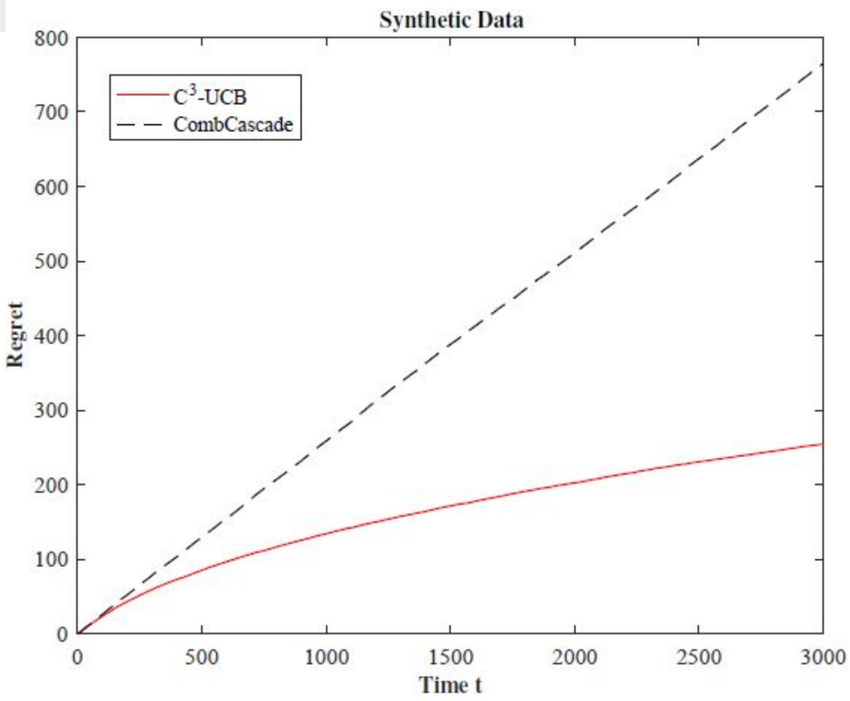


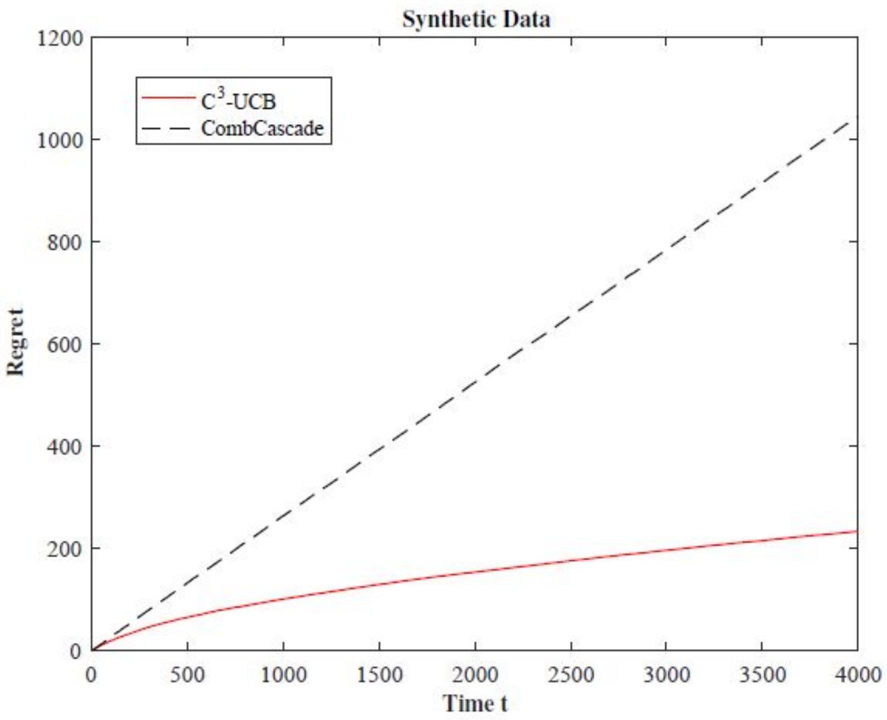
Disjunctive, $\gamma = 1$, 9.77%



Disjunctive, $\gamma = 0.9^{k-1}$, 24.6%

The above two settings are of disjunctive objective where the learning agent chooses a set of K items out of L ground items and observes a prefix of the chosen K items until the first one with weight 1;





Conjunctive, $\gamma = 1$, 3.33%

Conjunctive, $\gamma = 0.9^{k-1}$, 22.4%

The above two settings are of conjunctive objective where the learning agent chooses a set of K items out of L ground items and observes from the first item until the first one with weight 0.

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Experiment 2

Movie Recommendation: evaluate C3-UCB algorithm with dataset MovieLens (Lam & Herlocker, 2015) of 2015

Problem:

1. There is a big sparse matrix $A \in \{0, 1\}^{N1 \times N2}$ where **A(i; j) = 1 denotes user i has watched movie j**.
2. Split A as H + F according to a Bernoulli distribution ~Ber(p) for some fixed p. (H : known history "what users have watched" and F as future criterion.)
3. Derive **feature vectors** of both users and movies via SVD, $\mathbf{U = (u_1, . . ., u_{N1}) \text{ and } M = (m_1, . . ., m_{N2})}$.
4. At every time t, we randomly choose a user $I_t \in [N_1]$. From Li et al., 2010, use $\mathbf{x_{t;j} = u_{I_t} m^T_j}$ as the contextual information for each movie **j**.
5. User provides cascading feedback, stopping at a movie they like and Agent receives discounted reward based on stop position.
6. The real weight of movie **j** at time t, $\mathbf{w_t(j)}$**, is** $\mathbf{F(I_t, m_j)}$.

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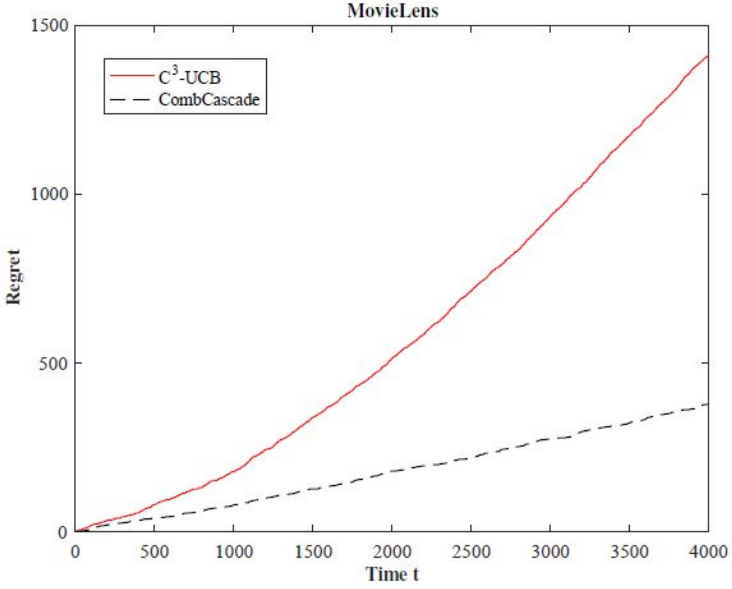
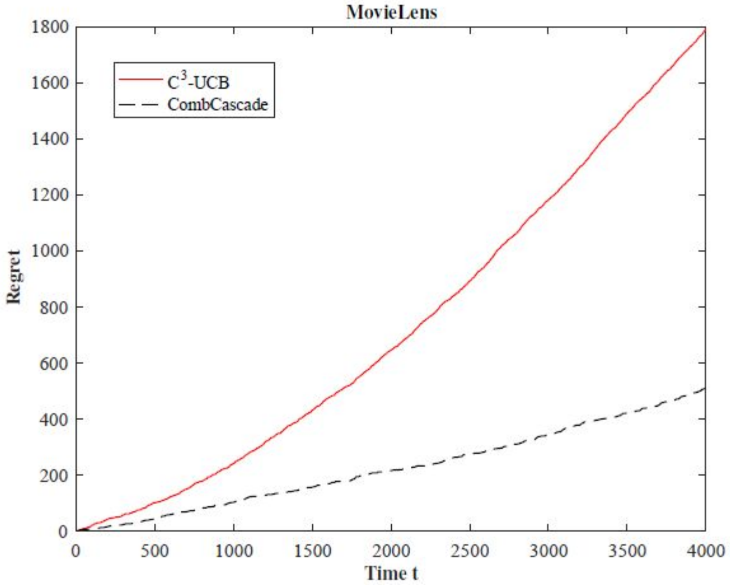
Setup:

1. The problem is a contextual cascading bandit with L = 400 movies and K = 4, d = 400 where at each time t.
2. We experiment with both $\gamma = 1$ (no position discount) and $\gamma = 0{:}9$, and compare our algorithm with CombCascade.



γ=1



γ=0.9

Result: The rewards of our algorithms are 3.52 and 3.736 times of those of CombCascade (for $\gamma = 1$ and 0:9, respectively), which demonstrate the advantage to involve contextual information in real applications.

Experiment 3

Network Routing Problem: evaluate C^3-UCB with RocketFuel dataset (Spring et al., 2004).

Setup:

1. The ground set E is the set of links in the network.
2. Before learning, the environment randomly chooses a d-dimensional vector $\boldsymbol{\theta} \in [0, 1]^d$ **(d = 5)**
3. At each time t, a pair of source and destination nodes are randomly chosen
4. The feasible action set $\mathbf{S_t}$ at time t contains all simple paths, paths without cycles, between the source and destination.
5. Any edge a in the set $\mathbf{S_t}$ is assigned with a random d-dimensional contextual information vector $\mathbf{x_{t,a}}$.

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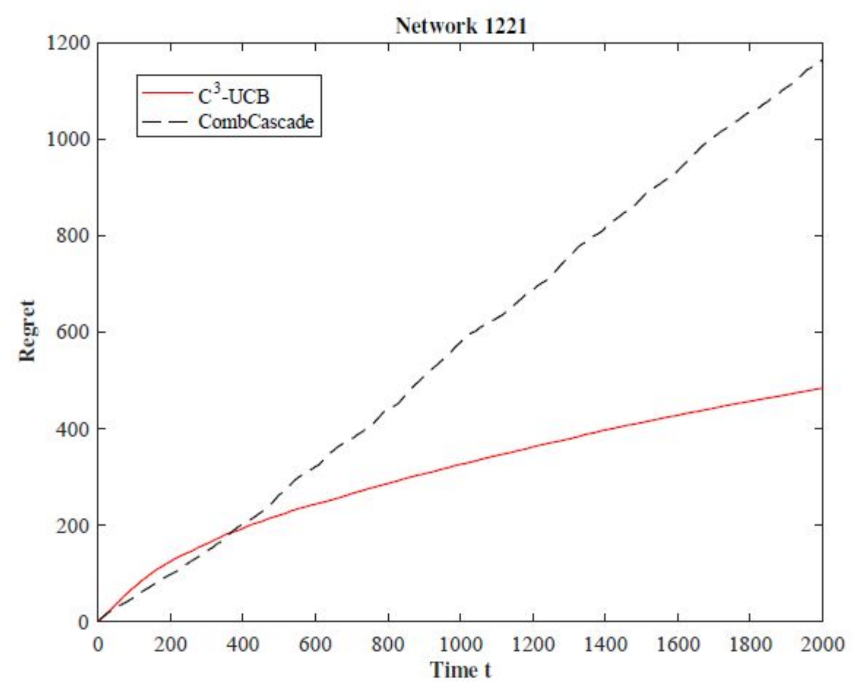
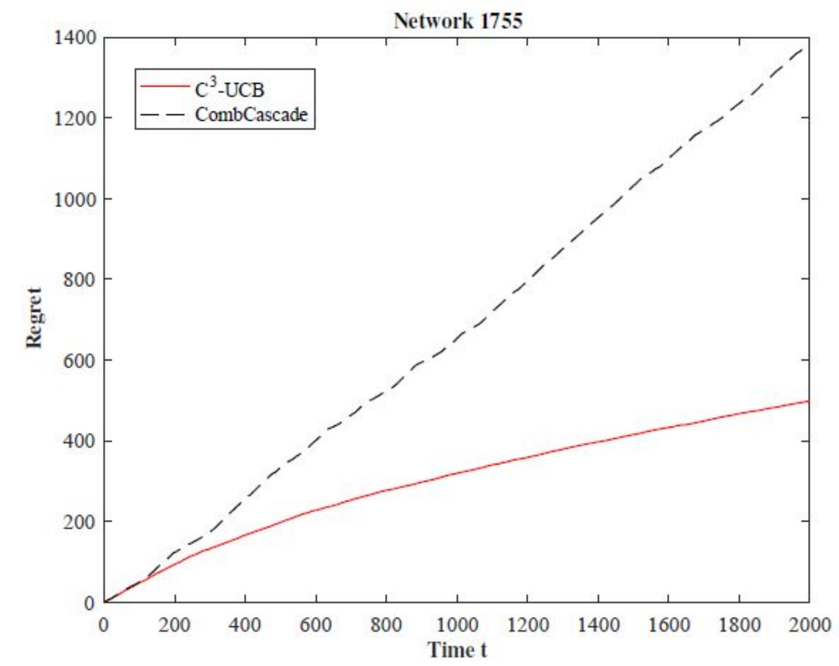


Network 1221, γ=1



Network 1221, γ=0.9

Result: Both θ and x have been processed same as experiment 1, such that $\boldsymbol{\theta_*}^{\mathbf{T}} \boldsymbol{x} \in [0, 1]$. The weight for each edge a is a sample from Bernoulli distribution with mean $\boldsymbol{\theta_*}^{\mathbf{T}} \boldsymbol{x_{t,a}}$. Then the learning agent recommends a feasible path A to maximize the expected reward in the conjunctive objective. This experiment is on different position discounts.

Conclusion:

- Incorporating contextual information to cascading bandit with position discounts
- Each action is an ordered list and only a prefix of the action is observed each time.
- demonstrate the advantage to involve contextual information and position discounts.
- Application potential
    - Any sequential list recommendation (search, ads, mobile recommendations)
        - Need online (real time) feedback

Future Work:

- investigate on lower bounds of the regret and cascading on general graphs

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