Math 365/Comp 365: Homework 3

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Problem 1

In class, we showed how to construct a sequence of unit lower-triangular matrices $\{L_k\}_{k=1,2,\ldots,n-1}$ **of the form**

$$L_k = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & -\ell_{k+1,k} & 1 & & \\ & & \vdots & & \ddots & \\ & & -\ell_{n,k} & & & 1 \end{pmatrix}, \text{ where } \ell_{j,k} = \frac{A_{j,k}}{A_{k,k}}, \quad \text{for } k \le j \le n,$$

such that

$$L_{n-1}L_{n-2}\ldots L_2L_1A = U,$$

with U **being an upper-triangular matrix. We now want to show in two steps that**

$$L := L_1^{-1}L_2^{-1}\ldots L_{n-2}^{-1}L_{n-1}^{-1}$$

is a unit lower-triangular matrix.

(a) Show that

$$L_k^{-1} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \ell_{k+1,k} & 1 & & \\ & & \vdots & & \ddots & \\ & & \ell_{n,k} & & & 1 \end{pmatrix}.$$

Hint: **Start by showing that** $L_k = I - \ell_k e_k^\top$**, where** e_k **is an** $n \times 1$ **vector with a 1 in the** k^{th} **entry, and zeros elsewhere, and**

$$\ell_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \ell_{k+1,k} \\ \vdots \\ \ell_{n,k} \end{pmatrix}.$$

Solution:

$$\begin{aligned}
&L_k = I - \ell_k e_k^\top \\
&L_k(I - \ell_k e_k^\top) = (I - \ell_k e_k^\top)(I + \ell_k e_k^\top) \\
&= I + \ell_k e_k^\top - \ell_k e_k^\top - \ell_k e_k^\top \ell_k e_k^\top \\
&\because e_k^\top \ell_k = 0 \\
&\therefore L_k \left(I + \ell_k e_k^\top\right) = I \\
&L_k^{-1} = I + \ell_k e_k^\top
\end{aligned}$$

$$= \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \ell_{k+1,k} & 1 & & \\ & & \ddots & & \ddots & \\ & & \ell_{n,k} & & & 1 \end{pmatrix}$$

(b) Now show that

$$L := L_1^{-1} L_2^{-1} \ldots L_{n-2}^{-1} L_{n-1}^{-1} = \begin{pmatrix} 1 & & & & \\ \ell_{21} & 1 & & & \\ \ell_{31} & \ell_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{n,n-1} & 1 \end{pmatrix}$$

Solution:

$$\because \forall k \le j \le n, \quad e_k^\top \ell_j = 0$$

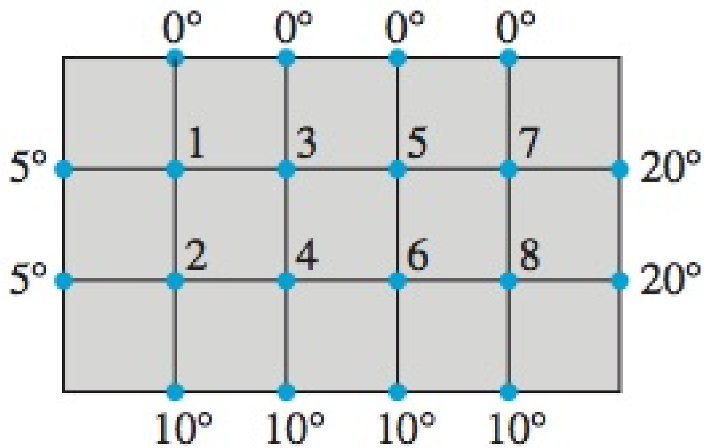
$$\begin{aligned}
\therefore L_k^{-1} L_{k+1}^{-1} &= \left(I + \ell_k e_k^\top\right)\left(I + \ell_{k+1} e_{k+1}^\top\right) = I + \ell_k e_k^\top + \ell_{k+1} e_{k+1}^\top + \ell_k e_k^\top \ell_{k+1} e_{k+1}^\top \\
&= I + \ell_k e_k^\top + \ell_{k+1} e_{k+1}^\top \\
L_k^{-1} L_{k+1}^{-1} L_{k+2}^{-1} &= \left(I + \ell_k e_k^\top + \ell_{k+1} e_{k+1}^\top\right)\left(I + \ell_{k+2} e_{k+2}^\top\right) \\
&= I + l_k e_k^\top + l_{k+1} e_{k+1}^\top + l_{k+2}^\top k_{k+2}^\top + 0 \\
&\ldots\ldots
\end{aligned}$$

$$\therefore L := L_1^{-1} L_2^{-1} \ldots L_{n-2}^{-1} L_{n-1}^{-1} = I + \ell_1 e_1^\top + \ell_2 e_2^\top + \ldots + \ell_{n-1} e_{n-1}^\top = \begin{pmatrix} 1 & & & \\ \ell_{31} & \ell_{32} & 1 & \\ \vdots & \vdots & \ddots & \ddots \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{n,n-1} \end{pmatrix}$$

Problem 2

This was Exercise 2 on the *LU* Activity

This problem is from Section 2.5, page 131 of Linear Algebra and Its Applications, by David Lay, the textbook many of you used for MATH 236.



An important concern in the study of heat transfer is to determine the steady-state temperature distribution of a thin plate when the temperature around the boundary is known. Assume the plate shown in the figure above represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let the variables $x_1, x_2, \ldots, x_8$ denote the temperatures at nodes 1 through 8 in the picture. In steady state, the temperature at a node is approximately equal to the average of the four nearest nodes (to the left, above, right, below).

a) The solution to the approximate steady-state heat flow problem for this plate can be written as a system of linear equations $Ax = b$, where $x = [x_1, x_2, \ldots, x_8]$ is the vector of temperatures at nodes 1 through 8. Find the 8×8 matrix A and the vector b. Hint: A should be a banded matrix with many zeros in the top right and bottom left parts of A.

Solution:

$$x_1 = 1/4(0 + 5 + x_3 + x_2)$$
$$x_2 = 1/4(10 + 5 + x_1 + x_4)$$
$$x_3 = 1/4(0 + x_1 + x_4 + x_5)$$
$$x_4 = 1/4(10 + x_6 + x_2 + x_3)$$
$$x_5 = 1/4(0 + x_7 + x_3 + x_6)$$
$$x_6 = 1/4(10 + x_5 + x_4 + x_8)$$
$$x_7 = 1/4(0 + 20 + x_5 + x_8)$$
$$x_8 = 1/4(10 + 20 + x_6 + x_7)$$

$$x_1 \begin{bmatrix} 4 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 4 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 4 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ -1 \\ 4 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 4 \\ -1 \\ -1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 4 \\ 0 \\ -1 \end{bmatrix} + x_7 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 4 \\ -1 \end{bmatrix} + x_8 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \\ 0 \\ 10 \\ 0 \\ 10 \\ 20 \\ 30 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 4 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 5 \\ 15 \\ 0 \\ 10 \\ 0 \\ 10 \\ 20 \\ 30 \end{bmatrix}$$

b) Use your function from Exercise 1 (on the activity) to perform an LU factorization of A. Do you notice anything special about the structures of L and U?

Solution:

```
myLU = function(A,tol=10^-20) {
  n = nrow(A)
  L = diag(x=1,nrow=n)
  U=A
  for ( k in 1:(n-1) ) {
    pivot = U[k,k]
    if (abs(pivot) < tol) stop('zero pivot encountered')
    for ( j in (k+1):n ) {
      mult =  U[j,k]/pivot
      U[j,] = U[j,] - mult * U[k,]
      L[j,k] = mult
    }
  }
  return(list(L=L,U=U))
}
```

```
A=cbind(c(4,-1,-1,0,0,0,0,0), c(-1,4,0,-1,0,0,0,0),c(-1,0,4,-1,-1,0,0,0),
        c(0,-1,-1,4,0,-1,0,0),c(0,0,-1,0,4,-1,-1,0),c(0,0,0,-1,-1,4,0,-1),
        c(0,0,0,0,-1,0,4,-1),c(0,0,0,0,0,-1,-1,4))
(out=myLU(A))
```

```
## $L
##       [,1]        [,2]       [,3]        [,4]       [,5]        [,6]
## [1,]  1.00  0.00000000  0.0000000  0.00000000  0.0000000  0.00000000
## [2,] -0.25  1.00000000  0.0000000  0.00000000  0.0000000  0.00000000
## [3,] -0.25 -0.06666667  1.0000000  0.00000000  0.0000000  0.00000000
## [4,]  0.00 -0.26666667 -0.2857143  1.00000000  0.0000000  0.00000000
## [5,]  0.00  0.00000000 -0.2678571 -0.08333333  1.0000000  0.00000000
## [6,]  0.00  0.00000000  0.0000000 -0.29166667 -0.2921348  1.00000000
## [7,]  0.00  0.00000000  0.0000000  0.00000000 -0.2696629 -0.08612836
## [8,]  0.00  0.00000000  0.0000000  0.00000000  0.0000000 -0.29482402
##             [,7] [,8]
## [1,]  0.0000000    0
## [2,]  0.0000000    0
## [3,]  0.0000000    0
## [4,]  0.0000000    0
## [5,]  0.0000000    0
## [6,]  0.0000000    0
## [7,]  1.0000000    0
## [8,] -0.2931381    1
##
## $U
```

```
##      [,1]  [,2]      [,3]      [,4]       [,5]      [,6]       [,7]
## [1,]    4 -1.00 -1.000000  0.000000  0.0000000  0.000000  0.0000000
## [2,]    0  3.75 -0.250000 -1.000000  0.0000000  0.000000  0.0000000
## [3,]    0  0.00  3.733333 -1.066667 -1.0000000  0.000000  0.0000000
## [4,]    0  0.00  0.000000  3.428571 -0.2857143 -1.000000  0.0000000
## [5,]    0  0.00  0.000000  0.000000  3.7083333 -1.083333 -1.0000000
## [6,]    0  0.00  0.000000  0.000000  0.0000000  3.391854 -0.2921348
## [7,]    0  0.00  0.000000  0.000000  0.0000000  0.000000  3.7051760
## [8,]    0  0.00  0.000000  0.000000  0.0000000  0.000000  0.0000000
##           [,8]
## [1,]  0.000000
## [2,]  0.000000
## [3,]  0.000000
## [4,]  0.000000
## [5,]  0.000000
## [6,] -1.000000
## [7,] -1.086128
## [8,]  3.386790
```

L and U are banded matrices with many zeros in the top right and bottom left parts of L and U.

c) Once you have an LU factorization for a matrix A, you need to do the two-step procedure to complete the back substitution to solve $Ax = b$. Here is code for that:

```
mySolve=function(L,U,b,tol=1e-10){

  n=nrow(L)

  # First solve Ly=b
  y = rep(0,n)         # pre-allocate a vector y with 0s in it.
  if(abs(L[1,1])<tol) stop('There is a zero on the diagonal of L')
  y[1] = b[1]/L[1,1]  # Fill in the 1st value of y
  for (j in 2:n ) {
    if(abs(L[j,j])<tol) stop('There is a zero on the diagonal of L')
    y[j] = (b[j] - L[j,1:(j-1)]%*%y[1:(j-1)])/L[j,j]
  }

  # Then solve Ux=y
  x = rep(0,n)         # pre-allocate a vector x with 0s in it.
  if(abs(U[n,n])<tol) stop('There is a zero on the diagonal of U')
  x[n] = y[n]/U[n,n]  # Fill in the nth value of x
  for (j in (n-1):1 ) {
    if(abs(U[j,j])<tol) stop('There is a zero on the diagonal of U')
    x[j] = (y[j] - U[j,(j+1):n]%*%x[(j+1):n])/U[j,j]
  }
  return(x)
}
```

Make sure you understand what the code is doing, and then use it to find the steady-state temperatures at nodes 1 through 8.

Solution:

```
b=c(5,15,0,10,0,10,20,30)
(x=mySolve(out$L,out$U,b))
```

```
## [1]  3.956938  6.588517  4.239234  7.397129  5.602871  8.760766  9.411483
```

```
## [8] 12.043062
```

d) The temperature on the right-hand side of the plate was measured incorrectly. It is actually 30 degrees. Find the new steady-state temperatures. Hint: you should not do another LU factorization!

Solution

```
b[7]=30
b[8]=40
(x=mySolve(out$L,out$U,b))
```

```
## [1]  4.138756  6.770335  4.784689  7.942584  7.057416 10.215311 13.229665
## [8] 15.861244
```

e) Use the R function `solve(A)` to compute A^{-1}. Note that A^{-1} is a dense matrix (without many zeros). When A is large, L and U can be stored in much less space than A^{-1}. This fact is another reason for preferring the LU factorization of A to A^{-1} itself.

Solution:

```
solve(A)
```

```
##             [,1]        [,2]       [,3]       [,4]       [,5]       [,6]
## [1,] 0.295264808 0.086553374 0.09450586 0.05094869 0.03180993 0.02273552
## [2,] 0.086553374 0.295264808 0.05094869 0.09450586 0.02273552 0.03180993
## [3,] 0.094505857 0.050948688 0.32707474 0.10928890 0.10450421 0.05913216
## [4,] 0.050948688 0.094505857 0.10928890 0.32707474 0.05913216 0.10450421
## [5,] 0.031809932 0.022735522 0.10450421 0.05913216 0.32707474 0.10928890
## [6,] 0.022735522 0.031809932 0.05913216 0.10450421 0.10928890 0.32707474
## [7,] 0.009998350 0.008183468 0.03180993 0.02273552 0.09450586 0.05094869
## [8,] 0.008183468 0.009998350 0.02273552 0.03180993 0.05094869 0.09450586
##             [,7]        [,8]
## [1,] 0.009998350 0.008183468
## [2,] 0.008183468 0.009998350
## [3,] 0.031809932 0.022735522
## [4,] 0.022735522 0.031809932
## [5,] 0.094505857 0.050948688
## [6,] 0.050948688 0.094505857
## [7,] 0.295264808 0.086553374
## [8,] 0.086553374 0.295264808
```

Problem 3

This problem is taken from the in-class portion of an old midterm.

You are trying to solve a linear system of four equations:

$$\begin{bmatrix} & & \\ & A & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 4 \\ 2.25 \end{bmatrix},$$

but unfortunately A is a black box and you cannot see the contents. Oh my! Luckily, from the physics of the problem, you do have three pieces of information about the system and the solution:

- $||x||_\infty = 2$
- **The condition number of A, using the ∞-norm, is 32**

- $\begin{bmatrix} & & \\ & A & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 2.0750 \\ 0.1625 \\ 0.4500 \\ -0.1875 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 4 \\ 2.1875 \end{bmatrix}$

Is this information enough to determine the solution x to $Ax = b$? If yes, find x. If no, say as much as you can about x.

Solution:

$$Cond(A) = ||A||_\infty ||A^{-1}||_\infty = 32 = \max \frac{\text{relative forward error}}{\text{relative backward error}} \geq \frac{\frac{\|x-x_a\|_\infty}{\|x\|_\infty}}{\frac{\|Ax_a-b\|_\infty}{\|b\|_\infty}} \geq \frac{\frac{\|x-x_a\|_\infty}{2}}{\frac{|2.25-2.1875|}{4}}$$

```
32*(2.25-2.1875)/2
```

```
## [1] 1
```

$$\therefore ||x - x_a||_\infty \leq 1$$

$$\because ||x||_\infty = 2 = \max\{|\bar{x}|\} \text{ and } x_a = \begin{bmatrix} 2.0750 \\ 0.1625 \\ 0.4500 \\ -0.1875 \end{bmatrix}$$

$$\therefore |x_i - x_{ai}| \leq 1 \; i = 1, 2, 3, 4$$

$$\therefore |x_1| = 2$$

$$\begin{aligned} & -0.8375 \leq x_2 \leq 1.1625 \\ \text{and:} \quad & -0.55 \leq x_3 \leq 1.45 \\ & -1.1875 \leq x_4 \leq 0.8125 \end{aligned}$$

I believe this is what we can find for x

Problem 4

A real $n \times n$ matrix Q is *orthonormal* if

$$Q^\top Q = QQ^T = I; \; i.e., Q^{-1} = Q^T.$$

Note: sometimes you will see these matrices just called *orthogonal matrices*. The analagous matrices that contain complex entries ($Q^*Q = QQ^* = I$, where the * is a conjugate transpose) are called *unitary*.

Show that $||Q||_2 = ||Q^{-1}||_2 = 1$, and therefore the 2-norm condition number of any orthonormal matrix Q is $\kappa_2(Q) = ||Q||_2||Q^{-1}||_2 = 1$.

Proof:

$$\begin{aligned} \|Q\|_2 &= \max_{\|x\|\neq 0} \frac{\|Qx\|_2}{\|x\|_2} = \max_{\|x\|\neq 0} (\frac{\|Qx\|_2^2}{\|x\|_2^2})^{1/2} = \max_{\|x\|\neq 0} (\frac{(Qx)^\top Qx}{x^\top x})^{1/2} = \max_{\|x\|\neq 0} (\frac{x^\top Q^\top Qx}{x^\top x})^{1/2} \\ &= \max_{\|x\|\neq 0} (\frac{x^\top x}{x^\top x})^{1/2} = \max_{\|x\|\neq 0} (\frac{x^\top QQ^\top x}{x^\top x})^{1/2} = \max_{\|x\|\neq 0} (\frac{x^\top (Q^{-1})^\top Q^{-1}x}{x^\top x})^{1/2} \\ &= \max_{\|x\|\neq 0} (\frac{(Q^{-1}x)^\top Q^{-1}x}{x^\top x})^{1/2} = \max_{\|x\|\neq 0} \frac{\|Q^{-1}x\|_2}{\|x\|_2} = \|Q^{-1}\|_2 = 1 \end{aligned}$$

$$\therefore ||Q||_2 = ||Q^{-1}||_2 = 1$$
$$\therefore \kappa_2(Q) = ||Q||_2||Q^{-1}||_2 = 1$$

Problem 5

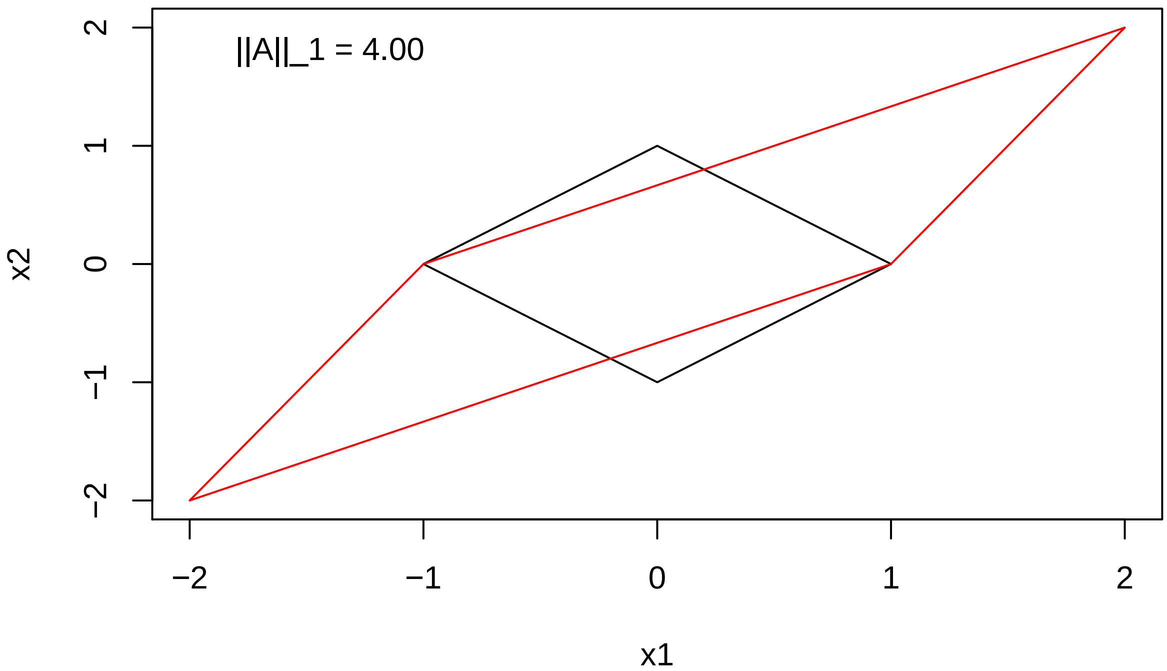
This problem has 3 parts.

(a) Modify my `UnitCircleMap` function, which shows the image of the unit ball under a linear mapping A and computes the matrix norm of A. We want to let the user choose the norm to be 1, 2, or ∞ ("I"). The code is below, and there are a few places where you need to insert a few lines.

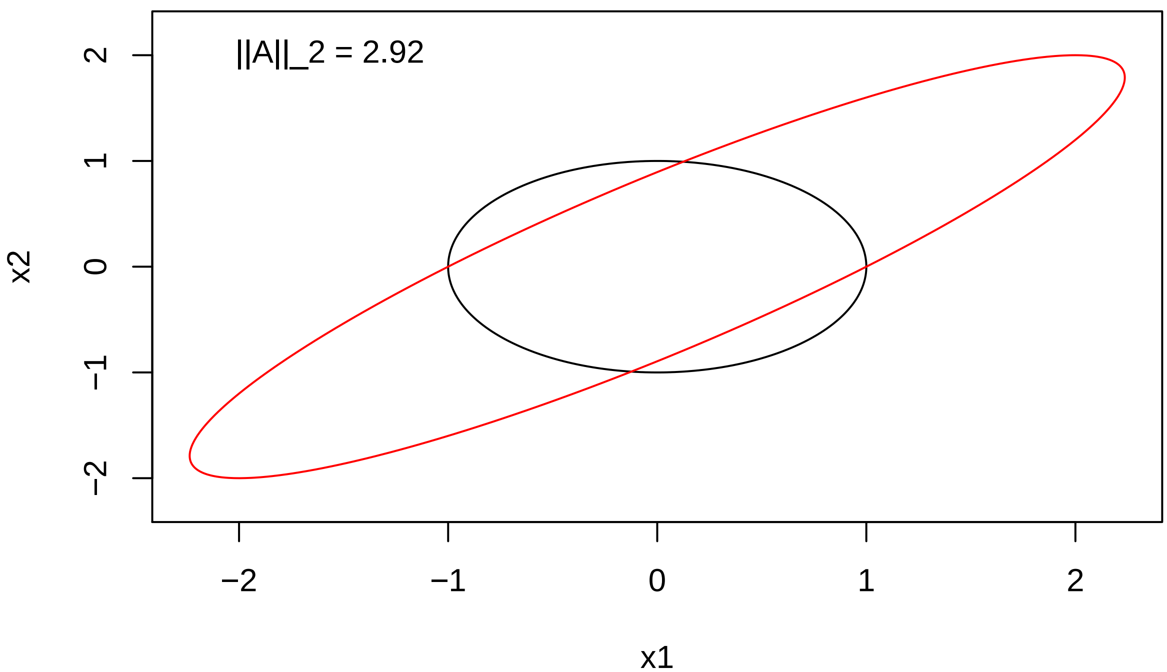
```
UnitCircleMap = function(A,p=2, name="A") {

  if (p==2) {
    t = seq(0,2*pi,len=1000)
    x = cos(t)
    y = sin(t)
    nn=norm(A,type="2")
    ni=as.character(p)
  }
  else if (p==1){
    # Insert some code here to create points along the unit 1-norm circle
    x=c(seq(1,0,len=500),seq(0,-1,len=500),seq(-1,0,len=500),seq(0,1,len=500))
    y=c(seq(0,1,len=500),seq(1,0,len=500),seq(0,-1,len=500),seq(-1,0,len=500))
    nn=norm(A,type="1")
    ni=as.character(p)
  }
  else if (p=="I"){
    # Insert some code here to create points along the unit infinity-norm circle
    x=c(rep(1,500),seq(1,-1,len=500),rep(-1,500),seq(-1,1,len=500))
    y=c(seq(-1,1,len=500),rep(1,500),seq(1,-1,len=500),rep(-1,500))
    nn=norm(A,type="I")
    ni="Inf"
  }

  pts = A %*% t(cbind(x,y))

  newx = pts[1,]
  newy = pts[2,]

  M = max(c(newx,newy,1.5))
  m = min(c(newx,newy,-1.5))

  plot(x,y,type='l',col='black',xlim=c(m,M),ylim=c(m,M),xlab="x1",ylab="x2")
  lines(newx,newy,col='red')

  normSize=sprintf("||%s||_%s = %1.2f",name,ni,nn)
  text(-.7*M,.9*M,normSize)
}
```

Here is an example of how the function should be called, and the output it should display. In this case, $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$. Once you've filled in the missing lines of the function, you can also test your code on this A with the 1 norm and ∞-norm.

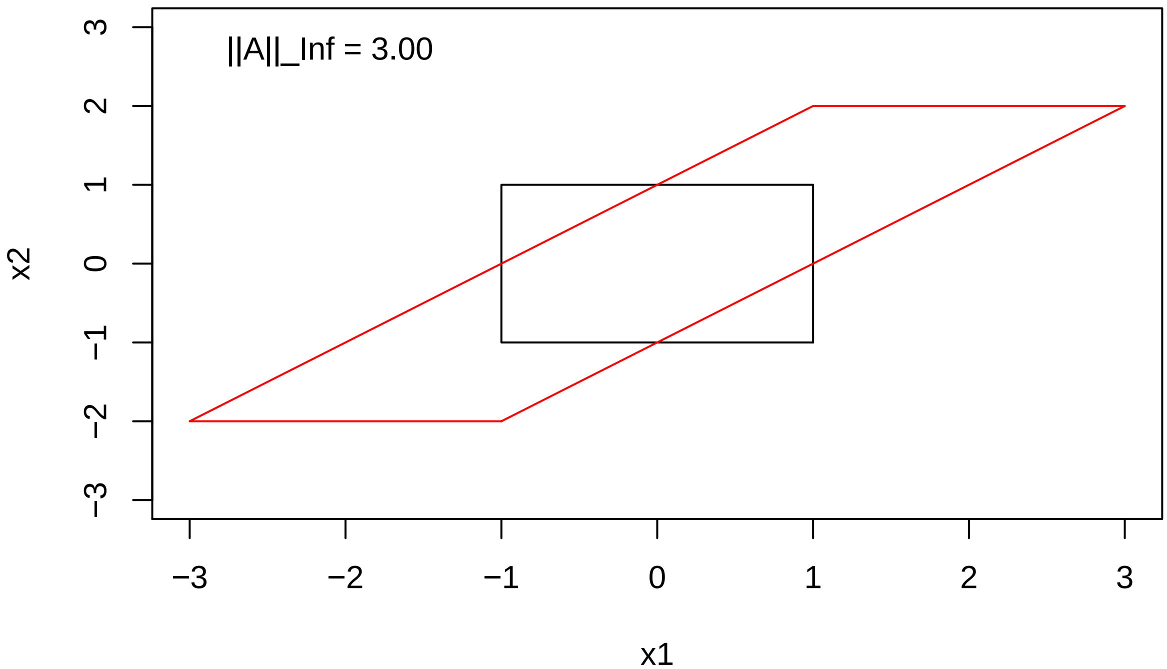
```
A=cbind(c(1,0),c(2,2))
UnitCircleMap(A,p=1)
```




```
UnitCircleMap(A,p=2)
```




```
UnitCircleMap(A,p='I')
```



(b) Compute the condition number of the matrix A above using `Cond` (using the $p = 1, 2, \infty$-norms). Visually explain why the condition number for each p is the value you found.

Solution:

```
Cond = function(A,p=2) {
  if (p == 2) {  # by default use the
    s = svd(A)$d
    s = s[s>0]
    return(max(s)/min(s))
  }
  if (p == 1) {  # use the 1 norm
    Ainv = solve(A)
    return(max(colSums(abs(A)))*max(colSums(abs(Ainv))))
  }
  if (p == 'I') {   # use the infinity norm
    Ainv = solve(A)
    return(max(rowSums(abs(A)))*max(rowSums(abs(Ainv))))
  }
}
```

```
A=cbind(c(1,0),c(2,2))
Cond(A,1)
```

```
## [1] 6
```

```
Cond(A,2)
```

```
## [1] 4.265564
```

```
Cond(A,"I")
```

```
## [1] 6
```

```
Ainv = solve(A)
norm(Ainv,"1")
```

```
## [1] 1.5
```

```
norm(Ainv,"2")
```

```
## [1] 1.460405
```

```
norm(Ainv,"I")
```

```
## [1] 2
```

$$\text{Cond }(A) = \|A\|_p \cdot \left\|A^{-1}\right\|_p = \frac{\left(\max_{\|x\|_p=1} \|A\vec{x}\|_p\right)}{\left(\min_{\|x\|_p=1} \|A\vec{x}\|_p\right)}$$

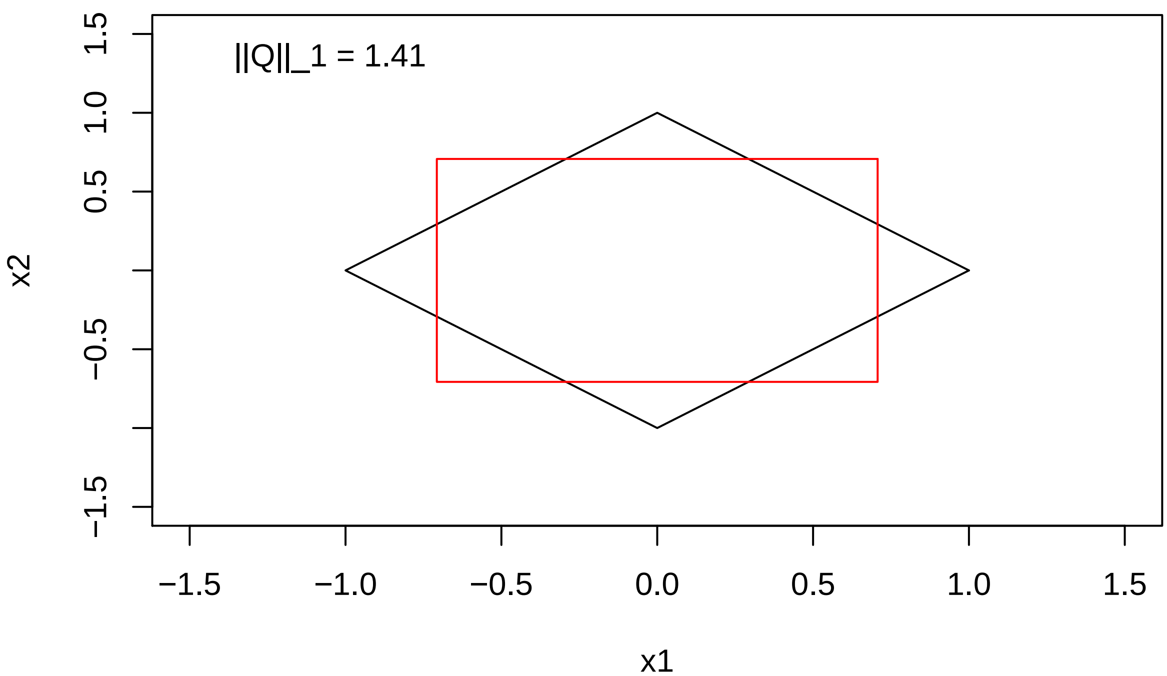
The product of the norm of matrix A and inverse of A is the condition number. Geometrically, it's the longest vector from the origin over the shortest. When $p = 1$, which is the sum of absolute of cordinate geometrically, the norm of A could be got by $2 + 2 = 4$, where (2,2) is the coridinate of the farest points from the origin. And $\|A^{-1}\|_1$ equals to $\frac{1}{2/3}$, which is the sum of vector closest to the origin. Similarly, when $p = 2$, which is the length of vector geometrically, it's the longest length over the shortest one, which is 2.92 over 1/1.46 approximately equals to 4.265564 ($\|A\|_2 = 2.92, \|A^{-1}\|_2 = 1.46$). And when $p = \infty$, it's the largest absolute value of cordinates, so the condition number is $3/(1/2) = 6$ ($\|A\|_\infty = 3, \|A^{-1}\|_\infty = 1/2$)

(c) Use your code to determine whether $||Q||_1 = ||Q||_\infty = ||Q||_2 = 1$ for all orthonormal matrices Q.

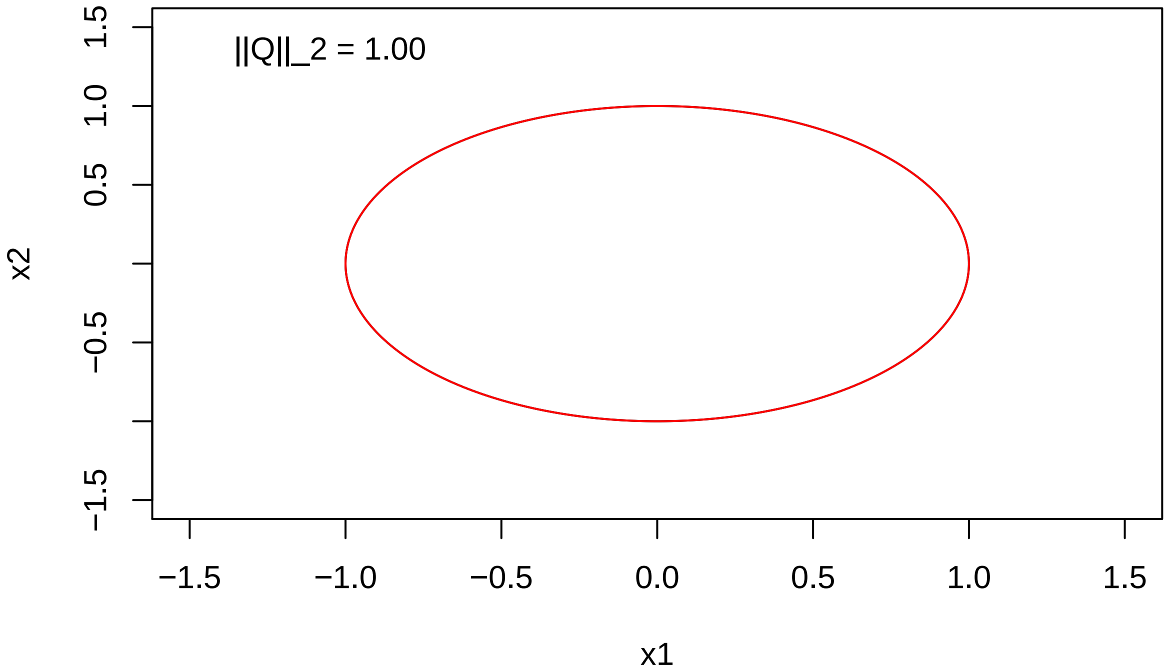
Hint: Try $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, which is a rotation matrix that rotates vectors counter-clockwise by $\frac{\pi}{4}$.

Solution:

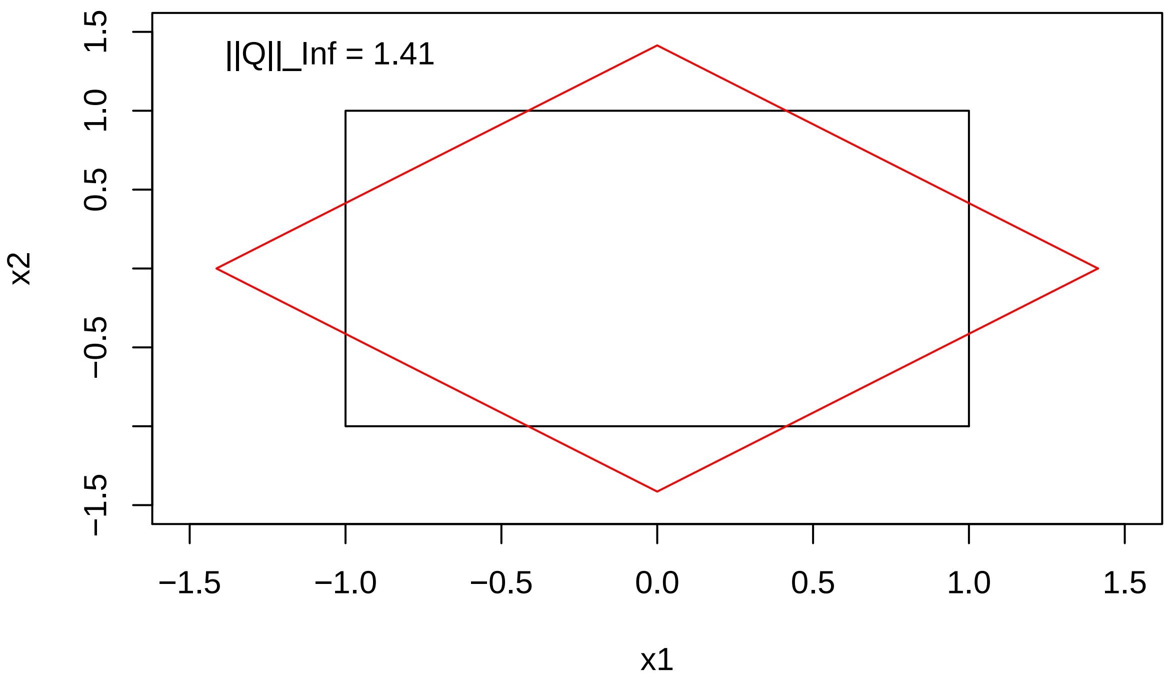
```
Q = cbind(c(1/sqrt(2),1/sqrt(2)),c(-1/sqrt(2),1/sqrt(2)))
UnitCircleMap(Q,p=1, name="Q")
```




```
UnitCircleMap(Q,p=2, name="Q")
```




```r
UnitCircleMap(Q,p="I", name="Q")
```



Therefore, they are not necessarily equal to 1.

Problem 6

This was Question 5 on the Activity on norms.

Prove that

$$||A||_\infty = \max_{1\leq i\leq n}\left\{\sum_{j=1}^{n}|A_{ij}|\right\} = \text{maximum absolute row sum.}$$

Proof:

$$\because ||x||_\infty = \max_{1\le i\le n}\{|\bar{x}_i|\}$$

$$\|A\|_\infty = \max_{\|x\|_\infty=1}\|Ax\|_\infty = \max_{\|x\|_\infty=1}\left\{\max_{1\le i\le n}\left\{\left|\sum_{i=1}^{n} A_{ij}x_j\right|\right\}\right\} \leqslant \max_{\|x\|_\infty=1}\max_{1\le i\le n}\sum_{j=1}^{n}|A_{ij}|\,|x_j|$$

$$\leqslant \max_{1\le j\le n}\sum_{i=1}^{n}|A_{ij}| = \text{max absolute row sum}$$

$$\text{Assume } \bar{y}\in\mathbb{R}^n,\ \text{and } y_j = \begin{cases} 1 & \text{if } A_{kj}\ge 0 \\ -1 & \text{if } A_{kj}<0 \end{cases}\ \text{for } 1\le i\le n$$

$$\because y \subseteq \{x|\|x\|_\infty = 1\}$$

$$\therefore \|A\|_\infty = \max_{\|x\|=1}\|Ax\|_\infty \geqslant \|Ay\|_\infty$$

$$\geqslant \max_{1\le i\le n}\left|\sum_{j=1}^{n}A_{ij}y_j\right| = \max_{1\le i\le n}\sum_{j=1}^{n}|A_{ij}| = \text{max absolute row sum}$$

$$\therefore \|A\|_\infty \le \max_{1\le i\le n}\sum_{j=1}^{n}|A_{ij}|,\ \text{and}\ \|A\|_\infty \ge \max_{1\le i\le n}\sum_{j=1}^{n}|A_{ij}|$$

$$\therefore \|A\|_\infty = \text{max absolute row sum}$$

Problem 7

Solve the following system by finding the $PA = LU$ factorization and then carrying out the two-step back substitution (all by hand, that is, not using any R functions. Of course, you can type your solution, and check your work in R, though.):

$$Ax = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 5 \end{pmatrix} = b.$$

Solution:

$$U = A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix},\ P = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{exchange the first two rows: } U = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix},\ P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix} \frac{row2 = \frac{1}{3}row1 + row2}{row3 = row3 - \frac{2}{3}row1} \begin{pmatrix} 3 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{4}{3} \\ 0 & -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ \frac{2}{3} & 0 & 1 \end{pmatrix}$$

$$\text{exchange the last two rows: } U = \begin{pmatrix} 3 & 1 & 1 \\ 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{4}{3} \end{pmatrix} \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix}$$

$$row3 = \frac{1}{2}row2 + row3 : U = \begin{pmatrix} 3 & 1 & 1 \\ 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{3}{2} \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{2} & 1 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$Ly = Pb:$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ -\frac{1}{3} & -\frac{1}{2} & 1 \end{pmatrix} y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 5 \end{pmatrix}$$

$$\therefore y = \begin{pmatrix} 5 \\ \frac{5}{3} \\ \frac{9}{2} \end{pmatrix}$$

$$Ux = y:$$

$$\begin{pmatrix} 3 & 1 & 1 \\ 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{3}{2} \end{pmatrix} x = \begin{pmatrix} 5 \\ \frac{5}{3} \\ \frac{9}{2} \end{pmatrix} \therefore x = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

```r
# Test
P = cbind(c(0,1,0),c(1,0,0),c(0,0,1))
A = rbind(c(-1,0,1),c(3,1,1),c(2,0,1))
L = cbind(c(1,2/3,-1/3),c(0,1,-1/2),c(0,0,1))
U = cbind(c(3,0,0),c(1,-2/3,0),c(1,1/3,3/2))
P %*% A
```

```
##      [,1] [,2] [,3]
## [1,]    3    1    1
## [2,]   -1    0    1
## [3,]    2    0    1
```

```r
L %*% U
```

```
##      [,1] [,2] [,3]
## [1,]    3    1    1
## [2,]    2    0    1
## [3,]   -1    0    1
```

```r
b = c(2,5,5)
solve(A, b)
```

```
## [1]  1 -1  3
```