Geometric Measure Theory Research

Daily Report
in reverse chronological order

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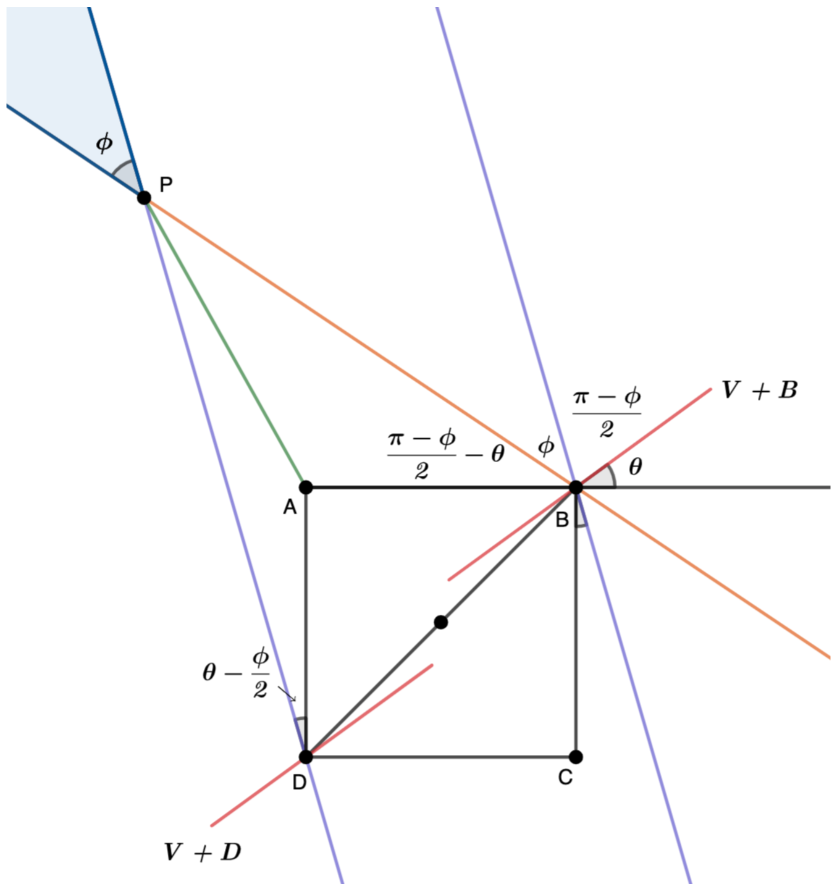
1 Research

1.1 July 13 Distance between Bad Cone at a Cube to the Cube



Lemma 1.1.1 *For a dyadic cube $Q \in \mathbb{R}^2$, $V \in G(1,2)$, $\alpha \in (0,1)$, assume that the inclination angle of V is θ, then* $\operatorname{dist}(C^2_{\mathcal{B}}(Q,V,\alpha),Q) = s \cdot \operatorname{side} Q$ *where s is*

$$\sqrt{\left(\alpha+\sqrt{1-\alpha^2}\right)|\cos\theta-\sin\theta|\left(\frac{\left(\alpha+\sqrt{1-\alpha^2}\right)|\cos\theta-\sin\theta|}{4\alpha^2(1-\alpha^2)}-\frac{\sin\theta}{\alpha}-\frac{\cos\theta}{\sqrt{1-\alpha^2}}\right)+1}$$

and $\operatorname{dist}(C^2_{\mathcal{B}}(Q,V,\alpha),\operatorname{center} Q) = s' \cdot \operatorname{side} Q$ *where s' is*

$$\sqrt{\left(\alpha+\sqrt{1-\alpha^2}\right)|\cos\theta-\sin\theta|\left(\frac{\left(\alpha+\sqrt{1-\alpha^2}\right)|\cos\theta-\sin\theta|}{4\alpha^2(1-\alpha^2)}-\frac{\cos\theta+\sin\theta}{2\alpha}-\frac{\cos\theta-\sin\theta}{2\sqrt{1-\alpha^2}}\right)+\frac{1}{2}}$$

Proof. For convenience, assume that A,B,C,D are four corners of Q(note that point $A,B,C \notin Q$, but exist in the cube system and ∂Q), P is one of vertices of $C^2_{\mathcal{B}}(Q,V,\alpha)$, ϕ is the aperture of $Q^2_{\mathcal{B}}(Q,V,\alpha)$. Then for any $y \in \partial C_{\mathcal{B}}(B,V,\alpha)$, $\operatorname{dist}(y-B,V) = \alpha|y-B| =$

$\sin\frac{\pi-\phi}{2}|y-B|$, so $\phi = \pi - 2\arcsin\alpha$. Note that $\angle PBA = (\pi-\phi)/2 - \theta = \arcsin\alpha - \theta$, $\angle PDA = \pi/2 - ((\pi-\phi)/2 - \theta) - \phi = \theta - \phi/2 = \theta + \arcsin\alpha - \pi/2$. By the *Law of Sines*,

$$\begin{aligned}
|P-D| &= \frac{|B-D|}{\sin\angle BPD}\cdot\sin\angle PBD \\
&= \frac{\sqrt{2}\sin(\arcsin(\alpha)-\theta+\pi/4)}{\sin(2\arcsin\alpha)}\cdot\operatorname{side}Q \\
&= \frac{\left(\alpha+\sqrt{1-\alpha^2}\right)|\cos\theta-\sin\theta|}{2\alpha\sqrt{1-\alpha^2}}\cdot\operatorname{side}Q
\end{aligned}$$

Then applying the *Law of Cosines*, denoting $|P-D|$ as $t\cdot\operatorname{side}Q$,

$$\begin{aligned}
\operatorname{dist}(C^2_{\mathcal{B}}(Q,V,\alpha),Q) &= |P-A| \\
&= \sqrt{|P-D|^2+|A-D|^2-2|P-D||A-D|\cos\angle ADP} \\
&= \sqrt{t^2+1-2t\cdot\cos(\theta+\arcsin\alpha-\pi/2)}\cdot\operatorname{side}Q \\
&= \sqrt{t^2+1-2t\left(\sqrt{1-\alpha^2}\sin\theta+\alpha\cos\theta\right)}\cdot\operatorname{side}Q \\
&= s\cdot\operatorname{side}Q
\end{aligned}$$

Similarly,

$$\begin{aligned}
\operatorname{dist}(C^2_{\mathcal{B}}(Q,V,\alpha),Q),\operatorname{center}Q) &= |P-\operatorname{center}Q| \\
&= \sqrt{\frac{1}{4}|B-D|^2+|P-D|^2-|B-D||P-D|\cos\angle PDB} \\
&= \sqrt{\frac{1}{2}+t^2-\sqrt{2}t\cos(\theta+\arcsin\alpha-\pi/4)}\cdot\operatorname{side}Q \\
&= \sqrt{\frac{1}{2}+t^2-t\left(\sqrt{1-\alpha^2}(\cos\theta+\sin\theta)+\alpha(\cos\theta-\sin\theta)\right)}\cdot\operatorname{side}Q \\
&= s'\cdot\operatorname{side}Q
\end{aligned}$$

1.2 July 8 Complete and Modify our Lemma

Corollary 1.2.1 (**[Nap20, Corollary 7.1]**) *Let μ be a Radon measure on H, V be an m-dimensional linear plane in H, $\alpha \in (0,1)$, and $0 < r < \infty$. If for μ-a.e. $x \in H$*

$$\mu\left(C_{\mathcal{B}}(x, r, V, \alpha)\right) = 0$$

then μ is carried by m-Lipschitz graphs.

Lemma 1.2.1 *For $0 < \alpha_2 < \alpha_1 < 1$, $V \in G(1,2)$, a dyadic cube Q in $\mathbb{R}^2$, and $s \geq \frac{3\sqrt{2}(1+\alpha_2)}{\alpha_1 - \alpha_2}$, a dilation cube $2R$ such that R is the dyadic cube with the same side length as Q and*

$$\operatorname{dist}(Q, 2R) \geq s \cdot \operatorname{side} Q$$

then

(i) if $2R \subset C^{2,1}_{\mathcal{B}}(Q, V, \alpha_1)$, then $2R \subset C^{2,2}_{\mathcal{B}}(Q, V, \alpha_2)$.

(ii) if for $n = n(s) \in \mathbb{N}$ such that $2^n \geq 2(\sqrt{2}+1)s + 3\sqrt{2}$, then $2sQ \subset 2^n R$.

(iii) if for $m = m(s) \in \mathbb{N}$ such that $2^m \geq (3\sqrt{2}+2)s + 3\sqrt{2}$, then $3sQ \subset 2^m R$.

Proof.

(i) When $2R \subset C^{2,1}_{\mathcal{B}}(Q, V, \alpha_1)$, there exists some $r \in 2R \cap C^1_{\mathcal{B}}(Q, V, \alpha_1)$ and hence for any $q \in Q$ such that $r \in C_{\mathcal{B}}(q, V, \alpha_1)$, so $\operatorname{dist}(r-q, V) > \alpha_1|r-q|$. Besides, arbitrary $r' \in 2R$ and $q' \in Q$ must satisfy $|r - r'| \leq 2\sqrt{2} \cdot \operatorname{side} R$ and $|q - q'| \leq \sqrt{2} \cdot \operatorname{side} Q$, respectively. Assume that $|r - q| = s' \cdot \operatorname{side} Q$ then $s' \geq s$. Since $|r - r'| \geq |\operatorname{dist}(r' - q', V) - \operatorname{dist}(r - q, V)| - |q - q'|$, no matter $\operatorname{dist}(r' - q', V) \leq \operatorname{dist}(r - q, V)$ or $\operatorname{dist}(r' - q', V) \geq \operatorname{dist}(r - q, V)$,

$$\begin{aligned}
\operatorname{dist}(r' - q', V) &\geq \operatorname{dist}(r - q, V) - |q - q'| - |r - r'| \\
&> \alpha_1|r - q| - 3\sqrt{2} \cdot \operatorname{side} Q \\
&\geq (\alpha_1 s' - 3\sqrt{2}) \cdot \operatorname{side} Q \\
&\geq \alpha_2(s' + 3\sqrt{2}) \cdot \operatorname{side} Q & \left(s' \geq s \geq \frac{3\sqrt{2}(1+\alpha_2)}{\alpha_1 - \alpha_2}\right) \\
&\geq \alpha_2(|r - q| + |r - r'| + |q - q'|) \\
&\geq \alpha_2|r' - q'|
\end{aligned}$$

Therefore, $2R \subset \{R : R \subset \bigcap_{q' \in Q} C_{\mathcal{B}}(q', V, \alpha)\} = C^{2,2}_{\mathcal{B}}(Q, V, \alpha_2)$.

(ii) Assume for all $q'' \in 2sQ$, then

$$\begin{aligned}
|\operatorname{center} 2R - q''| &\leq \left(\frac{1}{2}\operatorname{diam} 2R + \operatorname{dist}(2R, Q) + \frac{1}{2}\operatorname{diam} Q + \frac{1}{2}\operatorname{diam} 2sQ\right) \\
&\leq (\sqrt{2} + s + \frac{\sqrt{2}}{2} + \sqrt{2}s) \operatorname{side} Q \\
&\leq \frac{2^n}{2}\operatorname{side} R = \frac{1}{2}\operatorname{side} 2^n R
\end{aligned}$$

which implies that $2sQ \subset B(\text{center } R, \frac{1}{2}\,\text{side } 2^n R) \subset 2^n R$.

(iii) Similarly, assume $\forall q''' \in 3sQ$, then

$$\begin{aligned}
|\,\text{center } 2R - q'''| &\le (\frac{1}{2}\,\text{diam } 2R + \text{dist}(2R, Q) + \frac{1}{2}\,\text{diam } Q + \frac{1}{2}\,\text{diam } 3sQ) \\
&\le (\sqrt{2} + s + \frac{\sqrt{2}}{2} + \frac{3}{2}\sqrt{2}s)\,\text{side } Q \\
&\le \frac{2^m}{2}\,\text{side } R = \frac{1}{2}\,\text{side } 2^m R
\end{aligned}$$

which implies that $3sQ \subset B(\text{center } R, \frac{1}{2}\,\text{side } 2^m R) \subset 2^m R$.

Lemma 1.2.2 *If there exist $K \ge 1$, such that for μ-a.e. $x \in \mathbb{R}^2, 0 < r < \infty$,*

$$\mu(B(x, 2r)) \le K\mu(B(x, r)) \tag{1}$$

then for any dyadic cube Q containing x where (1) holds, $s, n \in \mathbb{N}, n \ge 2$, $s \in \mathbb{N}$

$$\mu(2^n sQ) \le K^{3n-3}\mu(2sQ) \tag{2}$$

Proof. By the half open definition of the dyadic cube, for μ-a.e. $x \in \mathbb{R}^2$, x must be in only one cube Q. Let $r = s \cdot \text{side}\, Q/2$. We claim that $2^n sQ \subset B(x, 2^{3n-3}r)$ and $B(x, r) \subset 2sQ$. For any $q \in 2^n sQ$, $|\,\text{center}\, Q - x| \le \sqrt{2}/2 \cdot \text{side}\, Q$ and $|\,\text{center}\, 2^n sQ - q| \le \sqrt{2}/2 \cdot \text{side}\, 2^n sQ = 2^{n-1}\sqrt{2}s \cdot \text{side}\, Q$. Note that $n \ge 2$, so

$$|x - q| \le \left(\frac{\sqrt{2}}{2} + 2^{n-1}\sqrt{2}s\right)\text{side}\, Q < \frac{2^{3n-3}s}{2}\,\text{side}\, Q = 2^{3n-3}r$$

Then $q \in B(x, 2^{3n-3}r)$ and thus $2^n sQ \subset B(x, 2^{3n-3}r)$. Consider that for all $p \in \bigcup_{q \in Q} B(q, r) \setminus sQ$, $\text{dist}(p, \partial sQ) \le r = \text{dist}(\partial 2sQ, sQ)$, so $B(x, r) \subset \bigcup_{q \in Q} B(q, r) \subset 2sQ$. Therefore, by the containments and (1),

$$\mu(2^n sQ) \le \mu(B(x, 2^{3n-3}r)) \le \prod_{i=1}^{3n-3} K \cdot \mu(B(x, r)) \le K^{3n-3}\mu(B(x, r)) \le K^{3n-3}\mu(2sQ)$$

Lemma 1.2.3 *Let μ be a measure on $\mathbb{R}^2$ and E denote the set of points μ-a.e. $x \in \mathbb{R}^2$.For $x \in E$, there exists $K \ge 1$ such that $\mu(B(x, 2r)) \le K\mu(B(x, r))$. For $0 < \alpha_2 < \alpha_1 < 1$, $V \in G(1, 2)$, any dyadic cube Q_k contained μ-a.e. x in E with the side length $2^{-k}, k \in \mathbb{N}$, and a dilation constant $s \in \mathbb{N}$ such that $s \ge \frac{6\sqrt{2}(1+\alpha_2)}{\alpha_1 - \alpha_2}$, when $Q \downarrow x$ (i.e. $k \to \infty \Rightarrow 2^{-k} \downarrow 0$),*

$$\frac{\mu(C_{\mathcal{B}}^{2,1}(Q_k, V, \alpha_2) \cap 2sQ_k)}{\mu(2sQ_k)} \to 0 \tag{3}$$

if and only if E is μ-carried by Lipschitz graphs with respect to V and the Lipschitz constant at most $\sqrt{\alpha_2^2/(1 - \alpha_2^2)}$

Proof. We first show the sufficient condition holds. By (3) there exists a large enough k such that for any $\delta > 0$,

$$\mu(C_{\mathcal{B}}^{2,1}(Q_k, V, \alpha_2) \cap 2sQ) < \delta\mu(2sQ_k) \tag{4}$$

Fix a k and $Q_k \ni x$. Now construct sets:

$$S_{Q_k} := \{2R_k : \operatorname{dist}(2R_k, Q_k) \geq \frac{1}{2}s \cdot \operatorname{side} Q_k, 2R_k \cap (2sQ_k \cap C_{\mathcal{B}}^{2,1}(Q_k, V, \alpha_1)) \neq \emptyset\}$$
$$S'_{Q_k} := \{2R_k : \operatorname{dist}(2R_k, Q_k) \geq \frac{1}{2}s \cdot \operatorname{side} Q_k, 2R_k \subset 2sQ_k \cap C_{\mathcal{B}}^{2,1}(Q_k, V, \alpha_1)\}$$
$$S''_{Q_k} := S_{Q_k} \setminus S'_{Q_k}$$

where $2R_k$ is the dilation cube of the dyadic cube R_k with the same side length as Q_k. Then for any $2R_k \in S'_{Q_k}$, by Lemma 1.2.1 (i), $2R_k \subset C_{\mathcal{B}}^{2,2}(Q_k, V, \alpha_2)$ and apparently $2R_k \subset C_{\mathcal{B}}^{2,1}(Q_k, V, \alpha_2) \cap 2sQ_k$ as well($2R_k \subset C_{\mathcal{B}}^{2}(Q_k, V, \alpha_2) \cap 2sQ_k$, too). Then, fix $2^n = n(s), n \in \mathbb{N}$ such that $2^n \geq (\sqrt{2}+1)s + 3\sqrt{2}$, then by Lemma 1.2.1 (ii), $2sQ_k \subset 2^n R_k$. Now assume that there are some μ-a.e. x are contained in $2R_k$, then by (4) and Lemma 1.2.2 (2),

$$\mu(2^n R_k) \leq K^{3n-3}\mu(2R_k) \leq K^{3n-3}\mu(C_{\mathcal{B}}^{2,1}(Q, V, \alpha_2) \cap 2sQ_k) < \delta K^{3n-3}\mu(2sQ_k)$$

which is a contradiction if we choose $\delta < K^{-3n+3}$. Thus,

$$\mu(S'_{Q_k}) = \mu(\bigcup_{2R_k \in S'_{Q_k}} 2R_k) = \sum_{2R_K \in S'_{Q_k}} \mu(2R_k) = 0$$

Similarly, we will consider any $2R'_k \in S''_{Q_k}$. Now fix $2^m = m(s), m \in \mathbb{N}$ such that $2^m \geq (3\sqrt{2}/2 + 1)s + 3\sqrt{2}$ and then by Lemma 1.2.1 (ii), $2R'_k \subset 3sQ_k \subset 2^m R_k$. Then with the similar contradiction proof noting that $\mu(2sQ_k) \leq \mu(3sQ_k)$, we can say $\mu(S''_{Q_k}) = 0$. Hence, $\mu(S_{Q_k}) = \mu(S'_{Q_k} \cup S''_{Q_k}) = 0$. Note that this will hold for all dyadic cubes with the smaller side length than Q_k.

Assume that $r = (s - \sqrt{2}/2) \cdot \operatorname{side} Q_k$ then for μ-a.e. $x \in Q_k$, any $y \in C_{\mathcal{B}}(x, r, V, \alpha_1)$, we have $\operatorname{dist}(x - y, V) > \alpha_1|x - y|$. Choose $k' > k$, and then there exists dyadic cube $Q_{k'} \ni x$ such that $(\sqrt{2} + s/2) \cdot 2^{-k'} \leq |y - x|$. Then $\operatorname{dist}(y, Q_{k'})| \geq |y - x| - \operatorname{dist}(x, \partial Q_{k'}) \geq (\sqrt{2} + s/2 - \sqrt{2}) \operatorname{side} Q_{k'} = s/2 \cdot \operatorname{side} Q_{k'}$, so that $y \in \bigcup_{i=k}^{\infty}\{S_{Q_i} : x \in Q_i \subset Q_k\}$. Hence, $C_{\mathcal{B}}(x, r, V, \alpha_1) \subset \bigcup_{i=k}^{\infty}\{S_{Q_i} : x \in Q_i \subset Q_k\}$. Thus, for μ-a.e. $x \in E$ where x must be contained in only one Q_k,

$$\mu(C_{\mathcal{B}}(x, r, V, \alpha_1)) \leq \mu\left(\bigcup_{i=k}^{\infty}\{S_{Q_i} : x \in Q_i \subset Q_k)\}\right) = 0$$

It follows immediately that for μ-a.e. $x \in E$, $\mu(C_{\mathcal{B}}(x, r, V, \alpha_1)) = 0$. Applying Corollary 1.2.1, the sufficient condition holds then.

To prove the necessary condition, suppose that E is contained in a collection of Lipschitz graphs $\{\Gamma_i\}$ where $\Gamma_i \cap E = \{(v, f_i(v)) : (v, f_i(v)) \in E\} \subset V \times V^{\perp} = \mathbb{R}^2$ where f_i is a Lipschitz function $f_i : V \to V^{\perp}$ with Lipschitz constant L_i at most $\sqrt{\alpha_2^2/(1 - \alpha_2^2)}$. Let any point $p = (v_1, f_i(v_1)), q = (v_2, f_i(v_2)) \in E \cap \Gamma_i$, then we have $|f_i(v_1) - f_i(v_2)| \leq L_i|v_1 - v_2|$. Consider the good cone $C_{\mathcal{G}}(p, V, \alpha_2)$, then,

$$|p - q| = \sqrt{|v_1 - v_2|^2 + |f_i(v_1) - f_i(v_2)|^2} \geq \sqrt{\frac{1}{L_i^2} + 1}|f_i(v_1) - f_i(v_2)| \geq \frac{1}{\alpha_2} \operatorname{dist}(p - q, V)$$

Thus, $q \in C_{\mathcal{G}}(p, V, \alpha_2)$. As p, q, Γ_i arbitrary, $E \subset C_{\mathcal{G}}(x, V, \alpha_2), \forall x \in E$. Now, for $Q_k \ni x$,

$$C^2_{\mathcal{B}}(Q_k, V, \alpha_2) \cap 2sQ_k \subset C_{\mathcal{B}}(x, V, \alpha_2) \cap 2sQ_k \subset 2sQ_k \setminus C_{\mathcal{G}}(x, V, \alpha_2) \subset 2sQ_k \setminus E \tag{5}$$

Next, we claim that

$$\lim_{k\to\infty} \frac{\mu(2sQ_k \setminus E)}{\mu(2sQ_k)} = 0 \tag{6}$$

We choose $r_k = (\sqrt{2}s + \sqrt{2}/2) \cdot 2^{-k}$ such that apparently $2sQ_k \subset B(x, r_k) \subset 4sQ_k$ for any $x \in Q_k$. Note that by our choice of r_k, when $k \to \infty$, then $r_k \to 0$. By Lemma 1.2.2 and Lemma 1.4.1,

$$\lim_{k\to\infty} \frac{\mu(2sQ_k \setminus E)}{\mu(2sQ_k)} \leq \lim_{k\to\infty} \frac{\mu(2sQ_k \setminus E)}{K^{-3}\mu(4sQ_k)} \leq K^3 \lim_{k\to\infty} \frac{\mu(B(x, r_k) \setminus E)}{\mu(B(x, r_k))} = 0$$

Therefore (6) holds. Combining (5),

$$\lim_{k\to\infty} \frac{\mu(C^2_{\mathcal{B}}(Q_k, V, \alpha_2) \cap 2sQ_k)}{\mu(2sQ_k)} = 0$$

This completes the proof of the necessary condition.

1.3 July 7 Complete Necessary Condition for the Lemma

Lemma 1.3.1 (Sufficient Condition for Geometric Lemma) *Let $V \in G(1,2)$, $\alpha \in (0,1)$, non-empty set $E \subset \mathbb{R}^2$. If there exists a Lipschitz function $f : V \to V^\perp$ with Lipschitz constant at most α, such that E is contained in the Lipschitz graph $\Gamma = \{(x, f(x)) : x \in V\} \subset V \times V^\perp = \mathbb{R}^2$, then $E \subset C_{\mathcal{G}}(p, V, \alpha), \forall p \in E$.*

Proof. Assume the Lipschitz constant is $L \le \alpha$. Let any point $p = (x_1, f(x_1)), q = (x_2, f(x_2)) \in E \subset \Gamma$, then we have $|f(x_1) - f(x_2)| \le L|x_1 - x_2|$. Consider the good cone $C_{\mathcal{G}}(p, V, \alpha)$. Then,

$$\operatorname{dist}(p - q, V) = |f(x_1) - f(x_2)| \le L|x_1 - x_2| \le \alpha\sqrt{|x_1 - x_2|^2 + |f(x_1) - f(x_2)|^2} = \alpha|p - q|$$

Thus, $q \in C_{\mathcal{G}}(p, V, \alpha)$. As p, q arbitrary, $E \subset C_{\mathcal{G}}(p, V, \alpha), \forall p \in E$

Lemma 1.3.2 *Let μ be a Radon measure on $\mathbb{R}^2$ and $E \subset \mathbb{R}^2$ measurable. For $x \in E$, there exists $K \ge 1$ such that $\mu(B(x, 2r)) \le K\mu(B(x, r))$. For any dyadic cube Q_k with side length 2^{-k} contained μ-a.e. $x \in E$, $s > 3\sqrt{2}$(this is just simply set for this proof)*

$$\lim_{k\to\infty} \frac{\mu(2sQ_k \setminus E)}{\mu(2sQ_k)} = 0$$

Proof. We choose r where $(\sqrt{2}s + \sqrt{2}/2) \cdot 2^{-k} \le r \le (2\sqrt{2}s - \sqrt{2}/2) \cdot 2^{-k}$ such that $2sQ_k \subset B(x, r) \subset 4sQ_k$ for any $x \in Q_k$. Note that by our choice of r, when k is large enough, $r < \epsilon, \forall \epsilon > 0$. Recall that by Lemma 1.4.1, for existed $\epsilon > 0$, whenever $r < \epsilon$,

$$\frac{\mu(E \setminus B(x, r))}{\mu(B(x, r))} < \frac{\delta}{K^3}, \quad \forall \delta > 0 \tag{7}$$

Besides, considering Lemma 1.2.2 and containment, and combining (7), there exists a large enough k such that

$$\frac{\mu(2sQ_k \setminus E)}{\mu(2sQ_k)} \le \frac{\mu(2sQ_k \setminus E)}{K^{-3}\mu(4sQ_k)} \le K^3\frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} < K^3\frac{\delta}{K^3} = \delta$$

Therefore the conclusion holds.

Alternative Proof. In the fixed dyadic cube system, we can divide E into countable many E_i such that $\bigcup_i E_i = E$ and E_i is contained in only one dilation cube $2sQ_{E_i,k}$ with side $Q_{E_i,k} = 2^{-k}$(E_i may not be disjoint). Define the Radon measure λ_i by $\lambda_i(E_i) = \int_{E_i} \chi_{E_i} d\mu$ such that $\mu(E_i) = \lambda_i(E_i)$, $\mu(2sQ_{E_i,k} \setminus E_i) = \lambda_i(2sQ_{E_i,k} \setminus E_i) = 0$. Then $\lambda_i \ll \mu$ and by [Mat99, Theorem 2.12(2)], considering μ-a.e. $x \in E_i$

$$\int_{2sQ_{E_i,k}} D(\lambda_i, \mu, x) d\mu x = \lambda(2sQ_{E_i,k}) = \int_{2sQ_{E_i,k}} \chi_{E_i} d\mu$$

Hence, for any dyadic Q_k conatined $x \in E$,

$$\begin{aligned}\lim_{k\to\infty}\frac{\mu(E\cap 2sQ_k)}{\mu(2sQ_k)} &= \lim_{k\to\infty}\frac{1}{\mu(2sQ_{E_i,k})}\int_{2sQ_{E_i,k}}\chi_{E_i}d\mu \\ &= \lim_{k\to\infty}\frac{\lambda_i(2sQ_{E_i,k})}{\mu(2sQ_{E_i,k})} \\ &= \lim_{k\to\infty}\frac{\lambda_i(E_i\cup(2sQ_{E_i,k}\setminus E_i))}{\mu(E_i\cup(2sQ_{E_i,k}\setminus E_i))} \\ &= \lim_{k\to\infty}\frac{\lambda_i(E_i)+0}{\mu(E_i)+0} \\ &= 1\end{aligned}$$

Therefore,

$$\lim_{k\to\infty}\frac{\mu(2sQ_k\setminus E)}{\mu(2sQ_k)} = \lim_{k\to\infty}\frac{\mu(2sQ_k\setminus(2sQ_k\cap E))}{\mu(2sQ_k)} = \lim_{k\to\infty}\frac{\mu(2sQ_k)-\mu(2sQ_k\cap E)}{\mu(2sQ_k)} = 1-1=0$$

Lemma 1.3.3 *Let μ be a measure on $\mathbb{R}^2$ and E denote the set of points μ-a.e. $x\in\mathbb{R}^2$. For $x\in E$, there exists $K\geq 1$ such that $\mu(B(x,2r))\leq K\mu(B(x,r))$. For $0<\alpha_2<\alpha_1<1$, $V\in G(1,2)$, any dyadic cube Q_k contained μ-a.e. x in $\mathbb{R}^2$ with the side length $2^{-k}, k\in\mathbb{N}$, and a dilation constant $s\in\mathbb{N}$ such that $s\geq\frac{3\sqrt{2}(1+\alpha_2)}{\alpha_1-\alpha_2}$, when $Q\downarrow x$ (i.e. $k\to\infty\Rightarrow 2^{-k}\downarrow 0$),*

$$\frac{\mu(C^2_{\mathcal{B}}(Q_k,V,\alpha_2)\cap 2sQ_k)}{\mu(2sQ_k)}\to 0 \tag{8}$$

if and only if E is μ-carried by Lipschitz graphs with respect to V and the Lipschitz constant at most α_2

Proof. To prove the necessary condition, suppose that E is conatined in the Lipschitz graph $\Gamma=\{(x,f(x)):x\in V\}\subset V\times V^\perp=\mathbb{R}^2$ where f is a Lipschitz function $f:V\to V^\perp$ with Lipschitz constant at most α_2. Assume the Lipschitz constant is $L\leq\alpha_2$. Let any point $p=(x_1,f(x_1)), q=(x_2,f(x_2))\in E\subset\Gamma$, then we have $|f(x_1)-f(x_2)|\leq L|x_1-x_2|$. Consider the good cone $C_{\mathcal{G}}(p,V,\alpha_2)$, then,

$$\operatorname{dist}(p-q,V)=|f(x_1)-f(x_2)|\leq L|x_1-x_2|\leq\alpha_2\sqrt{|x_1-x_2|^2+|f(x_1)-f(x_2)|^2}=\alpha|p-q|$$

Thus, $q\in C_{\mathcal{G}}(p,V,\alpha)$. As p,q arbitrary, $E\subset C_{\mathcal{G}}(p,V,\alpha),\forall p\in E$. Now, for $Q_k\ni x$,

$$C^2_{\mathcal{B}}(Q_k,V,\alpha_2)\cap 2sQ_k\subset C_{\mathcal{B}}(x,V,\alpha_2)\cap 2sQ_k\subset 2sQ_k\setminus C_{\mathcal{G}}(x,V,\alpha_2)\subset 2sQ_k\setminus E \tag{9}$$

Next, we claim that

$$\lim_{k\to\infty}\frac{\mu(2sQ_k\setminus E)}{\mu(2sQ_k)}=0 \tag{10}$$

We choose r where $(\sqrt{2}s+\sqrt{2}/2)\cdot 2^{-k}\leq r\leq(2\sqrt{2}s-\sqrt{2}/2)\cdot 2^{-k}$ such that $2sQ_k\subset B(x,r)\subset 4sQ_k$ for any $x\in Q_k$. Note that by our choice of r, when k is large enough, $r<\epsilon,\forall\epsilon>0$. Recall that by Lemma 1.4.1, for existed $\epsilon>0$, whenever $r<\epsilon$,

$$\frac{\mu(E\setminus B(x,r))}{\mu(B(x,r))}<\frac{\delta}{K^3},\quad\forall\delta>0 \tag{11}$$

Besides, considering Lemma 1.2.2 and containment, and combining (7), there exists a large enough k such that

$$\frac{\mu(2sQ_k \setminus E)}{\mu(2sQ_k)} \leq \frac{\mu(2sQ_k \setminus E)}{K^{-3}\mu(4sQ_k)} \leq K^3 \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} < K^3 \frac{\delta}{K^3} = \delta$$

Therefore (6) holds. Combining (5),

$$\lim_{k\to\infty} \frac{\mu(C^2_{\mathcal{B}}(Q_k, V, \alpha_2) \cap 2sQ_k)}{\mu(2sQ_k)} = 0$$

This completes the proof of the necessary condition.

1.4 July 6 Lemma for $C^{2,1}_{\mathcal{B}}(Q,V,\alpha)$ and Notes for [Mat99]

Definition 1.4.1 (Derivative of μ) *Let μ and λ be locally finite Borel measures on $\mathbf{R}^n$. The upper and lower derivatives of μ with respect to λ at a point $x \in \mathbf{R}^n$ are defined by*

$$\overline{D}(\mu,\lambda,x) = \limsup_{r\downarrow 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))}$$
$$\underline{D}(\mu,\lambda,x) = \liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))}$$

At the points x where the limit exists we define the derivative of μ by

$$D(\mu,\lambda,x) = \overline{D}(\mu,\lambda,x) = \underline{D}(\mu,\lambda,x)$$

Definition 1.4.2 (absolutely continuous) *Let μ and λ be measures on $\mathbf{R}^n$. We say that μ is absolutely continuous with respect to λ if $\lambda(A)=0 \quad$ implies $\mu(A)=0 \quad$ for all $A \subset \mathbf{R}^n$ In this case we write*

$$\mu \ll \lambda$$

Theorem 1.4.1 ([Mat99, Theorem 2.8]) *Let μ be a Radon measure on $\mathbf{R}^n$, $A \subset \mathbf{R}^n$ and $\mathcal{B}$ a family of closed balls such that each point of A is the centre of arbitrarily small balls of $\mathcal{B}$, that is,*

$$\inf\{r : B(x,r) \in \mathcal{B}\} = 0 \quad \text{for } x \in A$$

Then there are disjoint balls $B_i \in \mathcal{B}$ such that

$$\mu\left(A \backslash \bigcup_i B_i\right) = 0$$

Lemma 1.4.1 ([Mat99, Lemma 2.13]) *Let μ and λ be Radon measures on $\mathbf{R}^n$, $0 < t < \infty$ and $A \subset \mathbf{R}^n$.*

(1) If $\underline{D}(\mu,\lambda,x) \le t$ for all $x \in A$, then $\mu(A) \le t\lambda(A)$

(2) If $\overline{D}(\mu,\lambda,x) \ge t$ for all $x \in A$, then $\mu(A) \ge t\lambda(A)$

Proof. (1) Let $\varepsilon > 0$. Using Definition 1.5(4) we find an open set U such that $A \subset U$ and $\lambda(U) \le \lambda(A) + \varepsilon$. An application of Theorem 2.8 gives disjoint closed balls $B_i \subset U$ such that

$$\mu(B_i) \le (t+\varepsilon)\lambda(B_i) \quad \text{and} \quad \mu\left(A \backslash \bigcup_i B_i\right) = 0$$

Then

$$\begin{aligned}\mu(A) &\le \sum_i \mu(B_i) \le (t+\varepsilon)\sum_i \lambda(B_i) \\ &\le (t+\varepsilon)\lambda(U) \le (t+\varepsilon)(\lambda(A)+\varepsilon)\end{aligned}$$

Letting $\varepsilon \downarrow 0$, we get $\mu(A) \le t\lambda(A)$, which proves (1).(2) can be proven in the same way.

Theorem 1.4.2 ([Mat99, Theorem 2.12(2)]) *Let μ and λ be Radon measures on $\mathbb{R}^n$. For all Borel sets $B \subset \mathbb{R}^n$,*

$$\int_B D(\mu, \lambda, x) d\lambda x \leq \mu(B)$$

with equality if $\mu \ll \lambda$

Corollary 1.4.1 ([Mat99, Corollary 2.14])

1. *If $A \subset \mathbf{R}^n$ is λ measurable, then the limit*

$$\lim_{r \downarrow 0} \frac{\lambda(A \cap B(x,r))}{\lambda(B(x,r))}$$

exists and equals 1 for λ almost all $x \in A$ and equals 0 for λ almost all $x \in \mathbf{R}^n \backslash A$.

2. *If $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ is locally λ integrable, then $\lim_{r \downarrow 0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f d\lambda = f(x)$ for λ almost all $x \in \mathbf{R}^n$.*

Proof. (1) follows from (2) with $f = \chi_A$. To prove (2) we may assume $f \geq 0$. Define the Radon measure μ by $\mu(A) = \int_A f d\lambda$. Then $\mu \ll \lambda$ and Theorem 2.12(2) gives

$$\int_B D(\mu, \lambda, x) d\lambda x = \mu(B) = \int_B f d\lambda$$

for all Borel sets B. Obviously this means that $f(x) = D(\mu, \lambda, x)$ for λ almost all $x \in \mathbf{R}^n$, which proves (2).

Corollary 1.4.2 ([Nap20, Corollary 7.1]) *Let μ be a Radon measure on H, V be an m-dimensional linear plane in H, $\alpha \in (0,1)$, and $0 < r < \infty$. If for μ-a.e. $x \in H$*

$$\mu\left(C_{\mathcal{B}}(x, r, V, \alpha)\right) = 0$$

then μ is carried by m-Lipschitz graphs.

Lemma 1.4.2 *For $0 < \alpha_2 < \alpha_1 < 1$, $V \in G(1,2)$, a dyadic cube Q in $\mathbb{R}^2$, and $s \geq \frac{3\sqrt{2}(1+\alpha_2)}{\alpha_1 - \alpha_2}$, a dilation cube $2R$ such that R is the dyadic cube with the same side length as Q and*

$$\operatorname{dist}(Q, 2R) \geq s \cdot \operatorname{side} Q$$

then

(i) *if $2R \subset C_{\mathcal{B}}^{2,1}(Q, V, \alpha_1)$, then $2R \subset C_{\mathcal{B}}^{2,2}(Q, V, \alpha_2)$.*

(ii) *if for $n = n(s) \in \mathbb{N}$ such that $2^n \geq 2(\sqrt{2}+1)s + 3\sqrt{2}$, then $2sQ \subset 2^n R$.*

(iii) *if for $m = m(s) \in \mathbb{N}$ such that $2^m \geq (3\sqrt{2}+2)s + 3\sqrt{2}$, then $3sQ \subset 2^m R$.*

Proof.

(i) When $2R \subset C_{\mathcal{B}}^{2,1}(Q, V, \alpha_1)$, there exists some $r \in 2R \cap C_{\mathcal{B}}^{1}(Q, V, \alpha_1)$ and hence for any $q \in Q$ such that $r \in C_{\mathcal{B}}(q, V, \alpha_1)$, so $\operatorname{dist}(r-q, V) > \alpha_1|r-q|$. Besides, arbitrary $r' \in 2R$ and $q' \in Q$ must satisfy $|r - r'| \leq 2\sqrt{2} \cdot \operatorname{side} R$ and $|q - q'| \leq \sqrt{2} \cdot \operatorname{side} Q$, respectively. Assume that $|r - q| = s' \cdot \operatorname{side} Q$ then $s' \geq s$. Since $|r - r'| \geq |\operatorname{dist}(r' - q', V) - \operatorname{dist}(r - q, V)| - |q - q'|$, no matter $\operatorname{dist}(r' - q', V) \leq \operatorname{dist}(r - q, V)$ or $\operatorname{dist}(r' - q', V) \geq \operatorname{dist}(r - q, V)$,

$$\begin{aligned}
\operatorname{dist}(r' - q', V) &\geq \operatorname{dist}(r - q, V) - |q - q'| - |r - r'| \\
&> \alpha_1 |r - q| - 3\sqrt{2} \cdot \operatorname{side} Q \\
&\geq (\alpha_1 s' - 3\sqrt{2}) \cdot \operatorname{side} Q \\
&\geq \alpha_2 (s' + 3\sqrt{2}) \cdot \operatorname{side} Q \qquad\qquad (s' \geq s \geq \frac{3\sqrt{2}(1+\alpha_2)}{\alpha_1 - \alpha_2}) \\
&\geq \alpha_2 (|r - q| + |r - r'| + |q - q'|) \\
&\geq \alpha_2 |r' - q'|
\end{aligned}$$

Therefore, $2R \subset \{R : R \subset \bigcap_{q' \in Q} C_{\mathcal{B}}(q', V, \alpha)\} = C_{\mathcal{B}}^{2,2}(Q, V, \alpha_2)$.

(ii) Assume for all $q'' \in 2sQ$, then

$$\begin{aligned}
|\operatorname{center} 2R - q''| &\leq \left(\frac{1}{2} \operatorname{diam} 2R + \operatorname{dist}(2R, Q) + \frac{1}{2} \operatorname{diam} Q + \frac{1}{2} \operatorname{diam} 2sQ\right) \\
&\leq (\sqrt{2} + s + \frac{\sqrt{2}}{2} + \sqrt{2}s) \operatorname{side} Q \\
&\leq \frac{2^n}{2} \operatorname{side} R = \frac{1}{2} \operatorname{side} 2^n R
\end{aligned}$$

which implies that $2sQ \subset B(\operatorname{center} R, \frac{1}{2} \operatorname{side} 2^n R) \subset 2^n R$.

(iii) Similarly, assume $\forall q''' \in 3sQ$, then

$$\begin{aligned}|\operatorname{center} 2R - q'''| &\le (\frac{1}{2}\operatorname{diam} 2R + \operatorname{dist}(2R, Q) + \frac{1}{2}\operatorname{diam} Q + \frac{1}{2}\operatorname{diam} 3sQ) \\ &\le (\sqrt{2} + s + \frac{\sqrt{2}}{2} + \frac{3}{2}\sqrt{2}s)\operatorname{side} Q \\ &\le \frac{2^m}{2}\operatorname{side} R = \frac{1}{2}\operatorname{side} 2^m R\end{aligned}$$

which implies that $3sQ \subset B(\operatorname{center} R, \frac{1}{2}\operatorname{side} 2^m R) \subset 2^m R$.

Lemma 1.4.3 *If there exist $K \ge 1$, such that for μ-a.e. $x \in \mathbb{R}^2, 0 < r < \infty$,*

$$\mu(B(x, 2r)) \le K\mu(B(x, r)) \tag{12}$$

then for any dyadic cube Q containing x where (1) holds, $n \in \mathbb{N}, n \ge 2$,

$$\mu(2^n Q) \le K^{3n-3}\mu(2Q) \tag{13}$$

Proof. By the half open definition of the dyadic cube, for μ-a.e. $x \in \mathbb{R}^2$, x must be in only one cube Q. Let $r = \operatorname{side} Q/2$. We claim that $2^n Q \subset B(x, 2^{3n-3}r)$ and $B(x, r) \subset 2Q$. For any $q \in 2^n Q$, $|\operatorname{center} Q - x| \le \sqrt{2}/2 \cdot \operatorname{side} Q$ and $|\operatorname{center} 2^n Q - q| \le \sqrt{2}/2 \cdot \operatorname{side} 2^n Q = 2^{n-1}\sqrt{2} \cdot \operatorname{side} Q$. Note that $n \ge 2$, so

$$|x - q| \le \left(\frac{\sqrt{2}}{2} + 2^{n-1}\sqrt{2}\right)\operatorname{side} Q < \frac{2^{3n-3}}{2}\operatorname{side} Q = 2^{3n-3}r$$

Then $q \in B(x, 2^{3n-3}r)$ and thus $2^n Q \subset B(x, 2^{3n-3}r)$. Consider that for all $p \in \bigcup_{q\in Q} B(q, r) \setminus Q$, $\operatorname{dist}(p, \partial Q) \le r = \operatorname{dist}(\partial 2Q, Q)$, so $B(x, r) \subset \bigcup_{q\in Q} B(q, r) \subset 2Q$. Therefore, by the containments and (1),

$$\mu(2^n Q) \le \mu(B(x, 2^{3n-3}r)) \le \prod_{i=1}^{3n-3} K \cdot \mu(B(x, r)) \le K^{3n-3}\mu(B(x, r)) \le K^{3n-3}\mu(2Q)$$

Lemma 1.4.4 *Let μ be a measure on $\mathbb{R}^2$ and E denote the set of points μ-a.e. $x \in \mathbb{R}^2$.For $x \in E$, there exists $K \ge 1$ such that $\mu(B(x, 2r)) \le K\mu(B(x, r))$. For $0 < \alpha_2 < \alpha_1 < 1$, $V \in G(1, 2)$, any dyadic cube Q_k contained μ-a.e. x in $\mathbb{R}^2$ with the side length $2^{-k}, k \in \mathbb{N}$, and a dilation constant $s \in \mathbb{N}$ such that $s \ge \frac{3\sqrt{2}(1+\alpha_2)}{\alpha_1 - \alpha_2}$, when $Q \downarrow x$ (i.e. $k \to \infty \Rightarrow 2^{-k} \downarrow 0$),*

$$\frac{\mu(C_{\mathcal{B}}^{2,1}(Q_k, V, \alpha_2) \cap 2sQ_k)}{\mu(2sQ_k)} \to 0 \tag{14}$$

if and only if E is μ-carried by Lipschitz graphs with respect to V and the Lipschitz constant α_1

Proof. We first show the sufficient condition holds. By (3) there exists a large enough k such that for any $\delta > 0$,

$$\mu(C_{\mathcal{B}}^{2,1}(Q_k, V, \alpha_2) \cap 2sQ) < \delta\mu(2sQ_k) \tag{15}$$

Fix a k and $Q_k \ni x$. Now construct sets:

$$\begin{aligned}
S_{Q_k} &:= \{2R_k : \operatorname{dist}(2R_k, Q_k) \geq s \cdot \operatorname{side} Q_k, 2R_k \cap (2sQ_k \cap C_{\mathcal{B}}^{2,1}(Q_k, V, \alpha_1)) \neq \emptyset\} \\
S'_{Q_k} &:= \{2R_k : \operatorname{dist}(2R_k, Q_k) \geq s \cdot \operatorname{side} Q_k, 2R_k \subset 2sQ_k \cap C_{\mathcal{B}}^{2,1}(Q_k, V, \alpha_1)\} \\
S''_{Q_k} &:= S_{Q_k} \setminus S'_{Q_k}
\end{aligned}$$

where $2R_k$ is the dilation cube of the dyadic cube R_k with the same side length as Q_k. Then for any $2R_k \in S'_{Q_k}$, by Lemma 1.2.1 (i), $2R_k \subset C_{\mathcal{B}}^{2,2}(Q_k, V, \alpha_2)$ and apparently $2R_k \subset C_{\mathcal{B}}^{2,1}(Q_k, V, \alpha_2) \cap 2sQ_k$ as well. Then, fix $2^n = n(s), n \in \mathbb{N}$ such that $2^n \geq 2(\sqrt{2}+1)s + 3\sqrt{2}$, then by Lemma 1.2.1 (ii), $2sQ_k \subset 2^n R_k$. Now assume that there are some μ-a.e. x are contained in $2R_k$, then by (4) and Lemma 1.2.2 (2),

$$\mu(2^n R_k) \leq K^{3n-3}\mu(2R_k) \leq K^{3n-3}\mu(C_{\mathcal{B}}^{2,1}(Q, V, \alpha_2) \cap 2sQ_k) < \delta K^{3n-3}\mu(2sQ_k)$$

which is a contradiction if we choose $\delta < K^{-3n+3}$. Thus,

$$\mu(S'_{Q_k}) = \mu(\bigcup_{2R_k \in S'_{Q_k}} 2R_k) = \sum_{2R_K \in S'_{Q_k}} \mu(2R_k) = 0$$

Similarly, we will consider any $2R'_k \in S''_{Q_k}$. Now fix $2^m = m(s), m \in \mathbb{N}$ such that $2^m \geq (3\sqrt{2}+2)s + 3\sqrt{2}$ and then by Lemma 1.2.1 (ii), $2R'_k \subset 3sQ_k \subset 2^m R_k$. Then with the similar contradiction proof noting that $\mu(2sQ_k) \leq \mu(3sQ_k)$, we can say $\mu(S''_{Q_k}) = 0$. Hence, $\mu(S_{Q_k}) = \mu(S'_{Q_k} \cup S''_{Q_k}) = 0$. Note that this will hold for all dyalic cubes with the smaller side length than Q_k.

Assume that for μ-a.e. $x \in Q_k$, any $y \in C_{\mathcal{B}}(x, V, \alpha_1)$, we have $\operatorname{dist}(x-y, V) > \alpha_1|x-y|$. There exists dyadic cube $Q_{k'} \ni x$ such that $(\sqrt{2}+s) \cdot 2^{-k'} \leq |y-x|$. Then $\operatorname{dist}(y, Q_{k'})| \geq |y-x| - \operatorname{dist}(x, \partial Q_{k'}) \geq (\sqrt{2}+s-\sqrt{2}) \operatorname{side} Q_{k'} = s \cdot \operatorname{side} Q^{k'}$, so that $y \in S_{Q_{k'}}$.

Hence, $C_{\mathcal{B}}(x, V, \alpha_1) \subset \bigcup_{k=1}^{\infty}\{S_{Q_k} : x \in Q_k\}$. Thus, for μ-a.e. $x \in E$ where x must be contained in only one Q_k,

$$\mu(C_{\mathcal{B}}(x, V, \alpha_1)) \leq \mu\left(\bigcup_{k=1}^{\infty}\{S_{Q_k} : x \in Q_k)\}\right) = 0$$

It follows immediately that for μ-a.e. $x \in E$, $\mu(C_{\mathcal{B}}(x, V, \alpha_1)) = 0$. Applying Corollary 1.2.1, the sufficient condition holds then.

To prove the necessary condition, for any Q_k, suppose that $r = \sqrt{2}s \cdot 2^{-k}$ such that $2sQ \subset B(x,r), x \in Q$. By the assumption and Corollary 1.4.1(1), it follows that

$$\lim_{r \downarrow 0} \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} = 0$$

Furthermore, applying Lemma 1.3.1, for $Q_k \ni x$, $C_{\mathcal{B}}^{2,1}(Q_k, V, \alpha) \cap 2sQ_k \subset C_{\mathcal{B}}(x, r, V, \alpha) \subset B(x, r) \setminus E$. Then, problem appears

$$\lim_{r \downarrow 0} \frac{C_{\mathcal{B}}^{2,1}(Q_k, V, \alpha) \cap 2sQ_k}{\mu(2sQ_k)} = 0$$

(need to be proved)

$$\lim_{k \to \infty} \frac{\mu(2sQ_k \setminus E)}{\mu(2sQ_k)} = 0$$

(Another Draft for this proof)

To prove the necessary condition, suppose that E is conatined in the Lipschitz graph $\Gamma = \{(x, f(x)) : x \in V\} \subset V \times V^{\perp} = \mathbb{R}^2$ where f is a Lipschitz function $f : V \to V^{\perp}$ with Lipschitz constant α_2. Then if we choose any $x_1, x_2 \in E \subset \Gamma$, $\operatorname{dist}(x_2 - x_1, V) = |f(x_2) - f(x_1)| \leq \alpha_2 |x_2 - x_1|$. Thus, $x_1 \in C_{\mathcal{G}}(x_2, V, \alpha_2)$. As x_1, x_2 arbitrary, $E \subset C_{\mathcal{G}}(x, V, \alpha_2), \forall x \in E$. Now, for $Q_k \ni x$,

$$C_{\mathcal{B}}^{2,1}(Q_k, V, \alpha_2) \cap 2sQ_k \subset C_{\mathcal{B}}(x, V, \alpha_2) \cap 2sQ_k \subset 2sQ_k \setminus C_{\mathcal{G}}(x, V, \alpha_2) \subset 2sQ_k \setminus E \tag{16}$$

By SOMESOMESOME Lemma,

$$\lim_{k \to \infty} \frac{\mu(2sQ_k \setminus E)}{\mu(2sQ_k)} = 0$$

Combining (5),

$$\lim_{k \to \infty} \frac{\mu(C_{\mathcal{B}}^{2,1}(Q_k, V, \alpha_2) \cap 2sQ_k)}{\mu(2sQ_k)} = 0$$

This completes the proof of the necessary condition.

Lemma 1.4.5 *Let μ be a Radon measure on $\mathbb{R}^n$ and $E \subset \mathbb{R}^n$ measurable. For any dyadic cube Q_k with side length 2^{-k} contained μ-a.e. $x \in E$,*

$$\lim_{k \to \infty} \frac{\mu(2sQ_k \setminus E)}{\mu(2sQ_k)} = 0$$

Lemma 1.4.6 (Sufficient Condition for Geometric Lemma) *Let $V \in G(1, 2)$, $\alpha \in (0, 1)$, non-empty set $E \subset \mathbb{R}^2$. If there exists a Lipschitz function $f : V \to V^{\perp}$ with Lipschitz constant α such that E is contained in the Lipschitz graph $\Gamma = \{(x, f(x)) : x \in V\} \subset V \times V^{\perp} = \mathbb{R}^2$, then $E \subset C_{\mathcal{G}}(x, V, \alpha), \forall x \in E$.*

Proof. Since $\Gamma \subset V \times V^{\perp}$ and if we choose any $x_1, x_2 \in E \subset \Gamma$, then $\operatorname{dist}(x_2 - x_1, V) = |f(x_2) - f(x_1)| \leq \alpha |x_2 - x_1|$. Thus, $x_1 \in C_{\mathcal{G}}(x_2, V, \alpha)$. As x_1, x_2 arbitrary, $E \subset C_{\mathcal{G}}(x, V, \alpha), \forall x \in E$.

1.5 July 1-5 Necessary Condition for the Lemma

Lemma 1.5.1 *Let μ be a measure on $\mathbb{R}^2$ and E denote the set of points μ-a.e. $x \in \mathbb{R}^2$.For $x \in E$, there exists $K \geq 1$ such that $\mu(B(x, 2r)) \leq K\mu(B(x, r))$. For $0 < \alpha_2 < \alpha_1 < 1$, $V \in G(1, 2)$, any dyadic cube Q_k contained μ-a.e. x in $\mathbb{R}^2$ with the side length $2^{-k}, k \in \mathbb{N}$, and a dilation constant $s \in \mathbb{N}$ such that $s \geq \frac{3\sqrt{2}(1+\alpha_2)}{\alpha_1 - \alpha_2}$, when $Q \downarrow x$ (i.e. $k \to \infty \Rightarrow 2^{-k} \downarrow 0$),*

$$\frac{\mu(C_{\mathcal{B}}^{1,1}(Q, V, \alpha_2) \cap 2sQ)}{\mu(2sQ)} \to 0 \tag{17}$$

if and only if E is μ-carried by Lipschitz graphs.

Proof of Necessary Condition. verified that the neccesary condition cannot hold.

1.6 June 30-31 Wrapping everything up

Definition 1.6.1 (Distance) *We define the distance* $\operatorname{dist}(X, Y)$ *from a set(or a point)* X *to another set* Y *by:*

$$\operatorname{dist}(X, Y) := \inf\{|x - y| : x \in X, y \in Y\}$$

Lemma 1.6.1 *For* $0 < \alpha_2 < \alpha_1 < 1$*,* $V \in G(1, 2)$*, a dyadic cube* Q *in* $\mathbb{R}^2$*, and a dilation cube* $2R$ *where* R *is the dyadic cube with the same side length as* Q*, Let* $s \in \mathbb{N}$ *satisify*

$$\operatorname{dist}(Q, 2R) \geq s \cdot \operatorname{side} Q \geq \frac{3\sqrt{2}(1 + \alpha_2)}{\alpha_1 - \alpha_2} \cdot \operatorname{side} Q \tag{18}$$

Then

(i) if $2R \subset C_{\mathcal{B}}^{1,1}(Q, V, \alpha_1)$*, then* $2R \subset C_{\mathcal{B}}^{1,2}(Q, V, \alpha_2)$*.*

(ii) suppose a dilation constant $t = 2^n, n \in \mathbb{N}$*, if* $t \geq 2(\sqrt{2} + 1)s + 3\sqrt{2}$*, then* $2sQ \subset tR$*.*

(iii) another dilation constant $t' = 2^m, m \in \mathbb{N}$*, if* $t' \geq (3\sqrt{2} + 2)s + 3\sqrt{2}$*, then* $3sQ \subset t'R$*.*

Proof.

(i) When $2R \subset C_{\mathcal{B}}^{1,1}(Q, V, \alpha_1)$, for some $r \in 2R \cap C_{\mathcal{B}}^{1}(Q, V, \alpha_1)$, there exists $q \in Q$ such that $r \in C_{\mathcal{B}}(q, V, \alpha_1)$, so $\operatorname{dist}(r - q, V) > \alpha_1 |r - q|$. Besides, for any $r' \in 2R$, consider some $q' \in Q$ which must satisfy $|r - r'| \leq 2\sqrt{2} \cdot \operatorname{side} R$ and $|q - q'| \leq \sqrt{2} \cdot \operatorname{side} Q$. Assume that $|r - q| = s' \cdot \operatorname{side} Q$ then $s' \geq s$. Since $|r - r'| \geq |\operatorname{dist}(r' - q', V) - \operatorname{dist}(r - q, V)| - |q - q'|$, no matter $\operatorname{dist}(r' - q', V) \lesseqgtr \operatorname{dist}(r - q, V)$,

$$\begin{aligned}
\operatorname{dist}(r' - q', V) &\geq \operatorname{dist}(r - q, V) - |q - q'| - |r - r'| \\
&> \alpha_1 |r - q| - 3\sqrt{2} \cdot \operatorname{side} Q \\
&\geq (\alpha_1 s' - 3\sqrt{2}) \cdot \operatorname{side} Q \\
&\geq \alpha_2 (s' + 3\sqrt{2}) \cdot \operatorname{side} Q \\
&\geq \alpha_2 (|r - q| + |r - r'| + |q - q'|) \\
&\geq \alpha_2 |r' - q'|
\end{aligned}$$

Therefore, $2R \subset C_{\mathcal{B}}^{1,2}(Q, V, \alpha_2)$.

(ii) Assume $\forall q'' \in 2sQ$, then

$$\begin{aligned}
|\operatorname{center} 2R - q''| &\leq (\frac{1}{2} \operatorname{diam} 2R + \operatorname{dist}(2R, Q) + \frac{1}{2} \operatorname{diam} Q + \frac{1}{2} \operatorname{diam} 2sQ) \\
&\leq (\sqrt{2} + s + \frac{\sqrt{2}}{2} + \sqrt{2}s) \operatorname{side} Q \\
&\leq \frac{t}{2} \operatorname{side} R = \frac{1}{2} \operatorname{side} tR
\end{aligned}$$

which implies that $2sQ \subset B(\operatorname{center} R, \frac{1}{2} \operatorname{side} tR) \subset tR$.

(iii) Similarly, assume $\forall q''' \in 3sQ$, then

$$\begin{aligned}|\operatorname{center} 2R - q'''| &\leq (\frac{1}{2}\operatorname{diam} 2R + \operatorname{dist}(2R, Q) + \frac{1}{2}\operatorname{diam} Q + \frac{1}{2}\operatorname{diam} 3sQ)\\ &\leq (\sqrt{2} + s + \frac{\sqrt{2}}{2} + \frac{3}{2}\sqrt{2}s)\operatorname{side} Q\\ &\leq \frac{t'}{2}\operatorname{side} R = \frac{1}{2}\operatorname{side} t'R\end{aligned}$$

which implies that $3sQ \subset B(\operatorname{center} R, \frac{1}{2}\operatorname{side} t'R) \subset t'R$.

Lemma 1.6.2 *If for μ-a.e. $x \in \mathbb{R}^2, 0 < r < \infty, \mu(B(x, 2r)) \leq K\mu(B(x, r)), K \in \mathbb{R}^+$, then for $n \in \mathbb{N}, n \geq 2$,*

$$\mu(2^n Q) \leq K^{3n-3}\mu(2Q) \tag{19}$$

where Q is the dyadic cube containing μ-a.e. x, and $2^n Q$, $2Q$ are dilation cubes.

Proof. By the half open definition of the dyadic cube, for μ-a.e. $x \in \mathbb{R}^2$, x must in only one cube Q. Let $r = \operatorname{side} Q/2$. We claim that $2^n Q \subset B(x, 8^{n-1}r)$ and $B(x, r) \subset 2Q$. For any $q \in 2^n Q$, $|\operatorname{center} Q - x| \leq \sqrt{2}/2 \cdot \operatorname{side} Q$ and $|\operatorname{center} 2^n Q - q| \leq \sqrt{2}/2 \cdot \operatorname{side} 2^n Q = 2^{n-1}\sqrt{2} \cdot \operatorname{side} Q$. Note that $n \geq 2$, so

$$|x - q| \leq \left(\frac{\sqrt{2}}{2} + 2^{n-1}\sqrt{2}\right)\operatorname{side} Q < \frac{8^{n-1}}{2}\operatorname{side} Q = 8^{n-1}r$$

Then $q \in B(x, 8^{n-1}r)$ and thus $2^n Q \subset B(x, 8^{n-1}r)$. Consider that for $\forall p \in \bigcup_{q \in Q} B(q, r) \setminus Q$, $\operatorname{dist}(p, \partial Q) \leq r = \operatorname{dist}(\partial 2Q, Q)$, so $B(x, r) \subset \bigcup_{q \in Q} B(q, r) \subset 2Q$. Therefore, by the containments and the doubling measure condition,

$$\mu(2^n Q) \leq \mu(B(x, 2^{3n-3}r)) \leq \prod_{i=1}^{3n-3} K \cdot \mu(B(x, r)) \leq K^{3n-3}\mu(B(x, r)) \leq K^{3n-3}\mu(2Q)$$

Lemma 1.6.3 *Let μ be a measure on $\mathbb{R}^2$. Let $0 < \alpha_2 < \alpha_1 < 1$, $V \in G(1, 2)$, any dyadic cube Q_k contained μ-a.e. x in $\mathbb{R}^2$ with the side length $2^{-|k|}, k \in \mathbb{Z}$, dilation constants $s \in \mathbb{N}$, parameter $K > 0$. Let E denote the set of points $x \in \mathbb{R}^2$ then*

(i) When $Q \downarrow x$ (i.e. $|k| \to \infty \Rightarrow 2^{-|k|} \downarrow 0$),

$$\frac{\mu(C_{\mathcal{B}}^{1,1}(Q, V, \alpha_2) \cap 2sQ)}{\mu(2sQ)} \to 0 \tag{20}$$

(ii) s satisfies

$$s \geq \frac{3\sqrt{2}(1 + \alpha_2)}{\alpha_1 - \alpha_2} \tag{21}$$

(iii) For μ-a.e $x \in E$, $0 < r < \infty$

$$\mu(B(x, 2r)) \leq K\mu(B(x, r)) \tag{22}$$

if and only if E is μ-carried by Lipschitz graphs.

Proof. We first show the sufficient condition holds. By (3) and $\lim_{|k|\to\infty} \operatorname{side} Q_k = 0$, there exists a large enough k such that for any $\delta > 0$,

$$\mu(C^{1,1}_{\mathcal{B}}(Q_k, V, \alpha_2) \cap 2sQ) < \delta\mu(2sQ_k) \tag{23}$$

Fix a $|k|$ and $Q_k \ni x$. Now construct sets:

$$\begin{aligned}
S_{Q_k} &:= \{2R_k : \operatorname{dist}(2R_k, Q_k) \geq s \cdot \operatorname{side} Q_k, 2R_k \cap (2sQ_k \cap C^{1,1}_{\mathcal{B}}(Q_k, V, \alpha_1)) \neq \emptyset\} \\
S'_{Q_k} &:= \{2R_k : \operatorname{dist}(2R_k, Q_k) \geq s \cdot \operatorname{side} Q_k, 2R_k \subset 2sQ_k \cap C^{1,1}_{\mathcal{B}}(Q_k, V, \alpha_1)\} \\
S''_{Q_k} &:= S_{Q_k} \setminus S'_{Q_k}
\end{aligned}$$

where $2R_k$ is the dilation cube of the dyadic cube R_k with the same side length as Q_k. Then for any $2R_k \in S'_{Q_k}$, by Lemma 1.2.1 (i), $2R_k \subset C^{1,2}_{\mathcal{B}}(Q_k, V, \alpha_2)$ and apparently $2R_k \subset C^{1,1}_{\mathcal{B}}(Q_k, V, \alpha_2) \cap 2sQ_k$ as well. Then, fix $t = 2^n, n \in \mathbb{N}$ such that $t \geq 2(\sqrt{2}+1)s + 3\sqrt{2}$ then by Lemma 1.2.1 (ii), $tR_k \subset 2sQ_k$. Now assume that there are some μ-a.e. x are contained in $2R_k$, then by Lemma 1.2.2 and (4),

$$\mu(tR_k) \leq K^{3n-3}\mu(2R_k) \leq K^{3n-3}\mu(C^{1,1}_{\mathcal{B}}(Q, V, \alpha_2) \cap 2sQ_k) < \delta K^{3n-3}\mu(2sQ_k)$$

which is a contradiction if we choose $\delta < K^{-3n+3}$. Thus, $0 = \mu(2R_k) = \mu(\bigcup_{2R_k \in S'_{Q_k}} 2R_k) = \mu(S'_{Q_k})$. Moreover, we will consider any $2R'_k \in S''_{Q_k}$. Now fix $t' = 2^m, m \in \mathbb{N}$ such that $t \geq (3\sqrt{2}+2)s + 3\sqrt{2}$ and then by Lemma 1.2.1 (ii), $2R'_k \subset 3sQ_k \subset t'R_k$. Then with the similar contradiction proof noting that $\mu(2sQ_k) \leq \mu(3sQ_k)$, we can say $0 = \mu(2R'_k) = \mu(\bigcup_{2R'_k \in S''_{Q_k}} 2R'_k) = \mu(S''_{Q_k})$. Hence, $\mu(S_{Q_k}) = \mu(S'_{Q_k} \cup S''_{Q_k}) = 0$.

Assume that for μ-a.e. $x \in Q_k$, any $y \in C_{\mathcal{B}}(x, V, \alpha_1)$, we have $\operatorname{dist}(x-y, V) > \alpha_1|x-y|$. If $|x-y| \geq (\sqrt{2}+s) \operatorname{side} Q_k$ then $\operatorname{dist}(y, Q_k)| \geq |y-x| - \operatorname{dist}(x, Q_k) \geq (\sqrt{2}+s-\sqrt{2}) \operatorname{side} Q_k = s \cdot \operatorname{side} Q$, so that $y \in S_{Q_k}$. On the other hand, if $|y-x| < (\sqrt{2}+s) \operatorname{side} Q_k$, then there exists $\epsilon_y > 0$ such that $\epsilon_y \leq |y-x|$. There also exists a $|k'| >> |k|$ such that for the dilation cube $Q_{k'}, x \in Q_{k'} \subset Q_k$, it satisfies $\epsilon_y \geq (\sqrt{2}+s) \cdot 2^{-|k'|}$, which similarly implies that $y \in S_{Q'_k}$. Hence, $C_{\mathcal{B}}(x, V, \alpha_1) \subset \bigcup_{i=k}^{\infty}\{S_{Q_i} : x \in Q_i \subseteq Q_k\}$. Thus, for each μ-a.e. $x \in E$ where x must be contained in only one Q_k,

$$\mu(C_{\mathcal{B}}(x, V, \alpha_1)) \leq \mu\left(\bigcup_{i=k}^{\infty}\{S_{Q_i} : x \in Q_i \subseteq Q_k\}\right) = 0$$

It follows immediately that for μ-a.e. $x \in E$, $\mu(C_{\mathcal{B}}(x, V, \alpha_1)) = 0$. Applying Corollary 1.2.1, the sufficient condition holds then.

1.7 June 29 $\lim_{Q\to x}\mu(C^{1,1}_{\mathcal{B}}(Q,V,\alpha)\cap 2sQ)/\mu(2sQ)=0$

Lemma 1.7.1 *Suppose $0<\alpha_2<\alpha_1<1$, $V\in G(1,2)$, a dyadic cube Q in $\mathbb{R}^2$ with the side length $2^{-|k|}$, and a dilation constant $s\in\mathbb{N}, s>1$. For doubling measure μ-a.e. x with the doubling constant $K\in\mathbb{Z}^+$, if*

$$\lim_{Q\to x}\frac{\mu(C^{1,1}_{\mathcal{B}}(Q,V,\alpha_2)\cap 2sQ)}{\mu(2sQ)}=0 \tag{24}$$

then μ is carried by Lipschitz graphs.

Proof. Fix a $Q\ni x$ with side length $2^{-|k|}$. For a dilation cube $2R\subset C^{1,1}_{\mathcal{B}}(Q,V,\alpha_1)$ with the double side length as Q, we can say that for some $x\in 2R\cap C^1_{\mathcal{B}}(Q,V,\alpha_1)$, there exists $q\in Q$ such that $x\in C_{\mathcal{B}}(q,V,\alpha_1)$. Now if we choose

$$s'=|x-p|\geq\frac{3\sqrt{2}(1+\alpha_2)}{\alpha_1-\alpha_2} \tag{25}$$

we claim that $2R\in C^{1,2}_{\mathcal{B}}(Q,V,\alpha_2)$. For any $y\in 2R$, assume some $q'\in Q$. Now we have $|x-y|\leq 2\sqrt{2}\,\mathrm{side}\,Q$, $|q-q'|\leq\sqrt{2}\,\mathrm{side}\,Q$. Since $|x-y|\geq|\mathrm{dist}(y-q',V)-\mathrm{dist}(x-q,V)|-|q-q'|$, no matter $\mathrm{dist}(y-q',V)\lesseqgtr\mathrm{dist}(x-q,V)$,

$$\begin{aligned}\mathrm{dist}(y-q',V)&\geq\mathrm{dist}(x-q,V)-|q-q'|-|x-y|\\&>\alpha_1|x-q|-3\sqrt{2}\cdot\mathrm{side}\,Q\\&\geq(\alpha_1 s'-3\sqrt{2})\cdot\mathrm{side}\,Q\\&\geq\alpha_2(s'+3\sqrt{2})\,\mathrm{side}\,Q\\&\geq\alpha_2(|x-q|+|y-x|+|q-q'|)\\&\geq\alpha_2|y-q'|\end{aligned}$$

Thus, there exists $q'\in Q$ such that $y\in C_{\mathcal{B}}(q',V,\alpha_2)$ and $2R\in C^{1,2}_{\mathcal{B}}(Q,V,\alpha_2)$ holds immediately. In particular, in this case,

$$d:=|\,\mathrm{center}\,Q-\mathrm{center}\,2R|\leq|q-x|+|\,\mathrm{center}\,2R-x|+|q-\mathrm{center}\,Q|\leq s'+\frac{3}{2}\sqrt{2} \tag{26}$$

Next, let the $s\in\mathbb{N}, s\geq s'$. With a $t\in\mathbb{N}$ in the form of $2^n, n\in\mathbb{N}$, we claim that

$$sQ\cap 2R=\emptyset\quad\text{ and }\quad 2R\subset 2sQ\subset tR \tag{27}$$

if $t\geq 2(\sqrt{2}+1)s+3\sqrt{2}$. It is obvious for the choice of t, since assume $\forall q''\in 2sQ$, then

$$|\,\mathrm{center}\,2R-q''|\leq(d+\sqrt{2}s)\,\mathrm{side}\,Q\leq\frac{t}{2}\,\mathrm{side}\,R$$

which implies that $2sQ\subset B(\mathrm{center}\,R,\frac{1}{2}\,\mathrm{side}\,tR)\subset tR$. Then, (27) for t is concluded. Assume that for any $p\in sQ, p'\in 2R$, SOMETHING SOMETHING SOMETHING

By (3), there exists a large enough k such that

$$\mu(C_{\mathcal{B}}^{1,1}(Q,V,\alpha_2)\cap 2sQ) < \delta\mu(2sQ) \tag{28}$$

As $2R \subset \mu(C_{\mathcal{B}}^{1,1}(Q,V,\alpha_2)\cap 2sQ)$, $\mu(tR) \leq K^{3^{t/2}}\mu(2R) \leq K^{3^{t/2}}\mu(C_{\mathcal{B}}^{1,1}(Q,V,\alpha_2)\cap 2sQ) < \delta K^{3^{t/2}}\mu(2sQ)$, which will be a contradiction if we choose $\delta < K^{-3^{t/2}}$ as $2sQ \subset tR$. Thus, $\mu(tR) = \mu(2R) = 0$ when fix the side Q and s.

Finally, we claim that $\mu(C_{\mathcal{B}}^{1,1}(Q,V,\alpha_1)) = 0$

After that, for μ-a.e. x in each Q, it follows immediately that $C_{\mathcal{B}}(x,V,\alpha) = 0$. Therefore, by Lemma 1.2.1, μ is carried by Lipschitz graphs.

Lemma 1.7.2 *For $\alpha_1, \alpha_2 \in (0,1), \alpha_1 > \alpha_2$, $V \in G(1,2)$, a dyadic cube Q in $\mathbb{R}^2$, and a dilation cube $2R$ with the double side length as Q, assume that for some $x \in 2R \cap C_{\mathcal{B}}^1(Q,V,\alpha_1)$, there exists $q \in Q$ such that $x \in C_{\mathcal{B}}(q,V,\alpha_1)$(i.e. $2R \in C_{\mathcal{B}}^{1,1}(Q,V,\alpha_1)$). Now if we choose*

$$s = |x-p| \geq \frac{3\sqrt{2}(1+\alpha_2)}{\alpha_1-\alpha_2} \tag{29}$$

then $2R \in C_{\mathcal{B}}^{1,2}(Q,V,\alpha_2)$.

Proof. For any $y \in 2R$, assume some $q' \in Q$. Now we have $|x-y| \leq 2\sqrt{2}\,\mathrm{side}\,Q$, $|q-q'| \leq \sqrt{2}\,\mathrm{side}\,Q$. Since $|x-y| \geq |\operatorname{dist}(y-q',V) - \operatorname{dist}(x-q,V)| - |q-q'|$, no matter $\operatorname{dist}(y-q',V) \lesseqgtr \operatorname{dist}(x-q,V)$,

$$\begin{aligned}
\operatorname{dist}(y-q',V) &\geq \operatorname{dist}(x-q,V) - |q-q'| - |x-y| \\
&> \alpha_1|x-q| - 3\sqrt{2}\cdot\mathrm{side}\,Q \\
&\geq (\alpha_1 s - 3\sqrt{2})\cdot\mathrm{side}\,Q \\
&\geq \alpha_2(s+3\sqrt{2})\,\mathrm{side}\,Q \\
&\geq \alpha_2(|x-q| + |y-x| + |q-q'|) \\
&\geq \alpha_2|y-q'|
\end{aligned}$$

Thus, there exists $q' \in Q$ such that $y \in C_{\mathcal{B}}(q',V,\alpha_2)$ and $2R \in C_{\mathcal{B}}^{1,2}(Q,V,\alpha_2)$ holds immediately. In particularly, in this case,

$$d := |\,\mathrm{center}\,Q - \mathrm{center}\,2R| \leq |q-x| + |\,\mathrm{center}\,2R - x| + |q - \mathrm{center}\,Q| \leq s + \frac{3}{2}\sqrt{2} \tag{30a}$$

$$d \geq ||q-x| - |\,\mathrm{center}\,2R - x| - |q - \mathrm{center}\,Q|| \geq s - \frac{3}{2}\sqrt{2} \tag{30b}$$

1.8 June 28 tR cover 2sQ

Lemma 1.8.1 *For $\alpha_1, \alpha_2 \in (0,1), \alpha_1 > \alpha_2$, $V \in G(1,2)$, a cube $Q \in \mathbb{R}^2$ with side length $2^{-|k|}$, and a cube R with the same side length as Q satisfies*

$$R \in C^{1,1}_{\mathcal{B}}(Q, V, \alpha_1) \quad \textit{and} \quad R \in C^{1,2}_{\mathcal{B}}(Q, V, \alpha_2) \tag{31}$$

And if $s, t \in \mathbb{N}$ are coefficients of the dilation cube and $d \cdot \operatorname{side} Q = |\operatorname{center} Q - \operatorname{center} R|$ such that

$$d + \frac{\sqrt{2}}{2} \leq s \leq \sqrt{2}d - 1 \quad \textit{and} \quad t \geq 2d + 2\sqrt{2}s \tag{32}$$

Then

$$sQ \cap R = \emptyset \quad \textit{and} \quad R \subset 2sQ \subset tR \tag{33}$$

Proof. Note that by Lemma 1.10.1, to satisfy (31),

$$d \geq \left| \frac{2 - 2\alpha_2}{\alpha_1 - \alpha_2} - \sqrt{2} \right|$$

For any $q \in sQ, q' \in R$, by (32),

$$\begin{aligned}
|\operatorname{center} sQ - q| \leq \frac{\sqrt{2}s}{2} \operatorname{side} Q &\leq \left| \left(d - \frac{\sqrt{2}}{2} \right) \right| \operatorname{side} Q \\
&\leq ||\operatorname{center} Q - \operatorname{center} R| - \frac{\sqrt{2}}{2} \operatorname{side} R| \\
&\leq ||\operatorname{center} Q - \operatorname{center} R| - |\operatorname{center} R - q'|| \\
&\leq |\operatorname{center} Q - q'| \qquad \text{(34a)} \\
&\leq \left(d + \frac{\sqrt{2}}{2} \right) \operatorname{side} Q \\
&\leq s \cdot \operatorname{side} Q \\
&= \frac{1}{2} \operatorname{side} 2sQ \qquad \text{(34b)}
\end{aligned}$$

Equation (34a) implies that $sQ \subset B(\operatorname{center} Q, \frac{\sqrt{2}}{2} \operatorname{side} 2sQ)$, $q' \notin B(\operatorname{center} Q, \frac{\sqrt{2}}{2} \operatorname{side} 2sQ)$ $\Rightarrow sQ \cap R = \emptyset$. Besides, (34a) to (34b) implies that $R \subset B(\operatorname{center} Q, \frac{1}{2} \operatorname{side} 2sQ) \subset 2sQ$. For the choice of t, assume $\forall q'' \in 2sQ$, then

$$|\operatorname{center} R - q''| \leq (d + \sqrt{2}s) \operatorname{side} Q \leq \frac{t}{2} \operatorname{side} R$$

which implies that $2sQ \subset B(\operatorname{center} R, \frac{1}{2} \operatorname{side} tR) \subset tR$. Then, (33) is concluded.

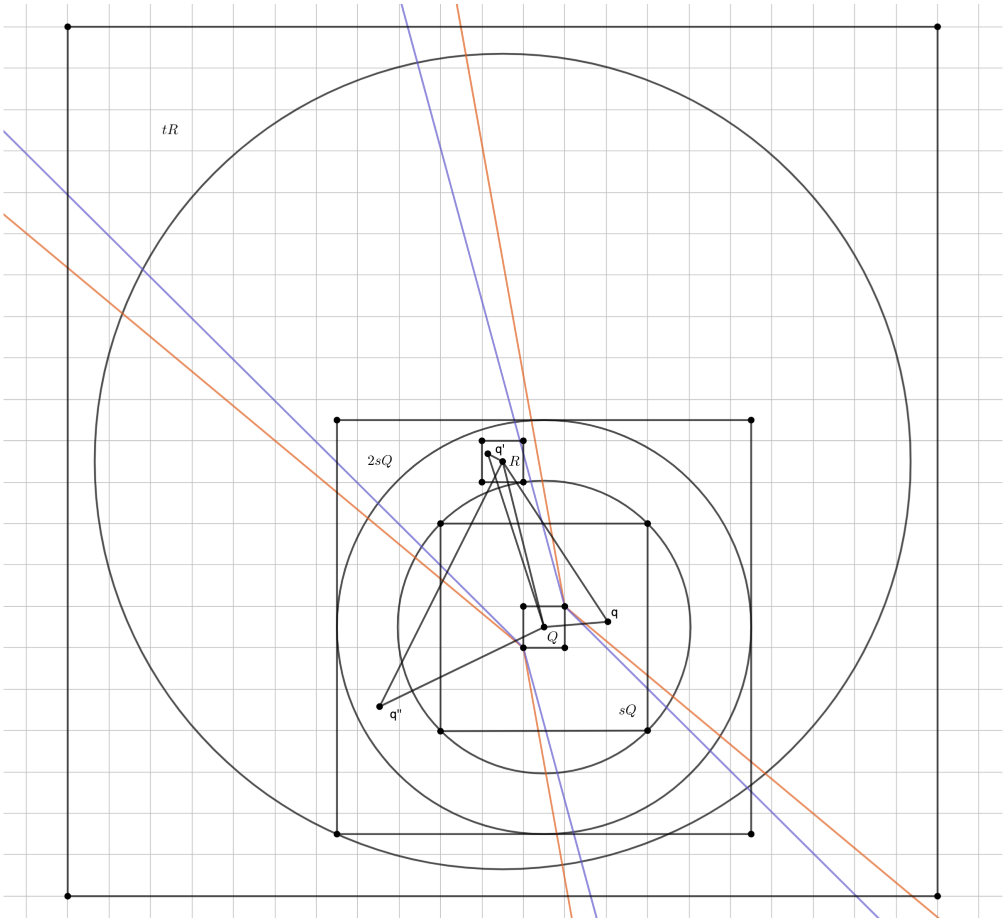


Figure 1: An example in the proof on $\mathbb{R}^2$

1.9 June 25-27 Modified Lemmas and New Lemmas

Lemma 1.9.1 *If for μ-a.e. $x \in \mathbb{R}^2, \mu(B(x, 2r)) \leq K\mu(B(x, r)), K \in \mathbb{R}^+$, then*

$$\mu(5Q) \leq K^4 \mu(3Q)$$

where Q is the dyadic cube, and $5Q$, $3Q$ are dilation cubes such that $\operatorname{side} 3Q = 3 \operatorname{side} Q$, $\operatorname{side} 5Q = 5 \operatorname{side} Q$, *and* $\operatorname{center} 5Q = \operatorname{center} 3Q = \operatorname{center} Q$.

Proof. By the half open definition of the dyatic cube, for μ-a.e. $x \in \mathbb{R}^2$, x must in only one cube Q. Let $r = \operatorname{side} Q/2$(in $\mathbb{R}^2$). We first claim that $5Q \subset B(x, 16r)$ and $B(x, r) \subset 3Q$. As for any $q \in 5Q$, $|\operatorname{center} Q - x| \leq \sqrt{2} \cdot \operatorname{side} Q/2$ and $|\operatorname{center} 5Q - q| \leq \sqrt{2} \cdot \operatorname{side} 5Q/2 = 5\sqrt{2} \cdot \operatorname{side} Q/2$,

$$|x - q| = |x - \operatorname{center} Q + \operatorname{center} 5Q - q| \leq 3\sqrt{2} \cdot \operatorname{side} Q < \frac{16 \operatorname{side} Q}{2} = 16r$$

Then $q \in B(x, 16r)$ and thus $5Q \subset B(x, 16r)$. Similarly, for any $q' \in B(x, r)$,

$$|\operatorname{center} 3Q - q'| = |\operatorname{center} Q - x + x - q'| \leq \frac{\sqrt{2} \cdot \operatorname{side} Q}{2} + r < \frac{3 \operatorname{side} Q}{2} = \frac{1}{2} \operatorname{side} 3Q$$

Then $q' \in 3Q$ and thus $B(x, r) \subset 3Q$. Therefore, by the containments and the doubling measure condition,

$$\mu(5Q) \leq \mu(B(x, 16r)) \leq K^4 \mu(B(x, r)) \leq K^4 \mu(3Q) \tag{35}$$

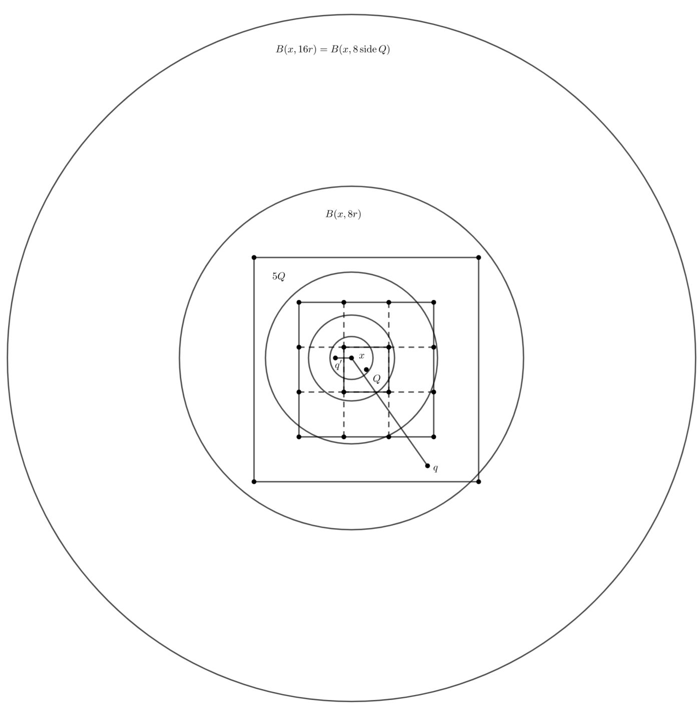


Figure 2: An example of containment in the proof on $\mathbb{R}^2$

Lemma 1.9.2 *Let μ be a Radon measure on $\mathbb{R}^n$, V be an m-dimensional linear plane in $\mathbb{R}^n$, $\alpha \in (0,1)$. Suppose $E \subset \mathbb{R}^n$ such that for μ-a.e. $x \in E$, and for every dyadic cube Q containing y,*

$$\mu(C^2_{\mathcal{B}}(Q, V, \alpha)) = 0 \tag{36}$$

Then μ is carried by m-Lipschitz graphs(i.e. E is contained μ-a.e. in an m-Lipschitz graph.)

Proof. we first claim that if every dyadic cube $Q_k \ni x$ in the form of (50) with side length $2^{-|k|}$ satisfies (36), then

$$\mu(C_{\mathcal{B}}(x, V, \alpha)) = 0 \tag{37}$$

Fix arbitrary $q_1 \in Q_k$, if arbitrary $p \in C_{\mathcal{B}}(q_1, V, \alpha)$, we have $\operatorname{dist}(p - q_1, V) > \alpha|p - q_1|$ by definition of bad cones. Now choose $\epsilon_p > 0$ that satisfies $\operatorname{dist}(p - q_1, V) \geq \alpha(|p - q_1| + \epsilon_p)$. Then, for another point q_2 in $Q_k \ni q_1$, we have $|q_1 - q_2| \leq \sqrt{n} \cdot 2^{-|k|}$. Hence, there exists a large enough $|k_p|$ such that $|q_1 - q_2| < \alpha\epsilon_p/2 < \epsilon_p/2$. Then,

$$\begin{aligned}
\operatorname{dist}(p - q_2, V) &\geq \operatorname{dist}(p - q_1, V) - |q_1 - q_2| \\
&\geq \alpha(|p - q_1| + \epsilon_p) - \alpha(\epsilon_p/2) \\
&= \alpha(|p - q_1| + \epsilon_p/2) \\
&> \alpha\left(|p - q_1| + |q_2 - q_1|\right) \\
&\geq \alpha\left(|p - q_2|\right)
\end{aligned}$$

It follows that $p \in C^2_{\mathcal{B}}(q_2, V, \alpha)$ and thus $p \in \bigcap_{q_i \in Q} C_{\mathcal{B}}(q_i, V, \alpha) = C^2_{\mathcal{B}}(Q_{k_p}, V, \alpha)$. Accordingly, for any $x \in E$, if any $p \in C_{\mathcal{B}}(x, V, \alpha)$, then $p \in C_{\mathcal{B}}(Q_k, V, \alpha)$, $x \in Q_k$ for some $|k|$, which implies

$$p \in \bigcup_{|k|=1}^{\infty} C^2_{\mathcal{B}}(Q_k, V, \alpha) \Rightarrow C_{\mathcal{B}}(x, V, \alpha) \subset \bigcup_{|k|=1}^{\infty} \left\{C^2_{\mathcal{B}}(Q_k, V, \alpha) : Q_k \ni x\right\} \tag{38}$$

Combining (36) and (38), for μ-a.e. $x \in E$,

$$\mu(C_{\mathcal{B}}(x, V, \alpha)) \leq \mu\left(\bigcup_{k=1}^{\infty} \left\{C^2_{\mathcal{B}}(Q_k, V, \alpha) : Q_k \ni x\right\}\right) = 0$$

Therefore, (37) holds immediately and after applying corollary 1.2.1, we obtain the desired result.

1.10 June 23-24 Lemma of Guarantee Distance Between Cubes

Lemma 1.10.1 *For $\alpha_1, \alpha_2 \in (0,1), \alpha_1 > \alpha_2$, a m-plane V in $\mathbb{R}^n$, a cube Q with side length $2^{-|k|}$, and a cube R with the same side length as Q, the distance bewteen the center of R and the center of Q has to satisfy*

$$|\operatorname{center} Q - \operatorname{center} R| \geq \left| \frac{2\sqrt{n} - 2\alpha_2}{\alpha_1 - \alpha_2} - \sqrt{n} \right| \cdot 2^{-|k|}$$

to guarantee

$$R \in C^{1,1}_{\mathcal{B}}(Q, V, \alpha_1) \quad \textit{and} \quad R \in C^{1,2}_{\mathcal{B}}(Q, V, \alpha_2) \tag{39}$$

Proof. By assumption, there are some $x \in R \cap C^{1,1}_{\mathcal{B}}$ such that there exists $q \in Q$ and $x \in C_{\mathcal{B}}(q, V, \alpha_1)$. Then $\operatorname{dist}(x - q, V) > \alpha_1 |x - q|$. Now assume that $|x - q| = s \cdot 2^{-|k|}$. Then to satisfy (39), for any $y \in R$, there exists $q' \in Q$ such that $y \in C_{\mathcal{B}}(q', V, \alpha)$. Now we have $|x-y| \leq \sqrt{n} 2^{-|k|}$, $|q - q'| \leq \sqrt{n} 2^{-|k|}$. Since $|x-y| \geq |\operatorname{dist}(y-q', V) - \operatorname{dist}(x-q, V)| - |q-q'|$, no matter $\operatorname{dist}(y - q', V) \lesseqgtr \operatorname{dist}(x - q, V)$,

$$\begin{aligned}
\operatorname{dist}(y - q', V) &\geq \operatorname{dist}(x - q, V) - |q - q'| - |x - y| \\
&> \alpha_1 |x - q| - \sqrt{n} 2^{-|k|+1} \\
&= \alpha_1 s 2^{-|k|} - \sqrt{n} 2^{-|k|+1}
\end{aligned}$$

To guarantee that $\operatorname{dist}(y - q', V) > \alpha_2 |y - q'|$, we let $\alpha_1 s 2^{-|k|} - \sqrt{n} 2^{-|k|+1} \geq \alpha_2 |y - q'|$ then the assumption holds. Equivalently,

$$\begin{aligned}
\alpha_1 s 2^{-|k|} - \sqrt{n} 2^{-|k|+1} &\geq \alpha_2 |y - q'| \\
&= \alpha_2 |x - q + y - x + q - q'| \\
&\geq \alpha_2 ||x - q| - |y - x| - |q - q'|| \\
&\geq \alpha_2 |s 2^{-|k|} - 2^{-|k|+1}|
\end{aligned}$$

And we have

$$s \geq \frac{2\sqrt{n} - 2\alpha_2}{\alpha_1 - \alpha_2}$$

As $|q - \operatorname{center} Q| \leq \sqrt{n} 2^{-|k|-1}$ and $|x - \operatorname{center} R| \leq \sqrt{n} 2^{-|k|-1}$,

$$\begin{aligned}
|\operatorname{center} Q - \operatorname{center} R| &\geq ||x - q| - |x - \operatorname{center} R| - |q - \operatorname{center} Q|| \\
&\geq \left| \frac{2\sqrt{n} - 2\alpha_2}{\alpha_1 - \alpha_2} - \sqrt{n} \right| \cdot 2^{-|k|}
\end{aligned}$$

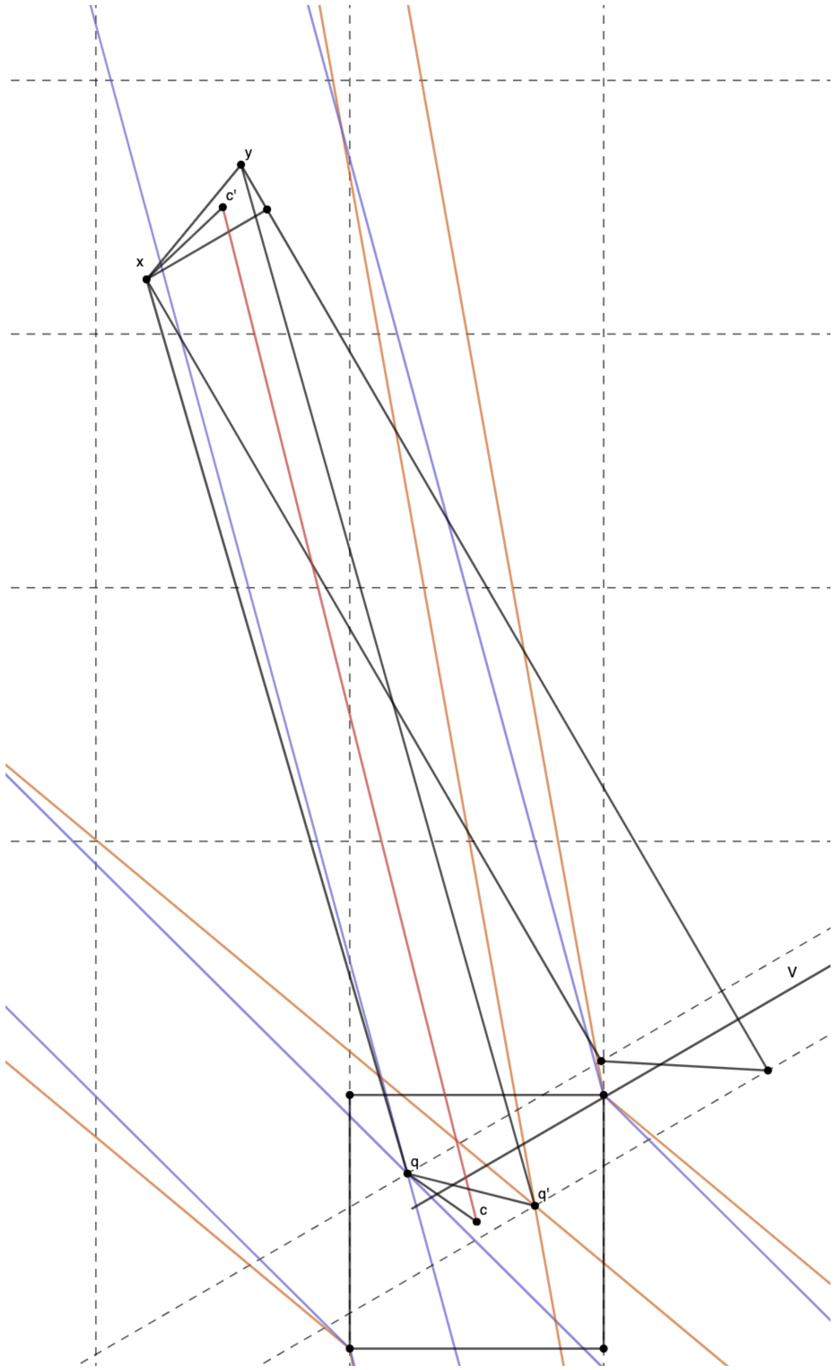


Figure 3: An example of containment in the proof on $\mathbb{R}^2$

1.11 June 22 Doubling Measure for Cubes and other Lemmas

Lemma 1.11.1 *Let μ be a pointwise doubling measure on $\mathbb{R}^n$. For μ-a.e. $x \in \mathbb{R}^n$, Q with side length $2^{-|k|}$ containing x, there is an m-plane V and an $\alpha \in (0,1)$ such that with a scaling constant s,*

$$\lim_{|k|\to\infty} \frac{\mu\left(C_{\mathcal{B}}(Q, V, \alpha)\right)}{\mu(sQ))} = 0 \tag{40}$$

if and only if μ is carried by Lipschitz graphs.

Proof for Sufficient Condition. With the same notation in Lemma 1.13.1, where $E = \bigcup_{i=1}^{\infty} E_i$ and for μ-a.e. $y \in E$, $E_i = Q_{E_i} \cap E$. Fix a Q_{E_i}, by (40), for any $\epsilon > 0$, there exists $N_i \in \mathbb{N}$ such that when $k \geq N_i$,

$$\left|\frac{\mu\left(C^2_{\mathcal{B}}(Q_{E_i}, V, \alpha)\right)}{\mu(sQ_{E_i}))}\right| < \epsilon$$

Define $r = 2^{-|k|}s$, then $\lim_{|k|\to\infty} r = 0$. Thus, there exists $N' \in \mathbb{N}$ such that whenever $k \geq N'$, $|r| < \epsilon$. Thus, choosing $N = \max\{\{N_i\} \cup \{N'\}$, two inequalities hold for every Q_{E_i} in the meanwhile. Then for μ-a.e. $y \in Q_{E_i}$ and by (49), $s \cdot Q_{E_i} \subset B(y,r)$ and $C_{\mathcal{B}}(y, r, V, \alpha) \subset C_{\mathcal{B}}(y, V, \alpha) \subset C^2_{\mathcal{B}}(Q_{E_i}, V, \alpha)$. Thus, let $\delta = \epsilon$, we have for any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $|r - 0| < \delta$, by containment,

$$\left|\frac{\mu(C_{\mathcal{B}}(y, r, V, \alpha))}{\mu(B(y,r))}\right| \leq \left|\frac{\mu\left(C^2_{\mathcal{B}}(Q_{E_i}, V, \alpha)\right)}{\mu(sQ_{E_i}))}\right| < \epsilon \tag{41}$$

Therefore

$$\lim_{r\downarrow 0} \frac{\mu(C_{\mathcal{B}}(y, r, V, \alpha))}{\mu(B(y,r))} = 0 \tag{42}$$

Applying [Nap20, Theorem D], the conclusion holds.

Theorem [Nap20, Theorem D] *Let μ be a pointwise doubling measure on a separable, finite or infinite dimensional Hilbert space H. For μ-a.e. $x \in H$ there is an m -plane V and an $\alpha \in (0,1)$ such that*

$$\lim_{r\downarrow 0} \frac{\mu\left(C_{\mathcal{B}}(x, r, V, \alpha)\right)}{\mu(B(x,r))} = 0$$

if and only if μ is carried by Lipschitz graphs.

Lemma 1.11.2 *If μ-a.e. $x \in \mathbb{R}^n, \mu(B(x,2r)) \leq K\mu(B(x,r)), K \in \mathbb{Z}^+$, then $\mu(2Q) \leq K^3\mu(Q)$ where Q is the dyadic cube, $2Q$ is the cube with double dilation of Q.*

Proof. By the half open definition of the dyatic cube, for μ-a.e. $x \in \mathbb{R}^n$, x must in only one cube, named Q with side $Q = 2^{-|k|}$. Let $r = \frac{1}{2}$ side Q, then by the containment

$$\mu(4Q) \leq \mu(B(x, 8r)) \leq K^3\mu(B(x,r)) \leq K^3\mu(2Q) \tag{43}$$

$$\mu(5Q) \leq \mu(B(x, 16r)) \leq K^4\mu(B(x,r)) \leq K^4\mu(3Q) \tag{44}$$

There always exists k such that any μ-a.e. x must in $1/2Q$ with side length $2^{-|k|}$. Then similarly we have $\mu(2Q) \leq K^3\mu(Q)$.

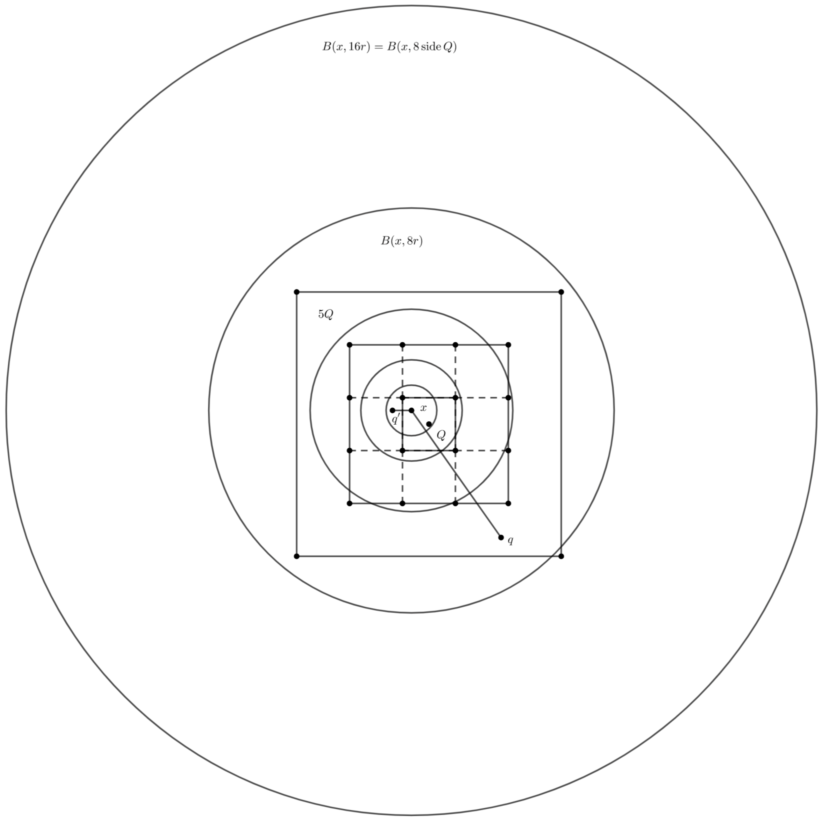


Figure 4: An example of containment in the proof on $\mathbb{R}^2$

1.12 June 21 $\lim_{Q\to x}\mu(C_{\mathcal{B}}(Q,V,\alpha))/\mu(sQ)=0$

Definition 1.12.1 (Alternative Definition for Bad Cones at a Cube) *For the dyadic cube Q, let R denote the dyadic cube with the same side length as Q in the cube system. Then definitions of bad cones at a cube can be*

$$C_{\mathcal{B}}^{k,1}(Q,V,\alpha)=\{R:R\cap C_{\mathcal{B}}^{k}(Q,V,\alpha)\neq\emptyset\},\quad C_{\mathcal{B}}^{k,2}(Q,V,\alpha)=\{R:R\subset C_{\mathcal{B}}^{k}(Q,V,\alpha)\}$$

for $k=1,2$.

Theorem A *Let μ be a pointwise doubling measure on $\mathbb{R}^n$. For μ-a.e. $x\in\mathbb{R}^n$, Q that contains x, there is an m-plane V and an $\alpha\in(0,1)$ such that with a scaling constant s,*

$$\lim_{Q\downarrow x}\frac{\mu\left(C_{\mathcal{B}}(Q,V,\alpha)\right)}{\mu(sQ))}=0 \tag{45}$$

if and only if μ is carried by Lipschitz graphs.

Proof for Sufficient Condition. With the same notation in Lemma 1.13.1, where $E=\bigcup_{i=1}^{\infty}E_i$ and for some $y_i\in E$, $E_i=\{y_i\}\subset Q_{E_i}$. Fix $y_0\in Q_{E_i}$ and by (49), when $|k|\downarrow 0$, $C_{\mathcal{B}}(y_0,V,\alpha)\subset C_{\mathcal{B}}^2(Q_{E_i},V,\alpha)$. Then if $r=s\cdot 2^{-|k|}$, we have $s\cdot Q_{E_i}\subset B(y_0,r)$ and $C_{\mathcal{B}}(y_0,r,V,\alpha)\subset C_{\mathcal{B}}(y_0,V,\alpha)\subset C_{\mathcal{B}}^2(Q_{E_i},V,\alpha)$. Now as $|k|\downarrow 0\Rightarrow r\downarrow 0$ and by (45),

$$\lim_{r\downarrow 0}\frac{\mu(C_{\mathcal{B}}(y_i,r,V,\alpha))}{\mu(B(y_i,r))}\leq\lim_{Q\downarrow x}\frac{\mu\left(C_{\mathcal{B}}(Q,V,\alpha)\right)}{\mu(sQ))}=0 \tag{46}$$

Applying 46 for every y in every Q_{E_i}, and [Nap20, Theorem D], the conclusion holds.

1.13 June 17-20 $\mu(C^2_{\mathcal{B}}(Q,V,\alpha)) = 0 \Rightarrow \mu$ Carried by Lipschitz Graph

Lemma 1.13.1 *Let μ be a Radon measure on $\mathbb{R}^n$, V be an m-dimensional linear plane in $\mathbb{R}^n$, $\alpha \in (0,1)$. Suppose $E \subset \mathbb{R}^n$ such that for μ-a.e. $y \in E$, and for every dyadic cube Q containing y,*

$$\mu(C^2_{\mathcal{B}}(Q,V,\alpha)) = 0 \tag{47}$$

Then μ is carried by m-Lipschitz graphs(i.e. E is contained μ-a.e. in an m-Lipschitz graph.)

Proof. we first claim that if every dyadic cube $Q \ni y$ in the form of (50) with side length $2^{-|k|}$ satisfies (47), then

$$\mu(C_{\mathcal{B}}(y,V,\alpha)) = 0 \tag{48}$$

Note that each $y \in E$ is contained in no more than one half open Q. Let Q_{E_i} denote the dyadic cube that contains $E_i \subset E$ such that E_i consists of all $y \in E \cap Q_{E_i}$ and E is the countable union of disjoint E_i. For any two points $x_1 \in Q_{E_i}, y_1 \in E_i$, we have $|x_1 - y_1| \le \sqrt{n}\cdot 2^{-|k|}$. Consider that if any $p \in C_{\mathcal{B}}(y_1,V,\alpha)$, we have $\operatorname{dist}(p-y_1,V) > \alpha|p-y_1|$ by definition of bad cones. Now let $\epsilon > 0$ such that $\operatorname{dist}(p-y_1,V) \ge \alpha(|p-y_1|+\epsilon)$. When $|k| \to \infty, 2^{-|k|} \to 0$, so there exists $N \in \mathbb{N}$ such that whenever $|k| \ge N$, for $x_1, y_1 \in Q_{E_i}$ we have $|x_1 - y_1| < \alpha\epsilon/2 < \epsilon/2$, recalling that $\alpha \in (0,1)$. Then

$$\begin{aligned}
\operatorname{dist}(p - x_1, V) &\ge \operatorname{dist}(p-y_1,V) - |x_1 - y_1| \\
&\ge \alpha(|p-y_1|+\epsilon) - \alpha(\epsilon/2) \\
&= \alpha(|p-y_1|+\epsilon/2) \\
&> \alpha\left(|p-y_1| + |x_1-y_1|\right) \\
&\ge \alpha\left(|p-x_1|\right)
\end{aligned}$$

It follows that $p \in C_{\mathcal{B}}(x_1,V,\alpha)$, and as x_1 is arbitrary point in Q_{E_i}, $p \in C^2_{\mathcal{B}}(Q_{E_i},V,\alpha)$. Thus, when $|k| \to \infty$

$$C_{\mathcal{B}}(y_1,V,\alpha) \subset C_{\mathcal{B}}(Q_{E_i},V,\alpha), \quad \forall y_1 \in E_i \subset Q_{E_i} \tag{49}$$

Then applying (47),

$$\mu(\bigcup_{y\in E} C_{\mathcal{B}}(y,V,\alpha)) = \lim_{|k|\to\infty} \mu\left(\bigcup_{E_i\in E}\bigcup_{y\in Q_{E_i}} C_{\mathcal{B}}(y,V,\alpha)\right) \le \lim_{|k|\to\infty} \mu\left(\bigcup_{E_i\in E} C^2_{\mathcal{B}}(Q_{E_i},V,\alpha)\right) = 0$$

Therefore, we have (48) for each $y \in E$. Then applying corollary 1.2.1, we obtain the desired result.

An example of this proof illustrated in $\mathbb{R}^2$ is shwon in Figure 5.

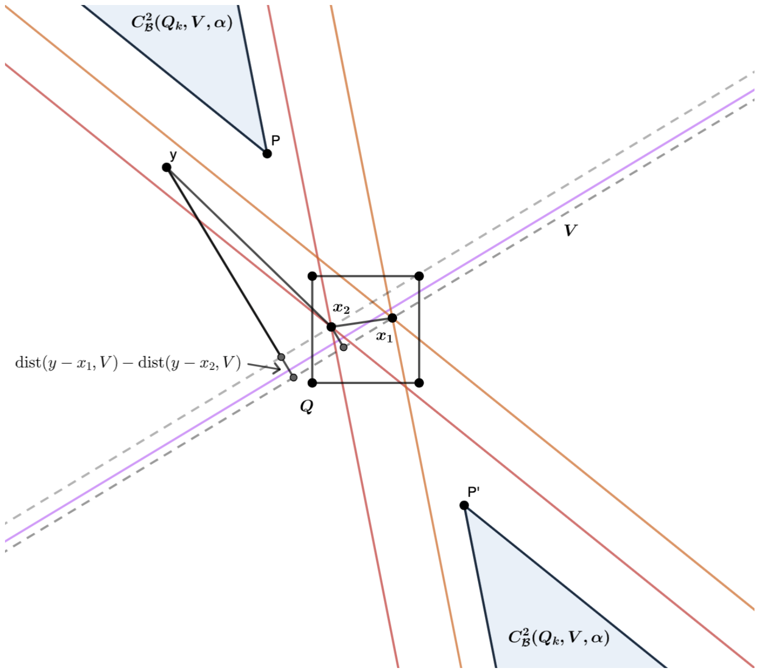


Figure 5: Example visulization of proof in $\mathbb{R}^2$

Corollary 1.13.1 ([Nap20, Corollary 7.1]) *Let μ be a Radon measure on H, V be an m-dimensional linear plane in H, $\alpha \in (0,1)$, and $0 < r < \infty$. If for μ -a.e. $x \in H$*

$$\mu\left(C_{\mathcal{B}}(x, r, V, \alpha)\right) = 0$$

then μ is carried by m-Lipschitz graphs.

1.14 June 16 New Cone Definition for Dyadic Cubes

Definition 1.14.1 (Dyadic Cubes) *A dyadic cube Q is a set of the form*

$$Q = \left[\frac{j_1}{2^k}, \frac{j_1+1}{2^k}\right) \times \cdots \times \left[\frac{j_n}{2^k}, \frac{j_n+1}{2^k}\right), \quad k, j_1, \ldots, j_n \in \mathbb{Z} \tag{50}$$

Definition 1.14.2 (Bad Cone at a Cube(definition 1)) *Let Q be the dyadic cube, then we can have the first definition of bad cone at Q with respect to V and α by:*

$$C^1_{\mathcal{B}}(Q, V, \alpha) := \bigcup_{x \in Q} C_{\mathcal{B}}(x, V, \alpha), \quad C^1_{\mathcal{B}}(Q, r, V, \alpha) = C^1_{\mathcal{B}}(Q, V, \alpha) \cap B(x, r)$$

where V is a m-dimensional linear plane through the origin.

Definition 1.14.3 (Bad Cone for Cube(definition 2)) *We can have the second definition of bad cone at Q with respect to V and α with the similar notation:*

$$C^2_{\mathcal{B}}(Q, V, \alpha) := \bigcap_{x \in Q} C_{\mathcal{B}}(x, V, \alpha), \quad C^2_{\mathcal{B}}(Q, r, V, \alpha) = C^2_{\mathcal{B}}(Q, V, \alpha) \cap B(x, r)$$

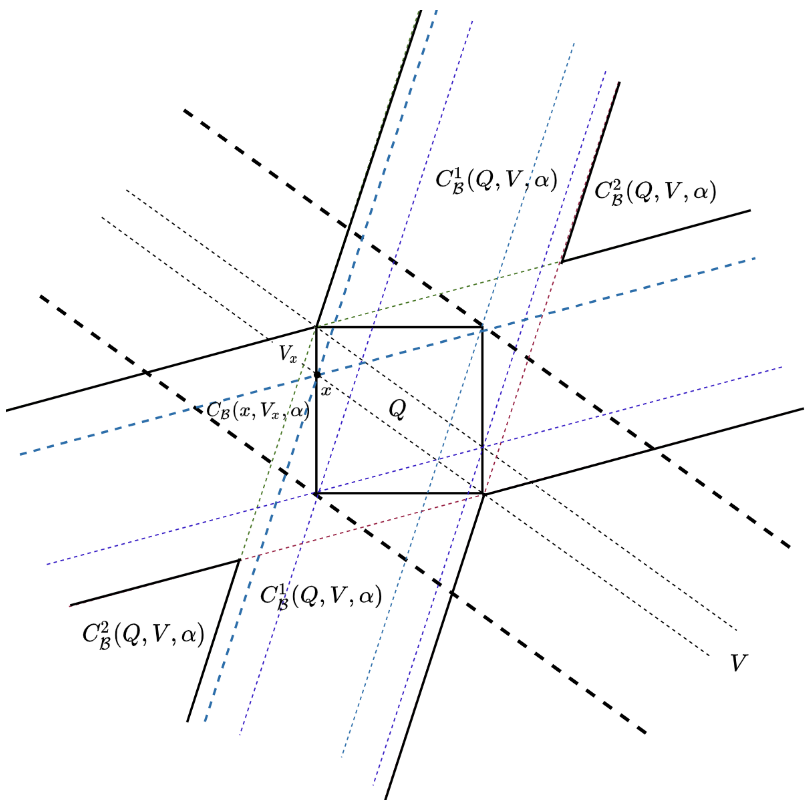


Figure 6: Visulize two definitions of bad cone at a cube in $\mathbb{R}^2$

Problem 1.14.1 *Suppose that at μ-a.e. x, $\alpha \in (0,1)$, for all k, for every cube $Q \ni x$ of side length 2^{-k} satisfies*

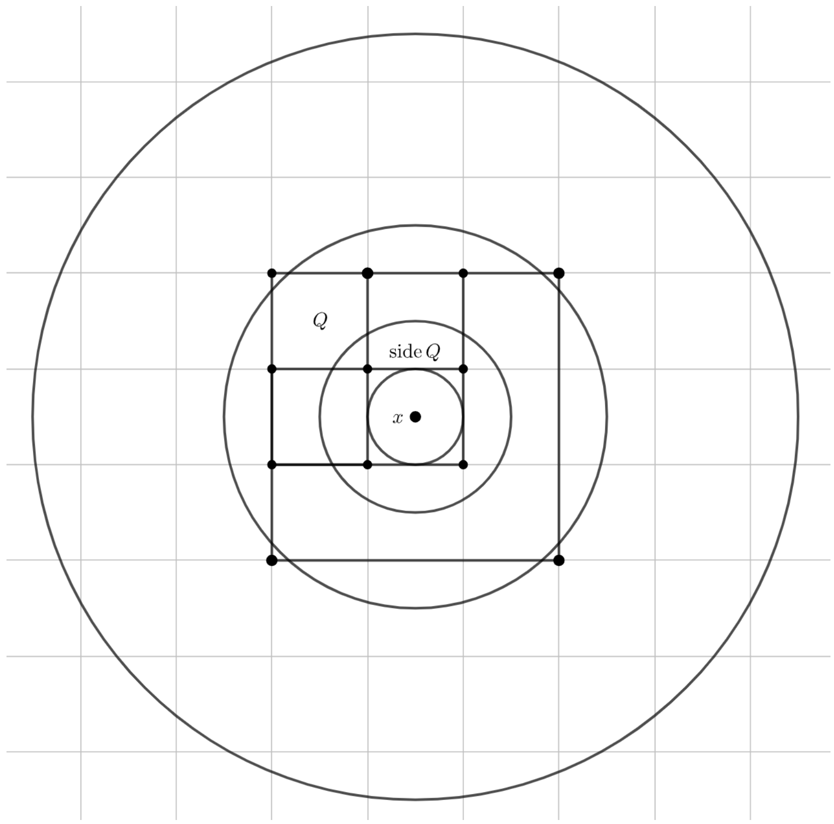
$$\mu(C^2_{\mathcal{B}}(Q, V, \alpha)) = 0$$

is μ-carried Lipschitz by Lipschitz graphs?

1.15 June 15 A Proposition of Doubling Measure for Cubes

Proposition 1.15.1 *If $\forall x \in \mathbb{R}^2, \mu(B(x,2r)) \leq K\mu(B(x,r)), K \in \mathbb{Z}^+$, then $\exists R \in \mathbb{Z}^+$ s.t. $\mu(3Q) \leq R\mu(Q)$ where Q is the cube, 3Q is the cube with triple side length with same center as Q.*

Proof. $\mu(3Q) \leq \mu(B(x, 4\,\mathrm{side}\,Q)) \leq K^3\mu(B(x, \frac{1}{2}\,\mathrm{side}\,r)) \leq K^3\mu(Q)$ due to doubling measure and containment.



1.16 June 14 Paper [Nap20] Sec. 7 Graph Rectifiable Measures

Definition 1.16.1 (Good Cone) *Let V be a m-dimensional plane in Hilbert space H, then we can define the good cone at x with respect to V and α by:*

$$C_{\mathcal{G}}(x,V,\alpha) := \{y \in H : \operatorname{dist}(y-x,V) \leq \alpha|x-y|\}$$

another notation:

$$C_{\mathcal{G}}(x,r,V,\alpha) = C_{\mathcal{G}}(x,V,\alpha) \cap B(x,r)$$

Definition 1.16.2 (Bad Cone) *The bad cone at x with respect to V and α:*

$$C_{\mathcal{B}}(x,V,\alpha) := H \setminus C_{\mathcal{G}(x,V,\alpha)}, \quad C_{\mathcal{B}}(x,r,V,\alpha) = C_{\mathcal{B}}(x,V,\alpha) \cap B(x,r)$$

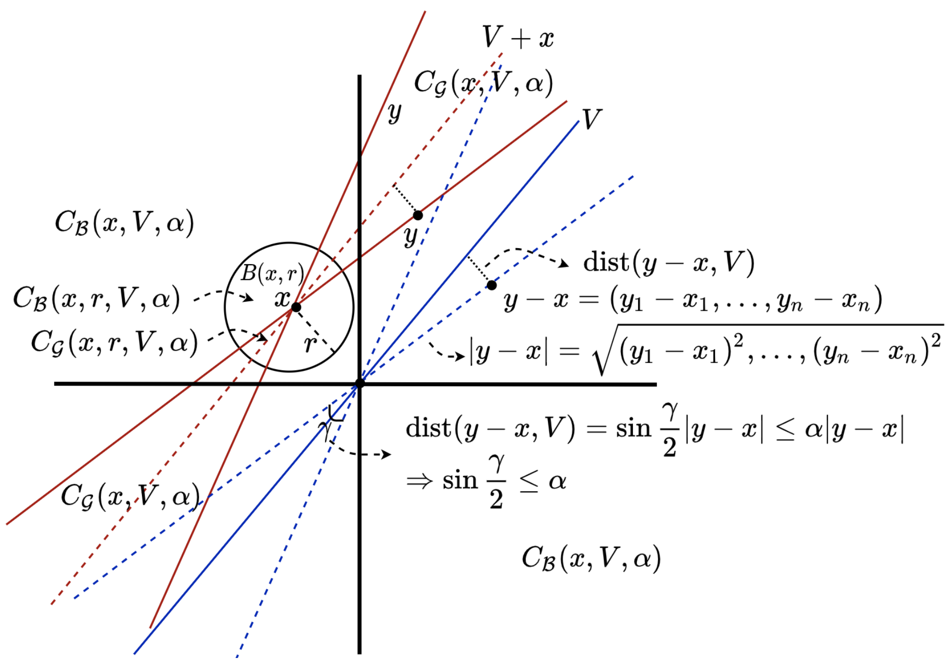


Figure 7: Example visualization of definitions related to cones in $\mathbb{R}^2$

Definition 1.16.3 (Carried & Singular) *Let $(\mathbb{X},\mathcal{M})$ be a measurable space, and let $\mathcal{N} \subset \mathcal{M}$ be a family of measurable sets. We say*

(1) μ is carried by $\mathcal{N}$ if there exist countably many $N_i \in \mathcal{N}$ such that $\mu\left(\mathbb{X} \setminus \bigcup_i N_i\right) = 0$;

(2) μ is singular to $\mathcal{N}$ if $\mu(N) = 0$ for every $N \in \mathcal{N}$.

A σ -finite measure μ on $(\mathbb{X},\mathcal{M})$ can be decomposed uniquely as

$$\mu = \mu_{\mathcal{N}} + \mu_{\mathcal{N}}^{\perp}$$

where $\mu_{\mathcal{N}}$ is carried by $\mathcal{N}$ and $\mu_{\mathcal{N}}^{\perp}$ is singular to $\mathcal{N}$.

Theorem 1.16.1 (Theorem 7.1 Geometric Lemma)
Let $F \subset H$, let V be an m-dimensional linear plane in H, and let $\alpha \in (0,1)$. If

$$F \backslash C_{\mathcal{G}}(x, V, \alpha) = \emptyset \text{ for all } x \in F$$

then F is contained in an m -Lipschitz graphs. In particular, $F \subset \Gamma$ where Γ is a Lipschitz graph with respect to V and the Lipschitz constant corresponding to Γ is at most $1 + 1/\left(1-\alpha^2\right)^{1/2}$.

Proof. Let $x \in F$. Let $P_V : H \to V$ denote standard projection onto the m -plane V. Suppose that $|P_V x - P_V y| < \left(1-\alpha^2\right)^{1/2}|x-y|$. Then $y \in C_{\mathcal{B}}(x, V, \alpha)$, and by assumption of F this means that $y \notin F$. Thus we may assume that if $x, y \in F$ then

$$|P_V x - P_V y| \geq \left(1-\alpha^2\right)^{1/2}|x-y|$$

From this inequality we see that $P_V \mid F$ is one-to-one with Lipschitz inverse $f = (P_V \mid F)^{-1}$ and $\mathrm{Lip}(f) \leq \left(1-\alpha^2\right)^{-1/2}$. Note that $F = f\left(P_V \mid F\right)$. Then there exists a Lipschitz extension $\tilde{f} : V \to H$ so that $F \subset \tilde{f}(V)$. Thus the desired result holds.

Corollary 1.16.1 (Corollary 7.1) *Let μ be a Radon measure on H, V be an m-dimensional linear plane in H, $\alpha \in (0,1)$, and $0 < r < \infty$. If for μ -a.e. $x \in H$*

$$\mu\left(C_{\mathcal{B}}(x, r, V, \alpha)\right) = 0$$

then μ is carried by m -Lipschitz graphs.

Proof. Let F denote the set of $x \in H$ that satisfy (18). We may assume $F \subset B(0, r/2)$; otherwise we may write F as a union of countably many sufficiently small sets and show that each one is m -graph rectifiable. Let $\{x_i\}$ be a countable dense subset of F. It follows from (18) and the containment $F \subset B(0, r/2)$ that for each x_i there exists $F_i \subset F$ such that

$$F_i \cap C_{\mathcal{B}}\left(x_i, r, V, \alpha\right) = F_i \cap C_{\mathcal{B}}\left(x_i, V, \alpha\right) = \emptyset$$

and $\mu\left(F \backslash F_i\right) = 0$. Define $F' := \bigcap_{i=1}^{\infty} F_i$. Then

$$\mu\left(F \backslash F'\right) = \mu\left(F \backslash \bigcap_{i=1}^{\infty} F_i\right) = \mu\left(\bigcup_{i=1}^{\infty} F \backslash F_i\right) \leq \sum_{i=1}^{\infty} \mu\left(F \backslash F_i\right) = 0$$

We claim that $F' \cap C_{\mathcal{B}}(x, V, \alpha) = \emptyset$ for every $x \in F'$. Fix $x \in F'$, and let $y \in C_{\mathcal{B}}(x, V, \alpha)$. By definition of bad cone we have that $\mathrm{dist}(y-x, V) > \alpha|y-x|$. Now let $\epsilon > 0$ such that $\mathrm{dist}(y-x, V) \geq \alpha(|y-x|+\epsilon)$. Recalling that $0 < \alpha < 1$, choose x_i such that $|x_i - x| < \alpha\epsilon/2 < \epsilon/2$. Then

$$\begin{aligned}
\mathrm{dist}\left(y-x_i, V\right) &\geq \mathrm{dist}(y-x, V) - |x-x_i| \\
&\geq \alpha(|y-x|+\epsilon) - \alpha(\epsilon/2) \\
&= \alpha(|y-x|+\epsilon/2) \\
&> \alpha\left(|y-x| + |x_i - x|\right) \\
&\geq \alpha\left(|y-x_i|\right).
\end{aligned}$$

In particular, we conclude that $y \in C_{\mathcal{B}}\left(x_i, V, \alpha\right)$. Since $F_i \cap C_{\mathcal{B}}\left(x_i, V, \alpha\right) = \emptyset$, it must be that case that $y \notin F_i$. It follows that $y \notin F'$, and thus $F' \cap C_{\mathcal{B}}(x, V, \alpha) = \emptyset$ for all $x \in F'$. By an application of Theorem 7.1 we conclude that there exists an m -Lipschitz graph Γ such that $F' \subset \Gamma$, so $\mu(F \backslash \Gamma) = 0$.

Lemma 1.16.1 (Lemma 7.2) *Let μ be a Radon measure on H. For $x_0 \in H, V$ an m-dimensional linear plane, $\alpha \in (0,1)$, and parameter $K > 0$, let E denote the set of points $x \in H$ such that (i) The sequence of functions*

$$f_r(x) := \frac{\mu\left(C_{\mathcal{B}}(x, r, V, \alpha)\right)}{\mu(B(x, r))}$$

converges to 0 uniformly on E, and (ii) there exists $r_1 > 0$ such that at every $x \in E$,

$$\mu(B(x, 2r)) \leq K \mu(B(x, r)) \text{ for all } r \in (0, r_1]$$

Then E is μ -carried by m -Lipschitz graphs with Lipschitz constants depending on at most K and α.

Proof. Fix $\delta > 0$. By uniform convergence, choose $r_\delta \leq r_1$ such that for all $r < r_\delta$ and for all $x \in E$ (19)

$$\frac{\mu\left(C_{\mathcal{B}}(x, 2r, V, \alpha)\right)}{\mu(B(x, 2r))} < \delta$$

Fix $x \in E$, and define $S := E \cap C_{\mathcal{B}}(x, r, V, 2\alpha)$. Assuming the set is non-empty, fix $y_0 \in S$ such that $|x - y_0| = \max_{y \in S} |a - y| =: \lambda r$ for some $0 < \lambda \leq 1$. As an application of Lemma 7.1 choose η_α such that $B\left(y_0, \eta_\alpha \lambda r\right) \subset C_{\mathcal{B}}(x, 2r, V, \alpha)$. Let $d = \log_2\left(\frac{\lambda+2}{\eta_\alpha \lambda}\right)$. Then

$$2^d \eta_\alpha \lambda r = \frac{\lambda+2}{\eta_\alpha \lambda} \eta_\alpha \lambda r = (\lambda+2) r = \left|x - y_0\right| r + 2r.$$

In particular, for the specified value of $d, B(x, 2r) \subset B\left(y_0, 2^d \eta_\alpha \lambda r\right)$. Applying condition (ii) of the set E at the point y_0 we see that (20) $\mu\left(C_{\mathcal{B}}(x, 2r, V, \alpha)\right) \geq \mu\left(B\left(x, \eta_\alpha \lambda r\right)\right) \geq K^{-d} \mu\left(B\left(y_0, 2^d \eta_\alpha \lambda r\right)\right) \geq K^{-d} \mu(B(x, 2r))$ Combining inequalities (19) and (20), we get the density ratio bounds

$$\delta > \frac{\mu\left(C_{\mathcal{B}}(x, 2r, V, \alpha)\right)}{\mu(B(x, 2r))} \geq K^{-d}$$

for all $r < r_\delta$. In particular, this implies that $d > \frac{-\log(\delta)}{\log K}$. Equivalently,

$$\log\left(\frac{\lambda+2}{\eta_\alpha \lambda}\right) > \frac{-\log \delta}{\log K}$$

so that if δ is chosen to be less than $2^{-\log K \log\left(\frac{5}{\eta_\alpha}\right)}$ then $\lambda < \frac{1}{2}$. From this result we conclude that for $r < r_\delta$ and for all $y \in S, |x - y| < \frac{1}{2} r$. Letting $r \downarrow 0$ we conclude that $\mu\left(E \cap C_{\mathcal{B}}\left(x, r_\delta, V, 2\alpha\right)\right) = 0$. Thus we can apply Corollary 7.1, and we obtain the desired conclusion.

2 Reading Notes of [Fal14]

2.1 June 9 Local Structure of Fractals

2.1.1 Densities(5.1)

Definition 2.1.1 (s-sets)
Borel sets of Hausdorff dimension s with positive finite s-dimensional Hausdorff measure.

Definition 2.1.2 (Density of an s-set) *We define the lower and upper densities of an s -set F at a point $x \in \mathbb{R}^n$ as*

$$\underline{D}^s(F,x) = \varliminf_{r\to 0} \frac{\mathcal{H}^s(F \cap B(x,r))}{(2r)^s}$$

and

$$\bar{D}^s(F,x) = \varlimsup_{r\to 0} \frac{\mathcal{H}^s(F \cap B(x,r))}{(2r)^s}$$

respectively (note that $|B(x,r)| = 2r$). If $\underline{D}^s(F,x) = \bar{D}^s(F,x)$, we say that the density of F at x exists and we write $D^s(F,x)$ for the common value.

Definition 2.1.3 (Regular Point) *A regular point x of F is a point at which*

$$\underline{D}^s(F,x) = \overline{D}^s(F,x) = 1$$

otherwise, x is an irregular point.

Definition 2.1.4 (Regular of s-set) *An s-set is termed regular if $\mathcal{H}^s$-almost all of its points (i.e. all of its points except for a set of $\mathcal{H}^s$−measure 0) are regular and irregular if $\mathcal{H}^s$-almost all of its points are irregular.*

Note:

1. 'irregular' does not mean 'not regular'
2. s-set F must be irregular unless s is an integers
3. a regular 1-set consists of portions of rectifiable curves of finite length, whereas an irregular 1-set is totally disconnected and dust-like and typically of fractalform.

Proposition 2.1.1 *Let μ be a mass distribution on $\mathbb{R}^n$, let $F \subset \mathbb{R}^n$ be a Borel set and let $0 < c < \infty$ be a constant.*

a. If $\varlimsup_{r\to 0} \mu(B(x,r))/r^s < c$ for all $x \in F$, then $\mathcal{H}^s(F) \geq \mu(F)/c$.

b. If $\varlimsup_{r\to 0} \mu(B(x,r))/r^s > c$ for all $x \in F$, then $\mathcal{H}^s(F) \leq 2^s \mu(\mathbb{R}^n)/c$

Proposition 2.1.2 *Let F be an s-set in $\mathbb{R}^n$. Then*

a. $\underline{D}^s(F,x) = \bar{D}^s(F,x) = 0$ for $\mathcal{H}^s$-almost all $x \notin F$

b. $2^{-s} \leq \bar{D}^s(F,x) \leq 1$ *for* $\mathcal{H}^s$*-almost all* $x \in F$.

Partial Proof:

a. If F is closed and $x \notin F$, then $B(x,r) \cap F = \emptyset$ if r is small enough. Hence, $\lim_{r\to 0} \mathcal{H}^s(F \cap B(x,r))/(2r)^s = 0$. If F is not closed, the proof is a little more involved and we omit it here.

b. This follows quickly from Proposition 2.1.1 a by taking μ as the restriction of $\mathcal{H}^s$ to F, that is, $\mu(A) = \mathcal{H}^s(F \cap A)$: if

$$F_1 = \left\{ x \in F : \bar{D}^s(F,x) = \overline{\lim_{r\to 0}} \frac{\mathcal{H}^s(F \cap B(x,r))}{(2r)^s} < 2^{-s}c \right\},$$

then $\mathcal{H}^s(F_1) \geq \mathcal{H}^s(F)/c \geq \mathcal{H}^s(F_1)/c$. If $0 < c < 1$, this is only possible if $\mathcal{H}^s(F_1) = 0$; thus, for almost all $x \in F$, we have $\bar{D}^s(F,x) \geq 2^{-s}$. The upper bound follows in essentially the same way using Proposition 2.1.1 b.

Proposition 2.1.3 *Let* F *be an s-set and let* E *be a Borel set of* F*, then:*

$$\frac{\mathcal{H}^s(F \cap B(x,r))}{(2r)^s} = \frac{\mathcal{H}^s(E \cap B(x,r))}{(2r)^s} + \frac{\mathcal{H}^s((F \backslash E) \cap B(x,r))}{(2r)^s}$$

for almost all x *in* E*, we have*

$$\frac{\mathcal{H}^s((F \backslash E) \cap B(x,r))}{(2r)^s} \to 0 \qquad as\ r \to 0$$

by Proposition 2.1.2 a., so letting $r \to 0$ *gives*

$$\underline{D}^s(F,x) = \underline{D}^s(E,x); \quad \bar{D}^s(F,x) = \bar{D}^s(E,x)$$

for $\mathcal{H}^s$*-almost all* x *in* E.

Note:

(i) if E is a subset of an $s-$set F with $\mathcal{H}^s(E) > 0$, then E is regular if F is regular and vice versa.

(ii) intersection of a regular and an irrgular set has $\mathcal{H}^s$-measuer zero.

Theorem 2.1.1 *Let* F *be an s-set in* $\mathbb{R}^2$*. Then* F *is irregular unless* s *is an integer.*

2.1.2 Structure of 1-sets(5.2)

Theorem 2.1.2 (Decomposition Theorem) *Let F be a 1 -set. The set of regular points of F forms a regular set, the set of irregular points forms an irregular set.*

Proof. : By Proposition 2.1.3 taking E as the set of regular and irregular points.

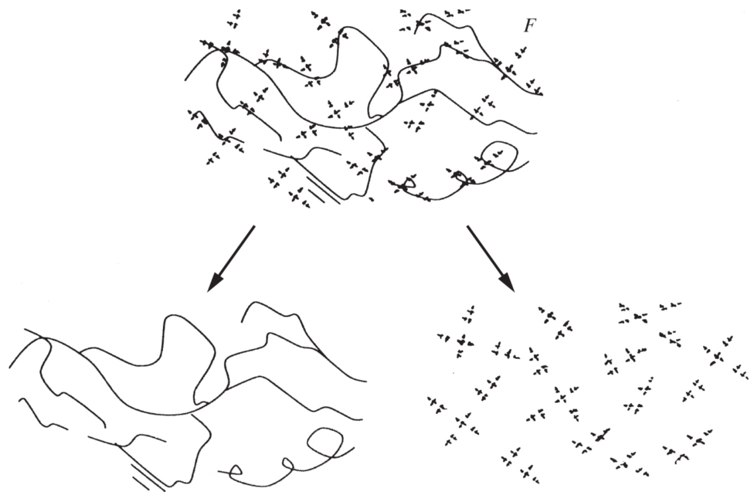


Figure 8: Decomposition of a 1-set into a regular 'curve-like' part and an irregular 'curve-free' part

Note: Regular 1-sets are made up from *pieces of curve*, whereas irregular 1-sets are dust-like and *'curve-free'*, that is, intersect any (finite length) curve in length zero, illustrated in Figure 8.

Definition 2.1.5 (Curve(Jordan Curve)) *A Jordan curve C is the image of a continuous injection(one-to-one) $\psi : [a, b] \to \mathbb{R}^2$, where $[a, b] \subset \mathbb{R}$ is a proper closed interval.*

Length:

$$\mathcal{L}(C) = \sup \sum_{i=1}^{\infty} |x_i - x_{i-1}|$$

where the supremum is taken over all dissections of C by points $x_0, x_1, \ldots, x_m$ in that order along the curve.

Definition 2.1.6 (Rectifiable Curve) *If $\mathcal{L}(C)$ is positive and finite, C is a rectifiable curve.*

Lemma 2.1.1 *If C is a rectifiable curve, then $\mathcal{H}^1(C) = \mathcal{L}(C)$*

Proof. : For $x, y \in C$, let $C_{x,y}$ denote that part of C between x and y. As orthogonal projection onto the line through x and y does not increase distances, Proposition 2.2.1 gives

$\mathcal{H}^1\left(C_{x,y}\right) \geq \mathcal{H}^1[x,y] = |x-y|$, where $[x,y]$ is the straight-line segment joining x to y. Hence, for any dissection $x_0, x_1, \ldots, x_m$ of C

$$\sum_{i=1}^{m}\left|x_i - x_{i-1}\right| \leq \sum_{i=1}^{m} \mathcal{H}^1\left(C_{x_i, x_{i-1}}\right) \leq \mathcal{H}^1(C)$$

so taking the supremum over all dissections gives $\mathcal{L}(C) \leq \mathcal{H}'(C)$. On the other hand, let $f : [0, \mathcal{L}(C)] \to C$ be the mapping that takes to the point on C at distance t along the curve from one of its ends. Clearly, $|f(t) - f(u)| \leq |t-u|$ for $0 \leq t, u \leq \mathcal{L}(C)$, that is, f is Lipschitz with $\mathcal{H}^1(C) \leq \mathcal{H}^1[0, \mathcal{L}(C)] = \mathcal{L}(C)$ by Proposition 2.2.1 as required.

Lemma 2.1.2 *A rectifiable curve is a regular 1-set.*

Proof. : As C is rectifiable $\mathcal{L}(C) < \infty$, and because C has distinct end points p and q, it is clear that $\mathcal{L}(C) \geq |p-q| > 0$. By Lemma 2.1.1, $0 < \mathcal{H}^1(C) < \infty$, so C is a 1-set.

A point x of C that is not an end point divides C into two parts $C_{p,x}$ and $C_{x,q}$. If r is sufficiently small, then moving away from x along the curve $C_{x,q}$, we reach a first point y on C with $|x-y| = r$. Then $C_{x,v} \subset B(x,r)$ and

$$r = |x-y| \leq \mathcal{L}\left(C_{x,y}\right) = \mathcal{H}^1\left(C_{x,y}\right) \leq \mathcal{H}^1\left(C_{x,q} \cap B(x,r)\right).$$

Similarly, $r \leq \mathcal{H}^1\left(C_{p,x} \cap B(x,r)\right)$, so, adding, $2r \leq \mathcal{H}^1(C \cap B(x,r))$, if r is small enough. Thus

$$\underline{D}^1(C,x) = \lim_{r \to 0} \frac{\mathcal{H}^1(C \cap B(x,r))}{2r} \geq 1.$$

By Proposition 2.1.2(a) $\underline{D}^1(C,x) \leq \bar{D}^1(C,x) \leq 1$ for $\mathcal{H}^1$ -almost all x, so $D^1(C,x)$ exists and equals 1 for almost all $x \in C$, so C is regular.

Definition 2.1.7 (Curve-like Curve) *A 1-set is curve-like if it is contained in a countable union of rectifiable curves.*

Proposition 2.1.4 *A curve like 1-set is a regular 1-set.*

Proof. : If F is a curve-like 1 -set, then $F \subset \bigcup_{i=1}^{\infty} C_i$ where the C_i are rectifiable curves. For each i and $\mathcal{H}^1$ -almost all $x \in F \cap C_i$, we have, using Lemma 2.1.2 and 2.1.1,

$$1 = \underline{D}^1\left(C_i, x\right) = \underline{D}^1\left(F \cap C_i, x\right) \leq \underline{D}^1(F, x)$$

and, hence, $1 \leq \underline{D}^1(F,x)$ for almost all $x \in F$. But for almost all $x \in F$, we have $\underline{D}^1(F,x) \leq \bar{D}^1(F,x) \leq 1$ by Proposition 2.1.2, so $D^1(F,x) = 1$ almost everywhere, and F is regular.

Definition 2.1.8 (Curve-free) *A 1-set is called curve-free if its intersection with every rectifiable curve has $\mathcal{H}^1$-measure zero.*

Proposition 2.1.5 *Let F be a curve-free 1 -set in $\mathbb{R}^2$. Then $\underline{D}^1(F,x) \leq \frac{3}{4}$ at almost all $x \in F$.*

Theorem 2.1.3

a. A 1*-set in $\mathbb{R}^2$ is irregular if and only if it is curve-free.*

b. A 1-set in $\mathbb{R}^2$ is regular if and only if it is the union of a curve-like set and a set of $\mathcal{H}^1$-measure zero.

2.1.3 Tangents to s-sets(5.3)

2.2 June 7 Hausdorff and packing measures and dimensions

2.2.1 Hausdorff Measure(3.1)

Definition 2.2.1 (Hausdorff Measure) *Suppose that F is a subset of $\mathbb{R}^n$ and $s \geq 0$. For each $\delta > 0$,*

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}.$$

And we write

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F)$$

as the s-dimensional Hausdorff measure on F.

Note: Thus, we look at all covers of F by sets of diameter at most δ and seek to minimise the sum of the s^{th} powers of the diameters.

It can be shown as a measure as: $\mathcal{H}^s(\emptyset) = 0$; if E is contained in F, then $\mathcal{H}^s(E) \leq \mathcal{H}^s(F)$; if $\{F_i\}$ is any countable collection of sets, then $\mathcal{H}^s\left(\bigcup_{i=1}^{\infty} F_i\right) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(F_i)$.

Property 2.2.1 (Equivalence of Hausdorff Measure)

(i) with $n-$dimensional Lebesgue measure: $\mathcal{H}^n(F) = c_n^{-1} vol^n(F)$ where c_n is the volume of an n-dimensional ball of diameter 1, so that $c_n = \pi^{n/2}/2^n(n/2)!$ if n is even and $c_n = \pi^{n/2}/2^n(n/2)!$ if odd.

(ii) $\mathcal{H}^0(F)$ is the number of points

(iii) $\mathcal{H}^1(F)$ is the length of a smooth curve F

(iv) $\mathcal{H}^2(F) = (4/\pi) \times \text{area}(F)$ if F is a smooth surface

(v) $\mathcal{H}^3(F) = (6/\pi) \times \text{vol}(F)$

(vi) $\mathcal{H}^m(F) = c_m^{-1} \times \text{vol}^m(F)$ if F is a smooth $m-$ dimensional submanifold of $\mathbb{R}^n$ (i.e. an m-dimensional surface in the classical sense).

Proposition 2.2.1 (Hölder condition of exponent α) *Let $F \subset \mathbb{R}^n$ and $f : F \to \mathbb{R}^m$ be a mapping such that*

$$|f(x) - f(y)| \leq c|x-y|^\alpha \quad (x, y \in F)$$

for constants $\alpha > 0$ and $c > 0$. Then for each s

$$\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha}\mathcal{H}^s(F)$$

In particular, if f is a Lipschitz mapping, that is, if $\alpha = 1$, then

$$\mathcal{H}^s(f(F)) \leq c^s\mathcal{H}^s(F)$$

Proof: If $\{U_i\}$ is a δ-cover of F, then since $|f(F\cap U_i)| \leq c\,|F\cap U_i|^{\alpha} \leq c\,|U_i|^{a}$, it follows that $\{f(F\cap U_i)\}$ is a $c\delta^{a}$-cover of $f(F)$. Thus, $\sum_i |f(F\cap U_i)|^{s/\alpha} \leq c^{s/\alpha}\sum_i |U_i|^{s}$, so that $\mathcal{H}^{s/\alpha}_{c\delta'}(f(F)) \leq c^{s/\alpha}\mathcal{H}^{s}_{\delta}(F)$. Letting $\delta \to 0$ gives the result. The result for the Lipschitz case is immediate on setting $\alpha = 1$.

Property 2.2.2 (Scaling Property) *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a similarity transformation of scale factor $\lambda > 0$. If $F \subset \mathbb{R}^n$, then*

$$\mathcal{H}^s(f(F)) = \lambda^s \mathcal{H}^s(F)$$

Proof: $|f(x) - f(y)| = \lambda|x - y|$ and so $|f^{-1}(x) - f^{-1}(y)| = \lambda^{-1}|x - y| \quad (x, y \in F)$ and apply Proposition 2.2.1 for $c = \lambda$.

Note: property above can be considered as scaling of a length, area, or a volume by mutiplying $\lambda, \lambda^2, \lambda^3$ respectively. And if f is congruence or isometry, that is, $|f(x) - f(y)| = |x - y|$, then $\mathcal{H}^s(f(F)) = \mathcal{H}^s(F)$. Thus, Hausdorff measures are *translation invarient*(i.e. $\mathcal{H}^s(F + z) = \mathcal{H}^s(F)$ where $F + z = \{x + z : x \in F\}$) and *rotation invariant.*

2.2.2 Hausdorff dimension(3.2)

Property 2.2.3 ($\mathcal{H}^s(F)$ is non-increasing) *If $t < s$ and $\{U_i\}$ is a δ cover of F, then*

$$\sum_i |U_i|^t \leq \sum_i |U_i|^{|-s} |U_i|^s \leq \sum_i |U_i|^s \delta^{t-s} = \delta^{t-s} \sum_i |U_i|^s$$

taking infima over all δ-covers,

$$\mathcal{H}^t_\delta(F) \leq \delta^{t-s} \mathcal{H}^s_\delta(F)$$

which is non-increasing when $\delta \leq 1$.

Definition 2.2.2 (Hausdorff Dimension)

$$\dim_{\mathrm{H}} F = \inf \{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup \{s : \mathcal{H}^s(F) = \infty\}$$

(taking the supremum of the empty set to be o), so that

$$\mathcal{H}^s(F) = \begin{cases} \infty & \text{if } 0 \leq s < \dim_{\mathrm{H}} F \\ 0 & \text{if } s > \dim_{\mathrm{H}} F \end{cases}$$

If $s = \dim_{\mathrm{H}} F$, then $\mathcal{H}^s(F)$ may be zero or infinite or may satisfy

$$0 < \mathcal{H}^s(F) < \infty.$$

Note: Conisder that when $\delta \to 0$, if $\mathcal{H}^s < \infty$ then $\mathcal{H}^t(F) = 0$ for $t > s$, s.t. there is a critical value of s at which measure "jumps" from ∞ to 0, named *Hausdorff dimension.*

Proposition 2.2.2 (Hausdorff Dimension under Lipschitz Mapping)

a. *Let $F \subset \mathbb{R}^n$ and suppose that $f : F \to \mathbb{R}^m$ satisfies the Hölder condition*

$$|f(x) - f(y)| \leq c|x - y|^\alpha \quad (x, y \in F).$$

Then $\dim_{\mathrm{H}} f(F) \leq (1/\alpha) \dim_{\mathrm{H}} F$. In particular, if f is a Lipschitz mapping, that is, if $\alpha = 1$, then $\dim_H f(F) \leq \dim_H F$.

b. *If $f : F \to \mathbb{R}^m$ is a bi-Lipschitz transformation, that is,*

$$c_1|x - y| \leq |f(x) - f(y)| \leq c|x - y| \quad (x, y \in F),$$

where $0 < c_1 \leq c < \infty$, then $\dim_{\mathrm{H}} f(F) = \dim_{\mathrm{H}} F$.

2.3 June 2-3 Box-Counting Dimension

2.3.1 Box-Counting Dimensions(2.1)

Definition 2.3.1 (δ-mesh) *The family of cubes of the form*

$$[m_1\delta, (m_1+1)\,\delta] \times \cdots \times [m_n\delta, (m_n+1)\,\delta]\,,$$

where $m_1, \ldots, m_n$ are integers, is called the δ-mesh or δ-grid of $\mathbb{R}^n$.

Definition 2.3.2 (Diameter) *any subset of $n-$dimensional Euclidean space, $\mathbb{R}^n$, the **diameter** of U is defined as $|U| = \sup\{|x-y| : x, y \in U\}$*

Definition 2.3.3 (Box-Counting Dimension) *The lower and upper box-counting dimensions of a subset F of $\mathbb{R}^n$ are given by*

$$\underline{\dim}_{\mathrm{B}} F = \varliminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

$$\overline{\dim}_{\mathrm{B}} F = \varlimsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

and the box-counting dimension of F by

$$\dim_{\mathrm{B}} F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

(if this limit exists), where $N_\delta(F)$ is any of the following:

i the smallest number of sets of diameter at most δ that cover F

ii the smallest number of closed balls of radius δ that cover F

iii the smallest number of cubes of side δ that cover F;

iv the number of δ-mesh cubes that intersect F;

v the largest number of disjoint balls of radius δ with centres in F

Figure 9 illustrates Five ways of finding the box dimension of F.

Motivation: If $N_\delta(F)$ obeys, at least approximately, a power law

$$N_\delta(F) \simeq c\delta^{-s}$$

for positive constant c and s, we say that F has box dimension s.
Algorithms to solve for s:

$$\log N_\delta(F) \simeq \log c - s \log \delta$$

so

$$s \simeq \frac{\log N_\delta(F)}{-\log \delta} + \frac{\log c}{\log \delta}$$

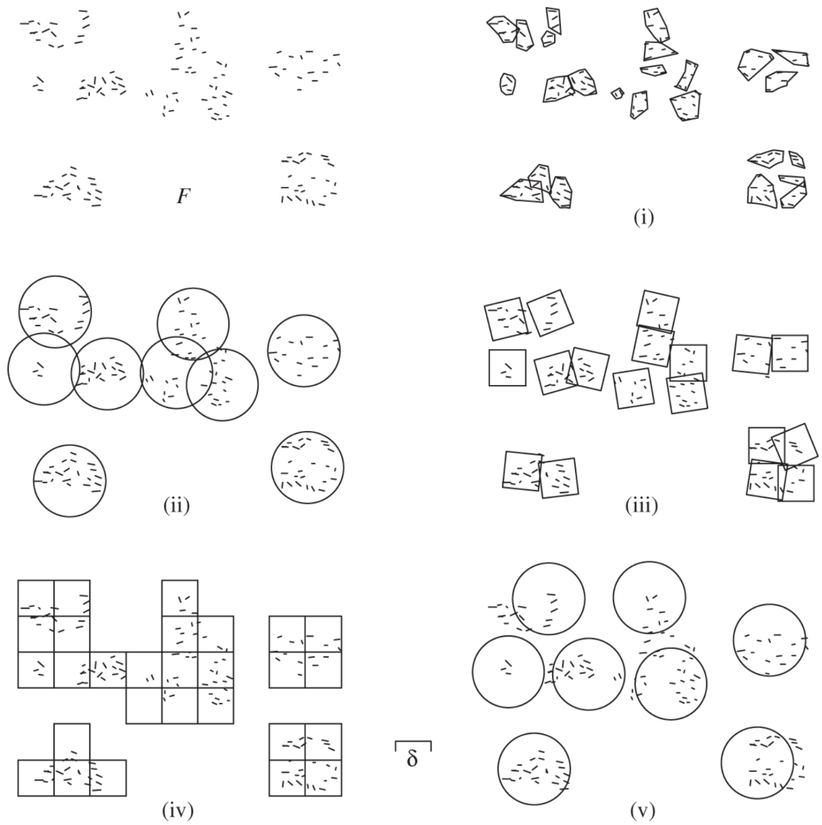


Figure 9: Five ways of finding the box dimension of F

and we might hope to obtain s as

$$s = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta},$$

with the second term disapperaing in the limit. And this implies that we assume that δ is sufficiently small.

Roughly speaking, equation for dimension says that $N_\delta(F) \simeq c\delta^{-s}$ for small δ, where $s = \dim_{\mathrm{B}} F$, or, more precisely, that

$$N_\delta(F)\delta^s \to \infty \quad \text{if } s < \dim_{\mathrm{B}} F$$

and

$$N_\delta(F)\delta^s \to 0 \quad \text{if } s > \dim_{\mathrm{B}} F.$$

Proof of Equivalence of Definition 2.3.3:
e.g.: (i)⇔(iv): Let $N_\delta(F)$ for the smallest number of sets of diameter δ that can cover F whereas $N'_\delta(F)$ be the number of δ-mesh cubes that intersect F. Note that these cubes provide a collection of $N'_\delta(F)$ sets of diameter $\delta\sqrt{n}$(diagonal for "diameter at most δ") that cover F,

$$N_{\delta\sqrt{n}}(F) \leq N'_\delta(F)$$

On the other hand, set of diameter at most δ is(or must be) contained in 3^n mesh cubes of side δ(e.g. choosing a cube containing some points of the set with its all neighbouring cubes, like the middle one and its 8 neighnors when $n = 2$). Then we have:

$$N'_\delta(F) \leq 3^n N_\delta(F)$$

Combining these inequalities and dividing by $-\log\delta$,

$$\frac{\log N_{\delta\sqrt{n}}(F)}{-\log(\delta\sqrt{n}) + \log\sqrt{n}} \leq \frac{\log N'_\delta(F)}{-\log\delta} \leq \frac{\log 3^n + \log N_\delta(F)}{-\log\delta}$$

so taking lower limits as $\delta \to 0$,

$$\underline{\lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \leq \underline{\lim}_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log\delta} \leq \underline{\lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}$$

with the other terms diappearing in the limit. Thus, lower limit works as sama as for both $N_\delta(F)$, and similar for upper box dimension.

Calculations: See Exercise 2.4

Note: More generally a set F made up of m similar disjoint copies of itself at scale r has

$$\dim_{\mathrm{B}} F = \log m / -\log r$$

Consider the neighborhood: $F_\delta = \{x \in \mathbb{R}^n : |x - y| \leq \delta \quad \text{for some } y \in F\}$. The rate at which the n-dimensional volume, that is, n-dimensional Lebesgue measure $\mathcal{L}^n$, of F_δ shrinks as $\delta \to 0$

This idea extends to fractional dimensions. If F is a subset of $\mathbb{R}'$ and $\lim_{\delta\to 0}\left(\mathcal{L}^n\left(F_\delta\right)/\delta^{n-s}\right) = c$ for some $s > 0$ and $0 < c < \infty$, it makes sense to regard F as s -dimensional, and it turns out that s is just the boxcounting dimension. The number c is called the *s-dimensional Minkowski content* of F — a quantity that is useful in some concepts but has the disadvantages that it does not exist for many standard fractals and that it is not necessarily additive on disjoint subsets, that is, is not a measure. Even if this limit does not exist, we can take lower and upper limits, and these are related to the box dimensions.

Proposition 2.3.1 *If F is a subset of $\mathbb{R}^n$, then*

$$\underline{\dim}_{\mathrm{B}} F = n - \overline{\lim}_{\delta\to 0} \frac{\log \mathcal{L}^n\left(F_\delta\right)}{\log\delta}$$

$$\overline{\dim}_{\mathrm{B}} F = n - \underline{\lim}_{\delta\to 0} \frac{\log \mathcal{L}^n\left(F_\delta\right)}{\log\delta}$$

where F_δ is the δ -neighbourhood of F.

Proof: If F can be covered by $N_\delta(F)$ balls of radius $\delta < 1$, then F_δ can be covered by the concentric balls of radius 2δ. Hence,

$$\mathcal{L}^n\left(F_\delta\right) \leq N_\delta(F) c_n (2\delta)^n,$$

where c_n is the volume of the unit ball in $\mathbb{R}^n$. Taking logarithms,

$$\frac{\log \mathcal{L}^n\left(F_\delta\right)}{-\log \delta} \leq \frac{\log 2^n c_n + n \log \delta + \log N_\delta(F)}{-\log \delta}$$

so

$$\varliminf_{\delta \to 0} \frac{\log \mathcal{L}^n\left(F_\delta\right)}{-\log \delta} \leq -n + \underline{\dim}_{\mathrm{B}} F$$

with a similar inequality for the upper limits. On the other hand, if there are $N_\delta(F)$ disjoint balls of radius δ with centres in F, then by adding their volumes,

$$N_\delta(F) c_n \delta^n \leq \mathcal{L}^n\left(F_\delta\right)$$

Taking logarithms and letting $\delta \to 0$ gives the opposite inequality to the third inequality, using Equivalent definition (v).

Note: In the context of Proposition above, box dimension is sometimes referred to as Minkowski dimension or Minkowski–Bouligand dimension.

2.3.2 Properties and Problems of Box-Counting Dimension(2.2)

Proposition 2.3.2

a. *If* $F \subset \mathbb{R}^n$ *and* $f : F \to \mathbb{R}^m$ *is a Lipschitz transformation, that is,*

$$|f(x) - f(y)| \leq c|x - y| \quad (x, y \in F),$$

then $\underline{\dim}_{\mathrm{B}} f(F) \leq \underline{\dim}_{\mathrm{B}} F$ *and* $\overline{\dim}_{\mathrm{B}} f(F) \leq \overline{\dim}_{\mathrm{B}} F$.

b. *If* $F \subset \mathbb{R}^n$ *and* $f : F \to \mathbb{R}^m$ *is a bi-Lipschitz transformation, that is,*

$$c_1|x - y| \leq |f(x) - f(y)| \leq c|x - y| \quad (x, y \in F),$$

where $0 < c_1 \leq c < \infty$, *then* $\underline{\dim_B} f(F) = \underline{\dim}_B F$ *and* $\overline{\dim}_{\mathrm{B}} f(F) = \overline{\dim}_{\mathrm{B}} F$.

2.4 May 27 Mathematical Background

2.4.1 Measures and Mass Distributions(1.3)

Definition 2.4.1 (Measure) *We call μ a measure on $\mathbb{R}^n$ if μ assigns a non-negative number, possibly ∞, to each subset of $\mathbb{R}^n$ such that*
(a) $\mu(\emptyset) = 0$
(b) $\mu(A) \leq \mu(B)$ *if* $A \subset B$
(c) if $A_1, A_2, \ldots$ is a countable (or finite) sequence of sets, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu\left(A_i\right)$$

with equality in above, that is

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu\left(A_i\right),$$

if the A_i are disjoint Borel sets.

Condition (a) says that the ***empty set has zero measure***, condition (b) says ***'the larger the set, the larger the measure'*** and condition (c) says that if a set is a union of a countable number of pieces (which may overlap), then the sum of the measure of the pieces is at least equal to the measure of the whole. If ***a set is decomposed into a countable number of disjoint Borel sets***, then the ***total measure of the pieces equals the measure of the whole***.

Property 2.4.1 (Measure)

1. *if $B \subset A$ A and B are Borel sets with $\mu(B)$ finite,*

$$\mu(A \setminus B) = \mu(A) - \mu(B)$$

as $A = B \cup (A \setminus B)$ and using Definition 2.4.1 (c).

2. *if $A_1 \subset A_2 \subset \cdots$ is an increasing sequence of Borel sets, then*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu\left(A_i\right).$$

as $\bigcup_{i=1}^{\infty} A_i = A_1 \cup (A_2 \backslash A_1) \cup (A_3 \backslash A_2) \cup \ldots,$

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \mu\left(A_1\right) + \sum_{i=1}^{\infty}\left(\mu\left(A_{i+1}\right) - \mu\left(A_i\right)\right) \\
&= \mu\left(A_1\right) + \lim_{k \to \infty} \sum_{i=1}^{k}\left(\mu\left(A_{i+1}\right) - \mu\left(A_i\right)\right) \\
&= \lim_{k \to \infty} \mu\left(A_k\right).
\end{aligned}$$

3. *A simple extension of above is that if, for $\delta > 0, A_\delta$ are Borel sets that are increasing as δ decreases, that is, $A_{\delta'} \subset A_\delta$ for $0 < \delta < \delta'$, then*

$$\mu\left(\bigcup_{\delta>0} A_\delta\right) = \lim_{\delta\to 0} \mu\left(A_\delta\right) \ .$$

Definition 2.4.2 (Support of μ)
spt μ, is the smallest closed set X such that $\mu(\mathbb{R}^n \setminus X) = 0$.

By above, x is in the support if and only if $\forall r > 0, \mu(B(x,r)) > 0$. We say that μ ***is a measure on a set*** A ***if*** A ***contains the support of*** μ.

Definition 2.4.3 (Mass Distributions) *A measure on a bounded subset of $\mathbb{R}^n$ for which $0 < \mu\left(\mathbb{R}^n\right) < \infty$ will be called a mass distribution, and we think of $\mu(A)$ as the mass of the set A.*

Definition 2.4.4 (Lebsgue Measure on $\mathbb{R}$)

$$\mathcal{L}^1(A) = \inf\left\{\sum_{i=1}^{\infty}(b_i - a_i) : A \subset \bigcup_{i=1}^{\infty}[a_i, b_i]\right\}$$

2.5 May 25 Mathematical Background

2.5.1 Basic Set Theory(1.1)

Review and summary of some definitions and theorems:

Definition 2.5.1 (Countable) *An infinite set A is countable if its elements can be listed in the form $x_1, x_2, ...$ with every element of appearing at a specific place in the list; otherwise, the set is uncountable*

Definition 2.5.2 (Open) $A \subset \mathbb{R}^n$ *is open if,* $\forall x \in A$*,* $\exists B(x, r) \in A$ *where* $r > 0$.

Definition 2.5.3 (Closed) $A \subset \mathbb{R}^n$ *is closed if, whenever* $\{x_k\} \in A$*,* $x_k \to x \in \mathbb{R}^n$*, then* $x \in A$.

Definition 2.5.4 (Closure) $\bar{A}$ *is the intersection of all the closed sets containing a set A.*

Definition 2.5.5 (Interior) $int(A)$ *is the union of all open sets contained in A.*

Definition 2.5.4 and 2.5.5 shows that The *closure* of A is thought of as the **smallest closed set** containing A, and the *interior* as the **largest open set** contained in A.

Definition 2.5.6 (Boundary) $\partial A = \bar{A} \setminus int(A)$

Theorem 2.5.1 $x \in \partial A \Leftrightarrow \forall r > 0, B(x, r) \cap A \neq \emptyset, B(x, r) \cap A^C \neq \emptyset$

Definition 2.5.7 (Dense) *Set B is a dense in A if* $A \subset \bar{B}$*, that is, if there are points of B arbitrarily close to each point of A.*

Definition 2.5.8 (Compact) *A is compact if any collection of open sets that covers A has a finite subcollection which also covers A.*

Theorem 2.5.2 *A compact subset of* $\mathbb{R}^n$ *is both closed and bounded.*

Theorem 2.5.3 *The intersection of any collection of compact sets is compact.*

Definition 2.5.9 (Connected) $A \subset \mathbb{R}^n$ *is connected if there not exists open sets U and V s.t.* $A \in U \cap V$ *with disjoint and nonempty* $A \cap U$ *and* $A \cap V$.

Definition 2.5.10 (Connected Component) *Connected component of x is the largest connected subset of A containing a point x.*

Definition 2.5.11 (Disconnect) *The set A is totally disconnected if the connected component of each point consists of just that point.*

The definition of *disconnect* also can be as: $\exists$ open sets U and V s.t. $x \in U, y \in V$ and $A \subset U \cap V$.

Definition 2.5.12 (Borel Set) *Borel Sets is the smallest collection fo subsets of* $\mathbb{R}^n$ *with the following properties:*

1. *Every open set and every closed set is a Borel set.*
2. *The union of every finite or countable collection of Borel sets is a Borel set, and the intersection of every finite or countable collection of Borel sets is a Borel set.*

In short, Any set that can be constructed using a sequence of countable unions or intersections starting with the open sets or closed sets will certainly be Borel.

2.5.2 Functions and Limits(1.2)

Definition 2.5.13 (Congruence) *The transformation* $S : \mathbb{R}^n \to \mathbb{N}^n$ *is congruence or isometry if it preserves distances i.e. if* $|S(x) - S(y)| = |x - y|$ *for* $x, y \in \mathbb{R}^n$

Special cases include *translations*, which are of the form $S(x) = x + a$ and have the effect of shifting points parallel to the vector a, *rotations* which have a centre a such that $|S(x) - a| = |x - a|$ for all x (for convenience, we also regard the identity transformation given by $I(x) = x$ as a rotation) and *reflections*, which maps points to their mirror images in some $(n-1)$-dimensional plane. A congruence that may be achieved by a combination of a rotation and a translation, that is, does not involve reflection, is called a *rigid motion* or *direct congruence*. A transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ is a *similarity* of *ratio* or *scale* $c > 0$ if $|S(x) - S(y)| = c|x - y|$ for all x, y in $\mathbb{R}^n$. A similarity transforms sets into geometrically similar ones with all lengths multiplied by the factor c.

Definition 2.5.14 (Linear Transformation) *A transformation* $T : \mathbb{R}^n \to \mathbb{R}^n$ *is linear if* $\forall x, y \in \mathbb{R}^n, T(x + y) = T(x) + T(y)$ *and* $T(\lambda x) = \lambda T(x), \lambda \in \mathbb{R}$

Such a linear transformation is *non-singular* if $T(x) = 0$ if and only if $x = 0$. If $S : \mathbb{R}^n \to \mathbb{R}^n$ is of the form $S(x) = T(x) + a$, where T is a non-singular linear transformation and a is a vector in $\mathbb{R}^n$, then S is called an *affine transformation or an affinity.* An affinity may be thought of as a shearing transformation; its contracting or expanding effect need not be the same in every direction. However, if T is orthonormal, then s is a congruence, and if T is a scalar multiple of an orthonormal transformation, then T is a similarity.

Definition 2.5.15 (Hölder Function) *A function* $f : X \to Y$ *is called a Hölder function of exponent* α *if*

$$|f(x) - f(y)| \leq c|x - y|^\alpha \quad (x, y \in X)$$

for some constant $c \geq 0$.

Definition 2.5.16 (Lipschitz Function) *The function* f *is called Lipschitz if*

$$|f(x) - f(y)| \leq c|x - y| \quad (x, y \in X)$$

and bi-Lipschitz if

$$c_1|x - y| \leq |f(x) - f(y)| \leq c_2|x - y| \quad (x, y \in X)$$

for $0 < c_1 \leq c_2 < \infty$, *in which case both* f *and* $f^{-1} : f(X) \to X$ *are Lipschitz functions.*

Definition 2.5.17 (Lower Limit)

$$\underline{\lim}_{x\to 0} f(x) \equiv \lim_{r\to 0}(inf\{f(x) : 0 < x < r\})$$

Note: $inf\{f(x) : 0 < x < r\}$ is either $-\infty$ for all positive r or else increases as r decreases, $\underline{\lim}_{x\to 0} f(x)$ always exists.

Definition 2.5.18 (Upper Limit)

$$\overline{\lim_{x \to 0}} f(x) \equiv \lim_{r \to 0}(sup\{f(x) : 0 < x < r\})$$

Note: The lower and upper limits exist as real numbers or $-\infty$ or ∞ for every function f and are indicative of the variation of f for x close to 0, shown in Figure 10.

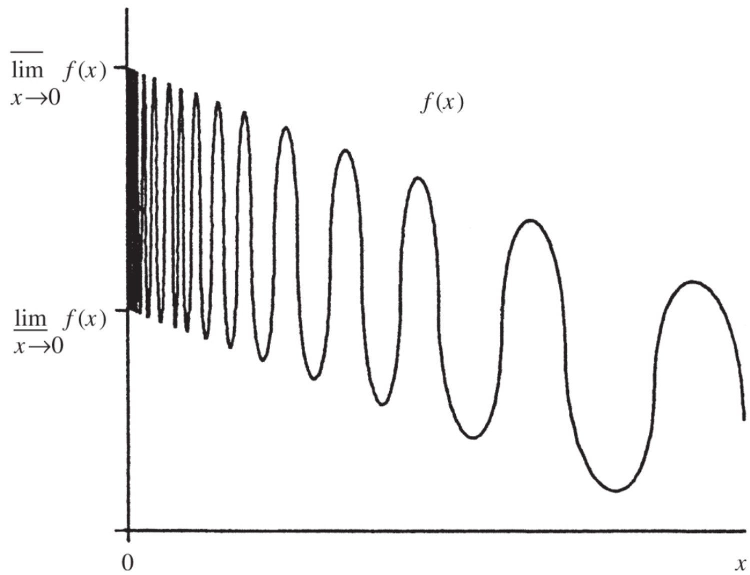


Figure 10: The upper and lower limits of a function.

We write $f(x) \sim g(x)$ to mean that $f(x)/g(x) \to 1$ as $x \to 0$.

Theorem 2.5.4 (Lipschitz functions are continuous)

Proof: Assum that the function $f : X \to Y$ is a Lipschitz function s.t. $|f(x) - f(y)| \leq c|x - y|(x, y \in X)$ for some constant $c \geq 0$. Then, $\forall \epsilon > 0$, let $\delta = \dfrac{\epsilon}{c}$, and we have $\forall x, y \in X, |x - y| < \delta \Rightarrow |x - y| < \dfrac{\epsilon}{c} \Rightarrow |f(x) - f(y)| \leq c|x - y| \leq c \cdot \dfrac{\epsilon}{c} = \epsilon \Rightarrow$ Lipschitz functions are continuous.

Definition 2.5.19 (Homeomorphism) *If $f : X \to Y$ is a continuous bijection with continuous inverse $f^{-1} : Y \to X$, then f is called a homeomorphism, and X and Y are termed homeomorphic sets.*

Corollary 2.5.1 *Congruences, similarities and affine transformations on $\mathbb{R}^n$ are examples of homeomorphisms.*

Definition 2.5.20 (Differentiable) *If $f : \mathbb{R}^n \to \mathbb{R}^n$, we say that f is differentiable at x and has derivative given by the linear mapping $f'(x) : \mathbb{R}^n \to \mathbb{R}^n$ if*

$$\lim_{h \to 0} \frac{|f(x + h) - f(x) - f'(x)h|}{|h|} = 0.$$

Theorem 2.5.5 (Mean Value Theorem) *Given $x < y$ and a real-value function f that is differentiable over an interval containing x and y, there exists w with $x < w < y$ s.t.*

$$\frac{f(y) - f(x)}{y - x} = f'(w)$$

Note: A consequence of the mean value theorem is that if $|f'(x)|$ is bounded over an interval, then f is Lipschitz over that interval.

Definition 2.5.21 (Pointwise Convergence) *For a sequence of functions: $f_k : X \to Y$ where X and Y are subsets of Euclidean spaces. f_k converge pointwise to a function $f : X \to Y$ if $f_k(x) \to f(x)$ as $k \to \infty$.*

Definition 2.5.22 (Unifrom Convergence) *For a sequence of functions: $f_k : X \to Y$ where X and Y are subsets of Euclidean spaces. f_k converge uniformly to a function $f : X \to Y$ if $sup_{x \in X}|f_k(x) - f(x)| \to 0$ as $k \to \infty$.*

Note: Uniform convergence is a stronger property than pointwise convergence i.e. Uniform convergence implies pointwise convergence, but not the other way around

Definition 2.5.23 (Another Definition of Pointwise Convergence) *For each $x \in D$, $\forall \delta > 0$, $\exists k_{x,\delta} > 0$, s.t. whenever $k > k_{x,\delta}$, $|f_k(x) - f(x)| < \delta$.*

Definition 2.5.24 (Another Definition of Uniform Convergence) *$\forall \delta > 0$, $\exists k_\delta > 0$ s.t. whenever $k > k_\delta$, $|f_k(x) - f(x)| < \delta$.*

Note: the main difference between pointwise and uniform convergence is that pointwise convergence is for each x in the domain, whereas uniform convergence is for all x in domain. And this is also the reason why sup shown in the definition in the textbook.

Theorem 2.5.6 *If the functions f_k are continuous and converge uniformly to f, then f is continuous.*

Theorem 2.5.7 (Logarithms) *Apparently, $a^c = b^{c \log a / \log b}$*

3 Exercises and Solutions of [Fal14]

3.1 June 11-13 Local structure of fractals(5)

3.1.1 Exercises and Solutions

Exercise 5.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $0 < c_1 \leq f'(x) \leq c_2$ for all x. Show that if F is an s-set in $\mathbb{R}$, then $\underline{D}^s(f(F), f(x)) = \underline{D}^s(F, x)$ for all x in $\mathbb{R}$, with a similar result for upper densities.*

Solution 5.2 *As f is continuously differentiable, f' is also continuous. Then,*

$$\lim_{w \to x} f'(w) = f'(x) = c, \text{ assuming } f'(x) = c, \forall x \in \mathbb{R}$$

$$\Rightarrow \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. whenever } |w - x| < \delta, |f'(w) - c| < \epsilon$$

$$\Rightarrow c - \epsilon < f'(w) < c + \epsilon$$

Besides, by Mean Value Theorem, for $y < w < z$ where $y, w, z \in B(x, \delta)$, chosing $0 < \epsilon < c$,

$$|f(y) - f(z)| = |f'(w)||y - z|$$

$$\Rightarrow (c - \epsilon)|y - z| \leq |f(y) - f(z)| \leq (c + \epsilon)|y - z|$$

Thus, f is bi-Lipschitz on $B(x, \delta)$.
Consider a Lemma for Proposition 2.2.1: if $p|x - y| \leq |g(x) - g(y)| \leq q|x - y|$,

$$p^s \mathcal{H}^s(g(U)) \leq \mathcal{H}^s(U) \leq q^s \mathcal{H}^s(g(U))$$

where p, q are constants and $g : U \subset \mathbb{R} \to \mathbb{R}^m$. The proof for this is just adding the left side for the proof of Proposition 2.2.1.
Then, let $0 < r < \delta$ s.t. $F \cap B(x, r) \subset B(x, r) \subset B(x, \delta)$ and let $f = g, p = c - \epsilon, q = c + \epsilon, U = F \cap B(x, r)$ which satisfied the lemma above. Hence,

$$(c - \epsilon)^s \mathcal{H}^s(F \cap B(x, r)) \leq \mathcal{H}^s(f(F \cap B(x, r))) \leq (c + \epsilon)^s \mathcal{H}^s(F \cap B(x, r))$$

$$\Leftrightarrow \frac{\mathcal{H}^s(f(F \cap B(x, r)))}{(c + \epsilon)^s} \leq \mathcal{H}^s(F \cap B(x, r)) \leq \frac{\mathcal{H}^s(f(F \cap B(x, r)))}{(c - \epsilon)^s}$$

Next, we claim that:

$$B(f(x), (c - \epsilon)r) \subset f(B(x, r)) \subset B(f(x), (c + \epsilon)r)$$

$$\Rightarrow f(F) \cap B(f(x), (c - \epsilon)r) \subset f(F \cap B(x, r)) \subset f(F) \cap B(f(x), (c + \epsilon)r)$$

Proof: if $f(m) \in B(f(x), (c - \epsilon)r), (c - \epsilon)|x - m| \leq |f(x) - f(m)| \leq |c - \epsilon|r \Rightarrow m \in B(x, r) \Rightarrow f(m) \in f(B(x, r)) \Rightarrow B(f(x), (c - \epsilon)r) \subset f(B(x, r))$. On the other hand, if $n \in f(B(x, r)), \exists s \in B(x, r)$ s.t. $f(s) = n$, so $|f(x) - n| \leq (c + \epsilon)|x - s| \leq (c + \epsilon)r \Rightarrow n \in B(f(x), (c + \epsilon)r) \Rightarrow f(B(x, r)) \subset B(f(x), (c + \epsilon)r)$. Then, by adding intersection for *preimage and image, the containment will not changed. Then, by property of measure, we have:*

$$\frac{\mathcal{H}^s(f(F)\cap B(f(x),(c-\epsilon)r))}{(c+\epsilon)^s} \le \frac{\mathcal{H}^s(f(F\cap B(x,r)))}{(c+\epsilon)^s}$$
$$\le \mathcal{H}^s(F\cap B(x,r))$$
$$\le \frac{\mathcal{H}^s(f(F\cap B(x,r)))}{(c-\epsilon)^s} \le \frac{\mathcal{H}^s(f(F)\cap B(f(x),(c+\epsilon)r))}{(c-\epsilon)^s}$$
$$\Rightarrow \left(\frac{c-\epsilon}{c+\epsilon}\right)^s \frac{\mathcal{H}^s(f(F)\cap B(f(x),(c-\epsilon)r))}{(2\cdot((c-\epsilon)r))^s}$$
$$\le \frac{\mathcal{H}^s(F\cap B(x,r))}{(2r)^s}$$
$$\le \left(\frac{c+\epsilon}{c-\epsilon}\right)^s \frac{\mathcal{H}^s(f(F)\cap B(f(x),(c+\epsilon)r))}{(2\cdot(c+\epsilon)r))^s}$$

Taking lower limit for $r \to 0$, we have:

$$\left(\frac{c-\epsilon}{c+\epsilon}\right)^s \underline{D}^s(f(F),f(x)) \le \underline{D}^s(F,x) \le \left(\frac{c+\epsilon}{c-\epsilon}\right)^s \underline{D}^s(f(F),f(x))$$

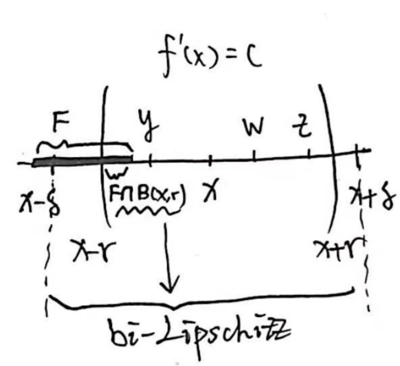
This inequality holds for all chosen $0<\epsilon<c$,

$$\left(\frac{c-\epsilon}{c+\epsilon}\right)^s \in (0,1) \le \frac{\underline{D}^s(F,x)}{\underline{D}^s(f(F),f(x))} \le (1,\infty) \ni \left(\frac{c+\epsilon}{c-\epsilon}\right)^s$$

Therefore,

$$\underline{D}^s(f(F),f(x)) = \underline{D}^s(F,x)$$

Similar for upper densities. A figure for varaibles relationship is shown below.



Exercise 5.8 *Let $F_1, F_2, \ldots$ be 1-sets in the plane such that $F = \bigcup_{k=1}^{\infty} F_k$ is a 1-set. Show that if F_k is regular for all k, then F is regular, and if F_k is irregular for all k, then F is irregular.*

Solution 5.8

i. *If each F_k is regular and 1-set for all k, by Theorem 2.1.3 b., it is the union of a curve-like set(i.e. contained in a countable union of rectifiable curves) and a set of $\mathcal{H}^1$-measure zero. Then, a 1-set $F = \displaystyle\bigcup_{k=1}^{\infty} F_k$ is union of a curve-like set which is contained in a countable union of rectifiable curves, and, a countable union of a set of $\mathcal{H}^1$-measure zero which is a set of $\mathcal{H}^1$-measure zero. By Theorem 2.1.3 b., F is regular.*

ii. *If each F_k is irrgular, by definition of curve-free, $\mathcal{H}^1(C \cap F_k) = 0$ where C is a rectifiable curve. Then, $0 \leq \mathcal{H}^1(C \cap \displaystyle\bigcup_{k=1}^{\infty} F_k) = \mathcal{H}^1(\bigcup_{k=1}^{\infty}(C \cap F_k)) \leq \sum_{k=1}^{\infty} \mathcal{H}^1(C \cap F_k) = 0$, such that F is curve-free. By Theorem 2.1.3 b., F is irregular.*

3.2 June 8 Hausdorff and packing measures and dimensions(3)

3.2.1 Exercises and Solutions

Exercise 3.7 *Let $f : [0,1] \to \mathbb{R}$ be a Lipschitz function. Writing* $\operatorname{graph} f = \{(x, f(x)) : 0 \le x \le 1\}$*, show that* $\dim_{\mathrm{H}}$ *graph* $f = 1$*. Note, in particular, that this is true if f is continuously differentiable, see Exercise 1.13(Rademacher's theorem).*

Solution 3.7 *Let $g(x) = (x, f(x))$ where $g : [0,1] \to \operatorname{graph} f$. Then g is bi-Lipschitz, since:*

$$|g(x) - g(y)|^2 = |x-y|^2 + |f(x) - f(y)|^2$$

Then,

$$|x-y|^2 \le |g(x) - g(y)|^2 \le |x-y|^2 + c^2|x-y|^2 = \left(1 + c^2\right)|x-y|^2$$

since $|f(x) - f(y)| \le c|x-y|$ for some $c > 0$. Thus by taking the square root, g is bi-Lipschitz. Therefore, by Proposition 2.2.2 b.,

$$\dim_{\mathrm{H}} \operatorname{graph} f = \dim_{\mathrm{H}} g([0,1]) = \dim_{\mathrm{H}}([0,1]) = 1$$

Note: If f is Lipschitz(or f is continuously differentiable), then graph f is bi-Lipschitz.

Exercise *Prove Proposition 2.2.2:*

(a) Let $F \subset \mathbb{R}^n$ and suppose that $f : F \to \mathbb{R}^m$ satisfies the Hölder condition

$$|f(x) - f(y)| \le c|x-y|^\alpha \quad (x, y \in F).$$

Then $\dim_{\mathrm{H}} f(F) \le (1/\alpha) \dim_{\mathrm{H}} F$. In particular, if f is a Lipschitz mapping, that is, if $\alpha = 1$, then $\dim_H f(F) \le \dim_H F$.

(b) If $f : F \to \mathbb{R}^m$ is a bi-Lipschitz transformation, that is,

$$c_1|x-y| \le |f(x) - f(y)| \le c|x-y| \quad (x, y \in F),$$

where $0 < c_1 \le c < \infty$, then $\dim_{\mathrm{H}} f(F) = \dim_{\mathrm{H}} F$.

Solution

(a) If $s > \dim_{\mathrm{H}} F$, then by Proposition 3.1 $\mathcal{H}^{s/\alpha}(f(F)) \le c^{s/\alpha}\mathcal{H}^s(F) = 0$, implying that $\dim_{\mathrm{H}} f(F) \le s/\alpha$ for all $s > \dim_{\mathrm{H}} F$. The conclusion for Lipschitz mappings is immediate on taking $\alpha = 1$.

(b) For the bi-Lipschitz case, just as in Proposition 2.5 for box dimension, applying the Lipschitz result to $f^{-1} : f(F) \to F$ yields the reverse inequality $\dim_{\mathrm{H}} F \le \dim_{\mathrm{H}} f(F)$.

3.3 June 4-6 Box-counting dimension(2)

3.3.1 Exercises and Solutions

Exercise 2.1 *Verify directly from the definitions that Equivalent definitions 2.3.3(ii) and (iv) give the same values for box dimension.*

Solution 2.1 *Let F be a subset of $\mathbb{R}^n$, let $N_\delta(F)$ denote the smallest number of closed balls of radius δ that cover F and let $N'_\delta(F)$ denote the number of δ-mesh cubes that intersect F. Consider that the closed ball of radius $\delta\sqrt{n}$ will definitely contains the δ-mesh cube when centers are the same. On the other hand, any closed ball of radius δ is intersect(and contained if center of the ball is the center of all cubes) at most 4^n δ-mesh cubes. Thus,*

$$N_{\delta\sqrt{n}}(F) \leq N'_\delta(F) \leq 4^n N_\delta(F)$$

Combining this inequality and dividing by $-\log\delta$,

$$\frac{\log N_{\delta\sqrt{n}}(F)}{-\log(\delta\sqrt{n}) + \log\sqrt{n}} \leq \frac{\log N'_\delta(F)}{-\log\delta} \leq \frac{\log 4^n + \log N_\delta(F)}{-\log\delta}$$

so taking lower limits as $\delta \to 0$ where $\delta\sqrt{n} \to 0$ as well,

$$\varliminf_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \leq \varliminf_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log\delta} \leq \varliminf_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta},$$

with the other terms disappearing in the limit. Thus, the definition of lower box dimension is the same working with either $N_\delta(F)$ or $N'_\delta(F)$. Taking upper limits, we get a similar conclusion for upper box dimension.

Exercise 2.2 *Generalise Proposition 2.3.2 by showing that if $f : F \to \mathbb{R}^n$ satisfies the Hölder condition $|f(x) - f(y)| \leq c|x - y|^\alpha$ where $c > 0$ and $0 < \alpha \leq 1$, then $\underline{\dim}_B f(F) \leq (1/\alpha)\underline{\dim}_{\mathrm{B}} F$ and $\overline{\dim}_{\mathrm{B}} f(F) \leq (1/\alpha)\overline{\dim}_{\mathrm{B}} F$.*

Solution 2.2 *Note that if $\{U_i\}$ is a δ-cover of F, then so $\{U_i \cap F\}$. Then, as*

$$|f(U_i \cap F)| \leq c|U_i \cap F|^\alpha \leq c|U_i|^\alpha \leq c\delta^\alpha$$

(this can be understood as taking $x, y \in U_i \cap F$ or $x, y \in U_i$ where $|x - y| < \delta$ by construction) s.t. $\{f(U_i \cap F)\}$ is a $c\delta^\alpha$ cover of $f(F)$, and hence $N_{c\delta^\alpha}(f(F)) \leq N_\delta(F), \forall\delta > 0$(consider f is injective, or, we may have a better cover when considered overall for the image). Thus,

$$\begin{aligned}
\underline{\dim}_B f(F) &= \varliminf_{c\delta^\alpha\to 0} \frac{\log N_{c\delta^\alpha}(f(F))}{-\log c\delta^\alpha} \\
&\leq \varliminf_{\delta\to 0} \frac{\log N_\delta(F)}{-\alpha\log\delta - \log c} \\
&= \frac{1}{\alpha} \varliminf_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \\
&= \frac{1}{\alpha}\underline{\dim}_B F
\end{aligned}$$

and same process can be applied for upper box-counting dimension.

Exercise 2.4 *Verify that the Cantor dust depicted in Figure 11, has box dimension 1 (take E_0 to have side length 1).*

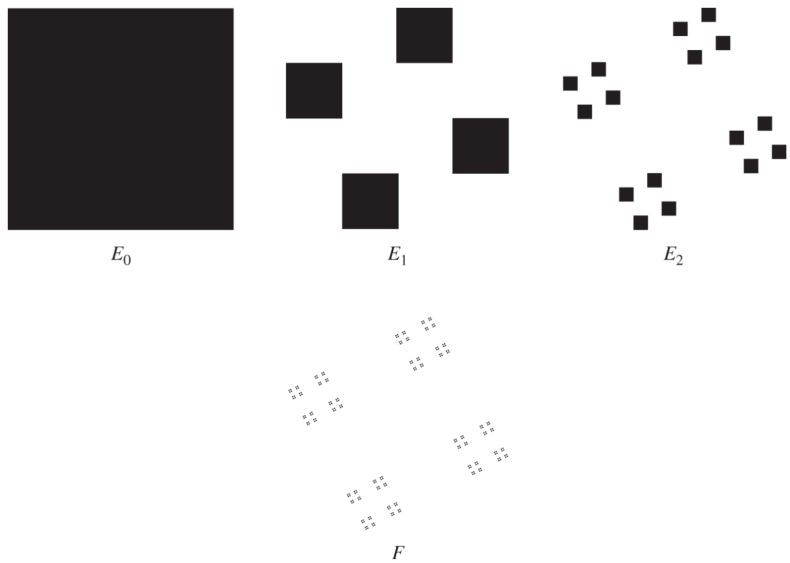


Figure 11: Construction of a 'Cantor dust'

Solution 2.4 *k^{th} stage of the construction consists of 4^k squares of side length 4^{-k}. Thus, if $4^{-k} < \delta \leq 4^{-k+1}$, the 4^k squares of E_k give a δ cover of F, so $N_\delta(F) \leq 4^k$. Then:*

$$\overline{\dim}_{\mathrm{B}} F = \overline{\lim_{\delta \to 0}} \frac{\log N_\delta(F)}{-\log \delta} \leq \overline{\lim_{k \to \infty}} \frac{\log 4^k}{-\log 4^{-k+1}} = 1$$

On the other hand, for $4^{-k-1} \leq \delta < 4^{-k}$, the cube can intersect at most two of the squares of E_k. There are 4^k squares in E_k, all containing points of F, so at least $4^k/2$ squares of side δ are required to cover F. Then, $N_\delta(F) \geq 4^k/2$, so:

$$\underline{\dim}_{\mathrm{B}} F = \underline{\lim_{\delta \to 0}} \frac{\log N_\delta(F)}{-\log \delta} \geq \underline{\lim_{k \to \infty}} \frac{\log 4^k/2}{-\log 4^{-k-1}} = 1$$

Therefore, the box-counting dimension of Cantor dust is 1.

Exercise 2.5 *Use Equivalent definition 2.3.3(i) to check that the upper box dimension of the von Koch curve(shown in Figure 12) is at most $\log 4/\log 3$ and 2.3.3(v) to check that the lower box dimension is at least this value.*

Solution 2.5 *Let $\delta_k = 3^{-k}$, for E_k, there are 4^k line segments so taking any plane set with diameter at most δ_k centered at the midpoint of each line segment can cover all points in F so $N_{\delta_k}(F) \leq 4^k$. Then,*

$$\overline{\dim}_{\mathrm{B}} F = \overline{\lim_{k \to \infty}} \frac{\log N_{\delta_k(F)}}{-\log \delta_k} \leq \overline{\lim_{k \to \infty}} \frac{\log 4^k}{\log 3^k} = \frac{\log 4}{\log 3}$$

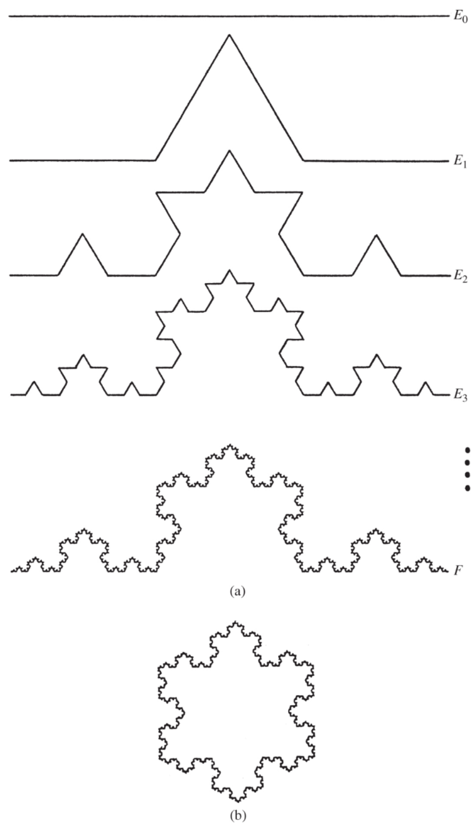


Figure 12: (a) Construction of the von Koch curve F. At each stage, the middle third of each interval is replaced by the other two sides of an equilateral triangle. (b) Three von Koch curves fitted together to form a snowflake curve.

On the other hand, there are $4^k + 1$ *vertices of* E_k *and if we take each vertices as centers of balls with radius* $\delta_k = 3^{-k}/2$*(it is also sufficient to take* $\delta_k = 3^{-k-1}$*), there would be at least* $4^k + 1$ *disjoint balls of radius* δ_k *with centers in* F*. Then,*

$$
\begin{aligned}
\underline{\dim}_B F &= \underline{\lim}_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k} \\
&\geq \underline{\lim}_{k\to\infty} \frac{\log\left(4^k+1\right)}{\log 3^{k+1}} \\
&\geq \underline{\lim}_{k\to\infty} \frac{\log\left(4^k\right)}{\log 3^{k+1}} \\
&= \underline{\lim}_{k\to\infty} \frac{k\log 4}{(k+1)\log 3} \\
&= \frac{\log 4}{\log 3}
\end{aligned}
$$

3.4 May 28-31 Measures(1.3)

3.4.1 Exercises and Solutions

Exercise 1.18 *Let $A_1, A_2, \ldots$ be a decreasing sequence of Borel subsets of $\mathbb{R}^n$ and let $A = \bigcap_{k=1}^{\infty} A_k$. If μ is a measure on $\mathbb{R}^n$ with $\mu(A_1) < \infty$, show using (1.6) that $\mu(A_k) \to \mu(A)$ as $k \to \infty$.*

Solution 1.18 *Consider that $\{A_1 \setminus A_k\}$ is an increasing sequence as $\{A_k\}$ decreasing. Then:*

$$\mu\left(\bigcup_{k=1}^{\infty}(A_1 \setminus A_k)\right) = \mu\left(A_1 \setminus \bigcap_{k=1}^{\infty} A_i\right) = \mu(A_1) - \mu(A)$$
$$= \lim_{k\to\infty} \mu(A_1 \setminus A_k) = \lim_{k\to\infty} (\mu(A_1) - \mu(A_k)) = \mu(A_1) - \lim_{k\to\infty} \mu(A_k)$$

As $\mu(A_1) < \infty$, $\lim_{k\to\infty} \mu(A_k) = \mu(\bigcap_{k=1}^{\infty} A_i)$

Conlusion: For a ***decreasing sequence*** A_k of Borel subsets of $\mathbb{R}^n$,

$$\lim_{k\to\infty} \mu(A_k) = \mu(\bigcap_{k=1}^{\infty} A_i)$$

Exercise 1.23 *Let D be a Borel subset of $\mathbb{R}^n$ and let μ be a measure on D with $\mu(D) < \infty$. Let $f_k : D \to \mathbb{R}$ be a sequence of functions such that $f_k(x) \to f(x)$ for all x in D. Prove **Egoroff's theorem**: that given $\varepsilon > 0$ there exists a Borel subset A of D with $\mu(D \backslash A) < \varepsilon$ such that $f_k(x)$ converges to $f(x)$ uniformly for x in A.*

Solution 1.23 *Assume that for $k, n \in \mathbb{Z}^+$, $A_{k,n} = \{x \in D : |f_l(x) - f(x)| < 1/n, \forall l \geq k\}$(so we consider $\delta = 1/n$ here), then we have $\bigcup_{k=1}^{\infty} A_{k,n} = D$ and $A_{1,n} \subset A_{2,n} \subset A_{3,n} \subset \ldots$. Next, by Property of measure 2.4.1:*

$$\mu(D) = \mu(\bigcup_{k=1}^{\infty} A_{k,n}) = \lim_{k\to\infty} \mu(A_{k,n}) < \infty$$

Hence,

$$\lim_{k\to\infty} \mu(D \setminus A_{k,n}) = \mu(D) - \lim_{k\to\infty} \mu(A_{k,n}) = 0$$

Then, $\exists k' \in \mathbb{N}$ s.t. whenever $k \geq k'$, $\mu(D \backslash A_{k,n}) < \frac{\epsilon}{2^n}$. Next, we can construct $A = \bigcap_{n=1}^{\infty} A_{k',n}$, which is a Borel subset of D and satisfies:

$$\mu(D \setminus A) = \mu(D \setminus \bigcap_{n=1}^{\infty} A_{k',n}) = \mu(\bigcup_{n=1}^{\infty} D \setminus A_{k',n}) = \sum_{n=1}^{\infty} \mu(D \setminus A_{k',n}) < \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$$

As $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n} = 1$, *and* A *exists for the question. Finally,* $\forall\delta > 0, n > 1/\delta, \forall x \in A$ *where* $x \in A_{k',n}$ *as well, such that whenever* $k \in \mathbb{N}, k > k'$, $|f_k(x) - f(x)| < 1/n < \delta \Rightarrow f_k(x)$ *converges to* $f(x)$ *uniformly for* x *in* A.

Note: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n} = 1$ is always used to construct ϵ in analysis proofs.

Exercise 1.24 *Prove that if* μ *is a measure on* D *and* $f : D \to \mathbb{R}$ *satisfies* $f(x) \geq 0$ *for all* x *in* D *and* $\int_D f \, \mathrm{d}\mu = 0$ *then* $f(x) = 0$ *for* μ *-almost all* x.

Solution 1.24 *Suppose* $f(x) \geq \epsilon > 0$ *on a set* $E_\epsilon \subset D$ *given* $\epsilon > 0$, *we have for* $x \in D \setminus E_\epsilon, f(x) = 0$ *and then:*

$$
\begin{aligned}
0 &= \int_D f(x)d\mu \\
&= \int_{E_\epsilon} f(x)d\mu + \int_{D\setminus E_\epsilon} f(x)d\mu \\
&\text{As } \epsilon\chi_{E_\epsilon}(x) \text{ is a simple function and by the integral of more general functions} \\
&\geq \int \epsilon\chi_{E_\epsilon}(x)d\mu + 0 \\
&= \epsilon\mu(E_\epsilon) + 0
\end{aligned}
$$

As $\epsilon\mu(E_\mu) \leq 0$ *while* $\epsilon > 0$, *we have* $\mu(E_\epsilon) = 0 \Rightarrow \mu\left(\bigcup_{\epsilon\in\mathbb{R}^+} E_\epsilon\right) = \mu\left(\{x : f(x) > 0\}\right) = 0 \Rightarrow f(x) = 0$ *for* μ*-a.e..*

Note:

1. $f(x) = 0$ for $\mu-$a.e $\Leftrightarrow$ The set of points where $f(x)! = 0(f(x) > 0$ in this case) has measure zero.
2. $\displaystyle\int_E 1d\mu = \mu(E)$

Definition 3.4.1 (Simple Function) *Simple functions are sums of linear combination of characteristic functions, e.g.* $f(x) = \sum a_i\chi_{A_i}(x)$

3.5 May 26 Functions and Limits(1.2)

3.5.1 Exercises and Solutions

Exercise 1.12 *Let $f, g : [0,1] \to \mathbb{R}$ be Lipschitz functions. Show that the functions defined on $[0,1]$ by $f(x) + g(x)$ and $f(x)g(x)$ are also Lipschitz.*

Solution:

(i) As f, g are Lipschitz function, we have $|f(x) - f(y)| \leq c_1|x-y|$ and $|g(x) - g(y)| \leq c_2|x-y|$ where $\forall x, y \in [0,1]$ and $c_1, c_2 \geq 0$. Then, $|(f(x)+g(x)) - (f(y)+g(y))| = |f(x)-f(y)+g(x)-g(y)| \leq |f(x)-f(y)|+|g(x)-g(y)| \leq (c_1+c_2)\cdot|x-y|, \forall x, y \in [0,1]$. Since $(c_1+c_2) \geq 0$, the condition is satisfied and therefore the functions defined on $[0,1]$ by $f(x)+g(x)$ is Lipschitz.

(ii) Consider that $|f(x) - f(0)| \leq c_1|x| \leq c_1, x \in [0,1]$, so we have non-negative $c_3 = |f(0)| + c_1 \geq |f(x)|$. Similarly, we have non-negative $c_4 \geq |g(x)|$

$|f|, |g| < 1$, $\forall x, y \in [0,1]$

$$
\begin{aligned}
&|f(x)g(x) - f(y)g(y)| \\
=&|f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
=&|f(x)(g(x)-g(y)) + g(y)(f(x)-f(y))| \\
\leq&|g(y)||(f(x)-f(y))| + |f(x)||(g(x)-g(y))| \\
\leq&c_1c_4||x-y| + c_2c_3|x-y| \\
\leq&(c_1c_4 + c_2c_3)|x-y|
\end{aligned}
$$

Since $c_1c_4 + c_2c_3 \geq 0$, the condition is satisfied and therefore the functions defined on $[0,1]$ by $f(x)g(x)$ is Lipschitz.

Exercise 1.13(Rademacher's theorem) *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with $|f'(x)| \leq c$ for all x. Show, using the mean value theorem, that f is a Lipschitz function.*

Solution: $\forall x, y \in \mathbf{R}, x \neq y$, by mean-value theorem, $\exists w \in (x,y)$ such that

$$
\begin{aligned}
&\frac{f(y)-f(x)}{y-x} = f'(w) \\
\Rightarrow&\left|\frac{f(y)-f(x)}{y-x}\right| = |f'(w)| \leq c \\
\Rightarrow&|f(x)-f(y)| \leq c|x-y| \quad (x, y \in \mathbb{R})
\end{aligned}
$$

Therefore, f is a Lipschitz function.

Exercise 1.14 *Show that every Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is continuous.*

Solution: See proof wrote for Theorem 2.5.4.

Exercise 1.15 *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + x$. Find (i) $f^{-1}(2)$, (ii) $f^{-1}(-2)$ and (iii) $f^{-1}([2,6])$.*

Solution: As $f(x) = x^2 + x$, $x = -\frac{1}{2} \pm \frac{\sqrt{1+4y}}{2}$

(i) $f^{-1}(2) = \{-2, 1\}$

(ii) $f^{-1}(-2) = \emptyset$

(iii) As $x = -\frac{1}{2} + \frac{\sqrt{1+4y}}{2}$ is increasing and $x = -\frac{1}{2} - \frac{\sqrt{1+4y}}{2}$ is decreasing while y increasing, $f^{-1}([2,6]) = [-3,-2] \cup [1,2]$

Exercise 1.16 *Show that $f(x) = x^2$ is Lipschitz on $[0,2]$, bi-Lipschitz on $[1,2]$ and not Lipschitz on $\mathbb{R}$.*

Solution:

(i) As $\forall x, y \in [0,2]$, $|x+y| \leq 4$, we have $|f(x)-f(y)| = |x^2-y^2| = |x+y||x-y| \leq 4|x-y|$. Thus, f is Lipschitz on $[0,2]$.

(ii) Apparently, $2|x-y| \leq |f(x)-f(y)| \leq 4|x-y|$ by above. As $f([1,2]) = [1,4]$, $\forall x, y \in [1,4]$, $\frac{1}{\sqrt{x}+\sqrt{y}} \leq \frac{1}{2}$, we have:

$$\left|f^{-1}(x) - f^{-1}(y)\right| = |\sqrt{x} - \sqrt{y}| = \left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right| \leq \frac{1}{2}|x-y|$$

$\Rightarrow$ so f^{-1} is Lipschitz on $[1,4]$.

$\Rightarrow f$ is bi-Lipschitz on $[1,2]$.

(iii) Let $x = ky, k \in \mathbb{R} \setminus \{0\}$, then $\frac{|f(x)-f(y)|}{|x-y|} = \frac{|k^2y^2-y^2|}{|ky-y|} = \left|\frac{k^2-1}{k-1}\right||y|$, which is unbounded on $\mathbb{R}$. Therefore, the Lipschitz constant does not exist and f is not Lipschitz on $\mathbb{R}$

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