Fractal Geometry
Mathematical Foundations and Applications
Third Edition

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Solutions to Exercises

Acknowledgement: Grateful thanks are due to Gwyneth Stallard for providing solutions for many of the exercises.

For many of the exercises, drawing a diagram will be found extremely helpful.

Chapter 1

1.1 (i) *The triangle inequaility.*

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then

$$\begin{aligned}
|x+y|^2 &= \sum_{i=1}^n (x_i + y_i)^2 = \sum_{i=1}^n x_i^2 + 2\sum_{i=1}^n x_i y_i + \sum_{i=1}^n y_i^2 \\
&\leq \sum_{i=1}^n x_i^2 + 2\left(\sum_{i=1}^n x_i^2\right)^{1/2}\left(\sum_{i=1}^n y_i^2\right)^{1/2} + \sum_{i=1}^n y_i^2 \\
&= \left(\left(\sum_{i=1}^n x_i^2\right)^{1/2} + \left(\sum_{i=1}^n y_i^2\right)^{1/2}\right)^2 = (|x| + |y|)^2
\end{aligned}$$

where we have used Cauchy's inequality.

(ii) *The reverse triangle inequality.*

Write $y = z - x$ so $x = z - y$. Then (i) becomes $|z| \leq |z-y| + |y|$ or $|z| - |y| \leq |z-y|$. Interchanging the roles of y and z we also have $|y| - |z| \leq |y - z| = |z - y|$. Thus $||z| - |y|| = \max\{|z| - |y|, |y| - |z|\} = |z - y|$, which is the desired inequality.

(iii) *Triangle inequality - metric form.*

We have

$$|x - y| = |(x - z) + (z - y)| \leq |x - z| + |z - y|$$

using triangle inequality (i).

1.2 We may assume A is closed since $A_\delta = \overline{A}_\delta$. Let $x \in A_{\delta+\delta'}$. Then there exists $a \in A$ such that $|x - a| \leq \delta + \delta'$. If $x = a$, then clearly $x \in (A_\delta)_{\delta'}$. Otherwise let y be the point on the line segment $[a, x]$ distance δ from a. Thus $y = a + \delta(x - a)/|x - a|$, so $|y - a| = \delta|x - a|/|x - a| = \delta$, so $y \in A_\delta$. Moreover, $x - y = x - a - \delta(x - a)/|x - a| = (x - a)[1 - \delta/|x - a|]$, so $|x - y| = |x - a| - \delta \leq \delta + \delta' - \delta = \delta'$. As $y \in A_\delta, x \in (A_\delta)_{\delta'}$, so $A_{\delta+\delta'} \subseteq (A_\delta)_{\delta'}$.

Now let $x \in (A_\delta)_{\delta'}$. We may find $y \in A_\delta$ such that $|x-y| \leq \delta'$, and then we may find $a \in A$ such that $|y-a| \leq \delta$. By the triangle inequality, Exercise 1.1(iii), $|x-a| \leq |x-y| + |y-a| \leq \delta' + \delta$, so $x \in A_{\delta+\delta'}$. Thus $(A_\delta)_{\delta'} \subseteq A_{\delta+\delta'}$. We conclude $(A_\delta)_{\delta'} = A_{\delta+\delta'}$.

1.3 Let A be bounded, that is A has finite diameter, so $\sup_{x,y\in A}|x-y| = d < \infty$, where d is the diameter of A. Let a be any point of A. Then for all $x \in A$, $|x-a| \leq d$, so that $|x| = |a+(x-a)| \leq |a|+|x-a| \leq |a|+d$, using the triangle inequality, Exercise 1.1(i). Thus, setting $r = |a|+d$, we have $x \in B(0,r)$. We conclude $A \subseteq B(0,r)$.

If $A \subseteq B(0,r)$ and $x, y \in A$, then $|x-y| \leq |x|+|y| \leq r+r = 2r$, so $\text{diam}A \leq 2r$, and in particular A is of finite diameter.

1.4 (i) A non-empty finite set is closed but not open, with $\overline{A} = A$, and $\text{int}A = \emptyset$.

(ii) The interval $(0,1)$ is open but not closed, with $\overline{(0,1)} = [0,1]$ and $\text{int}(0,1) = (0,1)$.

(iii) The interval $[0,1]$ is closed but not open, with $\overline{[0,1]} = [0,1]$ and $\text{int}[0,1] = (0,1)$.

(iv) The half-open interval $[0,1)$ is neither open or closed, with $\overline{[0,1)} = [0,1]$ and $\text{int}[0,1) = (0,1)$.

(v) The set $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ is closed but not open, with $\overline{A} = A$ and $\text{int}A = \emptyset$.

1.5 Following the usual construction, the middle third Cantor set may be written $F = \bigcap_{k=0}^{\infty} E_k$, where E_k consists of the union of 2^k disjoint closed intervals in $[0,1]$, each of length 3^{-k}. For each k, E_k is closed since it is the union of finitely many closed sets. Since the intersection of any collection of closed sets is closed (see Exercise 1.6), we conclude that F is closed. F is a subset of $[0,1]$ so it is bounded, and hence F is compact.

To show that F is totally disconnected, suppose $x, y \in F$ with $x < y$. Then we can find an E_k such that x and y belong to different intervals $[a,b]$ and $[c,d]$ of E_k with $b < c$. Let $b < p < c$. Then F is contained in the union of the disjoint open intervals $(-1,p)$ and $(p,2)$, with $x \in (-1,p)$ and $y \in (p,2)$. Thus F is totally disconnected.

Since F is closed, $\overline{F} = F$. Since F contains no open interval, $\text{int}F = \emptyset$, and thus $\partial F = \overline{F} \setminus \text{int}F = F$.

1.6 Let $\{A_i : i \in I\}$ be a collection of open subsets of $\mathbb{R}^n$ and let $A = \bigcup_{i\in I} A_i$. If $x \in A$, then x belongs to one of the sets, A_j, say. Since A_j is open, there exists $r > 0$ such that $B(x,r) \subset A_j \subset A$, and hence A is open.

Now let $\{A_1, A_2, \ldots, A_k\}$ be a finite collection of open subsets of $\mathbb{R}^n$ and let $A = \bigcap_{i=1}^{k} A_i$. If $x \in A$, then x belongs to each of the open sets A_i and hence, for each $i = 1, \ldots, k$, there exists $r_i > 0$ such that $B(x,r_i) \subset A_i$. Letting $r = \min_{1\leq i\leq k} r_i > 0$, then $B(x,r) \subset B(x,r_i) \subset A_i$ for all i, so that $B(x,r) \subset A$ and hence A is open.

Let $A \subset \mathbb{R}^n$ and let $B = \mathbb{R}^n \setminus A$ be the complement of A. First assume that B is not open. Then there exists $x \in B$ such that, for every positive integer k, the ball $B(x, 1/k)$ is not contained in B and we may choose a sequence $x_k \in B(x,1/k) \setminus B$, so $x_k \in A$ and $x_k \to x \notin A$, so A is not closed. Thus if A is closed then B must be open.

Now suppose that A is not closed so that there exists a sequence of points $x_k \in A$ with $x_k \to x \in B = \mathbb{R}^n \setminus A$. It follows that, for every $r > 0$, there is some $x_k \in B(x,r) \setminus B$ so that $B(x,r) \not\subset B$, giving that B is not open. Thus if B is open then $A = \mathbb{R}^n \setminus B$ must be closed.

Now let $\{B_i : i \in I\}$ be a collection of closed subsets of $\mathbb{R}^n$ and let $B = \bigcap_{i\in I} B_i$. Each of the sets $A_i = \mathbb{R}^n \setminus B_i$ is open. Thus

$$A = \bigcup_{i\in I} A_i = \bigcup_{i\in I}(\mathbb{R}^n \setminus B_i) = \mathbb{R}^n \setminus \bigcap_{i\in I} A_i = \mathbb{R}^n \setminus B$$

is open and hence B is closed.

Similarly, if $\{B_i : i = 1, \ldots, k\}$ is a finite collection of closed subsets of $\mathbb{R}^n$ and $B = \bigcup_{i=1}^k B_i$, then each of the sets $A_i = \mathbb{R}^n \setminus B_i$ is open and hence

$$A = \bigcap_{i=1}^{k} A_i = \bigcap_{i=1}^{k}(\mathbb{R}^n \setminus B_i) = \mathbb{R}^n \setminus \bigcap_{i=1}^{k} B_i = \mathbb{R}^n \setminus B$$

is open so that B is closed.

1.7 Recall that a subset of $\mathbb{R}^n$ is compact if and only if it is both closed and bounded. Exercise 1.6 showed that the intersection of any collection of closed subsets of $\mathbb{R}^n$ is closed. Thus, if $A_1 \supset A_2 \supset \cdots$ is a decreasing sequence of non-empty compact subsets of $\mathbb{R}^n$ then $A = \bigcap_{k=1}^\infty A_k$ is certainly closed. It is also bounded, since it is a subset of A_1 which is bounded, so A is compact.

To show that A is non-empty we argue by contradiction. Suppose that $\bigcap_{k=1}^\infty A_k = \emptyset$ so that $\mathbb{R}^n = \mathbb{R}^n \setminus \bigcap_{k=1}^\infty A_k = \bigcup_{k=1}^\infty (\mathbb{R}^n \setminus A_k)$. Then $A_1 \subset \bigcup_{k=1}^\infty (\mathbb{R}^n \setminus A_k)$. Since A_1 is compact, it follows that A_1 is contained in the union of finitely many of the open sets $\mathbb{R}^n \setminus A_k$. Since $\mathbb{R}^n \setminus A_1 \subset \mathbb{R}^n \setminus A_2 \subset \cdots$, it follows that $A_1 \subset (\mathbb{R}^n \setminus A_k)$ for some k. This is impossible, since $A_k \subset A_1$ and $A_k \neq \emptyset$, for each k.

1.8 The half-open interval $[0,1)$ is a Borel subset of $\mathbf{R}$ since, for example,

$$[0,1) = [0,2] \cap (-1,1).$$

where $[0,2]$ is closed and hence a Borel set and $(-1,1)$ is open and hence a Borel set.

1.9 Let A_k be the set of numbers in $[0,1]$ whose kth digit is 5. Then A_k is union of 10^{k-1} half open intervals, so is Borel. Then

$$F = \bigcap_{j=1}^{\infty}\bigcup_{k=j}^{\infty} A_k,$$

as $x \in F$ if and only if $x \in A_k$ for arbitraily large k. Thus F is formed as the countable intersection of a countable union of Borel sets, so is Borel.

1.10 Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $a = (a_1, a_2)$. We may write the transformation S as

$$S(x_1, x_2) = (cx_1 \cos\theta - cx_2 \sin\theta + a_1,\ cx_1 \sin\theta + cx_2 \cos\theta + a_2)$$

so

$$\begin{aligned}
&|S(x_1,x_2) - S(y_1, y_2)|^2 \\
&\quad = c^2\big|\big((x_1 - y_1)\cos\theta - (x_2 - y_2)\sin\theta, (x_1 - y_1)\sin\theta + (x_2 - y_2)\cos\theta\big)\big|^2 \\
&\quad = c^2\big((x_1 - y_1)^2\cos^2\theta + (x_2 - y_2)^2\sin^2\theta - 2(x_1 - y_1)(x_2 - y_2)\sin\theta\cos\theta \\
&\qquad + (x_1 - y_1)^2\sin^2\theta + (x_2 - y_2)^2\cos^2\theta + 2(x_1 - y_1)(x_2 - y_2)\sin\theta\cos\theta\big) \\
&\quad = c^2((x_1 - y_1)^2 + (x_2 - y_2)^2) = c^2|(x_1, x_2) - (y_1, y_2)|^2,
\end{aligned}$$

using that $\cos^2\theta + \sin^2\theta = 1$. Thus $|S(x) - S(y)| = c|x - y|$, so S is a similarity of ratio c.

Note that $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ gives the vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ rotated about the origin by an anticlockwise angle θ. Thus the geometrical effect of the similarity S is a dilation about the origin of scale c, followed by a rotation through angle θ, followed by a translation by the vector $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

1.11(i) Since $\sin x \to \sin 0 = 0$ as $x \to 0$, we have

$$\underline{\lim}_{x\to 0} \sin x = \overline{\lim}_{x\to 0} \sin x = \lim_{x\to 0} \sin x = 0.$$

(ii) We know that

$$-1 \le \sin(1/x) \le 1, \quad \text{for } x > 0$$

so that

$$-1 \le \underline{\lim}_{x\to 0} \sin(1/x) \le \overline{\lim}_{x\to 0} \sin(1/x) \le 1.$$

Moreover, for each $n = 1, 2, \ldots$,

$$\sin(1/x_n) = -1, \quad \text{for } x_n = 1/(2n + 3/2)\pi \to 0$$

and

$$\sin(1/y_n) = 1, \quad \text{for } y_n = 1/(2n + 1/2)\pi.$$

Thus

$$\underline{\lim}_{x\to 0} \sin(1/x) \le -1 \text{ and } \overline{\lim}_{x\to 0} \sin(1/x) \ge 1,$$

so $\underline{\lim}_{x\to 0} \sin(1/x) = -1$ and $\overline{\lim}_{x\to 0} \sin(1/x) = 1$.

(iii) We have

$|x^2 + x\sin(1/x)| \le |x^2| + |x| \to 0$ as $x \to 0$. Thus

$$\begin{aligned}
\underline{\lim}_{x\to 0}(x^2 + (3 + x)\sin(1/x)) &= \underline{\lim}_{x\to 0}(x^2 + x\sin(1/x)) + \underline{\lim}_{x\to 0} 3\sin(1/x) \\
&= 0 - 3 = -3
\end{aligned}$$

using part (ii). Similarly

$$\begin{aligned}
\overline{\lim}_{x\to 0}(x^2 + (3 + x)\sin(1/x)) &= \overline{\lim}_{x\to 0}(x^2 + x\sin(1/x)) + \overline{\lim}_{x\to 0} 3\sin(1/x) \\
&= 0 + 3 = 3.
\end{aligned}$$

1.12 If $f, g : [0,1] \to \mathbf{R}$ are Lipschitz functions, then there exist $c_1, c_2 > 0$ such that

$$|f(x) - f(y)| \le c_1|x - y| \text{ and } |g(x) - g(y)| \le c_2|x - y| \quad (x, y \in [0,1]).$$

It follows that

$$\begin{aligned} |f(x) + g(x) - (f(y) + g(y))| &\le |f(x) - f(y)| + |g(x) - g(y)| \\ &\le (c_1 + c_2)|x - y| \quad (x, y \in [0,1]) \end{aligned}$$

and so the function defined by $f(x) + g(x)$ is also Lipschitz.

For $x, y \in [0,1]$,

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)| \\ &\le |f(x)g(x) - f(y)g(x)| + |f(y)g(x) - f(y)g(y)| \\ &= |g(x)||f(x) - f(y)| + |f(y)||g(x) - g(y)|. \end{aligned}$$

Moreover, for $x \in [0,1]$, we have $|f(x) - f(0)| \le c_1|x| \le c_1$, so that $|f(x)| \le |f(0)| + c_1 = c_1'$, say. Similarly $|g(x)| \le c_2'$. Thus

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\le |c_1'||f(x) - f(y)| + |c_2'||g(x) - g(y)| \\ &\le (c_1c_1' + c_2c_2')|x - y| \end{aligned}$$

so $f(x)g(x)$ is Lipschitz.

1.13 Given $x, y \in \mathbf{R}$ with $y \ne x$, it follows from the mean-value theorem that there exists $a \in (y, x)$ with

$$\frac{f(x) - f(y)}{x - y} = f'(a).$$

Thus

$$\left|\frac{f(x) - f(y)}{x - y}\right| = |f'(a)| \le c$$

and hence

$$|f(x) - f(y)| \le c|x - y| \; (x, y \in \mathbb{R})$$

so that f is a Lipschitz function.

1.14 If $f : X \to Y$ is a Lipschitz function, then

$$|f(x) - f(y)| \le c|x - y| \quad (x, y \in \mathbb{R}),$$

for some $c > 0$. Thus, given $\epsilon > 0$ and $y \in \mathbb{R}$, it follows that $|f(x) - f(y)| < \epsilon$, whenever

$$c|x - y| < \epsilon,$$

that is, whenever

$$|x - y| < \epsilon/c.$$

So, on taking $\delta = \epsilon/c > 0$, it follows that f is continuous at y, using the 'epsilon-delta' definition of continuity.

1.15 Note that if $y = f(x) = x^2 + x$, then solving the quadratic equation for x, we get $x = \frac{1}{2}(-1 \pm (1+4y)^{1/2})$, taking real values only. Thus (i) $f^{-1}(2) = \{-2, 1\}$. (ii) $f^{-1}(-2) = \emptyset$. (iii) As y increaces from 2 to 6, $(1+4y)^{1/2}$ increases from 3 to 5, so x runs over two ranges $[1,2]$ and $[-3,-2]$. Hence $f^{-1}([2,6]) = [-3,-2] \cup [1,2]$.

1.16 For $0 \le x, y \le 2$,

$$|f(x) - f(y)| = |x^2 - y^2| = |x+y||x-y| \le 4|x-y|$$

so f is also Lipschitz on $[0,2]$.

Thus f is also Lipschitz on $[1,2]$, with $f([1,2]) = [1,4]$. For $1 \le x, y \le 4$,

$$|f^{-1}(x) - f^{-1}(y)| = |\sqrt{x} - \sqrt{y}| = |\frac{x-y}{\sqrt{x}+\sqrt{y}}| \le \frac{1}{2}|x-y|$$

so f^{-1} is Lipschitz on $[1,4]$, so f is bi-Lipschitz on $[1,2]$.

For $x > 0$,

$$\frac{|f(2x) - f(x)|}{|2x - x|} = \frac{4x^2 - x^2}{x} = 3x.$$

Thus $|f(x) - f(y)|/|x-y|$ is not bounded on $\mathbb{R}$ so f is not Lipschitz on $\mathbb{R}$.

1.17 We use the 'open cover' definition of compactness. Let E be compact, f continuous, and $f(E) \subset \bigcup U_i$, a cover of $f(E)$ by open sets. Since f is continuous, the sets $f^{-1}(U_i)$ are open, so $E \subset \bigcup f^{-1}(U_i)$ is a cover of E by open sets. By compactness of E this has a finite subcover, say $E \subset \bigcup_{r=1}^{m} f^{-1}(U_{i(r)})$, so $f(E) \subset \bigcup_{r=1}^{m} U_{i(r)}$, which gives a cover of $f(E)$ by a finite subset of the U_i. Thus $f(E)$ is compact.

1.18 We take complements in A_1. Thus $A_1 \setminus A_2, A_1 \setminus A_3, \ldots$ is an increasing sequence of sets, so by (1.6)

$$\mu(A_1 \setminus \bigcap_{i=1}^{\infty} A_i) = \mu(\bigcup_{i=1}^{\infty}(A_1 \setminus A_i)) = \lim_{i\to\infty} \mu(A_1 \setminus A_i).$$

Since $\mu(A_1) < \infty$, this gives $\mu(A_1) - \mu(\bigcap_{i=1}^{\infty} A_1) = \lim_{i\to\infty}(\mu(A_1) - \mu(A_i)) = \mu(A_1) - \lim_{i\to\infty} \mu(A_i)$, so $\mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$.

1.19 We show that μ satisfies conditions (1.1)–(1.4) and is hence a measure.

First, since $a \notin \emptyset$, $\mu(\emptyset) = 0$ and thus (1.1) is satisfied.

Secondly, suppose that $A \subset B$. If $a \in A$, then a also belongs to B and hence $\mu(A) = \mu(B) = 1$. If $a \notin A$, then $\mu(A) = 0 \le \mu(B)$. Thus, in both of the two possible cases, (1.2) is satisfied.

Finally, suppose that $A_1, A_2, \ldots$ is a sequence of sets. If $a \notin A_i$, for each $i \in \mathbf{N}$, then $a \notin \bigcup_{i=1}^{\infty} A_i$ so that

$$\mu(\bigcup_{i=1}^{\infty} A_i) = 0 = \sum_{i=1}^{\infty} \mu(A_i).$$

On the other hand, if $a \in A_j$, for some integer j, then $a \in \bigcup_{i=1}^{\infty} A_i$ so that

$$\mu(\bigcup_{i=1}^{\infty} A_i) = 1 = \mu(A_j) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

If the sets A_i are disjoint, then $a \notin A_i$ for $i \neq j$ so that $\mu(A_i) = 0$ for $i \neq j$ and hence

$$\mu(\bigcup_{i=1}^{\infty} A_i) = 1 = \sum_{i=1}^{\infty} \mu(A_i).$$

Thus, in both of the two possible cases, (1.3) and (1.4) are satisfied.

1.20 With the construction of the middle third Cantor set F as indicated in figure 0.1, the kth stage of the construction E_k is the union of 2^k intervals each of length 3^{-k}, with $E_0 \supset E_1 \supset E_2 \supset \ldots$ and $f = \bigcap_{k=1}^{\infty} E_k$.

Define a mass distribution μ by starting with unit mass on $E_0 = [0, 1]$, splitting this equally between the two intervals of E_1, splitting the mass on each of these intervals equally between the two sub-intervals in E_2, etc. Thus we construct a mass distribution μ on F by repeated subdivision, splitting the mass in as uniform a way as possible at each stage. For each interval I in E_k we have $\mu(I) = 2^{-k}$, and this allows us to calculate the mass of any combination of intervals from the E_k and defines μ on every subset of $\mathbb{R}$.

1.21 For all $\epsilon > 0$, $\emptyset \subset [0, \epsilon]$ so $\mathcal{L}^1(\emptyset) \leq \mathcal{L}^1([0, \epsilon]) = \epsilon$. This is true for arbitraily small $\epsilon > 0$, so $\mathcal{L}(\emptyset) = 0$, as required for (1.1).

Let $A \subset B$. Given $\epsilon > 0$ we may find a countable collection of intervals $[a_i, b_i]$ such that $A \subset B \subset \bigcup_{i=1}^{\infty} [a_i, b_i]$ with $\sum_{i=1}^{\infty} (b_i - a_i) < \mathcal{L}^1(B) + \epsilon$. It follows that $\mathcal{L}^1(A) \leq \mathcal{L}^1(B) + \epsilon$ for all $\epsilon > 0$, so that $\mathcal{L}^1(A) \leq \mathcal{L}^1(B)$ for (1.2).

For (1.3), assume that $\mathcal{L}^1(A_i) < \infty$ for each i, since the result is clearly true otherwise. For each $\epsilon > 0$ and $i = 1, 2, \ldots$, there exist intervals $[a_{i,j}, b_{i,j}]$ such that

$$A_i \subset \bigcup_{j=1}^{\infty} [a_{i,j}, b_{i,j}] \text{ and } \sum_{j=1}^{\infty} (b_{i,j} - a_{i,j}) < \mathcal{L}^1(A_i) + \frac{\epsilon}{2^i}.$$

Clearly $\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} [a_{i,j}, b_{i,j}]$ and so

$$\mathcal{L}^1(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (b_{i,j} - a_{i,j}) \leq \sum_{i=1}^{\infty} (\mathcal{L}^1(A_i) + \frac{\epsilon}{2^i}) = \sum_{i=1}^{\infty} \mathcal{L}^1(A_i) + \epsilon.$$

It follows that (1.3) holds.

1.22 We begin by showing that μ satisfies conditions (1.1)–(1.4) and is hence a measure.

First,

$$\mu(\emptyset) = \mathcal{L}^1(\{x : (x, f(x)) \in \emptyset\}) = \mathcal{L}^1(\emptyset) = 0$$

and so (1.1) is satisfied.

Second, if $A \subset B$, then $\{x : (x, f(x)) \in A\} \subset \{x : (x, f(x)) \in B\}$ and so, since $\mathcal{L}^1$ is a measure,

$$\mu(A) = \mathcal{L}^1(\{x : (x, f(x)) \in A\}) \leq \mathcal{L}^1(\{x : (x, f(x)) \in B\}) = \mu(B)$$

so that (1.2) is satisfied.

Finally, if $A_1, A_2, \ldots$ is a sequence of sets, then, since $\mathcal{L}^1$ is a measure,

$$\begin{aligned} \mu(\bigcup_{i=1}^{\infty} A_i) &= \mathcal{L}^1(\{x : (x, f(x)) \in \bigcup_{i=1}^{\infty} A_i\}) = \mathcal{L}^1(\bigcup_{i=1}^{\infty}\{x : (x, f(x)) \in A_i\}) \\ &\leq \sum_{i=1}^{\infty} \mathcal{L}^1(\{x : (x, f(x)) \in A_i\}) = \sum_{i=1}^{\infty} \mu(A_i) \end{aligned}$$

so that (1.3) is satisfied. If the sets A_i are disjoint Borel sets then, since $\mathcal{L}^1$ is a measure,

$$\begin{aligned} \mu(\bigcup_{i=1}^{\infty} A_i) &= \mathcal{L}^1(\{x : (x, f(x)) \in \bigcup_{i=1}^{\infty} A_i\}) = \mathcal{L}^1(\bigcup_{i=1}^{\infty}\{x : (x, f(x)) \in A_i\}) \\ &= \sum_{i=1}^{\infty} \mathcal{L}^1(\{x : (x, f(x)) \in A_i\}) = \sum_{i=1}^{\infty} \mu(A_i) \end{aligned}$$

so that (1.4) is satisfied.

Thus μ is a measure on $\mathbb{R}^2$. We now show that μ is supported by the graph of f. We begin by noting that, since $[0, 1]$ is compact (that is, closed and bounded) and the map F defined by $F(x) = (x, f(x))$ is continuous, then the graph of f which is equal to $F([0, 1])$ is also compact and hence closed. Clearly,

$$\mu(\mathbb{R}^2 \setminus \mathrm{graph} f) = \mathcal{L}^1(\{x : (x, f(x)) \in \mathbb{R}^2 \setminus \mathrm{graph} f\}) = \mathcal{L}^1(\emptyset) = 0.$$

Now let $a \in [0, 1]$ and let $r > 0$. Since f is continuous, a belongs to a non-trivial interval $I_r \subset [0, 1]$ such that, for each $x \in I_r$, we have $(x, f(x)) \in B((a, f(a)), r)$ and hence

$$\mu(B((a, f(a)), r)) = \mathcal{L}^1(\{x : (x, f(x)) \in B((a, f(a)), r)\}) \geq \mathcal{L}^1(I_r) > 0.$$

Thus the graph of f is the smallest closed set X such that $\mu(\mathbb{R}^2 \setminus X) > 0$; that is, the graph of f is the support of μ.

Finally,

$$\mu(\mathrm{graph} f) = \mathcal{L}^1([0, 1]) = 1$$

so that $0 < \mu(\mathrm{graph} f) < \infty$ and hence μ is a mass distribution.

1.23 For positive integers m, n define sets

$$A_{m,n} = \{x \in D : |f_k(x) - f(x)| < \frac{1}{m} \text{ for all } k \geq n\}.$$

For each m the sequence of sets $A_{m,1} \subset A_{m,2} \subset A_{m,2} \subset \ldots$ is increasing with $\bigcup_{n=1}^{\infty} A_{m,n} = D$, so by (1.6) there is a positive integer n_m such that $\mu(D \setminus A_{m,n}) < 2^{-m}\epsilon$ for all $n \geq n_m$. Define $A = \bigcap_{m=1}^{\infty} A_{m,n_m}$. Then

$$\mu(D \setminus A) < \mu(\bigcup_{m=1}^{\infty} D \setminus A_{m,n_m}) \leq \sum_{m=1}^{\infty} \frac{\epsilon}{2^m} \leq \epsilon$$

.

Let $\delta > 0$ and take $m > 1/\delta$. If $x \in A$, then $x \in A_{m,n_m}$, so $|f_k(x) - f(x)| < \frac{1}{m} < \delta$ for all $k \geq n_m$, so $f_k(x) \to f(x)$ uniformly on A.

1.24 For $n = 1, 2, \ldots$ let $D_n = \{x : f(x) \geq 1/n\}$. Then

$$0 = \int_D f d\mu \geq \int_D \chi_{D_n} \frac{1}{n} d\mu = \frac{1}{n}\mu(D_n),$$

since $\frac{1}{n}\chi_{D_n}$ is a simple function. Thus $\mu(D_n) = 0$ for all n. Since $\{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} D_n$, it follows that $\mu\{x : f(x) > 0\} = 0$, that is $f(x) = 0$ for almost all x.

1.25 $\mathsf{E}((X - \mathsf{E}(X))^2) = \mathsf{E}(X^2 - 2X\mathsf{E}(X) + \mathsf{E}(X)^2) = \mathsf{E}(X^2) - 2\mathsf{E}(X)\mathsf{E}(X) + \mathsf{E}(X)^2 = \mathsf{E}(X^2) - \mathsf{E}(X)^2$.

1.26 The uniform distribution on $[a, b]$ has p.d.f. $f(u) = 1/(a - b)$ for $a \leq u \leq b$ and $f(u) = 0$ otherwise. Thus

$$\mathsf{E}(X) = (a-b)^{-1}\int_a^b u du = (a-b)^{-1}\left[\frac{1}{2}u^2\right]_a^b = \frac{1}{2}(b^2 - a^2)/(b-a) = \frac{1}{2}(a+b).$$

$$\begin{aligned}\mathsf{E}(X^2) &= (a-b)^{-1}\int_a^b u^2 du = (a-b)^{-1}\left[\frac{1}{3}u^3\right]_a^b \\ &= \frac{1}{3}(b^3 - a^3)/(b-a) = \frac{1}{3}(a^2 + ab + b^2).\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{var}(X) = E(X^2) - E(X)^2 &= \frac{1}{3}(a^2 + ab + b^2) - \frac{1}{4}(a+b)^2 \\ &= \frac{1}{12}(a^2 - 2ab + b^2) = \frac{1}{12}(a-b)^2.\end{aligned}$$

1.27 Define random variables X_k by $X_k = 0$ if $\omega \notin A_k$ and $X_k = 1$ if $\omega \in A_k$. Then $N_k = X_1 + \ldots + X_k$, so by the strong law of large numbers (1.25), $N_k/k \to \mathsf{E}(X_k) = p$. Thus taking A_k to be the event that the kth trial is sucessful, N_k/k is the proportion of sucesses, which converges to p, the probability of sucess.

1.28 With $X_k = 1$ if a six is scored on he kth throw and 0 otherwise, and $S_k = X_1 + \ldots + X_k$ as the number of sixes in the first k throws, X_k has mean $m = \frac{1}{6}$ and variance $\sigma^2 = \frac{1}{6}(\frac{5}{6})^2 + \frac{5}{6}(\frac{1}{6})^2 = \frac{5}{36}$. By (1.26)

$$\mathsf{P}(S_k \geq 1050) = \mathsf{P}\left(\frac{S_k - 1000}{\sqrt{5/36}\sqrt{6000}} \geq \sqrt{3}\right) \simeq \int_{\sqrt{3}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right) du = 0.075.$$

Chapter 2

2.1 Let F be a subset of $\mathbb{R}^n$, let $N_\delta(F)$ denote the smallest number of closed balls of radius δ that cover F and let $N'_\delta(F)$ denote the number of δ-mesh cubes that intersect F.

For each δ-mesh cube that intersects F, take a closed ball of radius $\delta\sqrt{n}$ whose centre is at the centre of the cube; the ball clearly contains the cube (whose diagonal is of length $\delta\sqrt{n}$) and so $N_{\delta\sqrt{n}}(F) \leq N'_\delta(F)$. On the other hand, any closed ball of radius δ intersects at most 4^n δ-mesh cubes and so $N'_\delta(F) \leq 4^n N_\delta(F)$. Combining:

$$N_{\delta\sqrt{n}}(F) \leq N'_\delta(F) \leq 4^n N_\delta(F)$$

so that if $\delta\sqrt{n} < 1$, then

$$\frac{\log N_{\delta\sqrt{n}}(F)}{-\log\delta} \leq \frac{\log N'_\delta(F)}{-\log\delta} \leq \frac{\log 4^n N_\delta(F)}{-\log\delta}$$

so

$$\frac{\log N_{\delta\sqrt{n}}(F)}{-\log\delta\sqrt{n} + \log\sqrt{n}} \leq \frac{\log N'_\delta(F)}{-\log\delta} \leq \frac{\log 4^n + \log N_\delta(F)}{-\log\delta}.$$

Taking lower limits as $\delta \to 0$, so that also $\delta\sqrt{n} \to 0$, we get that

$$\underline{\lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta} \leq \underline{\lim}_{\delta\to 0}\frac{\log N'_\delta(F)}{-\log\delta}\underline{\lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta},$$

so these terms are equal; in other words the value of the expression for lower box-counting dimension is the same for both $N_\delta(F)$ (using definition (ii) of lower box dimension), and $N'_\delta(F)$ (using definition (iv)).

The correspondance of the two definitions of upper box dimension follows in exactly the same way but taking upper limits.

2.2 Let f satisfy the Hölder condition

$$|f(x) - f(y)| \leq c|x-y|^\alpha \quad (x, y \in F).$$

Suppose that F can be covered by $N_\delta(F)$ sets of diameter at most δ. Then the $N_\delta(F)$ images of these sets under f form a cover of $f(F)$ by sets of diameter at most $c\delta^\alpha$. Thus

$$\begin{aligned}\overline{\dim}_B f(F) &= \overline{\lim}_{c\delta^\alpha\to 0}\frac{\log N_{c\delta^\alpha}(f(F))}{-\log c\delta^\alpha} \leq \overline{\lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\alpha\log\delta - \log c} \\ &= \frac{1}{\alpha}\overline{\lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta} = \frac{1}{\alpha}\overline{\dim}_B F.\end{aligned}$$

A similar argument gives the result for lower box dimensions.

2.3 Let E_k denote those numbers in $[0,1]$ whose expansions do not contain the digit 5 in the fist k decimal places. Then $F = \bigcap_{k=1}^\infty E_k$. Let $N_\delta(F)$ denote the least number of intervals of length δ that can cover F. Let k be the integer such that

$10^{-k} \leq \delta < 10^{-k-1}$. Since E_k may be regarded as the union of 9^k intervals of lengths 10^{-k}, we get $N_\delta(F) \leq 9^k$, so

$$\overline{\dim}_B F = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \leq \overline{\lim}_{k\to\infty} \frac{\log 9^k}{-\log 10^{-k-1}} \leq \overline{\lim}_{k\to\infty} \frac{k \log 9}{(k+1)\log 10} = \frac{\log 9}{\log 10}.$$

Now let $0 < \delta < 1$ and let k be the integer such that $10^{-k+1} \leq \delta < 10^{-k}$. Since any set of diameter δ can intersect at most two of the component intervals of E_k of length 10^{-k} and each such component interval contains points of F, at least $\frac{1}{2}9^k$ intervals of length δ are needed to cover F. Thus $N_\delta(F) \geq \frac{1}{2}9^k$, so

$$\underline{\dim}_B F = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \geq \underline{\lim}_{k\to\infty} \frac{\log \frac{1}{2}9^k}{-\log 10^{-k+1}} \geq \underline{\lim}_{k\to\infty} \frac{k \log 9}{(k-1)\log 10} = \frac{\log 9}{\log 10}.$$

We conclude that the box dimension of F exists, with $\dim_{\mathrm{B}} F = \log 9/\log 10$.

2.4 Let $N_\delta(F)$ denote the smallest number of squares (that is, 2-dimensional cubes) of side δ that cover F. We will use the fact (see (2.10)) that, if $\delta_k = 4^{-k}$, then

$$\dim_B F = \lim_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k},$$

if this limit exists.

It follows from the construction of F shown in figure 0.4 that $N_{\delta_k}(F) \leq 4^k$ and so

$$\overline{\dim}_B F = \overline{\lim}_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k} \leq \overline{\lim}_{k\to\infty} \frac{\log 4^k}{\log 4^k} = 1.$$

On the other hand, any square of side $\delta_k = 4^{-k}$ intersects at most two of the squares of side δ_k in E_k. Since F meets every one of the 4^k squares which comprise E_k, it follows that $N_{\delta_k}(F) \geq \frac{1}{2}4^k$, so

$$\begin{aligned}
\underline{\dim}_B F &= \underline{\lim}_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k} \geq \underline{\lim}_{k\to\infty} \frac{\log \frac{1}{2}4^k}{\log 4^k} \\
&= \underline{\lim}_{k\to\infty} \frac{k\log 4 - \log 2}{k \log 4} = 1.
\end{aligned}$$

Thus $\dim_B F = 1$.

2.5 Let $N_\delta(F)$ denote the smallest number of sets of diameter at most δ that cover F and let $\delta_k = 3^{-k}$. For each of the straight line segments that makes up E_k, take a closed disc of diameter δ_k, centred at the midpoint of the line. There are 4^k such discs and they cover F, so that (see Equivalent definitions 3.1 and comments following)

$$\overline{\dim}_B F = \overline{\lim}_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k} \leq \overline{\lim}_{k\to\infty} \frac{\log 4^k}{\log 3^k} = \frac{\log 4}{\log 3}.$$

Now let $N_\delta(F)$ denote the largest number of disjoint balls of radius δ with centres in F. The 4^k straight line segments that make up E_k have $4^k + 1$ distinct endpoints, each of which belongs to F. Balls of radius $1/3^{k+1}$ centred at these endpoints are mutually disjoint and so, putting $\delta_k = 3^{-(k+1)}$, we have by Equivalent definition (v), that

$$\begin{aligned}\underline{\dim}_B F &= \underline{\lim}_{k\to\infty}\frac{\log N_{\delta_k}(F)}{-\log\delta_k} \geq \underline{\lim}_{k\to\infty}\frac{\log(4^k+1)}{\log 3^{k+1}} \\ &\geq \underline{\lim}_{k\to\infty}\frac{\log(4^k)}{\log 3^{k+1}} = \underline{\lim}_{k\to\infty}\frac{k\log 4}{(k+1)\log 3} = \frac{\log 4}{\log 3}.\end{aligned}$$

2.6 Let $N_\delta(F)$ denote the smallest number of squares (that is, 2-dimensional cubes) of side δ that cover F. For $k = 1, 2, \ldots$, the Sierpiński triangle F can be covered by 3^k squares of side 2^{-k} and so, putting $\delta_k = 2^{-k}$, we have

$$\overline{\dim}_B F = \overline{\lim}_{k\to\infty}\frac{\log N_{\delta_k}(F)}{-\log\delta_k} \leq \overline{\lim}_{k\to\infty}\frac{\log 3^k}{\log 2^k} = \frac{\log 3}{\log 2}.$$

Now let $N_\delta(F)$ denote the largest number of disjoint balls of radius δ with centres in F. The top vertex of each of the 3^k triangles in E_k belongs to F and balls of radius $1/2^{k+1}$ centered at these vertices are mutually disjoint. So, putting $\delta_k = 2^{-(k+1)}$, we have

$$\underline{\dim}_B F = \underline{\lim}_{k\to\infty}\frac{\log N_{\delta_k}(F)}{-\log\delta_k} \geq \underline{\lim}_{k\to\infty}\frac{\log 3^k}{\log 2^{k+1}} = \frac{\log 3}{\log 2}.$$

Thus $\dim_B F = \log 3/\log 2$.

2.7 The middle-third Cantor set has 2^k gaps of length 3^{-k-1} for $k = 0, 1, 2, \ldots$. If $\frac{1}{2}3^{-k} < \delta \leq \frac{1}{2}3^{-k-1}$ the δ-neighbourhood fills the gaps of lengths 3^{-k} or less, and has two parts of length δ in the gaps of length 3^{-k-1} or more. Summing these lengths over all gaps, and noting that the parts of F_δ at each end of F have length δ, gives

$$\begin{aligned}\mathcal{L}(F_\delta) &= \sum_{i=k}^{\infty} 2^{i-1}3^{-i} + 2\delta\sum_{i=1}^{k-1} 2^{i-1} + 2\delta \\ &= \left(\frac{2}{3}\right)^{k-1} + 2^k\delta\end{aligned}$$

on summing the geometric series. Hence

$$2^k\delta \leq \mathcal{L}(F_\delta) \leq 2^k\delta\left(\frac{2}{3}\right)^{k-1} \leq\leq 4\times 2^k\delta$$

or

$$c_1\delta^{1-\log 2/log3}\delta \leq \mathcal{L}(F_\delta) \leq c_21\delta^{1-\log 2/log3}.$$

Hence Proposition 2.4 gives that $\dim_{\mathrm{P}} F = \log 2/log3$

2.8 The idea is to construct a set such at some scales a relatively large number of boxes are needed in a covering and at other scales one can manage with relatively few. We adapt the middle third Cantor set by deleting the middle 3/5 of intervals at certain scales rather than the middle 1/3. Thus set $k_n = 10^n$, for $n = 0, 1, 2, \ldots$ and let $E = \bigcap_{k=0}^{\infty} E_k$, where $E_0 = [0, 1]$, and

- for $k_0 \le k \le k_1$, $k_2 < k \le k_3, \ldots$, E_k is obtained by deleting the middle 1/3 of each interval in E_{k-1};
- for $k_1 < k \le k_2$, $k_3 < k \le k_4, \ldots$, E_k is obtained by deleting the middle 3/5 of each interval in E_{k-1}.

We estimate the lower and upper box dimensions of E by estimating $N_\delta(E)$, the least number of closed intervals of length δ that can cover E.
(i) If n is even, then E_{k_n} is made up of 2^{k_n} intervals of length

$$\delta_n = \left(\frac{1}{3}\right)^{k_1}\left(\frac{1}{5}\right)^{k_2-k_1}\cdots\left(\frac{1}{3}\right)^{k_{n-1}-k_{n-2}}\left(\frac{1}{5}\right)^{k_n-k_{n-1}} < \left(\frac{1}{5}\right)^{k_n-k_{n-1}}.$$

Taking these intervals as a cover

$$\begin{aligned}\underline{\dim}_B E &\le \underline{\lim}_{n\to\infty}\frac{\log N_{\delta_n}(E)}{-\log\delta_n} \le \underline{\lim}_{n\to\infty}\frac{\log 2^{k_n}}{\log 5^{k_n-k_{n-1}}}\\ &= \underline{\lim}_{n\to\infty}\frac{k_n\log 2}{(k_n-k_{n-1})\log 5} = \underline{\lim}_{n\to\infty}\frac{10k_{n-1}\log 2}{9k_{n-1}\log 5} = \frac{10\log 2}{9\log 5}.\end{aligned}$$

(ii) If n is odd, then E_{k_n} is made up of 2^{k_n} intervals of length

$$\delta_n = \left(\frac{1}{3}\right)^{k_1}\left(\frac{1}{5}\right)^{k_2-k_1}\cdots\left(\frac{1}{5}\right)^{k_{n-1}-k_{n-2}}\left(\frac{1}{3}\right)^{k_n-k_{n-1}} > \left(\frac{1}{5}\right)^{k_{n-1}}\left(\frac{1}{3}\right)^{k_n-k_{n-1}}.$$

Any interval of length δ_n meets at most two of the intervals in E_{k_n} and so, since E has points in every interval in E_{k_n},

$$\begin{aligned}\overline{\dim}_B E &\ge \overline{\lim}_{n\to\infty}\frac{\log N_{\delta_n}(E)}{-\log\delta_n} \ge \overline{\lim}_{n\to\infty}\frac{\log(2^{k_n}/2)}{\log(5^{k_{n-1}}3^{k_n-k_{n-1}})}\\ &= \overline{\lim}_{n\to\infty}\frac{k_n\log 2-\log 2}{k_{n-1}\log 5+(k_n-k_{n-1})\log 3}\\ &= \overline{\lim}_{n\to\infty}\frac{10k_{n-1}\log 2-\log 2}{k_{n-1}\log 5+9k_{n-1}\log 3}\\ &= \frac{10\log 2}{\log 5+9\log 3} \ge \frac{10\log 2}{11\log 3}.\end{aligned}$$

Since

$$\frac{10\log 2}{9\log 5} < \frac{10\log 2}{11\log 3}$$

$$\underline{\dim}_B E < \overline{\dim}_B E,$$

as required.

2.9 The idea is to construct sets E and F such that at every scale one of E or F looks 'large' and the other looks 'small'. Let E be the set described in the Solution to Exercise 3.8. We construct a set F in a similar way, except that the scaling of intervals is complementary and the set is positioned to be disjoint from E. Thus set $k_n = 10^n$, for $n = 0, 1, 2, \ldots$ and let $F = \bigcap_{k=0}^\infty F_k$, where $F_0 = [2, 3]$, and

- for $k_0 \le k \le k_1$, $k_2 < k \le k_3, \ldots$, F_k is obtained by deleting the middle 3/5 of each interval in F_{k-1};
- for $k_1 < k \le k_2$, $k_3 < k \le k_4, \ldots$, F_k is obtained by deleting the middle 1/3 of each interval in F_{k-1}.

As in the solution to Exercise 2.8 we get that

$$\underline{\dim}_B E, \underline{\dim}_B F \le \frac{10 \log 2}{9 \log 5}.$$

For each $k = 1, 2, \ldots$, let δ_k denote the length of the longest intervals in $E_k \cup F_k$ — there are 2^k such intervals, each of which meets $E \cup F$. Since any other interval of length δ_k meets at most two of these intervals, it follows that the smallest number of closed intervals of length δ_k that cover $E \cup F$ satisfies $N_{\delta_k}(E \cup F) \ge 2^k/2$. Now $\delta_k \ge \left(\frac{1}{3}\right)^{k/2} \left(\frac{1}{5}\right)^{k/2}$ and $\delta_k \ge (1/5)\delta_{k-1}$, so by the note after Definitions 3.1

$$\begin{aligned}
\underline{\dim}_B E \cup F &= \underline{\lim}_{\delta_k \to 0} \frac{\log N_{\delta_k}(E \cup F)}{-\log \delta_k} \ge \underline{\lim}_{k \to \infty} \frac{\log 2^k/2}{\log 5^{k/2} \log 3^{k/2}} \\
&= \underline{\lim}_{k \to \infty} \frac{k \log 2 - \log 2}{(k/2) \log 5 + (k/2) \log 3} = \frac{2 \log 2}{\log 5 + \log 3} > \frac{10 \log 2}{9 \log 5}.
\end{aligned}$$

2.10 Since F is a countable set, $\dim_H F = 0$.

The box dimension calculation is similar to Example 2.7. Let $N_\delta(F)$ be the smallest number of sets of diameter at most δ that cover F. If $|U| = \delta < 1/2$ and k is the integer satisfying

$$\frac{2k-1}{k^2(k-1)^2} = \frac{1}{(k-1)^2} - \frac{1}{k^2} > \delta \ge \frac{1}{k^2} - \frac{1}{(k+1)^2} = \frac{2k+1}{(k+1)^2 k^2},$$

then U can cover at most one of the points $\{1, \frac{1}{4}, \ldots, \frac{1}{k^2}\}$. Thus $N_\delta(F) \ge k$ and hence

$$\begin{aligned}
\underline{\dim}_B F &= \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \ge \underline{\lim}_{k \to \infty} \frac{\log k}{\log \frac{(k+1)^2 k^2}{2k+1}} \\
&= \underline{\lim}_{k \to \infty} \frac{\log k}{2 \log(k+1) + 2 \log k - \log 2 - \log(k + 1/2)} = \frac{1}{3}.
\end{aligned}$$

On the other hand, if

$$\frac{2k-1}{k^2(k-1)^2} > \delta \ge \frac{2k+1}{(k+1)^2 k^2},$$

then $k+1$ intervals of length δ cover $[0, 1/k^2]$ leaving $k-1$ points of F which can be covered by $k-1$ intervals of length δ. Thus

$$\begin{aligned}
\overline{\dim}_B F &= \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \le \overline{\lim}_{k \to \infty} \frac{\log 2k}{\log \frac{(k-1)^2 k^2}{2k-1}} \\
&= \overline{\lim}_{k \to \infty} \frac{\log 2k}{2 \log(k-1) + 2 \log k - \log 2 - \log(k - 1/2)} = \frac{1}{3}.
\end{aligned}$$

Thus $\dim_B F = 1/3$.

2.11 The von Koch curve F has (upper and lower) box dimensions equal to $\log 4/\log 3$. Moreover, by virtue of the self-similarity of F, $\dim_{\rm B}(F\cap V)=\log 4/\log 3$ for every open set V that intersects F. By Proposition 2.8, $\dim_{\rm MB} F = \dim_{\rm B} F = \log 4/\log 3$.

2.12 Recall that the divider dimension of a curve C is defined as $\lim_{\delta\to 0}\log M_\delta(C)/-\log\delta$ (assuming that this limit exists), where $M_\delta(C)$ is the maximum number of points $x_0,x_1,\ldots,x_m$ on C, in that order, with $|x_i-x_{i-1}|\geq\delta$ for $i=1,2,\ldots,m$.

By inspection of the von Koch curve C, taken to have of baselength 1, (see Figure 0.2), we have that if k is the integer such that $3^{-k-1}\leq\delta<3^{-k}$, then $4^k<4^k+1\leq M_\delta(C)\leq 4^{k+1}+1<4^{k+2}$. Then

$$\frac{k\log 4}{(k+1)\log 3}=\frac{\log(4^k)}{-\log(3^{-k-1})}\leq\frac{\log M_\delta(C)}{-\log\delta}\leq\frac{\log(4^{k+2})}{-\log(3^{-k})}\frac{(k+2)\log 4}{\log 3}.$$

As $\delta\to 0$, $k\to\infty$, so taking limits gives that

$$\text{divider dimension } =\lim_{\delta\to 0}\frac{\log M_\delta(C)}{-\log\delta}=\frac{\log 4}{-\log 3}$$

(which, of course equals the Hausdorff and box dimensions of C).

2.13 Recall that the divider dimension of a curve C is defined as $\lim_{\delta\to 0}\log M_\delta(C)/-\log\delta$ (assuming that this limit exists), where $M_\delta(C)$ is the maximum number of points $x_0,x_1,\ldots,x_m$ on C, in that order, with $|x_i-x_{i-1}|\geq\delta$ for $i=1,2,\ldots,m$.

Consider Equivalent definition 3.1(v) of box dimension, taking $N_\delta(C)$ to be the greatest number of disjoint balls of radius δ with centres on C. Then if $B_1,\ldots,B_{N_\delta(C)}$ is a maximal collection of disjoint balls of radii δ with centres on C, every ball B_i must contain at least one point x_j in any maximal sequence of points $x_0,x_1,\ldots,x_1$ for the divider dimension, otherwise the centre of B_i may be added to the sequence to increase its length. Thus $N_\delta(C)\leq M_\delta(C)$, so

$$\frac{N_\delta(C)}{-\log\delta}\leq\frac{\log M_\delta(C)}{-\log\delta},$$

and taking limits as $\delta\to 0$ gives that the box dimension is less than or equal to the divider dimension, assumming both exist. (If not a similar inequality holds for lower and upper box and divider dimensions.)

2.14 The middle λ Cantor set F may be constructed from the unit interval by removing 2^k open intervals of lengths $\lambda(\frac{1}{2}(1-\lambda))^k$ for $k=0,1,2,\ldots$. Thus, denoting these complementary intervals by I_i, we have

$$\sum_i |I_i|^s=\sum_{k=0}^{\infty}2^k\lambda^s\left(\frac{1}{2}(1-\lambda)\right)^{ks}.$$

This is a geometric series which converges if and only if the common ratio $2(\frac{1}{2}(1-\lambda))^s<1$, that is if $s>\log 2/\log(2/(1-\lambda))$, a number equal to box dimensions (and Hausdorff dimension) of F, see also Exercise 3.14.

Chapter 3

3.1 Put

$$\overline{\mathcal{H}}^s_\delta(F) = \inf\{\sum_i |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \text{ by closed sets}\}.$$

Since we have reduced the class of permissible covers by restricting to covers by closed sets, we must have $\overline{\mathcal{H}}^s_\delta(F) \geq \mathcal{H}^s_\delta(F)$. Now suppose that $\{U_i\}$ is a δ-cover of F. Since the closure $\overline{U}_i$ of U_i satisfies $|\overline{U}_i| = |U_i|$, it follows that $\{\overline{U}_i\}$ is a δ-cover of F by closed sets with $\sum_i |\overline{U}_i|^s = \sum_i |U_i|^s$. Since this is true for every δ-cover of F, it follows that $\overline{\mathcal{H}}^s_\delta(F) \leq \mathcal{H}^s_\delta(F)$. Thus $\overline{\mathcal{H}}^s_\delta(F) = \mathcal{H}^s_\delta(F)$ for all $\delta > 0$ and so the value of $\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F)$ is unaltered if we only consider δ-covers by closed sets.

3.2 Suppose that $\{U_i\}$ is a δ-cover of F. For any set U_i in the cover we have $|U_i|^0 = 1$ and so $\sum_i |U_i|^0$ is equal to the number of sets in the cover. Thus $\mathcal{H}^0_\delta(F)$ is the smallest number of sets that form a δ-cover of F.

If F has k points, $x_1, x_2, \ldots, x_k$, then the k balls of radius $\delta/2$ with centers at $x_1, x_2, \ldots, x_k$ form a δ-cover of F and so $\mathcal{H}^0_\delta(F) \leq k$. Moreover, if $\delta > 0$ is so small that $|x_i - x_j| > \delta$ for all $i \neq j$, then any δ-cover of F must contain at least k sets and so $\mathcal{H}^0_\delta(F) \geq k$. So, for δ small enough, we have $\mathcal{H}^0_\delta(F) = k$ and hence $\mathcal{H}^0(F) = \lim_{\delta\to 0} \mathcal{H}^0_\delta(F) = k$.

Finally, if F has infinitely many points, then for each positive integer k, we can take a set $F_k \subset F$ such that F_k has k points. Then $\mathcal{H}^0(F) \geq \mathcal{H}^0(F_k) = k$ for all k, so $\mathcal{H}^0(F) = \infty$.

3.3 Clearly, for every $0 < \epsilon \leq \delta$, we may cover the empty set with a single set of diameter ϵ, so $0 \leq \mathcal{H}^s_\delta(\emptyset) \leq \epsilon^s$ for all $\epsilon > 0$, giving $\mathcal{H}^s_\delta(\emptyset) = 0$. Thus $\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F) = 0$.

If $E \subset F$, every δ-cover of F is also a δ-cover of E, so taking infima over all δ-covers gives $\mathcal{H}^s_\delta(E) \leq \mathcal{H}^s_\delta(F)$ for all $\delta > 0$. Letting $\delta \to 0$ gives $\mathcal{H}_\delta(E) \leq \mathcal{H}_\delta(F)$.

Now let $F_1, F_2, \ldots$ be subsets of $\mathbb{R}^n$. Without loss of generality, we may assume that $\sum_{i=1}^\infty \mathcal{H}^s_\delta(F_i) < \infty$. For $\epsilon > 0$ let $\{U_{i,j} : j = 1, 2, \ldots\}$ be a δ-cover of F_i such that $\sum_{j=1}^\infty |U_{i,j}|^s \leq \mathcal{H}^s_\delta(F_i) + 2^{-i}\epsilon$. Then $\{U_{i,j} : i = 1, 2, \ldots, j = 1, 2, \ldots\}$ is a δ-cover of $\bigcup_{i=1}^\infty F_i$ and

$$\mathcal{H}^s_\delta\left(\bigcup_{i=1}^\infty F_i\right) \leq \sum_{i=1}^\infty \sum_{j=1}^\infty |U_{i,j}|^s \leq \sum_{i=1}^\infty \left(\mathcal{H}^s_\delta(F_i) + \frac{\epsilon}{2^i}\right) = \epsilon + \sum_{i=1}^\infty \mathcal{H}^s_\delta(F_i) \leq \epsilon + \sum_{i=1}^\infty \mathcal{H}^s(F_i).$$

Since this is true for every $\epsilon > 0$, it follows that

$$\mathcal{H}^s(\bigcup_{i=1}^\infty F_i) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(\bigcup_{i=1}^\infty F_i) \leq \sum_{i=1}^\infty \mathcal{H}^s(F_i)$$

as required.

3.4 Note that in calculating $\mathcal{H}^s_\delta([0,1])$ it is enough to consider coverings by intervals. If $0 \le s < 1$ and $\{U_i\}$ is a δ-cover of $[0,1]$ by intervals, then

$$1 \le \sum_i |U_i| = \sum_i |U_i|^{1-s}|U_i|^s \le \delta^{1-s}\sum_i |U_i|^s.$$

Hence $\mathcal{H}^s_\delta([0,1]) \ge \delta^{s-1}$, so letting $\delta \to 0$ gives $\mathcal{H}^s([0,1]) = \infty$.

For $s > 1$, we may cover $[0,1]$ by at most $(1+1/\delta)$ intervals of length δ, so

$$\mathcal{H}^s_\delta([0,1]) \le (1+1/\delta)\delta^s \to 0$$

as $\delta \to 0$, so $\mathcal{H}^s([0,1]) = 0$.

For $s = 1$, if $\{U_i\}$ is a δ-cover of $[0,1]$ by intervals, then $1 \le \sum_i |U_i|$, so $\mathcal{H}^1_\delta([0,1]) \ge 1$, and letting $\delta \to 0$ gives $\mathcal{H}^1([0,1]) \ge 1$.

Taking a cover $[0,1]$ by at most $(1+1/\delta)$ intervals of length δ,

$$\mathcal{H}^1_\delta([0,1]) \le (1+1/\delta)\delta \to 1$$

as $\delta \to 0$, so $\mathcal{H}^1([0,1]) \le 1$. We conclude that $\mathcal{H}^1([0,1]) = 1$.

3.5 First suppose that F is bounded, say $F \subset [-m,m]$. By the mean value theorem, for some $z \in [-m,m]$,

$$|f(x) - f(y)| = |f'(z)||x-y| \le (\sup_{z\in[-m,m]} |f'(z)|)|x-y|$$

Since $f'(z)$ is continuous it is bounded on $[-m,m]$. Thus f is Lipschitz on F, so $\dim_{\mathrm{H}} f(F) \le \dim_{\mathrm{H}}(F)$ by Proposition 3.3. For arbitrary $F \subset \mathbb{R}$, $f(F) = \bigcup_{m=1}^\infty f(F_n \cap [-m,m])$, so by countable stability

$$\dim_{\mathrm{H}} f(F) = \sup_m \dim_{\mathrm{H}} f(F_n \cap [-m,m]) \le \sup_m \dim_{\mathrm{H}}(F_n \cap [-m,m]) \le \dim_{\mathrm{H}} F,$$

by the bounded case.

3.6 Let $F_k = F \cap [1/k, k]$. If $x, y \in F_k$, then

$$|f(x) - f(y)| = |x^2 - y^2| = |x-y||x+y|$$

and so

$$\frac{2}{k}|x-y| \le |f(x) - f(y)| \le 2k|x-y|.$$

Thus f is a bi-Lipschitz map on F_k and so, by Proposition 3.3, $\dim_H f(F_k) = \dim_H F_k$. Similarly, if $G_k = F \cap [-k, -1/k]$, then $\dim_H f(G_k) = \dim_H G_k$.

Now $F = (F \cap \{0\}) \cup \bigcup_{k=1}^\infty (F_k \cup G_k)$ and $f(F) = f(F \cap \{0\}) \cup \bigcup_{k=1}^\infty (f(F_k) \cup f(G_k))$. Since $F \cap \{0\}$ and $f(F \cap \{0\})$ contain at most one point, they both have zero dimension. Thus, by countable stability,

$$\begin{aligned} \dim_H F &= \sup\{\dim_H F_k, \dim_H G_k : k = 1, 2, \ldots\} \\ &= \sup\{\dim_H f(F_k), \dim_H f(G_k) : k = 1, 2, \ldots\} = \dim_H f(F). \end{aligned}$$

[Note that this result is not true for box dimension. For example, using Example 2.7 and Exercise 2.10 we see that $\dim_B f(F) \ne \dim_B F$ when $F = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$.]

3.7 Define $g : [0,1] \to \mathrm{graph} f$ by $g(x) = (x, f(x))$. We claim that g is bi-Lipschitz. For:

$$|g(x) - g(y)|^2 = |x-y|^2 + |f(x) - f(y)|^2$$

so

$$|x-y|^2 \leq |g(x) - g(y)|^2 \leq |x-y|^2 + c^2|x-y|^2 = (1+c^2)|x-y|^2$$

since $|f(x) - f(y)| \leq c|x-y|$ for some $c > 0$. Thus g is bi-Lipschitz, so $1 = \dim_{\mathrm{H}}([0,1]) = \dim_{\mathrm{H}} g([0,1]) = \dim_{\mathrm{H}} \mathrm{graph} f$.

3.8 Both $\{0,1,2,3,\ldots\}$ and $\{0,1,\frac{1}{2},\frac{1}{3},\ldots\}$ are countable sets, so have Hausdorff dimension 0.

3.9 Note that F splits into 9 parts $F_i = F \cap [i/10, (i+1)/10]$ for $i = 0,1,2,3,4,6,7,8,9$, these parts disjoint except possibly for endpoints which have s-dimensional measure 0 if $s > 0$. It follows from Scaling Property 2.1 that, for $s > 0$, $\mathcal{H}^s(F_i) = 10^{-s}\mathcal{H}^s(F)$ for all i. Summing, and using that F is essentially a disjoint union of the F_i, it follows that, for $s > 0$,

$$\mathcal{H}^s(F) = \sum_{i=0,1,2,3,4,6,7,8,9} \mathcal{H}^s(F_i) = 9 \times 10^{-s}\mathcal{H}^s(F).$$

If we assume that when $s = \dim_H F$ we have $0 < \mathcal{H}^s(F) < \infty$, then, for this value of s, we may divide through by $\mathcal{H}^s(F)$ to obtain $1 = 9 \times 10^{-s}$ and hence $s = \dim_H F = \log 9/\log 10$.

3.10 Note that, for $i,j = 0,1,2,3,4,6,7,8,9$ the sets $F \cap ([i/10,(i+1)/10] \times [j/10,(j+1)/10])$ are scale $1/10$ similar copies of F. By the addition and scaling properties of Hausdorff measure,

$$\mathcal{H}^s(F) = \sum_{i,j \neq 5} \mathcal{H}^s(F \cap ([i/10,(i+1)/10] \times [j/10,(j+1)/10]) = 9^2 10^{-s}\mathcal{H}^s(F),$$

provided $0 < \mathcal{H}^s(F) < \infty$ when $s = \dim_{\mathrm{H}} F$, in which case $1 = 9^2 10^{-s}$, giving $s = 2\log 9/\log 10$.

3.11 The set F comprises one similar copy of itself at scale $\frac{1}{2}$, say F_0, and four similar copies at scale $\frac{1}{4}$, say F_1, F_2, F_3, F_4. By the additive and scaling properties of Hausdorff measure, noting that the F_i intersect only in single points,

$$\mathcal{H}^s(F) = \mathcal{H}^s(F_0) + \sum_{i=1}^{4} \mathcal{H}^s(F_i) = \left(\frac{1}{2}\right)^s \mathcal{H}^s(F) + 4\left(\frac{1}{4}\right)^s \mathcal{H}^s(F)$$

for $s > 0$. Provided $0 < \mathcal{H}^s(F) < \infty$ when $s = \dim_{\mathrm{H}} F$, we have $1 = (\frac{1}{2})^s + 4(\frac{1}{2})^s$. Thus $4(\frac{1}{2}^s)^2 + (\frac{1}{2})^s - 1 = 0$; solving this quadratic equation in $(\frac{1}{2})^s$ gives $(\frac{1}{2})^s = (-1+\sqrt{17})/8$ as the positive solution, so $s = (\log 8 - \log(\sqrt{17}-1))/\log 2$.

3.12 F is the union of countably many translates of the middle third Cantor set, all of wich have Hausdorff dimension $\log 2/\log 3$, so $\dim_{\mathrm{H}} F = \log 2/\log 3$ using countable stability.

3.13 F is the union, over all finite sequences $a_1, a_2, \ldots, a_k$ of the digits $0, 1, 2$, of similar copies of the middle-third Cantor set scaled by a factor 3^{-k} and translated to have left hand end at $0.a_1a_2\ldots a_k$, to base 3. Thus F is the union of countably many similar copies of the Cantor set, so $\dim_{\mathrm{H}} F = \log 2/\log 3$ using countable stability.

3.14 The set F is the union of two disjoint similar copies of itself, F_L, F_R, say, at scales $\frac{1}{2}(1-\lambda)$. By the additive and scaling properties of Hausdorff measure

$$\mathcal{H}^s(F) = \mathcal{H}^s(F_L) + \mathcal{H}^s(F_R) = 2\left(\frac{1}{2}(1-\lambda)\right)^s \mathcal{H}^s(F)$$

for $s \geq 0$. Provided $0 < \mathcal{H}^s(F) < \infty$ when $s = \dim_{\mathrm{H}} F$, we have $1 = 2(\frac{1}{2}(1-\lambda))^s$, giving that $\dim_{\mathrm{H}} F = \log 2/\log(2/(1-\lambda))$.

The set E is the union of four disjoint similar copies of itself, E_1, E_2, E_3, E_4, say, at scales $\frac{1}{2}(1-\lambda)$. By the additive and scaling properties of Hausdorff measure

$$\mathcal{H}^s(F) = \sum_{i=1}^{4} \mathcal{H}^s(F_i) = 4\left(\frac{1}{2}(1-\lambda)\right)^s \mathcal{H}^s(F)$$

for $s \geq 0$. Provided $0 < \mathcal{H}^s(F) < \infty$ when $s = \dim_{\mathrm{H}} F$, we have $1 = 4(\frac{1}{2}(1-\lambda))^s$, giving that $\dim_{\mathrm{H}} F = \log 4/\log(2/(1-\lambda)) = 2\log 2/\log(2/(1-\lambda))$.

3.15 Take the unit square E_0 and divide it into 16 squares of side $1/4$. Now take $0 < r < 1/4$, put a square of side r in the middle of each of the 16 small squares and discard everything that is not inside one of these squares, to get a set E_1.

Keep on repeating this process so that, at the k-th stage, there is a collection E_k of 16^k disjoint squares of side r^k. Then $F_r = \bigcap_k E_k$ is a totally disconnected subset of $\mathbb{R}^2$. (If two points x, y are in the same component of F_r, then they must belong to the same square in E_k, for all $k = 1, 2, \ldots$. Thus $|x-y| \leq \sqrt{2}r^k$, for each $k = 1, 2, \ldots$, and hence $|x-y| = 0$ so that $x = y$.)

The set F_r is made up of 16 disjoint similar copies of itself, each scaled by a factor r, denote these sets as $F_{r,1}, \ldots, F_{r,16}$. It follows from Scaling property 2.1 that, for $s \geq 0$,

$$\mathcal{H}^s(F_r) = \sum_{i=1}^{16} \mathcal{H}^s(F_{r,i}) = \sum_{i=1}^{16} r^s \mathcal{H}^s(F_r).$$

Assuming that when $s = \dim_H F_r$ we have $0 < \mathcal{H}^s(F_r) < \infty$ (using the heuristic method), then, for this value of s we may divide by $\mathcal{H}^s(F_r)$ to obtain $1 = 16r^s$ and so $s = \dim_H F_r = -\log 16/\log r$. As r increases from 0 to $1/4$, $\dim_H F_r$ increases from 0 to 2, taking every value in between.

A set consisting of a single point gives a totally disconnected subset of $\mathbb{R}^2$ of Hausdorff dimension zero. It remains to show that there exists a totally disconnected subset of $\mathbb{R}^2$ of Hausdorff dimension two. For one of many ways to do this, let $G = \bigcup_{k=5}^{\infty} G_k$, where $G_k = F_{1/4-1/k} + (k, 0)$. The sets G_k are disjoint and hence G is a totally disconnected subset of $\mathbb{R}^2$. By countable stability, we have

$$\dim_H G = \sup_{5\le k\le\infty} \dim_H G_k = \sup_{5\le k\le\infty} \dim_H(F_{1/4-1/k}) = 2,$$

using that G_k is congruent to $F_{1/4-1/k}$, whose dimension tends to 2 as $k \to \infty$.

3.16 Note that F is just a copy of the middle-third Cantor set scaled by $\frac{1}{3}\pi$. Thus $\dim_{\mathrm{H}} F = \log 2/\log 3$.

3.17 Let $E = [0,1] \cap \mathbb{Q}$ and $F = \{x \in [0,1] : x - \sqrt{2} \in \mathbb{Q}\}$, so that E and F are disjoint dense subsets of $[0,1]$. If $\{B_i\}$ is any collection of disjoint balls (i.e. intervals) with centres in E and radii at most δ, then, by considering the lengths of the B_i, we see that $\sum_i |B_i| \le 1 + \delta$. Moreover, taking B_i as nearly abutting intervals of lengths 2δ we can get $\sum_i |B_i| \ge 1$. Thus, since

$$\mathcal{P}^1_\delta(E) = \sup\left\{\sum_i |B_i| : \{B_i\}\text{are disjoint balls of radii} \le \delta \text{ with centres in } F\right\},$$

we get $1 \le \mathcal{P}^1_\delta(E) \le 1 + \delta$. Letting $\delta \to 0$ gives $\mathcal{P}^1_0(E) = 1$. In a similar way, $\mathcal{P}^1_0(E) = 1$ and $\mathcal{P}^1_0(E \cup F) = 1$. In particular $\mathcal{P}^1_0(E \cup F) \ne \mathcal{P}^1_0(E) + \mathcal{P}^1_0(F)$.

3.18 We use the notation of section 2.5. Let U_i be a δ-cover of F. Then

$$\sum h(|U_i|) = \sum (h(|U_i|)/g(|U_i|))g(|U_i|) \le \eta(\delta) \sum g(|U_i|)$$

where $\eta(\delta) = \sup_{0<t\le\delta} h(t)/g(t)$. Taking infima, $\mathcal{H}^h_\delta(F) \le \eta(\delta)\mathcal{H}^g_\delta(F)$. Letting $\delta \to 0$, then $\eta(\delta) \to 0$, and $\mathcal{H}^g_\delta(F) \to \mathcal{H}^g(F) < \infty$, so $\mathcal{H}^h_\delta(F) \to 0$, that is $\mathcal{H}^h(F) = 0$.

3.19 If $F_1 \subset F_2$ then any δ-cover of F_2 by rectangles is also a δ-cover of F_1, so that from the definition, $\mathcal{H}^{s,t}_\delta(F_1) \le \mathcal{H}^{s,t}_\delta(F_2)$, and letting $\delta \to 0$ gives $\mathcal{H}^{s,t}(F_1) \le \mathcal{H}^{s,t}(F_2)$. In particular, if $(s,t) \in \mathrm{print} F_1$ then $0 < \mathcal{H}^{s,t}(F_1)$ so $0 < \mathcal{H}^{s,t}(F_2)$ giving $(s,t) \in \mathrm{print} F_2$. Thus $\mathrm{print} F_1 \subset \mathrm{print} F_2$.

It follows at once that $\mathrm{print} F_k \subset \mathrm{print}(\bigcup_{i=1}^\infty F_i)$ for all k, so that $\bigcup_{i=1}^\infty \mathrm{print} F_i \subset \mathrm{print}(\bigcup_{i=1}^\infty F_i)$.

Now suppose $(s,t) \notin \mathrm{print} F_i$ for all i. Then $\mathcal{H}^{s,t}(F_i) = 0$ for all i, so $\mathcal{H}^{s,t}(\bigcup_{i=1}^\infty F_i) = 0$ since $\mathcal{H}^{s,t}$ is a measure. We conclude that $(s,t) \notin \mathrm{print}(\bigcup_{i=1}^\infty F_i)$. Thus $\bigcup_{i=1}^\infty \mathrm{print} F_i = \mathrm{print}(\bigcup_{i=1}^\infty F_i)$.

Suppose now that $s' + t' \le s + t$ and $t' \le t$. For any δ-cover of a set F by rectangles U_i with sides $a(U_i) \ge b(U_i)$, we have

$$\begin{aligned}
\sum_i a(U_i)^s b(U_i)^t &\le \sum_i a(U_i)^{s-s'} a(U_i)^{s'} b(U_i)^{t-t'} b(U_i)^{t'} \\
&\le \sum_i a(U_i)^{s'} b(U_i)^{t'} a(U_i)^{s-s'+t-t'} \\
&\le \delta^{(s+t)-(s'+t')} \sum_i a(U_i)^{s'} b(U_i)^{t'}.
\end{aligned}$$

It follows from the definition that if $0 < \delta < 1$ then $\mathcal{H}^{s,t}_\delta(F) \leq \mathcal{H}^{s',t'}_\delta(F)$, so $\mathcal{H}^{s,t}(F) \leq \mathcal{H}^{s',t'}(F)$. Thus if $(s,t) \in \mathrm{print}F$ then $0 < \mathcal{H}^{s,t}(F) \leq \mathcal{H}^{s',t'}(F)$, so $(s',t') \in \mathrm{print}F$.

Since $\mathrm{print}(F_1 \cup F_2) = \mathrm{print}F_1 \cup \mathrm{print}F_2$, taking F_1 and F_2 such that the union of their dimension prints is not convex will give a set $F_1 \cup F_2$ with non-convex dimension print. Taking F_1 a circle and F_2 the product of uniform Cantor sets of dimensions $\frac{1}{3}$ and $\frac{3}{4}$, will achieve this, see figure 3.4.

3.20 Let F be the Sierpiński triangle of side 1. Let $x \in F$ and let

$$2^{-k-1}\sqrt{3} \leq r \leq 2^{-k}\sqrt{3}$$

for some positive integer $k \geq 1$. From the geometry of the Sierpiński triangle $B(x,r) \supseteq B(x, 2^{-k-1}\sqrt{3}$ contains an empty equilateral triangle of side 2^{-k-1} which contains an inscribed disc of radius $2^{-k-1}\sqrt{3}/4 \geq r/8$. Thus $\mathrm{por}(F,x,r) \geq \frac{1}{8}$, so F is $\frac{1}{8}$-uniformly porous (and so $\frac{1}{9}$-uniformly porous).