Graph Rectifiability Summer Research with Lisa Naples

Daily Report

By

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Summer 2021

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1 May 26

1.1 Exercises and Solutions

Exercise 1.12 *Let $f, g : [0, 1] \to \mathbb{R}$ be Lipschitz functions. Show that the functions defined on $[0, 1]$ by $f(x) + g(x)$ and $f(x)g(x)$ are also Lipschitz.*

Solution:

(i) As f, g are Lipschitz function, we have $|f(x) - f(y)| \leq c_1|x - y|$ and $|g(x) - g(y)| \leq c_2|x - y|$ where $\forall x, y \in [0, 1]$ and $c_1, c_2 \geq 0$. Then, $|(f(x) + g(x)) - (f(y) + g(y))| = |f(x) - f(y) + g(x) - g(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| \leq (c_1 + c_2) \cdot |x - y|, \forall x, y \in [0, 1]$. Since $(c_1 + c_2) \geq 0$, the condition is satisfied and therefore the functions defined on $[0, 1]$ by $f(x) + g(x)$ is Lipschitz.

(ii) Consider that $|f(x) - f(0)| \leq c_1|x| \leq c_1, x \in [0, 1]$, so we have non-negative $c_3 = |f(0)| + c_1 \geq |f(x)|$. Similarly, we have non-negative $c_4 \geq |g(x)|$

$|f|, |g| < 1$, $\forall x, y \in [0, 1]$

$$
\begin{aligned}
&|f(x)g(x) - f(y)g(y)| \\
=&|f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
=&|f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))| \\
\leq&|g(y)||(f(x) - f(y))| + |f(x)||(g(x) - g(y))| \\
\leq&c_1c_4||x - y| + c_2c_3|x - y| \\
\leq&(c_1c_4 + c_2c_3)|x - y|
\end{aligned}
$$

Since $c_1c_4 + c_2c_3 \geq 0$, the condition is satisfied and therefore the functions defined on $[0, 1]$ by $f(x)g(x)$ is Lipschitz.

Exercise 1.13 *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with $|f'(x)| \leq c$ for all x. Show, using the mean value theorem, that f is a Lipschitz function.*

Solution: $\forall x, y \in \mathbf{R}, x \neq y$, by mean-value theorem, $\exists w \in (x, y)$ such that

$$
\begin{aligned}
&\frac{f(y) - f(x)}{y - x} = f'(w) \\
\Rightarrow &\left|\frac{f(y) - f(x)}{y - x}\right| = |f'(w)| \leq c \\
\Rightarrow &|f(x) - f(y)| \leq c|x - y| \quad (x, y \in \mathbb{R})
\end{aligned}
$$

Therefore, f is a Lipschitz function.

Exercise 1.14 *Show that every Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is continuous.*

Solution: See proof wrote for Theorem 2.2.1.

Exercise 1.15 *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + x$. Find (i) $f^{-1}(2)$, (ii) $f^{-1}(-2)$ and (iii) $f^{-1}([2, 6])$.*

Solution: As $f(x) = x^2 + x$, $x = -\frac{1}{2} \pm \frac{\sqrt{1+4y}}{2}$

(i) $f^{-1}(2) = \{-2, 1\}$

(ii) $f^{-1}(-2) = \emptyset$

(iii) As $x = -\frac{1}{2} + \frac{\sqrt{1+4y}}{2}$ is increasing and $x = -\frac{1}{2} - \frac{\sqrt{1+4y}}{2}$ is decreasing while y increasing, $f^{-1}([2,6]) = [-3,-2] \cup [1,2]$

Exercise 1.16 *Show that $f(x) = x^2$ is Lipschitz on $[0,2]$, bi-Lipschitz on $[1,2]$ and not Lipschitz on $\mathbb{R}$.*

Solution:

(i) As $\forall x, y \in [0,2]$, $|x+y| \leq 4$, we have $|f(x)-f(y)| = |x^2 - y^2| = |x+y||x-y| \leq 4|x-y|$. Thus, f is Lipschitz on $[0,2]$.

(ii) Apparently, $2|x-y| \leq |f(x) - f(y)| \leq 4|x-y|$ by above. As $f([1,2]) = [1,4]$, $\forall x, y \in [1,4]$, $\frac{1}{\sqrt{x}+\sqrt{y}} \leq \frac{1}{2}$, we have:

$$\left|f^{-1}(x) - f^{-1}(y)\right| = |\sqrt{x} - \sqrt{y}| = \left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right| \leq \frac{1}{2}|x-y|$$

$\Rightarrow$ so f^{-1} is Lipschitz on $[1,4]$.

$\Rightarrow f$ is bi-Lipschitz on $[1,2]$.

(iii) Let $x = ky, k \in \mathbb{R} \setminus \{0\}$, then $\frac{|f(x)-f(y)|}{|x-y|} = \frac{|k^2y^2 - y^2|}{|ky-y|} = \left|\frac{k^2-1}{k-1}\right||y|$, which is unbounded on $\mathbb{R}$. Therefore, the Lipschitz constant does not exist and f is not Lipschitz on $\mathbb{R}$

2 May 25

2.1 Basic Set Theory

Review and summary of some definitions and theorems:

Definition 2.1.1 (Countable) *An infinite set A is countable if its elements can be listed in the form $x_1, x_2, ...$ with every element of appearing at a specific place in the list; otherwise, the set is uncountable*

Definition 2.1.2 (Open) *$A \subset \mathbb{R}^n$ is open if, $\forall x \in A$, $\exists B(x, r) \in A$ where $r > 0$.*

Definition 2.1.3 (Closed) *$A \subset \mathbb{R}^n$ is closed if, whenever $\{x_k\} \in A$, $x_k \to x \in \mathbb{R}^n$, then $x \in A$.*

Definition 2.1.4 (Closure) *$\bar{A}$ is the intersection of all the closed sets containing a set A.*

Definition 2.1.5 (Interior) *$int(A)$ is the union of all open sets contained in A.*

Definition 2.1.4 and 2.1.5 shows that The *closure* of A is thought of as the **smallest closed set** containing A, and the *interior* as the **largest open set** contained in A.

Definition 2.1.6 (Boundary) $\partial A = \bar{A} \setminus int(A)$

Theorem 2.1.1 $x \in \partial A \Leftrightarrow \forall r > 0, B(x, r) \cap A \neq \emptyset, B(x, r) \cap A^C \neq \emptyset$

Definition 2.1.7 (Dense) *Set B is a dense in A if $A \subset \bar{B}$, that is, if there are points of B arbitrarily close to each point of A.*

Definition 2.1.8 (Compact) *A is compact if any collection of open sets that covers A has a finite subcollection which also covers A.*

Theorem 2.1.2 *A compact subset of $\mathbb{R}^n$ is both closed and bounded.*

Theorem 2.1.3 *The intersection of any collection of compact sets is compact.*

Definition 2.1.9 (Connected) *$A \subset \mathbb{R}^n$ is connected if there not exists open sets U and V s.t. $A \in U \cap V$ with disjoint and nonempty $A \cap U$ and $A \cap V$.*

Definition 2.1.10 (Connected Component) *Connected component of x is the largest connected subset of A containing a point x.*

Definition 2.1.11 (Disconnect) *The set A is totally disconnected if the connected component of each point consists of just that point.*

The definition of *disconnect* also can be as: $\exists$ open sets U and V s.t. $x \in U, y \in V$ and $A \subset U \cap V$.

Definition 2.1.12 (Borel Set) *Borel Sets is the smallest collection fo subsets of $\mathbb{R}^n$ with the following properties:*

1. *Every open set and every closed set is a Borel set.*
2. *The union of every finite or countable collection of Borel sets is a Borel set, and the intersection of every finite or countable collection of Borel sets is a Borel set.*

In short, Any set that can be constructed using a sequence of countable unions or intersections starting with the open sets or closed sets will certainly be Borel.

2.2 Functions and Limits

Definition 2.2.1 (Congruence) *The transformation $S : \mathbb{R}^n \to \mathbb{N}^n$ is congruence or isometry if it preserves distances i.e. if $|S(x) - S(y)| = |x - y|$ for $x, y \in \mathbb{R}^n$*

Special cases include *translations*, which are of the form $S(x) = x + a$ and have the effect of shifting points parallel to the vector a, *rotations* which have a centre a such that $|S(x) - a| = |x - a|$ for all x (for convenience, we also regard the identity transformation given by $I(x) = x$ as a rotation) and *reflections*, which maps points to their mirror images in some $(n-1)$-dimensional plane. A congruence that may be achieved by a combination of a rotation and a translation, that is, does not involve reflection, is called a *rigid motion* or *direct congruence.* A transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ is a *similarity* of *ratio* or *scale* $c > 0$ if $|S(x) - S(y)| = c|x - y|$ for all x, y in $\mathbb{R}^n$. A similarity transforms sets into geometrically similar ones with all lengths multiplied by the factor c.

Definition 2.2.2 (Linear Transformation) *A transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear if $\forall x, y \in \mathbb{R}^n, T(x + y) = T(x) + T(y)$ and $T(\lambda x) = \lambda T(x), \lambda \in \mathbb{R}$*

Such a linear transformation is *non-singular* if $T(x) = 0$ if and only if $x = 0$. If $S : \mathbb{R}^n \to \mathbb{R}^n$ is of the form $S(x) = T(x) + a$, where T is a non-singular linear transformation and a is a vector in $\mathbb{R}^n$, then S is called an *affine transformation or an affinity.* An affinity may be thought of as a shearing transformation; its contracting or expanding effect need not be the same in every direction. However, if T is orthonormal, then s is a congruence, and if T is a scalar multiple of an orthonormal transformation, then T is a similarity.

Definition 2.2.3 (Hölder Function) *A function $f : X \to Y$ is called a Hölder function of exponent α if*

$$|f(x) - f(y)| \leq c|x - y|^\alpha \quad (x, y \in X)$$

for some constant $c \geq 0$.

Definition 2.2.4 (Libschitz Function) *The function f is called Lipschitz if*

$$|f(x) - f(y)| \leq c|x - y| \quad (x, y \in X)$$

and bi-Lipschitz if

$$c_1|x - y| \leq |f(x) - f(y)| \leq c_2|x - y| \quad (x, y \in X)$$

for $0 < c_1 \leq c_2 < \infty$, in which case both f and $f^{-1} : f(X) \to X$ are Lipschitz functions.

Definition 2.2.5 (Lower Limit)

$$\underline{\lim}_{x \to 0} f(x) \equiv \lim_{r \to 0}(inf\{f(x) : 0 < x < r\})$$

Note: $inf\{f(x) : 0 < x < r\}$ is either $-\infty$ for all positive r or else increases as r decreases, $\underline{\lim}_{x \to 0} f(x)$ always exists.

Definition 2.2.6 (Upper Limit)

$$\overline{\lim_{x\to 0}} f(x) \equiv \lim_{r\to 0}(sup\{f(x) : 0 < x < r\})$$

Note: The lower and upper limits exist as real numbers or $-\infty$ or ∞ for every function f and are indicative of the variation of f for x close to 0, shown in Figure 1.

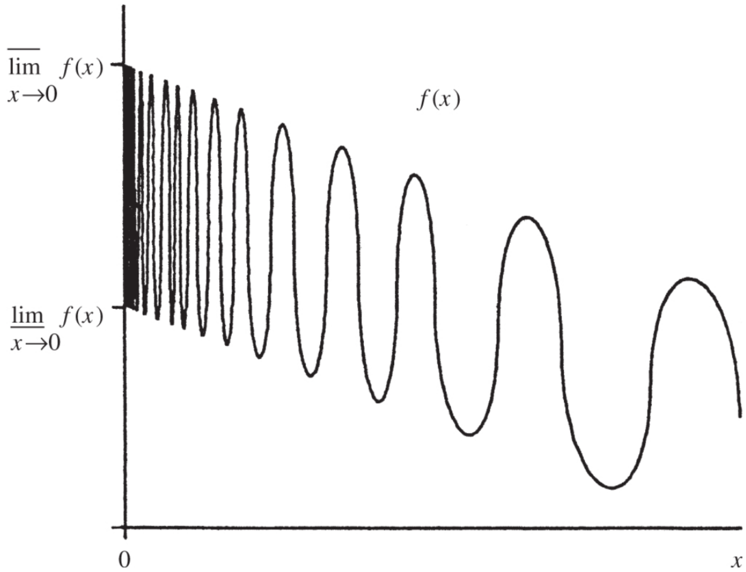

Figure 1: The upper and lower limits of a function.

We write $f(x) \sim g(x)$ to mean that $f(x)/g(x) \to 1$ as $x \to 0$.

Theorem 2.2.1 (Lipschitz functions are continuous)

Proof: Assum that the function $f : X \to Y$ is a Lipschitz function s.t. $|f(x) - f(y)| \leq c|x - y|(x, y \in X)$ for some constant $c \geq 0$. Then, $\forall \epsilon > 0$, let $\delta = \dfrac{\epsilon}{c}$, and we have $\forall x, y \in X, |x - y| < \delta \Rightarrow |x - y| < \dfrac{\epsilon}{c} \Rightarrow |f(x) - f(y)| \leq c|x - y| \leq c \cdot \dfrac{\epsilon}{c} = \epsilon \Rightarrow$ Lipschitz functions are continuous.

Definition 2.2.7 (Homeomorphism) *If $f : X \to Y$ is a continuous bijection with continuous inverse $f^{-1} : Y \to X$, then f is called a homeomorphism, and X and Y are termed homeomorphic sets.*

Corollary 2.2.1 *Congruences, similarities and affine transformations on $\mathbb{R}^n$ are examples of homeomorphisms.*

Definition 2.2.8 (Differentiable) *If $f : \mathbb{R}^n \to \mathbb{R}^n$, we say that f is differentiable at x and has derivative given by the linear mapping $f'(x) : \mathbb{R}^n \to \mathbb{R}^n$ if*

$$\lim_{h\to 0} \frac{|f(x + h) - f(x) - f'(x)h|}{|h|} = 0.$$

Definition 2.2.9 (Pointwise Convergence) *For a sequence of functions:* $f_k : X \to Y$ *where* X *and* Y *are subsets of Euclidean spaces.* f_k *converge pointwise to a function* $f : X \to Y$ *if* $f_k(x) \to f(x)$ *as* $k \to \infty$.

Definition 2.2.10 (Unifrom Convergence) *For a sequence of functions:* $f_k : X \to Y$ *where* X *and* Y *are subsets of Euclidean spaces.* f_k *converge uniformly to a function* $f : X \to Y$ *if* $sup_{x \in X}|f_k(x) - f(x)| \to 0$ *as* $k \to \infty$.

Note: Uniform convergence is a stronger property than pointwise convergence i.e. Uniform convergence implies pointwise convergence, but not the other way around

Theorem 2.2.2 *If the functions* f_k *are continuous and converge uniformly to* f*, then* f *is continuous.*

proof: TODO

Theorem 2.2.3 (Logarithms) *Apparently,* $a^c = b^{c \log a / \log b}$