Graph Rectifiability Summer Research

Daily Report
(in reverse chronological order)

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Contents

1 May 28-30

1.1 Exercises and Solutions

Exercise 1.18 *Let $A_1, A_2, \ldots$ be a decreasing sequence of Borel subsets of $\mathbb{R}^n$ and let $A = \bigcap_{k=1}^{\infty} A_k$. If μ is a measure on $\mathbb{R}^n$ with $\mu(A_1) < \infty$, show using (1.6) that $\mu(A_k) \to \mu(A)$ as $k \to \infty$.*

Solution 1.18 *Consider that $\{A_1 \setminus A_k\}$ is an increasing sequence as $\{A_k\}$ decreasing. Then:*

$$\mu\left(\bigcup_{k=1}^{\infty}(A_1 \setminus A_k)\right) = \mu\left(A_1 \setminus \bigcap_{k=1}^{\infty} A_i\right) = \mu(A_1) - \mu(A)$$
$$= \lim_{k\to\infty} \mu(A_1 \setminus A_k) = \lim_{k\to\infty}(\mu(A_1) - \mu(A_k)) = \mu(A_1) - \lim_{k\to\infty}\mu(A_k)$$

As $\mu(A_1) < \infty$, $\lim_{k\to\infty} \mu(A_k) = \mu(\bigcap_{k=1}^{\infty} A_i)$

Conlusion: For a ***decreasing sequence*** A_k of Borel subsets of $\mathbb{R}^n$,

$$\lim_{k\to\infty} \mu(A_k) = \mu(\bigcap_{k=1}^{\infty} A_i)$$

Exercise 1.23 *Let D be a Borel subset of $\mathbb{R}^n$ and let μ be a measure on D with $\mu(D) < \infty$. Let $f_k : D \to \mathbb{R}$ be a sequence of functions such that $f_k(x) \to f(x)$ for all x in D. Prove* ***Egoroff's theorem****: that given $\varepsilon > 0$ there exists a Borel subset A of D with $\mu(D\setminus A) < \varepsilon$ such that $f_k(x)$ converges to $f(x)$ uniformly for x in A.*

Solution 1.23 *Assume that for $k, n \in \mathbb{Z}^+$, $A_{k,n} = \{x \in D : |f_l(x) - f(x)| < 1/n, \forall l \geq k\}$(so we consider $\delta = 1/n$ here), then we have $\bigcup_{k=1}^{\infty} A_{k,n} = D$ and $A_{1,n} \subset A_{2,n} \subset A_{3,n} \subset \ldots$. Next, by Property of measure 2.1.1:*

$$\mu(D) = \mu(\bigcup_{k=1}^{\infty} A_{k,n}) = \lim_{k\to\infty} \mu(A_{k,n}) < \infty$$

Hence,

$$\lim_{k\to\infty} \mu(D \setminus A_{k,n}) = \mu(D) - \lim_{k\to\infty}\mu(A_{k,n}) = 0$$

Then, $\exists k' \in \mathbb{N}$ s.t. whenever $k \geq k'$, $\mu(D\setminus A_{k,n}) < \frac{\epsilon}{2^n}$. Next, we can construct $A = \bigcap_{n=1}^{\infty} A_{k',n}$, which is a Borel subset of D and satisfies:

$$\mu(D \setminus A) = \mu(D \setminus \bigcap_{n=1}^{\infty} A_{k',n}) = \mu(\bigcup_{n=1}^{\infty} D \setminus A_{k',n}) = \sum_{n=1}^{\infty} \mu(D \setminus A_{k',n}) < \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$$

As $\sum_{n=1}^{\infty}\frac{1}{2^n} = 1$*, and* A *exists for the question. Finally,* $\forall\delta > 0, n > 1/\delta, \forall x \in A$ *where* $x \in A_{k',n}$ *as well, such that whenever* $k \in \mathbb{N}, k > k'$*,* $|f_k(x) - f(x)| < 1/n < \delta \Rightarrow f_k(x)$ *converges to* $f(x)$ *uniformly for* x *in* A*.*

Note: $\sum_{n=1}^{\infty}\frac{1}{2^n} = 1$ is always used to construct ϵ in analysis proofs.

Exercise 1.24 *Prove that if* μ *is a measure on* D *and* $f : D \to \mathbb{R}$ *satisfies* $f(x) \geq 0$ *for all* x *in* D *and* $\int_D f \, \mathrm{d}\mu = 0$ *then* $f(x) = 0$ *for* μ *-almost all* x*.*

Solution 1.24 *Suppose* $f(x) \geq \epsilon > 0$ *on a set* $E_\epsilon \subset D$ *given* $\epsilon > 0$*, we have for* $x \in D \setminus E_\epsilon, f(x) = 0$ *and then:*

$$\begin{aligned}
0 &= \int_D f(x)d\mu \\
&= \int_{E_\epsilon} f(x)d\mu + \int_{D\setminus E_\epsilon} f(x)d\mu \\
&\text{As } \epsilon\chi_{E_\epsilon}(x) \text{ is a simple function and by the integral of more general functions} \\
&\geq \int \epsilon\chi_{E_\epsilon}(x)d\mu + 0 \\
&= \epsilon\mu(E_\epsilon) + 0
\end{aligned}$$

As $\epsilon\mu(E_\mu) \leq 0$ *while* $\epsilon > 0$*, we have* $\mu(E_\epsilon) = 0 \Rightarrow \mu\left(\bigcup_{\epsilon\in\mathbb{R}^+} E_\epsilon\right) = \mu\left(\{x : f(x) > 0\}\right) = 0 \Rightarrow f(x) = 0$ *for* μ*-a.e..*

Note:

1. $f(x) = 0$ for $\mu-$a.e $\Leftrightarrow$ The set of points where $f(x)! = 0(f(x) > 0$ in this case) has measure zero.
2. $\int_E 1d\mu = \mu(E)$

Definition 1.1.1 (Simple Function) *Simple functions are sums of linear combination of characteristic functions, e.g.* $f(x) = \sum a_i\chi_{A_i}(x)$

2 May 27

2.1 Measeres and Mass Distributions(1.3)

Definition 2.1.1 (Measure) *We call μ a measure on $\mathbb{R}^n$ if μ assigns a non-negative number, possibly ∞, to each subset of $\mathbb{R}^n$ such that*

(a) $\mu(\emptyset) = 0$

(b) $\mu(A) \leq \mu(B) \quad$ *if* $A \subset B$

(c) if $A_1, A_2, \ldots$ *is a countable (or finite) sequence of sets, then*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu\left(A_i\right)$$

with equality in above, that is

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu\left(A_i\right),$$

if the A_i *are disjoint Borel sets.*

Condition (a) says that the ***empty set has zero measure***, condition (b) says ***'the larger the set, the larger the measure'*** and condition (c) says that if a set is a union of a countable number of pieces (which may overlap), then the sum of the measure of the pieces is at least equal to the measure of the whole. If ***a set is decomposed into a countable number of disjoint Borel sets***, then the ***total measure of the pieces equals the measure of the whole***.

Property 2.1.1 (Measure)

1. *if* $B \subset A$ A *and* B *are Borel sets with* $\mu(B)$ *finite,*

$$\mu(A \setminus B) = \mu(A) - \mu(B)$$

as $A = B \cup (A \setminus B)$ *and using Definition 2.1.1 (c).*

2. *if* $A_1 \subset A_2 \subset \cdots$ *is an increasing sequence of Borel sets, then*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu\left(A_i\right).$$

as $\bigcup_{i=1}^{\infty} A_i = A_1 \cup (A_2 \backslash A_1) \cup (A_3 \backslash A_2) \cup \ldots,$

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \mu\left(A_1\right) + \sum_{i=1}^{\infty}\left(\mu\left(A_{i+1}\right) - \mu\left(A_i\right)\right) \\
&= \mu\left(A_1\right) + \lim_{k \to \infty} \sum_{i=1}^{k}\left(\mu\left(A_{i+1}\right) - \mu\left(A_i\right)\right) \\
&= \lim_{k \to \infty} \mu\left(A_k\right).
\end{aligned}$$

3. *A simple extension of above is that if, for $\delta > 0, A_\delta$ are Borel sets that are increasing as δ decreases, that is, $A_{\delta'} \subset A_\delta$ for $0 < \delta < \delta'$, then*

$$\mu\left(\bigcup_{\delta>0} A_\delta\right) = \lim_{\delta\to 0} \mu\left(A_\delta\right) \ .$$

Definition 2.1.2 (Support of μ)
spt μ, is the smallest closed set X such that $\mu(\mathbb{R}^n \setminus X) = 0$.

By above, x is in the support if and only if $\forall r > 0, \mu(B(x,r)) > 0$. We say that μ ***is a measure on a set*** A ***if*** A ***contains the support of*** μ.

Definition 2.1.3 (Mass Distributions) *A measure on a bounded subset of $\mathbb{R}^n$ for which $0 < \mu(\mathbb{R}^n) < \infty$ will be called a mass distribution, and we think of $\mu(A)$ as the mass of the set A.*

Definition 2.1.4 (Lebsgue Measure on $\mathbb{R}$)

$$\mathcal{L}^1(A) = \inf\left\{\sum_{i=1}^{\infty}(b_i - a_i) : A \subset \bigcup_{i=1}^{\infty}[a_i, b_i]\right\}$$

3 May 26

3.1 Exercises and Solutions

Exercise 1.12 *Let $f, g : [0,1] \to \mathbb{R}$ be Lipschitz functions. Show that the functions defined on $[0,1]$ by $f(x) + g(x)$ and $f(x)g(x)$ are also Lipschitz.*

Solution:

(i) As f, g are Lipschitz function, we have $|f(x) - f(y)| \leq c_1|x-y|$ and $|g(x) - g(y)| \leq c_2|x-y|$ where $\forall x, y \in [0,1]$ and $c_1, c_2 \geq 0$. Then, $|(f(x)+g(x)) - (f(y)+g(y))| = |f(x)-f(y)+g(x)-g(y)| \leq |f(x)-f(y)|+|g(x)-g(y)| \leq (c_1+c_2)\cdot|x-y|, \forall x, y \in [0,1]$. Since $(c_1+c_2) \geq 0$, the condition is satisfied and therefore the functions defined on $[0,1]$ by $f(x)+g(x)$ is Lipschitz.

(ii) Consider that $|f(x) - f(0)| \leq c_1|x| \leq c_1, x \in [0,1]$, so we have non-negative $c_3 = |f(0)| + c_1 \geq |f(x)|$. Similarly, we have non-negative $c_4 \geq |g(x)|$

$|f|, |g| < 1$, $\forall x, y \in [0,1]$

$$
\begin{aligned}
&|f(x)g(x) - f(y)g(y)| \\
=&|f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
=&|f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))| \\
\leq&|g(y)||(f(x) - f(y))| + |f(x)||(g(x) - g(y))| \\
\leq&c_1c_4||x-y| + c_2c_3|x-y| \\
\leq&(c_1c_4 + c_2c_3)|x-y|
\end{aligned}
$$

Since $c_1c_4 + c_2c_3 \geq 0$, the condition is satisfied and therefore the functions defined on $[0,1]$ by $f(x)g(x)$ is Lipschitz.

Exercise 1.13 *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with $|f'(x)| \leq c$ for all x. Show, using the mean value theorem, that f is a Lipschitz function.*

Solution: $\forall x, y \in \mathbf{R}, x \neq y$, by mean-value theorem, $\exists w \in (x, y)$ such that

$$
\begin{aligned}
&\frac{f(y) - f(x)}{y - x} = f'(w) \\
\Rightarrow&\left|\frac{f(y) - f(x)}{y - x}\right| = |f'(w)| \leq c \\
\Rightarrow&|f(x) - f(y)| \leq c|x - y| \quad (x, y \in \mathbb{R})
\end{aligned}
$$

Therefore, f is a Lipschitz function.

Exercise 1.14 *Show that every Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is continuous.*

Solution: See proof wrote for Theorem 4.2.1.

Exercise 1.15 *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + x$. Find (i) $f^{-1}(2)$, (ii) $f^{-1}(-2)$ and (iii) $f^{-1}([2,6])$.*

Solution: As $f(x) = x^2 + x$, $x = -\frac{1}{2} \pm \frac{\sqrt{1+4y}}{2}$

(i) $f^{-1}(2) = \{-2, 1\}$

(ii) $f^{-1}(-2) = \emptyset$

(iii) As $x = -\frac{1}{2} + \frac{\sqrt{1+4y}}{2}$ is increasing and $x = -\frac{1}{2} - \frac{\sqrt{1+4y}}{2}$ is decreasing while y increasing, $f^{-1}([2,6]) = [-3,-2] \cup [1,2]$

Exercise 1.16 *Show that $f(x) = x^2$ is Lipschitz on $[0,2]$, bi-Lipschitz on $[1,2]$ and not Lipschitz on $\mathbb{R}$.*

Solution:

(i) As $\forall x, y \in [0,2]$, $|x+y| \leq 4$, we have $|f(x)-f(y)| = |x^2 - y^2| = |x+y||x-y| \leq 4|x-y|$. Thus, f is Lipschitz on $[0,2]$.

(ii) Apparently, $2|x-y| \leq |f(x) - f(y)| \leq 4|x-y|$ by above. As $f([1,2]) = [1,4]$, $\forall x, y \in [1,4]$, $\frac{1}{\sqrt{x}+\sqrt{y}} \leq \frac{1}{2}$, we have:

$$\left|f^{-1}(x) - f^{-1}(y)\right| = |\sqrt{x} - \sqrt{y}| = \left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right| \leq \frac{1}{2}|x-y|$$

$\Rightarrow$ so f^{-1} is Lipschitz on $[1,4]$.

$\Rightarrow f$ is bi-Lipschitz on $[1,2]$.

(iii) Let $x = ky, k \in \mathbb{R} \setminus \{0\}$, then $\frac{|f(x)-f(y)|}{|x-y|} = \frac{|k^2y^2 - y^2|}{|ky-y|} = \left|\frac{k^2-1}{k-1}\right||y|$, which is unbounded on $\mathbb{R}$. Therefore, the Lipschitz constant does not exist and f is not Lipschitz on $\mathbb{R}$

4 May 25

4.1 Basic Set Theory(1.1)

Review and summary of some definitions and theorems:

Definition 4.1.1 (Countable) *An infinite set A is countable if its elements can be listed in the form $x_1, x_2, ...$ with every element of appearing at a specific place in the list; otherwise, the set is uncountable*

Definition 4.1.2 (Open) *$A \subset \mathbb{R}^n$ is open if, $\forall x \in A$, $\exists B(x, r) \in A$ where $r > 0$.*

Definition 4.1.3 (Closed) *$A \subset \mathbb{R}^n$ is closed if, whenever $\{x_k\} \in A$, $x_k \to x \in \mathbb{R}^n$, then $x \in A$.*

Definition 4.1.4 (Closure) *$\bar{A}$ is the intersection of all the closed sets containing a set A.*

Definition 4.1.5 (Interior) *$int(A)$ is the union of all open sets contained in A.*

Definition 4.1.4 and 4.1.5 shows that The *closure* of A is thought of as the **smallest closed set** containing A, and the *interior* as the **largest open set** contained in A.

Definition 4.1.6 (Boundary) $\partial A = \bar{A} \setminus int(A)$

Theorem 4.1.1 $x \in \partial A \Leftrightarrow \forall r > 0, B(x, r) \cap A \neq \emptyset, B(x, r) \cap A^C \neq \emptyset$

Definition 4.1.7 (Dense) *Set B is a dense in A if $A \subset \bar{B}$, that is, if there are points of B arbitrarily close to each point of A.*

Definition 4.1.8 (Compact) *A is compact if any collection of open sets that covers A has a finite subcollection which also covers A.*

Theorem 4.1.2 *A compact subset of $\mathbb{R}^n$ is both closed and bounded.*

Theorem 4.1.3 *The intersection of any collection of compact sets is compact.*

Definition 4.1.9 (Connected) *$A \subset \mathbb{R}^n$ is connected if there not exists open sets U and V s.t. $A \in U \cap V$ with disjoint and nonempty $A \cap U$ and $A \cap V$.*

Definition 4.1.10 (Connected Component) *Connected component of x is the largest connected subset of A containing a point x.*

Definition 4.1.11 (Disconnect) *The set A is totally disconnected if the connected component of each point consists of just that point.*

The definition of *disconnect* also can be as: $\exists$ open sets U and V s.t. $x \in U, y \in V$ and $A \subset U \cap V$.

Definition 4.1.12 (Borel Set) *Borel Sets is the smallest collection fo subsets of $\mathbb{R}^n$ with the following properties:*

1. *Every open set and every closed set is a Borel set.*
2. *The union of every finite or countable collection of Borel sets is a Borel set, and the intersection of every finite or countable collection of Borel sets is a Borel set.*

In short, Any set that can be constructed using a sequence of countable unions or intersections starting with the open sets or closed sets will certainly be Borel.

4.2 Functions and Limits(1.2)

Definition 4.2.1 (Congruence) *The transformation $S : \mathbb{R}^n \to \mathbb{N}^n$ is congruence or isometry if it preserves distances i.e. if $|S(x) - S(y)| = |x - y|$ for $x, y \in \mathbb{R}^n$*

Special cases include *translations*, which are of the form $S(x) = x + a$ and have the effect of shifting points parallel to the vector a, *rotations* which have a centre a such that $|S(x) - a| = |x - a|$ for all x (for convenience, we also regard the identity transformation given by $I(x) = x$ as a rotation) and *reflections*, which maps points to their mirror images in some $(n-1)$-dimensional plane. A congruence that may be achieved by a combination of a rotation and a translation, that is, does not involve reflection, is called a *rigid motion* or *direct congruence*. A transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ is a *similarity* of *ratio* or *scale* $c > 0$ if $|S(x) - S(y)| = c|x - y|$ for all x, y in $\mathbb{R}^n$. A similarity transforms sets into geometrically similar ones with all lengths multiplied by the factor c.

Definition 4.2.2 (Linear Transformation) *A transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear if $\forall x, y \in \mathbb{R}^n, T(x + y) = T(x) + T(y)$ and $T(\lambda x) = \lambda T(x), \lambda \in \mathbb{R}$*

Such a linear transformation is *non-singular* if $T(x) = 0$ if and only if $x = 0$. If $S : \mathbb{R}^n \to \mathbb{R}^n$ is of the form $S(x) = T(x) + a$, where T is a non-singular linear transformation and a is a vector in $\mathbb{R}^n$, then S is called an *affine transformation or an affinity*. An affinity may be thought of as a shearing transformation; its contracting or expanding effect need not be the same in every direction. However, if T is orthonormal, then s is a congruence, and if T is a scalar multiple of an orthonormal transformation, then T is a similarity.

Definition 4.2.3 (Hölder Function) *A function $f : X \to Y$ is called a Hölder function of exponent α if*

$$|f(x) - f(y)| \leq c|x - y|^\alpha \quad (x, y \in X)$$

for some constant $c \geq 0$.

Definition 4.2.4 (Libschitz Function) *The function f is called Lipschitz if*

$$|f(x) - f(y)| \leq c|x - y| \quad (x, y \in X)$$

and bi-Lipschitz if

$$c_1|x - y| \leq |f(x) - f(y)| \leq c_2|x - y| \quad (x, y \in X)$$

for $0 < c_1 \leq c_2 < \infty$, in which case both f and $f^{-1} : f(X) \to X$ are Lipschitz functions.

Definition 4.2.5 (Lower Limit)

$$\underline{\lim}_{x \to 0} f(x) \equiv \lim_{r \to 0}(inf\{f(x) : 0 < x < r\})$$

Note: $inf\{f(x) : 0 < x < r\}$ is either $-\infty$ for all positive r or else increases as r decreases, $\underline{\lim}_{x\to 0} f(x)$ always exists.

Definition 4.2.6 (Upper Limit)

$$\overline{\lim_{x\to 0}} f(x) \equiv \lim_{r\to 0}(sup\{f(x) : 0 < x < r\})$$

Note: The lower and upper limits exist as real numbers or $-\infty$ or ∞ for every function f and are indicative of the variation of f for x close to 0, shown in Figure 1.

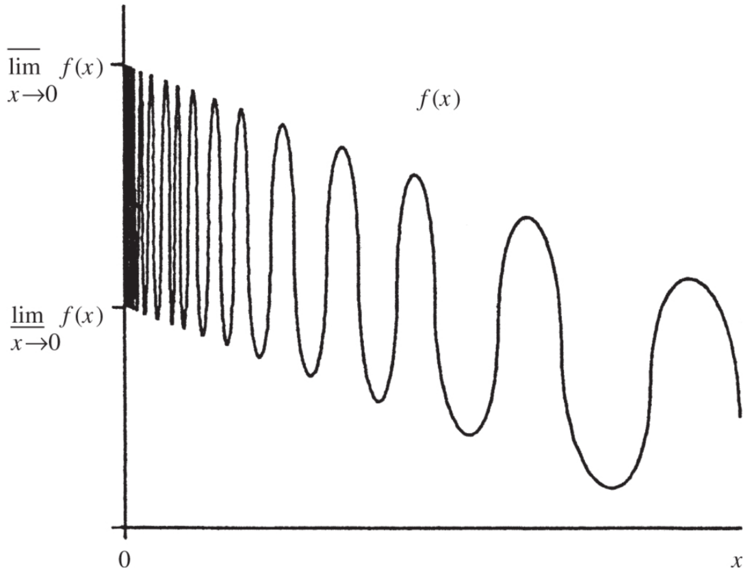

Figure 1: The upper and lower limits of a function.

We write $f(x) \sim g(x)$ to mean that $f(x)/g(x) \to 1$ as $x \to 0$.

Theorem 4.2.1 (Lipschitz functions are continuous)

Proof: Assum that the function $f : X \to Y$ is a Lipschitz function s.t. $|f(x) - f(y)| \leq c|x - y|(x, y \in X)$ for some constant $c \geq 0$. Then, $\forall \epsilon > 0$, let $\delta = \dfrac{\epsilon}{c}$, and we have $\forall x, y \in X, |x - y| < \delta \Rightarrow |x - y| < \dfrac{\epsilon}{c} \Rightarrow |f(x) - f(y)| \leq c|x - y| \leq c \cdot \dfrac{\epsilon}{c} = \epsilon \Rightarrow$ Lipschitz functions are continuous.

Definition 4.2.7 (Homeomorphism) *If $f : X \to Y$ is a continuous bijection with continuous inverse $f^{-1} : Y \to X$, then f is called a homeomorphism, and X and Y are termed homeomorphic sets.*

Corollary 4.2.1 *Congruences, similarities and affine transformations on $\mathbb{R}^n$ are examples of homeomorphisms.*

Definition 4.2.8 (Differentiable) *If $f : \mathbb{R}^n \to \mathbb{R}^n$, we say that f is differentiable at x and has derivative given by the linear mapping $f'(x) : \mathbb{R}^n \to \mathbb{R}^n$ if*

$$\lim_{h\to 0} \frac{|f(x + h) - f(x) - f'(x)h|}{|h|} = 0.$$

Definition 4.2.9 (Pointwise Convergence) *For a sequence of functions:* $f_k : X \to Y$ *where* X *and* Y *are subsets of Euclidean spaces.* f_k *converge pointwise to a function* $f : X \to Y$ *if* $f_k(x) \to f(x)$ *as* $k \to \infty$*.*

Definition 4.2.10 (Unifrom Convergence) *For a sequence of functions:* $f_k : X \to Y$ *where* X *and* Y *are subsets of Euclidean spaces.* f_k *converge uniformly to a function* $f : X \to Y$ *if* $sup_{x \in X}|f_k(x) - f(x)| \to 0$ *as* $k \to \infty$*.*

Note: Uniform convergence is a stronger property than pointwise convergence i.e. Uniform convergence implies pointwise convergence, but not the other way around

Definition 4.2.11 (Another Definition of Pointwise Convergence) *For each* $x \in D$*,* $\forall \delta > 0$*,* $\exists k_{x,\delta} > 0$*, s.t. whenever* $k > k_{x,\delta}$*,* $|f_k(x) - f(x)| < \delta$*.*

Definition 4.2.12 (Another Definition of Uniform Convergence) $\forall \delta > 0$*,* $\exists k_\delta > 0$ *s.t. whenever* $k > k_\delta$*,* $|f_k(x) - f(x)| < \delta$*.*

Note: the main difference between pointwise and uniform convergence is that pointwise convergence is for each x in the domain, whereas uniform convergence is for all x in domain. And this is also the reason why sup shown in the definition in the textbook.

Theorem 4.2.2 *If the functions* f_k *are continuous and converge uniformly to* f*, then* f *is continuous.*

Theorem 4.2.3 (Logarithms) *Apparently,* $a^c = b^{c \log a / \log b}$