Graph Rectifiability Summer Research with Lisa Naples

Daily Report

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Contents

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1.1 Basic Set Theory

Review and summary of some definitions and theorems:

Definition 1.1.1 (Countable) *An infinite set A is countable if its elements can be listed in the form $x_1, x_2, ...$ with every element of appearing at a specific place in the list; otherwise, the set is uncountable*

Definition 1.1.2 (Open) *$A \subset \mathbb{R}^n$ is open if, $\forall x \in A$, $\exists B(x,r) \in A$ where $r > 0$.*

Definition 1.1.3 (Closed) *$A \subset \mathbb{R}^n$ is closed if, whenever $\{x_k\} \in A$, $x_k \to x \in \mathbb{R}^n$, then $x \in A$.*

Definition 1.1.4 (Closure) *$\bar{A}$ is the intersection of all the closed sets containing a set A.*

Definition 1.1.5 (Interior) *$int(A)$ is the union of all open sets contained in A.*

Definition 1.1.4 and 1.1.5 shows that The *closure* of A is thought of as the **smallest closed set** containing A, and the *interior* as the **largest open set** contained in A.

Definition 1.1.6 (Boundary) $\partial A = \bar{A} \setminus int(A)$

Theorem 1.1.1 $x \in \partial A \Leftrightarrow \forall r > 0, B(x,r) \cap A \neq \emptyset, B(x,r) \cap A^C \neq \emptyset$

Definition 1.1.7 (Dense) *Set B is a dense in A if $A \subset \bar{B}$, that is, if there are points of B arbitrarily close to each point of A.*

Definition 1.1.8 (Compact) *A is compact if any collection of open sets that covers A has a finite subcollection which also covers A.*

Theorem 1.1.2 *A compact subset of $\mathbb{R}^n$ is both closed and bounded.*

Theorem 1.1.3 *The intersection of any collection of compact sets is compact.*

Definition 1.1.9 (Connected) *$A \subset \mathbb{R}^n$ is connected if there not exists open sets U and V s.t. $A \in U \cap V$ with disjoint and nonempty $A \cap U$ and $A \cap V$.*

Definition 1.1.10 (Connected Component) *Connected component of x is the largest connected subset of A containing a point x.*

Definition 1.1.11 (Disconnect) *The set A is totally disconnected if the connected component of each point consists of just that point.*

The definition of *disconnect* also can be as: $\exists$ open sets U and V s.t. $x \in U, y \in V$ and $A \subset U \cap V$.

Definition 1.1.12 (Borel Set) *Borel Sets is the smallest collection fo subsets of $\mathbb{R}^n$ with the following properties:*

1. *Every open set and every closed set is a Borel set.*
2. *The union of every finite or countable collection of Borel sets is a Borel set, and the intersection of every finite or countable collection of Borel sets is a Borel set.*

In short, Any set that can be constructed using a sequence of countable unions or intersections starting with the open sets or closed sets will certainly be Borel.

1.2 Functions and Limits

Definition 1.2.1 (Congruence) *The transformation* $S : \mathbb{R}^n \to \mathbb{N}^n$ *is congruence or isometry if it preserves distances i.e. if* $|S(x) - S(y)| = |x - y|$ *for* $x, y \in \mathbb{R}^n$

Special cases include *translations*, which are of the form $S(x) = x + a$ and have the effect of shifting points parallel to the vector a, *rotations* which have a centre a such that $|S(x) - a| = |x - a|$ for all x (for convenience, we also regard the identity transformation given by $I(x) = x$ as a rotation) and *reflections*, which maps points to their mirror images in some $(n-1)$-dimensional plane. A congruence that may be achieved by a combination of a rotation and a translation, that is, does not involve reflection, is called a *rigid motion* or *direct congruence.* A transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ is a *similarity* of *ratio* or *scale* $c > 0$ if $|S(x) - S(y)| = c|x - y|$ for all x, y in $\mathbb{R}^n$. A similarity transforms sets into geometrically similar ones with all lengths multiplied by the factor c.

Definition 1.2.2 (Linear Transformation) *A transformation* $T : \mathbb{R}^n \to \mathbb{R}^n$ *is linear if* $\forall x, y \in \mathbb{R}^n, T(x + y) = T(x) + T(y)$ *and* $T(\lambda x) = \lambda T(x), \lambda \in \mathbb{R}$

Such a linear transformation is *non-singular* if $T(x) = 0$ if and only if $x = 0$. If $S : \mathbb{R}^n \to \mathbb{R}^n$ is of the form $S(x) = T(x) + a$, where T is a non-singular linear transformation and a is a vector in $\mathbb{R}^n$, then S is called an *affine transformation or an affinity.* An affinity may be thought of as a shearing transformation; its contracting or expanding effect need not be the same in every direction. However, if T is orthonormal, then s is a congruence, and if T is a scalar multiple of an orthonormal transformation, then T is a similarity.

Definition 1.2.3 (Hölder Function) *A function* $f : X \to Y$ *is called a Hölder function of exponent* α *if*

$$|f(x) - f(y)| \leq c|x - y|^\alpha \quad (x, y \in X)$$

for some constant $c \geq 0$.

Definition 1.2.4 (Libschitz Function) *The function* f *is called Lipschitz if*

$$|f(x) - f(y)| \leq c|x - y| \quad (x, y \in X)$$

and bi-Lipschitz if

$$c_1|x - y| \leq |f(x) - f(y)| \leq c_2|x - y| \quad (x, y \in X)$$

for $0 < c_1 \leq c_2 < \infty$*, in which case both* f *and* $f^{-1} : f(X) \to X$ *are Lipschitz functions.*

Definition 1.2.5 (Lower Limit)

$$\underline{\lim}_{x \to 0} f(x) \equiv \lim_{r \to 0} (inf\{f(x) : 0 < x < r\})$$

Note: $inf\{f(x) : 0 < x < r\}$ is either $-\infty$ for all positive r or else increases as r decreases, $\underline{\lim}_{x \to 0} f(x)$ always exists.

Definition 1.2.6 (Upper Limit)

$$\overline{\lim_{x\to 0}} f(x) \equiv \lim_{r\to 0}(sup\{f(x) : 0 < x < r\})$$

Note: The lower and upper limits exist as real numbers or $-\infty$ or ∞ for every function f and are indicative of the variation of f for x close to 0, shown in Figure 1.

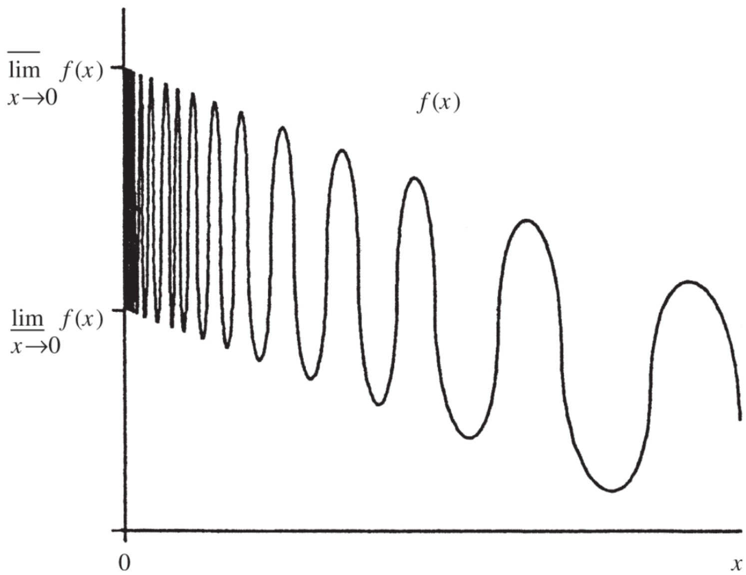

Figure 1: The upper and lower limits of a function.

We write $f(x) \sim g(x)$ to mean that $f(x)/g(x) \to 1$ as $x \to 0$.

Theorem 1.2.1 (Lipschitz and Hölder functions are continuous)

Proof: for the function $f : X \to Y$ is a Hölder function of exponent α s.t. $|f(x) - f(y)| \leq c|x - y|^{\alpha} \quad (x, y \in X)$ for some constant $c \geq 0$. Then, $\forall \epsilon > 0$, let $\delta = \frac{\epsilon}{c}$. Hence, $\forall x, y \in X, |x-y| < \delta \Rightarrow |x-y| < \frac{\epsilon}{c} \Rightarrow |f(x)-f(y)| \leq c|x-y| \leq c \cdot \frac{\epsilon}{c} = \epsilon \Rightarrow$ Hölder functions are continous, and let exponent $\alpha = 1$, then we have Lipschitz functions are continuous .

Definition 1.2.7 (Homeomorphism) *If $f : X \to Y$ is a continuous bijection with continuous inverse $f^{-1} : Y \to X$, then f is called a homeomorphism, and X and Y are termed homeomorphic sets.*

Corollary 1.2.1 *Congruences, similarities and affine transformations on $\mathbb{R}^n$ are examples of homeomorphisms.*

Definition 1.2.8 (Differentiable) *If $f : \mathbb{R}^n \to \mathbb{R}^n$, we say that f is differentiable at x and has derivative given by the linear mapping $f'(x) : \mathbb{R}^n \to \mathbb{R}^n$ if*

$$\lim_{h\to 0} \frac{|f(x+h) - f(x) - f'(x)h|}{|h|} = 0.$$

Definition 1.2.9 (Pointwise Convergence) *For a sequence of functions:* $f_k : X \to Y$ *where* X *and* Y *are subsets of Euclidean spaces.* f_k *converge pointwise to a function* $f : X \to Y$ *if* $f_k(x) \to f(x)$ *as* $k \to \infty$.

Definition 1.2.10 (Unifrom Convergence) *For a sequence of functions:* $f_k : X \to Y$ *where* X *and* Y *are subsets of Euclidean spaces.* f_k *converge uniformly to a function* $f : X \to Y$ *if* $sup_{x \in X} |f_k(x) - f(x)| \to 0$ *as* $k \to \infty$.

Note: Uniform convergence is a stronger property than pointwise convergence i.e. Uniform convergence implies pointwise convergence, but not the other way around

Theorem 1.2.2 *If the functions* f_k *are continuous and converge uniformly to* f*, then* f *is continuous.*

proof: TODO

Theorem 1.2.3 (Logarithms) *Apparently,* $a^c = b^{c \log a / \log b}$