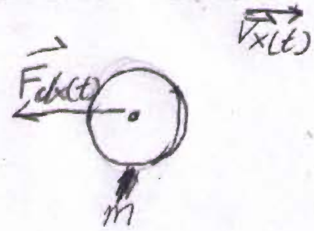
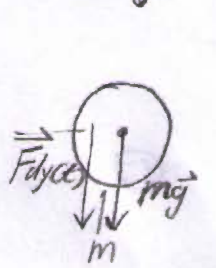
1.

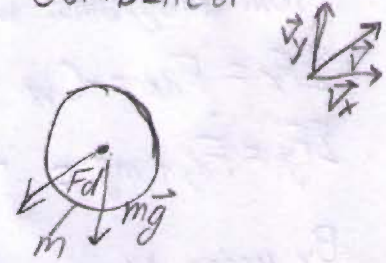
Forces in x direction

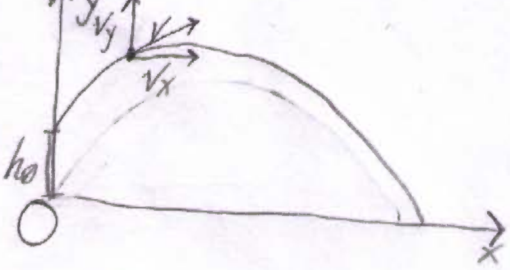


Forces in y direction



Combined





Location described by point on unspecified trajectory

$$F_d = \frac{1}{2} \rho C_d A v^2 \quad \cdots \text{①}$$

Where ρ is the density of medium, C_d is the coefficient of drag, and A is the cross sectional area

When ① is decomposed into x and y directions, we have

$$F_{dx} = \frac{1}{2} \rho C_{dx} A_x v_x^2 \quad \cdots \text{(2.1)}$$

$$F_{dy} = \frac{1}{2} \rho C_{dy} A_y v_y^2 \quad \cdots \text{(2.2)}$$

Since the object is sperical,

$C_{dx} = C_{dy} = C_d \cdots$

$A_x = A_y = A$

Let $C = \frac{1}{2} \rho C_d A$

Then (2.1) and (2.2) can be expressed as

$$F_{dx} = C v_x^2 \quad \cdots \text{(2.1*)}$$

$$F_{dy} = C v_y^2 \quad \cdots \text{(2.2*)}$$

2.

From diagrams, we can see:

$$\Sigma F_x = F_{dx} = -Cv_x^2 \quad \cdots (3.1)$$

$$\Sigma F_y = F_{dy} + m\vec{g} = -Cv_y^2 - mg \quad \cdots (3.2)$$

By using Newton's law, $\Sigma F = ma$

$$\frac{dv_x}{dt} = a_x = \frac{\Sigma F_x}{m} = \frac{-Cv_x^2}{m} \quad \cdots (4.1)$$

$$\frac{dv_y}{dy} = a_y = \frac{\Sigma F_y}{m} = \frac{-Cv_y^2 - mg}{m} \quad \cdots (4.2)$$

To find the flight time of the ball, we can use (4.2) and find y, we can solve the differential equation

$$\frac{dv_y}{dt} = \frac{-Cv_y^2 - mg}{m}$$

$$\int_0^t \frac{1}{m}\,dt = \int_{v_{y0}}^{v_y} \frac{1}{-Cv_y^2 - mg}\,dv_y$$

$$\frac{t}{m} = -\frac{1}{C}\int_{v_{y0}}^{v_y} \frac{1}{v_y^2 - \frac{mg}{C}}\,dv_y$$

$$\frac{t}{m} = -\frac{1}{C}\left(\sqrt{\frac{C}{mg}}\tan^{-1}\left(\sqrt{\frac{C}{mg}}\,v_y\right)\Big|_{v_{y0}}^{v_y}\right)$$

$$\frac{t}{m} = -\frac{1}{C}\sqrt{\frac{C}{mg}}\left[\tan^{-1}\left(\sqrt{\frac{C}{mg}}\,v_y\right) - \tan^{-1}\left(\sqrt{\frac{C}{mg}}\,v_{y0}\right)\right]$$

$$\frac{t}{m} = -\frac{1}{\sqrt{Cmg}}\left[\tan^{-1}\left(\sqrt{\frac{C}{mg}}\,v_y\right) - \tan^{-1}\left(\sqrt{\frac{C}{mg}}\,v_{y0}\right)\right]$$

$$-\sqrt{\frac{Cg}{m}}\,t = \tan^{-1}\left(\sqrt{\frac{C}{mg}}\,v_y\right) - \tan^{-1}\left(\sqrt{\frac{C}{mg}}\,v_{y0}\right)$$

$$\tan^{-1}\left(\sqrt{\frac{C}{mg}}\,v_y\right) = \tan^{-1}\left(\sqrt{\frac{C}{mg}}\,v_{y0}\right) - \sqrt{\frac{Cg}{m}}\,t$$

$$\sqrt{\frac{C}{mg}}\,v_y = \tan\left[\tan^{-1}\left(\sqrt{\frac{C}{mg}}\,v_{y0}\right) - \sqrt{\frac{Cg}{m}}\,t\right]$$

$$\boxed{v_y = \sqrt{\frac{mg}{C}}\tan\left[\tan^{-1}\left(\sqrt{\frac{C}{mg}}\,v_{y0}\right) - \sqrt{\frac{Cg}{m}}\,t\right]} \quad \cdots (5)$$

3. (5) can be manipulated into a form that we can solve for y

$$\frac{dy}{dt} = V_y = \sqrt{\frac{mg}{c}} \tan\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right) - \sqrt{\frac{cg}{m}} t\right]$$

$$\int_{h_0}^{y} dy = \int_0^t \sqrt{\frac{mg}{c}} \tan\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right) - \sqrt{\frac{cg}{m}} t\right] dt$$

$$y - h_0 = \sqrt{\frac{mg}{c}} \int_0^t \tan\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right) - \sqrt{\frac{cg}{m}} t\right] dt$$

let $u = \tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right) - \sqrt{\frac{cg}{m}} t$

$$du = -\sqrt{\frac{cg}{m}} dt$$

$$dt = -\sqrt{\frac{m}{cg}} du$$

then

$$y - h_0 = -\sqrt{\frac{mg}{c}} \sqrt{\frac{m}{cg}} \int_{\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right)}^{\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right) - \sqrt{\frac{cg}{m}} t} \tan(u)\, du$$

$$y - h_0 = -\frac{m}{c} \ln\left|\sec(u)\right| \Big|_{\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right)}^{\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right) - \sqrt{\frac{cg}{m}} t}$$

$$\boxed{y - h_0 = \frac{m}{c} \ln\left|\frac{\sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right)\right]}{\sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right) - \sqrt{\frac{cg}{m}} t\right]}\right|} \quad \cdots\cdots (6)$$

To solve for flight time, we set y in 6 to 0

$$-h_0 = \frac{m}{c} \ln\left|\frac{\sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right)\right]}{\sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right) - \sqrt{\frac{cg}{m}} t\right]}\right|$$

$$-\frac{ch_0}{m} = \ln\left|\frac{\sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right)\right]}{\sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right) - \sqrt{\frac{cg}{m}} t\right]}\right|$$

$$\frac{1}{e^{\frac{ch_0}{m}}} = \left|\frac{\sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right)\right]}{\sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right) - \sqrt{\frac{cg}{m}} t\right]}\right|$$

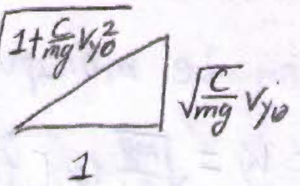
$$\sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right)\right] > 0$$

$$\frac{1}{e^{\frac{ch_0}{m}}} = \frac{\sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right)\right]}{\left|\sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right) - \sqrt{\frac{cg}{m}} t\right]\right|}$$

$$\frac{1}{e^{\frac{ch_0}{m}} \sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right)\right]} = \frac{1}{\sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right) - \sqrt{\frac{cg}{m}} t\right]}$$

$$e^{\frac{ch_0}{m}} \sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0}\right)\right] = \sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}} V_{y0} - \sqrt{\frac{cg}{m}} t\right)\right]$$

$$\pm e^{\frac{Ch_0}{m}}\sqrt{1+\frac{c}{mg}v_{y0}^2} = \sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}}v_{y0}\right) - \sqrt{\frac{Cg}{m}}\,t\right]$$



$$\sec^{-1}\left[\pm e^{\frac{Ch_0}{m}}\sqrt{1+\frac{c}{mg}v_{y0}^2}\right] = \tan^{-1}\left(\sqrt{\frac{c}{mg}}v_{y0}\right) - \sqrt{\frac{Cg}{m}}\,t$$

$$\boxed{\sqrt{\frac{m}{Cg}}\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}}v_{y0}\right) - \cos^{-1}\left(\frac{1}{\pm e^{\frac{Ch_0}{m}}\sqrt{1+\frac{c}{mg}v_{y0}^2}}\right)\right] = t} \quad \ldots (7)$$

Find whichever makes more sense. Probably +

And to find x as a function of t, we solve the differential equation (4.1)

$$\frac{dv_x}{dt} = -\frac{Cv_x^2}{m}$$

$$\int_0^t \frac{1}{m}dt = \int_{v_{x0}}^{v_x} -\frac{1}{Cv_x^2}dv_x$$

$$\frac{t}{m} = \frac{1}{C}\left(\frac{1}{v_x}\Big|_{v_{x0}}^{v_x}\right)$$

$$\frac{Ct}{m} = \frac{1}{v_x} - \frac{1}{v_{x0}}$$

$$\frac{1}{v_x} = \frac{Ct}{m} + \frac{1}{v_{x0}}$$

$$\frac{1}{v_x} = \frac{v_{x0}Ct + m}{mv_{x0}}$$

$$\boxed{v_x = \frac{mv_{x0}}{v_{x0}Ct + m}} \quad \ldots (8)$$

To find x, we integrate (8)

$$\frac{dx}{dt} = \frac{mv_{x0}}{v_{x0}Ct+m}$$

$$\int_0^x dx = \int_0^t \frac{mv_{x0}}{v_{x0}Ct+m}dt$$

$$x = mv_{x0}\int_0^t \frac{1}{v_{x0}Ct+m}dt$$

let $u = v_{x0}Ct + m$

$$du = v_{x0}C\,dt$$

$$dt = \frac{1}{v_{x0}C}du$$

$$x = \frac{m}{C}\int_m^{v_{x0}Ct+m}\frac{1}{u}du$$

5. $x = \frac{m}{c}\left(\ln|u|_m^{v_{x0}ct+m}\right)$

$$\boxed{x = \frac{m}{c}\ln\left(1+\frac{v_{x0}ct}{m}\right)} \cdots ⑨$$

Proofs for v_y, v_x, y, ~~x~~, and t

When Air Resistance is neglegible ($c \to 0$):

$$\lim_{c\to 0} v_y = \sqrt{\frac{mg}{c}}\tan\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}}v_{y0}\right) - \sqrt{\frac{cg}{m}}t\right]$$

Since $\sqrt{\frac{c}{mg}}v_{y0}$ is very close to 0, a taylor expansion can be used until $O(c^3)$

Substitute Ⓣ4, we get:

$$\lim_{c\to 0} v_y = \sqrt{\frac{mg}{c}}\tan\left[\sqrt{\frac{c}{mg}}v_{y0} - \cancel{O(c^3)} - \sqrt{\frac{cg}{m}}t\right]$$

Next we substitute in Ⓣ5, and we get

$$\sqrt{\frac{mg}{c}} * \sqrt{\frac{c}{mg}}v_{y0} - \sqrt{\frac{mg}{c}} * \sqrt{\frac{cg}{m}}t$$

$$= v_{y0} - gt$$

Which is consistent with the uniform acceleration equation

Taylor Series:

$\sin(x) \approx x - \frac{x^3}{3!} + O(x^5) \cdots$ Ⓣ1

$\cos(x) \approx 1 - \frac{x^2}{2!} + O(x^4) \cdots$ Ⓣ2

$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4) \cdots$ Ⓣ3

$\tan^{-1}(x) \approx x - \frac{x^3}{3} + O(x^5) \cdots$ Ⓣ4

$\tan(x) \approx x + \frac{x^3}{3!} + O(x^5) \cdots$ Ⓣ5

$\sin^{-1}(x) \approx x + \frac{1}{6}x^3 + O(x^5) \cdots$ Ⓣ6

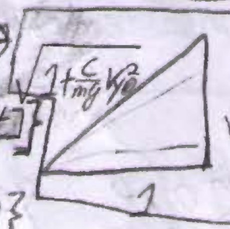
$\cos^{-1}(x) \approx \frac{\pi}{2}x - x - \frac{1}{6}x^3 + O(x^5) \cdots$ Ⓣ7

$$\lim_{c\to 0}(y - h_0) = \frac{m}{c}\left|\underbrace{\ln\left\{\sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}}v_{y0}\right)\right]\right\}}_{⑩} - \underbrace{\ln\left\{\sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}}v_{y0}\right) - \sqrt{\frac{cg}{m}}t\right]\right\}}_{⑪}\right|$$

$$⑩ = \ln\left(\sqrt{1+\frac{c}{mg}v_{y0}^2}\right)$$

$$= \frac{1}{2}\ln\left(1+\frac{c}{mg}v_{y0}^2\right)$$

$$⑪ = -\ln\left\{\sec\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}}v_{y0}\right) - \sqrt{\frac{cg}{m}}t\right]\right\}$$



$$= \ln\left\{\cos\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}}v_{y0}\right) - \sqrt{\frac{cg}{m}}t\right]\right\}$$

$$= \ln\left\{\cos\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}}v_{y0}\right)\right]\sin\left(\sqrt{\frac{cg}{m}}t\right) + \sin\left[\tan^{-1}\left(\sqrt{\frac{c}{mg}}v_{y0}\right)\right]\cos\left(\sqrt{\frac{cg}{m}}t\right)\right\}$$

$$= \ln\left[\frac{\sin\left(\sqrt{\frac{cg}{m}}t\right)}{\sqrt{1+\frac{c}{mg}v_{y0}^2}} + \frac{\sqrt{\frac{c}{mg}}v_{y0}\cos\left(\sqrt{\frac{cg}{m}}t\right)}{\sqrt{1+\frac{c}{mg}v_{y0}^2}}\right]$$

$$= \ln\left[\sqrt{\frac{c}{mg}}v_{y0}\sin\left(\sqrt{\frac{cg}{m}}t\right) + \cos\left(\sqrt{\frac{cg}{m}}t\right)\right] - \frac{1}{2}\ln\left(1+\frac{c}{mg}v_{y0}^2\right)$$

By combining ⑪ and ⑩, we get

$$\frac{m}{c}\left|\cancel{\frac{1}{2}\ln\left(1+\frac{c}{mg}v_{y0}^2\right) - \frac{1}{2}\ln\left(1+\frac{c}{mg}v_{y0}^2\right)} + \ln\left[\sqrt{\frac{c}{mg}}v_{y0}\sin\left(\sqrt{\frac{cg}{m}}t\right) + \cos\left(\sqrt{\frac{cg}{m}}t\right)\right]\right|$$

$$= \frac{m}{c}\left|\ln\left[\sqrt{\frac{c}{mg}}v_{y0}\sin\left(\sqrt{\frac{cg}{m}}t\right) + \cos\left(\sqrt{\frac{cg}{m}}t\right)\right]\right|$$

Substitute in T2 and T2, we get

$$= \frac{m}{c}\ln\left[\sqrt{\frac{c}{mg}}v_{y0}\sqrt{\frac{cg}{m}}t - \cancel{O(c^3)} + 1 - \frac{cg}{2m}t^2 + \cancel{O(c^4)}\right]$$

$$= \frac{m}{c}\ln\left(1 + \frac{c}{m}v_{y0}t - \frac{cg}{2m}t^2\right)$$

Substitute in T3, we get

$$y - h_0 = \frac{m}{c}\left[\frac{c}{m}v_{y0}t - \frac{cg}{2m}t^2 + O(c^2)\right]$$

$$y - h_0 = v_{y0}t - \frac{1}{2}gt^2$$

$$y = v_{y0}t - \frac{1}{2}gt^2 + h_0$$

Which is consistent with the uniform acceleration equations.

$$\lim_{c\to 0} v_x = \frac{mv_{x0}}{v_{x0}ct + m} = \frac{mv_{x0}}{m} = v_{x0}$$

Consistent with uniform acceleration equations.

$$\lim_{c\to 0} x = \frac{m}{c}\ln\left(1 + \frac{v_{x0}ct}{m}\right)$$

Substitute in T3, we get

$$\frac{m}{c} * \frac{v_{x0}ct}{m} = v_{x0}t$$

Which is consistent with uniform acceleration equations.

I hope you could read this mess.
If not, ask me whenever you see me.