A first-order differential equation $\frac{dy}{dx} = f(x,y)$ is said to be in differential form when it is written in the form $M(x,y)\,dx + N(x,y)\,dy = 0$ for functions M and N.

Ex - 1) $\frac{dy}{dx} = 6xy$

$\Rightarrow dy = 6xy\,dx$

$\Rightarrow -6xy\,dx + dy = 0$

$\Rightarrow M(x,y) = -6xy, \quad N(x,y) = 1$

2) $3y\frac{dy}{dx} = -e^x$

3) $\frac{dy}{dx} = \frac{\sin(x) - \ln(y)}{\sqrt{y} + 7x}$

2) $3y\frac{dy}{dx} = -e^x \Rightarrow 3y\,dy = -e^x\,dx$

$\Rightarrow e^x\,dx + 3y\,dy = 0$

$\Rightarrow M(x,y) = e^x, \quad N(x,y) = 3y$

3) $$\frac{dy}{dx} = \frac{\sin(x) - \ln(y)}{\sqrt{y} + 7x}$$

$\Rightarrow (\sqrt{y} + 7x)\,dy = (\sin(x) - \ln(y))\,dx$

$\Rightarrow (\ln(y) - \sin(x))\,dx + (\sqrt{y} + 7x)\,dy = 0$

$\Rightarrow M(x,y) = \ln(y) - \sin(x), \quad N(x,y) = \sqrt{y} + 7x$

When finding solutions to DEs, we've focused almost exclusively on explicit solutions ie solutions of the form $y=f(x)$.

Let us now consider implicit solutions $F(x,y) = C$ where C is a constant and

y is a function of x.

Then $F(x,y) = C$

$$\Rightarrow \frac{d}{dx}\left[F(x,y)\right] = 0$$

$$\Rightarrow \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \quad \left(\text{Multivariable Chain Rule}\right)$$

$$\Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

Thus, implicit solutions to DEs can be obtained from DEs in differential form where $M(x,y) = \frac{\partial F}{\partial x}$ and $N(x,y) = \frac{\partial F}{\partial y}$

Definition - A DE in differential form $M(x,y)\,dx + N(x,y)\,dy = 0$ is called <u>exact</u>

If there exists a function $F(x,y)$ such that $M(x,y) = \frac{\partial F}{\partial x}$ and $N(x,y) = \frac{\partial F}{\partial y}$

Ex- 1) $2x\,dx + 2y\,dy = 0$ is exact

since if we take $F(x,y) = x^2 + y^2$,

then $M(x,y) = 2x = \frac{\partial F}{\partial x}$ and

$N(x,y) = 2y = \frac{\partial F}{\partial y}$

2) $2y\,dx + 2x\,dy = 0$ is exact

since if we take $F(x,y) = 2xy$,

then $M(x,y) = 2y = \frac{\partial F}{\partial x}$ and

$N(x,y) = 2x = \frac{\partial F}{\partial y}$

3) $(e^y - \sin(x) + 2xy^2)dx + (xe^y + 2x^2y - 6)dy = 0$

is exact since if we take

$F(x,y) = xe^y + \cos(x) + x^2y^2 - 6y$

then $\frac{\partial F}{\partial x} = e^y - \sin(x) + 2xy^2 = M(x,y)$

and $\frac{\partial F}{\partial y} = xe^y + 2x^2y - 6 = N(x,y)$

Once we know the DE $M(x,y)dx + N(x,y)dy = 0$

is exact (ie there exists $F(x,y)$ with

$M(x,y) = \frac{\partial F}{\partial x}$ and $N(x,y) = \frac{\partial F}{\partial y}$), we can write

the general solution implicitly as $F(x,y) = C$

for arbitrary constant C.

Ex - Solution to $2x\,dx + 2y\,dy = 0$ is

$x^2 + y^2 = C$

Solution to $2y\,dx + 2x\,dy = 0$ is

$2xy = C$

Two questions - 1) How can we determine if DE is exact without knowing F?

2) How can we find F, once exactness has been shown?

If $M(x,y)dx + N(x,y)dy = 0$ is exact, then there exists F such that $M = F_x$ and $N = F_y$.

Then $M_y = F_{xy}$ and $N_x = F_{yx}$

Since $F_{xy} = F_{yx}$, it follows that $M_y = N_x$.

Fact - $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact if and only if $M_y = N_x$

Ex - 1) $2x\,dx + 2y\,dy = 0$

2) $2y\,dx + 2x\,dy = 0$

3) $(e^y - \sin(x) + 2xy^2)\,dx + (xe^y + 2x^2y - 6)\,dy = 0$

4) $y\sin(x)\,dx + \cos(x)\,dy = 0$

1) $M(x,y) = 2x$ and $N(x,y) = 2y$

$\Rightarrow M_y = 0$ and $N_x = 0$ so exact

2) $M(x,y) = 2y$ and $N(x,y) = 2x$

$\Rightarrow M_y = 2$ and $N_x = 2$ so exact

3) $M(x,y) = e^y - \sin(x) + 2xy^2$

$N(x,y) = xe^y + 2x^2y - 6$

$\Rightarrow M_y = e^y + 4xy$ and $N_x = e^y + 4xy$

So exact

4) $M(x,y) = y\sin(x)$ and $N(x,y) = \cos(x)$

$\Rightarrow M_y = \sin(x)$ and $N_x = -\sin(x)$ so not exact

Once exactness is determined, find F using partial integration.

Ex - $2x\,dx + 2y\,dy = 0$ is exact

We want F such that $F_x = M = 2x$ and $F_y = N = 2y$

$F_x = 2x \Rightarrow F = \int 2x\,dx = x^2 + g(y)$

where $g(y)$ is arbitrary function of y

Then $F_y = g'(y)$ and this must equal $2y$

So $g'(y) = 2y \Rightarrow g(y) = y^2$

Thus $F(x,y) = x^2 + y^2$

Ex - $2y\,dx + 2x\,dy = 0$ is exact

We want F such that $F_x = M = 2y$

and $F_y = N = 2x$

$F_x = 2y \Rightarrow F = \int 2y\,dx = 2xy + g(y)$

Then $F_y = 2x + g'(y)$ and this must equal $2x$

So $g'(y) = 0 \Rightarrow g(y) = 0 \Rightarrow F(x,y) = 2xy$

Ex - $(e^y - \sin(x) + 2xy^2)dx + (xe^y + 2x^2y - 6)dy = 0$

is exact. We want F such that

$F_x = M = e^y - \sin(x) + 2xy^2$ and $F_y = xe^y + 2x^2y - 6$.

$F_x = e^y - \sin(x) + 2xy^2 \Rightarrow F = \int e^y - \sin(x) + 2xy^2\,dx$

$= xe^y + \cos(x) + x^2y^2 + g(y)$

Then $F_y = xe^y + 2x^2y + g'(y)$ and this must equal $xe^y + 2x^2y - 6$

So $g'(y) = -6 \Rightarrow g(y) = -6y$

Thus, $F(x,y) = xe^y + \cos(x) + x^2y^2 - 6y$

Ex - $\dfrac{dy}{dx} = \dfrac{\cos(x) - 2xy}{x^2 + e^y}$

1) Check exact

2) Find general solution

$\dfrac{dy}{dx} = \dfrac{\cos(x) - 2xy}{x^2 + e^y} \Rightarrow (x^2 + e^y)\,dy = (\cos(x) - 2xy)\,dx$

$\Rightarrow \quad (2xy - \cos(x))dx + (x^2 + e^y)dy = 0$

$\Rightarrow \quad M(x,y) = 2xy - \cos(x)$ and $N(x,y) = x^2 + e^y$

$\Rightarrow \quad M_y = 2x$ and $N_x = 2x$ so exact

Want F such that $F_x = M = 2xy - \cos(x)$ and

$F_y = N = x^2 + e^y$

$F_x = 2xy - \cos(x) \Rightarrow \quad F = \int 2xy - \cos(x)\, dx$

$= x^2y - \sin(x) + g(y)$

$\Rightarrow F_y = x^2 + g'(y)$ and this must equal $x^2 + e^y$

So $g'(y) = e^y \Rightarrow g(y) = e^y$

Thus $F(x,y) = x^2y - \sin(x) + e^y$

General Solution is then $x^2y - \sin(x) + e^y = C$

Consider a first-order linear DE

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Putting in differential form, we have

$$dy + P(x)y\,dx = Q(x)\,dx$$

$$\Rightarrow \left(P(x)y - Q(x)\right)dx + dy = 0$$

$$\Rightarrow M = P(x)y - Q(x) \quad \text{and} \quad N = 1$$

$$\Rightarrow M_y = P(x) \quad \text{and} \quad N_x = 0 \quad \text{so not exact}$$

When we multiply by the integrating factor $e^{\int P(x)\,dx}$, we get

$$\left(P(x)e^{\int P(x)dx}y - e^{\int P(x)dx}Q(x)\right)dx + e^{\int P(x)dx}dy = 0$$

Now $M = P(x)e^{\int P(x)dx}y - e^{\int P(x)dx}Q(x)$

and $N = e^{\int P(x)dx}$

$$\Rightarrow M_y = P(x)e^{\int P(x)dx} \quad \text{and} \quad N_x = P(x)e^{\int P(x)dx}$$

So now exact

Can we do this in general?

That is, given $M(x,y)dx + N(x,y)dy = 0$

not exact, can we find integrating factor

$u(x,y)$ such that $M(x,y)u(x,y)dx + N(x,y)u(x,y)dy = 0$

is exact?

If so, then $\frac{\partial}{\partial y}\Big(M(x,y)u(x,y)\Big) = \frac{\partial}{\partial x}\Big(N(x,y)u(x,y)\Big)$

$\Rightarrow M u_y + M_y u = N u_x + N_x u \quad *$

Finding u that satisfies this equation is challenging.

Special case 1: u depends only on x

Then $u_y = 0$ and $u_x = \frac{du}{dx}$

Then $*$ becomes $M_y u = N \frac{du}{dx} + N_x u$

$\Rightarrow N \frac{du}{dx} = M_y u - N_x u$

$\Rightarrow \frac{1}{u} \frac{du}{dx} = \frac{M_y - N_x}{N}$

Since u depends only on x, we must have

$\frac{M_y - N_x}{N}$ depend only on x.

In this case $\frac{1}{u} du = \frac{M_y - N_x}{N} dx$

$$\Rightarrow \ln(u) = \int \frac{M_y - N_x}{N} dx$$

$$\Rightarrow u = e^{\int \frac{M_y - N_x}{N} dx}$$

Fact - If $M(x,y)dx + N(x,y)dy = 0$ is not exact, but $\frac{M_y - N_x}{N}$ depends only on x, then multiplying the DE by the integrating factor $u = e^{\int \frac{M_y - N_x}{N} dx}$ will make it exact.

Special case 2: u depends only on y

Then $u_x = 0$ and $u_y = \frac{du}{dy}$

Then $*$ becomes $M \frac{du}{dy} + M_y u = N_x u$

$$\Rightarrow \quad M \frac{du}{dy} = N_x u - M_y u$$

$$\Rightarrow \quad \frac{1}{u} \frac{du}{dy} = \frac{N_x - M_y}{M}$$

Since u depends only on y, we must have

$\frac{N_x - M_y}{M}$ depend only on y

In this case $\quad \frac{1}{u} du = \frac{N_x - M_y}{M} dy$

$$\Rightarrow \quad \ln(u) = \int \frac{N_x - M_y}{M} dy$$

$$\Rightarrow \quad u = e^{\int \frac{N_x - M_y}{M} dy}$$

Fact - If $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact, but $\frac{N_x - M_y}{M}$ depends only on y, then multiplying the DE by the integrating factor $u = e^{\int \frac{N_x - M_y}{M} dy}$ will make it exact.

Ex - $y\sin(x)\,dx + \cos(x)\,dy = 0$

$M = y\sin(x)$ and $N = \cos(x)$

$\Rightarrow M_y = \sin(x)$ and $N_x = -\sin(x)$

So not exact

But $\dfrac{M_y - N_x}{N} = \dfrac{\sin(x) - (-\sin(x))}{\cos(x)} = \dfrac{2\sin(x)}{\cos(x)} = 2\tan(x)$

ie a function of x only.

So $u = e^{\int 2\tan(x)\,dx} = e^{-2\ln(\cos(x))} = e^{\ln(\cos(x)^{-2})}$

$= \cos(x)^{-2}$

$= \dfrac{1}{\cos^2(x)}$

Is an integrating factor for the DE.

Multiplying the DE by $\dfrac{1}{\cos^2(x)}$ yields

$\dfrac{y\sin(x)}{\cos^2(x)}\,dx + \dfrac{1}{\cos(x)}\,dy = 0$

$\Rightarrow y\sec(x)\tan(x)\,dx + \sec(x)\,dy = 0$

$\Rightarrow M = y\sec(x)\tan(x)$ and $N = \sec(x)$

$\Rightarrow$ $M_y = \sec(x)\tan(x)$ and $N_x = \sec(x)\tan(x)$

So now exact

We want F such that $F_x = M = y\sec(x)\tan(x)$

and $F_y = \sec(x)$

$F_y = \sec(x) \Rightarrow F = \int \sec(x)\,dy = y\sec(x) + g(x)$

$\Rightarrow F_x = y\sec(x)\tan(x) + g'(x)$ and this must equal $y\sec(x)\tan(x)$

So $g'(x) = 0 \Rightarrow g(x) = 0$

Thus $F(x,y) = y\sec(x)$

$\Rightarrow$ General solution is $y\sec(x) = C$

Go back to $y\sin(x)dx + \cos(x)dy = 0$

$\Rightarrow M = y\sin(x)$ and $N = \cos(x)$

$\Rightarrow M_y = \sin(x)$ and $N_x = -\sin(x)$

Not exact, but since $\frac{N_x - M_y}{M} = \frac{-\sin(x) - \sin(x)}{y\sin(x)}$

$$= \frac{-2\sin(x)}{y\sin(x)}$$

$$= -\frac{2}{y}$$

is a function of y only, an integrating factor is $u = e^{\int -\frac{2}{y} dy} = e^{-2\ln(y)} = e^{\ln(y^{-2})}$

$$= \frac{1}{y^2}$$

Multiplying the DE by this yields

$$\frac{\sin(x)}{y} dx + \frac{\cos(x)}{y^2} dy = 0$$

$$\Rightarrow \quad M = \frac{\sin(x)}{y} \quad \text{and} \quad N = \frac{\cos(x)}{y^2}$$

$$\Rightarrow \quad M_y = \frac{-\sin(x)}{y^2} \quad \text{and} \quad N_x = \frac{-\sin(x)}{y^2}$$

So now exact

We want F such that $F_x = M = \frac{\sin(x)}{y}$ and

$F_y = N = \frac{\cos(x)}{y^2}$

$$F_x = \frac{\sin(x)}{y} \Rightarrow F = \int \frac{\sin(x)}{y}\,dx = \frac{-\cos(x)}{y} + g(y)$$

$\Rightarrow F_y = \frac{\cos(x)}{y^2} + g'(y)$ and this must equal

$\frac{\cos(x)}{y^2}$ so $g'(y) = 0 \Rightarrow g(y) = 0$

Thus, $F(x,y) = \frac{-\cos(x)}{y}$

$\Rightarrow$ General solution is $\frac{-\cos(x)}{y} = C$

Solve the IVP

$$x_1' = 4x_1 - 2x_2$$
$$x_2' = -x_1 + 3x_2$$
$$x_1(0) = 11, \quad x_2(0) = -1$$

using Laplace transforms

Take Laplace of both sides of both equations

$$L(x_1') = L(4x_1 - 2x_2)$$
$$L(x_2') = L(-x_1 + 3x_2)$$

$\Downarrow$

$$sL(x_1) - x_1(0) = 4L(x_1) - 2L(x_2)$$
$$sL(x_2) - x_2(0) = -L(x_1) + 3L(x_2)$$

$$\Downarrow$$

$$sL(x_1) - 11 = 4L(x_1) - 2L(x_2)$$
$$sL(x_2) + 1 = -L(x_1) + 3L(x_2)$$

$$\Downarrow$$

$$(s-4)L(x_1) + 2L(x_2) = 11$$
$$L(x_1) + (s-3)L(x_2) = -1$$

$$\Downarrow$$

$$(s-4)(s-3)L(x_1) + 2(s-3)L(x_2) = 11s - 33$$
$$2L(x_1) + 2(s-3)L(x_2) = -2$$

$$\Downarrow$$

$$[s^2 - 7s + 12 - 2]L(x_1) = 11s - 31$$

$$\Rightarrow L(x_1) = \frac{11s-31}{s^2-7s+10} = \frac{11s-31}{(s-2)(s-5)} = \frac{3}{s-2} + \frac{8}{s-5}$$

$$\Rightarrow x_1 = L^{-1}\left(\frac{3}{s-2}\right) + L^{-1}\left(\frac{8}{s-5}\right)$$

$$\Rightarrow \quad X_1 = 3e^{2t} + 8e^{5t}$$

$$(s-4)L(x_1) + 2L(x_2) = \cancel{+}11$$

$$L(x_1) + (s-3)L(x_2) = -1$$

$$\Downarrow$$

$$(s-4)L(x_1) + 2L(x_2) = 11$$

$$(s-4)L(x_1) + (s-3)(s-4)L(x_2) = -s+4$$

$$\Downarrow$$

$$[2-(s^2-7s+12)]L(x_2) = s+7$$

$$\Rightarrow \quad (-s^2+7s-10)L(x_2) = s+7$$

$$\Rightarrow \quad (s^2-7s+10)L(x_2) = -s-7$$

$$\Rightarrow \quad L(x_2) = \frac{-s-7}{s^2-7s+10} = \frac{-s-7}{(s-2)(s-5)} = \frac{3}{s-2} - \frac{4}{s-5}$$

$$\Rightarrow \quad X_2 = L^{-1}\left(\frac{3}{s-2}\right) - L\left(\frac{4}{s-5}\right)$$

$$\Rightarrow \quad X_2 = 3e^{2t} - 4e^{5t}$$

Repeated eigenvalues for systems

If an eigenvalue has multiplicity $K > 1$ in the characteristic polynomial $\det(A-\lambda I)$, and there are K linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_K$ corresponding to λ, then $e^{\lambda t}\vec{v}_1, \ldots, e^{\lambda t}\vec{v}_K$ are independent solutions to $\vec{x}\,' = A\vec{x}$

Ex.
$$\begin{aligned} x_1' &= 2x_1 + x_2 + x_3 \\ x_2' &= x_1 + 2x_2 + x_3 \\ x_3' &= x_1 + x_2 + 2x_3 \end{aligned}$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{so} \quad A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix}$$

$$\Rightarrow \det(A-\lambda I) = (2-\lambda)\det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} - \det\begin{bmatrix} 1 & 1 \\ 1 & 2-\lambda \end{bmatrix} + \det\begin{bmatrix} 1 & 2-\lambda \\ 1 & 1 \end{bmatrix}$$

$$= (2-\lambda)\left((2-\lambda)(2-\lambda) - 1\right) - (2-\lambda-1) + 1 - (2-\lambda)$$

$$= (2-\lambda)(\lambda^2 - 4\lambda + 3) - 2 + \lambda + 1 + 1 - 2 + \lambda$$

$$= (2-\lambda)(\lambda-1)(\lambda-3) + 2\lambda - 2$$

$$= (2-\lambda)(\lambda-1)(\lambda-3) + 2(\lambda-1)$$

$$= (\lambda-1)\left[(2-\lambda)(\lambda-3) + 2\right]$$

$$= (\lambda-1)(-\lambda^2 + 5\lambda - 4)$$

$$= -(\lambda-1)(\lambda^2-5\lambda+4)$$
$$= -(\lambda-1)(\lambda-1)(\lambda-4)$$
$$= -(\lambda-1)^2(\lambda-4)$$ So eigenvalues are $\lambda=1$, $\lambda=4$

For $\lambda=4$, $A-4I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

Want $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ such that $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[-R_1+R_3\to R_3]{2R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \xrightarrow{-\frac{1}{3}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$$\xrightarrow[2R_2+R_1\to R_1]{-3R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow$$

$v_1 - v_3 = 0$

$v_2 - v_3 = 0$

Let $v_3 = s$

Then $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s \\ s \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} s$

For $\lambda=1$, $A-I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow v_1 + v_2 + v_3 = 0$$

$\rightarrow v_1 = -v_2 - v_3$. Let $v_2 = s$, $v_3 = t$

Then $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t$

General solution: $\vec{x} = c_1 e^{4t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Consider now the case when eigenvalue λ has multiplicity > 1 but only one eigenvector

Ex. $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) + 1$$
$$= \lambda^2 - 4\lambda + 4$$
$$= (\lambda - 2)^2 \Rightarrow \lambda = 2$$

$A - 2I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

In this case, the matrix is <u>defective</u>

How, then do we find additional independent solution to $\vec{x}' = A\vec{x}$

In general, if λ has multiplicity 2 but only 1 eigenvector $\vec{v}$, then $e^{\lambda t}\vec{v}$ is a solution. We guess that $\vec{x} = te^{\lambda t}\vec{v}$ is another

Then $\vec{x}' = t\lambda e^{\lambda t}\vec{v} + e^{\lambda t}\vec{v}$

So $\vec{x}' = A\vec{x}$ becomes $t\lambda e^{\lambda t}\vec{v} + e^{\lambda t}\vec{v} = te^{\lambda t}A\vec{v}$

$$\Rightarrow t\lambda e^{\lambda t}\vec{v} + e^{\lambda t}\vec{v} = te^{\lambda t}\lambda\vec{v}$$

We have the extra term $e^{\lambda t}\vec{v}$

So we guess instead $\vec{x} = te^{\lambda t}\vec{v} + e^{\lambda t}\vec{u}$ for some constant vector $\vec{u}$.

Then $\vec{x}' = t\lambda e^{\lambda t}\vec{v} + e^{\lambda t}\vec{v} + \lambda e^{\lambda t}\vec{u}$

So $\vec{x}' = A\vec{x}$ becomes $t\lambda e^{\lambda t}\vec{v} + e^{\lambda t}\vec{v} + \lambda e^{\lambda t}\vec{u} = te^{\lambda t}A\vec{v} + e^{\lambda t}A\vec{u}$

$$\Rightarrow t\lambda e^{\lambda t}\vec{v} + e^{\lambda t}\vec{v} + \lambda e^{\lambda t}\vec{u} = te^{\lambda t}\lambda\vec{v} + e^{\lambda t}A\vec{u}$$

$$\Rightarrow \vec{v} + \lambda u = A\vec{u}$$

$$\Rightarrow \vec{v} = A\vec{u} - \lambda\vec{u}$$

$$\Rightarrow \vec{v} = (A - \lambda I)\vec{u}$$

Definition. if $\vec{v}$ is an eigenvector corresponding to λ, then a vector $\vec{u}$ satisfying $(A-\lambda I)\vec{u} = \vec{v}$ is called a generalized eigenvector corresponding to λ

Ex. $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ had $\lambda = 2$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The generalized eigenvector $\vec{u}$ satisfies $(A-\lambda I)\vec{u} = \vec{v}$

ie $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow \quad -u_1 + u_2 = 1 \quad \Rightarrow \quad u_1 = 0,\ u_2 = 1 \Rightarrow \vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Note, infinitely many possibilities for $\vec{u}$.

Fact. if $\vec{x}' = A\vec{x}$ is a 2x2 system, and A has eigenvalue λ with multiplicity 2, only one eigenvector $\vec{v}$, and generalized eigenvector $\vec{u}$, then the general solution is $\vec{x} = c_1 e^{\lambda t}\vec{v} + c_2(te^{\lambda t}\vec{v} + e^{\lambda t}\vec{u})$

Ex. $\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= -x_1 + 3x_2 \end{aligned}$ ie $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

We showed A has $\lambda = 2$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Thus, $\vec{x} = c_1 e^{2t}\begin{bmatrix}1\\1\end{bmatrix} + c_2\left(te^{2t}\begin{bmatrix}1\\1\end{bmatrix} + e^{2t}\begin{bmatrix}0\\1\end{bmatrix}\right)$

$$\Rightarrow \quad x_1 = c_1 e^{2t} + c_2(te^{2t})$$
$$x_2 = c_1 e^{2t} + c_2(te^{2t} + e^{2t})$$

Ex - $\begin{aligned} x_1' &= 2x_1 + x_2 \\ x_2' &= -x_1 + 4x_2 \end{aligned} \Rightarrow \begin{bmatrix}x_1'\\x_2'\end{bmatrix} = \underbrace{\begin{bmatrix}2 & 1\\-1 & 4\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$$\det(A-\lambda I) = \det\begin{bmatrix}2-\lambda & 1\\-1 & 4-\lambda\end{bmatrix} = (2-\lambda)(4-\lambda)+1$$
$$= \lambda^2 - 6\lambda + 9$$
$$= (\lambda-3)^2$$

So $\lambda = 3$. Thus, $A - 3I = \begin{bmatrix}-1 & 1\\-1 & 1\end{bmatrix} \Rightarrow \vec{v} = \begin{bmatrix}1\\1\end{bmatrix}$

Look for $\vec{u}$ such that $\begin{bmatrix}-1 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

$$\Rightarrow \quad -u_1 + u_2 = 1 \quad \Rightarrow \quad u_1 = -1, \quad u_2 = 0$$
$$\Rightarrow \quad \vec{u} = \begin{bmatrix}-1\\0\end{bmatrix}$$

So $\vec{x} = c_1 e^{3t}\begin{bmatrix}1\\1\end{bmatrix} + c_2\left(te^{3t}\begin{bmatrix}1\\1\end{bmatrix} + e^{3t}\begin{bmatrix}-1\\0\end{bmatrix}\right)$

$$\Rightarrow \quad x_1 = c_1 e^{3t} + c_2(te^{3t} - e^{3t})$$
$$x_2 = c_1 e^{3t} + c_2 te^{3t}$$