A curve $\vec{r}$ such that $\|\vec{r}\,'\| = 1$ is a <u>unit speed curve</u>.

Can we always reparametrize to make a curve unit speed? Yes

The <u>arclength reparametrization</u> of a smooth curve $\vec{r}(t)$ works as follows:

1) Compute the arclength function $s(t)$.
2) Invert the function i.e. solve for t in terms of s.
3) Substitute this expression for t into $\vec{r}$ to get a reparametrized curve $\vec{r}(s)$.

The resulting curve will have unit speed ie $\|\vec{r}\,'(s)\| = 1$

Example - $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), 4t \rangle \quad t \geq 0$

Last time, we computed $s = 5t$

Thus, $t = \dfrac{s}{5}$

Then $\vec{r}(s) = \langle 3\cos\left(\frac{s}{5}\right), 3\sin\left(\frac{s}{5}\right), \frac{4s}{5} \rangle$

The <u>curvature</u> of a smooth curve $\vec{r}(t)$ is $K(t) = \dfrac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$

Example- $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), 4t \rangle$

$\vec{T}(t) = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$. $\qquad \vec{r}'(t) = \langle -3\sin(t), 3\cos(t), 4 \rangle$

$\|\vec{r}'(t)\| = 5 \qquad$ So $\vec{T}(t) = \langle -\frac{3}{5}\sin(t), \frac{3}{5}\cos(t), \frac{4}{5} \rangle$

$\vec{T}'(t) = \langle -\frac{3}{5}\cos(t), -\frac{3}{5}\sin(t), 0 \rangle$

$\|\vec{T}'(t)\| = \sqrt{\frac{9}{25}\cos^2(t) + \frac{9}{25}\sin^2(t)} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

Thus, $K(t) = \dfrac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \dfrac{3/5}{5} = \dfrac{3}{25}$

The <u>unit normal vector</u> of a smooth curve $\vec{r}(t)$ is $\vec{N}(t) = \dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$

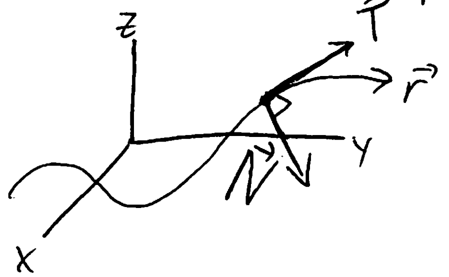
Example- $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), 4t \rangle$

$\vec{T}'(t) = \langle -\frac{3}{5}\cos(t), -\frac{3}{5}\sin(t), 0 \rangle$ and $\|\vec{T}'(t)\| = \frac{3}{5}$

Then $\vec{N}(t) = \dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \langle -\cos(t), -\sin(t), 0 \rangle$

Observe $\vec{T}(t) \cdot \vec{N}(t) = 0$

Fact- $\vec{T}(t)$ and $\vec{N}(t)$ are orthogonal



The <u>binormal vector</u> of a smooth curve $\vec{r}(t)$ is the unit vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

Example - $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), 4t \rangle$

$\vec{T}(t) = \langle -\frac{3}{5}\sin(t), \frac{3}{5}\cos(t), \frac{4}{5} \rangle$

$\vec{N}(t) = \langle -\cos(t), -\sin(t), 0 \rangle$

$\vec{B}(t) = \langle \frac{4}{5}\sin(t), -\frac{4}{5}\cos(t), \frac{3}{5} \rangle$

Let $\vec{r}(t)$ be a smooth parametrized curve.
The <u>osculating plane</u> for the curve at $t=t_0$ is the plane spanned by $\vec{T}(t_0)$ and $\vec{N}(t_0)$

It contains the point on $\vec{r}$ at which $t = t_0$ and has normal vector $\vec{B}(t_0)$.

Example - Write the general equation for the osculating plane of $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), 4t \rangle$ at $t = 0$.

$\vec{r}(0) = \langle 3, 0, 0 \rangle$ so point is $(3, 0, 0)$

$\vec{B}(t) = \langle \frac{4}{5}\sin(t), -\frac{4}{5}\cos(t), \frac{3}{5} \rangle$

So $\vec{n} = \vec{B}(0) = \langle 0, -\frac{4}{5}, \frac{3}{5} \rangle$

Thus, we have $0(x-3) - \frac{4}{5}(y-0) + \frac{3}{5}(z-0) = 0$

$$-\frac{4}{5}y + \frac{3}{5}z = 0$$

Fact - For a smooth parametrized curve $\vec{r}(t)$, its acceleration vector $\vec{a}(t) = \vec{r}\,''(t)$ lies in the osculating plane.

Thus, we can write $\vec{a}(t)$ as a linear combination of $\vec{T}(t)$ and $\vec{N}(t)$, say $\vec{a}(t) = a_T(t)\vec{T}(t) + a_N(t)\vec{N}(t)$ where $a_T(t)$ and $a_N(t)$ are real-valued functions.

The function $a_T(t)$ is the <u>scalar tangent component of acceleration</u>, and is defined to be $a_T(t) = v'(t)$ where $v(t) = \|\vec{r}\,'(t)\|$

The function $a_N(t)$ is the <u>scalar normal component of acceleration</u>, and is defined to be

$$a_N(t) = \kappa(t)\|\vec{r}\,'(t)\|^2$$

Example. $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), 4t \rangle$

$\vec{r}\,'(t) = \langle -3\sin(t), 3\cos(t), 4 \rangle$ so $v(t) = \|\vec{r}\,'(t)\| = 5$

Thus $a_T(t) = v'(t) = 0$

$\kappa(t) = \frac{3}{25}$ so $a_N(t) = \kappa(t)\|\vec{r}\,'(t)\|^2 = \frac{3}{25}(5)^2 = 3$

Let $\vec{r}(t)$ be a smooth parametrized curve. The <u>osculating circle</u> for the curve at $t = t_0$ is the circle tangent to $\vec{r}$ at $t = t_0$, lying in the osculating plane.

The radius of the osculating circle is called the <u>radius of curvature</u>, and is given by $\rho(t_0) = \frac{1}{\kappa(t_0)}$

The center of the circle is $\vec{c}(t_0) = \vec{r}(t_0) + \rho(t_0)\vec{N}(t_0)$

Example- $\vec{r}(t) = \langle 3\cos(t),\ 3\sin(t),\ 4t \rangle$ at $t=0$

$K(t) = \frac{3}{25}$ so $\rho(t) = \frac{1}{K(t)} = \frac{25}{3}$

Hence $\rho(0) = \frac{25}{3} \leftarrow$ radius of curvature

$\vec{r}(0) = \langle 3, 0, 0 \rangle$

$\vec{N}(t) = \langle -\cos(t), -\sin(t), 0 \rangle$

So $\vec{N}(0) = \langle -1, 0, 0 \rangle$

Center is $\vec{c}(0) = \vec{r}(0) + \rho(0)\vec{N}(0)$

$= \langle 3, 0, 0 \rangle + \frac{25}{3}(-1, 0, 0)$

$= \left(-\frac{16}{3}, 0, 0\right)$

$\vec{L} = \langle x_0 + at,\ y_0 + bt,\ z_0 + ct \rangle$

$x = x_0 + at,\quad y = y_0 + bt,\quad z = z_0 + ct$

$\frac{x - x_0}{a} = t,\quad \frac{y - y_0}{b} = t,\quad \frac{z - z_0}{c} = t$

$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$ (symmetric equations)