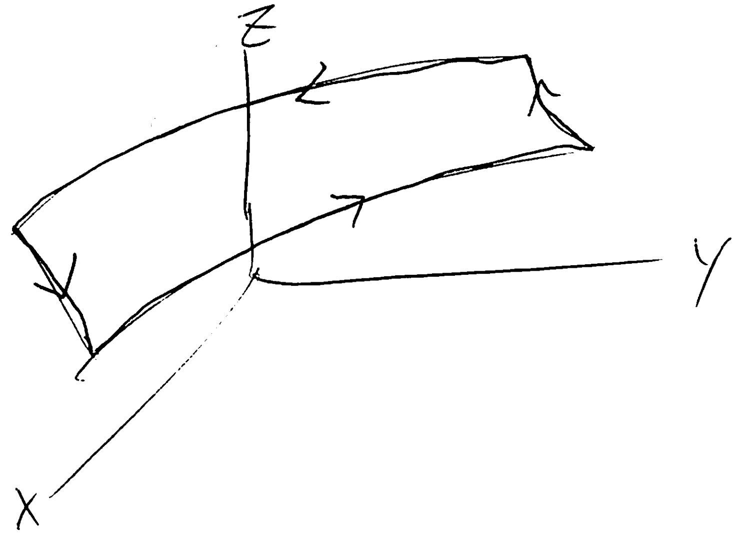
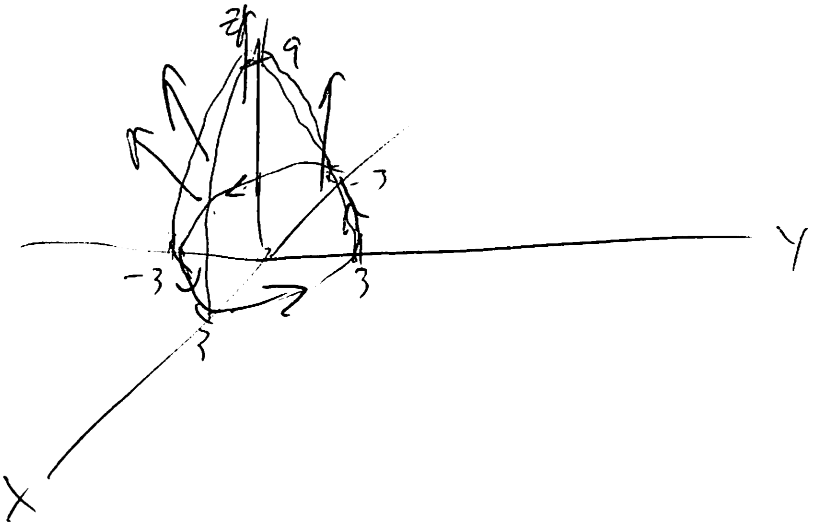
Consider a smooth surface in $\mathbb{R}^3$.



We can think of the boundary of the surface as a smooth simple closed curve in $\mathbb{R}^3$.

We want the orientation of the curve to "match" the orientation of the surface. To insure this, use right hand rule.

Ex - Let S be the portion of the graph of $f(x,y) = 9 - x^2 - y^2$ above the xy plane



The boundary curve is the circle of radius 3 with $z=0$

Stoke's Theorem - Let S be a smooth surface, and let C be the simple closed smooth boundary curve of S. Assume the orientations of C and S "match". Then for any vector field $\vec{F}$ in $\mathbb{R}^3$,

$$\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

Ex. $f(x,y) = 9-x^2-y^2$ above xy plane

$\vec{F}(x,y,z) = \langle x-y,\ x+y,\ z+1 \rangle$

Do $\oint_C \vec{F} \cdot d\vec{r}$

In this case, C is circle of radius 3 with $z=0$. So $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), 0 \rangle$

$0 \le t \le 2\pi$

$\vec{F}(\vec{r}(t)) = \langle 3\cos(t) - 3\sin(t), 3\cos(t) + 3\sin(t), 1 \rangle$

$\vec{r}\,'(t) = \langle -3\sin(t), 3\cos(t), 0 \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = -9\sin(t)\cos(t) + 9\sin^2(t)$

$+ 9\cos^2(t) + 9\sin(t)\cos(t) + 0 = 9$

Then $\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 9\, dt = 18\pi$

Do $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$

S is portion of $f(x,y) = 9 - x^2 - y^2$ above xy plane.

So $\vec{r}(u,v) = \langle u, v, 9 - u^2 - v^2 \rangle$, $u^2 + v^2 \le 9$

$\text{curl}(\vec{F}) = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \langle x - y, x + y, z + 1 \rangle$

$= \langle 0, 0, 2 \rangle$

So $\text{curl}(\vec{F})(\vec{r}(u,v)) = \langle 0, 0, 2 \rangle$

$\vec{r}_u = \langle 1, 0, -2u \rangle$, $\quad \vec{r}_v = \langle 0, 1, -2v \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 2u, 2v, 1 \rangle$

Then $\text{curl}(\vec{F})(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = 2$

Thus, $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \iint_{u^2+v^2 \leq 9} 2 \, du\,dv$

$$= 2 \iint_{u^2+v^2 \leq 9} 1 \, du\,dv$$

$$= 2\,(\text{Area of circle of radius } 3)$$

$$= 2(9\pi) = 18\pi$$

Example- $\vec{F}(x,y,z) = \langle z^2, x^2, y^2 \rangle$

Let C be the boundary curve of the plane $x+y+z=7$ above the region $R = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq 2\}$

Compute $\oint_C \vec{F} \cdot d\vec{r}$

By Stoke's Theorem, $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$

Parametrize S: $\vec{r}(u,v) = \langle u, v, 7-u-v \rangle$

$0 \le u \le 1, \quad 0 \le v \le 2$

$$\text{Curl}(\vec{F}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle z^2, x^2, y^2 \rangle$$

$$= \langle 2y, 2z, 2x \rangle$$

$$\text{Curl}(\vec{F})(\vec{r}(u,v)) = \langle 2v, 14-2u-2v, 2u \rangle$$

$\vec{r}_u = \langle 1, 0, -1 \rangle \qquad \vec{r}_v = \langle 0, 1, -1 \rangle$

$$\vec{r}_u \times \vec{r}_v = \langle 1, 1, 1 \rangle$$

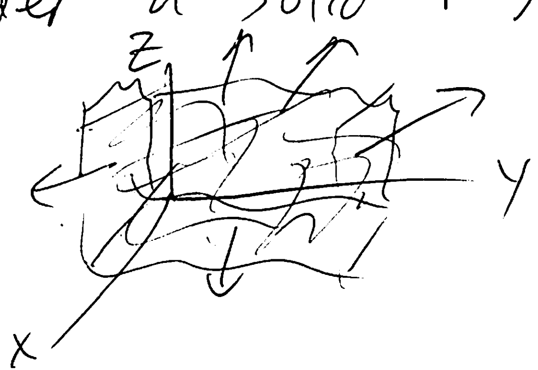
$$\text{Curl}(\vec{F})(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = 2v + 14 - 2u - 2v + 2u$$

$$= 14$$

So $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_0^2 \int_0^1 14 \, du \, dv$

$$= 28$$

Consider a solid region in $\mathbb{R}^3$,



We can think of the boundary of the region as a smooth surface in $\mathbb{R}^3$

Divergence Theorem - Let R be a solid region in $\mathbb{R}^3$, and let S be the smooth boundary surface of R, chosen with outward pointing orientation $\vec{n}$. Then for any vector field $\vec{F}$ in $\mathbb{R}^3$,

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV$$

Example - Let R be the region on and inside the unit sphere. Then S is the sphere itself. Verify Divergence Theorem for 1) $\vec{F}(x,y,z) = \langle x, y, z \rangle$.

2) $\vec{F}(x,y,z) = \langle 5x^3, 5y^3, 5z^3 \rangle$

Set up both, integrate one.

1) Do $\iint_S \vec{F} \cdot d\vec{S}$

$\vec{r}(u,v) = \langle \cos(u)\sin(v),\ \sin(u)\sin(v),\ \cos(v) \rangle$

$0 \leq u \leq 2\pi,\ 0 \leq v \leq \pi$

$\vec{F}(\vec{r}(u,v)) = \langle \cos(u)\sin(v),\ \sin(u)\sin(v),\ \cos(v) \rangle$

$\vec{r}_u = \langle -\sin(u)\sin(v),\ \cos(u)\sin(v),\ 0 \rangle$

$\vec{r}_v = \langle \cos(u)\cos(v),\ \sin(u)\cos(v),\ -\sin(v) \rangle$

$\vec{r}_u \times \vec{r}_v = \langle -\sin^2(v)\cos(u),\ -\sin^2(v)\sin(u),\ -\sin(v)\cos(v) \rangle$

Want outward orientation so choose

$\vec{r}_u \times \vec{r}_v = \langle \sin^2(v)\cos(u),\ \sin^2(v)\sin(u),\ \sin(v)\cos(v) \rangle$

$\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = \sin^3(v)\cos^2(u) + \sin^3(v)\sin^2(u) + \sin(v)\cos^2(v)$

$= \sin(v)$

Then $\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^{\pi} \sin(v)\, dv\, du$

$\int_0^{\pi} \sin(v)\, dv = -\cos(v) \Big|_0^{\pi} = 1 + 1 = 2$

$\int_0^{2\pi} 2\, du = 4\pi$

Now do $\iiint_R \operatorname{div}(\vec{F})\, dV$

$\vec{F} = \langle x, y, z \rangle \implies div(\vec{F}) = 3$

So $\iiint_R div(\vec{F})\, dV = \iiint_R 3\, dV = 3\iiint_R 1\, dV$

$= 3(\text{Volume of unit sphere})$

$= 3 \cdot \frac{4}{3}\pi = 4\pi$

2) Set up $\iint_S \vec{F} \cdot d\vec{S}$

$\vec{F}(\vec{r}(u,v)) = \langle 5\cos^3(u)\sin^3(v),\ 5\sin^3(u)\sin^3(v),\ 5\cos^3(v) \rangle$

$\vec{r}_u \times \vec{r}_v = \langle \sin^2(v)\cos(u),\ \sin^2(v)\sin(u),\ \sin(v)\cos(v) \rangle$

$\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = 5\cos^4(u)\sin^5(v) + 5\sin^4(u)\sin^5(v)$
$+ 5\sin(v)\cos^4(v)$

Hence $\iint_S \vec{F} \cdot dS = \int_0^{2\pi}\int_0^{\pi} 5\cos^4(u)\sin^5(v) + 5\sin^4(u)\sin^5(v) + 5\sin(v)\cos^4(v)\, dv\, du$

Now do $\iiint_R div(\vec{F})\, dV$

$\vec{F} = \langle 5x^3, 5y^3, 5z^3 \rangle \implies div(\vec{F}) = 15x^2 + 15y^2 + 15z^2$

So $\iiint_R \operatorname{div}(\vec{F})\, dV = \iiint_R 15x^2 + 15y^2 + 15z^2\, dV$

In spherical coordinates, this becomes

$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 15\rho^4 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$\int_0^1 15\rho^4 \sin(\phi)\, d\rho = 3\rho^5 \sin(\phi)\Big|_0^1 = 3\sin(\phi)$$

$$\int_0^{\pi} 3\sin(\phi)\, d\phi = -3\cos(\phi)\Big|_0^{\pi} = 6$$

$$\int_0^{2\pi} 6\, d\theta = 12\pi$$

Example. cube $R = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$

$\vec{F} = \langle -5x, 4y, 2z \rangle$

Let S be the boundary of R.

Compute $\iint_S \vec{F} \cdot d\vec{S}$

By the divergence theorem, we have

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV$$

$$\operatorname{div}(\vec{F}) = 1$$

So $$\iiint_R \operatorname{div}(\vec{F})\, dV = \iiint_R 1\, dV$$

$$= \text{volume of cube}$$

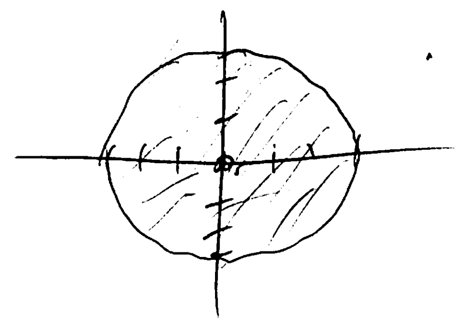
$$= 1$$

$$\frac{(T_1 + T_2 + T_3 + T_4 + WA)(.15) + (\text{Maple})(.05)}{8 \quad c}$$

Absolute max/min

$f(x,y) = x^2 + 2y^3 - 7$ over the circle

$x^2 + y^2 \leq 9$



~~Check~~ Evaluate f ~~on~~ the boundary $x^2+y^2=9$ and at critical points inside the region

Largest = max Smallest = min

$f_x = 2x$ $f_y = 6y^2$

Critical point is $(0,0)$ which is inside the region

On the boundary, $x^2 = 9 - y^2$ with $-3 \leq y \leq 3$

So $f(x,y) = x^2 + 2y^3 - 7$

$$= 9 - y^2 + 2y^3 - 7$$

$$= 2y^3 - y^2 + 2 \qquad \text{with} \quad -3 \leq y \leq 3$$

$f'(y) = 6y^2 - 2y = 2y(3y-1)$

This is 0 when $y = 0, \frac{1}{3}$

$f(0) = 2$ $\qquad f\left(\frac{1}{3}\right) = \frac{2}{27} - \frac{1}{9} + 2$ $\qquad f(3) = 47$

$\qquad\qquad = \frac{53}{27}$ $\qquad f(-3) = -61$

At the CP, $f(0,0) = -7$

Max at $(0, 3, 47)$ $\qquad$ Min at $(0, -3, -61)$

$\vec{F}(x,y,z) = \left\langle xz - yz,\ xz + yz,\ \frac{z}{\sqrt{x^3 + y^3}} \right\rangle$

C is circle of radius 4 at $z = 3$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$\vec{r}(t) = \langle 4\cos(t),\ 4\sin(t),\ 3 \rangle,\quad 0 \le t \le 2\pi$

$\vec{F}(\vec{r}(t)) = \left\langle 12\cos(t) - 12\sin(t),\ 12\cos(t) + 12\sin(t),\ \frac{3}{\sqrt{64\cos^2(t) + 64\sin^2(t)}} \right\rangle$

$\vec{r}\,'(t) = \langle -4\sin(t),\ 4\cos(t),\ 0 \rangle$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \cancel{-48\sin(t)\cos(t)} + 48\sin^2(t) + 48\cos^2(t) + \cancel{48\sin(t)\cos(t)} + 0$$

$= 48$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 48\,dt = 48t\Big|_0^{2\pi} = 96\pi$$

$\vec{F}(x,y,z) =$

$\langle ye^z + 2x,\ xe^z - 3,\ xye^z + \cos(z) \rangle$

$\vec{F}$ conservative if and only if $\text{curl}(\vec{F}) = \vec{0}$

$$\text{curl}(\vec{F}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$\times \langle ye^z + 2x,\ xe^z - 3,\ xye^z + \cos(z) \rangle$$

$$= \left\langle \frac{\partial}{\partial y}(xye^z + \cos(z)) - \frac{\partial}{\partial z}(xe^z - 3), \right.$$

$$- \left(\frac{\partial}{\partial x}(xye^z + \cos(z)) - \frac{\partial}{\partial z}(ye^z + 2x) \right),$$

$$\left. \frac{\partial}{\partial x}(xe^z - 3) - \frac{\partial}{\partial y}(ye^z + 2x) \right\rangle$$

$$= \langle xe^z - xe^z, -(ye^z - ye^z), e^z - e^z \rangle$$

$$= \langle 0, 0, 0 \rangle$$ Thus, conservative.

We can thus find $f(x,y,z)$ such that

$$\nabla f = \langle ye^z + 2x, \ xe^z - 3, \ xye^z + \cos(z) \rangle$$

ie $f_x = ye^z + 2x$, $f_y = xe^z - 3$, $f_z = xye^z + \cos(z)$.

$$f_x = ye^z + 2x \Rightarrow f(x,y,z) = \int ye^z + 2x \, dx = xye^z + x^2 + h(y,z)$$

$$\Rightarrow f_y = xe^z + h_y(y,z)$$ This should equal $xe^z - 3$

$$\text{So } h_y = -3 \Rightarrow h(y,z) = \int -3 \, dy = -3y + g(z)$$

$$\Rightarrow f(x,y,z) = xye^z + x^2 - 3y + g(z)$$

$\Rightarrow f_z = xye^z + g'(z)$

This should equal $xye^z + \cos(z)$ so $g'(z) = \cos(z)$

$\Rightarrow g(z) = \sin(z) + K$ (K constant)

$\Rightarrow f(x,y,z) = xye^z + x^2 - 3y + \sin(z) + K$

$f_x = ye^z + 2x$, $f_y = xe^z - 3$, $f_z = xye^z + \cos(z)$

$f = \int ye^z + 2x \, dx$ $= xye^z + x^2$

$f = \int xe^z - 3 \, dy$ $= xye^z - 3y$

$f = \int xye^z + \cos(z) \, dz$ $= xye^z + \sin(z)$

$f(x,y,z) = xye^z + x^2 - 3y + \sin(z) + K$

If I start at $(1,1,0)$ and end at $(5,6,0)$

compute $\int_C \vec{F} \cdot d\vec{r}$

Since $\vec{F}$ is <u>conservative</u>, by the <u>Fund. Thm of Line Int.</u>,

we have $\int_C \vec{F} \cdot d\vec{r} = \underline{f(5,6,0) - f(1,1,0)}$

$= (30 + 25 - 18) - (1 + 1 - 3)$

$= 37 - (-1) = 38$

$f(x,y,z) = x^2yz - e^y\cos(z) + \ln(x) - y^3z^4 + 8$

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2xyz + \frac{1}{x},\ x^2z - e^y\cos(z) - 3y^2z^4,\ x^2y + e^y\sin(z) - 4y^3z^3 \rangle$

$\Delta f = f_{xx} + f_{yy} + f_{zz}$

$= 2yz - \frac{1}{x^2} - e^y\cos(z) - 6yz^4 + e^y\cos(z) - 12y^3z^2$

Let R be the region on and inside the sphere of radius 3. Let S be the sphere itself.

Let $\vec{F}(x,y,z) = \langle x, y, z \rangle$

1) Compute $\iint_S \vec{F} \cdot d\vec{S}$, using

$\vec{r}(u,v) = \langle 3\cos(u)\sin(v),\ 3\sin(u)\sin(v),\ -3\cos(v) \rangle$

$0 \le v \le \pi$

$0 \le u \le 2\pi$

2) Compute $\iiint_R \operatorname{div}(\vec{F})\, dV$

3) They should be equal. Why?

1) $\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi} \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dv\, du$

$\vec{r}_u = \langle -3\sin(u)\sin(v),\ 3\cos(u)\sin(v),\ 0 \rangle$

$\vec{r}_v = \langle 3\cos(u)\cos(v),\ 3\sin(u)\cos(v),\ 3\sin(v) \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 9\cos(u)\sin^2(v),\ 9\sin(u)\sin^2(v),\ -9\sin(v)\cos(v) \rangle$

$\vec{F}(\vec{r}(u,v)) = \langle 3\cos(u)\sin(v),\ 3\sin(u)\sin(v),\ -3\cos(v) \rangle$

$$\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = 27\cos^2(u)\sin^3(v) + 27\sin^2(u)\sin^3(v) + 27\sin(v)\cos^2(v)$$
$$= 27\sin^3(v) + 27\sin(v)\cos^2(v)$$
$$= 27\sin(v)$$

So $\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi} 27\sin(v)\,dv\,du = -108\pi$

2) $\operatorname{div}(\vec{F}) = 3$

So $\iiint_R \operatorname{div}(\vec{F})\,dV = \iiint_R 3\,dV$
$$= 3\,(\text{volume of } R)$$
$$= 3\left(\tfrac{4}{3}\pi(3)^3\right)$$
$$= 108\pi$$

3) Divergence Theorem