Recall, a linear second-order DE has the form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

Our focus in this class will be when $a_2(x)$, $a_1(x)$, $a_0(x)$ are constant functions. That is, the DE has the form $ay'' + by' + cy = f(x)$ where $a, b, c \in \mathbb{R}$, $a \neq 0$.

We will start by looking for solutions to the homogeneous version of the DE $ay'' + by' + cy = 0$

Specifically, we will look for two linearly independent solutions to the DE.

Definition - Two functions $y_1(x)$ and $y_2(x)$ are linearly independent if neither of them can be written as a constant multiple of the other

We can check if two functions are linearly independent in the following way:

Two functions $y_1(x)$ and $y_2(x)$ are linearly independent on $(-\infty, \infty)$ if $y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0$ for all $x \in (-\infty, \infty)$

The expression $y_1(x)y_2'(x) - y_1'(x)y_2(x)$ is called the Wronskian of y_1 and y_2

Ex - 1) $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$ with $r_1 \neq r_2$

2) $y_1 = e^{rx}$, $y_2 = xe^{rx}$

Check for linear independence on $(-\infty, \infty)$

1) $y_1' = r_1 e^{r_1 x}$ $\qquad y_2' = r_2 e^{r_2 x}$

So $y_1 y_2' - y_1' y_2 = e^{r_1 x} r_2 e^{r_2 x} - r_1 e^{r_1 x} e^{r_2 x}$

$= r_2 e^{(r_1+r_2)x} - r_1 e^{(r_1+r_2)x}$

$= \underbrace{(r_2 - r_1)}_{\neq 0} \underbrace{e^{(r_1+r_2)x}}_{\neq 0} \neq 0$

Thus, y_1, y_2 lin. independent on $(-\infty, \infty)$

2) $y_1' = r e^{rx}$, $\qquad y_2' = rxe^{rx} + e^{rx}$

So $y_1 y_2' - y_1' y_2 = e^{rx}(rxe^{rx} + e^{rx}) - re^{rx} xe^{rx}$

$= rxe^{2rx} + e^{2rx} - rxe^{2rx}$

$= e^{2rx} \neq 0$

Thus y_1, y_2 lin. independent on $(-\infty, \infty)$

Theorem - If $y_1(x)$ and $y_2(x)$ are two solutions to the DE $ay'' + by' + cy = 0$, that are linearly independent on $(-\infty, \infty)$, then the general solution to the DE can be expressed as $y = c_1 y_1 + c_2 y_2$ for constants c_1, c_2.

Now, looking at $ay'' + by' + cy = 0$, we see that solutions to the DE have the property that y'' is a linear combination of y' and y. This motivates us to look for solutions of the form $y = e^{rx}$

If $y = e^{rx}$, then $y' = re^{rx}$ and $y'' = r^2 e^{rx}$

Plugging these into the DE yields

$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$ ie

$e^{rx}(ar^2 + br + c) = 0$ ie

$ar^2 + br + c = 0$

Thus, $y = e^{rx}$ is a solution to the DE $ay'' + by' + cy = 0$ if and only if

$ar^2 + br + c = 0$

The equation $ar^2 + br + c = 0$ is called the <u>auxiliary equation</u> for the DE.

Case 1: two distinct real solutions to the auxiliary equation

If the auxiliary equation has two distinct real solutions r_1, r_2 ($r_1 \neq r_2$) then $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are both solutions to the homogeneous DE.

We previously showed that such functions are linearly independent on $(-\infty, \infty)$

Thus, we have the following:

Fact: If the auxiliary equation $ar^2+br+c=0$ for the homogeneous DE $ay''+by'+cy=0$ has two distinct real solutions r_1, r_2, then the general solution to the DE is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case 2: One repeated solution to the auxiliary equation

If the auxiliary equation has one repeated solution r, then $y_1 = e^{rx}$ is a solution to the DE. We need a second solution that is linearly independent to y_1

Ex- Show that if r is a repeated solution of $ar^2+br+c=0$, then $y_2 = xe^{rx}$ is a solution of $ay''+by'+cy=0$

Observe that solutions of $ar^2+br+c=0$

are $r = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

If r is a repeated root, $b^2-4ac=0$

so $r = \dfrac{-b}{2a}$ ie $2ar = -b$ ie

$2ar + b = 0$

$y_2 = xe^{rx} \Rightarrow y_2' = rxe^{rx} + e^{rx}$

$\Rightarrow y_2'' = r^2xe^{rx} + 2re^{rx}$

Then $ay_2'' + by_2' + cy_2' = a(r^2xe^{rx} + 2re^{rx})$
$+ b(rxe^{rx} + e^{rx}) + cxe^{rx}$

$= ar^2xe^{rx} + 2are^{rx} + brxe^{rx} + be^{rx} + cxe^{rx}$

$= \underbrace{(ar^2 + br + c)}_{=0}xe^{rx} + \underbrace{(2ar+b)}_{=0}e^{rx}$

$= 0$ ✓

Thus, $y_2 = xe^{rx}$ is a solution to the DE as well

We previously showed that $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent on $(-\infty, \infty)$, so we have the following:

Fact - If the auxillary equation $ar^2 + br + c = 0$ for the DE $ay'' + by' + cy = 0$ has one repeated solution r, then the general solution to the DE is $y = c_1 e^{rx} + c_2 x e^{rx}$

Case 3: complex conjugate solutions to the auxillary equation

Recall Euler's formula which states $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Note this implies $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$
$= \cos(\theta) - i\sin(\theta)$
since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$

Now, suppose the auxiliary equation has complex conjugate solutions $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, $\beta \neq 0$

Then $r_1 \neq r_2$ and so $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are two linearly independent solution to the DE $ay'' + by' + cy = 0$

Observe then that the general solution becomes

$$
\begin{aligned}
y &= c_1 e^{r_1 x} + c_2 e^{r_2 x} \\
&= c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} \\
&= c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{-i\beta x} \\
&= e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right) \\
&= e^{\alpha x}\left[c_1\left(\cos(\beta x) + i\sin(\beta x)\right) + c_2\left(\cos(\beta x) - i\sin(\beta x)\right)\right] \\
&= e^{\alpha x}\left[(c_1 + c_2)\cos(\beta x) + (c_1 i - c_2 i)\sin(\beta x)\right] \\
&= e^{\alpha x}\left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right)
\end{aligned}
$$

Thus, we have the following:

Fact- If the auxillary equation $ar^2 + br + c = 0$ for the DE $ay'' + by' + cy = 0$ has complex conjugate solutions $\alpha \pm \beta i$, then the general solution to the DE is $y = e^{\alpha x}\left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right)$

Ex- 1) $y'' + 2y' - 8y = 0$

2) $y'' - 3y' - 10y = 0$

3) $y'' + 6y' + 9y = 0$

4) $y'' - 2y' + y = 0$

5) $y'' - 4y' + 13y = 0$

6) $y'' + 6y' + 10y = 0$

7) $y'' + 25y = 0$

1) Auxillary equation is

$$r^2 + 2r - 8 = 0$$

$(r+4)(r-2) = 0$ so $r_1 = -4,\ r_2 = 2$

Thus $y = c_1 e^{-4x} + c_2 e^{2x}$

2) Auxiliary equation is $r^2 - 3r - 10 = 0$

ie $(r-5)(r+2) = 0$ so $r_1 = 5,\ r_2 = -2$

Thus $y = c_1 e^{5x} + c_2 e^{-2x}$

3) Auxiliary equation is $r^2 + 6r + 9 = 0$

ie $(r+3)^2 = 0$ so $r = -3$ repeated

Thus $y = c_1 e^{-3x} + c_2 x e^{-3x}$

4) Auxiliary equation is $r^2 - 2r + 1 = 0$

ie $(r-1)^2 = 0$ so $r = 1$ repeated

Thus $y = c_1 e^{x} + c_2 x e^{x}$

5) Auxilliary equation is $r^2 - 4r + 13 = 0$

so $r = \dfrac{4 \pm \sqrt{16 - 4(13)}}{2} = \dfrac{4 \pm \sqrt{-36}}{2}$

$= \dfrac{4 \pm 6i}{2} = 2 \pm 3i$

$\Rightarrow$ $y = e^{2x}(c_1 \cos(3x) + c_2 \sin(3x))$

6) Auxilliary equation is $r^2 + 6r + 10 = 0$

so $r = \dfrac{-6 \pm \sqrt{36 - 4(10)}}{2} = \dfrac{-6 \pm \sqrt{-4}}{2}$

$= \dfrac{-6 \pm 2i}{2} = -3 \pm i$

$\Rightarrow$ $y = e^{-3x}(c_1 \cos(x) + c_2 \sin(x))$

7) Auxilliary equation is $r^2 + 25 = 0$

so $r = \pm 5i$

Thus $y = c_1 \cos(5x) + c_2 \sin(5x)$

Ex - $y'' + 2y' - 8y = 0$

$y(0) = 5, \quad y'(0) = -2$

General solution is $y = c_1 e^{-4x} + c_2 e^{2x}$

$5 = y(0) = c_1 + c_2$

$y' = -4c_1 e^{-4x} + 2c_2 e^{2x}$

$-2 = y'(0) = -4c_1 + 2c_2$

$5 = c_1 + c_2 \Rightarrow c_1 = 5 - c_2$

So $-2 = -4c_1 + 2c_2 = -4(5 - c_2) + 2c_2$

ie $-2 = -20 + 6c_2 \Rightarrow c_2 = 3$

Then $c_1 = 5 - c_2 = 2$

Therefore $y = 2e^{-4x} + 3e^{2x}$

Recall the Wronskian of two functions y_1, y_2:

$$y_1 y_2' - y_1' y_2$$

We can think of this as the determinant of the <u>Wronskian matrix</u> of y_1 and y_2:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

In general, the determinant of a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad-bc$