A <u>vector</u> is a list of objects (usually numbers or functions), represented as either a row or column.

The number of objects in the list determine the <u>dimension</u> of the vector

Ex - $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$ are 3-dim vectors.

If two vectors have the same dimension, we can add/subtract them component-wise ie

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \pm \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 \pm v_1 \\ u_2 \pm v_2 \\ \vdots \\ u_n \pm v_n \end{bmatrix}$$

We can multiply a vector by a constant by multiplying each component by the constant ie $k\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} ku_1 \\ ku_2 \\ \vdots \\ ku_n \end{bmatrix}$

If two vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

have the same dimension, we can compute their <u>dot product</u> as follows:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Ex. $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

1) $3\vec{u} - 2\vec{v}$

2) $\vec{u} + 4\vec{v}$

3) $\vec{u} \cdot \vec{v}$

1) $3\vec{u} - 2\vec{v} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} - \begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} = \begin{bmatrix} -5 \\ -4 \\ -3 \end{bmatrix}$

2) $\vec{u} + 4\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 16 \\ 20 \\ 24 \end{bmatrix} = \begin{bmatrix} 17 \\ 22 \\ 27 \end{bmatrix}$

3) $\vec{u} \cdot \vec{v} = 1(4) + 2(5) + 3(6) = 32$

A <u>matrix</u> is an array of objects (numbers, functions, etc.) consisting of both rows and columns

We refer to a matrix with m rows and n columns as an $m \times n$ matrix

Ex - $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ has size 2×3

$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ has size 3×2

For a matrix A, we will use a_{ij} to denote the entry in the i-th row and j-th column

The i-th row will be denoted A_{i*}

The j-th column will be denoted A_{*j}

Ex - $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

1) $a_{12} = ?$

2) $a_{21} = ?$

3) $A_{3*} = ?$

4) $A_{*3} = ?$

1) $a_{12} = 2$

2) $a_{21} = 4$

3) $A_{3*} = \begin{bmatrix} 7 & 8 & 9 \end{bmatrix}$

4) $A_{*3} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$

If two matrices A, B have the same size, we can add/subtract them componentwise ie the ij-th entry of $A \pm B$ is $a_{ij} \pm b_{ij}$

We can multiply a matrix by a scalar by multiplying each entry by the scalar.

ie the ij-th entry of kA is ka_{ij}

Ex - $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $B = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

1) $3A - B$

2) $A + 2B$

1) $3A - B = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \\ 21 & 24 & 27 \end{bmatrix} - \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -2 & 2 \\ 6 & 10 & 14 \\ 18 & 22 & 26 \end{bmatrix}$

2) $A + 2B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 18 & 16 & 14 \\ 12 & 10 & 8 \\ 6 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 19 & 18 & 17 \\ 16 & 15 & 14 \\ 13 & 12 & 11 \end{bmatrix}$

Let A be an $m \times n$ matrix and $\vec{x}$ an n-dimensional column vector. The product $A\vec{x}$ is defined as follows:

$$A\vec{x} = \begin{bmatrix} A_{1*} \cdot \vec{x} \\ A_{2*} \cdot \vec{x} \\ \vdots \\ A_{m*} \cdot \vec{x} \end{bmatrix}$$

Observe output is m-dimensional column vector

Ex- 1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

1) $$A\vec{x} = \begin{bmatrix} [1\ 2\ 3] \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \\ [4\ 5\ 6] \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1-2+6 \\ 4-5+12 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

2) $$A\vec{x} \quad \begin{bmatrix} [1\ 4] \cdot \begin{bmatrix} 5 \\ -2 \end{bmatrix} \\ [2\ 5] \cdot \begin{bmatrix} 5 \\ -2 \end{bmatrix} \\ [3\ 6] \cdot \begin{bmatrix} 5 \\ -2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 5-8 \\ 10-10 \\ 15-12 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}$$

Let A be an $m \times n$ matrix and B an $n \times p$ matrix. Then the product AB is an $m \times p$ matrix and is computed as follows:

$$AB = \begin{bmatrix} AB_{*1} & AB_{*2} & \cdots & AB_{*p} \end{bmatrix}$$

Ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 7 \\ -1 & 5 \\ 2 & -3 \end{bmatrix}$

1) AB

2) BA

1) $AB = \begin{bmatrix} A\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} & A\begin{bmatrix} 7 \\ 5 \\ -3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} [1\ 2\ 3] \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} & [1\ 2\ 3] \cdot \begin{bmatrix} 7 \\ 5 \\ -3 \end{bmatrix} \\ [4\ 5\ 6] \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} & [4\ 5\ 6] \cdot \begin{bmatrix} 7 \\ 5 \\ -3 \end{bmatrix} \end{bmatrix}$

$$= \begin{bmatrix} 1 - 2 + 6 & 7 + 10 - 9 \\ 4 - 5 + 12 & 28 + 25 - 18 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 8 \\ 11 & 35 \end{bmatrix}$$

2) $BA = \begin{bmatrix} B\begin{bmatrix}1\\4\end{bmatrix} & B\begin{bmatrix}2\\5\end{bmatrix} & B\begin{bmatrix}3\\6\end{bmatrix} \end{bmatrix}$

$$= \begin{bmatrix} [1\ 7]\cdot\begin{bmatrix}1\\4\end{bmatrix} & [1\ 7]\cdot\begin{bmatrix}2\\5\end{bmatrix} & [1\ 7]\cdot\begin{bmatrix}3\\6\end{bmatrix} \\ [-1\ 5]\cdot\begin{bmatrix}1\\4\end{bmatrix} & [-1\ 5]\cdot\begin{bmatrix}2\\5\end{bmatrix} & [-1\ 5]\cdot\begin{bmatrix}3\\6\end{bmatrix} \\ [2\ -3]\cdot\begin{bmatrix}1\\4\end{bmatrix} & [2\ -3]\cdot\begin{bmatrix}2\\5\end{bmatrix} & [2\ -3]\cdot\begin{bmatrix}3\\6\end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1+28 & 2+35 & 3+42 \\ -1+20 & -2+25 & -3+30 \\ 2-12 & 4-15 & 6-18 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 37 & 45 \\ 19 & 23 & 27 \\ -10 & -11 & -12 \end{bmatrix}$$

Observe $AB \neq BA$ (not even same size)

If A is 4×3 and B is 3×5

AB exists but BA does not.

A matrix with the same number of rows and columns is called a <u>square matrix</u>

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

A square matrix whose only nonzero entries occur on the main diagonal is a <u>diagonal matrix</u> e.g $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

A diagonal matrix with all 1s on the main diagonal is called an <u>identity matrix</u>

e.g $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The $n \times n$ ~~diagona~~ identity matrix satisfies

$I\vec{x} = \vec{x}$ for all n-dim vectors $\vec{x}$

$IA = A$ for all matrices A with n rows

$AI = A$ for all matrices A with n columns

Properties of matrix algebra - For $m \times n$ matrices A, B, C and scalars k, r:

1) $A + B = B + A$

2) $A + (B+C) = (A+B) + C$

3) $k(A+B) = kA + kB$

4) $(k+r)A = kA + rA$

5) $k(rA) = (kr)A = r(kA)$

6) $(AB)C = A(BC)$ if defined

7) $A(B+C) = AB + AC$ if defined

8) $(A+B)C = AC + BC$ if defined

9) $(rA)B = r(AB) = A(rB)$ if defined

Note $AB \neq BA$ in general.

For an $m \times n$ matrix A, the $n \times m$ matrix obtained by interchanging the rows and columns of A is called the <u>transpose</u> of A, denoted A^T. In other words, the ij-th entry of A^T is a_{ji}.

Ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

For an $n \times n$ matrix A, the <u>Inverse</u> of A is the $n \times n$ matrix B satisfying $AB = I_n$ and $BA = I_n$. We denote this as $B = A^{-1}$

Ex - $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 7 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 6 & -11 & 3 \\ -1 & 1 & 0 \\ -1 & 3 & -1 \end{bmatrix}$

Show $B = A^{-1}$.

$$AB = \begin{bmatrix} 6-2-3 & -11+2+9 & 3+0-3 \\ 6-3-3 & -11+3+12 & 3+0-3 \\ 12-7-5 & -22+7+15 & 6+0-5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 6-11+6 & 12-33+21 & 18-33+15 \\ -1+1+0 & -2+3+0 & -3+3+0 \\ -1+3-2 & -2+9-7 & -3+9-5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the <u>determinant</u> of A is $\det(A) = a_{11}a_{22} - a_{21}a_{12}$

For a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

the determinant is

$$\det(A) = a_{11} \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$+ a_{13} \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Ex- 1) $A = \begin{bmatrix} 1 & 2 \\ -2 & 7 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

1) $\det(A) = 1(7) - (-2)(2) = 7 + 4 = 11$

2) $\det(A) = 1 \cdot \det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2\det\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3\det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$

$= 1(5(9) - 8(6)) - 2(4(9) - 7(6)) + 3(4(8) - 7(5))$

$$= -3 - 2(-6) + 3(3) = 0$$

Consider a linear system of n equations in n variables

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned} \quad (*)$$

Example -

$$\begin{aligned}
x_1 + 2x_2 + 3x_3 &= 8 \\
2x_1 + 5x_2 - x_3 &= -6 \\
-x_1 + x_2 + 2x_3 &= 4
\end{aligned}$$

$$\xrightarrow[\text{Eq1} + \text{Eq3}]{-2\cdot\text{Eq1} + \text{Eq2}} \quad \begin{aligned}
x_1 + 2x_2 + 3x_3 &= 8 \\
x_2 - 7x_3 &= -22 \\
3x_2 + 5x_3 &= 12
\end{aligned}$$

$$\xrightarrow{-3\cdot\text{Eq2} + \text{Eq3}} \quad \begin{aligned}
x_1 + 2x_2 + 3x_3 &= 8 \\
x_2 - 7x_3 &= -22 \\
26x_3 &= 78
\end{aligned}$$

So $26x_3 = 78 \Rightarrow x_3 = 3$

Then $x_2 - 7x_3 = -22$

$\Rightarrow x_2 - 21 = -22 \Rightarrow x_2 = -1$

Then $x_1 + 2x_2 + 3x_3 = 8$

$\Rightarrow x_1 - 2 + 9 = 8 \Rightarrow x_1 = 1$

Notice we can think of the system in this way

$$\begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + 5x_2 - x_3 \\ -x_1 + x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} [1\ 2\ 3] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ [2\ 5\ -1] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ [-1\ 1\ 2] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & -1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \\ 4 \end{bmatrix}$$

$$\Rightarrow \quad A\vec{x} = \vec{b}$$

In general, we can write the system $*$

as
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$
$$A \qquad\qquad \vec{x} \quad = \quad \vec{b}$$

The matrix $[A \mid \vec{b}]$ is called the <u>augmented coefficient matrix</u> for the system.

We can then do the elimination operations on this matrix to solve the system, keeping in mind that the jth column of A corresponds to the variable x_j

Ex - System above
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 2 & 5 & -1 & -6 \\ -1 & 1 & 2 & 4 \end{array}\right]$$

$$\xrightarrow[R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & 1 & -7 & -22 \\ 0 & 3 & 5 & 12 \end{array}\right]$$

$$\xrightarrow{-3R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & 1 & -7 & -22 \\ 0 & 0 & 26 & 78 \end{array}\right]$$

$$\xrightarrow{\frac{1}{26} R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & 1 & -7 & -22 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[-3R_3 + R_1 \to R_1]{7R_3 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{-2R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Ex - 1)

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 3x_1 + x_2 + 2x_3 &= 5 \\ x_1 + x_3 &= 1 \end{aligned}$$

2) $$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 4x_1 + 5x_2 + 6x_3 &= 7 \\ 7x_1 + 8x_2 + 9x_3 &= 11 \end{aligned}$$

3) $$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 4x_1 + 5x_2 + 6x_3 &= 7 \\ 7x_1 + 8x_2 + 9x_3 &= 13 \end{aligned}$$

1) $$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 3 & 1 & 2 & 5 \\ 1 & 0 & 1 & 1 \end{array}\right] \xrightarrow[-R_1 + R_3 = R_3]{-3R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 4 & -1 & 5 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 4 & -1 & 5 \end{array}\right] \xrightarrow{-4R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{-R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{-R_3 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

So $x_1 = 2$
$x_2 = 1$
$x_3 = -1$

2) $$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 11 \end{array}\right] \xrightarrow[-7R_1+R_3 \to R_3]{-4R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 3 \\ 0 & -6 & -12 & 4 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -6 & -12 & 4 \end{array}\right] \xrightarrow{6R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

Last row Implies $0 = -2$. False

So no solution.

3) $$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 7 \\ 7 & 8 & 9 & 13 \end{array}\right] \xrightarrow[-7R_1+R_3 \to R_3]{-4R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 3 \\ 0 & -6 & -12 & 6 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -6 & -12 & 6 \end{array}\right] \xrightarrow{6R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{aligned} x_1 - x_3 &= 3 \\ x_2 + 2x_3 &= -1 \end{aligned}$$

Let $x_3 = s$. Then

$$\begin{aligned} x_1 &= 3 + s \\ x_2 &= -1 - 2s \\ x_3 &= s \end{aligned} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Infinitely many solutions.