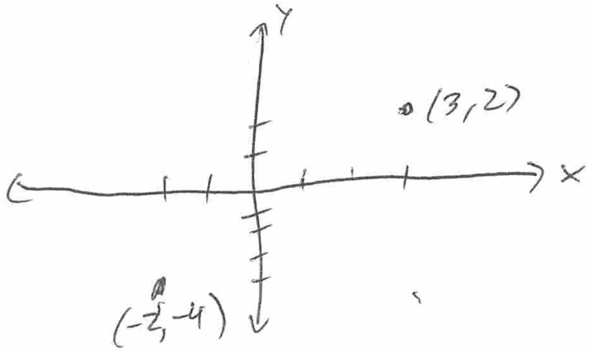
Cartesian Plane ($\mathbb{R}^2$, xy plane)

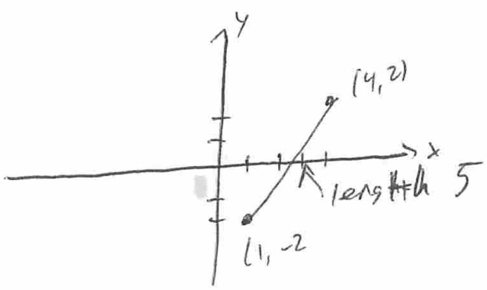


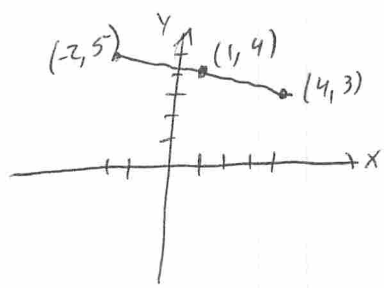
Distance formula for $\mathbb{R}^2$: The distance between (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

<u>Midpoint</u> of (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Example - a) Find distance between $(1, -2)$ and $(4, 2)$

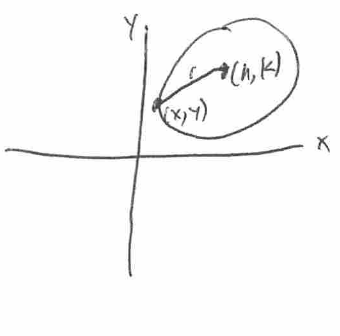
b) Find midpoint of $(4, 3)$ and $(-2, 5)$





a) $D = \sqrt{(4-1)^2 + (2+2)^2} = \sqrt{25} = 5$

b) $M = \left(\frac{4-2}{2}, \frac{3+5}{2}\right) = (1, 4)$



By distance formula

$r = \sqrt{(x-h)^2 + (y-k)^2}$

$\Rightarrow r^2 = (x-h)^2 + (y-k)^2$

Equation of a circle centered at (h, k) with radius r.

Example - Show that $x^2 - 6x + y^2 + 4y - 3 = 0$ is the equation of a circle. Find center/radius

Recall - to complete the square of an expression $x^2 + bx$, add and subtract $\frac{b^2}{4}$ ie $x^2 + bx = x^2 + bx + \frac{b^2}{4} - \frac{b^2}{4}$

$$= \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4}$$

$$x^2 - 6x + y^2 + 4y - 3 = 0$$

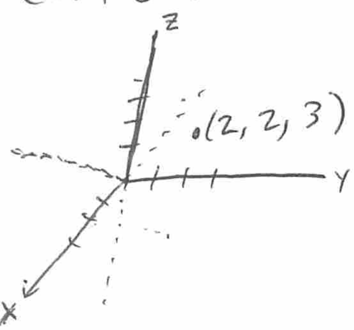
$$x^2 - 6x + 9 - 9 + y^2 + 4y + 4 - 4 - 3 = 0$$

$$(x-3)^2 - 9 + (y+2)^2 - 4 - 3 = 0$$

$$(x-3)^2 + (y+2)^2 = 16$$

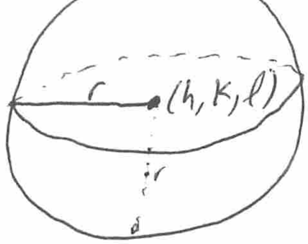
Center is $(3, -2)$, $r = 4$

Cartesian coordinates in three dimensions $(\mathbb{R}^3)$



Distance formula in $\mathbb{R}^3$: distance from (x_1, y_1, z_1) to (x_2, y_2, z_2) is $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

Midpoint formula between (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$



Equation of sphere with radius r, center (h, k, l)
is $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

Examples - 1) Show that $x^2 - 10x + y^2 + 2y + z^2 - 8z - 7 = 0$
is equation of a sphere.
Find radius / center

2) Find equation of the sphere that has $(7, 2, -4)$ and $(3, 10, 4)$ as endpoints of a diameter of the sphere.

1) $x^2 - 10x + 25 - 25 + y^2 + 2y + 1 - 1 + z^2 - 8x + 16 - 16 - 7 = 0$

$(x-5)^2 - 25 + (y+1)^2 - 1 + (z-4)^2 - 16 - 7 = 0$

$(x-5)^2 + (y+1)^2 + (z-4)^2 = 49$

Center is $(5, -1, 4)$, $r = 7$

2) Center is midpoint $= \left(\frac{7+3}{2}, \frac{2+10}{2}, \frac{-4+4}{2}\right) = (5, 6, 0)$

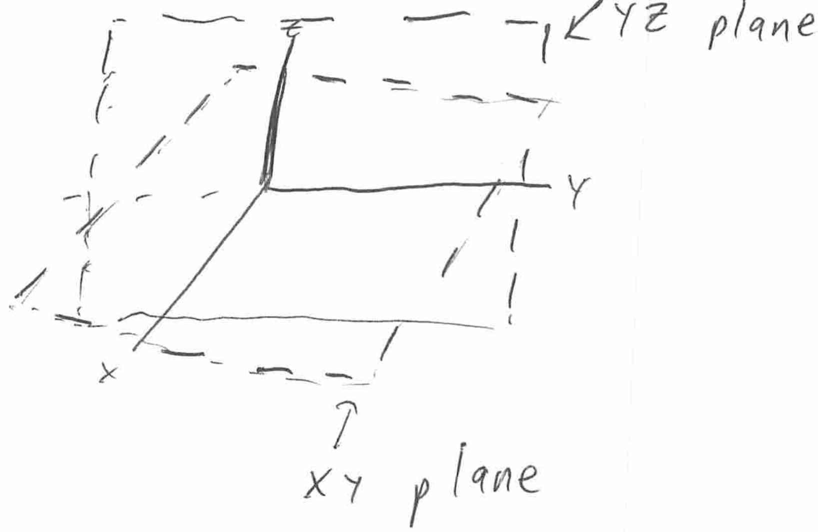
Radius is $\frac{\text{distance}}{2}$

$$\text{Distance} = \sqrt{(3-7)^2 + (10-2)^2 + (4+4)^2} = \sqrt{144}$$
$$= 12$$

So radius $= 6$

$$(x-5)^2 + (y-6)^2 + (z-0)^2 = 36$$

Planes

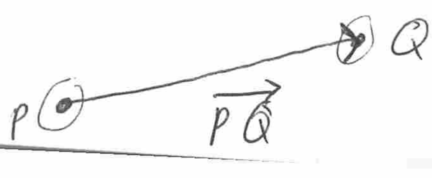


A <u>vector</u> is a geometric object with both magnitude and direction.

It is represented as a directed line segment between two points P and Q, denoted $\overrightarrow{PQ}$

The magnitude of $\overrightarrow{PQ}$ is the distance between P and Q.

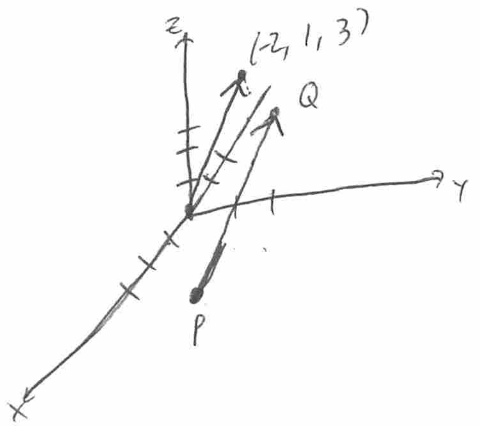
The direction of $\overrightarrow{PQ}$ is <u>from</u> P <u>to</u> Q



If $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$,
then $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

Example. Find $\overrightarrow{PQ}$ for $P = (3, 1, 0)$ and
$Q = (1, 2, 3)$

$$\overrightarrow{PQ} = \langle 1-3, 2-1, 3-0 \rangle = \langle -2, 1, \overset{3}{\cancel{0}} \rangle$$



A vector that begins at the origin $O = (0,0,0)$
is a <u>position vector</u>

Example: $P = (-2, 1, 3)$
Find the position vector defined by P

$$\overrightarrow{OP} = \langle -2-0, 1-0, 3-0 \rangle = \langle -2, 1, 3 \rangle$$

Two vectors with the same magnitude and direction are said to be <u>equivalent</u>.

Equivalent vectors have identical representations

Example - $P_1 = (3, 1, 0)$, $Q_1 = (1, 2, 3)$

$P_2 = (-1, 4, 1)$.

Find Q_2 so that $\overrightarrow{P_1Q_1}$ is equivalent to $\overrightarrow{P_2Q_2}$.

$\overrightarrow{P_1Q_1} = \langle -2, 1, 3 \rangle$

If $Q_2 = (x, y, z)$ is such that $\overrightarrow{P_2Q_2} = \langle -3, 1, 3 \rangle$

then $\langle x+1, y-4, z-1 \rangle = \langle -2, 1, 3 \rangle$

$\Rightarrow x = -3, \ y = 5, \ z = 4$

The sum of two vectors $\overrightarrow{PQ} = \langle a_1, a_2, a_3 \rangle$ and $\overrightarrow{RS} = \langle b_1, b_2, b_3 \rangle$ is $\overrightarrow{PQ} + \overrightarrow{RS} = \langle a_1+b_1, a_2+b_2, a_3+b_3 \rangle$

Triangular law of vector addition - if P, Q, R are all points in $\mathbb{R}^3$, then $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

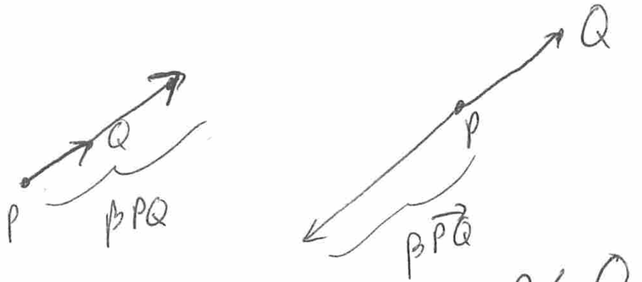
Example - Verify this law for $P = (3, 1, 0)$, $Q = (1, 2, 3)$ and $R = (4, 5, 6)$

$\overrightarrow{PQ} = \langle -2, 1, 3 \rangle$, $\overrightarrow{QR} = \langle 3, 3, 3 \rangle$, $\overrightarrow{PR} = \langle 1, 4, 6 \rangle$

$\overrightarrow{PQ} + \overrightarrow{QR} = \langle 1, 4, 6 \rangle = \overrightarrow{PR}$

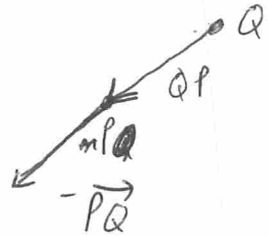
...he product of a scalar $\beta \in \mathbb{R}$ and a vector ...

$= \langle a_1, a_2, a_3 \rangle$ is $\beta \overrightarrow{PQ} = \langle \beta a_1, \beta a_2, \beta a_3 \rangle$



$\beta > 0$ $\beta < 0$

$-\overrightarrow{PQ} = \overrightarrow{QP}$



$P = (3, 1, 0)$ $Q = (1, 2, 3)$

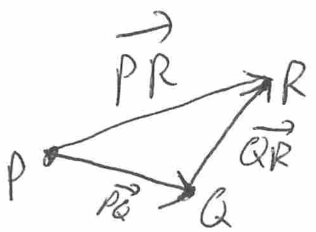
$\overrightarrow{QP} = \langle 2, -1, -3 \rangle$

$\overrightarrow{PQ} = \langle -2, 1, 3 \rangle$

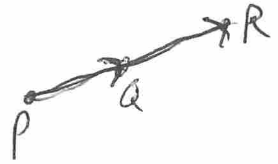
$-\overrightarrow{PQ} = (-1)\overrightarrow{PQ} = \langle 2, -1, -3 \rangle = \overrightarrow{QP}$

Ex - $\overrightarrow{PQ} - \overrightarrow{RQ} = ?$

$\overrightarrow{PQ} + (-\overrightarrow{RQ}) = \underbrace{\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}}_{\text{triangle law}}$



Fact - Let P, Q, R be points and $\beta \in \mathbb{R}$ a scalar such that $\overrightarrow{PR} = \beta \overrightarrow{PQ}$



Then $d(P,R) = |\beta|\, d(P,Q)$

where d represents distance between points.

Example. $P = (1,2,3)$ $\quad Q = (2,4,1)$

Find R such that $\overrightarrow{PR} = \beta \overrightarrow{PQ}$ and verify the above fact for

1) $\beta = 4$

2) $\beta = -2$

1) $\overrightarrow{PQ} = \langle 1, 2, -2 \rangle$

$4\overrightarrow{PQ} = \langle 4, 8, -8 \rangle$

Let $R = (x, y, z)$, then $\overrightarrow{PR} = \langle x-1, y-2, z-3 \rangle$

If this $= \langle 4, 8, -8 \rangle$, then $x-1 = 4$, $y-2 = 8$, $z-3 = -8$ so $R = (5, 10, -5)$

$d(P,Q) = \sqrt{(2-1)^2 + (4-2)^2 + (1-3)^2} = \sqrt{9} = 3$

$d(P,R) = \sqrt{(5-1)^2 + (10-2)^2 + (-5-3)^2} = \sqrt{144} = 12$

Let $P = (x, y, z)$, What is $\overrightarrow{PP}$??

$\overrightarrow{PP} = \langle x-x, y-y, z-z \rangle = \langle 0, 0, 0 \rangle$

The <u>zero vector</u> is $\vec{0} = \langle 0, 0, 0 \rangle$

Properties: For vectors $\vec{a}, \vec{b}, \vec{c}$ and scalars $\beta, \gamma \in \mathbb{R}$

1) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

2) $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

3. $\vec{a} + \vec{0} = \vec{a}$

4. $\vec{a} + (-\vec{a}) = \vec{0}$

5. $\beta(\vec{a} + \vec{b}) = \beta\vec{a} + \beta\vec{b}$

6) $(\beta + \gamma)\vec{a} = \beta\vec{a} + \gamma\vec{a}$

7) $(\beta\gamma)\vec{a} = \beta(\gamma\vec{a})$

8) $1\vec{a} = \vec{a}$

9) $0\vec{a} = \vec{0}$

The <u>magnitude</u> of $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Example - 1) Find $\|\vec{a}\|$ for $\vec{a} = \langle 1, 2, 2 \rangle$

2) For points P, Q what is $\|\overrightarrow{PQ}\|$?

3) If $\|\vec{a}\| = 0$, what is $\vec{a}$?

1) $\|\vec{a}\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$

2) $P = (x_1, y_1, z_1)$ $Q = (x_2, y_2, z_2)$

Then $\vec{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

So $\|\vec{PQ}\| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} = d(P,Q)$

Fact- $\|\vec{a}\| = 0$ if and only if $\vec{a} = \vec{0}$

A <u>unit vector</u> is a vector with magnitude 1.

Two vectors $\vec{a}, \vec{b}$ are <u>parallel</u> if there exists $\beta \in \mathbb{R}$ such that $\vec{b} = \beta \vec{a}$

If $\beta > 0$, $\vec{a}$ and $\vec{b}$ have same direction

If $\beta < 0$, $\vec{a}$ and $\vec{b}$ have opposite direction.

Ex- $\langle 1, 2, 3 \rangle$ and $\langle 2, 4, 6 \rangle$ are parallel

because $\langle 2, 4, 6 \rangle = 2 \langle 1, 2, 3 \rangle$

Fact- For vectors $\vec{a}$, and scalar $\beta \in \mathbb{R}$,

$\|\beta \vec{a}\| = |\beta| \|\vec{a}\|$

Claim- Every nonzero vector is parallel to a unit vector.

Proof- Consider $\vec{a}$. Want to find β such that $\beta \vec{a}$ is a unit vector ie $\|\beta \vec{a}\| = 1$. This means $|\beta| \|\vec{a}\| = 1$ and so $|\beta| = \frac{1}{\|\vec{a}\|}$

Thus, $\beta = \frac{1}{\|\vec{a}\|}$ or $\beta = \frac{-1}{\|\vec{a}\|}$

Fact- If $\vec{a}$ is nonzero, then $\frac{1}{\|\vec{a}\|}\vec{a}$ and $\frac{-1}{\|\vec{a}\|}\vec{a}$ are both unit vectors parallel to $\vec{a}$

Example - Find two unit vectors parallel to $\vec{a} = \langle 1, 2, 2 \rangle$

$\|\vec{a}\| = 3$

So $\frac{1}{\|\vec{a}\|}\vec{a} = \frac{1}{3}\langle 1, 2, 2\rangle = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$

and $\frac{-1}{\|\vec{a}\|}\vec{a} = \frac{-1}{3}\langle 1, 2, 2\rangle = \langle -\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3} \rangle$

The unit vector $\frac{1}{\|\vec{a}\|}\vec{a}$ is called the direction of $\vec{a}$.

Observe $\vec{a} = 1\vec{a} = \left(\|\vec{a}\| \frac{1}{\|\vec{a}\|}\right)\vec{a} = \underbrace{\|\vec{a}\|}_{\text{magnitude}} \underbrace{\left(\frac{1}{\|\vec{a}\|}\vec{a}\right)}_{\text{direction}}$

Three special unit vectors are

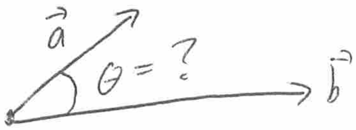
$\vec{i} = \langle 1, 0, 0 \rangle$ $\quad \vec{j} = \langle 0, 1, 0 \rangle$ $\quad \vec{k} = \langle 0, 0, 1 \rangle$

Observe $\vec{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle = a_1\langle 1, 0, 0\rangle + a_2\langle 0, 1, 0\rangle + a_3\langle 0, 0, 1\rangle$

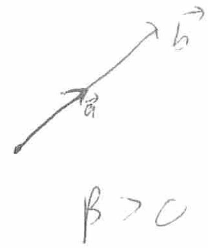
$= a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

So every vector in $\mathbb{R}^3$ can be written as linear combination of $\vec{i}, \vec{j}, \vec{k}$

Ex- $\langle 4, 5, 6 \rangle = 4\vec{i} + 5\vec{j} + 6\vec{k}$



Special case: $\vec{b} = \beta \vec{a}$



For nonzero vectors $\vec{a}, \vec{b}$, if $\vec{b} = \beta \vec{a}$ then the angle between them is 0 if $\beta > 0$ and π if $\beta < 0$

For $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, the <u>dot product</u> of $\vec{a}$ and $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example - 1) Compute $\vec{a} \cdot \vec{b}$ for $\vec{a} = \langle 1, 2, 3 \rangle$, $\vec{b} = \langle 4, 5, 6 \rangle$

2) What is $\vec{a} \cdot \vec{a}$ for $\vec{a} = \langle a_1, a_2, a_3 \rangle$?

1) $\langle 1, 2, 3 \rangle \cdot \langle 4, 5, 6 \rangle = 1(4) + 2(5) + 3(6) = 32$

2) $\langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = a_1^2 + a_2^2 + a_3^2$

$$= \left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)^2$$

$$= \|\vec{a}\|^2$$

The <u>angle</u> between nonzero vectors $\vec{a}, \vec{b}$ is the value $\theta \in [0, \pi]$ such that $\cos(\theta) = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \, \|\vec{b}\|}$

Example. 1) Find angle between $\vec{a} = \langle 1, \sqrt{11}, 2 \rangle$ and $\vec{b} = \langle 4, 0, 3 \rangle$

2) Find $\vec{a} \cdot \vec{b}$ if $\|\vec{a}\| = 5$, $\|\vec{b}\| = 6$, and the angle between $\vec{a}, \vec{b}$ is $\pi/6$

1) $\cos(\theta) = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \, \|\vec{b}\|} = \dfrac{10}{4(5)} = \dfrac{1}{2}$

$\vec{a} \cdot \vec{b} = \langle 1, \sqrt{11}, 2 \rangle \cdot \langle 4, 0, 3 \rangle = 10$

$\|\vec{a}\| = \sqrt{1^2 + (\sqrt{11})^2 + 2^2} = \sqrt{16} = 4$

$\|\vec{b}\| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5$

$\cos(\theta) = \frac{1}{2} \Rightarrow \theta = \pi/3$

2) $\underline{\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)} = 5(6)\cos(\pi/6) = 30\left(\frac{\sqrt{3}}{2}\right)$

$= 15\sqrt{3}$

Properties. For vectors $\vec{a}, \vec{b}, \vec{c}$ and scalar $\beta \in \mathbb{R}$

1) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

2) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

3) $(\beta \vec{a}) \cdot \vec{b} = \beta(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\beta \vec{b})$

4) $\vec{a} \cdot \vec{0} = 0$

5) $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

Two vectors are <u>orthogonal</u> (or perpendicular) to each other if the angle between them is $\frac{\pi}{2}$

Two vectors $\vec{a}, \vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$

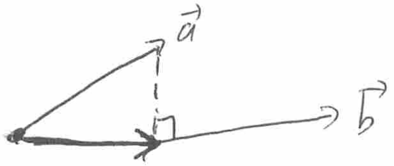
Example - Check for orthogonality

1) $\vec{a} = \langle 1, 2, -1 \rangle$, $\vec{b} = \langle -1, 4, 7 \rangle$

2) $\vec{a} = \langle 1, 2, 3 \rangle$, $\vec{b} = \langle 0, -4, 3 \rangle$

1) Yes b/c $\langle 1, 2, -1 \rangle \cdot \langle -1, 4, 7 \rangle = -1 + 8 - 7 = 0$

2) No b/c $\langle 1, 2, 3 \rangle \cdot \langle 0, -4, 3 \rangle = 0 - 8 + 9 = 1 \neq 0$



Dashed line is segment of minimal distance.

The resulting vector is the vector projection of $\vec{a}$ onto $\vec{b}$

The vector projection (or orthogonal projection) of $\vec{a}$ onto $\vec{b}$ is $\text{proj}_{\vec{b}}(\vec{a}) = \dfrac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \vec{b}$

The scalar projection of $\vec{a}$ onto $\vec{b}$ is $S_{\vec{b}}(\vec{a}) = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|}$

Observe $\text{proj}_{\vec{b}}(\vec{a}) = \dfrac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{b} = \underbrace{\dfrac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|}}_{\substack{\text{scalar}\\ \text{projection}\\ \text{of } \vec{a} \text{ onto } \vec{b}}} \underbrace{\left(\dfrac{1}{\|\vec{b}\|} \vec{b}\right)}_{\substack{\text{direction}\\ \text{of } \vec{b}}}$

Example - $\vec{a} = \langle -3, 3, 3 \rangle$ and $\vec{b} = \langle 2, 4, 2 \rangle$

1) Find scalar and vector projections of $\vec{a}$ onto $\vec{b}$

2) Find scalar and vector projections of $\vec{b}$ onto $\vec{a}$