$$\iiint_R x\,\sigma(x,y,z)\,dV = \int_0^4\int_0^2\int_0^2 x^2yz\,dx\,dy\,dz$$

$$\int_0^2 x^2yz\,dx = \frac{x^3yz}{3}\Big|_0^2 = \frac{8}{3}yz$$

$$\int_0^2 \frac{8}{3}yz\,dy = \frac{8}{3}z\cdot\frac{y^2}{2}\Big|_0^2 = \frac{16}{3}z$$

$$\int_0^4 \frac{16}{3}z\,dz = \frac{16z^2}{6}\Big|_0^4 = \frac{256}{6} = \frac{128}{3}$$

$$\iiint_R y\,\sigma(x,y,z)\,dV = \int_0^4\int_0^2\int_0^2 xy^2z\,dx\,dy\,dz$$

$$\int_0^2 xy^2z\,dx = \frac{x^2}{2}y^2z\Big|_0^2 = 2y^2z$$

$$\int_0^2 2y^2z\,dy = \frac{2y^3z}{3}\Big|_0^2 = \frac{16}{3}z$$

$$\int_0^4 \frac{16}{3}z\,dz = \frac{128}{3} \quad \text{(from above)}$$

$$\iiint_R z\,\sigma(x,y,z)\,dV = \int_0^4\int_0^2\int_0^2 xyz^2\,dx\,dy\,dz$$

$$\int_0^2 xyz^2\,dx = \frac{x^2yz^2}{2}\Big|_0^2 = 2yz^2$$

$$\int_0^2 2yz^2\,dy = y^2z^2\Big|_0^2 = 4z^2$$

$$\int_0^4 4z^2\,dz = \frac{4}{3}z^3\Big|_0^4 = \frac{256}{3}$$

<u>Center of mass</u> $= \left(\frac{128/3}{32}, \frac{128/3}{32}, \frac{256/3}{32}\right)$

centroid

$$= \left(\frac{4}{3}, \frac{4}{3}, \frac{8}{3}\right)$$

Suppose now the object defined by $R \subseteq \mathbb{R}^3$ is electrically charged. The net charge of the object, Q, depends on the charge density function $\gamma(x,y,z)$ ie the charge per unit volume.

The net charge is $Q = \iiint_R \gamma(x,y,z)\,dV$

Ex- $R = \{(x,y,z) \mid 0 \le x \le 2,\ 0 \le y \le 2,\ 0 \le z \le 4\}$

$$\gamma(x,y,z) = x + y + z$$

$$Q = \iiint_R \gamma(x,y,z)\,dV = \int_0^4\int_0^2\int_0^2 x+y+z\,dx\,dy\,dz$$

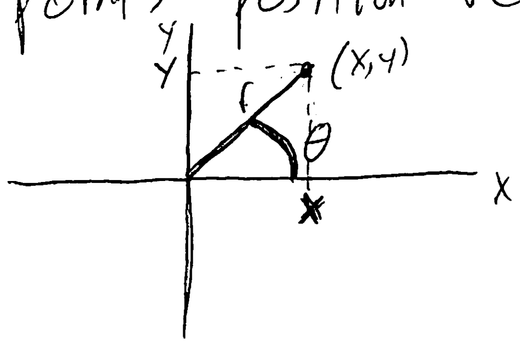
$$\int_0^2 x+y+z\,dx = \frac{x^2}{2} + xy + xz \Big|_0^2 = 2+2y+2z$$

$$\int_0^2 2+2y+2z\,dy = 2y + y^2 + 2zy \Big|_0^2 = 8+4z$$

$$\int_0^4 8+4z\,dz = 8z + 2z^2 \Big|_0^4 = 32+32 = 64$$

Usually, we represent points in $\mathbb{R}^2$ in Cartesian coordinates i.e. as ordered pairs (x,y) where we move x-units horizontally and y-units vertically from the origin

We also can represent points in $\mathbb{R}^2$ using <u>polar coordinates</u> i.e. as ordered pairs (r,θ) where $r \geq 0$ is the distance from the point to the origin, and $\theta \in [0, 2\pi)$ is the angle between the point's position vector and the positive x-axis



Using trig/Pythagoras, we see $x = r\cos(\theta)$ and $y = r\sin(\theta)$

We can relate Cartesian coordinates to polar coordinates using the equations:

$x = r\cos(\theta)$, $y = r\sin(\theta)$, $x^2 + y^2 = r^2$, $\tan(\theta) = \frac{y}{x}$

We can use these to represent functions $f(x,y)$ and regions $R \subseteq \mathbb{R}^2$ using polar coordinates.

Ex- 1) $f(x,y) = e^{x^2+y^2} - xy$

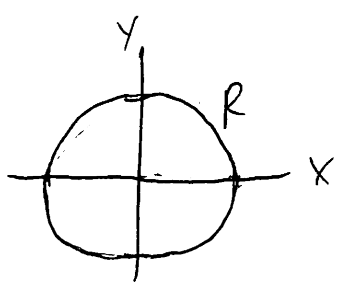
Ex- $R =$ 2) unit circle $= \{(x,y) \mid x^2+y^2 \leq 1\}$

3) top half of unit circle

4) top right quadrant of unit circle

1) $f(r\cos(\theta), r\sin(\theta)) = e^{r^2} - (r\cos(\theta))(r\sin(\theta))$

$= e^{r^2} - r^2\cos(\theta)\sin(\theta)$

2) 

$R = \{(r,\theta) \mid 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi\}$

3) ~~top~~ half $= \{(r,\theta) \mid 0 \leq r \leq 1,\ 0 \leq \theta \leq \pi\}$

4) top right quadrant = $\{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2}\}$

We can do partial derivatives / integrals in polar coordinates much like we did in Cartesian coordinates. That is, to differentiate / integrate with respect to r, hold θ constant (and vice versa)

Ex - for f above, do
1) f_r
2) $\int_0^{\pi/2} f \, d\theta$

$$f = e^{r^2} - r^2\cos(\theta)\sin(\theta)$$

1) $f_r = 2re^{r^2} - 2r\cos(\theta)\sin(\theta)$

2) $\int_0^{\pi/2} [e^{r^2} - r^2\cos(\theta)\sin(\theta)]\,d\theta = \theta e^{r^2} - \frac{r^2\sin^2(\theta)}{2}\Big|_0^{\pi/2}$

$= \frac{\pi}{2}e^{r^2} - \frac{r^2}{2}$

For the purposes of computing double integrals, $\iint_R f(x,y)\,dA$, using polar coordinates, we need to represent f, R, and dA in terms of r and θ.

Fact - in polar coordinates, $dA = r\,dr\,d\theta$

Thus, $\iint_R f(x,y)\,dA = \iint_R f(r\cos(\theta), r\sin\theta)\,r\,dr\,d\theta$

Ex- 1) Compute $\iint_R e^{x^2+y^2} - xy\,dA$ where

$R = \{(x,y) \mid x^2+y^2 \leq 1\}$ = unit circle

In polar coordinates, our function is $e^{r^2} - r^2\cos(\theta)\sin(\theta)$

and our region is $R = \{(r,\theta) \mid 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi\}$

So $\iint_R e^{x^2+y^2} - xy\,dA = \int_0^{2\pi}\int_0^1 (e^{r^2} - r^2\sin(\theta)\cos(\theta))\,r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 re^{r^2} - r^3\sin(\theta)\cos(\theta)\,dr\,d\theta$

$\int_0^1 re^{r^2} - r^3\sin(\theta)\cos(\theta)\,dr = \frac{1}{2}e^{r^2} - \frac{r^4\sin(\theta)\cos(\theta)}{4}\Big|_0^1$

$= \frac{1}{2}e - \frac{1}{4}\sin(\theta)\cos(\theta) - \frac{1}{2}$

$\int_0^{2\pi} \frac{1}{2}e - \frac{1}{2} - \frac{1}{4}\sin(\theta)\cos(\theta)\,d\theta = \frac{\theta}{2}e - \frac{\theta}{2} - \frac{1}{8}\sin^2(\theta)\Big|_0^{2\pi}$

$= \pi e - \pi$

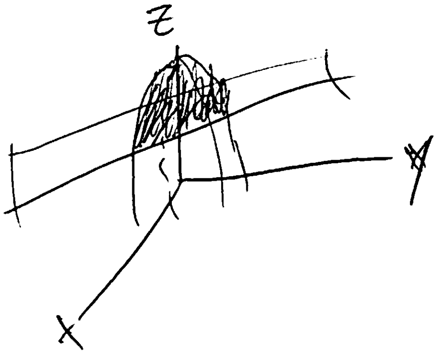
Ex - Use calculus to show that the area of the unit circle is π

$$\text{Area} = \iint_R 1 \, dA = \int_0^{2\pi}\int_0^1 r \, dr \, d\theta$$

$$\int_0^1 r \, dr = \frac{r^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\int_0^{2\pi} \frac{1}{2} \, d\theta = \frac{\theta}{2}\Big|_0^{2\pi} = \pi$$

Ex - Volume of the solid region bounded by $z = 9 - x^2 - y^2$ and $z = 5$



To get R, we equate the surfaces to get $9 - x^2 - y^2 = 5$ i.e. $x^2 + y^2 = 4$

$$V = \iint_R (9 - x^2 - y^2) - 5 \, dA = \iint_R 4 - x^2 - y^2 \, dA$$

In polar coordinates, our integrand is $4 - r^2$ and our region is $R = \{(r, \theta) \mid 0 \leq r \leq 2, \ 0 \leq \theta \leq 2\pi\}$

$$\iint_R 4-x^2-y^2\,dA = \int_0^{2\pi}\int_0^2 (4-r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 4r-r^3\,dr\,d\theta$$

$$\int_0^2 4r-r^3\,dr = 2r^2-\frac{r^4}{4}\Big|_0^2 = 8-4 = 4$$

$$\int_0^{2\pi} 4\,d\theta = 4\theta\Big|_0^{2\pi} = 8\pi$$

Fact - if a, b, α, β are all constants, then

$$\int_\alpha^\beta\int_a^b r f(r\cos(\theta), r\sin(\theta))\,dr\,d\theta = \int_a^b\int_\alpha^\beta r f(r\cos(\theta), r\sin(\theta))\,d\theta\,dr$$

We also can integrate over what are called r-simple regions, where r is bounded by functions of θ ie $R = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ g(\theta) \le r \le h(\theta)\}$

In this case, $\iint_R f(x,y)\,dA = \int_\alpha^\beta\int_{g(\theta)}^{h(\theta)} r f(r\cos(\theta), r\sin(\theta))\,dr\,d\theta$

Example- $\int_0^{\pi/2}\int_{-4\sin(\theta)}^{4} x\,dA$

$$= \int_0^{\pi/2}\int_{-4\sin(\theta)}^{4} r\cos(\theta)\,r\,dr\,d\theta = \int_0^{\pi/2}\int_{-4\sin(\theta)}^{4} r^2\cos(\theta)\,dr\,d\theta$$

$$\int_{-4\sin(\theta)}^{4} r^2\cos(\theta)\,dr = \frac{r^3\cos(\theta)}{3}\Big|_{-4\sin(\theta)}^{4} = \frac{64}{3}\cos(\theta) + \frac{64}{3}\sin^3(\theta)\cos(\theta)$$

$$\int_0^{\pi/2} \frac{64}{3}\cos(\theta) + \frac{64}{3}\sin^3(\theta)\cos(\theta)\, d\theta = \frac{64}{3}\sin(\theta) + \frac{64}{12}\sin^4(\theta) \Big|_0^{\pi/2}$$

$$= \frac{64}{3} + \frac{16}{3} = \frac{80}{3}$$

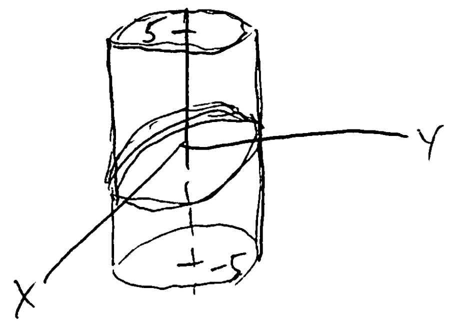
We can extend polar coordinates to represent points in $\mathbb{R}^3$ using <u>cylindrical coordinates</u>, which represents a point in $\mathbb{R}^3$ as (r, θ, z) where r and θ define $\mathbb{R}^2$ as before

We can express functions $f(x,y,z)$ and regions $R \subseteq \mathbb{R}^3$ using cylindrical coordinates by using the polar coordinate representations of x and y, and keeping z as usual.

Example - 1) $f(x,y,z) = e^{x^2+y^2} - xyz$

2) R = cylinder of height 10 and radius 2 centered at the origin in $\mathbb{R}^3$

1) $f(r\cos(\theta), r\sin(\theta), z) = e^{r^2} - r^2\cos(\theta)\sin(\theta)z$

2) 

$R = \{(r, \theta, z) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ -5 \le z \le 5\}$

Derivatives and integrals in cylindrical coordinates are done much like they were in Cartesian coordinates

Namely, to differentiate/integrate with respect to r, hold θ and z constant, and so on.

For the purposes of computing triple integrals, $\iiint_R f(x,y,z)\,dV$, in cylindrical coordinates, we need to represent f, R and dV in terms of r, θ, z.

Fact - in cylindrical coordinates, $dV = r\,dr\,d\theta\,dz$

Thus, $\iiint_R f(x,y,z)\,dV = \iiint_R f(r\cos(\theta), r\sin(\theta), z)\,r\,dr\,d\theta\,dz$

Example - Show that the cylinder above of height 10 and radius 2 has volume 40π

$$V = \iiint_R 1\,dV = \int_{-5}^{5}\int_0^{2\pi}\int_0^2 r\,dr\,d\theta\,dz$$

$$\int_0^2 r\,dr = \frac{r^2}{2}\Big|_0^2 = 2$$

$$\int_0^{2\pi} 2\,d\theta = 2\theta\Big|_0^{2\pi} = 4\pi$$

$$\int_{-5}^{5} 4\pi\,dz = 4\pi z\Big|_{-5}^{5} = 20\pi - (-20\pi) = 40\pi$$

Ex - Compute $\iiint_R e^{x^2+y^2} - xyz\,dV$ where R is the cylinder from above

In cylindrical coordinates, our function is $e^{r^2} - r^2\cos(\theta)\sin(\theta)z$ and $R = \{(r,\theta,z) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ -5 \le z \le 5\}$

So our integral is $\int_{-5}^{5}\int_0^{2\pi}\int_0^2 r(e^{r^2} - r^2\cos(\theta)\sin(\theta)z)\,dr\,d\theta\,dz$

$$\int_0^2 re^{r^2} - r^3\cos(\theta)\sin(\theta)z\,dr = \frac{1}{2}e^{r^2} - \frac{r^4}{4}\cos(\theta)\sin(\theta)z\Big|_0^2$$

$$= \frac{1}{2}e^4 - 4\cos(\theta)\sin(\theta)z - \frac{1}{2}$$

$$\int_0^{2\pi} \frac{1}{2}e^4 - 4\cos(\theta)\sin(\theta)z - \frac{1}{2}\,d\theta = \frac{e^4}{2}\theta - 4z\left(\frac{\sin^2(\theta)}{2}\right) - \frac{1}{2}\theta\Big|_0^{2\pi}$$

$$= \pi e^4 - \pi$$

$$\int_{-5}^{5} \pi e^4 - \pi \, dz = \pi e^4 z - \pi z \Big|_{-5}^{5}$$

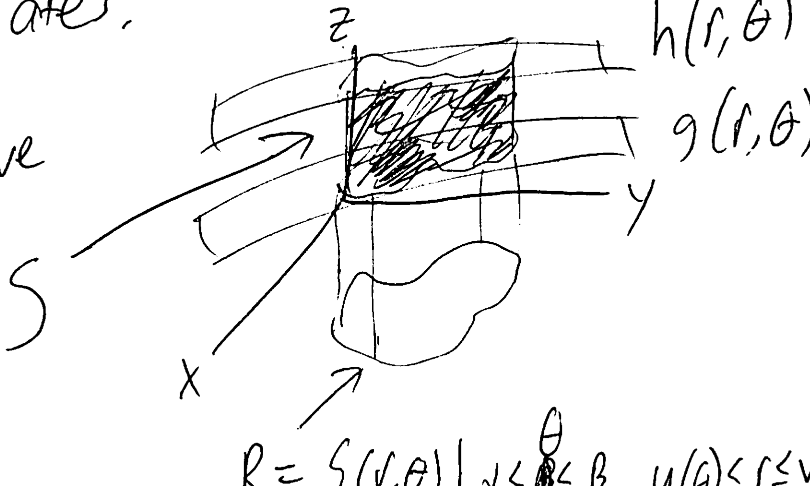
$$= (5\pi e^4 - 5\pi) - (-5\pi e^4 + 5\pi)$$

$$= 10\pi e^4 - 10\pi$$

Fact - If $R = \{(r, \theta, z) \mid a \leq r \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq z \leq d\}$ where $a, b, \alpha, \beta, c, d$ are all constants, then the order of integration does not matter, so long as the correct limits of integration go with the correct variable.

We can integrate over z-simple regions in cylindrical coordinates much like we did in Cartesian coordinates.

That is, if we have

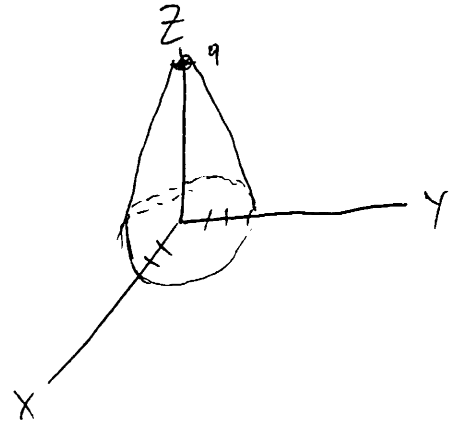


$R = \{(r, \theta) \mid \alpha \leq \theta \leq \beta,\ u(\theta) \leq r \leq v(\theta)\}$

Then $\iiint_S f(x,y,z)\,dV = \int_\alpha^\beta \int_{u(\theta)}^{v(\theta)} \int_{g(r,\theta)}^{h(r,\theta)} r f(r\cos(\theta), r\sin(\theta), z)\,dz\,dr\,d\theta$

The graph of a circular cone with height h and radius σ is $z = h - \frac{h}{\sigma}\sqrt{x^2+y^2}$ where $z \geq 0$

For example, the cone of height 9 and radius 3 is $z = 9 - 3\sqrt{x^2+y^2}$ where $z \geq 0$



Volume of a cone is $\dfrac{\pi h (\text{radius})^2}{3}$

Show that the volume of the cone $z = 9 - 3\sqrt{x^2+y^2}$ where $z \geq 0$ is 27π with calcllus

In cylindrical coordinates, the region defining the cone is

$R = \{ (r, \theta, z) \mid 0 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 9 - 3r \}$

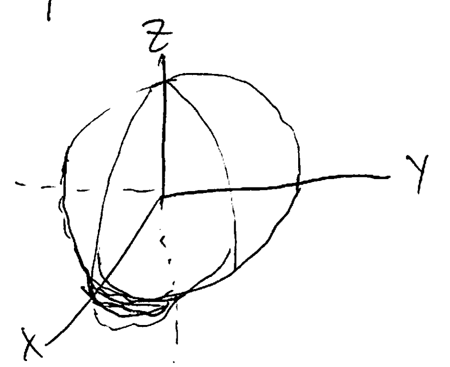
Then volume $= \iiint_R 1 \, dV = \int_0^{2\pi} \int_0^3 \int_0^{9-3r} r \, dz\,dr\,d\theta$

$$\int_0^{9-3r} r\,dz = rz\Big|_0^{9-3r} = r(9-3r) = 9r - 3r^2$$

$$\int_0^3 9r - 3r^2 \, dr = \frac{9}{2}r^2 - r^3\Big|_0^3 = \frac{81}{2} - 27 = \frac{27}{2}$$

$$\int_0^{2\pi} \frac{27}{2}\,d\theta = \frac{27}{2}\theta\Big|_0^{2\pi} = 27\pi$$

Example - Find volume of the unit sphere $x^2 + y^2 + z^2 = 1$ using calculus



In cylindrical coordinates, we can write this as $R = \{(r,\theta,z) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ -\sqrt{1-r^2} \le z \le \sqrt{1-r^2}\}$

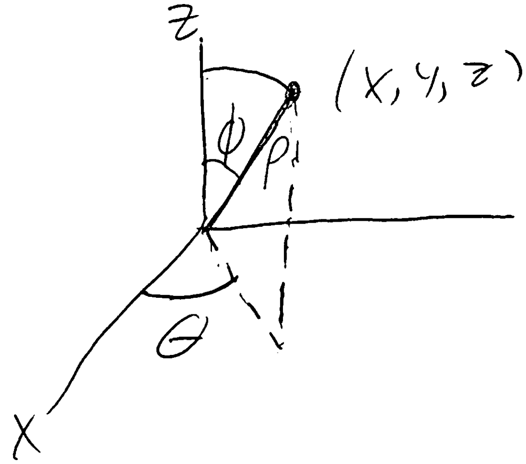
$$\iiint_R 1\, dV = \int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r\, dz\, dr\, d\theta$$

$$\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r\, dz = rz\Big|_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} = 2r\sqrt{1-r^2}$$

$$\int_0^1 2r\sqrt{1-r^2}\, dr = -\frac{2}{3}(1-r^2)^{3/2}\Big|_0^1 = \frac{2}{3}$$

$$\int_0^{2\pi} \frac{2}{3}\, d\theta = \frac{2}{3}\theta\Big|_0^{2\pi} = \frac{4\pi}{3}$$

Another way to represent points in $\mathbb{R}^3$ is with <u>spherical coordinates</u>. Each point (x, y, z) can be represented as (ρ, θ, ϕ) where $\rho \geq 0$ is the distance from the point to the origin, $\theta \in [0, 2\pi)$ is the same as before (ie the angle from the point (x, y) position vector to the positive x-axis) and $\phi \in [0, \pi]$ is the angle from the positive z axis outward.



We can relate cartesian coordinates to spherical coordinates as follows:

$x = \rho \sin(\phi)\cos(\theta)$, $y = \rho \sin(\phi)\sin(\theta)$,

$z = \rho\cos(\phi)$, $x^2 + y^2 + z^2 = \rho^2$

We can use these to express functions $f(x,y,z)$ and regions $R \subseteq \mathbb{R}^3$ in spherical coordinates.

Ex - 1) $f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}} + xyz$

2) $R =$ unit sphere
top half of unit sphere
first octant of unit sphere

1) $f = e^{(\rho^2)^{3/2}} + \rho\sin(\phi)\cos(\theta)\rho\sin(\phi)\sin(\theta)\rho\cos(\phi)$

$= e^{\rho^3} + \rho^3\sin^2(\phi)\cos(\phi)\cos(\theta)\sin(\theta)$

2) unit sphere $= \{(\rho,\theta,\phi) \mid 0\le\rho\le 1,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\pi\}$

top half $= \{(\rho,\theta,\phi) \mid 0\le\rho\le 1,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\frac{\pi}{2}\}$

first octant $= \{(\rho,\theta,\phi) \mid 0\le\rho\le 1,\ 0\le\theta\le\frac{\pi}{2},\ 0\le\phi\le\frac{\pi}{2}\}$

Partial derivative/integrals in spherical coordinates same as in cartesian.

That is, to differentiate/integrate with respect to ρ, hold θ and ϕ constant, and so on.

For the purposes of computing triple integrals, $\iiint_R f(x,y,z)\,dV$, in spherical coordinates, We need to express f, R, and dV in terms of ρ, θ, ϕ.

Fact - in spherical coordinates, $dV = \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$

Thus, $\iiint_R f(x,y,z)\, dV = \iiint_R f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\rho^2\sin(\phi)$
$d\rho\, d\theta\, d\phi$

Ex - show volume of unit sphere is $\frac{4}{3}\pi$ with spherical coordinates

$$\text{Volume} = \iiint_R 1\, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^1 \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$$

$$\int_0^1 \rho^2 \sin(\phi)\, d\rho = \frac{\rho^3}{3}\sin(\phi)\Big|_0^1 = \frac{1}{3}\sin(\phi)$$

$$\int_0^{2\pi} \frac{1}{3}\sin(\phi)\, d\theta = \frac{1}{3}\sin(\phi)\theta\Big|_0^{2\pi} = \frac{2\pi}{3}\sin(\phi)$$

$$\int_0^{\pi} \frac{2\pi}{3}\sin(\phi)\, d\phi = -\frac{2\pi}{3}\cos(\phi)\Big|_0^{\pi} = \frac{2\pi}{3} - \left(-\frac{2\pi}{3}\right)$$
$$= \frac{4\pi}{3}$$

Fact - if the limits of integration for ρ, θ, ϕ are all constants, order of integration does not matter.

We also can integrate over what are called ρ-simple regions, where ρ is bounded by functions of θ and ϕ.

That is, $R = \{(\rho,\theta,\phi) \mid \alpha \le \theta \le \beta,\ \gamma \le \phi \le \delta,\ g(\theta,\phi) \le \rho \le h(\theta,\phi)\}$

Ex - $\iiint_R \frac{1}{x^2+y^2+z^2}\, dV$ where

$R = \{(\rho,\theta,\phi) \mid \pi \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{2},\ \sin(\theta)\cos(\phi) \le \rho \le \cos(\theta)\cos(\phi)\}$

Our function in spherical coordinates is $\frac{1}{\rho^2}$

So integral is $\int_{\pi}^{2\pi}\int_0^{\pi/2}\int_{\sin(\theta)\cos(\phi)}^{\cos(\theta)\cos(\phi)} \frac{1}{\rho^2}\rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_{\sin(\theta)\cos(\phi)}^{\cos(\theta)\cos(\phi)} \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$\int_{\sin(\theta)\cos(\phi)}^{\cos(\theta)\cos(\phi)} \sin(\phi)\, d\rho = \rho \sin(\phi) \Big|_{\sin(\theta)\cos(\phi)}^{\cos(\theta)\cos(\phi)}$$

$$= \cos(\theta)\sin(\phi)\cos(\phi) - \sin(\theta)\sin(\phi)\cos(\phi)$$

$$\int_0^{\pi/2} \cos(\theta)\sin(\phi)\cos(\phi) - \sin(\theta)\sin(\phi)\cos(\phi)\, d\phi$$

$$= \frac{\cos(\theta)\sin^2(\phi)}{2} - \frac{\sin(\theta)\sin^2(\phi)}{2} \Big|_0^{\pi/2}$$

$$= \frac{\cos(\theta)}{2} - \frac{\sin(\theta)}{2}$$

$$\int_{\pi}^{2\pi} \frac{\cos(\theta)}{2} - \frac{\sin(\theta)}{2}\, d\theta = \frac{\sin(\theta)}{2} + \frac{\cos(\theta)}{2} \Big|_{\pi}^{2\pi}$$

$$= \frac{1}{2} - \left(-\frac{1}{2}\right)$$

$$= 1$$