Now, let us consider the nonhomogeneous problem

$$ay'' + by' + cy = f(x)$$

We will start with a solution technique called the <u>method of undetermined coefficients</u>

In this method, we look for a solution y_p of the DE that looks "similar" to $f(x)$ (with arbitrary constant coefficients) while also taking into account its derivatives.

For example, if $f(x)$ is a polynomial of degree n, we'd look for y_p of the form $y_p = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0$

where $A_n, \dots, A_0$ are arbitrary constants.

If $f(x)$ is a polynomial of degree n multiplied by e^{kx}, we'd look for y_p of the form $y_p = (A_n x^n + A_{n-1}x^{n-1} + \dots + A_1 x + A_0)e^{kx}$

If $f(x)$ is a polynomial of degree n multiplied by $e^{\alpha x}\cos(\beta x)$ and/or $e^{\alpha x}\sin(\beta x)$, we'd look for y_p of the form

$$y_p = (A_n x^n + \dots + A_0)e^{\alpha x}\cos(\beta x) + (B_n x^n + \dots + B_0)e^{\alpha x}\sin(\beta x)$$

We then plug y_p, y_p', and y_p'' into the DE and solve for the coefficients

Ex - $y'' + 2y' - 8y = -16x^2$

Let $y_p = A_2 x^2 + A_1 x + A_0$

Then $y_p' = 2A_2 x + A_1$

$y_p'' = 2A_2$

Plug these into DE to get

$$2A_2 + 2(2A_2x + A_1) - 8(A_2x^2 + A_1x + A_0) = -16x^2$$

$$-8A_2x^2 + (4A_2 - 8A_1)x + (2A_2 + 2A_1 - 8A_0) = -16x^2$$

Compare the x^2 terms: $-8A_2 = -16$

Compare the x terms: $4A_2 - 8A_1 = 0$

Compare the constant terms: $2A_2 + 2A_1 - 8A_0 = 0$

Then $A_2 = 2$, $A_1 = 1$, $A_0 = \frac{3}{4}$

Thus $y_p = 2x^2 + x + \frac{3}{4}$

Ex -
1) $y'' + 2y' - 8y = 14e^{3x}$
2) $y'' + 2y' - 8y = -25xe^{x}$
3) $y'' + 2y' - 8y = -40\sin(2x)$
4) $y'' + 2y' - 8y = e^{2x}$

1) $y_p = Ae^{3x}$

Then $y_p' = 3Ae^{3x}$ and $y_p'' = 9Ae^{3x}$

Plug these into the DE to get

$$9Ae^{3x} + 6Ae^{3x} - 8Ae^{3x} = 14e^{3x}$$

$$7Ae^{3x} = 14e^{3x} \quad \text{so} \quad A = 2$$

Thus $y_p = 2e^{3x}$

2) $y_p = (Ax + B)e^x$

$y_p' = (Ax + B)e^x + Ae^x$

$y_p'' = (Ax + B)e^x + 2Ae^x$

Plug these into the DE to get

$$Axe^x + Be^x + 2Ae^x + 2Axe^x + 2Be^x + 2Ae^x$$
$$- 8Axe^x - 8Be^x = -25xe^x$$

$$-5Axe^x + (-5B + 4A)e^x = -25xe^x$$

Compare the xe^x terms: $-5A = -25$

Compare the e^x terms: $-5B + 4A = 0$

So $A = 5$ and $B = 4$

Thus $y_p = (5x+4)e^x$

3) $y_p = A\sin(2x) + B\cos(2x)$

$y_p' = 2A\cos(2x) - 2B\sin(2x)$

$y_p'' = -4A\sin(2x) - 4B\cos(2x)$

Plug these into the DE to get

$-4A\sin(2x) - 4B\cos(2x) + 4A\cos(2x) - 4B\sin(2x)$
$-8A\sin(2x) - 8B\cos(2x) = -40\sin(2x)$

$\Rightarrow (-12A - 4B)\sin(2x) + (4A - 12B)\cos(2x) = -40\sin(2x)$

Compare $\sin(2x)$ terms: $-12A - 4B = -40$

Compare $\cos(2x)$ terms: $4A - 12B = 0$

$A = 3B$ so $-40B = -40$

ie $B = 1$ and $A = 3$

Thus, $y_p = 3\sin(2x) + \cos(2x)$

4) $y_p = Ae^{2x}$

$y_p' = 2Ae^{2x}$ and $y_p'' = 4Ae^{2x}$

Plug these into the DE to get

$4Ae^{2x} + 4Ae^{2x} - 8Ae^{2x} = e^{2x}$

$0 = e^{2x}$

Since $r=2$ is a solution to the auxiliary equation $r^2 + 2r - 8 = 0$ for the DE, e^{2x} is a solution to the homogeneous DE $y'' + 2y' - 8y = 0$

Let's try $y_p = Axe^{2x}$ instead.

Then $y_p' = 2Axe^{2x} + Ae^{2x}$

$y_p'' = 4Axe^{2x} + 4Ae^{2x}$

Plug these into the DE to get

$4Axe^{2x} + 4Ae^{2x} + 4Axe^{2x} + 2Ae^{2x}$

$- 8Axe^{2x} = e^{2x}$

$6Ae^{2x} = e^{2x}$ so $6A = 1$ ie $A = \frac{1}{6}$

Thus $y_p = \frac{1}{6}xe^{2x}$

Will this always work?

Ex- $y'' + y = \sin(x)$

$y_p = A\sin(x) + B\cos(x)$

$y_p' = A\cos(x) - B\sin(x)$

$y_p'' = -A\sin(x) - B\cos(x)$

Plug these into the DE to get

$$-A\sin(x) - B\cos(x) + A\sin(x) + B\cos(x) = \sin(x)$$

$$0 = \sin(x)$$

Let's try $y_p = x(A\sin(x) + B\cos(x))$

$$y_p' = x(A\cos(x) - B\sin(x)) + A\sin(x) + B\cos(x)$$

$$y_p'' = x(-A\sin(x) - B\cos(x)) + 2(A\cos(x) - B\sin(x))$$

Plug these into DE to get

$$x(-A\sin(x) - B\cos(x)) + 2A\cos(x) - 2B\sin(x) + x(A\sin(x) + B\cos(x))$$
$$= \sin(x)$$

$$\Rightarrow 2A\cos(x) - 2B\sin(x) = \sin(x)$$

Compare $\cos(x)$ terms: $2A = 0$

Compare $\sin(x)$ terms: $-2B = 1$

So $A = 0$ and $B = -\frac{1}{2}$

Thus $y_p = -\frac{1}{2}x\cos(x)$

Ex: $y'' - 4y' + 4y = e^{2x}$

Start with $y_p = Ae^{2x}$

$y_p' = 2Ae^{2x}$ and $y_p'' = 4Ae^{2x}$

Plug into DE to get

$$4Ae^{2x} - 8Ae^{2x} + 4Ae^{2x} = e^{2x}$$

$$0 = e^{2x}$$

Let's try $y_p = Axe^{2x}$

$y_p' = 2Axe^{2x} + Ae^{2x}$

$y_p'' = 4Axe^{2x} + 4Ae^{2x}$

Plug into DE to get

$$4Axe^{2x} + 4Ae^{2x} - 8Axe^{2x} - 4Ae^{2x} + 4Axe^{2x} = e^{2x}$$

$$0 = e^{2x}$$

Note Auxiliary equation for this DE

is $r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0$

So both e^{2x} and xe^{2x} solve the

homogeneous DE

Let's try $y_p = Ax^2e^{2x}$

$$y_p' = 2Ax^2e^{2x} + 2Axe^{2x}$$

$$y_p'' = 4Ax^2e^{2x} + 8Axe^{2x} + 2Ae^{2x}$$

Plug into the DE to get

$$4Ax^2e^{2x} + 8Axe^{2x} + 2Ae^{2x} - 8Ax^2e^{2x} - 8Axe^{2x} + 4Ax^2e^{2x} = e^{2x}$$

$$\Rightarrow 2Ae^{2x} = e^{2x} \quad \text{so} \quad A = \tfrac{1}{2}$$

Thus $y_p = \frac{1}{2}x^2e^{2x}$

Consider the DE $ay'' + by' + cy = f(x)$

If $f(x)$ is a degree n polynomial multiplied by e^{rx} (r a constant), then

i) If r is <u>not</u> a solution to the auxiliary equation $ar^2 + br + c = 0$, let

$$y_p = (A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_0) e^{rx}$$

ii) If r is a single (ie not repeated) solution to the auxiliary equation let

$$y_p = (A_n x^n + \dots + A_1 x + A_0) x e^{rx}$$
$$= (A_n x^{n+1} + \dots + A_1 x^2 + A_0 x) e^{rx}$$

iii) If r is a repeated solution to the auxiliary equation, let $y_p = (A_n x^n + \dots + A_1 x + A_0) x^2 e^{rx}$

$$= (A_n x^{n+2} + \dots + A_1 x^3 + A_0 x^2) e^{rx}$$

If $f(x)$ is a degree n polynomial multiplied by $e^{\alpha x} \sin(\beta x)$ and/or $e^{\alpha x} \cos(\beta x)$, then

i) If $\alpha \pm \beta i$ are <u>not</u> solutions to the auxiliary equation, let

$$y_p = (A_n x^n + \dots + A_1 x + A_0) e^{\alpha x} \sin(\beta x) + (B_n x^n + \dots + B_1 x + B_0) e^{\alpha x} \cos(\beta x)$$

ii) If $\alpha \pm \beta i$ are solutions to the auxiliary eq.

let $$y_p = (A_n x^{n+1} + \dots + A_1 x^2 + A_0 x) e^{\alpha x} \sin(\beta x) + (B_n x^{n+1} + \dots + B_1 x^2 + B_0 x) e^{\alpha x} \cos(\beta x)$$

Ex- 1) $y'' - 3y' + 2y = e^{8x}$

2) $y'' - 3y' + 2y = xe^x$

3) $y'' - 2y' + y = e^{3x} + \cos(2x)$

1) Start with $y_p = Ae^{8x}$

Since 8 is not a root of auxiliary equation

we keep $y_p = Ae^{8x}$

Then $y_p' = 8Ae^{8x}$ and $y_p'' = 64Ae^{8x}$

Plug into DE to get

$$64Ae^{8x} - 24Ae^{8x} + 2Ae^{8x} = e^{8x}$$

$$42Ae^{8x} = e^{8x}$$

$$A = \frac{1}{42}$$

Thus, $y_p = \frac{1}{42} e^{8x}$

2) Start with $y_p = (Ax + B)e^x$

Since 1 is a single root of auxiliary eq,

we modify $y_p = (Ax^2 + Bx)e^x$

$$y_p' = (Ax^2 + Bx)e^x + (2Ax + B)e^x$$

$$y_p'' = (Ax^2 + Bx)e^x + 2(2Ax + B)e^x + 2Ae^x$$

Plug into DE to get

$$Ax^2e^x + Bxe^x + 4Axe^x + 2Be^x + 2Ae^x - 3Ax^2e^x$$
$$- 3Bxe^x - 6Axe^x - 3Be^x + 2Ax^2e^x + 2Bxe^x = xe^x$$

$$-2Axe^x + (2A-B)e^x = xe^x$$

Compare xe^x terms: $-2A = 1$

Compare e^x terms: $2A - B = 0$

So $A = -\frac{1}{2}$ and $B = -1$

Thus $y_p = \left(-\frac{1}{2}x^2 - x\right)e^x$

3) $y_p = Ae^{3x} + B\cos(2x) + C\sin(2x)$

$y_p' = 3Ae^{3x} - 2B\sin(2x) + 2C\cos(2x)$

$y_p'' = 9Ae^{3x} - 4B\cos(2x) - 4C\sin(2x)$

Plug these into DE to get

$$9Ae^{3x} - 4B\cos(2x) - 4C\sin(2x) - 6Ae^{3x}$$
$$+ 4B\sin(2x) - 4C\cos(2x) + Ae^{3x} + B\cos(2x)$$
$$+ C\sin(2x) = e^{3x} + \cos(2x)$$

$$4Ae^{3x} + (-3B-4C)\cos(2x) + (4B-3C)\sin(2x)$$
$$= e^{3x} + \cos(2x).$$

Compare e^{3x} terms: $4A = 1$

" $\cos(2x)$ terms: $-3B-4C = 1$

" $\sin(2x)$ terms: $4B-3C = 0$

$A = \frac{1}{4}$, $B = -\frac{3}{25}$, $C = -\frac{4}{25}$

Thus, $y_p = \frac{1}{4}e^{3x} - \frac{3}{25}\cos(2x) - \frac{4}{25}\sin(2x)$

This last example illustrates the <u>superposition principle</u>, which states if y_{p_1} is a solution to $ay'' + by' + cy = f(x)$ and y_{p_2} is a solution to $ay'' + by' + cy = g(x)$ then $y_{p_1} + y_{p_2}$ is a solution to $ay'' + by' + cy = f(x) + g(x)$

We can thus apply MoUC simultaneously for both f and g.

Ex: $y'' - y' = x + e^{4x}$

Start with $y_p = Ax + B + Ce^{4x}$

$= (Ax + B)e^{0x} + Ce^{4x}$

Since 0 is a root of the auxiliary eq, we modify $y_p = (Ax^2 + Bx)e^{0x} + Ce^{4x}$

$= Ax^2 + Bx + Ce^{4x}$

$y_p' = 2Ax + B + 4Ce^{4x}$

$y_p'' = 2A + 16Ce^{4x}$

Plug into DE to get

$2A + 16Ce^{4x} - 2Ax - B - 4Ce^{4x} = x + e^{4x}$

$-2Ax + (2A - B) + 12Ce^{4x} = x + e^{4x}$

Compare x terms: $-2A = 1$

" constant terms: $2A - B = 0$

" e^{4x} terms: $12C = 1$

So $A = -\frac{1}{2}$, $B = -1$, $C = \frac{1}{12}$

Thus $y_p = -\frac{1}{2}x^2 - x + \frac{1}{12}e^{4x}$

The superposition principle also tells us how to find the general solution to $ay'' + by' + cy = f(x)$. If we write this as

$ay'' + by' + cy = 0 + f(x)$, then if y_h is the solution to $ay'' + by' + cy = 0$ and y_p is a solution to $ay'' + by' + cy = f(x)$, then $y = y_h + y_p$ is a solution to $ay'' + by' + cy = f(x)$ as well.

Thus, to find the general solution to

$ay'' + by' + cy = f(x)$, :

1) Find general solution y_h to homogeneous DE $ay'' + by' + cy = 0$

2) Find particular solution y_p to $ay'' + by' + cy = f(x)$

3) $y = y_h + y_p$

Ex: 1) $y'' + 6y' + 8y = 3e^x$

2) $y'' - 7y' + 10y = e^{2x} + e^{5x}$

1) Auxiliary eq is $r^2 + 6r + 8 = 0$ ie $(r+4)(r+2) = 0$

so $r = -4, -2$

Thus $y_h = c_1 e^{-4x} + c_2 e^{-2x}$

Look for $y_p = Ae^x$. Then $y_p' = Ae^x = y_p''$

Plug into DE to get

$Ae^x + 6Ae^x + 8Ae^x = 3e^x$ ie $15Ae^x = 3e^x$

So $A = \frac{1}{5}$ and $y_p = \frac{1}{5}e^x$

Therefore, $y = c_1 e^{-4x} + c_2 e^{-2x} + \frac{1}{5}e^x$

2) Auxillary eq is $r^2 - 7r + 10 = 0$

ie $(r-5)(r-2) = 0$ so $r = 5, 2$

Thus $y_h = c_1 e^{5x} + c_2 e^{2x}$

Start with $y_p = Ae^{5x} + Be^{2x}$

Since 5 and 2 are both roots of the auxiliary eq, multiply both by x

ie $y_p = Axe^{5x} + Bxe^{2x} = x(Ae^{5x} + Be^{2x})$

Then $y_p' = x(5Ae^{5x} + 2Be^{2x}) + Ae^{5x} + Be^{2x}$

$y_p'' = x(25Ae^{5x} + 4Be^{2x}) + 2(5Ae^{5x} + 2Be^{2x})$

Plug these into DE to get

$25xAe^{5x} + 4Bxe^{2x} + 10Ae^{5x} + 4Be^{2x} - 35Axe^{5x} - 14Bxe^{2x}$

$$-7Ae^{5x} - 7Be^{2x} + 10Axe^{5x} + 10Bxe^{2x} = e^{5x} + e^{2x}$$

$$3Ae^{5x} - 3Be^{2x} = e^{5x} + e^{2x}$$

Compare e^{5x} terms: $3A = 1$

" e^{2x} terms: $-3B = 1$

So $A = \frac{1}{3}$, $B = -\frac{1}{3}$, $y_p = \frac{1}{3}xe^{5x} - \frac{1}{3}xe^{2x}$

Hence, $y = C_1e^{5x} + C_2e^{2x} + \frac{1}{3}xe^{5x} - \frac{1}{3}xe^{2x}$

A second technique for solving nonhomogeneous second-order DEs is <u>variation of parameters</u>.

For this method, we will write our DEs as $y'' + by' + cy = f(x)$

Recall that to solve the DE $y'' + by' + cy = 0$, we find two linearly independent solutions y_1, y_2 and let $y_h = C_1y_1 + C_2y_2$

To solve the nonhomogeneous DE, we instead look for <u>function</u> combination of y_1 and y_2 ie $y_p = v_1 y_1 + v_2 y_2$ where v_1 and v_2 are functions.

Then
$$y_p' = v_1' y_1 + v_1 y_1' + v_2 y_2' + v_2' y_2$$
$$= v_1 y_1' + v_2 y_2' + (v_1' y_1 + v_2' y_2)$$

$$y_p'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2' + \frac{d}{dx}(v_1' y_1 + v_2' y_2)$$

Plug these into the DE to get

$$v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2' + \frac{d}{dx}(v_1' y_1 + v_2' y_2) + b v_1 y_1' + b v_2 y_2' + b(v_1' y_1 + v_2' y_2) + c v_1 y_1 + c v_2 y_2 = f(x)$$

$$v_1(\underbrace{y_1'' + b y_1' + c y_1}_{=0}) + v_2(\underbrace{y_2'' + b y_2' + c y_2}_{=0}) + (v_1' y_1' + v_2' y_2') + \frac{d}{dx}(v_1' y_1 + v_2' y_2) + b(v_1' y_1 + v_2' y_2) = f(x)$$

$$v_1' y_1' + v_2' y_2' + \frac{d}{dx}(v_1' y_1 + v_2' y_2) + b(v_1' y_1 + v_2' y_2) = f(x)$$

To solve the last equation, we can let

$$v_1' y_1 + v_2' y_2 = 0$$

$$v_1' y_1' + v_2' y_2' = f(x)$$

So $$y_2'(v_1' y_1 + v_2' y_2) - y_2 (v_1' y_1' + v_2' y_2') = -y_2 f(x)$$

$$v_1' y_1 y_2' - v_1' y_1' y_2 = -y_2 f(x)$$

$$v_1' = \frac{-y_2 f(x)}{y_1 y_2' - y_1' y_2}$$

$$v_1 = \int \frac{-y_2 f(x)}{y_1 y_2' - y_1' y_2} dx$$

Likewise $$y_1 (v_1' y_1' + v_2' y_2') - y_1'(v_1' y_1 + v_2' y_2) = y_1 f(x)$$

$$v_2' y_1 y_2' - v_2' y_1' y_2 = y_1 f(x)$$

$$v_2' = \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2}$$

$$v_2 = \int \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2} dx$$

Note $y_1 y_2' - y_1' y_2 \neq 0$ because y_1, y_2 are linearly independent (think Wronskian)

Thus, to solve $y'' + by' + cy = f(x)$ using Variation of parameters,

1) Find general solution $y_h = c_1 y_1 + c_2 y_2$ of the homogeneous DE $y'' + by' + cy = 0$

2) Find a particular solution to nonhomogeneous DE by letting $y_p = v_1 y_1 + v_2 y_2$ where
$$v_1 = \int \frac{-y_2 f(x)}{y_1 y_2' - y_1' y_2}\, dx \quad \text{and} \quad v_2 = \int \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2}\, dx$$

3) $y = y_h + y_p$

Ex -

1) $y'' + y = \csc(x)$

2) $y'' - 4y' + 4y = 30\sqrt{x}\, e^{2x}$

1) Auxiliary eq is $r^2+1=0$ so $r=\pm i$

Hence $y_h = C_1\cos(x) + C_2\sin(x)$

$y_1 = \cos(x) \Rightarrow y_1' = -\sin(x)$

$y_2 = \sin(x) \Rightarrow y_2' = \cos(x)$

So $y_1 y_2' - y_1' y_2 = \cos^2(x) + \sin^2(x) = 1$

$$V_1 = \int \frac{-y_2 f(x)}{1}\,dx = \int -\sin(x)\csc(x)\,dx$$

$$= \int -1\,dx = -x$$

$$V_2 = \int \frac{y_1 f(x)}{1}\,dx = \int \cos(x)\csc(x)\,dx = \int \frac{\cos(x)}{\sin(x)}\,dx$$

$$= \ln|\sin(x)|$$

$$y_p = V_1 y_1 + V_2 y_2 = -x\cos(x) + \ln|\sin(x)|\sin(x)$$

$$y = y_h + y_p = C_1\cos(x) + C_2\sin(x) - x\cos(x) + \ln|\sin(x)|\sin(x)$$

2) Auxiliary eq is $r^2-4r+4=0$ so $r=2$

Then $y_h = C_1 e^{2x} + C_2 x e^{2x}$

$$y_1 = e^{2x} \Rightarrow y_1' = 2e^{2x}$$

$$y_2 = xe^{2x} \Rightarrow y_2' = 2xe^{2x} + e^{2x}$$

So $y_1 y_2' - y_1' y_2 = e^{4x}$

$$v_1 = \int \frac{-y_2 f(x)}{e^{4x}}\,dx = \int \frac{-30\sqrt{x}e^{2x}\,xe^{2x}}{e^{4x}}\,dx$$

$$= \int -30x^{3/2}\,dx$$

$$= -12x^{5/2}$$

$$v_2 = \int \frac{y_1 f(x)}{e^{4x}}\,dx = \int \frac{e^{2x}\,30\sqrt{x}\,e^{2x}}{e^{4x}}\,dx$$

$$= \int 30\sqrt{x}\,dx = 20x^{3/2}$$

So $y_p = v_1 y_1 + v_2 y_2 = -12x^{5/2}e^{2x} + 20x^{3/2}xe^{2x}$

And $y = y_h + y_p$

$$= c_1 e^{2x} + c_2 xe^{2x} - 12x^{5/2}e^{2x} + 20x^{5/2}e^{2x}$$

$$= c_1 e^{2x} + c_2 xe^{2x} + 8x^{5/2}e^{2x}$$