Recall a real-valued function of a single variable is an expression f that assigns to each value x in its domain $D_f \subseteq \mathbb{R}$ one and only one value $f(x)$ in its range $R_f \subseteq \mathbb{R}$

A real-valued function in two variables is an expression f that assigns to each point (x, y) in its domain $D_f \subseteq \mathbb{R}^2$ one and only one value $f(x,y)$ in its range $R_f \subseteq \mathbb{R}$.

Likewise for functions of 3, 4, etc. variables

Examples - $f(x,y) = 2x + 3y + e^{\frac{x^2}{4} + y}$

$$g(x,y) = \sqrt{x^2 - y}$$

$$h(x,y) = \ln(x^2 + y^2)$$

$$p(x,y) = \frac{x^2 + y^2}{xy - 3x - 2y + 6} = \frac{x^2+y^2}{(x-2)(y-3)}$$

$$q(x,y,z) = x^3 + xy^2 + yz + x^2yz^2 + \cdots$$

1) $f(2,-1) = 2(2) + 3(-1) + e^{\frac{4}{4}-1} = 2$

$D_f = \mathbb{R}^2$

2) $D_g = \{(x,y) \mid x^2 - y \geq 0\}$

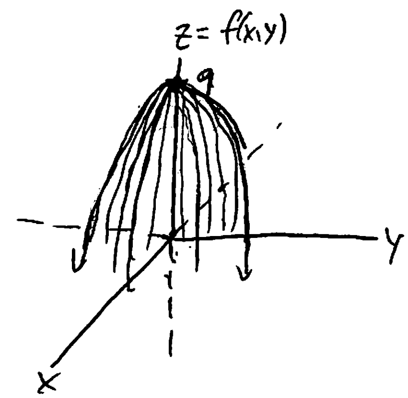
3) $D_h = \mathbb{R}^2 - \{(0,0)\} = \{(x,y) \mid x^2+y^2 > 0\}$

4) $D_p = \{(x,y) \mid x \neq 2,\ y \neq 3\}$

5) $D_q = \mathbb{R}^3$

Let f be a function of two variables with domain D_f. The graph of f is the set of all points of the form $(x, y, f(x,y))$ in $\mathbb{R}^3$, where (x,y) is in D_f.

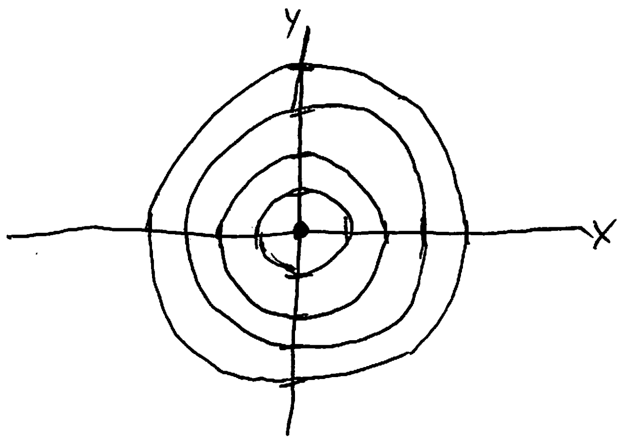
Example - $f(x,y) = 9 - x^2 - y^2$



Let f be a function of two variables, and let c be a value in its range R_f. The <u>c-level curve</u> of f is the set of points in $\mathbb{R}^2$ that satisfy the equation $f(x,y) = c$ in the xy plane. The set of all level curves of f is called the <u>contour diagram</u> of f.

Example - $f(x,y) = 9 - x^2 - y^2$ Draw the level curves for

1) $c = 0$
2) $c = \cancel{8}\ -7$
3) $c = 5$
4) $c = 8$



1) $9 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 9$ (circle with $r = 3$ center $(0,0)$)
2) $9 - x^2 - x^2 = -7 \Rightarrow x^2 + y^2 = 16$

Let f be a function of two variables. Then f has a limit L at the point (x_0, y_0) if, as (x,y) "gets close to" (x_0, y_0), $f(x,y)$ "gets close to" L. We write this as $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

Limit rules - Assume f and g have limits L_1 and L_2, respectively, at the point (x_0, y_0) in $\mathbb{R}^2$

ie $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L_1$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = L_2$.

Then

1) $\lim_{(x,y)\to(x_0,y_0)} (f \pm g)(x,y) = L_1 \pm L_2$

2) $\lim_{(x,y)\to(x_0,y_0)} (cf)(x,y) = cL_1$ for any constant c in $\mathbb{R}$

3) $\lim_{(x,y)\to(x_0,y_0)} (fg)(x,y) = L_1 L_2$

4) $\lim_{(x,y)\to(x_0,y_0)} \left(\frac{f}{g}\right)(x,y) = \frac{L_1}{L_2}$ provided $L_2 \neq 0$

5) $\lim_{(x,y)\to(x_0,y_0)} c = c$ for all constants c in $\mathbb{R}$

6) $\lim_{(x,y)\to(x_0,y_0)} x = x_0$

7) $\lim_{(x,y)\to(x_0,y_0)} y = y_0$

Example - $f(x,y) = x^2 + 3y^2 - 4xy + 2x - 7y + 1$

$$g(x,y) = \frac{3xy^2 + 4}{x^2 - 5y}$$

1) $\lim_{(x,y)\to(2,1)} f(x,y)$

2) $\lim_{(x,y)\to(2,1)} g(x,y)$

1) $\lim_{(x,y)\to(2,1)} (x^2 + 3y^2 - 4xy + 2x - 7y + 1)$

$$= \lim(x^2) + 3\lim(y^2) - 4\lim(xy) + 2\lim(x) - 7\lim(y) + \lim(1)$$

$$= \lim(x)\lim(x) + 3\lim(y)\lim(y) - 4\lim(x)\lim(y) + 2(2) - 7(1) + 1$$

$$= 2(2) + 3(1)(1) - 4(2)(1) + 4 - 7 + 1$$

$$= -3$$

Fact - If (x_0, y_0) is in the domain of f, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$$

Examples - $\displaystyle\lim_{(x,y)\to(2,4)} xy\cos(x^2-y) = 8$

$$\lim_{(x,y)\to(-2,8)} \left[\ln(x^3+y+1) + e^{x^3+y}\right] = 1$$

What if (x_0, y_0) is not in D_f?

Some cases generalize from single-variable calc.

Recall $\ln(x) \to -\infty$ as $x \to 0$. This still holds

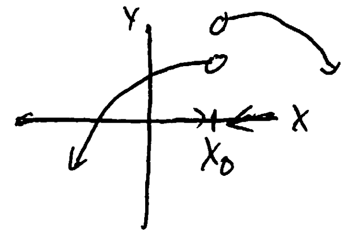
Also, if your limit is of the form $\frac{c}{0}$ where c is a nonzero constant, then limit is ∞ if $c>0$ and $-\infty$ if $c<0$. This still holds

Example- $\lim_{(x,y)\to(0,0)} \ln(x^2+y^2) = -\infty$

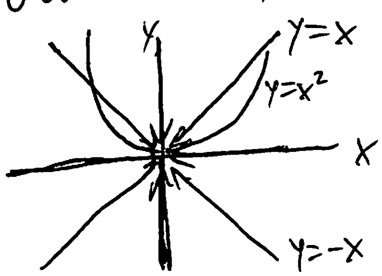
$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2+1}{3x+2y} = \infty$$

What about $\frac{0}{0}$, $0\cdot\infty$, and other indeterminate forms?

Recall that a single variable function $f(x)$ has a limit L at x_0 if and only if $f(x)$ has both a left hand and right hand limit at x_0 and both these limits equal L.



What is the analog for the multivariable case?
How many ways can you approach a point (x_0, y_0)?



How many directions can you approach $(0,0)$ from?

We can fix ~~x~~ $y=0$ and approach along the x-axis.
We can fix $x=0$ and approach along the y-axis
We can approach along the line $y=x$, $y=-x$, etc.
the parabola $y=x^2$, ...

There are infinitely many ways to approach a point (x_0, y_0).

A two variable function $f(x,y)$ has limit L at (x_0, y_0) if and only if f has a limit at (x_0, y_0) from all possible (infinite directions), and each of these limits is L.

Is this useful? Yes, for determining when f does not have a limit.

Example - $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$

Fix $y=0$ and take limit as $x \to 0$ to get

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$$

Fix $x=0$ and take limit as $y \to 0$ to get

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = \lim_{y\to 0} -1 = -1$$

Thus, limit does not exist.

Fact - If there exist two different directions approaching (x_0, y_0) such that $f(x,y)$ approaches a different value from each direction, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \text{ does not exist.}$$

Example - 1) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y^2}{x+y}$ 2) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$

1) Fix $y=0$ and take limit as $x \to 0$ to get

$$\lim_{(x,y)\to(0,0)} \frac{x+y^2}{x+y} = \lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$$

Fix $x=0$ and take limit as $y \to 0$ to get

$$\lim_{(x,y)\to(0,0)} \frac{x+y^2}{x+y} = \lim_{y\to 0} \frac{y^2}{y} = \lim_{y\to 0} y = 0$$

Thus, limit DNE.

2) Proceeding as in 1) we get limit $= 1$ for both $x=0$ and $y=0$

Let $y=x$ and take limit as $x \to 0$ to get

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2} = \lim_{x\to 0} \frac{(x+x)^2}{x^2+x^2} = \lim_{x\to 0} \frac{4x^2}{2x^2} = \lim_{x\to 0} 2 = 2$$

Thus, limit DNE