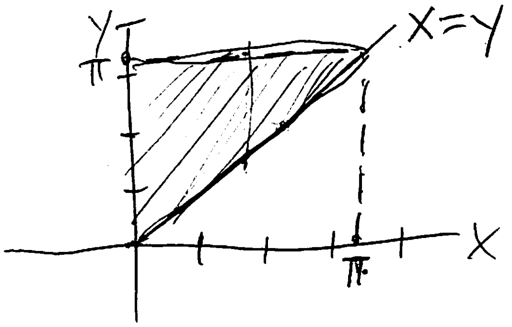
Sketch the region we are integrating over.



$0 \le x \le \pi$
$x \le y \le \pi$ Type 1

$0 \le y \le \pi$
$0 \le x \le y$ Type 2

Our integral becomes $\int_0^{\pi} \int_0^{y} x\cos(y^3)\, d\cancel{x}\, dy$

$$\int_0^{y} x\cos(y^3)\,dx = \frac{x^2\cos(y^3)}{2}\Big|_0^{y} = \frac{y^2\cos(y^3)}{2}$$

$$\int_0^{\pi} \frac{1}{2}y^2\cos(y^3)\,dy$$

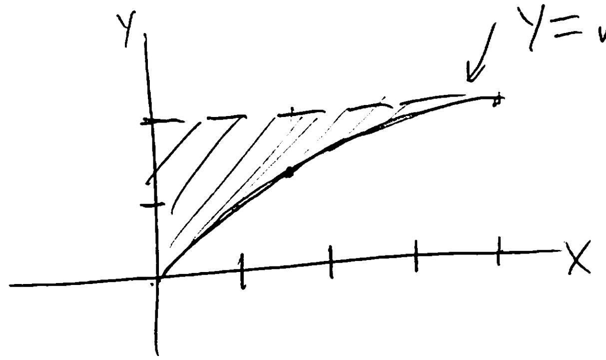
$u = y^3 \Rightarrow du = 3y^2dy$
$\Rightarrow \frac{1}{3}du = y^2dy$

$\downarrow$

$$\int \frac{1}{2}\cos(u)\frac{1}{3}du = \frac{1}{6}\int \cos(u)\,du = \frac{1}{6}\sin(u)$$

$$= \frac{1}{6}\sin(y^3)\Big|_0^{\pi} = \frac{1}{6}\sin(\pi^3)$$

Example- $\int_0^{4}\int_{\sqrt{x}}^{2} e^{y^3}\,dy\,dx$



$0 \le x \le 4$
$\sqrt{x} \le y \le 2$ Type 1

$0 \le y \le 2$
$0 \le x \le y^2$ Type 2

Our integral becomes $\displaystyle\int_0^2 \int_0^{y^2} e^{y^3}\,dx\,dy$

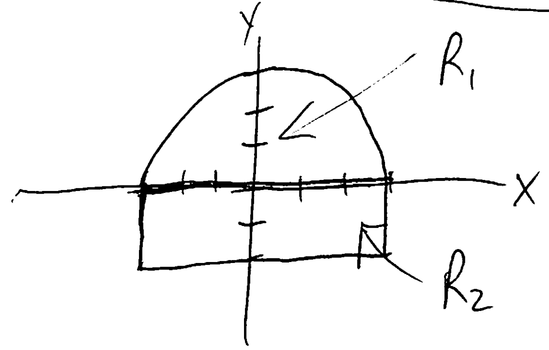
$$\int_0^{y^2} e^{y^3}\,dx = xe^{y^3}\Big|_0^{y^2} = y^2 e^{y^3}$$

$$\int_0^2 y^2 e^{y^3}\,dy \qquad u = y^3 \Rightarrow du = 3y^2\,dy \Rightarrow \tfrac{1}{3}du = y^2\,dy$$

$\downarrow$

$$\frac{1}{3}\int e^u\,du = \frac{1}{3}e^u = \frac{1}{3}e^{y^3}\Big|_0^2 = \frac{1}{3}e^8 - \frac{1}{3}$$

Let R be given by



Compute $\displaystyle\iint_R 2y\,dA$

Split R into $R_1 = \{(x,y) \mid -3 \le x \le 3,\ 0 \le y \le \sqrt{9-x^2}\}$

and $R_2 = \{(x,y) \mid -3 \le x \le 3,\ -2 \le y \le 0\}$

We know $\iint_{R_1} 2y\,dA = 36$ (from last time)

$$\iint_{R_2} 2y\,dA = \int_{-2}^{0}\int_{-3}^{3} 2y\,dx\,dy$$

$$\int_{-3}^{3} 2y\,dx = 2xy\Big|_{-3}^{3} = 6y - (-6y) = 12y$$

$$\int_{-2}^{0} 12y\,dy = 6y^2\Big|_{-2}^{0} = 0 - 24 = -24$$

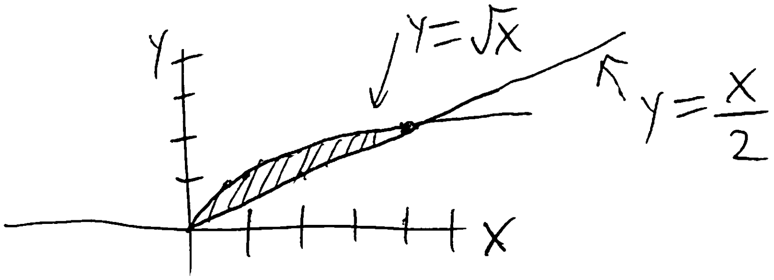
So $\iint_R 2y\,dA = 36 - 24 = 12$

Fact - if a region R can be split into two regions R_1 and R_2, then

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

Let R be the region bounded by $y = \sqrt{x}$ and $y = \frac{x}{2}$

Compute $\iint_R 3y\,dA$



$0 \le x \le 4$

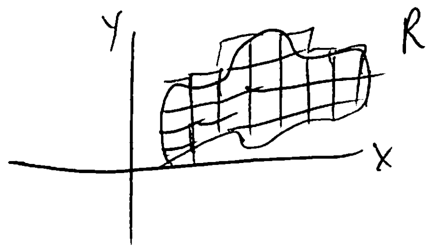
$\frac{x}{2} \le y \le \sqrt{x}$

$$\int_0^4 \int_{x/2}^{\sqrt{x}} 3y \, dy \, dx$$

$$\int_{x/2}^{\sqrt{x}} 3y \, dy = \frac{3}{2} y^2 \Big|_{\frac{x}{2}}^{\sqrt{x}} = \frac{3}{2}x - \frac{3}{8}x^2$$

$$\int_0^4 \frac{3}{2}x - \frac{3}{8}x^2 \, dx = \frac{3}{4}x^2 - \frac{1}{8}x^3 \Big|_0^4 = 12 - 8 = 4$$

Consider



Suppose we want to compute $\iint_R 1 \, dA$

Recall we estimate $\iint_R f(x,y)\, dA \approx \sum_{i=1}^{s} f(x_i^*, y_i^*) A_i$

where A_i = area of R_i

If $f(x,y) = 1$, then $\iint_R 1 \, dA \approx \sum_{i=1}^{s} A_i$ = area of R

Fact - The area of a region $R \subseteq \mathbb{R}^2$ is given by $\iint_R 1 \, dA$

Example - $R = \{(x,y) \mid 0 \le x \le 5,\ 0 \le y \le 4\}$

Area of $R = 20$. Show it with Calculus

$$\iint_R 1\, dA = \int_0^4 \int_0^5 1\, dx\, dy$$

$$\int_0^5 1\, dx = x \Big|_0^5 = 5$$

$$\int_0^4 5\, dy = 5y \Big|_0^4 = 20$$

Recall the average value of $f(x)$ on $[a,b]$

is $f_{avg} = \dfrac{\int_a^b f(x)\,dx}{b-a} = \dfrac{\int_a^b f(x)\,dx}{\int_a^b 1\,dx}$

The average value of $f(x,y)$ on $R \subseteq \mathbb{R}^2$ is

$$f_{avg} = \frac{\iint_R f(x,y)\,dA}{\iint_R 1\,dA}$$

Example - $f(x,y) = y$, $R = \{(x,y) \mid 0 \le x \le 5,\ 0 \le y \le 4\}$

We just showed $\iint_R 1\,dA = 20$

$$\iint_R f(x,y)\,dA = \int_0^4 \int_0^5 y\,dx\,dy$$

$$\int_0^5 y\,dx = xy\Big|_0^5 = 5y$$

$$\int_0^4 5y\,dy = \frac{5}{2}y^2\Big|_0^4 = 40$$

So $f_{avg} = \dfrac{40}{20} = 2$

Recall the mean value theorem for integrals says if $f(x)$ is continuous on $[a,b]$, then there exists $c \in [a,b]$ such that $f(c) = f_{avg}$

Mean Value Theorem. If $f(x,y)$ is continuous on $R \subseteq \mathbb{R}^2$, then there exists $(x_0, y_0) \in R$ such that $f(x_0, y_0) = f_{avg}$

For our example above ($f(x,y) = y$), we have $f(1,2) = 2 = f_{avg}$

Consider

Suppose we want to compute volume between f and g over R

V = volume under g over R − volume under f over R

$$= \iint_R g(x,y)\,dA - \iint_R f(x,y)\,dA$$

$$= \iint_R [g(x,y) - f(x,y)]\,dA$$

Fact - If f and g are defined on a region $R \subseteq \mathbb{R}^2$, with $f(x,y) \leq g(x,y)$ for all points in R, then the volume of the solid region between f and g over R is $\iint_R [g(x,y) - f(x,y)]\,dA$

Example - find volume between $x+y+3$ and $3x+5y+7$ over $R = \{(x,y) \mid 0 \leq x \leq 3,\ 0 \leq y \leq 2\}$

Here, $3x+5y+7 \geq x+y+3$ on R

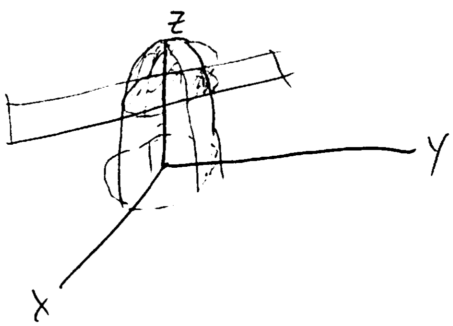
So $V = \iint_R [(3x+5y+7) - (x+y+3)]\,dA = \int_0^2 \int_0^3 (2x+4y+4)\,dx\,dy$

$$\int_0^3 2x+4y+4\,dx = x^2 + 4xy + 4x \Big|_0^3 = 21 + 12y$$

$$\int_0^2 21 + 12y\,dy = 21y + 6y^2 \Big|_0^2 = 42 + 24 = 66$$

Set up the integral for the volume between $z = 9 - x^2 - y^2$ and $z = 5$

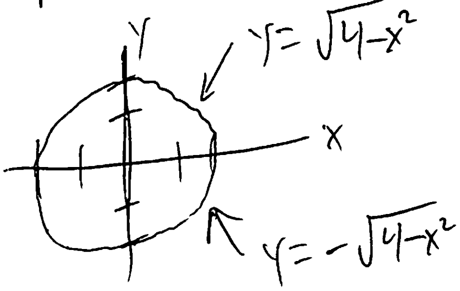
The region of integration is defined by the intersection of these two surfaces



To find curve of intersection, set equations equal to each other

$$9 - x^2 - y^2 = 5 \implies x^2 + y^2 = 4$$

Now represent this as either Type 1 or Type 2 region



$$\left.\begin{array}{l} -2 \le x \le 2 \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \end{array}\right\} R$$

Observe that $9 - x^2 - y^2 \geq 5$ for all points in R

So $V = \displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (9-x^2-y^2) - 5 \, dy\,dx$