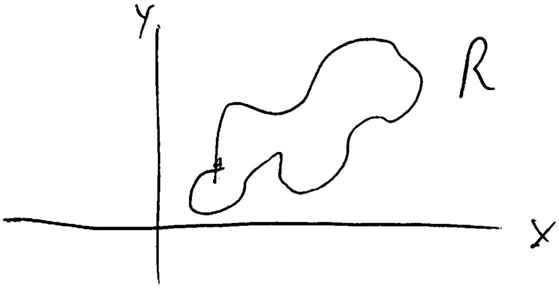
A lamina is the name given to any two-dimensional object in $\mathbb{R}^2$



The center of mass of a lamina is the point where the lamina balances.

Consider a lamina defined by a region $R \subseteq \mathbb{R}^2$. The mass of the lamina depends primarily on the mass density of the lamina (ie the mass per unit area),

If the density, denoted by σ, is constant, then the mass is $M = \sigma \cdot (\text{area of } R)$

If the mass density is a function $\sigma(x,y)$, then $M = \iint_R \sigma(x,y)\, dA$

Example - compute mass of the lamina defined by $R = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$, with mass density function $\sigma(x,y) = 6xy$

$$\iint_R \sigma(x,y)\,dA = \int_0^2 \int_0^3 6xy\,dx\,dy$$

$$\int_0^3 6xy\,dx = 3x^2y \Big|_0^3 = 27y$$

$$\int_0^2 27y\,dy = \frac{27y^2}{2}\Big|_0^2 = 54$$

The <u>center of mass</u> of a lamina with mass density function $\sigma(x,y)$ is

$$\left(\frac{\iint_R x\sigma(x,y)\,dA}{M}, \frac{\iint_R y\sigma(x,y)\,dA}{M} \right)$$

Example - R and $\sigma(x,y)$ as above

$M = 54$

$$\iint_R x\sigma(x,y)\,dA = \int_0^2 \int_0^3 6x^2y\,dx\,dy$$

$$\int_0^3 6x^2y\,dx = 2x^3y\Big|_0^3 = 54y$$

$$\int_0^2 54y\,dy = 27y^2\Big|_0^2 = 108$$

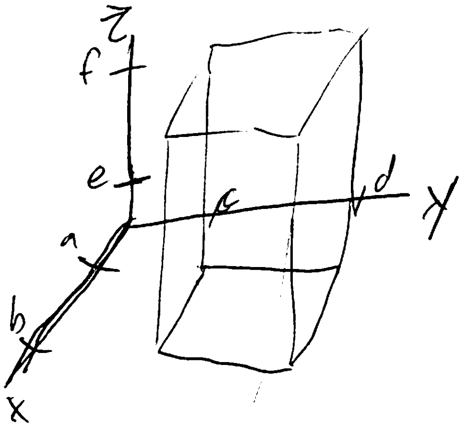
$$\iint_R y\sigma(x,y)\,dA = \int_0^2\int_0^3 6xy^2\,dx\,dy$$

$$\int_0^3 6xy^2\,dx = 3x^2y^2\Big|_0^3 = 27y^2$$

$$\int_0^2 27y^2\,dy = 9y^3\Big|_0^2 = 72$$

So center $= \left(\frac{108}{54}, \frac{72}{54}\right) = \left(2, \frac{4}{3}\right)$

Consider a rectangular prism region in $\mathbb{R}^3$ given by $R = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ e \le z \le f\}$



and consider a function $g(x,y,z)$

Divide $[a,b]$ into m subintervals, $[c,d]$ into n subintervals, and $[e,f]$ into p subintervals.

This yields $S = mnp$ sub-prism regions, call them $R_1, R_2, \ldots, R_S$.

The Riemann sum of g over R is $\sum_{i=1}^{S} g(x_i^*, y_i^*, z_i^*) V_i$

where V_i = volume of R_i and (x_i^*, y_i^*, z_i^*) is a sample point in R_i (vertice, center, etc.)

Take the limit as S goes to infinity to get

$$\iiint_R g(x,y,z)\,dV = \int_e^f \int_c^d \int_a^b g(x,y,z)\,dx\,dy\,dz$$

Fact - If a, b, c, d, e, f are all constants, then the order of integration does not matter, so long as the correct limits of integration go with the correct variable.

Example - $\iiint_R 2xy^2z^3\,dV$ where

$$R = \{(x,y,z) \mid 4 \le x \le 8,\ 1 \le y \le 2,\ 0 \le z \le 1\}$$

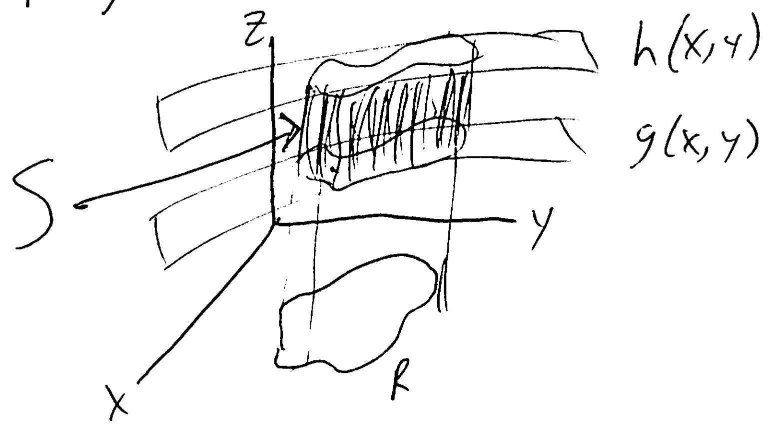
$$\int_0^1 \int_1^2 \int_4^8 2xy^2z^3 \, dx\, dy\, dz$$

$$\int_4^8 2xy^2z^3 \, dx = x^2y^2z^3 \Big|_4^8 = 64y^2z^3 - 16y^2z^3$$
$$= 48y^2z^3$$

$$\int_1^2 48y^2z^3 \, dy = 16y^3z^3 \Big|_1^2 = 128z^3 - 16z^3$$
$$= 112z^3$$

$$\int_0^1 112z^3 \, dz = 28z^4 \Big|_0^1 = 28$$

We can integrate over nonrectangular prism regions as well.



Then $\iiint_S f(x,y,z)\, dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\, dz \right) dA$

Note R could be a Type 1 or Type 2 region in the xy plane

Important: We are not finding the volume of S. We are integrating $f(x,y,z)$ over S.

This type of region is called z-simple.

Example: $\int_0^2 \int_1^x \int_{x-y}^{x+y} (z+3)\,dz\,dy\,dx$

$$\int_{x-y}^{x+y} z+3\,dz = \frac{z^2}{2} + 3z \Big|_{x-y}^{x+y}$$

$$= \left[\tfrac{1}{2}(x^2+2xy+y^2) + 3x+3y\right] - \left[\tfrac{1}{2}(x^2-2xy+y^2)+3x-3y\right]$$

$$= \tfrac{1}{2}x^2 + xy + \tfrac{1}{2}y^2 + 3x+3y - \tfrac{1}{2}x^2 + xy - \tfrac{1}{2}y^2 - 3x + 3y$$

$$= 2xy + 6y$$

$$\int_1^x (2xy+6y)\,dy = xy^2 + 3y^2 \Big|_1^x$$

$$= (x^3+3x^2) - (x+3)$$

$$= x^3 + 3x^2 - x - 3$$

$$\int_0^2 (x^3+3x^2-x-3)\,dx = \frac{x^4}{4} + x^3 - \frac{x^2}{2} - 3x \Big|_0^2$$

$$= 4 + 8 - 2 - 6$$

$$= 4$$

We can define y-simple and x-simple regions similarly

Namely, a y-simple region integral looks like

$$\iint_R \left(\int_{g(x,z)}^{h(x,z)} f(x,y,z) \right) dA \quad \text{where } R \subseteq xz \text{ plane}$$

An x-simple region integral looks like

$$\iint_R \left(\int_{g(y,z)}^{h(y,z)} f(x,y,z) \right) dA \quad \text{where } R \subseteq yz \text{ plane}$$

Examples - 1) $\int_0^2 \int_0^z \int_0^{\sqrt{3x+4z}} 2xy \, dy \, dx \, dz$

2) $\int_0^2 \int_{-z}^0 \int_2^{9y+6z} yz \, dx \, dy \, dz$

1) $\int_0^{\sqrt{3x+4z}} 2xy \, dy = xy^2 \Big|_0^{\sqrt{3x+4z}}$

$= x(3x+4z) = 3x^2 + 4xz$

$\int_0^z 3x^2 + 4xz \, dx = x^3 + 2x^2 z \Big|_0^z$

$= z^3 + 2z^3 = 3z^3$

$$\int_0^2 3z^3\,dz = \frac{3z^4}{4}\Big|_0^2 = 12$$

2) $$\int_2^{9y+6z} yz\,dx = xyz\Big|_2^{9y+6z}$$

$$= yz(9y+6z) - 2yz$$

$$= 9y^2z + 6yz^2 - 2yz$$

$$\int_{-z}^{0} (9y^2z + 6yz^2 - 2yz)\,dy = 3y^3z + 3y^2z^2 - y^2z\Big|_{-z}^{0}$$

$$= 0 - [-3z^4 + 3z^4 - z^3]$$

$$= z^3$$

$$\int_0^2 z^3\,dz = \frac{z^4}{4}\Big|_0^2 = 4$$

Recall that the area of a region $R \subseteq \mathbb{R}^2$ is given by $\iint_R 1\,dA$

The volume of a region $R \subseteq \mathbb{R}^3$ is given by $\iiint_R 1\,dV$

Ex - $R = \{(x,y,z) \mid 1 \leq x \leq 3,\ 1 \leq y \leq 4,\ 0 \leq z \leq 5\}$

Volume of R is 30. Show it with Calculus.

$$\iiint_R 1\, dV = \int_0^5 \int_1^4 \int_1^3 1\, dx\, dy\, dz$$

$$\int_1^3 1\, dx = x\Big|_1^3 = 3-1 = 2$$

$$\int_1^4 2\, dy = 2y\Big|_1^4 = 8-2 = 6$$

$$\int_0^5 6\, dz = 6z\Big|_0^5 = 30$$

Recall that the average value of $f(x,y)$ on $R \subseteq \mathbb{R}^2$ is $f_{avg} = \dfrac{\iint_R f(x,y)\, dA}{\iint_R 1\, dA}$

The average value of $f(x,y,z)$ on $R \subseteq \mathbb{R}^3$ is

$$f_{avg} = \frac{\iiint_R f(x,y,z)\, dV}{\iiint_R 1\, dV}$$

Consider a 3D object defined by $R \subseteq \mathbb{R}^3$

The center of mass of the object is the point where it balances.

The mass of R depends on its mass density (ie mass per unit volume), denoted by σ

If σ is constant, $M = \sigma(\text{Volume of } R)$

If $\sigma(x,y,z)$ is a function, $M = \iiint_R \sigma(x,y,z)\,dV$

Example - $R = \{(x,y,z) \mid 0 \le x \le 2,\ 0 \le y \le 2,\ 0 \le z \le 4\}$

$$\sigma(x,y,z) = xyz$$

$$M = \iiint_R \sigma(x,y,z)\,dV = \int_0^4 \int_0^2 \int_0^2 xyz\,dx\,dy\,dz$$

$$\int_0^2 xyz\,dx = \frac{1}{2}x^2yz \Big|_0^2 = 2yz$$

$$\int_0^2 2yz\,dy = y^2z \Big|_0^2 = 4z$$

$$\int_0^4 4z\,dz = 2z^2 \Big|_0^4 = 32$$

The center of mass of the object is

$$\left(\frac{\iiint_R x\,\sigma(x,y,z)\,dV}{M}, \frac{\iiint_R y\,\sigma(x,y,z)\,dV}{M}, \frac{\iiint_R z\,\sigma(x,y,z)\,dV}{M} \right)$$