1) $L^{-1}\left(\frac{9}{s}\right) = 9L^{-1}\left(\frac{1}{s}\right) = 9(1) = 9$

2) $L^{-1}\left(\frac{1}{s^2-5s+6}\right) = L^{-1}\left(\frac{1}{s-3} - \frac{1}{s-2}\right)$

$= L^{-1}\left(\frac{1}{s-3}\right) - L^{-1}\left(\frac{1}{s-2}\right)$

$= e^{3t} - e^{2t}$

3) $L^{-1}\left(\frac{s-5}{s^2+4s+3}\right) = L^{-1}\left(\frac{s-5}{(s+3)(s+1)}\right)$

$\frac{s-5}{(s+3)(s+1)} = \frac{A}{s+3} + \frac{B}{s+1} \Rightarrow s-5 = A(s+1) + B(s+3)$

$\Rightarrow A = 4, \quad B = -3$

Thus, we have $L^{-1}\left(\frac{s-5}{(s+3)(s+1)}\right) = L^{-1}\left(\frac{4}{s+3} - \frac{3}{s+1}\right)$

$= L^{-1}\left(\frac{4}{s+3}\right) - L^{-1}\left(\frac{3}{s+1}\right)$

$= 4L^{-1}\left(\frac{1}{s+3}\right) - 3L^{-1}\left(\frac{1}{s+1}\right)$

$= 4e^{-3t} - 3e^{-t}$

4) $\mathcal{L}^{-1}\left(\frac{4}{s^2+6s+25}\right) = \mathcal{L}^{-1}\left(\frac{4}{(s+3)^2+16}\right)$

$= e^{-3t}\sin(\cancel{4}t)$

5) $\mathcal{L}^{-1}\left(\frac{s-7}{s^2-4s+5}\right) = \mathcal{L}^{-1}\left(\frac{s-7}{(s-2)^2+1}\right)$

$= \mathcal{L}^{-1}\left(\frac{s-2}{(s-2)^2+1} - \frac{5}{(s-2)^2+1}\right)$

$= \mathcal{L}^{-1}\left(\frac{s-2}{(s-2)^2+1}\right) - 5\mathcal{L}^{-1}\left(\frac{1}{(s-2)^2+1}\right)$

$= e^{2t}\cos(t) - 5e^{2t}\sin(t)$

6) $\mathcal{L}^{-1}\left(\frac{8}{s^2-2s+1}\right) = \mathcal{L}^{-1}\left(\frac{8}{(s-1)^2}\right)$

$= 8\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right)$

$= 8te^{t}$

7) $\mathcal{L}^{-1}\left(\frac{10}{s^3}\right) = 5\mathcal{L}^{-1}\left(\frac{2}{s^3}\right) = 5t^2$

Ex - Compute the Laplace transform of $f'(t)$

$$L(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt$$

Let $u = e^{-st}$ and $dv = f'(t)\,dt$

Then $du = -se^{-st}$ and $v = f(t)$

So $\int e^{-st} f'(t)\,dt = e^{-st} f(t) + s\int e^{-st} f(t)\,dt$

Thus, $\int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$

$$= \lim_{N\to\infty}\left[e^{-st} f(t)\Big|_0^N\right] + sL(f(t))$$

$$= \lim_{N\to\infty}\left[e^{-sN} f(N) - f(0)\right] + sL(f(t))$$

If we assume f has exponential order K for some constant K, then for $s > K$,

$$\lim_{N\to\infty} e^{-sN} f(N) = 0$$

Thus, we are left with $sL(f) - f(0)$

Fact - If f is continuous on $[0, \infty)$, $f'(t)$ is piecewise continuous on $[0, \infty)$, and both have exponential order K, then for $s > K$, $L(f') = sL(f) - f(0)$

Ex - Compute $L(f'')$

$$L(f'') = sL(f') - f'(0)$$
$$= s(sL(f) - f(0)) - f'(0)$$
$$= s^2 L(f) - sf(0) - f'(0)$$

Consider a second-order DE $ay'' + by' + cy = f(t)$ with initial conditions $y(0)$, $y'(0)$.

Taking the Laplace transform of both sides of the DE yields $L(ay'' + by' + cy) = L(f(t))$

$$\Rightarrow \quad aL(y'') + bL(y') + cL(y) = L(f)$$

$$\Rightarrow \quad a\left(s^2L(y) - sy(0) - y'(0)\right) + b\left(sL(y) - y(0)\right) + cL(y) = L(f)$$

$$\Rightarrow \quad as^2L(y) - asy(0) - ay'(0) + bsL(y) - by(0) + cL(y) = L(f)$$

$$\Rightarrow \quad (as^2 + bs + c)L(y) = asy(0) + ay'(0) + by(0) + L(f)$$

$$\Rightarrow \quad L(y) = \frac{asy(0) + ay'(0) + by(0) + L(f)}{as^2 + bs + c}$$

$$\Rightarrow \quad y = L^{-1}\left(\frac{asy(0) + ay'(0) + by(0) + L(f)}{as^2 + bs + c}\right)$$

Ex - $y'' - 3y' + 2y = 0, \quad y(0) = 5, \quad y'(0) = 8$

$$L(y'' - 3y' + 2y) = L(0)$$

$$L(y'') - 3L(y') + 2L(y) = 0$$

$$s^2L(y) - sy(0) - y'(0) - 3\left(sL(y) - y(0)\right) + 2L(y) = 0$$

$$s^2L(y) - 5s - 8 - 3sL(y) + 15 + 2L(y) = 0$$

$$(s^2-3s+2)L(y) = 5s-7$$

$$L(y) = \frac{5s-7}{s^2-3s+2} = \frac{5s-7}{(s-2)(s-1)} = \frac{3}{s-2} + \frac{2}{s-1}$$

So $y = L^{-1}\left(\frac{3}{s-2} + \frac{2}{s-1}\right) = 3L^{-1}\left(\frac{1}{s-2}\right) + 2L^{-1}\left(\frac{1}{s-1}\right)$

$\Rightarrow$ $y = 3e^{2t} + 2e^{t}$

Ex - $y'' - 2y' + y = 0 \qquad y(0) = 3, \quad y'(0) = 7$

$$L(y'' - 2y' + y) = L(0)$$

$$L(y'') - 2L(y') + L(y) = 0$$

$$s^2L(y) - sy(0) - y'(0) - 2(sL(y) - y(0)) + L(y) = 0$$

$$s^2L(y) - 3s - 7 - 2sL(y) + 6 + L(y) = 0$$

$$(s^2-2s+1)L(y) = 3s+1$$

$$L(y) = \frac{3s+1}{s^2-2s+1} = \frac{3s+1}{(s-1)^2} = \frac{3(s-1)+4}{(s-1)^2}$$

$$\Rightarrow \quad L(y) = \frac{3}{s-1} + \frac{4}{(s-1)^2}$$

$$\Rightarrow \quad y = L^{-1}\left(\frac{3}{s-1} + \frac{4}{(s-1)^2}\right) = 3L^{-1}\left(\frac{1}{s-1}\right) + 4L^{-1}\left(\frac{1}{(s-1)^2}\right)$$

$$\Rightarrow \quad y = 3e^t + 4te^t$$

Ex - $y'' + 4y' + 5y = 0, \quad y(0) = 5, \quad y'(0) = -2$

$$L(y'' + 4y' + 5y) = L(0)$$

$$L(y'') + 4L(y') + 5L(y) = 0$$

$$s^2 L(y) - s\,y(0) - y'(0) + 4(sL(y) - y(0)) + 5L(y) = 0$$

$$s^2 L(y) - 5s + 2 + 4sL(y) - 20 + 5L(y) = 0$$

$$(s^2 + 4s + 5)L(y) = 5s + 18$$

$$L(y) = \frac{5s+18}{s^2+4s+5} = \frac{5s+18}{(s+2)^2+1}$$

$$= \frac{5(s+2) + 8}{(s+2)^2 + 1}$$

$$\Rightarrow \quad L(y) = \frac{5(s+2)}{(s+2)^2+1} + \frac{8}{(s+2)^2+1}$$

$$\Rightarrow \quad y = L^{-1}\left(\frac{5(s+2)}{(s+2)^2+1} + \frac{8}{(s+2)^2+1}\right)$$

$$= 5L^{-1}\left(\frac{s+2}{(s+2)^2+1}\right) + 8L^{-1}\left(\frac{1}{(s+2)^2+1}\right)$$

$$= 5e^{-2t}\cos(t) + 8e^{-2t}\sin(t)$$

Ex- $\quad y'' - 4y' + 4y = e^{3t}, \quad y(0) = 2, \; y'(0) = 11$

$$L(y'' - 4y' + 4y) = L(e^{3t})$$

$$L(y'') - 4L(y') + 4L(y) = \frac{1}{s-3}$$

$$s^2L(y) - sy(0) - y'(0) - 4(sL(y) - y(0)) + 4L(y) = \frac{1}{s-3}$$

$$s^2L(y) - 2s - 11 - 4sL(y) + 8 + 4L(y) = \frac{1}{s-3}$$

$$(s^2 - 4s + 4)L(y) = 2s + 3 + \frac{1}{s-3} = \frac{2s^2 - 3s - 8}{s-3}$$

$$L(y) = \frac{2s^2 - 3s - 8}{(s-3)(s^2 - 4s + 4)} = \frac{2s^2 - 3s - 8}{(s-3)(s-2)^2}$$

$$\frac{2s^2-3s-8}{(s-3)(s-2)^2} = \frac{A}{s-3} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$2s^2-3s-8 = A(s-2)^2 + B(s-2)(s-3) + C(s-3)$$

Plug in $s=3 \Rightarrow 1 = A$

Plug in $s=2 \Rightarrow -6 = -C$ so $C = 6$

Plug in $s=0 \Rightarrow -8 = 4 + 6B - 18$

$\Rightarrow B = 1$

So $\mathcal{L}(y) = \frac{1}{s-3} + \frac{1}{s-2} + \frac{6}{(s-2)^2}$

$$y = \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + 6\mathcal{L}^{-1}\left(\frac{1}{(s-2)^2}\right)$$

$$y = e^{3t} + e^{2t} + 6te^{2t}$$

Ex - $y'' - 8y' + 12y = 65\sin(3t)$, $y(0) = 4$

$y'(0) = 5$

$$L(y'' - 8y' + 12y) = L(65\sin(3t))$$

$$L(y'') - 8L(y') + 12L(y) = 65L(\sin(3t))$$

$$s^2L(y) - sy(0) - y'(0) - 8(sL(y) - y(0)) + 12L(y) = 65\left(\frac{3}{s^2+9}\right)$$

$$s^2L(y) - 4s - 5 - 8sL(y) + 32 + 12L(y) = \frac{195}{s^2+9}$$

$$(s^2 - 8s + 12)L(y) = 4s - 27 + \frac{195}{s^2+9}$$

$$(s^2 - 8s + 12)L(y) = \frac{4s^3 - 27s^2 + 36s - 48}{s^2+9}$$

$$L(y) = \frac{4s^3 - 27s^2 + 36s - 48}{(s^2 - 8s + 12)(s^2+9)}$$

$$L(y) = \frac{4s^3 - 27s^2 + 36s - 48}{(s-6)(s-2)(s^2+9)}$$

$$\frac{4s^3 - 27s^2 + 36s - 48}{(s-6)(s-2)(s^2+9)} = \frac{A}{s-6} + \frac{B}{s-2} + \frac{Cs+D}{s^2+9}$$

$$4s^3 - 27s^2 + 36s - 48 = A(s-2)(s^2+9) + B(s-6)(s^2+9) + (Cs+D)(s-6)(s-2)$$

Plug in $s=6 \Rightarrow 60 = A(180) \Rightarrow A = \frac{1}{3}$

Plug in $s=2 \Rightarrow -52 = -52B \Rightarrow B = 1$

Plug in $s=0 \Rightarrow -48 = -54 + 12D$

$\Rightarrow D = 1$

Plug in $s=1 \Rightarrow -35 = -\frac{10}{3} - 50 + (C+1)5$

$\Rightarrow 10 = -\frac{10}{3} + 5C$

$\Rightarrow \frac{40}{3} = 5C \Rightarrow C = \frac{8}{3}$

Thus, $$L(y) = \frac{\frac{1}{3}}{s-6} + \frac{1}{s-2} + \frac{\frac{8}{3}s+1}{s^2+9}$$

So $$y = \frac{1}{3}L^{-1}\left(\frac{1}{s-6}\right) + L^{-1}\left(\frac{1}{s-2}\right) + L^{-1}\left(\frac{\frac{8}{3}s+1}{s^2+9}\right)$$

$$y = \frac{1}{3}e^{6t} + e^{2t} + \frac{8}{3}L^{-1}\left(\frac{s}{s^2+9}\right) + \frac{1}{3}L^{-1}\left(\frac{3}{s^2+9}\right)$$

$$\boxed{y = \frac{1}{3}e^{6t} + e^{2t} + \frac{8}{3}\cos(3t) + \frac{1}{3}\sin(3t)}$$

Ex- $y'' - 6y' + 13y = 25e^{4t}$

$y(0) = 11, \quad y'(0) = 20$

$$\mathcal{L}(y'' - 6y' + 13y) = \mathcal{L}(25e^{4t})$$

$$\mathcal{L}(y'') - 6\mathcal{L}(y') + 13\mathcal{L}(y) = 25\mathcal{L}(e^{4t})$$

$$s^2\mathcal{L}(y) - sy(0) - y'(0) - 6(s\mathcal{L}(y) - y(0)) + 13\mathcal{L}(y) = \frac{25}{s-4}$$

$$s^2\mathcal{L}(y) - 11s - 20 - 6s\mathcal{L}(y) + 66 + 13\mathcal{L}(y) = \frac{25}{s-4}$$

$$(s^2 - 6s + 13)\mathcal{L}(y) = 11s - 46 + \frac{25}{s-4} = \frac{11s^2 - 90s + 209}{s-4}$$

$$\mathcal{L}(y) = \frac{11s^2 - 90s + 209}{(s-4)(s^2 - 6s + 13)}$$

$$\frac{11s^2 - 90s + 209}{(s-4)(s^2 - 6s + 13)} = \frac{A}{s-4} + \frac{Bs + C}{s^2 - 6s + 13}$$

$$11s^2 - 90s + 209 = A(s^2 - 6s + 13) + (Bs + C)(s-4)$$

$$11s^2 - 90s + 209 = As^2 - 6As + 13A + Bs^2 - 4Bs + Cs - 4C$$

Compare s^2 terms: $11 = A + B$

" s terms: $-90 = -6A - 4B + C$

" constant terms: $209 = 13A - 4C$

$B = 11 - A$ and $C = \frac{13}{4}A - \frac{209}{4}$

So $-90 = -6A - 44 + 4A + \frac{13}{4}A - \frac{209}{4}$

$$-360 = -24A - 176 + 16A + 13A - 209$$

$$25 = 5A \Rightarrow A = 5$$

Thus, $B = 6$ and $C = \frac{65 - 209}{4} = \frac{-144}{4} = -36$

So $\mathcal{L}(y) = \frac{5}{s-4} + \frac{6s - 36}{s^2 - 6s + 13}$

$$y = 5\mathcal{L}^{-1}\left(\frac{1}{s-4}\right) + \mathcal{L}^{-1}\left(\frac{6s-36}{s^2-6s+13}\right)$$

$$y = 5e^{4t} + \mathcal{L}^{-1}\left(\frac{6s-36}{(s-3)^2+4}\right)$$

$$y = 5e^{4t} + \mathcal{L}^{-1}\left(\frac{6(s-3) - 18}{(s-3)^2+4}\right)$$

$$y = 5e^{4t} + 6\mathcal{L}^{-1}\left(\frac{s-3}{(s-3)^2+4}\right) - 9\mathcal{L}^{-1}\left(\frac{2}{(s-3)^2+4}\right)$$

$$y = 5e^{4t} + 6e^{3t}\cos(2t) - 9e^{3t}\sin(2t)$$

Ex - A mass of 0.5 kg is attached to a spring with spring constant 8 N/m. There is a damping effect of 4 $N \cdot \frac{m}{s}$ and an external force of $20.5\cos(5t)$ N. If the mass is stretched 1 m from its equilibrium and released with a velocity of 7 m/s,

1) Set up a DE with ICs to model this
2) Solve with Laplace

1) $0.5y'' + 4y' + 8y = 20.5\cos(5t)$

$y(0) = 1$ and $y'(0) = 7$

2) $y'' + 8y' + 16y = 41\cos(5t)$

$$L(y'' + 8y' + 16y) = L(41\cos(5t))$$

$$L(y'') + 8L(y') + 16L(y) = 41L(\cos(5t))$$

$$s^2L(y) - sy(0) - y'(0) + 8(sL(y) - y(0)) + 16L(y) = 41\left(\frac{s}{s^2+25}\right)$$

$$s^2L(y) - s - 7 + 8sL(y) - 8 + 16L(y) = \frac{41s}{s^2+25}$$

$$(s^2 + 8s + 16)L(y) = s + 15 + \frac{41s}{s^2+25} = \frac{s^3 + 15s^2 + 66s + 375}{s^2+25}$$

$$L(y) = \frac{s^3 + 15s^2 + 66s + 375}{(s^2+8s+16)(s^2+25)} = \frac{s^3 + 15s^2 + 66s + 375}{(s+4)^2(s^2+25)}$$

$$\frac{s^3 + 15s^2 + 66s + 375}{(s+4)^2(s^2+25)} = \frac{A}{s+4} + \frac{B}{(s+4)^2} + \frac{Cs + D}{s^2+25}$$

$$s^3 + 15s^2 + 66s + 375 = A(s+4)(s^2+25) + B(s^2+25) + (Cs+D)(s+4)^2$$

Plug in $s=-4 \Rightarrow 287 = 41B \Rightarrow B = 7$

plug in $s=0 \Rightarrow 375 = 100A + 175 + 16D$

$\Rightarrow 200 = 100A + 16D$

Plug in $s=1 \Rightarrow 457 = 130A + 182 + 25C + 25D$

$\Rightarrow 275 = 130A + 25C + 25D$

Plug in $s=-1 \Rightarrow 323 = 78A + 182 - 9C + 9D$

$\Rightarrow 141 = 78A - 9C + 9D$

$C = \frac{78}{9}A + D - \frac{141}{9} = \frac{26}{3}A + D - \frac{47}{3}$

~~$275 = 130A$~~ $C = \frac{-130}{25}A - D + \frac{275}{25} = \frac{-26}{5}A - D + 11$

So $\frac{26}{3}A + D - \frac{47}{3} = -\frac{26}{5}A - D + 11$

$\Rightarrow 2D = \frac{-208}{15}A + \frac{80}{3} \Rightarrow D = \frac{-104}{15}A + \frac{40}{3}$

And $16D = -100A + 200 \Rightarrow D = \frac{-25}{4}A + \frac{25}{2}$

So $\frac{-104}{15}A + \frac{40}{3} = -\frac{25}{4}A + \frac{25}{2} \Rightarrow \frac{41}{60}A = \frac{5}{6} \Rightarrow A = \frac{50}{41}$

Then $D = -\frac{25}{4}\left(\frac{50}{41}\right) + \frac{25}{2} = \frac{200}{41}$

And $C = \frac{26}{3}\left(\frac{50}{41}\right) + \frac{200}{41} - \frac{47}{3} = -\frac{9}{41}$

So $\mathcal{L}(y) = \dfrac{50/41}{s+4} + \dfrac{7}{(s+4)^2} + \dfrac{-\frac{9}{41}s + \frac{200}{41}}{s^2+25}$

$$\Rightarrow y = \frac{50}{41}\mathcal{L}^{-1}\left(\frac{1}{s+4}\right) + 7\mathcal{L}^{-1}\left(\frac{1}{(s+4)^2}\right) - \frac{9}{41}\mathcal{L}^{-1}\left(\frac{s}{s^2+25}\right) + \frac{200}{41}\mathcal{L}^{-1}\left(\frac{1}{s^2+25}\right)$$

$$\Rightarrow y = \frac{50}{41}e^{-4t} + 7te^{-4t} - \frac{9}{41}\cos(5t) + \frac{40}{41}\mathcal{L}^{-1}\left(\frac{5}{s^2+25}\right)$$

$$\Rightarrow y = \frac{50}{41}e^{-4t} + 7te^{-4t} - \frac{9}{41}\cos(5t) + \frac{40}{41}\sin(5t)$$