Let $f(x,y)$ be a two-variable function with domain $D_f \subseteq \mathbb{R}^2$.

$f(x_0, y_0)$ is a local maximum value of $f$ at $(x_0, y_0) \in D_f$ if $f(x,y) \leq f(x_0, y_0)$ for all $(x,y)$ "near" $(x_0, y_0)$

$f(x_0, y_0)$ is a local minimum value of $f$ at $(x_0, y_0) \in D_f$ if $f(x,y) \geq f(x_0, y_0)$ for all $(x,y)$ "near" $(x_0, y_0)$

$f(x_0, y_0)$ is a global/absolute maximum value of $f$ on $D_f$ if $f(x,y) \leq f(x_0, y_0)$ for all $(x,y) \in D_f$

$f(x_0, y_0)$ is a global/absolute minimum value of $f$ on $D_f$ if $f(x,y) \geq f(x_0, y_0)$ for all $(x,y) \in D_f$

Let $f(x,y)$ be a two-variable function with domain $D_f \subseteq \mathbb{R}^2$. A point $(x_0, y_0) \in D_f$ is a <u>critical point</u> of $f$ if

1) $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

or 2) $f_x$ or $f_y$ is undefined at $(x_0, y_0)$

Example - 1) $f(x,y) = 8x + 10y - x^2 - y^2$

2) $f(x,y) = x^2 + 6xy + 10y^2 - 4x - 14y + 1$

3) $f(x,y) = x^3 + 3y^3 - 3x - 36y + 7$

1) $f_x = 8 - 2x \Rightarrow f_x = 0$ at $x = 4$

$f_y = 10 - 2y \Rightarrow f_y = 0$ at $y = 5$

Critical point is $(4, 5)$

2) $f_x = 2x + 6y - 4$

$f_y = 6x + 20y - 14$

$f_x = 0 = f_y$ means solving
$$2x + 6y = 4$$
$$6x + 20y = 14$$

Do $-3$ times first equation plus second equation to get $2y = 2$ so $y = 1$

Then $2x + 6(1) = 4$ so $2x = -2$ ie $x = -1$

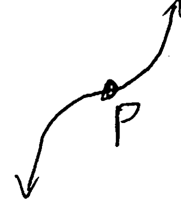
So critical point is $(-1, 1)$

3) $f_x = 3x^2 - 3 \Rightarrow f_x = 0$ at $x = 1, -1$

$f_y = 9y^2 - 36 \Rightarrow f_y = 0$ at $y = 2, -2$

So critical points are $(1,2)$, $(1,-2)$, $(-1,2)$, $(-1,-2)$

Critical points are candidates for possible locations of max/min values of a function.

Consider the case  from single variable calc.

A point $(x_0, y_0, f(x_0,y_0))$ is a <u>saddle point</u> of $f$ if "near" $(x_0,y_0)$ there are points $(x,y)$ where $f(x,y) > f(x_0,y_0)$ and points $(x,y)$ where $f(x,y) < f(x_0,y_0)$ and $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$.

Second derivative test - Let $f(x,y)$ be a continuously differentiable function, and let $(x_0,y_0)$ be a critical point of $f$. Consider the function $D(x,y) = f_{xx}f_{yy} - f_{xy}f_{yx}$. Then

1) If $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) > 0$ then $f(x_0,y_0)$ is a local minimum of $f$

2) If $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) < 0$ then $f(x_0,y_0)$ is a local maximum of $f$.

3) If $D(x_0,y_0)<0$, then $(x_0, y_0, f(x_0,y_0))$ is a saddle point of $f$

4) If $D(x_0, y_0) = 0$ then test is inconclusive.

Examples. 1) $f(x,y) = 8x + 10y - x^2 - y^2$

2) $f(x,y) = x^2 + 6xy + 10y^2 - 4x - 14y + 1$

3) $f(x,y) = x^3 + 3y^3 - 3x - 36y + 7$

1) Critical point is $(4, 5)$

$f_x = 8 - 2x$ $\qquad$ $f_y = 10 - 2y$

$f_{xx} = -2$ $\qquad$ $f_{yy} = -2$ $\qquad$ $f_{xy} = 0 = f_{yx}$

$D(x,y) = f_{xx}f_{yy} - f_{xy}f_{yx} = 4$

$D(4,5) = 4 > 0$ and $f_{xx}(4,5) = -2 < 0$

So $(4, 5, f(4,5)) = (4, 5, 41)$ is a local max.

2) Critical point is $(-1, 1)$

$f_x = 2x + 6y - 4$ $\qquad$ $f_y = 6x + 20y - 14$

$f_{xx} = 2$ $\qquad$ $f_{yy} = 20$ $\qquad$ $f_{xy} = 6 = f_{yx}$

$D(x,y) = 2(20) - 6(6) = 4$

$D(-1,1) = 4 > 0$ and $f_{xx}(-1,1) = 2 > 0$

So $(-1, 1, f(-1,1)) = (-1, 1, -4)$ is a local min

3) Critical points: $(1,2)$, $(1,-2)$, $(-1,2)$, $(-1,-2)$

$f_x = 3x^2 - 3$ $\qquad f_y = 9y^2 - 36$

$f_{xx} = 6x \qquad f_{yy} = 18y \qquad f_{xy} = 0 = f_{yx}$

$D(x,y) = 6x(18y) - 0(0) = 108xy$

$D(1,2) = 216 > 0$ and $f_{xx}(1,2) = 6 > 0$

So $(1, 2, f(1,2))$ is a local minimum.

$D(1,-2) = -216 < 0 \Rightarrow (1, -2, f(1,-2))$ is a saddle point

$D(-1,2) = -216 < 0 \Rightarrow (-1, 2, f(-1,2))$ is a saddle point

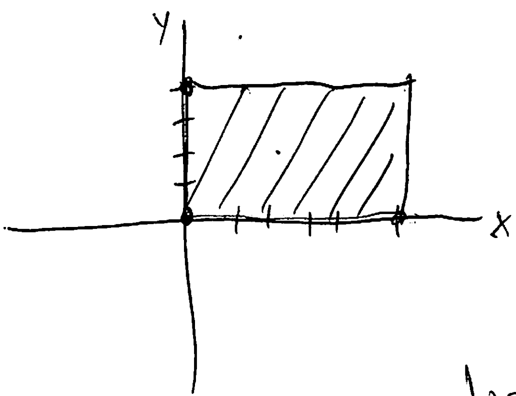
$D(-1,-2) = 216 > 0$ and $f_{xx}(-1,-2) = -6 < 0$

So $(-1, -2, f(-1,-2))$ is a local maximum.

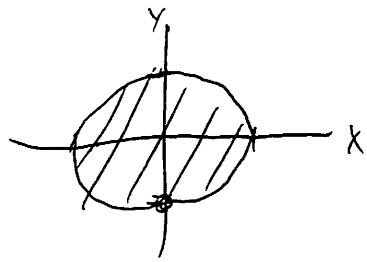
Recall that to find the absolute max/min of a single variable function $f(x)$ on a closed interval $[a,b]$, we evaluate $f$ at $x=a$, $x=b$, and all critical values of $f$ in $[a,b]$. The largest of these values is the absolute max and the smallest is the absolute min.

In the two-variable case, we consider $f(x,y)$ on a closed and bounded region $R \subseteq \mathbb{R}^2$

Examples - $R = \{(x,y) \mid 0 \le x \le 5, \ 0 \le y \le 4\}$




$R = \{(x,y) \mid x^2 + y^2 \le 1\}$




Candidates for locations of absolute max/min are critical points inside $R$ and the boundary of $R$

To find absolute max/min of $f(x,y)$ on a closed and bounded region $R \subseteq \mathbb{R}^2$:

1) Evaluate $f$ at all critical points of $f$ in $R$

2) Find the maximum and minimum values of $f$ on the boundary of $R$

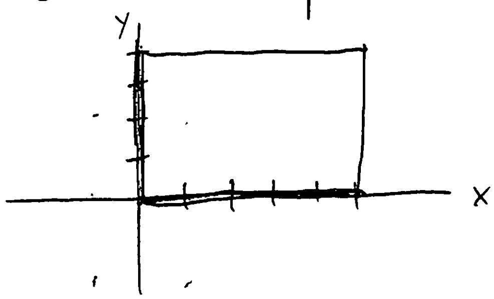
3) The largest value from steps 1) and 2) is the absolute max, and the smallest is the absolute min.

Example - $f(x,y) = x^3 + 3y^3 - 3x - 36y + 7$

$R = \{(x,y) \mid 0 \le x \le 5,\ 0 \le y \le 4\}$

Critical points: $(1,2), (1,-2), (-1,2), (-1,-2)$

outside $R$    outside $R$    outside $R$




$f(1,2) = -43$

On the bottom of $R$, $y=0$, and $x \in [0,5]$

Plug in $y=0$ to $f$ to get $x^3 - 3x + 7$

Find max/min of $f_1 = x^3 - 3x + 7$ on $[0,5]$

$f_1(0) = 7$, $f_1(5) = 117$, $f_1(1) = 5$

Note $x=1$ and $-1$ are the critical values of $f_1$

On the left of R, $x=0$ and $y \in [0,4]$

Plug in $x=0$ to f to get $3y^3-36y+7$

Find max/min of $f_2 = 3y^3-36y+7$ on $[0,4]$

$f_2(0)=7$ $\quad f_2(4)=55$ $\quad f_2(2)=-41$

Note $y=2$ and $-2$ are the critical values of $f_2$

On the top of R, $y=4$ and $x \in [0,5]$

Plug in $y=4$ to f to get $x^3-3x+55$

Find max/min of $f_3 = x^3 - 3x+55$ on $[0,5]$

$f_3(0)=55$ $\quad f_3(5)=165$ $\quad f_3(1)=53$

On the right of R, $x=5$ and $y \in [0,4]$

Plug in $x=5$ to f to get $3y^3-36y+117$

Find max/min of $f_4 = 3y^3-36y+117$ on $[0,4]$

$f_4(0)=117$ $\quad f_4(4)=165$ $\quad f_4(2)=69$

Looking at all the values, we see the absolute min is $-43$ and absolute max is 165.

To do constrained optimization of multivariable functions, we use <u>Lagrange multipliers</u>

Suppose we want to optimize a function $f(x,y)$ subject to a constraint $g(x,y) = c$, where $c$ is a constant.

1) Set up the polynomial $h(x, y, \lambda) = f(x,y) + \lambda(g(x,y) - c)$

   $\lambda$ is the Lagrange multiplier

2) Find values $x, y, \lambda$ such that $h_x = 0$, $h_y = 0$, and $h_\lambda = 0$

3) Compute $h_{xx}$, $h_{yy}$, $h_{xy}$, $h_{yx}$ and apply second derivative test to determine if values from step 2) produce maxima or minima.

Example- Find maxima and minima of

$6x + 12y - x^2 - y^2$ subject to the constraint

$x^2 + y^2 = 80$.

$$h(x, y, \lambda) = 6x + 12y - x^2 - y^2 + \lambda(x^2 + y^2 - 80)$$

$$h_x = 6 - 2x + 2x\lambda$$

$$h_y = 12 - 2y + 2y\lambda$$

$$h_\lambda = x^2 + y^2 - 80$$

$$h_x = 0 \implies 6 - 2x + 2x\lambda = 0 \implies \lambda = \frac{2x-6}{2x}$$

$$h_y = 0 \implies 12 - 2y + 2y\lambda = 0 \implies \lambda = \frac{2y-12}{2y}$$

So $\dfrac{2x-6}{2x} = \dfrac{2y-12}{2y}$

Cross multiply to get $4xy - 12y = 4xy - 24x$.

Thus $-12y = -24x$ i.e. $y = 2x$

$$h_\lambda = 0 \implies x^2 + y^2 = 80$$

Plug in $y = 2x$ to get $x^2 + (2x)^2 = 80$

$\implies 5x^2 = 80 \implies x^2 = 16 \implies x = 4, -4$

If $x = 4$, $y = 8$ and $\lambda = \frac{1}{4}$

If $x = -4$, $y = -8$ and $\lambda = \frac{7}{4}$

So critical points are $(4,8)$ with $\lambda = \frac{1}{4}$
and $(-4,-8)$ with $\lambda = \frac{7}{4}$

$h_{xx} = -2 + 2\lambda$ $\qquad$ $h_{yy} = -2 + 2\lambda$, $\quad h_{xy} = 0 = h_{yx}$

So $D(x,y,\lambda) = (-2+2\lambda)(-2+2\lambda)$

Then $D(4,8,\frac{1}{4}) = (-2+\frac{2}{4})(-2+\frac{2}{4}) = \frac{9}{4} > 0$

and $h_{xx}(4,8,\frac{1}{4}) = -2 + \frac{2}{4} = -\frac{3}{2} < 0$

So $(4,8,f(4,8)) = (4,8,40)$ is a local max

Also, $D(-4,-8,\frac{7}{4}) = (-2+\frac{14}{4})(-2+\frac{14}{4}) = \frac{9}{4} > 0$

and $h_{xx}(-4,-8,\frac{7}{4}) = -2 + \frac{14}{4} = \frac{3}{2} > 0$

So $(-4,-8,f(-4,-8)) = (-4,-8,-200)$ is a

local min.