Recall we find eigenvalues for A by solving the equation $\det(A-\lambda I)=0$ for λ. This equation is called the <u>characteristic equation</u> of A.

Let us now consider the case when this equation has complex solutions.

Ex- $A = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix}$

$$\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix} = (2-\lambda)(4-\lambda)+2$$
$$= \lambda^2 - 6\lambda + 10$$

So $\det(A-\lambda I)=0 \Rightarrow \lambda = \dfrac{6 \pm \sqrt{36-40}}{2} = \dfrac{6 \pm 2i}{2}$

$$\lambda = 3 \pm i$$

For $\lambda = 3+i$, $A-\lambda I = \begin{bmatrix} 2-(3+i) & 2 \\ -1 & 4-(3+i) \end{bmatrix} = \begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix}$

We want a veeter $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that

$$\begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (-1-i)v_1 + 2v_2 \\ -v_1 + (1-i)v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (-1-i)v_1 + 2v_2 = 0$$

$$\Rightarrow 2v_2 = (1+i)v_1$$

$$\Rightarrow v_2 = 1+i, \quad v_1 = 2 \quad \Rightarrow \vec{v} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For $\lambda = 3-i$, $A - \lambda I = \begin{bmatrix} 2-(3-i) & 2 \\ -1 & 4-(3-i) \end{bmatrix} = \begin{bmatrix} -1+i & 2 \\ -1 & 1+i \end{bmatrix}$

So $(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \begin{bmatrix} -1+i & 2 \\ -1 & 1+i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow (-1+i)v_1 + 2v_2 = 0$$

$$\Rightarrow 2v_2 = (1-i)v_1$$

$$\Rightarrow v_2 = 1-i, \quad v_1 = 2$$

$$\Rightarrow \vec{v} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Ex - 1) $A = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix}$

1) $\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 3 \\ -3 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda) + 9$

$= \lambda^2 - 2\lambda + 10$

So $\lambda = \dfrac{2 \pm \sqrt{4-40}}{2} = \dfrac{2 \pm 6i}{2} = 1 \pm 3i$

For $\lambda = 1+3i$, $A - \lambda I = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix}$

Want $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that $\begin{bmatrix} -3i & 3 \\ 3 & -3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -3iv_1 + 3v_2 = 0$

$\Rightarrow 3v_2 = 3iv_1$

$\Rightarrow v_2 = i,\ v_1 = 1 \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

For $\lambda = 1-3i$, $A - \lambda I = \begin{bmatrix} 3i & 3 \\ -3 & 3i \end{bmatrix}$

Want $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that $\begin{bmatrix} 3i & 3 \\ -3 & 3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$\Rightarrow 3iv_1 + 3v_2 = 0 \Rightarrow 3v_2 = -3iv_1$

$\Rightarrow v_2 = -i,\ v_1 = 1 \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2) $\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{pmatrix} = (1-\lambda)(-1-\lambda) + 5$

$$= \lambda^2 + 4$$

So $\lambda = \pm 2i$

For $\lambda = 2i$, ~~row reduce~~ $A - \lambda I = \begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix}$

Find $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that $\begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow (1-2i)v_1 - 5v_2 = 0$

$\Rightarrow (1-2i)v_1 = 5v_2$

$\Rightarrow v_1 = 5, \quad v_2 = 1-2i$

$\Rightarrow \vec{v} = \begin{bmatrix} 5 \\ 1-2i \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ -2 \end{bmatrix}$

For $\lambda = -2i$, $A - \lambda I = \begin{bmatrix} 1+2i & -5 \\ 1 & -1+2i \end{bmatrix}$

Find $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that $\begin{bmatrix} 1+2i & -5 \\ 1 & -1+2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow (1+2i)v_1 - 5v_2 = 0 \quad \Rightarrow (1+2i)v_1 = 5v_2$

$\Rightarrow v_1 = 5, \quad v_2 = 1+2i$

$\Rightarrow \vec{v} = \begin{bmatrix} 5 \\ 1+2i \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} - i\begin{bmatrix} 0 \\ -2 \end{bmatrix}$

Fact - For real matrices A, complex conjugate eigenvalues $\lambda = \alpha \pm i\beta$ correspond to complex conjugate eigenvectors $\vec{v} = \vec{u} \pm i\vec{w}$

It follows that if A is the matrix for a system of linear DEs, and A has complex eigenvalues $\lambda = \alpha \pm i\beta$ with eigenvectors $\vec{v} = \vec{u} \pm i\vec{w}$,

then
$$\vec{x} = c_1 e^{(\alpha+i\beta)t}(\vec{u}+i\vec{w}) + c_2 e^{(\alpha-i\beta)t}(\vec{u}-i\vec{w})$$

$$= c_1 e^{\alpha t} e^{i\beta t}(\vec{u}+i\vec{w}) + c_2 e^{\alpha t} e^{-i\beta t}(\vec{u}-i\vec{w})$$

$$= e^{\alpha t}\Big[c_1(\cos(\beta t) + i\sin(\beta t))(\vec{u}+i\vec{w}) + c_2(\cos(\beta t) - i\sin(\beta t))(\vec{u}-i\vec{w})\Big]$$

$$= e^{\alpha t}\Big[c_1(\cos(\beta t)\vec{u} + i\cos(\beta t)\vec{w} + i\sin(\beta t)\vec{u} - \sin(\beta t)\vec{w}) + c_2(\cos(\beta t)\vec{u} - i\cos(\beta t)\vec{w} - i\sin(\beta t)\vec{u} - \sin(\beta t)\vec{w})\Big]$$

$$= e^{\alpha t}\Big[\underbrace{(c_1+c_2)}_{C_1}\big(\cos(\beta t)\vec{u} - \sin(\beta t)\vec{w}\big) + \underbrace{i(c_1-c_2)}_{C_2}\big(\cos(\beta t)\vec{w} + \sin(\beta t)\vec{u}\big)\Big]$$

$$= e^{\alpha t}\Big[C_1\big(\cos(\beta t)\vec{u} - \sin(\beta t)\vec{w}\big) + C_2\big(\cos(\beta t)\vec{w} + \sin(\beta t)\vec{u}\big)\Big]$$

Ex- $$x_1' = 2x_1 + 2x_2$$
$$x_2' = -x_1 + 4x_2$$

$$\Rightarrow \quad \vec{X}' = \underbrace{\begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix}}_{A} \vec{X} \qquad \text{where } \vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We saw above that A has $\lambda = 3 \pm i$

with $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So $\alpha = 3$, $\beta = 1$, $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So $$\vec{X}(t) = e^{3t}\Big[C_1\Big(\cos(t)\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \sin(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\Big) + C_2\Big(\cos(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin(t)\begin{bmatrix} 2 \\ 1 \end{bmatrix}\Big)\Big]$$

Let us now consider the nonhomogeneous system $\vec{x}' = A\vec{x} + \vec{f}$ with $\vec{f} \neq \vec{0}$

Recall that general solution is

$\vec{x} = c_1\vec{x}_1 + \cdots + c_n\vec{x}_n + \vec{x}_p$ where $\vec{x}_1, \ldots, \vec{x}_n$ are lin. indep. solutions to $\vec{x}' = A\vec{x}$ and $\vec{x}_p$ is a particular solution to $\vec{x}' = A\vec{x} + \vec{f}$

It remains to find $\vec{x}_p$

We will start with Undetermined Coefficients

Ex.
$$x_1' = x_1 + 3x_2 - 7t$$
$$x_2' = 3x_1 - 7x_2 + 11t$$

$$\Rightarrow \underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}}_{\vec{x}'} = \underbrace{\begin{bmatrix} 1 & 3 \\ 3 & -7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} -7t \\ 11t \end{bmatrix}}_{\vec{f}}$$

Since $\vec{f} = \begin{bmatrix} -7t \\ 11t \end{bmatrix} = t\begin{bmatrix} -7 \\ 11 \end{bmatrix}$ is a constant vector times t, we look for a solution of the form $\vec{X}_p = \vec{a}t + \vec{b}$ for constant vectors $\vec{a}, \vec{b}$

Then $\vec{X}_p' = \vec{a}$

Plug $\vec{X}_p, \vec{X}_p'$ into original system to get $\vec{X}_p' = A\vec{X}_p + \vec{f}$

$$\Rightarrow \quad \vec{a} = A(\vec{a}t + \vec{b}) + t\begin{bmatrix} -7 \\ 11 \end{bmatrix}$$

$$\Rightarrow \quad \vec{a} = A\vec{a}t + A\vec{b} + t\begin{bmatrix} -7 \\ 11 \end{bmatrix}$$

$$\Rightarrow \quad A\vec{a} + \begin{bmatrix} -7 \\ 11 \end{bmatrix} = \vec{0} \quad \text{and} \quad A\vec{b} = \vec{a}$$

$$\Rightarrow \quad A\vec{a} = \begin{bmatrix} 7 \\ -11 \end{bmatrix} \quad \text{and} \quad A\vec{b} = \vec{a}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 3 & -7 & -11 \end{array}\right] \xrightarrow{-3R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -16 & -32 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{16}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{-3R_2 + R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

So $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Now solve $A\vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 3 & -7 & 2 \end{array}\right] \xrightarrow{-3R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & -16 & -1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{16}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & 1/16 \end{array}\right] \xrightarrow{-3R_2 + R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 13/16 \\ 0 & 1 & 1/16 \end{array}\right]$$

So $\vec{b} = \begin{bmatrix} 13/16 \\ 1/16 \end{bmatrix}$

Hence, $\vec{X}_p = \vec{a}t + \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 13/16 \\ 1/16 \end{bmatrix}$

Next, we look at Variation of Parameters

Recall that the homogeneous solution to

$\vec{X}' = A\vec{X} + \vec{f}$ is $\vec{X} = c_1\vec{X}_1 + \cdots + c_n\vec{X}_n = X\vec{c}$

where $X = [\vec{X}_1 \; \cdots \; \vec{X}_n]$ and $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

That is, we look for constant combinations of $\vec{X}_1, \dots, \vec{X}_n$.

For V.P., we consider <u>function</u> combinations

ie $\vec{X}_p = v_1 \vec{X}_1 + \cdots + v_n \vec{X}_n = X \vec{v}$ where the v_s are functions.

$$\vec{X}_p{}' = X \vec{v}\,' + X' \vec{v}$$

And $\underset{\displaystyle A\vec{x}_p}{\underset{\shortparallel}{AX\vec{v}}} = [A\vec{x}_1 \cdots A\vec{x}_n]\vec{v} = [\vec{x}_1{}' \cdots \vec{x}_n{}']\vec{v}$
$= X'\vec{v}$

Thus, $\vec{x}_p{}' = A\vec{x}_p + \vec{f}$ becomes

$$X\vec{v}\,' + X'\vec{v} = X'\vec{v} + \vec{f}$$

$$\Rightarrow X\vec{v}\,' = \vec{f}$$

$$\Rightarrow \vec{v}\,' = X^{-1}\vec{f}$$

$$\Rightarrow \vec{v} = \int X^{-1}\vec{f}\,dt$$

$$\Rightarrow \quad \vec{X}_p = X\vec{v} = \underline{X \int X^{-1} \vec{f}\, dt}$$

Ex - $\quad x_1' = x_1 + 3x_2 - 7t$

$\quad\quad x_2' = 3x_1 - 7x_2 + 11t$

$$\vec{x}\,' = \underbrace{\begin{bmatrix} 1 & 3 \\ 3 & -7 \end{bmatrix}}_{A} \vec{x} + \underbrace{\begin{bmatrix} -7t \\ 11t \end{bmatrix}}_{f}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 3 \\ 3 & -7-\lambda \end{pmatrix} = (1-\lambda)(-7-\lambda) - 9$$

$$= \lambda^2 + 6\lambda - 16$$

$$= (\lambda - 2)(\lambda + 8)$$

So $\lambda = 2, -8$

For $\lambda = 2$, $A - \lambda I = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}$

$\Rightarrow$ eigenvector is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

For $\lambda = -8$, $A - \lambda I = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$

$\Rightarrow$ eigenvector is $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$$\vec{X}_h = c_1 e^{2t}\begin{bmatrix}3\\1\end{bmatrix} + c_2 e^{-8t}\begin{bmatrix}-1\\3\end{bmatrix}$$

$$\vec{X}_1 = e^{2t}\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}3e^{2t}\\e^{2t}\end{bmatrix}, \quad \vec{X}_2 = e^{-8t}\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}-e^{-8t}\\3e^{-8t}\end{bmatrix}$$

$$X = \begin{bmatrix}3e^{2t} & -e^{-8t}\\ e^{2t} & 3e^{-8t}\end{bmatrix}$$

$$X^{-1} = \frac{1}{10e^{-6t}}\begin{bmatrix}3e^{-8t} & e^{-8t}\\ -e^{2t} & 3e^{2t}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{3}{10}e^{-2t} & \frac{1}{10}e^{-2t}\\ -\frac{1}{10}e^{8t} & \frac{3}{10}e^{8t}\end{bmatrix}$$

$$X^{-1}\vec{f} = \downarrow \begin{bmatrix}-7t\\11t\end{bmatrix} = \begin{bmatrix}-\frac{21}{10}te^{-2t} + \frac{11}{10}te^{-2t}\\ \frac{7}{10}te^{8t} + \frac{33}{10}te^{8t}\end{bmatrix}$$

$$= \begin{bmatrix}-te^{-2t}\\ 4te^{8t}\end{bmatrix}$$

$$\int X^{-1}\vec{f}\,dt = \begin{bmatrix}\int -te^{-2t}\,dt\\ \int 4te^{8t}\,dt\end{bmatrix} = \begin{bmatrix}\frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t}\\ \frac{1}{2}te^{8t} - \frac{1}{16}e^{8t}\end{bmatrix}$$

$$\vec{x}_p = X \int X^{-1} \vec{f}\, dt$$

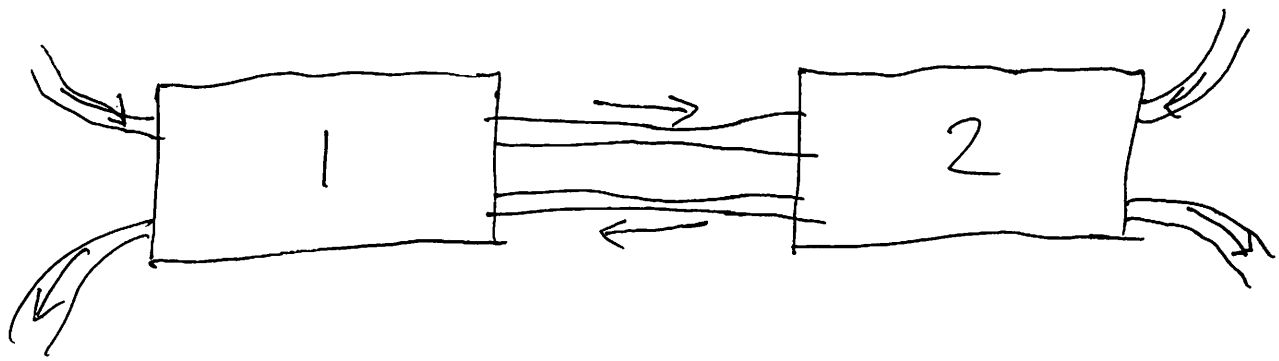
$$= \begin{bmatrix} 3e^{2t} & -e^{-8t} \\ e^{2t} & 3e^{-8t} \end{bmatrix} \begin{bmatrix} \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} \\ \frac{1}{2}te^{8t} - \frac{1}{16}e^{8t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}t + \frac{3}{4} - \frac{1}{2}t + \frac{1}{16} \\ \frac{1}{2}t + \frac{1}{4} + \frac{3}{2}t - \frac{3}{16} \end{bmatrix}$$

$$= \begin{bmatrix} t + 13/16 \\ 2t + 1/16 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 13/16 \\ 1/16 \end{bmatrix}$$

Consider two connected tanks, each containing a saltwater solution with a certain concentration. The tanks are well-stirred, and solution can flow into the tanks, out of the tanks, and/or between the tanks at various rates.



We can model this with a linear system of first-order DEs

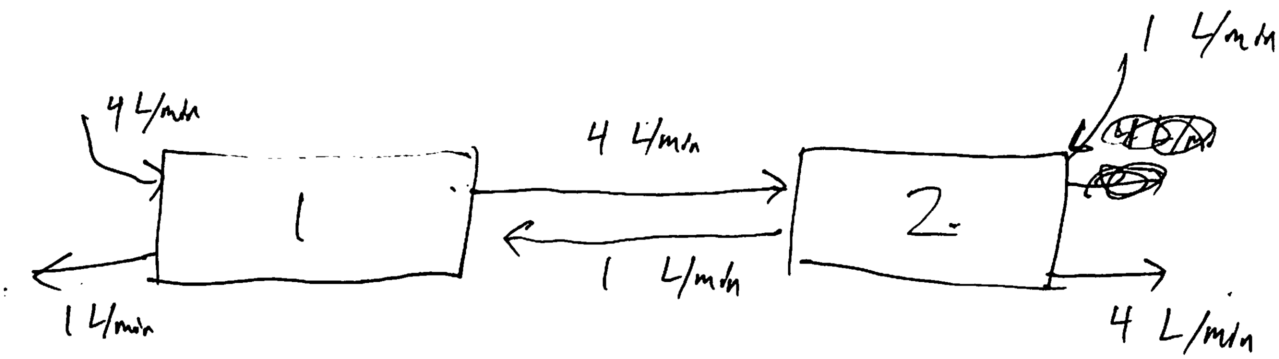
Let x_1 be the amount of salt in tank 1 at time t, and let x_2 be the amount in tank 2.

Then $X_1' = \underset{\text{tank 1}}{\text{rate in}} - \underset{\text{tank 1}}{\text{rate out}}$

$X_2' = \underset{\text{tank 2}}{\text{rate in}} - \underset{\text{tank 2}}{\text{rate out}}$

Ex - Consider two 50 L tanks of saltwater. Tank 1 initially contains 9 kg of salt, and tank 2 initially contains 6 kg. A saltwater solution with a concentration of .2 kg/L of salt flows into tank 1 at a rate of 4 L/min, while a saltwater solution with concentration .2 kg/L flows into tank 2 at a rate of 1 L/min. The tanks are kept well-stirred, and solution flows out of tank 1 at a rate of 1 L/min, while solution flows out of tank 2 at a rate of 4 L/min. Solution flows from tank 1 to tank 2 at a rate of 4 L/min, and Solution flows from tank 2 to tank 1 at a rate of 1 L/min

1) Set up system
2) Solve



$$X_1' = \text{rate in} - \text{rate out}$$
$$= 4(.2) + 1\left(\frac{X_2}{50}\right) - 5\left(\frac{X_1}{50}\right)$$
$$= 4/5 - \frac{X_1}{10} + \frac{X_2}{50} \qquad X_1(0) = 9$$

$$X_2' = \text{rate in} - \text{rate out}$$
$$= 1(.2) + 4\left(\frac{X_1}{50}\right) - 5\left(\frac{X_2}{50}\right)$$

$$= 1/5 + \frac{2x_1}{25} - \frac{x_2}{10} \qquad x_2(0) = 6$$

In normal form,

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1/10 & 1/50 \\ 2/25 & -1/10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4/5 \\ 1/5 \end{bmatrix}$$

$$\vec{x}' = A \quad \vec{x} \quad \vec{f}$$

$$\det(A - \lambda I) = \det\begin{bmatrix} -\frac{1}{10} - \lambda & \frac{1}{50} \\ 2/25 & -\frac{1}{10} - \lambda \end{bmatrix} = \left(-\tfrac{1}{10} - \lambda\right)\left(-\tfrac{1}{10} - \lambda\right) - \frac{2}{1250}$$

$$= \frac{1}{100} + \frac{1}{5}\lambda + \lambda^2 - \frac{2}{1250}$$

$$= \lambda^2 + \frac{1}{5}\lambda + \frac{21}{2500}$$

$$= \left(\lambda + \tfrac{3}{50}\right)\left(\lambda + \tfrac{7}{50}\right)$$

So $\lambda = -\frac{3}{50}$, $\lambda = -\frac{7}{50}$

For $\lambda = -\frac{3}{50}$, look at $A + \frac{3}{50}I = \begin{bmatrix} -\frac{2}{50} & \frac{1}{50} \\ \frac{2}{25} & -2/50 \end{bmatrix}$

Scale by 50: $\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$

$\Rightarrow$ eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $\lambda = -\frac{7}{50}$, look at $A + \frac{7}{50}I = \begin{bmatrix} 2/50 & 1/50 \\ 2/25 & 2/50 \end{bmatrix}$

Scale by 50: $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

$\Rightarrow$ eigenvector is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Then $\vec{X}_h = c_1 e^{-\frac{3}{50}t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-\frac{7}{50}t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\vec{X}_1 = \begin{bmatrix} e^{-\frac{3}{50}t} \\ 2e^{-3/50t} \end{bmatrix}$ $\quad\quad$ $\vec{X}_2 = \begin{bmatrix} -e^{-7/50t} \\ 2e^{-7/50t} \end{bmatrix}$

$$X = \begin{bmatrix} e^{-3/50t} & -e^{-7/50t} \\ 2e^{-3/50t} & 2e^{-7/50t} \end{bmatrix}$$

$$X^{-1} = \frac{1}{4e^{-10/50t}}\begin{bmatrix} 2e^{-\frac{7}{50}t} & e^{-7/50t} \\ -2e^{-3/50t} & e^{-3/50t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}e^{3/50t} & \frac{1}{4}e^{3/50t} \\ -\frac{1}{2}e^{7/50t} & \frac{1}{4}e^{7/50t} \end{bmatrix}$$

$\downarrow$

$$X^{-1}\vec{f} = \begin{bmatrix} 4/5 \\ 1/5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{5}e^{3/50t} + \frac{1}{20}e^{3/50t} \\ -\frac{2}{5}e^{7/50t} + \frac{1}{20}e^{7/50t} \end{bmatrix} = \begin{bmatrix} \frac{9}{20}e^{3/50t} \\ -\frac{7}{20}e^{7/50t} \end{bmatrix}$$

$$\int X^{-1}\vec{f}\,dt = \begin{bmatrix} \int \frac{9}{20}e^{3/50t}\,dt \\ \int -\frac{7}{20}e^{7/50t}\,dt \end{bmatrix} = \begin{bmatrix} \frac{15}{2}e^{3/50t} \\ -\frac{5}{2}e^{7/50t} \end{bmatrix}$$

$$\vec{X}_p = X \quad = \begin{bmatrix} e^{-3/50t} & -e^{-7/50t} \\ 2e^{-3/50t} & 2e^{-7/50t} \end{bmatrix} \begin{bmatrix} \frac{15}{2}e^{3/50t} \\ -\frac{5}{2}e^{7/50t} \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

Then $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{X}_h + \vec{X}_p = c_1 e^{-3/50t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-\frac{7}{50}t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 10 \\ 10 \end{bmatrix}$

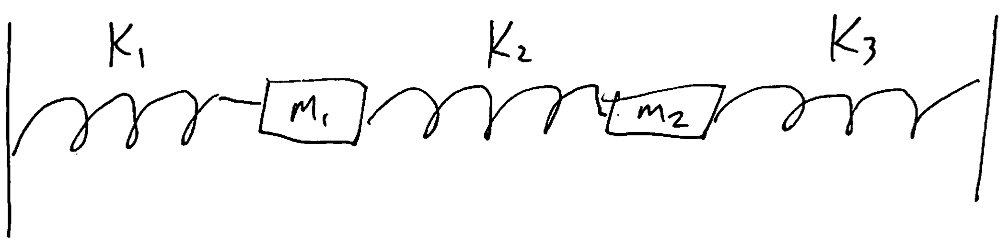
Apply IC $x_1(0) = 9$, $x_2(0) = 6$

$$\Rightarrow \begin{bmatrix} 9 \\ 6 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} c_1 - c_2 = -1 \\ 2c_1 + 2c_2 = -4 \end{array} \Rightarrow c_1 = -\frac{3}{2}, \quad c_2 = -\frac{1}{2}$$

$$\Rightarrow \vec{x} = -\frac{3}{2} e^{-3/50\, t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{2} e^{-7/50\, t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

Consider two frictionless masses m_1, m_2 connected to springs:



Let x_1 be displacement for m_1 and x_2 be displacement for m_2

Let k_1, k_2, k_3 be the spring constants $(k_2 \neq 0)$

Using Newton's 2nd law and Hooke's law and assuming no friction, we have

$$F_{m_1} = F_{\text{spring } m_1}$$

$$F_{m_2} = F_{\text{spring } m_2}$$

$$\Longrightarrow \quad m_1 x_1'' = -k_1 x_1 - k_2 x_1 + k_2 x_2$$

$$m_2 x_2'' = -k_2 x_2 - k_3 x_2 + k_2 x_1$$

$$\Longrightarrow \quad x_1'' = \frac{-(k_1+k_2)}{m_1} x_1 + \frac{k_2}{m_1} x_2$$

$$x_2'' = \frac{k_2}{m_2} x_1 - \frac{(k_2+k_3)}{m_2} x_2$$

$$\Longrightarrow \quad \begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-(k_2+k_3)}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\vec{x}'' = A \vec{x}$$

ie second derivative is "constant multiple" of $\vec{x}$

So assume $\vec{x} = e^{rt}\vec{v}$ where $\vec{v}$ is constant vector

Then $\vec{x}' = re^{rt}\vec{v}$ and $\vec{x}'' = r^2 e^{rt}\vec{v}$

So $\vec{x}'' = A\vec{x}$ becomes $r^2 e^{rt}\vec{v} = Ae^{rt}\vec{v}$

$$\Longrightarrow \quad r^2\vec{v} = A\vec{v}$$

$\Rightarrow \vec{v}$ is eigenvector for A with e-value r^2

Fact - when $K_2 \neq 0$, A has two real negative eigenvalues, $-\lambda_1$ and $-\lambda_2$.

Let $\vec{v}_1, \vec{v}_2$ be corresponding eigenvectors.

Then $r^2 = -\lambda_1$ and $r^2 = -\lambda_2$

$\Rightarrow \quad r = \pm\sqrt{-\lambda_1}$ and $r = \pm\sqrt{-\lambda_2}$

$\Rightarrow \quad r = \pm i\sqrt{\lambda_1}$ and $r = \pm i\sqrt{\lambda_2}$

Then $$\vec{X} = c_1 e^{\pm i\sqrt{\lambda_1}\,t}\vec{v}_1 + c_2 e^{\pm i\sqrt{\lambda_2}\,t}\vec{v}_2$$

$$= c_1\left(\cos(\sqrt{\lambda_1}\,t) \pm i\sin(\sqrt{\lambda_1}\,t)\right)\vec{v}_1$$
$$+ \; c_2\left(\cos(\sqrt{\lambda_2}\,t) \pm i\sin(\sqrt{\lambda_2}\,t)\right)\vec{v}_2$$

$$\Rightarrow \vec{X} = \left(c_1\cos(\sqrt{\lambda_1}\,t) + d_1\sin(\sqrt{\lambda_1}\,t)\right)\vec{v}_1$$
$$+ \left(c_2\cos(\sqrt{\lambda_2}\,t) + d_2\sin(\sqrt{\lambda_2}\,t)\right)\vec{v}_2$$

Ex - Consider two frictionless masses attached to three springs as above.

Mass 1 is 2 kg and Mass 2 is 3 kg

The leftmost spring has spring constant 8 N/m.

The middle spring has spring constant 6 N/m

The rightmost spring has spring constant 12 N/m

1) Set up system

2) Solve

$$m_1 x_1'' = -(k_1 + k_2) x_1 + k_2 x_2$$
$$m_2 x_2'' = k_2 x_1 - (k_2 + k_3) x_2$$

$$\Rightarrow \quad 2x_1'' = -14x_1 + 6x_2$$
$$3x_2'' = 6x_1 - 18x_2$$

$$\Rightarrow \quad x_1'' = -7x_1 + 3x_2$$
$$x_2'' = 2x_1 - 6x_2$$

$$\Rightarrow \underbrace{\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix}}_{\vec{x}''} = \underbrace{\begin{bmatrix} -7 & 3 \\ 2 & -6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}}$$

$$\det(A - \lambda I) = \det\begin{bmatrix} -7-\lambda & 3 \\ 2 & -6-\lambda \end{bmatrix} = (-7-\lambda)(-6-\lambda) - 6$$

$$= \lambda^2 + 13\lambda + 36$$

$$= (\lambda + 4)(\lambda + 9)$$

So $-\lambda_1 = -4$, $-\lambda_2 = -9$

For -4, look at $A + 4I = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}$

$\Rightarrow$ eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For -9, look at $A + 9I = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$

$\Rightarrow$ eigenvector is $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$

Thus, $\vec{X} = \left(c_1 \cos(\sqrt{\lambda_1}\, t) + d_1 \sin(\sqrt{\lambda_1}\, t)\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$+ \left(c_2 \cos(\sqrt{\lambda_2}\, t) + d_2 \sin(\sqrt{\lambda_2}\, t)\right)\begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\Rightarrow \underset{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}{\overset{\shortparallel}{\vec{X}}} = \left(c_1 \cos(2t) + d_1 \sin(2t)\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$+ \left(c_2 \cos(3t) + d_2 \sin(3t)\right)\begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\Rightarrow x_1(t) = c_1 \cos(2t) + d_1 \sin(2t) - 3c_2 \cos(3t) - 3d_2 \sin(3t)$$

$$x_2(t) = c_1 \cos(2t) + d_1 \sin(2t) + 2c_2 \cos(3t) + 2d_2 \sin(3t)$$