Test 3 will cover

1) Solving 2x2 homogeneous systems of first-order DEs

2) Using VoP to find particular solutions to nonhomogeneous systems

3) Solving 2x2 systems with complex e-values/vectors

4) Applications (tanks, springs)

Consider two masses connected to three springs. Mass 1 has mass 2 kg

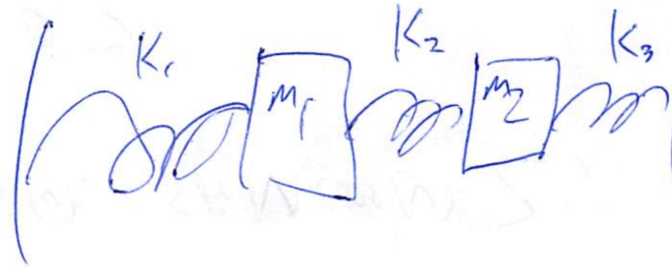
Mass 2 has mass 4 kg

The leftmost spring has stiffness 2 N/m

The middle spring has stiffness 4 N/m

The rightmost spring has " 4 N/m

1) Set up



2) Solve

In general, we have

$$m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2$$

$$m_2 x_2'' = k_2 x_1 - (k_2 + k_3) x_2$$

$$x_1'' = -\frac{6}{2}x_1 + \frac{4}{2}x_2 = -3x_1 + 2x_2$$

$$x_2'' = \frac{4}{4}x_1 - \frac{8}{4}x_2 = x_1 - 2x_2$$

Next, we find the matrix for the system and its e-values, e-vectors

$$A = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}$$

In normal form, $\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\vec{x}'' = A \vec{x}$$

Eigenvalues $\Rightarrow \det(A - \lambda I) = 0$

$$\Rightarrow \det\begin{pmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (-3-\lambda)(-2-\lambda) - 2 = 0$$

$$\Rightarrow \lambda^2 + 5\lambda + 4 = 0$$

$$\Rightarrow (\lambda+1)(\lambda+4) = 0$$

$$\Rightarrow \lambda = -1, -4$$

Eigenvectors $\Rightarrow (A - \lambda I)\vec{v} = \vec{0}$

For $\lambda = -1 \Rightarrow (A + I)\vec{v} = \vec{0}$

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0}.$$

$$\Rightarrow \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = -4 \quad \Rightarrow \quad (A + 4I)\vec{v} = \vec{0}$

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0}$$

$$\Rightarrow \quad \vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

In general, if a 2×2 spring system has e-values $-\lambda_1, -\lambda_2$ with corresponding e-vectors $\vec{v}_1, \vec{v}_2$ then

$$\vec{X} = \left(c_1 \cos(\sqrt{\lambda_1}\, t) + d_1 \sin(\sqrt{\lambda_1}\, t)\right)\vec{v}_1 + \left(c_2 \cos(\sqrt{\lambda_2}\, t) + d_2 \sin(\sqrt{\lambda_2}\, t)\right)\vec{v}_2$$

Here, $\lambda_1 = 1$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda_2 = 4$, $\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\vec{X} = \left(c_1 \cos(t) + d_1 \sin(t)\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_2 \cos(2t) + d_2 \sin(2t)\right)\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad x_1 = c_1 \cos(t) + d_1 \sin(t) + 2c_2 \cos(2t) + 2d_2 \sin(2t)$$

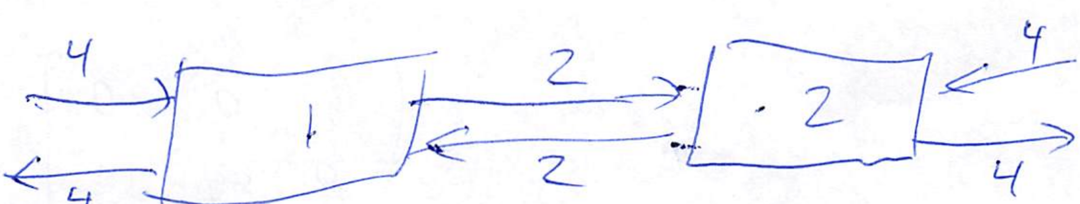
$$x_2 = c_1 \cos(t) + d_1 \sin(t) - c_2 \cos(2t) - d_2 \sin(2t)$$

Consider two interconnected tanks with volume 10 L

~~[illegible]~~ Tank 1 initially has 2 kg of salt

tank 2 initially has 3 kg of salt

At time $t=0$, ~~s~~olution with concentration .5 kg/L of salt starts flowing into tank 1 at a rate of 4 L/min. Solution with concentration .5 kg/L of salt flows into tank 2 at ~~a~~ rate of 4 L/min as well. Solution flows from tank 1 to tank 2 at rate of 2 L/min, and from tank 2 to tank 1 at 2 L/min. Solution flows out of tank 1 at rate of 4 L/min, and out of tank 2 at 4 L/min.



X_1 is amount of salt in tank 1 at time t

X_2 " " " tank 2 " "

$X_1' =$ amt into tank 1 $-$ amt out of tank 1

$= 4(.5) + 2\left(\frac{X_2}{10}\right) - 6\left(\frac{X_1}{10}\right)$

$= -\frac{3}{5}X_1 + \frac{1}{5}X_2 + 2 \qquad X_1(0) = 2$

$X_2' =$ amt into tank 2 $-$ amt out of tank 2

$= 4(.5) + 2\left(\frac{X_1}{10}\right) - 6\left(\frac{X_2}{10}\right)$

$X_2(0) = 3$

$= \frac{1}{5}X_1 - \frac{3}{5}X_2 + 2$

In normal form,

$$\begin{bmatrix} X_1' \\ X_2' \end{bmatrix} = \begin{bmatrix} -3/5 & 1/5 \\ 1/5 & -3/5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\vec{x}' = A \quad \vec{x} + \vec{f}$$

In general, for $\vec{x}' = A\vec{x} + \vec{f}$, we solve this by doing:

1) Use e-values, e-vectors of A, $\lambda_1, \lambda_2, \vec{v}_1, \vec{v}_2$ to form

$$\vec{X}_h = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

2) Form the matrix $X = \begin{bmatrix} e^{\lambda_1 t} \vec{v}_1 & e^{\lambda_2 t} \vec{v}_2 \end{bmatrix}$

3) $\vec{X}_p = X \int X^{-1} \vec{f} \, dt$

4) $\vec{X} = \vec{X}_h + \vec{X}_p$

1) $\det(A - \lambda I) = \det\begin{pmatrix} -3/5 - \lambda & 1/5 \\ 1/5 & -3/5 - \lambda \end{pmatrix}$

$$= \left(-\frac{3}{5} - \lambda\right)\left(-\frac{3}{5} - \lambda\right) - \frac{1}{25}$$

$$= \lambda^2 + \frac{6}{5}\lambda + \frac{8}{25}$$

$$= \left(\lambda + \frac{2}{5}\right)\left(\lambda + \frac{4}{5}\right)$$

So $\lambda_1 = -\frac{2}{5}$, $\lambda_2 = -\frac{4}{5}$

For $\lambda = -\frac{2}{5}$, $A - \lambda I = \begin{bmatrix} -1/5 & 1/5 \\ 1/5 & -1/5 \end{bmatrix}$

$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda = -\frac{4}{5}$, $A - \lambda I = \begin{bmatrix} 1/5 & 1/5 \\ 1/5 & 1/5 \end{bmatrix}$

$\Rightarrow \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\vec{X}_h = c_1 e^{-\frac{2}{5}t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-\frac{4}{5}t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

2) $$X = \begin{bmatrix} e^{-\frac{2}{5}t} & -e^{-4/5t} \\ e^{-\frac{2}{5}t} & e^{-\frac{4}{5}t} \end{bmatrix}$$

$$X^{-1} = \frac{1}{2e^{-6/5t}} \begin{bmatrix} e^{-4/5t} & e^{-4/5t} \\ -e^{-2/5t} & e^{-2/5t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}e^{2/5t} & \frac{1}{2}e^{2/5t} \\ -\frac{1}{2}e^{4/5t} & \frac{1}{2}e^{4/5t} \end{bmatrix}$$

$$X^{-1}\vec{f} = \begin{bmatrix} \frac{1}{2}e^{2/5 t} & \frac{1}{2}e^{2/5 t} \\ -\frac{1}{2}e^{4/5 t} & \frac{1}{2}e^{4/5 t} \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{2/5 t} \\ 0 \end{bmatrix}$$

$$\int X^{-1}\vec{f}\,dt = \begin{bmatrix} \int 2e^{2/5 t}\,dt \\ \int 0\,dt \end{bmatrix} = \begin{bmatrix} 5e^{2/5 t} \\ 0 \end{bmatrix}$$

$$\vec{X}_p = X\int X^{-1}\vec{f}\,dt = \begin{bmatrix} e^{-2/5 t} & -e^{-4/5 t} \\ e^{-2/5 t} & e^{-4/5 t} \end{bmatrix}\begin{bmatrix} 5e^{2/5 t} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

4) $\vec{X} = \vec{X}_n + \vec{X}_p$

$$\vec{X} = c_1 e^{-2/5 t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-4/5 t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad X_1(t) = c_1 e^{-2/5\, t} - c_2 e^{-4/5\, t} + 5$$

$$X_2(t) = c_1 e^{-2/5\, t} + c_2 e^{-4/5\, t} + 5$$

$$2 = X_1(0) = c_1 - c_2 + 5$$

$$3 = X_2(0) = c_1 + c_2 + 5$$

$$c_1 = -\frac{5}{2} \qquad c_2 = \frac{1}{2}$$

$$\Rightarrow \quad X_1(t) = -\frac{5}{2} e^{-2/5\, t} - \frac{1}{2} e^{-4/5\, t} + 5$$

$$X_2(t) = -\frac{5}{2} e^{-2/5\, t} + \frac{1}{2} e^{-4/5\, t} + 5$$

$$X_1' = 3x_1 - 4x_2$$

$$X_2' = 4x_1 + 3x_2$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \vec{x}' = A \vec{x}$$

$$\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{pmatrix}$$

$$= (3-\lambda)(3-\lambda) + 16$$

$$= \lambda^2 - 6\lambda + 25$$

$$\Rightarrow \lambda = \frac{6 \pm \sqrt{36-100}}{2} = \frac{6 \pm 8i}{2}$$

$$= 3 \pm 4i$$

For $\lambda = 3+4i$, $A-\lambda I = \begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix}$

Need $\begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ (2nd row) $4v_1 - 4iv_2 = 0$

$$4v_1 = 4iv_2$$

$\Rightarrow v_1 = i,\ v_2 = 1 \Rightarrow \vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Thus, we have $\lambda = 3 \pm 4i$

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In general, for $\lambda = \alpha \pm \beta i$

$$\vec{v} = \vec{u} \pm \vec{w} i$$

$$\vec{X} = e^{\alpha t}\left[c_1\left(\cos(\beta t)\vec{u} - \sin(\beta t)\vec{w}\right) + c_2\left(\cos(\beta t)\vec{w} + \sin(\beta t)\vec{u}\right)\right]$$

Here $\alpha = 3$, $\beta = 4$, $\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$!

$$\vec{X} = e^{3t}\left[c_1\left(\cos(4t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin(4t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + c_2\left(\cos(4t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(4t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right]$$

$$\Rightarrow \; X_1 = -c_1 e^{3t}\sin(4t) + c_2 e^{3t}\cos(4t)$$

$$X_2 = c_1 e^{3t}\cos(4t) + c_2 e^{3t}\sin(4t)$$