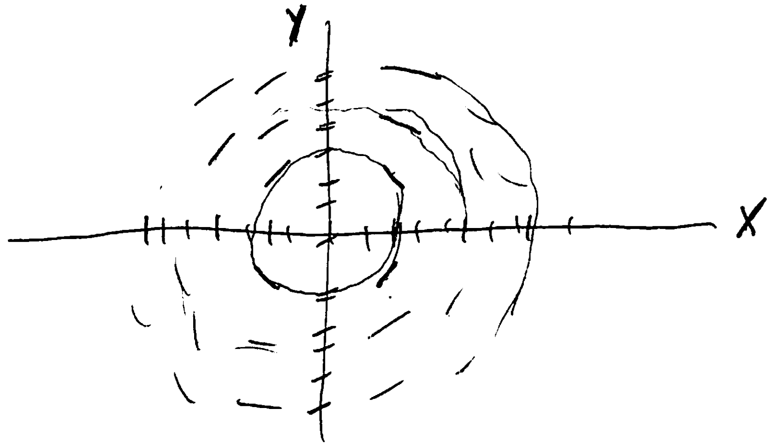
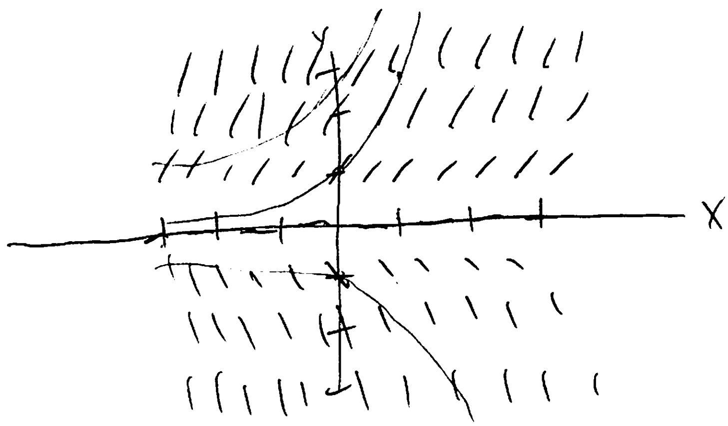
2)



10)



Note. We can use direction fields to visualize solutions to DEs and to predict values of y for given initial conditions

Tips for drawing/identifying direction fields:

1) Plug in $y=0$ and observe slopes along x-axis
2) Plug in $x=0$ and observe slopes along y-axis
3) Identify <u>equilibrium points</u> ie where $\frac{dy}{dx}=0$

4) Identify points where $\frac{dy}{dx}$ is undefined

5) Use Isoclines.

An <u>Isocline</u> for a DE $\frac{dy}{dx} = f(x,y)$ is a set of points that yield the same slope. That is, it is the set of points on the curve $f(x,y) = c$ for constant c.

Ex. 1) $\frac{dy}{dx} = y - x$

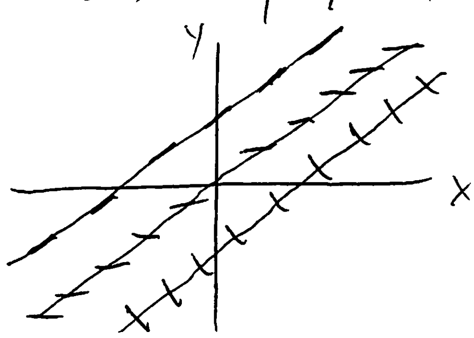
2) $\frac{dy}{dx} = x - y^2$

Draw isoclines for $c = -1, 0, 1$

1) $c = 0 \Rightarrow y - x = 0 \Rightarrow y = x$

$c = -1 \Rightarrow y - x = -1 \Rightarrow y = x - 1$

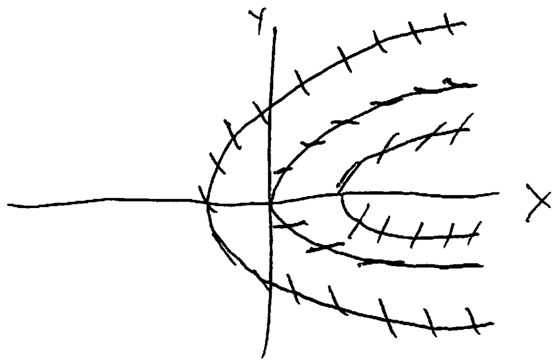
$c = 1 \Rightarrow y - x = 1 \Rightarrow y = x + 1$



2) $c=0 \Rightarrow x-y^2=0 \Rightarrow x=y^2$

$c=-1 \Rightarrow x-y^2=-1 \Rightarrow x=y^2-1$

$c=1 \Rightarrow x-y^2=1 \Rightarrow x=y^2+1$



The general form of an n-th order DE is $a_n(x,y)\frac{d^n y}{dx^n} + a_{n-1}(x,y)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x,y)\frac{dy}{dx} + a_0(x,y)y$
$= f(x)$

If the a_i depend only on x, the DE is linear.

If $f(x) \equiv 0$, the DE is homogeneous

Otherwise, the DE is nonhomogeneous.

Ex- 1) $\frac{dy}{dx} + y - e^x = 0$

2) $3x\frac{d^2y}{dx^2} = 6y$

3) $\frac{d^3y}{dx^3} + 7\frac{d^2y}{dx^2} = 11xy^2$

4) $6xy\frac{dy}{dx} = 4$

1) Write as $\frac{dy}{dx} + y = e^x \Rightarrow$ linear, nonhom.

2) Write as $3x\frac{d^2y}{dx^2} - 6y = 0 \Rightarrow$ linear, hom.

3) Write as $\frac{d^3y}{dx^3} + 7\frac{d^2y}{dx^2} - 11xy^2 = 0 \Rightarrow$ nonlinear hom.

4) nonlinear, nonhomogeneous

A DE that does not contain the independent variable is said to be <u>autonomous</u>.

Ex- $\frac{dy}{dx} = y$, $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 7y$, $\frac{dy}{dx} = y^3 + y^2 - 8$

A first order DE is <u>separable</u> if it can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

Ex-

1) $\frac{dy}{dx} = \frac{x}{y^2}$

2) $\frac{1}{x^2}\frac{dy}{dx} - 7 = y$

3) $\frac{dy}{dx} = e^{x+y}$

1) $\frac{dy}{dx} = x\left(\frac{1}{y^2}\right) \Rightarrow f(x) = x, \quad g(y) = \frac{1}{y^2}$

2) $\frac{1}{x^2}\frac{dy}{dx} - 7 = y \Rightarrow \frac{dy}{dx} = x^2(y+7)$

$\Rightarrow f(x) = x^2, \quad g(y) = y+7$

3) $\frac{dy}{dx} = e^x e^y \Rightarrow f(x) = e^x, \quad g(y) = e^y$

Solving separable DE

Special case: $g(y) \equiv 1$ ie $\frac{dy}{dx} = f(x)$

Then $y = \int f(x)\,dx + C = F(x) + C,$

where F is an antiderivative of f

If there is an initial condition $y(x_0) = y_0,$

then $y_0 = F(x_0) + C$ ie $C = y_0 - F(x_0)$

Then $y = F(x) + y_0 - F(x_0)$

$$\Rightarrow \quad y = y_0 + F(x) - F(x_0)$$

$$\Rightarrow \quad y = y_0 + \int_{x_0}^{x} f(t)\,dt$$

Fact - The general solution to $\frac{dy}{dx} = f(x)$ is $y = \int f(x)\,dx + C$.

The solution to the IVP $\frac{dy}{dx} = f(x)$, $y(x_0) = y_0$ is $y = y_0 + \int_{x_0}^{x} f(t)\,dt$

Ex- 1) $\frac{dy}{dx} = \cos(x)$, $y(0) = 7$

2) $\frac{dy}{dx} = e^{5x}$, $y(0) = 4$

1) $y = 7 + \int_0^x \cos(t)\,dt$

$y = 7 + [\sin(t)|_0^x$

$y = 7 + \sin(x)$

2) $y = 4 + \int_0^x e^{5t}\,dt$

$$y = 4 + \left[\tfrac{1}{5}e^{5t}\right]_0^x$$

$$y = 4 + \tfrac{1}{5}e^{5x} - \tfrac{1}{5}$$

$$y = \tfrac{19}{5} + \tfrac{1}{5}e^{5x}$$

To solve more general separable DE $\frac{dy}{dx} = f(x)\,g(y)$, we can

1) Separate ie $\frac{1}{g(y)}\,dy = f(x)\,dx$
2) Integrate $\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$

$$\Rightarrow \quad G(y) = F(x) + C$$

3) Solve for y (if needed/possible) or find implicit solution

Ex. 1) $\frac{dy}{dx} = \frac{x}{y^2}$

2) $\frac{1}{x^2}\frac{dy}{dx} - 7 = y$

3) $\frac{dy}{dx} = e^{x+y}$

4) $\frac{dy}{dx} = \frac{4x^3}{1+e^y}$

1) $\frac{dy}{dx} = \frac{x}{y^2} \Rightarrow y^2 dy = x\,dx$

$\Rightarrow \int y^2 dy = \int x\,dx$

$\Rightarrow \frac{y^3}{3} = \frac{x^2}{2} + C$

$\Rightarrow y^3 = \frac{3}{2}x^2 + C$

$\Rightarrow y = \sqrt[3]{\frac{3}{2}x^2 + C}$

2) $\frac{1}{x^2}\frac{dy}{dx} - 7 = y \Rightarrow \frac{1}{x^2}\frac{dy}{dx} = y+7$

$\Rightarrow \frac{1}{y+7}dy = x^2 dx$

$\Rightarrow \int \frac{1}{y+7} dy = \int x^2 dx$

$\Rightarrow \ln(y+7) = \frac{x^3}{3} + C$

$\Rightarrow y+7 = e^{x^3/3 + C}$

$\Rightarrow y = e^{\frac{x^3}{3}+C} - 7$

$$\Rightarrow y = e^{\frac{x^3}{3}} e^{c} - 7$$

$$\Rightarrow y = Ae^{x^3/3} - 7$$

3) $\frac{dy}{dx} = e^{x+y}$ $\Rightarrow \frac{dy}{dx} = e^x e^y$

$$\Rightarrow e^{-y}dy = e^x dx$$

$$\Rightarrow \int e^{-y}dy = \int e^x dx$$

$$\Rightarrow -e^{-y} = e^x + C$$

$$\Rightarrow e^{-y} = -e^x + C$$

$$\Rightarrow -y = \ln(-e^x + c)$$

$$\Rightarrow \boxed{y = -\ln(-e^x + c)}$$

4) $\frac{dy}{dx} = \frac{4x^3}{1+e^y}$ $\Rightarrow (1+e^y)dy = 4x^3 dx$

$$\Rightarrow \int 1 + e^y dy = \int 4x^3 dx$$

$$\Rightarrow y + e^y = x^4 + C$$

$$\Rightarrow y + e^y - x^4 = C$$

Recall, a first-order linear DE has the form $a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$

To solve a first-order linear DE:

1) Write the DE in <u>standard form</u>

$$\frac{dy}{dx} + P(x)y = Q(x)$$

2) Compute the <u>integrating factor</u>

$$u(x) = e^{\int P(x)\,dx}$$

3) Solve for y as

$$y = \frac{1}{u(x)}\left(\int Q(x)u(x)\,dx + C\right)$$

Ex -

1) $x\frac{dy}{dx} + y = 5$

2) $\frac{dy}{dx} + y - e^x = 0$

3) $\frac{1}{x^2}\frac{dy}{dx} - 7 = -3y$

4) $\frac{dy}{dx} - \frac{y}{x} = xe^x$

1) $x\frac{dy}{dx} + y = 5 \Rightarrow \frac{dy}{dx} + \frac{1}{x}y = \frac{5}{x}$

So $P(x) = \frac{1}{x}$ and $Q(x) = \frac{5}{x}$

Then $\int P(x)dx = \int \frac{1}{x} dx = \ln(x)$

Thus $u(x) = e^{\int P(x)dx} = e^{\ln(x)} = x$

Hence $\int Q(x)u(x)\,dx + C$

$= \int 5\,dx + C$

$= 5x + C$

Therefore $y = \frac{1}{u(x)}\left(\int Q(x)u(x)\,dx + C\right)$

$$\Rightarrow y = \frac{1}{x}(5x + C)$$

$$\Rightarrow y = 5 + \frac{C}{x}$$

2) $\frac{dy}{dx} + y - e^x = 0 \Rightarrow \frac{dy}{dx} + y = e^x$

So $P(x) = 1$ and $Q(x) = e^x$

Then $\int P(x)\,dx = \int 1\,dx = x$

Thus $u(x) = e^{\int P(x)\,dx} = e^x$

Hence $\int Q(x)u(x)\,dx = \int e^{2x}\,dx$

$$= \frac{1}{2}e^{2x}$$

Therefore $y = \frac{1}{u(x)}\left(\int Q(x)u(x)\,dx + C\right)$

$$\Rightarrow y = \frac{1}{e^x}\left(\frac{1}{2}e^{2x} + C\right)$$

$$\Rightarrow y = \frac{1}{2}e^x + \frac{C}{e^x}$$

$$\Rightarrow y = \frac{1}{2}e^x + Ce^{-x}$$

3) $\frac{1}{x^2}\frac{dy}{dx} - 7 = -3y$

$$\Rightarrow \frac{1}{x^2}\frac{dy}{dx} + 3y = 7$$

$$\Rightarrow \frac{dy}{dx} + 3x^2 y = 7x^2$$

So $P(x) = 3x^2$ and $Q(x) = 7x^2$

Then $u(x) = e^{\int P(x)dx} = e^{x^3}$

$$\int P(x)\,dx = \int 3x^2\,dx = x^3$$

Hence $\int Q(x)u(x)\,dx = \int 7x^2 e^{x^3} dx$

$$= \frac{7}{3}e^{x^3}$$

Therefore $y = \frac{1}{u(x)}\left(\int Q(x)u(x)dx + C\right)$

$$\Rightarrow y = \frac{1}{e^{x^3}}\left(\frac{7}{3}e^{x^3} + C\right)$$

$$\Rightarrow y = \frac{7}{3} + \frac{C}{e^{x^3}}$$

$$\Rightarrow y = \frac{7}{3} + Ce^{-x^3}$$

4) $\frac{dy}{dx} - \frac{y}{x} = xe^x$

So $P(x) = -\frac{1}{x}$ and $Q(x) = xe^x$

Then $\int P(x)dx = \int -\frac{1}{x}\,dx = -\ln(x)$

Hence $u(x) = e^{\int P(x)dx} = e^{-\ln(x)} = \frac{1}{e^{\ln(x)}}$

$$= \frac{1}{x}$$

Thus $\int Q(x)u(x)\,dx = \int e^x\,dx = e^x$

Therefore $y = \frac{1}{u(x)}\left(\int Q(x)u(x)dx + C\right)$

$$\Rightarrow y = x\left(e^x + C\right)$$

$$\Rightarrow y = xe^x + Cx$$

Why does this work?

First, let's verify that

$y = \frac{1}{u(x)}\left(\int Q(x)u(x)\,dx + C\right)$ satisfies

the DE $\frac{dy}{dx} + P(x)y = Q(x)$, where

$u(x) = e^{\int P(x)\,dx}$

Note that $u'(x) = P(x)e^{\int P(x)\,dx} = P(x)u(x)$

By the product rule,

$$\frac{dy}{dx} = \frac{1}{u(x)}Q(x)u(x) + \left(\int Q(x)u(x)\,dx + C\right)\left(\frac{-u'(x)}{u(x)^2}\right)$$

$$\frac{dy}{dx} = Q(x) + \left(\int Q(x)u(x)\,dx + C\right)\left(\frac{-P(x)u(x)}{u(x)^2}\right)$$

$$\frac{dy}{dx} = Q(x) - \underbrace{\frac{1}{u(x)}\left(\int Q(x)u(x)\,dx + C\right)}P(x)$$

$$\frac{dy}{dx} = Q(x) - P(x)y$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Now, starting with $\frac{dy}{dx} + P(x)y = Q(x)$,

let's derive $y = \frac{1}{u(x)}\left(\int Q(x)u(x)\,dx + C\right)$,

where $u(x) = e^{\int P(x)\,dx}$

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$$\Rightarrow \quad u(x)\frac{dy}{dx} + u(x)P(x)\,y = Q(x)\,u(x)$$

$$\Rightarrow \quad u(x)\frac{dy}{dx} + u'(x)\,y = Q(x)\,u(x)$$

$$\Rightarrow \quad \frac{d}{dx}\left[u(x)\,y\right] = Q(x)\,u(x)$$

$$\Rightarrow \quad u(x)\,y = \int Q(x)u(x)\,dx + C$$

$$\Rightarrow \quad y = \frac{1}{u(x)}\left[\int Q(x)u(x)\,dx + C\right]$$

Note, the Integrating factor $u(x) = e^{\int P(x)\,dx}$ was chosen specifically to satisfy $u'(x) = u(x)\,P(x)$.

That is, $u' = uP \Rightarrow \frac{du}{dx} = u\,P(x)$

$$\Rightarrow \frac{1}{u}\,du = P(x)\,dx$$

$$\Rightarrow \int \frac{1}{u}\,du = \int P(x)\,dx$$

$$\Rightarrow \ln(u) = \int P(x)\,dx$$

$$\Rightarrow u = e^{\int P(x)\,dx}$$