Recap: Variation of Parameters

Goal - Find general solution to

$$y'' + by' + cy = f(x)$$

1) Find homogeneous solution $y_h = c_1 y_1 + c_2 y_2$ to $y'' + by' + cy = 0$

2) Find particular solution to the nonhomogeneous DE of the form $y_p = v_1 y_1 + v_2 y_2$ where

$$v_1 = \int \frac{-y_2 f(x)}{y_1 y_2' - y_1' y_2}\,dx \quad \text{and} \quad v_2 = \int \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2}\,dx$$

3) $y = y_h + y_p$

Examples $y'' - 4y' + 4y = 30\sqrt{x}\,e^{2x}$

1) Find y_h Aux eq $r^2-4r+4=0$

$(r-2)^2=0$

$r=2$

$y_h = c_1 e^{2x} + c_2 x e^{2x}$

2) $y_1 = e^{2x} \Rightarrow y_1' = 2e^{2x}$

$y_2 = xe^{2x} \Rightarrow y_2' = 2xe^{2x} + e^{2x}$

So $y_1 y_2' - y_1' y_2 = e^{4x}$

$V_1 = \int \frac{-y_2 f(x)}{e^{4x}} dx = \int \frac{-xe^{2x} 30\sqrt{x} e^{2x}}{e^{4x}} dx$

$= \int -30x^{3/2} dx$

$= -12x^{5/2}$

$V_2 = \int \frac{y_1 f(x)}{e^{4x}} dx = \int \frac{e^{2x} 30\sqrt{x} e^{2x}}{e^{4x}} dx$

$= \int 30x^{1/2} dx$

$$= 20x^{3/2}$$

$$y_p = v_1 y_1 + v_2 y_2$$
$$= -12x^{5/2}e^{2x} + 20x^{3/2}xe^{2x}$$
$$= -12x^{5/2}e^{2x} + 20x^{5/2}e^{2x} = 8x^{5/2}e^{2x}$$

3) $y = y_h + y_p = c_1 e^{2x} + c_2 x e^{2x} + 8x^{5/2}e^{2x}$

One of the most common applications of second-order DEs is the mass-spring system. Imagine a mass attached to a hanging spring. It sits naturally at some equilibrium. We then stretch and release the mass-spring and observe the motion. Let y be the displacement (in meters) of the mass (in kg) from its equilibrium

Then $\frac{dy}{dt}$ is its velocity and $\frac{d^2y}{dt^2}$ is its acceleration.

We can measure the force on the mass as follows: $F = F_{spring} + F_{friction} + F_{ext}$

where F is the total force, F_{spring} is the force exerted by the spring, $F_{friction}$ is the force due to friction/air resistance/damping, and F_{ext} is any forces external to the system (all in newtons N)

We now use the following facts from Physics:

$F_{spring} = -ky$ where $k > 0$ is the spring constant (stiffness) in N/m (Hooke's Law)

$F_{friction} = -by'$ where $b > 0$ is the damping constant in $N \cdot \frac{m}{s}$ (Stokes' Law)

$F = my''$ (Newton's 2nd Law)

Thus $my'' = -ky - by' + F_{ext}$

$my'' + by' + ky = F_{ext}$ $\quad *$

This DE is homogeneous if $F_{ext} = 0$
nonhomogeneous if $F_{ext} \neq 0$

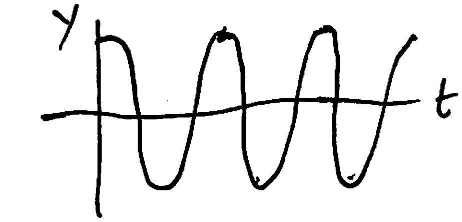
Consider first when $F_{ext} = 0$ ie $my'' + by' + ky = 0$

Case 1: no damping ie $b = 0$ ie $my'' + ky = 0$

Aux eq is $mr^2 + k = 0$ so $r = \pm\sqrt{\frac{-k}{m}}$

ie $r = \pm i\sqrt{\frac{k}{m}}$

Thus $y = c_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$

ie oscillitory motion 

Using difference rule for cosine, we can write y as $y = A\cos\left(\sqrt{\frac{K}{m}}\,t - \varphi\right)$

where $A = \sqrt{c_1^2 + c_2^2}$ and φ is the angle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan(\varphi) = \frac{c_2}{c_1}$

A is the <u>amplitude</u> of the oscillations

$\sqrt{\frac{K}{m}}$ is the <u>angular frequency</u> (radians/seconds)

$\frac{\sqrt{\frac{K}{m}}}{2\pi}$ is the <u>natural frequency</u> (cycles per sec)

$\frac{2\pi}{\sqrt{K/m}}$ is the <u>period</u> (in seconds)

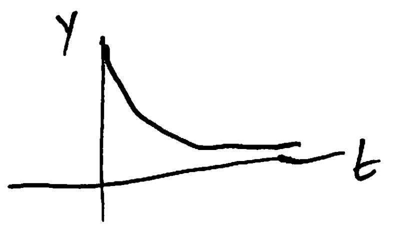
Case 2: $b \neq 0$ ie $my'' + by' + Ky = 0$

Aux eq is $mr^2 + br + K = 0$

so $r = \frac{-b \pm \sqrt{b^2 - 4mK}}{2m}$

If $b^2-4mk>0$, then we have two distinct <u>negative</u> real roots r_1, r_2

So $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ and $\lim_{t\to\infty} y = 0$



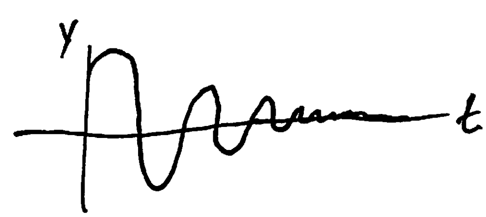
We call this <u>overdamping</u>

If $b^2-4mk<0$ we get complex roots

$r = \frac{-b}{2m} \pm i \frac{\sqrt{4mk-b^2}}{2m}$ with negative real part $\frac{-b}{2m}$

Then $y = e^{-\frac{b}{2m}t}\left(c_1 \cos\left(\frac{\sqrt{4mk-b^2}}{2m}t\right) + c_2 \sin\left(\frac{\sqrt{4mk-b^2}}{2m}t\right)\right)$

and $\lim_{t\to\infty} y = 0$

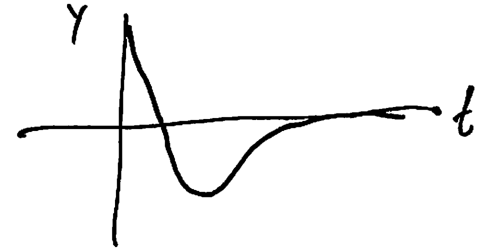


This is <u>underdamping</u>

If $b^2-4mk=0$ we get a repeated negative root $r = \frac{-b}{2m}$

So $y = c_1 e^{-\frac{b}{2m}t} + c_2 t e^{-\frac{b}{2m}t}$

and $\lim_{t \to \infty} y = 0$



This is <u>critical damping</u>

When $F_{ext} \neq 0$ we have $my'' + by' + ky = F(t)$

and a solution looks like $y = y_h + y_p$

In this context, y_h is the <u>transient solution</u>

and y_p is the <u>steady-state solution</u>.

For a function $f(t)$ defined on $[0, \infty)$,

the <u>Laplace transform</u> of f is

$$L(f) = \int_0^\infty e^{-st} f(t)\,dt = \lim_{N \to \infty} \int_0^N e^{-st} f(t)\,dt$$

Observe that $L(f)$ will be a function of s.

As such $L(f)$ is often denoted $F(s)$

Example - 1) $f(t) = 1$

2) $f(t) = e^{at}$ for constant a

3) $f(t) = t$

1) $$L(1) = \int_0^\infty e^{-st}\,dt = \lim_{N\to\infty} \int_0^N e^{-st}\,dt$$

$$= \lim_{N\to\infty} \left[-\frac{1}{s} e^{-st} \Big|_0^N \right]$$

$$= \lim_{N\to\infty} \left[-\frac{1}{s} e^{-sN} + \frac{1}{s} \right]$$

$$= \frac{1}{s} \qquad s > 0$$

2) $$L(e^{at}) = \int_0^\infty e^{-st} e^{at}\,dt = \lim_{N\to\infty} \int_0^N e^{-st+at}\,dt$$

$$= \lim_{N\to\infty} \int_0^N e^{-(s-a)t}\,dt$$

$$= \lim_{N\to\infty} \left[-\frac{1}{s-a} e^{-(s-a)t} \Big|_0^N \right]$$

$$= \lim_{N\to\infty} \left[-\frac{1}{s-a} e^{-(s-a)N} + \frac{1}{s-a} \right]$$

$$= \frac{1}{s-a} \qquad s > a$$

3) $L(t) = \int_0^\infty t e^{-st}\, dt = \lim_{N\to\infty} \int_0^N t e^{-st}\, dt$

For $\int t e^{-st}\, dt$, let $u = t$ and $dv = e^{-st}\,dt$

$du = dt$ and $v = -\frac{1}{s} e^{-st}$

So integral is $-\frac{1}{s} t e^{-st} + \int \frac{1}{s} e^{-st}\, dt$

$$= -\frac{1}{s} t e^{-st} - \frac{1}{s^2} e^{-st}$$

Thus $\lim_{N\to\infty} \left(-\frac{1}{s} t e^{-st} - \frac{1}{s^2} e^{-st} \Big|_0^N \right)$

$$= \lim_{N\to\infty} \left[\left(-\frac{1}{s} t e^{-sN} - \frac{1}{s^2} e^{-sN} \right) + \frac{1}{s^2} \right]$$

$$= \frac{1}{s^2} \qquad s > 0$$

Similarly, we can show:

$$L(t^n) = \frac{n!}{s^{n+1}} \qquad s > 0$$

$$L(\sin(bt)) = \frac{b}{s^2+b^2} \qquad s > 0$$

$$L(\cos(bt)) = \frac{s}{s^2+b^2} \qquad s > 0$$

$$L(e^{at}\sin(bt)) = \frac{b}{(s-a)^2+b^2} \qquad s > a$$

$$L(e^{at}\cos(bt)) = \frac{s-a}{(s-a)^2+b^2} \qquad s > a$$

$$L(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}} \qquad s > a$$

Ex -

1) $f(t) = t^2$

2) $f(t) = \cos(3t)$

3) $f(t) = \sin(4t)$

4) $f(t) = e^{6t}\cos(t)$

5) $f(t) = e^{t}\sin(5t)$

6) $f(t) = t^3 e^{-t}$

7) $f(t) = 8e^{2t} + t^4$

1) $L(f) = \frac{2!}{s^3} = \frac{2}{s^3} \qquad s > 0$

2) $L(f) = \frac{s}{s^2+9} \qquad s > 0$

3) $L(f) = \frac{4}{s^2+16} \qquad s > 0$

4) $L(f) = \frac{s-6}{(s-6)^2+1} \qquad s > 6$

5) $L(f) = \frac{5}{(s-1)^2+25} \qquad s > 1$

6) $L(f) = \frac{3!}{(s+1)^4} = \frac{6}{(s+1)^4} \qquad s > -1$

7) $L(f) = \int_0^\infty e^{-st}(8e^{2t} + t^4)\,dt$

$= \int_0^\infty 8e^{-st}e^{2t} + t^4 e^{-st}\,dt$

$= \int_0^\infty 8e^{-st}e^{2t}\,dt + \int_0^\infty e^{-st}t^4\,dt$

$= 8\int_0^\infty e^{-st}e^{2t}\,dt + \int_0^\infty e^{-st}t^4\,dt$

$$= 8\,L(e^{2t}) + L(t^4)$$

$$= 8 \cdot \frac{1}{s-2} + \frac{4!}{s^5}$$

$$= \frac{8}{s-2} + \frac{24}{s^5} \qquad s > 2$$

Fact - The Laplace transform is linear

That is, if f_1, f_2 are two functions whose Laplace transforms exist for $s > k$ then $L(f_1 + f_2) = L(f_1) + L(f_2)$ for $s > k$

Also, if f is a function whose Laplace transform exists for $s > k$, and if c is a constant, then $L(cf) = c\,L(f)$ for $s > k$.

Ex. 1) $f(t) = 3t - \cos(5t)$

2) $f(t) = 9\sin(t) + e^{8t}$

1) $L(f) = L(3t - \cos(5t))$

$= L(3t) - L(\cos(5t))$

$= 3L(t) - \dfrac{s}{s^2+25}$

$= 3 \cdot \dfrac{1}{s^2} - \dfrac{s}{s^2+25}$

$= \dfrac{3}{s^2} - \dfrac{s}{s^2+25} \qquad s > 0$

2) $L(f) = L(9\sin(t) + e^{8t})$

$= L(9\sin(t)) + L(e^{8t})$

$= 9L(\sin(t)) + \dfrac{1}{s-8}$

$= 9 \cdot \dfrac{1}{s^2+1} + \dfrac{1}{s-8}$

$= \dfrac{9}{s^2+1} + \dfrac{1}{s-8} \qquad s > 8$

For what functions does the Laplace transform exist? The obvious answer is the functions for which $\int_0^\infty e^{-st} f(t)\,dt$ exist on some interval (K, ∞)

$f(t)$ must be piecewise continuous on $[0, \infty)$

i.e. no vertical asymptotes on $[0, \infty)$

For example, $\ln(t)$, $\frac{1}{t}$, $\tan(t)$, $\sec(t)$, $\csc(t)$, $\cot(t)$ do NOT have Laplace transforms!

Ex - $f(t) = \begin{cases} 1 & 0 \leq t < 3 \\ 0 & 3 \leq t \leq 6 \\ e^t & 6 < t \end{cases}$

$$L(f) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^3 e^{-st}\,dt + \int_3^6 0\,dt + \int_6^\infty e^{-st} e^t\,dt$$

$$= -\frac{1}{s} e^{-st}\Big|_0^3 + 0 + \lim_{N \to \infty} \int_6^N e^{-(s-1)t}\,dt$$

$$= -\frac{1}{s} e^{-3s} + \frac{1}{s} + \lim_{N \to \infty} \left[-\frac{1}{s-1} e^{-(s-1)t} \Big|_6^N \right.$$

$$= -\frac{1}{s} e^{-3s} + \frac{1}{s} + \lim_{N \to \infty} \left[-\frac{1}{s-1} e^{-(s-1)N} + \frac{1}{s-1} e^{-6(s-1)} \right.$$

$$= -\frac{1}{s}e^{-3s} + \frac{1}{s} + \frac{1}{s-1}e^{-6(s-1)} \qquad s > 1$$

$f(t)$ must be of <u>exponential order.</u>

Formally, $f(t)$ is of exponential order K if there exists positive constants T and M such that $|f(t)| \leq Me^{Kt}$ for all $t \geq T$

Informally, $f(t)$ does not grow faster than e^{Kt} for all constants K

For example, e^{t^n} for $n > 1$ does NOT have a Laplace transform

e.g e^{t^2}, $e^{t^{3/2}}$, $e^{(t-1)^5}$

For a function $F(s)$, if there exists a function $f(t)$ defined on $[0, \infty)$ such that $L(f) = F(s)$, then $f(t)$ is the <u>inverse Laplace transform</u> of F and we write $f(t) = L^{-1}(F)$

Ex - 1) $F(s) = \frac{1}{s}$

2) $F(s) = \frac{1}{s-3}$

3) $F(s) = \frac{s}{s^2+4}$

4) $F(s) = \frac{5}{s^2+25}$

5) $F(s) = \frac{1}{(s+3)^2+1}$

6) $F(s) = \frac{s+3}{(s+3)^2+1}$

7) $F(s) = \frac{24}{s^5}$

1) $L^{-1}\left(\frac{1}{s}\right) = 1$

2) $L^{-1}(F) = e^{3t}$

3) $L^{-1}(F) = \cos(2t)$

4) $L^{-1}(F) = \sin(5t)$

5) $L^{-1}(F) = e^{-3t}\sin(t)$

6) $\mathcal{L}^{-1}(F) = e^{-3t}\cos(t)$

7) $\mathcal{L}^{-1}(F) = t^4 \qquad \left(\frac{24}{s^5} = \frac{4!}{s^{4+1}}\right)$

Fact - The inverse Laplace transform is linear

ie $\mathcal{L}^{-1}(F_1 + F_2) = \mathcal{L}^{-1}(F_1) + \mathcal{L}^{-1}(F_2)$ and

$\mathcal{L}^{-1}(cF) = c\mathcal{L}^{-1}(F)$

Ex - 1) $F(s) = \frac{9}{s}$

2) $F(s) = \frac{1}{s^2-5s+6}$

3) $F(s) = \frac{s-5}{s^2+4s+3}$

4) $F(s) = \frac{4}{s^2+6s+25}$

5) $F(s) = \frac{s-7}{s^2-4s+5}$

6) $F(s) = \frac{8}{s^2-2s+1}$

7) $F(s) = \frac{10}{s^3}$