Recall that if we differentiate $f(x,y)$ with respect to x, we hold y constant, and vice versa.

To antidifferentiate $f(x,y)$ with respect to x, hold y constant and antidifferent.. the x-terms as usual.

To antidifferentiate $f(x,y)$ with respect to y, hold x constant.

Likewise for functions of 3, 4, etc. variables

Example - 1) $\int_1^3 6x^2 y \, dx$

2) $\int_2^4 6x^2 y \, dy$

1) $$\int_1^3 6x^2y\,dx = 6y\int_1^3 x^2\,dx$$

$$= \frac{6yx^3}{3}\Big|_1^3$$

$$= 2yx^3\Big|_1^3$$

$$= 54y - 2y = 52y$$

2) $$\int_2^4 6x^2y\,dy = 6x^2\int_2^4 y\,dy$$

$$= \frac{6x^2y^2}{2}\Big|_2^4$$

$$= 3x^2y^2\Big|_2^4$$

$$= 48x^2 - 12x^2 = 36x^2$$

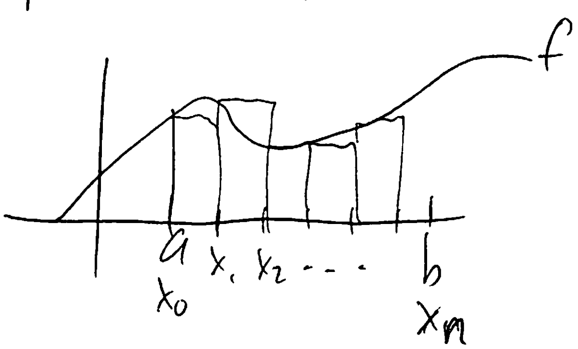
Notice we get functions as output, so we can integrate again.

$$\int_2^4 52y\,dy = 312 \qquad \int_1^3 36x^2\,dx = 312$$

In calc 1, we define definite integrals as limits of Riemann sums, and we use them to compute area.

In calc 3, we can still use Riemann sums, except now we compute volume.

Recall a Riemann sum for $f(x)$ on $[a,b]$

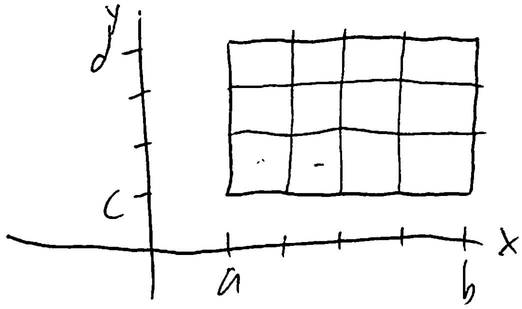


$$\sum_{i=1}^{n} f(x_i^*)\frac{b-a}{n}$$

where $x_i^* \in [x_{i-1}, x_i]$ was a sample point in each subinterval (right endpoint, left endpoint, midpoint, etc.)

Instead of a closed interval in $\mathbb{R}$, we consider a closed region in $\mathbb{R}^2$. We start with the simplest closed region, rectangles.

$$R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\} = [a,b] \times [c,d]$$



Divide $[a,b]$ into m subintervals

Divide $[c,d]$ into n subintervals

This yields $S = mn$ subregions.

Call these $R_1, R_2, \ldots, R_S$

The Riemann sum of $f(x,y)$ on R is

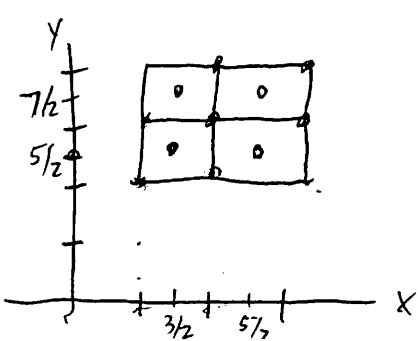
$$\sum_{i=1}^{S} f(x_i^*, y_i^*) A_i$$ where A_i = area of R_i

and (x_i^*, y_i^*) is a sample point in R_i

(lower left corner, lower right, upper left, upper right, center, etc.)

Ex- $f(x,y) = 6x^2y$ on

$$R = \{(x,y) \mid 1 \le x \le 3,\ 2 \le y \le 4\}$$



Our sample points are

$\left(\frac{3}{2}, \frac{5}{2}\right), \left(\frac{5}{2}, \frac{5}{2}\right), \left(\frac{3}{2}, \frac{7}{2}\right), \left(\frac{5}{2}, \frac{7}{2}\right)$

$A_i = 1$ for all i

R.S. is $\displaystyle\sum_{i=1}^{4} f(x_i^*, y_i^*)$

$= f\left(\frac{3}{2}, \frac{5}{2}\right) + f\left(\frac{5}{2}, \frac{5}{2}\right) + f\left(\frac{3}{2}, \frac{7}{2}\right) + f\left(\frac{5}{2}, \frac{7}{2}\right)$

$= 6\left[\left(\frac{9}{4}\right)\left(\frac{5}{2}\right) + \left(\frac{25}{4}\right)\left(\frac{5}{2}\right) + \left(\frac{9}{4}\right)\left(\frac{7}{2}\right) + \left(\frac{25}{4}\right)\left(\frac{7}{2}\right)\right]$

$= 6\left[\frac{45}{8} + \frac{125}{8} + \frac{63}{8} + \frac{175}{8}\right]$

$= 6\left(\frac{408}{8}\right) = 6(51) = 306$

We can make this more accurate by increasing the number of subregions. If we take the limit as the number of subregions goes to infinity, we get $\displaystyle\iint_R f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy$

= volume of region under f over R

Example - Find volume of the region under $f(x,y) = 4xy + 5$ over $R = \{(x,y) \mid 1 \le x \le 3,\ 1 \le y \le 2\}$

$$V = \int_1^2 \left(\int_1^3 (4xy+5)\, dx \right) dy$$

$$\int_1^3 4xy+5\, dx = 2x^2y + 5x \Big|_1^3$$

$$= (18y+15) - (2y+5) = 16y+10$$

$$\int_1^2 16y+10\, dy = 8y^2 + 10y \Big|_1^2$$

$$= (32+20) - (8+10) = 34$$

Example- 1) $\displaystyle\int_0^{\frac{\pi}{2}} \int_1^4 2x\cos(y)\, dx\, dy$

2) $\displaystyle\int_1^4 \int_0^{\pi/2} 2x\cos(y)\, dy\, dx$

1) $$\int_1^4 2x\cos(y)\, dx = x^2\cos(y) \Big|_1^4$$

$$= 16\cos(y) - \cos(y) = 15\cos(y)$$

$$\int_0^{\pi/2} 15\cos(y)\,dy = 15\sin(y)\Big|_0^{\pi/2} = 15 - 0 = 15$$

2) $$\int_0^{\pi/2} 2x\cos(y)\,dy = 2x\sin(y)\Big|_0^{\pi/2} = 2x - 0 = 2x$$

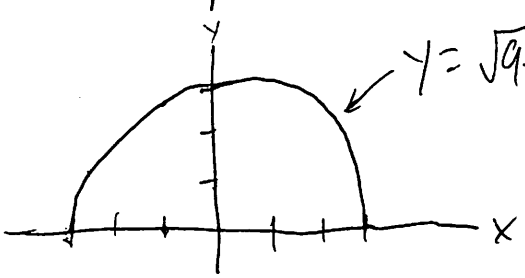
$$\int_1^4 2x\,dx = x^2\Big|_1^4 = 16 - 1 = 15$$

Fact - If a, b, c, d are all constants, then

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Suppose we want to ~~find~~ find volume of the region under $f(x,y)$ over a nonrectangular region.

Example - R = top half of the circle $x^2 + y^2 = 9$



$$R_1 = \{(x,y) \mid -3 \le x \le 3,\ 0 \le y \le \sqrt{9-x^2}\}$$

$$R_2 = \{(x,y) \mid 0 \le y \le 3,\ -\sqrt{9-y^2} \le x \le \sqrt{9-y^2}\}$$

Find Volume of region under $f(x,y) = 2y$ over R_1, R_2.

Over R_1: $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} 2y\,dy\,dx$

$$\int_0^{\sqrt{9-x^2}} 2y\,dy = y^2 \Big|_0^{\sqrt{9-x^2}} = 9-x^2$$

$$\int_{-3}^{3} 9-x^2\,dx = 9x - \frac{x^3}{3}\Big|_{-3}^{3}$$

$$= (27-9)-(-27+9) = 36$$

Over R_2: $\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} 2y\,dx\,dy$

$$\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} 2y\,dx = 2yx\Big|_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}$$

$$= 2y\sqrt{9-y^2} - \left(-2y\sqrt{9-y^2}\right) = 4y\sqrt{9-y^2}$$

$$\int_0^3 4y\sqrt{9-y^2}\,dy \qquad u = 9-y^2 \Rightarrow du = -2y\,dy$$

$$\Rightarrow -2du = 4y\,dy$$

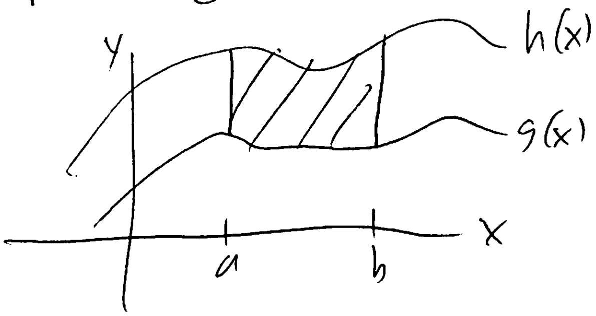
$\downarrow$

$$\int -2u^{1/2}\,du = -\frac{4}{3}u^{3/2} = -\frac{4}{3}(9-y^2)^{3/2}\Big|_0^3$$

$$= 0 - \left(-\frac{4}{3}(9)^{3/2}\right) = \frac{4}{3}(27) = 36$$

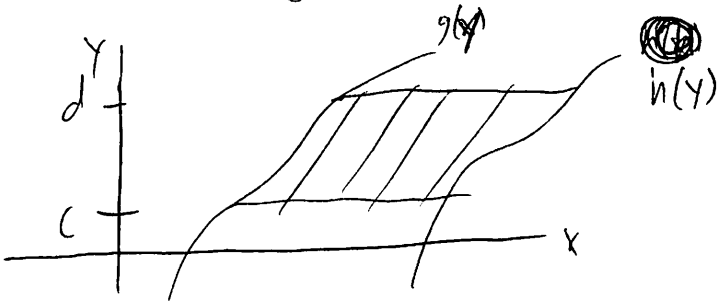
A type 1 region is a region of the form

$$R = \{(x,y) \mid a \le x \le b,\ g(x) \le y \le h(x)\}$$



A type 2 region is a region of the form

$$R = \{(x,y) \mid c \le y \le d,\ g(y) \le x \le h(y)\}$$



Switching from a type 1 region to its equivalent type 2 region (or vice versa) is called <u>reversing the order of integration</u>

Example - $\int_0^{\pi} \int_x^{\pi} x \cos(y^3)\, dy\, dx$