Last time, we looked at the matrix equation $A\vec{x} = \vec{b}$. We will often be interested in the case when $\vec{b} = \vec{0}$.

Such systems are homogeneous.

If $\vec{b} \neq \vec{0}$, system is nonhomogeneous.

$A\vec{x} = \vec{0}$ has one solution always $(\vec{x} = \vec{0})$

We want to know if there are nonzero solutions as well.

Ex. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 6 & 0 \end{array}\right] \xrightarrow[-R_1 + R_3 \to R_3]{-R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 5 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[-R_3 + R_1 \to R_1]{-2R_3 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_1\to R_1}\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]\Longrightarrow\begin{array}{l}x_1=0\\x_2=0\\x_3=0\end{array}$$

Ex:

$$\begin{aligned}x_1-x_2+x_3&=0\\2x_1+3x_2+12x_3&=0\\x_1+x_2+5x_3&=0\end{aligned}$$

$$\Rightarrow\begin{bmatrix}1&-1&1\\2&3&12\\1&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow\left[\begin{array}{ccc|c}1&-1&1&0\\2&3&12&0\\1&1&5&0\end{array}\right]\xrightarrow[-R_1+R_3\to R_3]{-2R_1+R_2\to R_2}\left[\begin{array}{ccc|c}1&-1&1&0\\0&5&10&0\\0&2&4&0\end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_2\to R_2}\left[\begin{array}{ccc|c}1&-1&1&0\\0&1&2&0\\0&2&4&0\end{array}\right]\xrightarrow{-2R_2+R_3\to R_3}\left[\begin{array}{ccc|c}1&-1&1&0\\0&1&2&0\\0&0&0&0\end{array}\right]$$

$$\xrightarrow{R_2+R_1\to R_1}\left[\begin{array}{ccc|c}1&0&3&0\\0&1&2&0\\0&0&0&0\end{array}\right]\Longrightarrow\begin{array}{l}x_1+3x_3=0\\x_2+2x_3=0\end{array}$$

Let $x_3 = s$. Then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -3s \\ -2s \\ s \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} s$

Infinitely many solutions.

We can use row operations to compute A^{-1} for $n > 2$.

For 2×2 matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we have

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

For 3×3 matrices, form the matrix $[A \mid I_3]$ and row reduce to $[I_3 \mid B]$.

Then $B = A^{-1}$

Ex. 1) $A = \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 7 & 5 \end{bmatrix}$

1) $A^{-1} = \frac{1}{6(4) - 3(5)} \begin{bmatrix} 4 & -5 \\ -3 & 6 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 & -5 \\ -3 & 6 \end{bmatrix}$

$= \begin{bmatrix} 4/9 & -5/9 \\ -1/3 & 2/3 \end{bmatrix}$

2) $\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 2 & 7 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1 + R_3 \to R_3]{-R_1 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 3 & -1 & -2 & 0 & 1 \end{array}\right]$

$\xrightarrow[-2R_2 + R_1 \to R_1]{-3R_2 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 3 & -2 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -3 & 1 \end{array}\right]$

$\xrightarrow{-R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 3 & -2 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 3 & -1 \end{array}\right] \xrightarrow{-3R_3 + R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -11 & 3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 3 & -1 \end{array}\right]$

Thus, $A^{-1} = \begin{bmatrix} 6 & -11 & 3 \\ -1 & 1 & 0 \\ -1 & 3 & -1 \end{bmatrix}$

Observe that in 2x2 case, we divide by det(A). If det(A) = 0, A^{-1} doesn't exist.

Observe that in 3x3 case, we row reduce

A to I. If this is not possible, then A^{-1} doesn't exist. Recall that for homogeneous systems $A\vec{x} = \vec{0}$, a unique solution ($\vec{x} = \vec{0}$) existed only when A reduced to I.

Theorem - The following are equivalent

1) A is invertible
2) $\det(A) \neq 0$
3) $A\vec{x} = \vec{0}$ has only the zero solution

Corollary - The following are equivalent

1) A is <u>not</u> invertible
2) $\det(A) = 0$
3) $A\vec{x} = \vec{0}$ has nonzero solutions

Note - If A is invertible, we can solve $A\vec{x} = \vec{b}$ as $\vec{x} = A^{-1}\vec{b}$

We will often be interested in finding vectors $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some constant λ.

Obviously, $\vec{x} = \vec{0}$ is one, but that's boring

A nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some constant λ is an <u>eigenvector</u> of A, and λ is the corresponding <u>eigenvalue</u>.

Ex - $A = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & -1 \\ 2 & 7 & -2 \end{bmatrix}$. Show that

1) $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and 2) $\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$ are eigenvectors of A, and find the eigenvalues.

1) $A\vec{x} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & -1 \\ 2 & 7 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = 2\vec{x}$

So eigenvalue is **2**.

2) $A\vec{x} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & -1 \\ 2 & 7 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix} = -\begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} = -\vec{x}$

So eigenvalue is -1.

Observe $A\vec{x} = \lambda\vec{x} \Rightarrow A\vec{x} = \lambda I\vec{x}$

$\Rightarrow A\vec{x} - \lambda I\vec{x} = \vec{0} \Rightarrow (A - \lambda I)\vec{x} = \vec{0}$

Ex - $A = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & -1 \\ 2 & 7 & -2 \end{bmatrix}$, Find the eigenvector associated with $\lambda = 3$

$A - 3I = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & -1 \\ 2 & 7 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 1 \\ 0 & 1 & -1 \\ 2 & 7 & -5 \end{bmatrix}$

$\left[\begin{array}{ccc|c} -1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 7 & -5 & 0 \end{array}\right] \xrightarrow{-R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 7 & -5 & 0 \end{array}\right]$

$$\xrightarrow{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \xrightarrow{-3R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_1\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \end{array}$$

Let $x_3 = s$ Then $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ s \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} s$

So eigenvector is $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

λ is an eigenvalue of A if and only if there are nonzero solutions to $(A-\lambda I)\vec{x}=\vec{0}$

From previous theorem, this means $\det(A-\lambda I)=0$

Thus, to find eigenvalues/eigenvectors:

1) Find λ such that $\det(A-\lambda I)=0$ These are e-values

2) For each λ, find $\vec{x}$ such that $(A-\lambda I)\vec{x}=\vec{0}$ these are e-vectors.

Ex - $A = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 6-\lambda & -3 \\ 2 & 1-\lambda \end{bmatrix}$$

So $\det(A - \lambda I) = (6-\lambda)(1-\lambda) - 2(-3)$

$$= 6 - 6\lambda - \lambda + \lambda^2 + 6$$

$$= \lambda^2 - 7\lambda + 12$$

$$= (\lambda - 3)(\lambda - 4)$$

Thus eigenvalues are $\lambda = 3$ and $\lambda = 4$

$$A - 3I = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix}$$

$\Rightarrow$ eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A - 4I = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix}$$

$\Rightarrow$ eigenvector is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Consider now vectors / matrices whose entries are single-variable functions.

e.g $\vec{x}(t) = \begin{bmatrix} t \\ \sin(t) \\ e^t \end{bmatrix}$, $A(t) = \begin{bmatrix} t^2 & \ln(t) \\ \cos(t) & 5 \end{bmatrix}$

A matrix $A(t)$ is continuous at t_0 if every entry $a_{ij}(t)$ is continuous at t_0.

e.g $A(t) = \begin{bmatrix} t^2 & \ln(t) \\ \cos(t) & 5 \end{bmatrix}$ is continuous for $t > 0$

$A(t) = \begin{bmatrix} \sqrt{t} & \frac{1}{t-3} \\ e^t & 6t \end{bmatrix}$ is continuous on $[0, 3) \cup (3, \infty)$

To differentiate a vector/matrix, differentiate each entry/component

Ex - 1) $\vec{x}(t) = \begin{bmatrix} t \\ \sin(t) \\ e^t \end{bmatrix}$, $\vec{x}'(t) =$, $\vec{x}'(0) =$

2) $A(t) = \begin{bmatrix} t^2+6 & \ln(t) \\ \cos(t-1) & 5 \end{bmatrix}$, $A'(t) = ?$ $A'(1) = ?$

1) $\vec{x}\,'(t) = \begin{bmatrix} 1 \\ \cos(t) \\ e^t \end{bmatrix} \longrightarrow \vec{x}\,'(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2) $A'(t) = \begin{bmatrix} 2t & \frac{1}{t} \\ -\sin(t-1) & 0 \end{bmatrix} \longrightarrow A'(1) = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

To integrate a vector/matrix, integrate each component/entry

Ex - 1) $\vec{x}(t) = \begin{bmatrix} 6t \\ 12e^{3t} \\ \frac{10}{2t+1} \end{bmatrix}$ $\quad \displaystyle\int_0^1 \vec{x}(t)\,dt$

2) $A(t) = \begin{bmatrix} 12t^2 - 7 & \frac{1}{t^2+1} \\ \sin(\pi t) & 6\sqrt{t} \end{bmatrix}$ $\quad \displaystyle\int_0^1 A(t)\,dt$

1) $$\int_0^1 \vec{x}(t)\,dt = \begin{bmatrix} \int_0^1 6t\,dt \\ \int_0^1 12e^{3t}\,dt \\ \int_0^1 \frac{10}{2t+1}\,dt \end{bmatrix} = \begin{bmatrix} 3t^2\big|_0^1 \\ 4e^{3t}\big|_0^1 \\ 5\ln(2t+1)\big|_0^1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4e^3 - 4 \\ 5\ln(3) \end{bmatrix}$$

2) $\int_0^1 A(t)\,dt = \begin{bmatrix} \int_0^1 12t^2-7\,dt & \int_0^1 \frac{1}{t^2+1}\,dt \\ \int_0^1 \sin(\pi t)\,dt & \int_0^1 6\sqrt{t}\,dt \end{bmatrix}$

$= \begin{bmatrix} 4t^3-7t \Big|_0^1 & \tan^{-1}(t)\Big|_0^1 \\ -\frac{1}{\pi}\cos(\pi t)\Big|_0^1 & 4t^{3/2}\Big|_0^1 \end{bmatrix}$

$= \begin{bmatrix} -3 & \pi/4 \\ \frac{2}{\pi} & 4 \end{bmatrix}$

Facts -

1) $\frac{d}{dt}[CA] = C\frac{dA}{dt}$ for constant matrices C

2) $\frac{d}{dt}[A \pm B] = \frac{dA}{dt} \pm \frac{dB}{dt}$

3) $\frac{d}{dt}[AB] = A\frac{dB}{dt} + \frac{dA}{dt}B$

Now consider a system of n linear <u>differential</u> equations:

$$x_1'(t) = a_{11}(t)x_1(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t)$$
$$x_2'(t) = a_{21}(t)x_1(t) + \cdots + a_{2n}(t)x_n(t) + f_2(t)$$
$$\vdots$$
$$x_n'(t) = a_{n1}(t)x_1(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t)$$

We can write this as

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t)x_1(t) + \cdots + a_{1n}(t)x_n(t) \\ a_{21}(t)x_1(t) + \cdots + a_{2n}(t)x_n(t) \\ \vdots \\ a_{n1}(t)x_1(t) + \cdots + a_{nn}(t)x_n(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

$$\vec{x}\,'(t) = A(t)\,\vec{x}(t) + \vec{f}(t)$$

This is the system in normal form

Ex- 1)
$$x_1' = t x_1 + \sin(t) x_2 + e^{7t}$$
$$x_2' = x_1 - \ln(t) x_2 + 3\cos(t)$$

2)
$$\frac{1}{t} x_1' = e^t x_1 - x_2 + \sqrt{t}\, x_3 + \tan^{-1}(t)$$
$$t x_2' = x_1 + \sin(t) x_2 - \cos(5t) x_3 + t$$
$$x_3' = (t^2 - 1) x_1 + 7\ln(t+4) x_2$$

1) $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} t & \sin(t) \\ 1 & -\ln(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^{7t} \\ 3\cos(t) \end{bmatrix}$

$\vec{x}' = A \quad \vec{x} + \vec{f}$

2) $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} te^t & -t & t^{3/2} \\ \frac{1}{t} & \frac{\sin(t)}{t} & -\frac{\cos(5t)}{t} \\ t^2-1 & 7\ln(t+4) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} t\tan^{-1}(t) \\ 1 \\ 0 \end{bmatrix}$

$\cancel{\vec{x}} = A \quad \vec{x} + \vec{f}$

We have the system of differential eqs. $\vec{x}'(t) = A(t)\vec{x}(t) + \vec{f}(t)$. If $\vec{f} = \vec{0}$, the system is homogeneous. Otherwise, nonhomogeneous.

We will start by describing a general solution to the homogeneous system $\vec{x}' = A\vec{x}$

A set of vectors $\vec{x}_1, \ldots, \vec{x}_n$ ~~is~~ <u>linearly dependent</u> if there exist constants $c_1, \ldots, c_n$
nonzero

such that $c_1\vec{X}_1 + \dots + c_n\vec{X}_n = \vec{0}$. In other words, one of the vectors can be written as a linear combination of the others.

If such nonzero constants do not exist, the vectors are <u>linearly independent</u>

The <u>Wronskian</u> of a set of n-dimensional vectors $\vec{X}_1, \dots, \vec{X}_n$ is the determinant of the matrix $\begin{bmatrix} \vec{X}_1 & \cdots & \vec{X}_n \end{bmatrix}$

Fact - a set of n-dim. vectors $\vec{X}_1, \dots, \vec{X}_n$ are linearly independent if and only if their Wronskian is nonzero on $(-\infty, \infty)$

Ex - 1) $\vec{X}_1(t) = \begin{bmatrix} 2\sin(t) \\ \cos(t)+\sin(t) \end{bmatrix}$ $\vec{X}_2 = \begin{bmatrix} \cos(t)+\sin(t) \\ \cos(t) \end{bmatrix}$

2) $\vec{X}_1(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$, $\vec{X}_2(t) = \begin{bmatrix} -e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}$, $\vec{X}_3 = \begin{bmatrix} -e^{4t} \\ e^{4t} \\ 0 \end{bmatrix}$

1) $W = \det\begin{bmatrix} 2\sin(t) & \cos(t)+\sin(t) \\ \cos(t)+\sin(t) & \cos(t) \end{bmatrix}$

$= 2\sin(t)\cos(t) - \left((\cos(t)+\sin(t))(\cos(t)+\sin(t))\right)$

$= 2\sin(t)\cos(t) - \left(\cos^2(t) + 2\sin(t)\cos(t) + \sin^2(t)\right)$

$= -1$ Thus, lin indep.

2) $W = \det\begin{bmatrix} e^{2t} & -e^{3t} & -e^{4t} \\ e^{2t} & 0 & e^{4t} \\ e^{2t} & e^{3t} & 0 \end{bmatrix}$

$= e^{2t}\det\begin{bmatrix} 0 & e^{4t} \\ e^{3t} & 0 \end{bmatrix} - (-e^{3t})\det\begin{bmatrix} e^{2t} & e^{4t} \\ e^{2t} & 0 \end{bmatrix} +$

$+ (-e^{4t})\det\begin{bmatrix} e^{2t} & 0 \\ e^{2t} & e^{3t} \end{bmatrix}$

$= e^{2t}(-e^{7t}) + e^{3t}(-e^{6t}) - e^{4t}(e^{5t})$

$= -3e^{9t} \neq 0$ for all t Thus, lin indep.

Theorem- Let $\vec{X}' = A\vec{X}$ be a homogeneous DE system. If $\vec{X}_1(t), \ldots, \vec{X}_n(t)$ are n linearly independent solution to this system, then the general solution is

$$\vec{X}(t) = c_1\vec{X}_1(t) + \cdots + c_n\vec{X}_n(t)$$

Oftentimes, we will write this as

$$\vec{X} = X(t)\vec{c} \quad \text{where} \quad X(t) = [\vec{X}_1(t) \ \cdots \ \vec{X}_n(t)]$$

$$\text{and} \quad \vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$X(t)$ is the <u>fundamental matrix</u> for the system

Ex - We showed above that $\vec{X}_1 = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$,

$\vec{X}_2 = \begin{bmatrix} -e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}$, $\vec{X}_3 = \begin{bmatrix} -e^{4t} \\ e^{4t} \\ 0 \end{bmatrix}$ are lin. indep. If they were all solutions to a system $\vec{X}' = A\vec{X}$, then the general solution would be

$$\vec{X}(t) = c_1\begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_2\begin{bmatrix} -e^{3t} \\ 0 \\ e^{3t} \end{bmatrix} + c_3\begin{bmatrix} -e^{4t} \\ e^{4t} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & -e^{3t} & -e^{4t} \\ e^{2t} & 0 & e^{4t} \\ e^{2t} & e^{3t} & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

To find a general solution to the nonhomogeneous system $\vec{x}\,' = A\vec{x} + \vec{f}$

1) Find n linearly independent solutions $\vec{x}_1, \dots, \vec{x}_n$ to

$$\vec{x}\,' = A\vec{x}$$

2) Find a particular solution $\vec{x}_p$ to

$$\vec{x}\,' = A\vec{x} + \vec{f}$$

3) General solution: $\vec{x}(t) = c_1\vec{x}_1(t) + \dots + c_n\vec{x}_n(t) + \vec{x}_p(t)$

$$= X(t)\vec{c} + \vec{x}_p(t)$$

Thus, we look for n linearly independent solutions to $\vec{x}\,' = A\vec{x}$. We will consider only the case when A is a constant matrix. In this case, $\vec{x}\,'$ is a "constant multiple" of $\vec{x}$.

So we look for solutions of the form

$\vec{x}(t) = e^{\lambda t}\vec{v}$ where $\vec{v}$ is a nonzero constant vector

Then $\vec{x}\,'(t) = \lambda e^{\lambda t}\vec{v}$ and $A\vec{x} = e^{\lambda t}A\vec{v}$

Thus $\vec{x}\,' = A\vec{x}$ implies $\lambda e^{\lambda t}\vec{v} = e^{\lambda t}A\vec{v}$

Cancel $e^{\lambda t}$ to get $\lambda\vec{v} = A\vec{v}$

That is, $\vec{v}$ is an eigenvector of A with eigenvalue λ.

Recap: If $\vec{v}$ is an eigenvector of A with corresponding eigenvalue λ, then $\vec{x}(t) = e^{\lambda t}\vec{v}$ is a solution to $\vec{x}\,'(t) = A\vec{x}(t)$

Theorem. If $\vec{v}_1, \ldots, \vec{v}_n$ are n linearly independent eigenvectors for the $n\times n$ matrix

A, with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$, then $e^{\lambda_1 t}\vec{v}_1, \dots, e^{\lambda_n t}\vec{v}_n$ are n linearly independent solutions to $\vec{x}' = A\vec{x}$. Hence, the general solution to the DE system is

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$$

Ex- 1) $x_1' = 6x_1 - 3x_2$

$x_2' = 2x_1 + x_2$

2) $x_1' = 8x_1 + x_2$

$x_2' = 10x_1 - x_2$

First write in normal form (identify A)

Find eigenvalues ie $\det(A - \lambda I) = 0$

Find eigenvectors ie $(A - \lambda I)\vec{v} = \vec{0}$

1) $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

We saw last time that A has eigenvectors

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ with corresponding e-values

3 and 4, respectively

So $\vec{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

2) $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 8 & 1 \\ 10 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{aligned} \det(A - \lambda I) = \det\begin{pmatrix} 8-\lambda & 1 \\ 10 & -1-\lambda \end{pmatrix} &= (8-\lambda)(-1-\lambda) - 10 \\ &= \lambda^2 - 7\lambda - 18 \\ &= (\lambda - 9)(\lambda + 2) \end{aligned}$$

So eigenvalues are $\lambda = 9$ and $\lambda = -2$

$$A - 9I = \begin{bmatrix} -1 & 1 \\ 10 & -10 \end{bmatrix} \Rightarrow \text{e-vector is } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A + 2I = \begin{bmatrix} 10 & 1 \\ 10 & 1 \end{bmatrix} \Rightarrow \text{e-vector is } \begin{bmatrix} -1 \\ 10 \end{bmatrix}$$

Thus, $\vec{X}(t) = c_1 e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -1 \\ 10 \end{bmatrix}$

$$\| $$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \quad x_1(t) = c_1 e^{9t} - c_2 e^{-2t}$$

$$x_2(t) = c_1 e^{9t} + 10 c_2 e^{-2t}$$