$$y' + \sin(t)\,y = g(t) = \begin{cases} \sin(t), & 0 \le t \le \pi \\ -\sin(t), & \pi \le t \le 2\pi \end{cases}$$

$$y(0) = 7$$

$$y' + \sin(t)\,y = \underline{\sin(t)} \qquad 0 \le t \le \pi$$

$$u(t) = e^{\int \sin(t)\,dt} = e^{-\cos(t)}$$

$$\int \sin(t) e^{-\cos(t)}\,dt = \int e^v\,dv \quad \text{where } v = -\cos(t)$$

$$= e^v + C$$

$$= e^{-\cos(t)} + C$$

Then $y = \frac{1}{e^{-\cos(t)}}\left(e^{-\cos(t)} + C\right)$

$$\Rightarrow y = 1 + Ce^{\cos(t)}$$

$$7 = y(0) = 1 + Ce \Rightarrow C = \frac{6}{e}$$

$$y = 1 + \frac{6}{e} e^{\cos(t)}, \quad 0 \le t \le \pi$$

Observe

$$y(\pi) = 1 + \frac{6}{e} e^{-1} = 1 + \frac{6}{e^2}$$

$$y' + \sin(t) y = -\sin(t), \quad \pi \le t \le 2\pi$$

with "initial" condition $y(\pi) = 1 + \frac{6}{e^2}$

$$u(t) = e^{\int \sin(t)\, dt} = e^{-\cos(t)}$$

$$\int -\sin(t) e^{-\cos(t)}\, dt = -\int e^v\, dv$$ where $v = -\cos(t)$.

$$= -e^v + C$$

$$= -e^{-\cos(t)} + C$$

Then $y = \frac{1}{e^{-\cos(t)}} \left(-e^{-\cos(t)} + C \right)$

$$\Rightarrow y = -1 + Ce^{\cos(t)}$$

$$1 + \frac{6}{e^2} = -1 + Ce^{-1}$$

$$2 + \frac{6}{e^2} = \frac{C}{e}$$

$$C = 2e + \frac{6}{e}$$

Hence $y = -1 + \left(2e + \frac{6}{e}\right)e^{\cos(t)}$

on $\pi \le t \le 2\pi$

A first-order DE is linear if $\frac{dy}{dx}$ and y appear only to the first power and are not multiplied together.

To solve a linear first-order DE, we put it in the form $\frac{dy}{dx} + P(x)y = Q(x)$

A 100 L tank of water initially holds 30 Kg of salt. Starting at time $t=0$, a salt water solution containing 0.1 Kg/L of salt is poured into the tank at a rate of 10 L/min. The tank is kept well-stirred, and the mixture flows out of the tank at the same rate.

1) Set up DE + IVP for this situation

2) Solve it

$$\frac{dA}{dt} = \begin{pmatrix}\text{inflow}\\\text{rate}\end{pmatrix}\begin{pmatrix}\text{inflow}\\\text{conc.}\end{pmatrix} - \begin{pmatrix}\text{outflow}\\\text{rate}\end{pmatrix}\begin{pmatrix}\text{outflow}\\\text{conc.}\end{pmatrix}$$

$$\frac{dA}{dt} = 10(.1) - 10\left(\frac{A}{100}\right)$$

$$\frac{dA}{dt} = 1 - \frac{1}{10}A \qquad A(0) = 30$$

$$\frac{dA}{dt} + \frac{1}{10}A = 1$$

$$u(t) = e^{\int \frac{1}{10}\,dt} = e^{t/10}$$

Then integrate $\int e^{t/10}\,dt = 10e^{t/10} + C$

Thus $\cancel{A} = \frac{1}{e^{t/10}}\left[10e^{t/10} + C\right]$

$\Rightarrow \cancel{A} = 10 + Ce^{-t/10}$

$30 = A(0) = 10 + C$ so $C = 20$

Thus $A(t) = 10 + 20e^{-t/10}$

Ex - Same set up as before except

inflow rate = 12 L/min

Outflow rate = 10 L/min

$$\frac{dA}{dt} = 12(.1) - 10\left(\frac{A}{100 + 2t}\right), \quad A(0) = 30$$

$$\frac{dA}{dt} = \frac{6}{5} - \frac{5}{50+t} A$$

$$\frac{dA}{dt} + \frac{5}{t+50} A = \frac{6}{5}$$

$$u(t) = e^{\int \frac{5}{t+50} dt} = e^{5\ln(t+50)} = e^{\ln(t+50)^5}$$

$$= (t+50)^5$$

Then integrate $\int \frac{6}{5}(t+50)^5 dt = \frac{1}{5}(t+50)^6 + C$

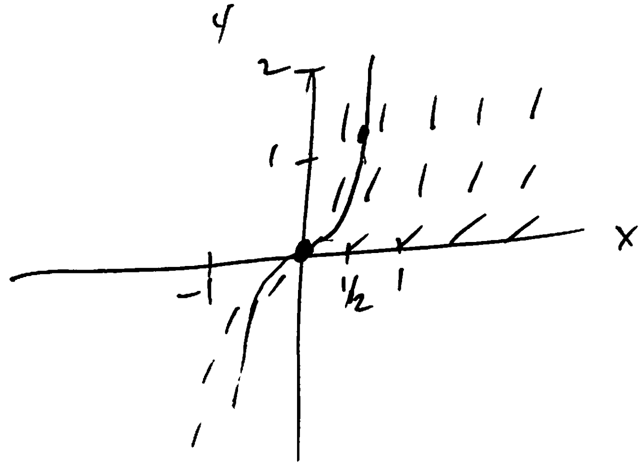
Thus $A = \frac{1}{(t+50)^5}\left(\frac{1}{5}(t+50)^6 + C\right)$

$$\Rightarrow A = \frac{1}{5}(t+50) + \frac{C}{(t+50)^5}$$

$$30 = A(0) = \frac{1}{5}(50) + \frac{C}{50^5}$$

$20 = \frac{C}{50^5}$ so $C = 20 \cdot 50^5$

Hence $A(t) = \frac{1}{5}(t+50) + \frac{20 \cdot 50^5}{(t+50)^5}$



1) $\frac{dy}{dx} = \frac{y^2}{x} = \frac{1}{x} \cdot y^2$

a) Separable or non-separable?
Linear of nonlinear?

b) Solve

$\frac{dy}{dx} = \frac{1}{x} \cdot y^2 \quad \Rightarrow \quad \frac{1}{y^2}\, dy = \frac{1}{x}\, dx$

$\Rightarrow \quad \int \frac{1}{y^2}\, dy = \int \frac{1}{x}\, dx$

$\Rightarrow \quad -\frac{1}{y} = \ln(x) + C$

$\Rightarrow \quad y = \frac{-1}{\ln(x) + C}$

2) $\frac{dy}{dx} = e^{x-y}$

a) Sep. or non sep

lin or non lin

$$\frac{dy}{dx} = e^x \cdot e^{-y}$$

$$\Rightarrow e^y\,dy = e^x\,dx$$

$$\Rightarrow \int e^y\,dy = \int e^x\,dx$$

$$\Rightarrow e^y = e^x + C$$

$$\Rightarrow y = \ln(e^x + C)$$

3) $\frac{dy}{dx} = 12e^{5x} - 7y$

Sep or non sep?

lin or non lin ?

$$\frac{dy}{dx} + 7y = 12e^{5x}$$

$u(x) = e^{\int 7\,dx} = e^{7x}$

Then integrate $\int 12e^{5x}e^{7x}\,dx$

$= \int 12e^{12x}\,dx$

$= e^{12x} + C$

Then $y = \frac{1}{e^{7x}}\left(e^{12x} + C\right)$

$\Rightarrow y = e^{5x} + \frac{C}{e^{7x}} = e^{5x} + Ce^{-7x}$

4) $2x\frac{dy}{dx} + 4y = 66x^9$

Sep or non sep?

Lin or non lin?

$\frac{dy}{dx} + \frac{2}{x}y = 33x^8$

$u(x) = e^{\int \frac{2}{x}dx} = e^{2\ln(x)} = e^{\ln(x^2)} = x^2$

Then integrate $\int 33x^8 x^2 \, dx$

$$= \int 33x^{10} \, dx$$

$$= 3x^{11} + C$$

Then $y = \frac{1}{x^2}(3x^{11} + C)$

$$y = 3x^9 + \frac{C}{x^2}$$

5) $3y'' + 5y' - 2y = 0$

Auxiliary equation: $3r^2 + 5r - 2 = 0$

$$r = \frac{-5 \pm \sqrt{25 - 4(3)(-2)}}{6}$$

$$r = \frac{-5 \pm \sqrt{49}}{6} = \frac{-5 \pm 7}{6}$$

$\Rightarrow$ $r = \frac{1}{3}$ and $r = -2$

$\Rightarrow$ $y = c_1 e^{\frac{1}{3}x} + c_2 e^{-2x}$

6) $y'' + 5y' + 9y = 0$

Auxiliary eq: $r^2 + 5r + 9 = 0$

$$\Rightarrow r = \frac{-5 \pm \sqrt{25 - 4(9)}}{2}$$

$$r = \frac{-5 \pm \sqrt{-11}}{2} = \frac{-5 \pm i\sqrt{11}}{2}$$

$$\Rightarrow r = -\frac{5}{2} \pm i\frac{\sqrt{11}}{2}$$

Then $y = e^{-\frac{5}{2}x}\left(c_1 \cos\left(\frac{\sqrt{11}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{11}}{2}x\right)\right)$