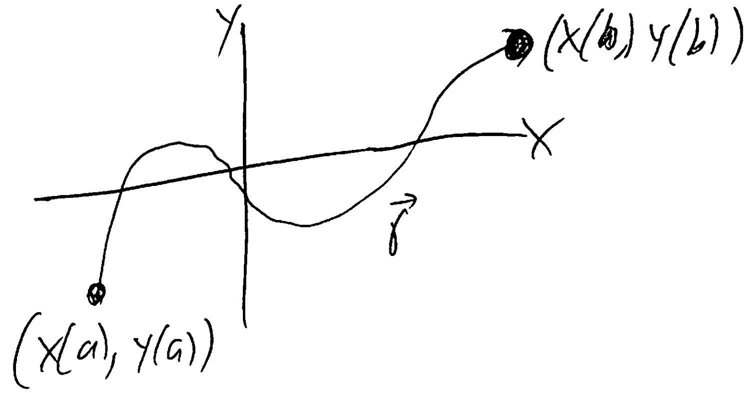
Recall that we can define a parametrized curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ using the vector-valued function

$\vec{r}(t) = \langle x(t), y(t) \rangle$ or $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

where $a \leq t \leq b$.



From here on out, we will say the curve C is given by $\vec{r}(t)$.

The curve C is <u>smooth</u> if $\vec{r}(t)$ has a continuous second derivative and $\vec{r}'(t) \neq \vec{0}$ for all t in (a, b)

The curve C is <u>simple</u> if it does not cross itself

Example - 1) $\vec{r}(t) = \langle t, t^2 \rangle$ with $0 \le t \le \sqrt{2}$

2) $\vec{r}(t) = \langle \cos(4t), \sin(4t), 1 \rangle$ with $0 \le t \le \frac{\pi}{8}$

Compute $\vec{r}\,'(t)$ and $\|\vec{r}\,'(t)\|$

1) $\vec{r}\,'(t) = \langle 1, 2t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2}$

2) $\vec{r}\,'(t) = \langle -4\sin(4t), 4\cos(4t), 0 \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{(-4\sin(4t))^2 + (4\cos(4t))^2 + 0^2}$

$= \sqrt{16\sin^2(4t) + 16\cos^2(4t)}$

$= \sqrt{16} = 4$

Now, let $f(x,y)$ be a two-variable function and $\vec{r}(t) = \langle x(t), y(t) \rangle$ define a curve C in $\mathbb{R}^2$.
The composition of f with $\vec{r}$ is
$f(\vec{r}(t)) = f(x(t), y(t))$

Likewise, the composition of $f(x,y,z)$ with $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is

$$f(\vec{r}(t)) = f(x(t), y(t), z(t))$$

Example - 1) $f(x,y) = \frac{12y}{x}$

$\vec{r}(t) = \langle t, t^2 \rangle$ with $0 \le t \le \sqrt{2}$

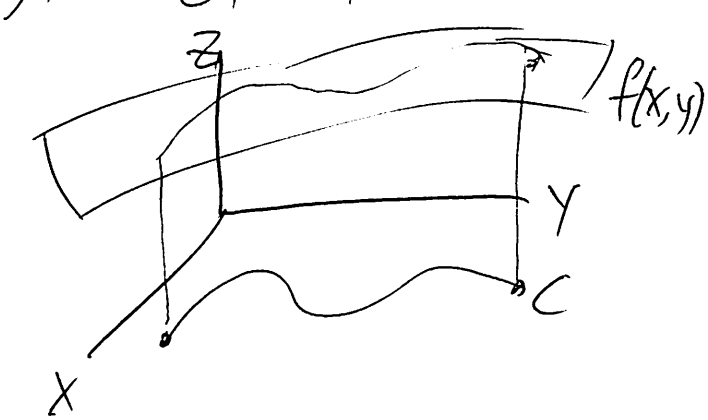
2) $f(x,y,z) = 2xyz$

$\vec{r}(t) = \langle \cos(4t), \sin(4t), 1 \rangle$, $0 \le t \le \frac{\pi}{8}$

1) $f(\vec{r}(t)) = f(t, t^2) = \frac{12t^2}{t} = 12t$

2) $f(\vec{r}(t)) = f(\cos(4t), \sin(4t), 1) = 2\cos(4t)\sin(4t)$

The composition $f(\vec{r}(t))$ defines the function f along the entire length of the curve C defined by $\vec{r}(t)$.



Integrating $f(\vec{r}(t))$ is thus equivalent to integrating f with respect to the arclength of C.

Recall the arclength function for a curve C defined by $\vec{r}(t)$ with $a \le t \le b$:

$$s(t) = \int_a^t \|\vec{r}\,'(p)\| \, dp$$

By the FToC, $\frac{ds}{dt} = \|\vec{r}\,'(t)\|$

$$\Longrightarrow \quad ds = \|\vec{r}\,'(t)\| \, dt$$

Let f be a multivariable function with domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let C be a simple smooth parametrized curve given by $\vec{r}(t)$, with $a \le t \le b$. The <u>line integral</u> of f along C with respect to arclength is $\int_C f \, ds = \int_a^b f(\vec{r}(t)) \, \|\vec{r}\,'(t)\| \, dt$

Ex - 1) $f(x,y) = \frac{12y}{x}$

$\vec{r}(t) = \langle t, t^2 \rangle$ with $0 \le t \le \sqrt{2}$

2) $f(x,y,z) = 2xyz$

$\vec{r}(t) = \langle \cos(4t), \sin(4t), 1 \rangle$ with $0 \le t \le \frac{\pi}{8}$

1) $f(\vec{r}(t)) = 12t$, $\|\vec{r}\,'(t)\| = \sqrt{1+4t^2}$

So $\int_C f\, ds = \int_0^{\sqrt{2}} 12t\sqrt{1+4t^2}\, dt$

$u = 1+4t^2$
$du = 8t\, dt$
$\frac{3}{2} du = 12t\, dt$

$= (1+4t^2)^{3/2} \Big|_0^{\sqrt{2}}$

$= 9^{3/2} - 1^{3/2} = 26$

$\int \frac{3}{2}\sqrt{u}\, du = \frac{\frac{3}{2}u^{3/2}}{3/2} = u^{3/2}$

2) $f(\vec{r}(t)) = 2\cos(4t)\sin(4t)$, $\|\vec{r}\,'(t)\| = 4$

So $\int_C f\, ds = \int_0^{\pi/8} 2\cos(4t)\sin(4t)\, 4\, dt$

$= \int_0^{\pi/8} 8\cos(4t)\sin(4t)\, dt$

$$= \sin^2(4t) \Big|_0^{\pi/8}$$

$$= 1 - 0 = 1$$

What do these integrals mean?

Suppose we have $f(x,y)$ and a curve C given by $\vec{r}(t) = \langle x(t), y(t) \rangle$

Then $\int_C f \, ds$ is the area bounded by C and the graph of f.

What about in three variables?

Suppose we have a wire or piece of string that we represent by a curve C in $\mathbb{R}^3$ given by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

If $f(x,y,z)$ is the mass density function of the wire/string, then the mass of the wire/string is $M = \int_C f\, ds$

The center of mass is $\left(\frac{\int_C x f(x,y,z)\, ds}{M}, \frac{\int_C y f(x,y,z)\, ds}{M}, \frac{\int_C z f(x,y,z)\, ds}{M} \right)$

Ex – $\vec{r}(t) = \langle \cos(4t), \sin(4t), 1 \rangle$ with $0 \le t \le \frac{\pi}{8}$

If the string has mass density function $f(x,y,z) = 2xyz$, then the mass of the string is $\int_C f\, ds = 1 = M$

Find the center of mass.

In terms of t, $x f(x,y,z) = x(t) f(\vec{r}(t))$

$$= 2\cos^2(4t)\sin(4t)$$

So $\int_C x f\, ds = \int_0^{\pi/8} 2\cos^2(4t)\sin(4t)\, 4\, dt$

$$= \int_0^{\pi/8} 8\cos^2(4t)\sin(4t)\, dt$$

$$= -\frac{2}{3}\cos^3(4t)\Big|_0^{\pi/8}$$

$$= 0 + \frac{2}{3} = \frac{2}{3}$$

And $yf(x,y,z) = y(t)f(\vec{r}(t)) = 2\cos(4t)\sin^2(4t)$

So $\int_C yf\,ds = \int_0^{\pi/8} 2\cos(4t)\sin^2(4t)\,4\,dt$

$$= \int_0^{\pi/8} 8\cos(4t)\sin^2(4t)\,dt$$

$$= \frac{2}{3}\sin^3(4t)\Big|_0^{\pi/8}$$

$$= \frac{2}{3} - 0 = \frac{2}{3}$$

And $zf(x,y,z) = z(t)f(\vec{r}(t)) = 2\cos(4t)\sin(4t)$

So $\int_C zf\,ds = \int_C f\,ds = 1$

Thus, center of mass is $\left(\frac{2}{3}, \frac{2}{3}, 1\right)$

A vector field on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point (x,y) in its domain $D_{\vec{F}} \subseteq \mathbb{R}^2$ a unique vector $\vec{F}(x,y)$ in its range $R_{\vec{F}} \subseteq \mathbb{R}^2$. We denote this as $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ where P and Q are real-valued two-variable functions.

Likewise for vector fields on $\mathbb{R}^3$ ie

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$$

Ex- 1) $\vec{F}(x,y) = \langle xy, 3\sin(x) + e^y \rangle$

Evaluate $\vec{F}(\pi, 0)$

2) $\vec{F}(x,y,z) = \langle x-z, z^2 + 4\cos(x-y), \ln(xyz) \rangle$

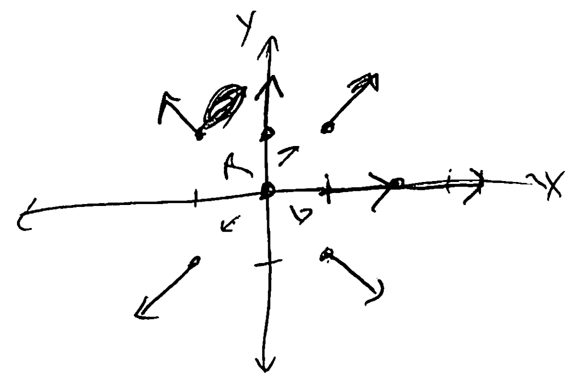
Evaluate $\vec{F}(1,1,1)$

1) $\vec{F}(\pi, 0) = \langle 0, 1 \rangle$

2) $\vec{F}(1,1,1) = \langle 0, 5, 0 \rangle$

We can represent vector fields graphically as such: At each point (x,y), we draw the vector $\vec{F}(x,y)$ with starting point (x,y). Likewise for $\mathbb{R}^3$.

Example - $\vec{F}(x,y) = \langle x, y \rangle$



The gradient of a function $f(x,y)$ or $f(x,y,z)$ is a special type of vector field.

A vector field $\vec{F}$ is <u>conservative</u> if it is the gradient of a function f; that is, there exists a real-valued function $f(x,y)$ or $f(x,y,z)$ such that

$\nabla f = \vec{F}$

Example - $\vec{F}(x,y) = \langle 2x, 2y \rangle$ is a conservative vector field because if we take $f(x,y) = x^2 + y^2$, then $\nabla f = \langle 2x, 2y \rangle = \vec{F}(x,y)$

The function f such that $\nabla f = \vec{F}$ is called the potential function for $\vec{F}$

In general, how do we check if a vector field is conservative?

Suppose we have a vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$

If $\vec{F}$ is conservative, then there exists a function $f(x,y)$ such that $\nabla f = \vec{F}$ ie $\langle f_x, f_y \rangle = \langle P(x,y), Q(x,y) \rangle$

ie $f_x = P(x,y)$ and $f_y = Q(x,y)$

Then $P_y = f_{xy}$ and $Q_x = f_{yx}$

Since $f_{xy} = f_{yx}$, we conclude that $P_y = Q_x$

Fact - A vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is conservative if and only if $P_y = Q_x$

A similar argument produces the following result for $\mathbb{R}^3$.

Fact - A vector field $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ is conservative if and only if

$$P_y = Q_x, \quad P_z = R_x, \quad Q_z = R_y$$

$$f'(x) = 3x^2 \rightarrow f(x) = x^3 + C$$

$$f_x(x,y) = 3x^2 \rightarrow f(x,y) = x^3 + g(y)$$

where g is some arbitary function of y

$$f'(y) = \cos(y) \rightarrow f(y) = \sin(y) + C$$

$$f_y(x,y) = \cos(y) \rightarrow f(x,y) = \sin(y) + h(x)$$

where h is some arbitrary function of x.

Example - 1) $\int 2x + 4y - 6x^2e^y \, dx$

2) $\int 2x + 4y - 6x^2e^y \, dy$

1) $x^2 + 4xy - 2x^3e^y + h(y)$

2) $2xy + 2y^2 - 6x^2e^y + g(x)$

Example - Check if the following are conservative vector fields

1) $\vec{F}(x,y) = \langle ye^x + y^2,\ 2xy + e^x - 5 \rangle$

2) $\vec{F}(x,y) = \langle x^2y - 7,\ \sin(y) + x^3 \rangle$

3) $\vec{F}(x,y,z) = \langle \sin(z) + y,\ z^2 + x,\ x\cos(z) + 2yz - 3 \rangle$

4) $\vec{F}(x,y,z) = \langle xy + yz,\ xz + zy,\ yx + xz \rangle$

1) $P = ye^x + y^2 \Rightarrow P_y = e^x + 2y$ $\quad =$ Yes

$Q = 2xy + e^x - 5 \Rightarrow Q_x = 2y + e^x$

2) $P = x^2y - 7 \Rightarrow P_y = x^2$ $\quad \neq$ No

$Q = \sin(y) + x^3 \Rightarrow Q_x = 3x^2$

3) $P = \sin(z) + y \Rightarrow P_y = 1 \qquad P_z = \cos(z)$

$Q = z^2 + x \Rightarrow Q_x = 1 \qquad Q_z = 2z$

$R = x\cos(z) + 2yz - 3 \Rightarrow R_x = \cos(z) \qquad R_y = 2z$

Yes

4) $P = xy + yz \Rightarrow P_y = x + z$ $\quad \neq$ No

$Q = xz + zy \Rightarrow Q_x = z$

Once we determine a vector field is conservative, how do we find its potential function.

Consider the conservative vector field

$\vec{F}(x,y) = \langle ye^x + y^2, 2xy + e^x - 5 \rangle$

There exists $f(x,y)$ such that $\nabla f = \vec{F}$ ie

$\langle f_x, f_y \rangle = \vec{F}$ ie $f_x = ye^x + y^2$
$f_y = 2xy + e^x - 5$

$f_x = ye^x + y^2 \Rightarrow f(x,y) = \int ye^x + y^2 \, dx$

$= ye^x + xy^2 + h(y)$

We need to find $h(y)$

$f(x,y) = ye^x + xy^2 + h(y) \Rightarrow f_y = e^x + 2xy + h'(y)$

This must equal $2xy + e^x - 5$ so $h'(y) = -5$

Thus $h(y) = -5y + K$ (K is a constant)

Hence, $f(x,y) = ye^x + xy^2 - 5y + K$

Example - $\vec{F}(x,y) = \langle x + y, x - y \rangle$

$P = x+y \Rightarrow P_y = 1$
$Q = x-y \Rightarrow Q_x = 1$ $=$ So conservative

Hence, there exists $f(x,y)$ such that $f_x = x+y$ and $f_y = x-y$

$f_x = x+y \Rightarrow f(x,y) = \int x+y \, dx = \frac{x^2}{2} + xy + h(y)$

$\Rightarrow f_y = x + h'(y)$

This must equal $x-y$ so $h'(y) = -y$

So $h(y) = \frac{-y^2}{2} + K$

So $f(x,y) = \frac{x^2}{2} + xy - \frac{y^2}{2} + K$

Now consider the conservative vector field
$\vec{F}(x,y,z) = \langle \sin(z)+y,\ z^2+x,\ x\cos(z) + 2yz - 3 \rangle$
There exists $f(x,y,z)$ such that $\nabla f = \vec{F}$ ie
$\langle f_x, f_y, f_z \rangle = \vec{F}$ ie $f_x = \sin(z)+y$, $f_y = z^2+x$
and $f_z = x\cos(z) + 2yz - 3$

$f_x = \sin(z) + y \Rightarrow f(x,y,z) = \int \sin(z) + y \, dx$
$= x\sin(z) + xy + h(y,z)$

We need to find $h(y,z)$

$f(x,y,z) = x\sin(z) + xy + h(y,z)$

$\Rightarrow f_y = x + h_y(y,z)$

This must equal $z^2 + x$ so $h_y(y,z) = z^2$

$h_y = z^2 \Rightarrow h(y,z) = \int z^2\,dy = yz^2 + g(z)$

$\Rightarrow f(x,y,z) = x\sin(z) + xy + yz^2 + g(z)$

We need to find $g(z)$

well, $f_z = x\cos(z) + 2yz + g'(z)$

This must equal $x\cos(z) + 2yz - 3$ so $g'(z) = -3$

Thus, $g(z) = -3z + K$ (K a constant)

So $f(x,y,z) = x\sin(z) + xy + yz^2 - 3z + K$

Example- $\vec{F}(x,y,z) = \langle y+z, x+z, x+y \rangle$

$P = y+z \Rightarrow P_y = 1$ and $P_z = 1$

$Q = x+z \Rightarrow Q_x = 1$ and $Q_z = 1$

$R = x+y \Rightarrow R_x = 1$ and $R_y = 1$

Hence conservative

Hence, there exists $f(x, y, z)$ such that

$f_x = y+z$, $f_y = x+z$, and $f_z = x+y$

$f_x = y+z \Rightarrow f(x,y,z) = \int y+z \, dx = xy + xz + h(y,z)$

$\Rightarrow f_y = x + h_y(y,z)$

This must equal $x+z$ so $h_y(y,z) = z$

$\Rightarrow h(y,z) = \int z \, dy = yz + g(z)$

$\Rightarrow f(x,y,z) = xy + xz + yz + g(z)$

$\Rightarrow f_z = x + y + g'(z)$

This must equal $x+y$ so $g'(z) = 0$ ie $g(z) = K$

Thus, $f(x, y, z) = xy + xz + yz + K$

Let $\vec{F}(x, y)$ be a vector field in $\mathbb{R}^2$ and let C be a parametrized curve in $\mathbb{R}^2$ given by $\vec{r}(t) = \langle x(t), y(t) \rangle$ with $a \le t \le b$. The composition of $\vec{F}$ and $\vec{r}$ is $\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t))$

Likewise the composition of $\vec{F}(x, y, z)$ with $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is $\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t), z(t))$

Example - 1) $\vec{F}(x,y) = \langle x+y, x-y \rangle$

$\vec{r}(t) = \langle t, t^2 \rangle$ with $0 \le t \le 2$

2) $\vec{F}(x,y,z) = \langle y+z, x+z, x+y \rangle$

$\vec{r}(t) = \langle 1, t, t^2 \rangle$ with $1 \le t \le 4$

1) $\vec{F}(\vec{r}(t)) = \vec{F}(t, t^2) = \langle t+t^2, t-t^2 \rangle$

2) $\vec{F}(\vec{r}(t)) = \vec{F}(1, t, t^2) = \langle t+t^2, 1+t^2, 1+t \rangle$

Recall that the line integral for real-valued f along C given by $\vec{r}(t)$ involved Integrating the real-valued function $f(\vec{r}(t))\|\vec{r}\,'(t)\|$ along C.

For the line integral of a vector field $\vec{F}$ along C given by $\vec{r}(t)$, we'd also like to Integrate a real-valued function involving $\vec{F}(\vec{r}(t))$ and $\vec{r}\,'(t)$

Let $\vec{F}$ be a vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let C be a simple smooth parametrized curve given by $\vec{r}(t)$ with $a \le t \le b$. The <u>line Integral</u> of $\vec{F}$ along C with respect to arclength is

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

Example - 1) $\vec{F}(x,y) = \langle x+y,\ x-y \rangle$
$\vec{r}(t) = \langle t, t^2 \rangle$ with $0 \le t \le 2$

2) $\vec{F}(x,y,z) = \langle y+z,\ x+z,\ x+y \rangle$
$\vec{r}(t) = \langle 1, t, t^2 \rangle$ with $1 \le t \le 4$

1)
$\vec{F}(\vec{r}(t)) = \langle t+t^2,\ t-t^2 \rangle$ $\qquad$ $\vec{r}\,'(t) = \langle 1, 2t \rangle$

So $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = t+t^2 + 2t(t-t^2) = t + 3t^2 - 2t^3$

Thus, $\int_C \vec{F} \cdot d\vec{r} = \int_0^2 t + 3t^2 - 2t^3 \, dt$

$= \frac{t^2}{2} + t^3 - \frac{t^4}{2} \Big|_0^2$

$= 2 + 8 - 8 = 2$

2) $\vec{F}(\vec{r}(t)) = \langle t+t^2,\ 1+t^2,\ 1+t \rangle$
$\vec{r}\,'(t) = \langle 0, 1, 2t \rangle$

So $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 1+t^2 + 2t(1+t) = 1 + 2t + 3t^2$

Thus, $\int_C \vec{F} \cdot d\vec{r} = \int_1^4 1 + 2t + 3t^2 \, dt$

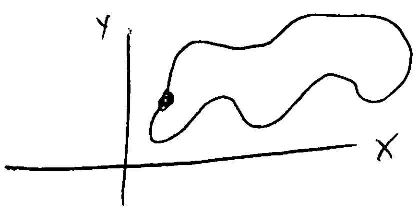
$= t + t^2 + t^3 \Big|_1^4$

$= (4+16+64) - (1+1+1) = 84 - 3 = 81$

What does this mean?

Let $\vec{F}$ be a force vector, and consider an object moving along a curve C. Then $\int_C \vec{F} \cdot d\vec{r}$ is the total work done by the force $\vec{F}$ acting on the object as it moves along C.

Let C be a parametrized curve given by $\vec{r}(t)$ with $a \le t \le b$. C is a <u>simple closed</u> curve if it does not intersect itself and $\vec{r}(a) = \vec{r}(b)$.



Example - circles in $\mathbb{R}^2$: $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$
$0 \le t \le 2\pi$

circles in $\mathbb{R}^3$: $\vec{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$
$0 \le t \le 2\pi$

The line integral of a vector field $\vec{F}$ along a parametrized simple closed curve C is denoted $\oint_C \vec{F} \cdot d\vec{r}$ and is called the circulation of the vector field around C.

Example - 1) $\vec{F}(x,y) = \langle x+y,\ x-y \rangle$

$\vec{r}(t) = \langle \cos(t),\ \sin(t) \rangle,\quad 0 \le t \le 2\pi$

2) $\vec{F}(x,y,z) = \langle y+z,\ x+z,\ x+y \rangle$

$\vec{r}(t) = \langle \cos(t),\ \sin(t),\ 2 \rangle,\quad 0 \le t \le 2\pi$

1) $\vec{F}(\vec{r}(t)) = \vec{F}(\cos(t), \sin(t)) = \langle \cos(t)+\sin(t),\ \cos(t)-\sin(t) \rangle$

$\vec{r}\,'(t) = \langle -\sin(t),\ \cos(t) \rangle$

So $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = -\sin(t)(\cos(t)+\sin(t)) + \cos(t)(\cos(t)-\sin(t))$

$= \cos^2(t) - \sin^2(t) - 2\sin(t)\cos(t)$

$= \cos(2t) - \sin(2t)$

So $\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \cos(2t) - \sin(2t)\, dt$

$= \frac{1}{2}\sin(2t) + \frac{1}{2}\cos(2t) \Big|_0^{2\pi}$

$= (0 + \frac{1}{2}) - (0 + \frac{1}{2}) = 0$

2) $\vec{F}(\vec{r}(t)) = \vec{F}(\cos(t),\ \sin(t),\ 2)$

$= \langle \sin(t)+2,\ \cos(t)+2,\ \cos(t)+\sin(t) \rangle$

$\vec{r}\,'(t) = \langle -\sin(t),\ \cos(t),\ 0 \rangle$

So $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = -\sin(t)(\sin(t)+2) + \cos(t)(\cos(t)+2)$

$$= \cos^2(t) - \sin^2(t) + 2\cos(t) - 2\sin(t)$$

$$= \cos(2t) + 2\cos(t) - 2\sin(t)$$

So $\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \cos(2t) + 2\cos(t) - 2\sin(t)\, dt$

$$= \frac{1}{2}\sin(2t) + 2\sin(t) + 2\cos(t) \Big|_0^{2\pi}$$

$$= (0+0+2) - (0+0+2) = 0$$

3) $\vec{F}(x,y) = \langle x-y,\ x+y \rangle$

$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle, \quad 0 \le t \le 2\pi$

$\vec{F}(\vec{r}(t)) = \vec{F}(\cos(t), \sin(t)) = \langle \cos(t) - \sin(t),\ \cos(t) + \sin(t) \rangle$

$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$

So $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = -\sin(t)(\cos(t) - \sin(t)) + \cos(t)(\cos(t) + \sin(t))$

$$= \sin^2(t) + \cos^2(t)$$

$$= 1$$

Thus $\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 1\, dt = t \Big|_0^{2\pi} = 2\pi$

Fundamental Theorem of Line Integrals

- Let f be a continuously differentiable function with domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, and let C be a simple smooth parametrized curve given by $\vec{r}(t)$, with $a \le t \le b$. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Corollary 1: If $\vec{F}$ is a conservative vector field with potential function f (ie $\nabla f = \vec{F}$), then $\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

Corollary 2: If $\vec{F}$ is a conservative vector field and C is a simple <u>closed</u> curve, then $\oint_C \vec{F} \cdot d\vec{r} = 0$.

Example -

1) $\vec{F}(x,y) = \langle x+y, x-y \rangle$

$\vec{r}(t) = \langle t, t^2 \rangle$, $0 \le t \le 2$

2) $F(x,y,z) = \langle y+z, x+z, x+y \rangle$

$\vec{r}(t) = \langle 1, t, t^2 \rangle, \quad 1 \le t \le 4$

1) Last time, we computed $\int_C \vec{F} \cdot d\vec{r} = 2$

We found the general potential function for $\vec{F}$ to be $f(x,y) = \frac{x^2}{2} + xy - \frac{y^2}{2} + K$

We can choose $K = 0$

Thus, according to the Fund. Thm,

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(2)) - f(\vec{r}(0))$$
$$= f(2,4) - f(0,0)$$
$$= 2 + 8 - 8 - 0 = 2$$

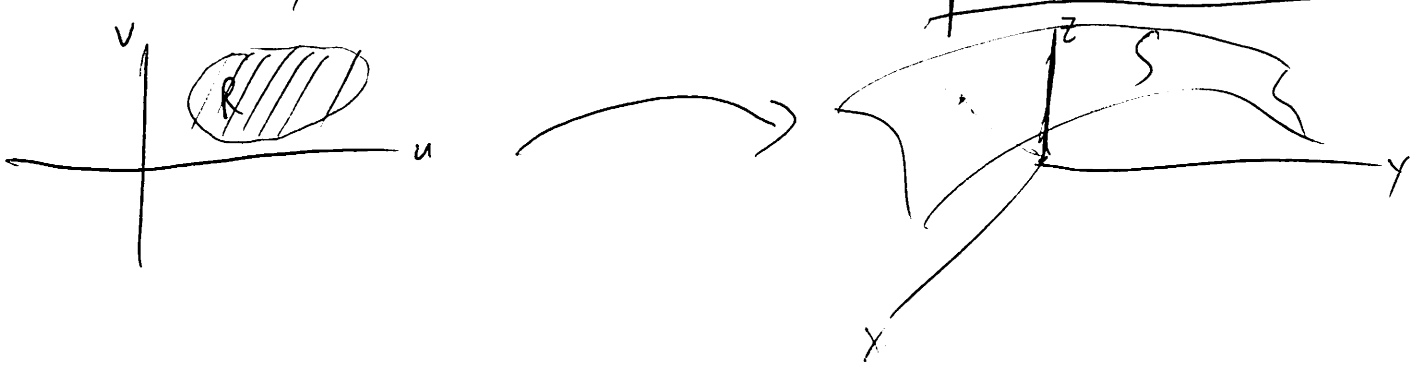
2) Last time, we showed $\int_C \vec{F} \cdot d\vec{r} = 81$

We found the potential function for $\vec{F}$ to be $f(x,y,z) = xy + xz + yz + K$

Thus, according to the Fund. Thm,

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(4)) - f(\vec{r}(1))$$
$$= f(1,4,16) - f(1,1,1)$$
$$= (4+16+64) - (1+1+1) = 81$$

Let R be a region in $\mathbb{R}^2$. Then we can define a multivariable vector-valued function $\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$ with $(u,v) \in R$. The set of points $S \subseteq \mathbb{R}^3$ swept out by $\vec{r}$ is called a parametric surface.



Ex- $\vec{r}(u,v) = \langle uv,\ e^u + \ln(v+1),\ u-v \rangle$
$$0 \le u \le 2, \quad 0 \le v \le 2$$

One common parametric surface is (part of) the graph of a function $f(x,y)$

In this case, $x(u,v) = u$, $y(u,v) = v$, and $z(u,v) = f(u,v)$ so $\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$

Example - 1) $f(x,y) = 9 - x^2 - y^2$ entire graph

2) $f(x,y) = 9 - x^2 - y^2$ above xy plane

3) The plane $3x + 4y + z = 1$ entire plane

4) The plane $3x + 4y + z = 1$ in first octant

1) $\vec{r}(u,v) = \langle u, v, 9 - u^2 - v^2 \rangle \quad (u,v) \in \mathbb{R}^2$

2) Above xy plane $\rightarrow z \geq 0 \rightarrow 9 - u^2 - v^2 \geq 0$

$\rightarrow u^2 + v^2 \leq 9$

So $\vec{r}(u,v) = \langle u, v, 9 - u^2 - v^2 \rangle \quad (u,v) \in \{(u,v) \mid u^2 + v^2 \leq 9\}$

3) $\vec{r}(u,v) = \langle u, v, 1 - 3u - 4v \rangle \quad (u,v) \in \mathbb{R}^2$

4) First octant $\rightarrow x \geq 0$ (ie $u \geq 0$), $y \geq 0$ (ie $v \geq 0$) and $z \geq 0$ (ie $1 - 3u - 4v \geq 0$ or $1 \geq 3u + 4v$)

So $\vec{r}(u,v) = \langle u, v, 1-3u-4v \rangle$ with

$(u,v) \in \{(u,v) \mid u \geq 0, v \geq 0, 3u+4v \leq 1\}$

Example- sphere of radius 3:

$\vec{r}(u,v) = \langle 3\cos(u)\sin(v), 3\sin(u)\sin(v), 3\cos(v) \rangle$

with $0 \leq u \leq 2\pi$, $0 \leq v \leq \pi$

cylinder of radius 2:

$\vec{r}(u,v) = \langle 2\cos(u), 2\sin(u), v \rangle$ with

$0 \leq u \leq 2\pi$, $a \leq v \leq b$

Let $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ define a parametric surface S, with $(u,v) \in R \subseteq \mathbb{R}^2$.

The <u>standard tangent vectors</u> of the surface are $\vec{r}_u = \langle x_u, y_u, z_u \rangle$ and $\vec{r}_v = \langle x_v, y_v, z_v \rangle$.

Example- $\vec{r}(u,v) = \langle u^2+v^2, u^2-v^2, uv \rangle$ with

$0 \leq u \leq 2$, $1 \leq v \leq 3$.

$\vec{r}_u = \langle 2u, 2u, v \rangle$

$\vec{r}_v = \langle 2v, -2v, u \rangle$

Let S be a parametrized surface given by $\vec{r}(u,v)$ with $(u,v) \in R \subseteq \mathbb{R}^2$. The surface S is <u>smooth</u> if $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ for all $(u,v) \in R$.

Example - $\vec{r}$ above

$\vec{r}_u \times \vec{r}_v = \langle 2u^2 + 2v^2, 2v^2 - 2u^2, -8uv \rangle$

Let S be a smooth parametrized surface given by $\vec{r}(u,v)$ with $(u,v) \in R \subseteq \mathbb{R}^2$. The <u>surface area</u> of S is $\iint_R \|\vec{r}_u \times \vec{r}_v\| \, du\, dv$

Ex - $\vec{r}(u,v) = \langle 3\cos(u)\sin(v), 3\sin(u)\sin(v), 3\cos(v) \rangle$

$0 \leq u \leq 2\pi, \quad 0 \leq v \leq \pi$

$\vec{r}_u = \langle -3\sin(u)\sin(v), 3\cos(u)\sin(v), 0 \rangle$

$\vec{r}_v = \langle 3\cos(u)\cos(v), 3\sin(u)\cos(v), -3\sin(v) \rangle$

$\vec{r}_u \times \vec{r}_v = \langle -9\cos(u)\sin^2(v), -9\sin(u)\sin^2(v), -9\sin^2(u)\sin(v)\cos(v) - 9\cos^2(u)\sin(v)\cos(v) \rangle$

$= \langle -9\cos(u)\sin^2(v), -9\sin(u)\sin^2(v), -9\sin(v)\cos(v) \rangle$

$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{81\cos^2(u)\sin^4(v) + 81\sin^2(u)\sin^4(v) + 81\sin^2(v)\cos^2(v)}$

$= \sqrt{81\sin^4(v) + 81\sin^2(v)\cos^2(v)}$

$= \sqrt{81\sin^2(v)} = 9\sin(v)$

Surface area $= \int_0^{2\pi}\int_0^{\pi} 9\sin(v)\,dv\,du$

$\int_0^{\pi} 9\sin(v)\,dv = -9\cos(v)\Big|_0^{\pi} = 9-(-9) = 18$

$\int_0^{2\pi} 18\,du = 18u\Big|_0^{2\pi} = 36\pi$

Ex - $\vec{r}(u,v) = \langle 2\cos(u), 2\sin(u), v \rangle$

$0 \le u \le 2\pi, \quad 0 \le v \le 10$

Cylinder SA is $2\pi rh + 2\pi r^2 \rightarrow 48\pi$

$\vec{r}_u = \langle -2\sin(u), 2\cos(u), 0 \rangle$

$\vec{r}_v = \langle 0, 0, 1 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 2\cos(u),\ 2\sin(u),\ 0 \rangle$

$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{4\cos^2(u) + 4\sin^2(u)} = \sqrt{4} = 2$

$\int_0^{2\pi} \int_0^{10} 2 \, dv\, du = 2(10)(2\pi) = 40\pi$

Let $f(x,y,z)$ be a three-variable function, and let $\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$ define a surface S. The composition of f and $\vec{r}$ is

$f(\vec{r}(u,v)) = f(x(u,v),\ y(u,v),\ z(u,v))$

Example - 1) $\vec{r}(u,v) = \langle 3\cos(u)\sin(v),\ 3\sin(u)\sin(v),\ 3\cos(v) \rangle$

$f(x,y,z) = x^2 + y^2 + z^2$

2) $\vec{r}(u,v) = \langle 2\cos(u),\ 2\sin(u),\ v \rangle$

$f(x,y,z) = x^2 + y^2 + z$

3) $\vec{r}(u,v) = \langle u^2+v^2,\ u^2-v^2,\ uv \rangle$

$f(x,y,z) = x + y + z$

1) $f(\vec{r}(u,v)) = f(3\cos(u)\sin(v),\ 3\sin(u)\sin(v),\ 3\cos(v))$

$$= 9\cos^2(u)\sin^2(v) + 9\sin^2(u)\sin^2(v) + 9\cos^2(v)$$

$$= 9\sin^2(v) + 9\cos^2(v)$$

$$= 9$$

2) $f(\vec{r}(u,v)) = f(2\cos(u), 2\sin(u), v)$

$$= 4\cos^2(u) + 4\sin^2(u) + v$$

$$= 4 + v$$

3) $f(\vec{r}(u,v)) = f(u^2+v^2, u^2-v^2, uv)$

$$= u^2+v^2+u^2-v^2+uv$$

$$= 2u^2 + uv$$

Let S be a smooth parametrized ~~function~~ surface given by $\vec{r}(u,v)$ with $(u,v) \in R \subseteq \mathbb{R}^2$, and let $f(x,y,z)$ be a three-variable function. The <u>surface integral</u> of f over S with respect to surface area is $\iint_S f\,dS = \iint_R f(\vec{r}(u,v))\,\|\vec{r}_u \times \vec{r}_v\|\,du\,dv$

Example. Do the sphere and cylinder from above

1) $f(\vec{r}(u,v)) = 9$, $\|\vec{r}_u \times \vec{r}_v\| = 9\sin(v)$

So $\int_S f \, dS = \int_0^{2\pi}\int_0^{\pi} 81\sin(v) \, dv\, du$

$$\int_0^{\pi} 81\sin(v)\, dv = -81\cos(v)\Big|_0^{\pi} = 81-(-81) = 162$$

$$\int_0^{2\pi} 162 \, du = 162(2\pi) = 324\pi$$

2) $f(\vec{r}(u,v)) = 4+v$, $\|\vec{r}_u \times \vec{r}_v\| = 2$

So $\int_S f \, dS = \int_0^{2\pi}\int_0^{10} 8+2v \, dv\, du$

$$\int_0^{10} 8+2v \, dv = 8v + v^2\Big|_0^{10} = 180$$

$$\int_0^{2\pi} 180 \, du = 180(2\pi) = 360\pi$$

Suppose we have a lamina R in $\mathbb{R}^3$ that we represent with a surface S given by $\vec{r}(u,v)$. If $f(x,y,z)$ is the mass density function for the lamina, then the mass of the lamina is

$$\int_S f\,dS = \iint_R f(\vec{r}(u,v))\,\|\vec{r}_u \times \vec{r}_v\|\,du\,dv = M.$$

The center of mass is $\left(\dfrac{\int_S xf\,dS}{M}, \dfrac{\int_S yf\,dS}{M}, \dfrac{\int_S zf\,dS}{M}\right)$

Recall that the cross product $\vec{r}_u \times \vec{r}_v$ yields a vector that is normal (ie orthogonal) to both $\vec{r}_u$ and $\vec{r}_v$

For a parametrized surface S given by $\vec{r}(u,v)$, the <u>unit normals</u> of $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ and $\vec{n} = -\dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$

The choice of unit normal for a surface is called an <u>orientation</u> for the surface.

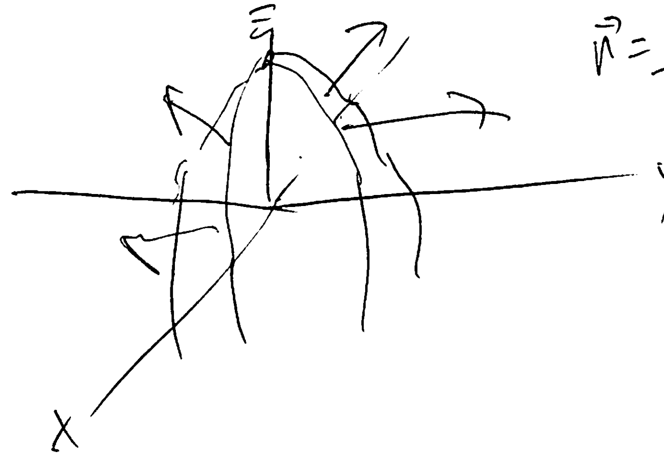
Example. $f(x,y) = 9 - x^2 - y^2$

Parametrically, this is $\vec{r}(u,v) = \langle u, v, 9-u^2-v^2 \rangle$

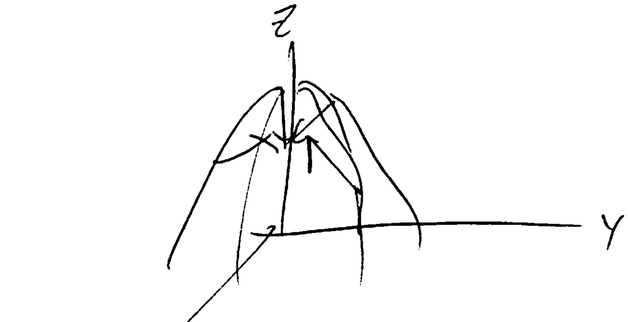
$\vec{r}_u = \langle 1, 0, -2u \rangle$ $\quad\quad$ $\vec{r}_v = \langle 0, 1, -2v \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 2u, 2v, 1 \rangle$, $\quad \|\vec{r}_u \times \vec{r}_v\| = \sqrt{4u^2+4v^2+1}$

$$\vec{n} = \frac{\langle 2u, 2v, 1 \rangle}{\sqrt{4u^2+4v^2+1}} \quad \text{or} \quad \vec{n} = \frac{\langle -2u, -2v, -1 \rangle}{\sqrt{4u^2+4v^2+1}}$$



$\vec{n} = \dfrac{\langle 2u, 2v, 1\rangle}{\sqrt{\sim}}$



$\vec{n} = \dfrac{\langle -2u, -2v, -1\rangle}{\sqrt{\sim}}$

Let $\vec{F}(x,y,z)$ be a vector field in $\mathbb{R}^3$, and let $\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ define a surface S. The composition of $\vec{F}$ and $\vec{r}$ is

$$\vec{F}(\vec{r}(u,v)) = \vec{F}(x(u,v),\ y(u,v),\ z(u,v))$$

Ex - $\vec{F}(x,y,z) = \langle x^2 + z,\ y^2 + z,\ xy^2 + x^2y\rangle$

surface defined by the graph of $f(x,y) = 9 - x^2 - y^2$

$$\vec{r}(u,v) = \langle u,\ v,\ 9 - u^2 - v^2\rangle$$

$$\vec{F}(\vec{r}(u,v)) = \vec{F}(u,\ v,\ 9-u^2-v^2)$$

$$= \langle u^2 + 9 - u^2 - v^2,\ v^2 + 9 - u^2 - v^2,\ uv^2 + u^2v\rangle$$

$$= \langle 9 - v^2,\ 9 - u^2,\ uv^2 + u^2v\rangle$$

Let S be a smooth parametrized surface given by $\vec{r}(u,v)$, with $(u,v) \in R \subseteq \mathbb{R}^2$. Let $\vec{F}$ be a vector field in $\mathbb{R}^3$. If the orientation of S is determined by the unit normal $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$, then the <u>surface integral</u> of $\vec{F}$ over S is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_R \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

Example. Compute $\iint_S \vec{F} \cdot d\vec{S}$ where

$\vec{F}(x,y,z) = \langle x^2 + z,\ y^2 + z,\ xy^2 + yx^2 \rangle$ and

S is the portion of the graph of $f(x,y) = 9 - x^2 - y^2$ with $0 \le x \le 1$ and $0 \le y \le 1$

$\vec{r}(u,v) = \langle u, v, 9 - u^2 - v^2 \rangle, \quad 0 \le u \le 1, \quad 0 \le v \le 1$

$\vec{F}(\vec{r}(u,v)) = \langle 9 - v^2,\ 9 - u^2,\ uv^2 + u^2 v \rangle$

$\vec{r}_u = \langle 1, 0, -2u \rangle \qquad \vec{r}_v = \langle 0, 1, -2v \rangle$

$$\vec{r}_u \times \vec{r}_v = \langle 2u, 2v, 1 \rangle$$

$$\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = \langle 9-v^2, 9-u^2, uv^2+u^2v \rangle \cdot \langle 2u, 2v, 1 \rangle$$

$$= 18u - 2uv^2 + 18v - 2u^2v + uv^2 + u^2v$$

$$= 18u + 18v - uv^2 - u^2v$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

$$= \int_0^1 \int_0^1 18u + 18v - uv^2 - u^2v \, du\, dv$$

$$\int_0^1 18u + 18v - uv^2 - u^2v \, du = \left[9u^2 + 18uv - \frac{u^2v^2}{2} - \frac{u^3v}{3} \right]\Big|_0^1$$

$$= 9 + 18v - \frac{v^2}{2} - \frac{v}{3}$$

$$\int_0^1 9 + 18v - \frac{v^2}{2} - \frac{v}{3}\, dv = \left[9v + 9v^2 - \frac{v^3}{6} - \frac{v^2}{6} \right]\Big|_0^1$$

$$= 9 + 9 - \frac{1}{6} - \frac{1}{6}$$

$$= 18 - \frac{1}{3} = \frac{53}{3}$$

Note, if we change orientation ie use $\vec{n} = -\frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$, we change the sign of the surface integral.

Flux is the amount of something passing through a surface in a unit of time (e.g. fluid, energy, etc.)

The vector field Surface Integral $\int_S \vec{F} \cdot \vec{n} \, dS$ tells us the flux through the surface S in the direction of $\vec{n}$.

Let $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ be a vector field in $\mathbb{R}^3$. The divergence of $\vec{F}$ is $div(\vec{F}) = P_x + Q_y + R_z$

Example - i) $\vec{F}(x,y,z) = \langle x^2 + yz - ze^x, \; 3xyz + \cos(z)\sin(y) - \ln(x), \; z - \sqrt{xy} \rangle$

2) $\vec{F}(x,y,z) = \langle xy,\ yz,\ zx \rangle$

3) For arbitrary $f(x,y,z)$, compute $div(\nabla f)$.

1)

$P = x^2 + yz - ze^x \Rightarrow P_x = 2x - ze^x$

$Q = 3xyz + \cos(z)\sin(y) - \ln(x)$

$\Rightarrow Q_y = 3xz + \cos(z)\cos(y)$

$R = z - \sqrt{xy} \Rightarrow R_z = 1$

So $div(\vec{F}) = 2x - ze^x + 3xz + \cos(z)\cos(y) + 1$

2) $P = xy \Rightarrow P_x = y$

$Q = yz \Rightarrow Q_y = z$

$R = zx \Rightarrow R_z = x$

So $div(\vec{F}) = y + z + x$

3) $f(x,y,z) \Rightarrow \nabla f = \langle f_x, f_y, f_z \rangle$

$P = f_x \Rightarrow P_x = f_{xx}$

$Q = f_y \Rightarrow Q_y = f_{yy}$

$R = f_z \Rightarrow R_z = f_{zz}$

So $\operatorname{div}(\nabla f) = f_{xx} + f_{yy} + f_{zz}$

For a function $f(x,y,z)$, the <u>Laplacian</u> of f is $\Delta f = \operatorname{div}(\nabla f) = f_{xx} + f_{yy} + f_{zz}$

Ex - $f(x,y,z) = 6x^3y + 8y^2z + z^2$

$f_x = 18x^2y \Rightarrow f_{xx} = 36xy$

$f_y = 6x^3 + 16yz \Rightarrow f_{yy} = 16z$

$f_z = 8y^2 + 2z \Rightarrow f_{zz} = 2$

So $\Delta f = 36xy + 16z + 2$

Let $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ be a vector field in $\mathbb{R}^3$. The <u>curl</u> of $\vec{F}$ is $\operatorname{curl}(\vec{F}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

We also can represent this as

$\operatorname{curl}(\vec{F}) = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \langle P, Q, R \rangle$

Examples - 1) $\vec{F}(x,y,z) = \langle xy, yz, zx \rangle$

2) $\vec{F}(x,y,z) = \langle xy+z, yz+x, zx+y \rangle$

3) For arbitrary $f(x,y,z)$, compute $\text{curl}(\nabla f)$

4) For arbitrary $\vec{F}(x,y,z) = \langle P, Q, R \rangle$, compute $\text{div}(\text{curl}(\vec{F}))$

1) $P = xy \Rightarrow P_y = x$ and $P_z = 0$

$Q = yz \Rightarrow Q_x = 0$ and $Q_z = y$

$R = zx \Rightarrow R_y = 0$ and $R_x = z$

So $\text{curl}(\vec{F}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

$= \langle -y, -z, -x \rangle$

2) $P = xy+z \Rightarrow P_y = x$ and $P_z = 1$

$Q = yz+x \Rightarrow Q_x = 1$ and $Q_z = y$

$R = zx+y \Rightarrow R_x = z$ and $R_y = 1$

So $\text{curl}(\vec{F}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

$= \langle 1-y,\ 1-z,\ 1-x \rangle$

3) $f(x,y,z) \Rightarrow \nabla f = \langle f_x, f_y, f_z \rangle$

$P = f_x \Rightarrow P_y = f_{xy}$ and $P_z = f_{xz}$

$Q = f_y \Rightarrow Q_x = f_{yx}$ and $Q_z = f_{yz}$

$R = f_z \Rightarrow R_x = f_{zx}$ and $R_y = f_{zy}$

So $\text{curl}(\nabla f) = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$

$= \langle f_{zy} - f_{yz},\ f_{xz} - f_{zx},\ f_{yx} - f_{xy} \rangle$

$= \langle 0, 0, 0 \rangle$

Recall a vector field $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ is conservative if there exists $f(x,y,z)$ such that $\nabla f = \vec{F}$. This occurs if and only if

$P_y = Q_x,\ P_z = R_x,\ Q_z = R_y$

$\Leftrightarrow Q_x - P_y = 0,\ P_z - R_x = 0,\ R_y - Q_z = 0$

$\Leftrightarrow \text{curl}(\vec{F}) = \vec{0}$

Fact - $\operatorname{curl}(\nabla f) = \vec{0}$ for all real-valued functions $f(x, y, z)$.

Fact - $\vec{F}(x, y, z)$ is conservative if and only if $\operatorname{curl}(\vec{F}) = \vec{0}$

4) $\vec{F}(x, y, z) = \langle P, Q, R \rangle$

$\operatorname{curl}(\vec{F}) = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle$

$$\begin{aligned}
\operatorname{div}(\operatorname{curl}(\vec{F})) &= (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z \\
&= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} \\
&= (P_{zy} - P_{yz}) + (Q_{xz} - Q_{zx}) + (R_{yx} - R_{xy}) \\
&= 0 + 0 + 0 = \boxed{0}
\end{aligned}$$

Fact - $\operatorname{div}(\operatorname{curl}(\vec{F})) = 0$ for all vector fields $\vec{F}(x, y, z)$