i.e $ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$

This is the general equation for the plane.

Example - 1) Find equation of plane passing through $p = (1,2,3)$ with normal vector $\vec{n} = \langle 4, 5, 6 \rangle$

Give general equation and find 3 other points in the plane

2) $\vec{L}_1 : \langle 1+t, 2-3t, 5+2t \rangle$ $\quad \vec{L}_2 = \langle 7-4s, 4s, s-1 \rangle$

Find the plane spanned by these two intersecting lines

1) $4(x-1) + 5(y-2) + 6(z-3) = 0$

$4x - 4 + 5y - 10 + 6z - 18 = 0$

$4x + 5y + 6z = 32$

To find points, plug in two values and solve for the third.

$x=0$ and $y=0 \Rightarrow z = \frac{32}{6} = \frac{16}{3}$ so $(0, 0, \frac{16}{3})$

$x=0$ and $z=0 \Rightarrow y = \frac{32}{5}$ so $(0, \frac{32}{5}, 0)$

$y=0$ and $z=0 \Rightarrow x = \frac{32}{4} = 8$ so $(8, 0, 0)$

2) First find point of intersection of $\vec{L}_1$ and $\vec{L}_2$

If $\vec{L}_1 = \vec{L}_2$, then $1+t = 7-4s$, $2-3t = 4s$, and $5+2t = s-1$

First equation implies $t = 6-4s$

Plug this into second equation to get $2-3(6-4s) = 4s$

$$\Rightarrow \quad -16 + 12s = 4s \quad \Rightarrow \quad -16 = -8s \quad \Rightarrow \quad s = 2$$

Then $t = 6 - 4s = 6 - 4(2) = -2$

Check. If $t = -2$ then $\vec{L}_1 = \langle -1, 8, 1 \rangle$
If $s = 2$ then $\vec{L}_2 = \langle -1, 8, 1 \rangle$ $\Big\} = \checkmark$

To get normal vector, cross product the direction vectors for $\vec{L}_1$ and $\vec{L}_2$.

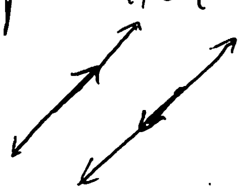
Direction vector for $\vec{L}_1$ is $\langle 1, -3, 2 \rangle$

Direction vector for $\vec{L}_2$ is $\langle -4, 4, 1 \rangle$

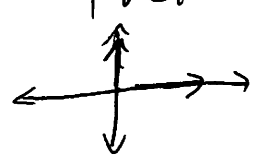
Cross product of these is $\langle -11, -9, -8 \rangle$

Hence, we have
$$-11(x+1) - 9(y-8) - 8(z-1) = 0$$
$$-11x - 11 - 9y + 72 - 8z + 8 = 0$$
$$-11x - 9y - 8z = -69$$

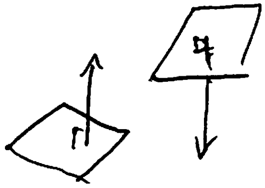
Two lines are parallel if their direction vectors are parallel



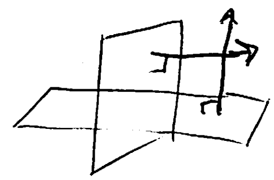
Two lines are perpendicular if their direction vectors are orthogonal.



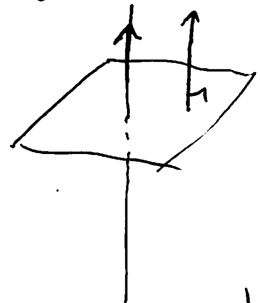
Two planes are parallel if their normal vectors are parallel.



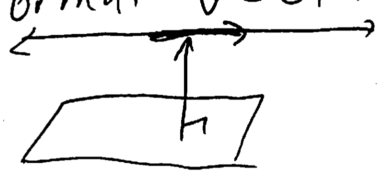
Two planes are perpendicular if their normal vectors are orthogonal



A line is perpendicular to a plane if the direction vector of the line and normal vector of the plane are parallel.



A line is parallel to a plane if the direction vector of the line and normal vector of the plane are orthogonal



Example - Find the vector equation of the line that passes through $(1,2,3)$ and is perpendicular to the plane $5x - y + 7z = 9$

The direction vector of the line is parallel to the normal vector of the plane, which is $\langle 5, -1, 7 \rangle$

We can take this to be our direction vector.

Then $\vec{L} = \langle 1+5t,\ 2-t,\ 3+7t \rangle$

Consider a plane containing point P and with normal vector $\vec{n}$. Then the shortest distance from a point Q to the plane is $\dfrac{|\vec{PQ}\cdot\vec{n}|}{\|\vec{n}\|}$

Example - Find the shortest distance from the point $Q = (3,4,-1)$ and the plane $2x-2y+z=0$

$\vec{n} = \langle 2,-2,1 \rangle$, $\quad P = (0,0,0)$

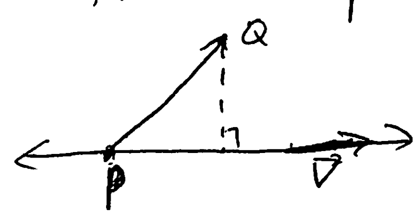
$\vec{PQ} = \langle 3,4,-1 \rangle$

$\vec{PQ}\cdot\vec{n} = \langle 3,4,-1 \rangle \cdot \langle 2,-2,1 \rangle = 6-8-1 = -3$

$\|\vec{n}\| = \sqrt{2^2+(-2)^2+1^2} = \sqrt{9} = 3$

So shortest distance is $\dfrac{|-3|}{3} = 1$

Consider a line through point P with direction vector $\vec{v}$. Then the shortest distance from a point Q to the line is $\dfrac{\|\vec{v}\times\vec{PQ}\|}{\|\vec{v}\|}$



Example - Find the shortest distance from the point $Q = (5,-1,2)$ to the line $\vec{L} = \langle 1+4t,\ 2-4t,\ -3+7t \rangle$

$\vec{v} = \langle 4,-4,7 \rangle$ $\quad P = (1,2,-3)$ $\quad \vec{PQ} = \langle 4,-3,5 \rangle$

$\vec{v} \times \vec{PQ} = \langle 4, -4, 7 \rangle \times \langle 4, -3, 5 \rangle = \langle 1, 8, 4 \rangle$

$\|\vec{v} \times \vec{PQ}\| = \sqrt{1^2 + 8^2 + 4^2} = \sqrt{81} = 9$

$\|\vec{v}\| = \sqrt{4^2 + (-4)^2 + 7^2} = \sqrt{81} = 9$

Shortest distance is then $\frac{9}{9} = 1$

A real-valued function of a single variable is an expression f that assigns to each value t in its domain $D_f \subseteq \mathbb{R}$ one and only one value $f(t)$ in its range $R_f \subseteq \mathbb{R}$.

Examples - $f(t) = t^2 + 2$ $\quad f(t) = e^t \quad f(t) = \ln(t)$

Domains - polynomials and exponential functions: $\mathbb{R}$

$\sqrt{t}$: $[0, \infty)$

$\ln(t)$: $(0, \infty)$

$\frac{h(t)}{g(t)}$: everywhere $g(t) \neq 0$

A vector-valued function is a vector $\vec{F}$ that assigns to each value t in its domain $D_{\vec{F}} \subseteq \mathbb{R}$ one and only one vector $\vec{F}(t)$ in its range $R_{\vec{F}} \subseteq \mathbb{R}^3$

We write these as $\vec{F}(t) = \langle P(t), Q(t), R(t) \rangle$ component functions where P, Q, and R are real-valued functions.

Examples - $\vec{F}(t) = \langle t^2 + 3t - 9, \sqrt{t-1}, \sin(4t) \rangle$

$\vec{G}(t) = \langle \frac{t+7}{t-5}, e^{2t}, \ln(t+1) \rangle$

The domain of a vector-valued function is the set of all values for which every component function is defined

Examples - $D_{\vec{F}}: [1, \infty)$

$D_{\vec{G}}: (-1, 5) \cup (5, \infty)$

Let $\vec{F}(t) = \langle P(t), Q(t), R(t) \rangle$ be a vector-valued function. $\vec{F}$ has limit $\vec{L} = \langle L_1, L_2, L_3 \rangle$ at a value t_0, denoted $\lim_{t \to t_0} \vec{F}(t) = \vec{L}$, if the following limits exist:

$\lim_{t \to t_0} P(t) = L_1 \qquad \lim_{t \to t_0} Q(t) = L_2 \qquad \lim_{t \to t_0} R(t) = L_3$

Example - $\vec{F}(t) = \langle t^2 + 3t - 9, e^{3t}, \frac{t^2-1}{t-1} \rangle$

1) Find $\lim_{t \to 0} \vec{F}(t)$

2) Find $\lim_{t \to 1} \vec{F}(t)$

1) $\lim_{t \to 0} (t^2 + 3t - 9) = -9 \qquad \lim_{t \to 0} e^{3t} = 1 \qquad \lim_{t \to 0} \frac{t^2-1}{t-1} = 1$

Thus $\lim_{t \to 0} \vec{F}(t) = \langle -9, 1, 1 \rangle$

2) $\lim_{t\to 1} (t^2+3t-9) = -5 \qquad \lim_{t\to 1} e^{3t} = e^3$

$$\lim_{t\to 1} \frac{t^2-1}{t-1} = \lim_{t\to 1} \frac{(t-1)(t+1)}{t-1} = \lim_{t\to 1} (t+1) = 2$$

So $\lim_{t\to 1} \vec{F}(t) = \langle -5, e^3, 2 \rangle$

A real-valued function f is continuous at t_0 if:

1) $f(t_0)$ exists
2) $\lim_{t\to t_0} f(t)$ exists
3) $\lim_{t\to t_0} f(t) = f(t_0)$

A vector-valued function $\vec{F}(t) = \langle P(t), Q(t), R(t) \rangle$ is continuous at t_0 if P, Q, and R are all continuous at t_0. In this case, $\lim_{t\to t_0} \vec{F}(t) = \vec{F}(t_0)$

Example- $\vec{F}(t) = \langle t^2+3t-9, \ e^{3t}, \ \frac{t^2-1}{t-1} \rangle$

1) Is $\vec{F}$ continuous at $t=0$? Yes
2) Is $\vec{F}$ continuous at $t=1$? No b/c $\frac{t^2-1}{t-1}$ is undefined at $t=1$.

A vector-valued function $\vec{F}(t) = \langle P(t), Q(t), R(t) \rangle$ is differentiable at t_0 if P, Q, R are all differentiable at t_0, in which case

$$\frac{d\vec{F}}{dt}(t_0) = \left\langle \frac{dP}{dt}(t_0),\ \frac{dQ}{dt}(t_0),\ \frac{dR}{dt}(t_0) \right\rangle$$

Example - $\vec{F}(t) = \langle t^2 + 3t - 9,\ e^{t^2-1},\ \sin(6t) \rangle$

Find $\frac{d\vec{F}}{dt}$ and $\frac{d\vec{F}}{dt}(0)$

$$\frac{d\vec{F}}{dt} = \langle 2t + 3,\ 2t\,e^{t^2-1},\ 6\cos(6t) \rangle$$

$$\frac{d\vec{F}}{dt}(0) = \langle 3,\ 0,\ 6 \rangle$$

Derivative Rules - For differentiable vector-valued functions $\vec{F}$ and $\vec{G}$ and differentiable real-valued functions g and h:

1) $\frac{d}{dt}\vec{K} = \vec{0}$ for all constant vectors $\vec{K}$
2) $\frac{d}{dt}(c\vec{F}(t)) = c\frac{d\vec{F}}{dt}$ for scalars $c \in \mathbb{R}$
3) $\frac{d}{dt}(\vec{F}(t) \pm \vec{G}(t)) = \frac{d\vec{F}}{dt} \pm \frac{d\vec{G}}{dt}$
4) $\frac{d}{dt}(h(t)\vec{F}(t)) = \frac{dh}{dt}\vec{F}(t) + h(t)\frac{d\vec{F}}{dt}$
5) $\frac{d}{dt}(\vec{F}(t) \cdot \vec{G}(t)) = \frac{d\vec{F}}{dt} \cdot \vec{G}(t) + \vec{F}(t) \cdot \frac{d\vec{G}}{dt}$
6) $\frac{d}{dt}(\vec{F}(t) \times \vec{G}(t)) = \frac{d\vec{F}}{dt} \times \vec{G}(t) + \vec{F}(t) \times \frac{d\vec{G}}{dt}$
7) $\frac{d}{dt}(\vec{F}(g(t))) = \vec{F}'(g(t))\frac{dg}{dt}$

Example - $\vec{F}(t) = \langle t^2, e^t, \ln(t) \rangle$

$g(t) = t^2+1$

$\vec{F}(g(t)) = \langle (t^2+1)^2, e^{t^2+1}, \ln(t^2+1) \rangle$

Let $\vec{F}(t) = \langle P(t), Q(t), R(t) \rangle$ be continuous on the interval $t_1 \le t \le t_2$. Then the <u>definite integral</u> of $\vec{F}$ over this interval is

$$\int_{t_1}^{t_2} \vec{F}(t)\,dt = \left\langle \int_{t_1}^{t_2} P(t)\,dt, \int_{t_1}^{t_2} Q(t)\,dt, \int_{t_1}^{t_2} R(t)\,dt \right\rangle$$

Example - $\vec{F}(t) = \langle 2t+1, \frac{1}{\sqrt{t}}, e^{3t} \rangle$

Find $\int_1^9 \vec{F}(t)\,dt$

$$\int_1^9 2t+1\,dt = [t^2+t|_1^9 = (81+9)-(1+1) = 88$$

$$\int_1^9 \frac{1}{\sqrt{t}}\,dt = 2\sqrt{t}\Big|_1^9 = 6-2 = 4$$

$$\int_1^9 e^{3t}\,dt = \frac{1}{3}e^{3t}\Big|_1^9 = \frac{1}{3}[e^{27}-e^3]$$

So $\int_1^9 \vec{F}(t)\,dt = \langle 88, 4, \frac{1}{3}[e^{27}-e^3] \rangle$

An <u>antiderivative</u> of a continuous vector-valued function $\vec{F}(t)$ is a vector-valued function $\vec{G}(t)$ such that $\frac{d\vec{G}}{dt} = \vec{F}(t)$.

Note if $\vec{G}(t)$ is an antiderivative of $\vec{F}(t)$, then so is $\vec{G}(t) + \vec{C}$ for all constant vectors $\vec{C}$.

We call $\vec{G}(t) + \vec{C}$ the <u>family of antiderivatives</u> of $\vec{F}(t)$

Example - Find the family of antiderivatives of $\vec{F}(t) = \langle 6t^2 + 5, e^{t-1}, \frac{1}{t} \rangle$

Then find an antiderivative $\vec{G}(t)$ such that $\vec{G}(1) = \langle 3, -1, 2 \rangle$

Family is $\int \vec{F}(t)\,dt + \vec{C} = \langle 2t^3 + 5t, e^{t-1}, \ln(t) \rangle + \vec{C}$

$\langle 3, -1, 2 \rangle = \vec{G}(1) = \langle 7, 1, 0 \rangle + \vec{C}$

So $\vec{C} = \langle -4, -2, 2 \rangle$

Thus. $\vec{G}(t) = \langle 2t^3 + 5t - 4, e^{t-1} - 2, \ln(t) + 2 \rangle$

A <u>parametrized curve</u> is a vector-valued function $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ with domain $D \subseteq \mathbb{R}$

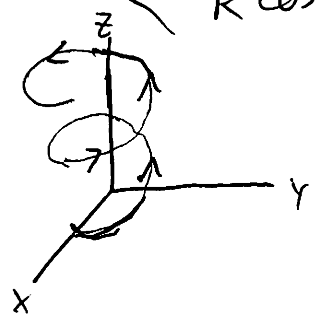
Examples - Lines $\vec{L}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$

for $-\infty < t < \infty$

If we specify a domain $a \leq t \leq b$, we get a line segment from $\vec{L}(a)$ to $\vec{L}(b)$.

A circular helix with radius R and pitch γ is

$\vec{r}(t) = \langle R\cos(t), R\sin(t), \gamma t \rangle$ for $t \geq 0$



A <u>smooth</u> curve in $\mathbb{R}^3$ is a parametrized curve $\vec{r}(t)$ with domain $D \subseteq \mathbb{R}$ such that

1) $\vec{r}(t)$ has a continuous second derivative for all t in D

2) $\vec{r}\,'(t) \neq \vec{0}$ for all t in D

Examples - $\vec{r}(t) = \langle \cos(t), e^{t^2}, \frac{t^2}{t^2-4} \rangle$

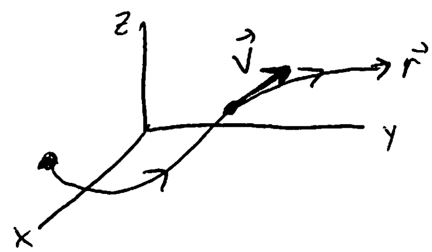
1) Is $\vec{r}$ smooth on $D = [5, 10]$
2) Is $\vec{r}$ smooth on $D = [-1, 1]$
3) Is $\vec{r}$ smooth on $D = [1, 4]$

$$\vec{r}\,'(t) = \left\langle -\sin(t),\ 2te^{t^2},\ \frac{-8t}{(t^2-4)^2} \right\rangle$$

$$\vec{r}\,''(t) = \left\langle -\cos(t),\ 4t^2e^{t^2} + 2e^{t^2},\ \frac{24t^2+32}{(t^2-4)^3} \right\rangle$$

1) Yes
2) $\vec{r}\,'(t) = \vec{0}$ at $t=0 \in [-1, 1]$ so No
3) $\vec{r}\,''(t)$ not continuous at $t=2 \in [1, 4]$ so No

The <u>velocity vector</u> (or principal tangent vector) to the parametrized curve $\vec{r}(t)$ at the point on $\vec{r}$ where $t = t_0$ is the vector $\vec{v}(t_0) = \vec{r}\,'(t_0)$. This vector is tangent to the curve $\vec{r}$ and in the direction of the curve.



Example - $\vec{r}(t) = \langle 4\sqrt{t},\ t^2-11,\ \frac{t^2}{t-2} \rangle$

Find $\vec{v}(t)$ and $\vec{v}(1)$.

$\vec{v}(t) = \vec{r}\,'(t) = \langle \frac{2}{\sqrt{t}},\ 2t,\ \frac{t^2-4t}{(t-2)^2} \rangle$

$\vec{v}(1) = \langle 2, 2, -3 \rangle$

The tangent line to the curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ at the point on $\vec{r}$ where $t = t_0$ is the line through point $(x(t_0), y(t_0), z(t_0))$ with direction vector $\vec{v}(t_0) = \vec{r}\,'(t_0)$

Example- $\vec{r}(t) = \langle 4\sqrt{t},\ t^2-11,\ \frac{t^2}{t-2} \rangle$

1) Find tangent line to $\vec{r}$ at $t=1$

2) Find tangent line to $\vec{r}$ at the point $(8, 5, 8)$

1) We know $\vec{v}(1) = \langle 2, 2, -3 \rangle$

Our point is $\vec{r}(1) = (4, -10, -1)$

$\vec{L} = \langle 4+2t, -10+2t, -1-3t \rangle$

2) $\vec{v}(t) = \langle \frac{2}{\sqrt{t}}, 2t, \frac{t^2-4t}{(t-2)^2} \rangle$

To find t, set $\vec{r} = (8, 5, 8)$ and solve for t. In this case, $t=4$

Then $\vec{v}(4) = \langle 1, 8, 0 \rangle$

$\vec{L} = \langle 8+t, 5+8t, 8 \rangle$

The unit tangent vector to the curve $\vec{r}$ at the point on $\vec{r}$ where $t=t_0$ is

$$\vec{T}(t_0) = \frac{1}{\|\vec{v}(t_0)\|}\vec{v}(t_0) = \frac{1}{\|\vec{r}\,'(t_0)\|}\vec{r}\,'(t_0)$$

Example - $\vec{r}(t) = \langle e^t + e^{-t}, 2\cos(t), 2\sin(t) \rangle$

Find $\vec{T}(t)$ and $\vec{T}(0)$

$\vec{v}(t) = \vec{r}\,'(t) = \langle e^t - e^{-t}, -2\sin(t), 2\cos(t) \rangle$

$$\|\vec{v}(t)\| = \sqrt{(e^t-e^{-t})^2 + (-2\sin(t))^2 + (2\cos(t))^2}$$

$$= \sqrt{e^{2t} - 2 + e^{-2t} + 4\sin^2(t) + 4\cos^2(t)}$$

$$= \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$$

Then $\vec{T}(t) = \frac{1}{\|\vec{v}(t)\|}\vec{v}(t) = \frac{1}{e^t+e^{-t}}\langle e^t-e^{-t}, -2\sin(t), 2\cos(t)\rangle$

$$= \left\langle \frac{e^t-e^{-t}}{e^t+e^{-t}}, \frac{-2\sin(t)}{e^t+e^{-t}}, \frac{2\cos(t)}{e^t+e^{-t}} \right\rangle$$

$\vec{T}(0) = \langle 0, 0, 1 \rangle$

The speed of a particle moving along a curve $\vec{r}$ is $v(t) = \|\vec{v}(t)\| = \|\vec{r}\,'(t)\|$

Example- For the example above, $v(t) = e^t + e^{-t}$

So the speed at $t=0$ is $v(0) = 2$

The <u>acceleration vector</u> of a curve $\vec{r}$ is

$\vec{a}(t) = \vec{r}\,''(t)$

Example- $\vec{r}(t) = \langle e^{t-1}, t^3, \ln(t) \rangle$

Find $\vec{a}(t)$ and $\vec{a}(1)$

$\vec{r}\,'(t) = \langle e^{t-1}, 3t^2, \frac{1}{t} \rangle$

$\vec{a}(t) = \vec{r}\,''(t) = \langle e^{t-1}, 6t, \frac{-1}{t^2} \rangle$

$\vec{a}(1) = \langle 1, 6, -1 \rangle$

The arclength of a smooth curve $\vec{r}$ on the interval $a \le t \le b$ is $L = \int_a^b v(t)\,dt$

$$= \int_a^b \|\vec{r}\,'(t)\|\,dt$$

Example - Find the arclength of

$\vec{r}(t) = \langle \frac{1}{2}t^2 - 3,\ \frac{4}{3}t^{3/2},\ 2t \rangle$ on the interval $4 \le t \le 8$

$$\vec{r}\,'(t) = \langle t,\ 2\sqrt{t},\ 2 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{t^2 + (2\sqrt{t})^2 + 2^2} = \sqrt{t^2 + 4t + 4}$$

$$= \sqrt{(t+2)^2} = t+2$$

$$L = \int_4^8 t+2\,dt = \frac{t^2}{2} + 2t \Big|_4^8 = (32+16) - (8+8)$$

$$= 32$$

The arclength function of a smooth curve $\vec{r}$ on the interval $a \le t \le b$ is $s(t) = \int_a^t \|\vec{r}\,'(p)\|\,dp$

Example - Find the arclength function of

$\vec{r}(t) = \langle 3\cos(t),\ 3\sin(t),\ 4t \rangle$ for $t \ge 0$

$$\vec{r}\,'(t) = \langle -3\sin(t),\ 3\cos(t),\ 4 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{(-3\sin(t))^2 + (3\cos(t))^2 + 4^2}$$

$$= \sqrt{9\sin^2(t) + 9\cos^2(t) + 16} = \sqrt{25} = 5$$

So $s(t) = \int_0^t 5\,dp = 5p\Big|_0^t = 5t - 0 = 5t$

Note $s(a) = 0$, $s(b) = L$, $s'(t) = \|\vec{r}\,'(t)\|$

(Fundamental Thm of Calc)

Consider $\vec{r}_1(t) = \langle \cos(\sqrt{t}),\ \sin(\sqrt{t}),\ 3 \rangle$ for ~~and~~ $1 \le t \le 4$

$\vec{r}_2(\tau) = \langle \cos(\tau),\ \sin(\tau),\ 3 \rangle$ for $1 \le \tau \le 2$

These parametrize the same curve.

We say $\vec{r}_2(\tau)$ is a reparametrization of $\vec{r}_1(t)$

with $t = \tau^2$

$$\vec{r}_1\,'(t) = \left\langle \frac{-\sin(\sqrt{t})}{2\sqrt{t}},\ \frac{\cos(\sqrt{t})}{2\sqrt{t}},\ 0 \right\rangle$$

$$\|\vec{r}_1\,'(t)\| = \sqrt{\frac{\sin^2(\sqrt{t})}{4t} + \frac{\cos^2(\sqrt{t})}{4t}} = \frac{1}{2\sqrt{t}}$$

$$\vec{r}_2\,'(\tau) = \langle -\sin(\tau),\ \cos(\tau),\ 0 \rangle$$

$$\|\vec{r}_2\,'(\tau)\| = \sqrt{\sin^2(\tau) + \cos^2(\tau)} = 1$$

A curve $\vec{r}$ such that $\|\vec{r}\,'\| = 1$ is a unit speed curve.

Can we always reparametrize to make a curve unit speed? Yes