1) $\text{proj}_{\vec{b}}(\vec{a}) = \frac{\vec{b}\cdot\vec{a}}{\vec{b}\cdot\vec{b}}\vec{b} = \frac{12}{24}\langle 2,4,2\rangle = \frac{1}{2}\langle 2,4,2\rangle = \langle 1,2,1\rangle$

$s_{\vec{b}}(\vec{a}) = \frac{\vec{b}\cdot\vec{a}}{\|\vec{b}\|} = \frac{12}{\sqrt{\vec{b}\cdot\vec{b}}} = \frac{12}{\sqrt{24}}$

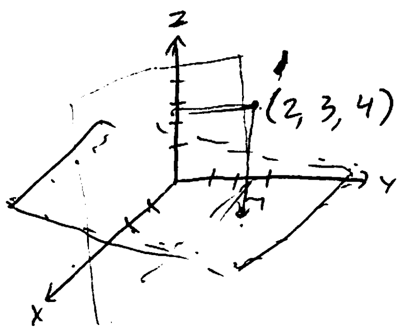
$\vec{a}\cdot\vec{b} = \langle -3,3,3\rangle \cdot \langle 2,4,2\rangle = -6+12+6 = 12$

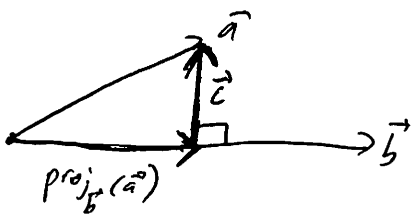
$\vec{b}\cdot\vec{b} = \langle 2,4,2\rangle \cdot \langle 2,4,2\rangle = 4+16+4 = 24$

2) $\text{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a} = \frac{12}{27}\langle -3,3,3\rangle = \frac{4}{9}\langle -3,3,3\rangle = \langle -\frac{4}{3},\frac{4}{3},\frac{4}{3}\rangle$

$s_{\vec{a}}(\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|} = \frac{12}{\sqrt{\vec{a}\cdot\vec{a}}} = \frac{12}{\sqrt{27}}$

$\vec{a}\cdot\vec{a} = \langle -3,3,3\rangle \cdot \langle -3,3,3\rangle = 9+9+9 = 27$





What can we say about $\vec{c}$?

1) $\vec{c}$ is orthogonal to $\vec{b}$

2) $\vec{a} = \text{proj}_{\vec{b}}(\vec{a}) + \vec{c}$

For nonzero vectors $\vec{a}, \vec{b}$, the orthogonal ~~decomposition~~ decomposition of $\vec{a}$ with respect to $\vec{b}$ is

$$\vec{a} = \underbrace{\text{proj}_{\vec{b}}(\vec{a})}_{\text{parallel to } \vec{b}} + \underbrace{\vec{a}_{\perp}}_{\text{orthogonal to } \vec{b}}, \quad \text{where} \quad \vec{a}_{\perp} = \vec{a} - \text{proj}_{\vec{b}}(\vec{a})$$

Ex - $\vec{a} = \langle -3, 3, 3 \rangle$ and $\vec{b} = \langle 2, 4, 2 \rangle$

Find orthogonal decomposition of $\vec{a}$ with respect to $\vec{b}$.

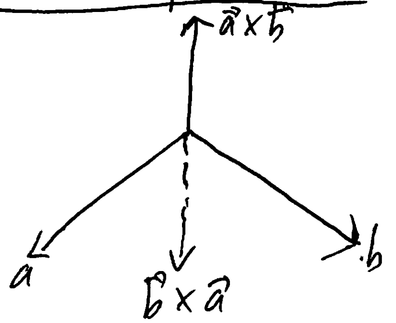
$\text{proj}_{\vec{b}}(\vec{a}) = \langle 1, 2, 1 \rangle$

$\vec{a}_\perp = \vec{a} - \langle 1, 2, 1 \rangle = \langle -4, 1, 2 \rangle$

Note $\vec{a}_\perp \cdot \vec{b} = \langle -4, 1, 2 \rangle \cdot \langle 2, 4, 2 \rangle = -8 + 4 + 4 = 0$

The minimal distance from $\vec{a}$ to $\vec{b}$ is $\|\vec{a}_\perp\|$

Cross product



$\vec{a} \times \vec{b}$

The cross product of $\vec{a}$ and $\vec{b}$ produces a vector orthogonal to both $\vec{a}$ and $\vec{b}$

The cross product of $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ is $\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$

Example - Compute $\vec{a} \times \vec{b}$ for $\vec{a} = \langle 1, 2, 3 \rangle$ $\vec{b} = \langle 4, 5, 6 \rangle$

$\vec{a} \times \vec{b} = \langle 2(6) - 3(5),\ 3(4) - 1(6),\ 1(5) - 2(4) \rangle$

$= \langle -3, 6, -3 \rangle$

The determinant of a 2x2 matrix $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is

$xw - yz$

We can write cross product as

$$\vec{a} \times \vec{b} = \det\begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix}\vec{i} - \det\begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix}\vec{j} + \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\vec{k}$$

Example. $\vec{a} \times \vec{b}$ for $\vec{a} = \langle 1, -3, 6 \rangle$, $\vec{b} = \langle 2, 5, 1 \rangle$

$\langle 1, -3, 6 \rangle$

$\langle 2, 5, 1 \rangle$

$$\det\begin{pmatrix} -3 & 6 \\ 5 & 1 \end{pmatrix}\vec{i} - \det\begin{pmatrix} 1 & 6 \\ 2 & 1 \end{pmatrix}\vec{j} + \det\begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix}\vec{k}$$

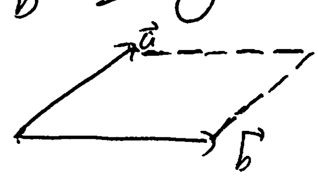
$$= \langle -33, 11, 11 \rangle$$

Properties. For vectors $\vec{a}, \vec{b}, \vec{c}$ and scalar $\beta \in \mathbb{R}$

1) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$
2) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
3) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
4) $(\beta \vec{a}) \times \vec{b} = \vec{a} \times (\beta \vec{b}) = \beta(\vec{a} \times \vec{b})$
5) $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \sin(\theta)$ where θ is the angle between $\vec{a}$ and $\vec{b}$

Two nonzero vectors $\vec{a}, \vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$

Consider



The vectors $\vec{a}, \vec{b}$ span a parallelogram

The area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$ is $\|\vec{a} \times \vec{b}\|$

Example- 1) Find area of parallelogram spanned by $\langle 6, 8, 5 \rangle$ and $\langle 2, 3, 2 \rangle$

2) Find area of parallelogram spanned by $\vec{a}, \vec{b}$ if $\|\vec{a}\| = 4$, $\|\vec{b}\| = 5$, and angle between them is $\pi/6$

1)

$$\vec{a} \times \vec{b} = \langle 6, 8, 5 \rangle \times \langle 2, 3, 2 \rangle = \langle 1, -2, 2 \rangle$$

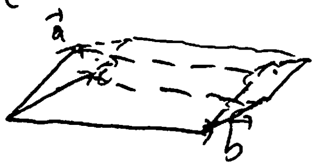
$$A = \|\vec{a} \times \vec{b}\| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$$

2) $A = \|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \sin(\theta) = 4(5)\sin\left(\frac{\pi}{6}\right) = 10$

The <u>scalar triple product</u> of $\vec{a}, \vec{b}, \vec{c}$ is $\vec{a} \cdot (\vec{b} \times \vec{c})$

The <u>vector triple product</u> of $\vec{a}, \vec{b}, \vec{c}$ is

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Consider 

The vectors $\vec{a}, \vec{b}, \vec{c}$ span a parallelepiped

The volume of the parallelepiped spanned by $\vec{a}$, $\vec{b}$, and $\vec{c}$ is $|\vec{a} \cdot (\vec{b} \times \vec{c})|$

Example - $\vec{a} = \langle 1, 2, 3 \rangle$ $\vec{b} = \langle 4, 5, 6 \rangle$ $\vec{c} = \langle 1, 5, 7 \rangle$

1) Find volume of parallelpiped spanned by $\vec{a}, \vec{b}, \vec{c}$

2) Find vector triple product $\vec{a} \times (\vec{b} \times \vec{c})$

1) $\vec{b} \times \vec{c} = \langle 4,5,6 \rangle \times \langle 1,5,7 \rangle = \langle 5, -22, 15 \rangle$

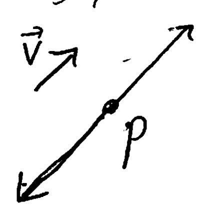
$\vec{a} \cdot (\vec{b} \times \vec{c}) = \langle 1,2,3 \rangle \cdot \langle 5,-22,15 \rangle = 5 - 44 + 45 = 6$

2) $\vec{a} \cdot \vec{b} = \langle 1,2,3 \rangle \cdot \langle 4,5,6 \rangle = 4 + 10 + 18 = 32$

$\vec{a} \cdot \vec{c} = \langle 1,2,3 \rangle \cdot \langle 1,5,7 \rangle = 1 + 10 + 21 = 32$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$
$$= 32\langle 4,5,6 \rangle - 32\langle 1,5,7 \rangle$$
$$= 32\langle 3, 0, -1 \rangle = \langle 96, 0, -32 \rangle$$

In $\mathbb{R}^3$, a line is determined by a point and a direction vector.



The <u>scalar parametric equations</u> of the line passing through $P = (x_0, y_0, z_0)$ with direction vector parallel to $\vec{v} = \langle a, b, c \rangle$ are $x = x_0 + at$, $y = y_0 + bt$, and $z = z_0 + ct$, where $t \in \mathbb{R}$ is a scalar, (or <u>parameter</u>)

The <u>vector parametric equation</u> of this line is

$$\vec{L} = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

Example- 1) Find scalar and vector parametric equations for line through $(1,2,3)$ with direction vector parallel to $\langle 4,5,6\rangle$

Then find 3 other points on the line

2) Find vector equation for line through $P=(-1,6,3)$ and $Q=(4,2,5)$

1) $x = 1+4t \qquad y = 2+5t \qquad z = 3+6t$

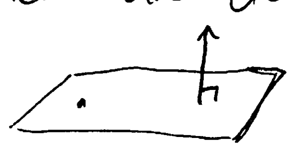
$\vec{L} = \langle 1+4t,\ 2+5t,\ 3+6t \rangle$

$t=1 \Rightarrow (5,7,9) \qquad t=-1 \Rightarrow (-3,-3,-3)$

2) Our direction vector is $\vec{PQ} = \langle 5,-4,2\rangle$

$\vec{L} = \langle -1+5t,\ 6-4t,\ 3+2t \rangle$

In $\mathbb{R}^3$, planes are determined by a point and a normal vector



A nonzero vector $\vec{n}$ is <u>normal</u> to a plane if it is orthogonal to all vectors in the plane, in which case $\vec{n}$ is a <u>normal vector</u> to the plane.

The equation of the plane passing through (x_0, y_0, z_0) with normal vector $\vec{n} = \langle a,b,c\rangle$ is $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

We can write this as $ax - ax_0 + by - by_0 + cz - cz_0 = 0$

i.e $\underline{ax + by + cz = d}$, where $d = ax_0 + by_0 + cz_0$

This is the general equation for the plane.

Example - 1) Find equation of plane passing through $P = (1, 2, 3)$ with normal vector $\vec{n} = \langle 4, 5, 6 \rangle$

Give general equation and find 3 other points in the plane

2) $\vec{L}_1: \langle 1+t,\ 2-3t,\ 5+2t \rangle$ $\quad \vec{L}_2 = \langle 7-4s,\ 4s,\ s-1 \rangle$

Find the plane spanned by these two intersecting lines