Recall, we say $y$ is directly proportional to $x$ if there exists some constant $k$ such that $y = kx$. We call $k$ the constant of proportionality.

Example: Newton's law of cooling states that the rate of change in the temperature of an object placed in a medium is directly proportional to the difference between the temperatures of the medium and the object

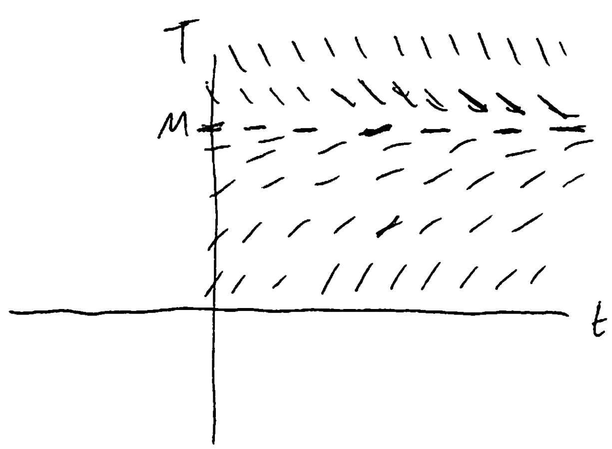
That is, $\frac{dT}{dt} = K(M - T)$, where $K > 0$, $M$ = temperature of medium, $T$ = temperature of object, $t$ = time.

1) Draw the direction field for this DE

2) Solve it as a separable DE

3) Solve it as a linear DE

1)




2) $\frac{dT}{dt} = K(M-T) \Rightarrow \frac{dT}{dt} = -K(T-M)$

$\Rightarrow \frac{1}{T-M} dT = -K dt$

$\Rightarrow \int \frac{1}{T-M} dT = \int -K dt$

$$\Rightarrow \ln(T-M) = -Kt + C_1$$

$$\Rightarrow T - M = e^{-Kt+C_1}$$

$$\Rightarrow T - M = e^{C_1} e^{-Kt}$$

$$\Rightarrow T = Ce^{-Kt} + M$$

Note $\lim_{t \to \infty} T(t) = M$

3) $\frac{dT}{dt} = K(M-T) \Rightarrow \frac{dT}{dt} = KM - KT$

$$\Rightarrow \frac{dT}{dt} + KT = KM$$

$P(t) = K$ and $Q(t) = KM$

Integrating factor is $u(t) = e^{\int P(t)dt} = e^{Kt}$

Then $T = \frac{1}{u(t)}\left(\int Q(t)u(t)\,dt + C\right)$

$$\Rightarrow T = \frac{1}{e^{kt}}\left(\int kMe^{kt} + C\right)$$

$$\Rightarrow T = \frac{1}{e^{kt}}\left(Me^{kt} + C\right)$$

$$\Rightarrow T = M + \frac{C}{e^{kt}}$$

$$\Rightarrow T = M + Ce^{-kt}$$

Ex - Suppose a $100°$ cup of coffee is placed in a $70°$ room. If the cup cools to $80°$ in 5 minutes, find $T(t)$.

General solution is $T(t) = M + Ce^{-kt}$

Since $M = 70$, $T(t) = 70 + Ce^{-kt}$

Since $T(0) = 100$, $100 = 70 + Ce^{0} = 70 + C$

$\Rightarrow C = 30 \Rightarrow T(t) = 70 + 30e^{-kt}$

Since $T(5) = 80$, $80 = 70 + 30e^{-5k}$

$\Rightarrow 10 = 30e^{-5k}$

$\Rightarrow \frac{1}{3} = e^{-5k}$

$\Rightarrow \ln\left(\frac{1}{3}\right) = -5k$

$\Rightarrow k = -\frac{1}{5}\ln\left(\frac{1}{3}\right)$

So $T(t) = 70 + 30e^{-\frac{1}{5}\ln(\frac{1}{3})t}$

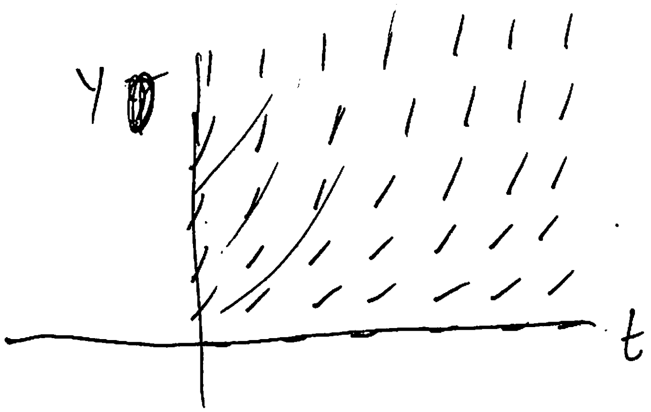
In population dynamics, the exponential growth model says that the rate of change in the population of a species is directly proportional to the population ie

$\frac{dy}{dt} = ky$, $k > 0$, $y$ = population, $t$ = time

1) Draw direction field

2) Solve as separable DE

3) Solve as linear DE

1) 


2) $\frac{dy}{dt} = ky \Rightarrow \frac{1}{y}dy = k\,dt$

$\Rightarrow \int \frac{1}{y}dy = \int k\,dt$

$\Rightarrow \ln(y) = kt + C_1$

$\Rightarrow y = e^{kt+C_1}$

$\Rightarrow y = e^{C_1}e^{kt}$

$\Rightarrow y = Ce^{kt}$

3) $\frac{dy}{dt} = ky \Rightarrow \frac{dy}{dt} - ky = 0$

So $P(t) = -K$ and $Q(t) = 0$

The integrating factor is $u(t) = e^{\int P(t)dt} = e^{-Kt}$

Then $y = \frac{1}{u(t)}\left(\int Q(t)u(t)\,dt + C\right)$

$\Rightarrow y = \frac{1}{e^{-Kt}}(C)$

$\Rightarrow y = Ce^{Kt}$

The limited growth model uses the fact that population growth is often limited by various factors (limited resources, predators, etc.) and so is capped by some value $M$.

The model says that the rate of change in the population is directly proportional to the difference between this limiting value and the population i.e

$$\frac{dy}{dt} = K(M-y) \quad \text{where } K>0,$$

$y=$ population, $t=$ time, and $M$ is the limiting population

We know from Newton's law that

$$y = M + Ce^{-Kt}$$

Observe $\lim_{t\to\infty} y = M$

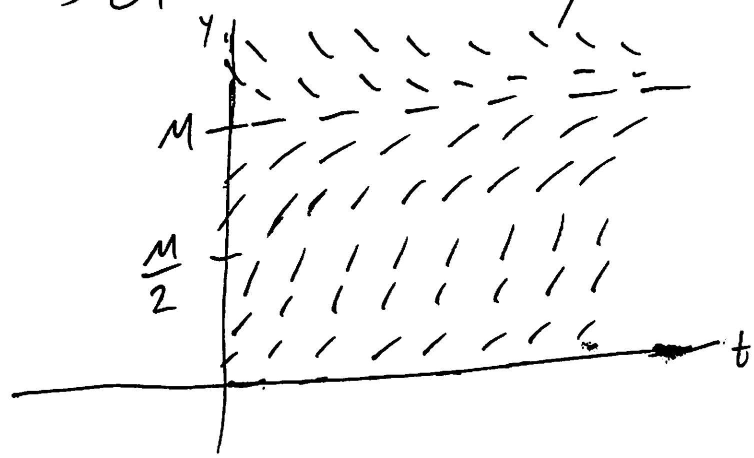
The logistic growth model incorporates dynamics from both exponential and limited growth. The DE for this model is $\frac{dy}{dt} = Ky(M-y)$

where $\mathbb{K} > 0$, $y =$ population, $t =$ time, and $M$ is the limiting population

1) Draw direction field

2) Solve for $y$

1)




2) $\frac{dy}{dt} = Ky(M-y) \Rightarrow \frac{dy}{dt} = -Ky(y-M)$

$\Rightarrow \frac{1}{y(y-M)} dy = -K dt$

$\Rightarrow \frac{1}{M}\left[\frac{1}{y-M} - \frac{1}{y}\right] dy = -K dt$

$\Rightarrow \frac{1}{M}\int \frac{1}{y-M} - \frac{1}{y} \, dy = \int -K dt$

$\Rightarrow \frac{1}{M}\left[\ln(y-M) - \ln(y)\right] = -Kt + C_1$

$$\Rightarrow \frac{1}{M}\ln\left(\frac{y-M}{y}\right) = -Kt + C_1$$

$$\Rightarrow \ln\left(\frac{y-M}{y}\right) = -KMt + MC_1$$

$$\Rightarrow \ln\left(\frac{y-M}{y}\right) = -KMt + C_2$$

$$\Rightarrow \frac{y-M}{y} = e^{-KMt + C_2}$$

$$\Rightarrow 1 - \frac{M}{y} = e^{C_2} e^{-KMt}$$

$$\Rightarrow 1 - \frac{M}{y} = Ce^{-KMt}$$

$$\Rightarrow \frac{M}{y} = 1 - Ce^{-KMt}$$

$$\Rightarrow y = \frac{M}{1 - Ce^{-KMt}}$$

Observe $\lim_{t \to \infty} y = M$

Another common use of differential equations is to model how the amount of "stuff" in a medium changes over time.

Example - the amount of salt in a solution of saltwater

Such DEs typically have the following form:

$$* \quad \frac{dA}{dt} = \text{rate coming in} - \text{rate going out}$$

where $A$ is the amount of "stuff" in the medium at time $t$.

When the medium in question is a liquid/solution, we can define this DE even further as:

$$\frac{dA}{dt} = (\text{inflow rate})(\text{concentration of inflow}) - (\text{outflow rate})(\text{concentration of outflow})$$

where concentration is defined as the amount of "stuff" per unit volume

ie $\text{concentration} = \frac{\text{amount}}{\text{volume}}$

Example - Consider a tank with 100 L of pure water (ie no salt). Suppose a saltwater solution (brine) containing 0.5 Kg/L of salt is being poured into the ~~mixer~~ tank at a rate of 8 L/min. The tank is kept well-stirred, and the mixture flows out of the tank at the same rate (ie 8 L/min)

1) Set up the DE modeling this scenario

2) Sketch direction field

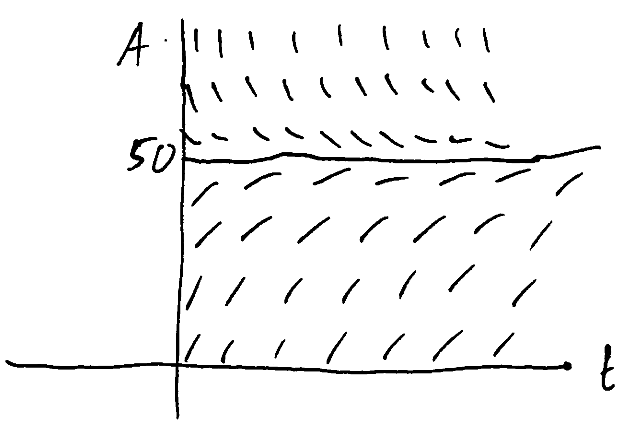
1) $$\frac{dA}{dt} = \binom{\text{inflow}}{\text{rate}}\binom{\text{inflow}}{\text{concentration}} - \binom{\text{outflow}}{\text{rate}}\binom{\text{outflow}}{\text{concentration}}$$

$$\frac{dA}{dt} = 8(.5) - 8\left(\frac{A}{100}\right)$$

$$\frac{dA}{dt} = 4 - \frac{2}{25}A, \quad A(0) = 0$$

3) Solve it as a separable equation

4) Solve it as a linear equation

2) 


3) $$\frac{dA}{dt} = 4 - \frac{2}{25}A \Rightarrow \frac{dA}{dt} = \frac{100-2A}{25}$$

$$\Rightarrow \frac{dA}{dt} = -\frac{2}{25}(A-50)$$

$$\Rightarrow \frac{1}{A-50}\,dA = -\frac{2}{25}\,dt$$

$$\Rightarrow \int \frac{1}{A-50}\,dA = \int -\frac{2}{25}\,dt$$

$$\Rightarrow \ln(A-50) = -\frac{2}{25}t + C_1$$

$$\Rightarrow A - 50 = e^{-\frac{2}{25}t} e^{C_1}$$

$$\Rightarrow A = 50 + Ce^{-2/25 t}$$

$0 = A(0) = 50 + C$ so $C = -50$

Thus $A(t) = 50 - 50e^{-2/25 t}$

4) $\frac{dA}{dt} = 4 - \frac{2}{25}A \Rightarrow \frac{dA}{dt} + \frac{2}{25}A = 4$

$P(t) = \frac{2}{25}$ and $Q(t) = 4$

Integrating factor is $u(t) = e^{\int P(t)dt} = e^{\frac{2}{25}t}$

Then $A = \frac{1}{e^{2/25 t}}\left(\int 4e^{\frac{2}{25}t}\,dt + C\right)$

$$\Rightarrow A = \frac{1}{e^{2/25 t}}\left(50 e^{\frac{2}{25}t} + C\right)$$

$$\Rightarrow A = 50 + Ce^{-2/25 t}$$

Ex - Consider a tank with 100 L of pure water. Suppose a brine solution containing 0.5 kg/L of salt is poured into the tank at a rate of 2 L/min. The tank is kept well-stirred, and the mixture flows out of the tank at a rate of 3 L/min

1) Set up DE to model this scenario

2) Solve

1) $$\frac{dA}{dt} = \begin{pmatrix}\text{inflow}\\ \text{rate}\end{pmatrix}\begin{pmatrix}\text{inflow}\\ \text{concentration}\end{pmatrix} - \begin{pmatrix}\text{outflow}\\ \text{rate}\end{pmatrix}\begin{pmatrix}\text{outflow}\\ \text{concentation}\end{pmatrix}$$

$$\frac{dA}{dt} = 2(0.5) - 3\left(\frac{A}{100-t}\right)$$

$$\frac{dA}{dt} = 1 - \frac{3A}{100-t}, \quad A(0) = 0$$

$$\frac{dA}{dt} + \frac{3}{100-t} A = 1$$

$$P(t) = \frac{3}{100-t} \qquad Q(t) = 1$$

Integrating factor is $u(t) = (100-t)^{-3}$

since $\int \frac{3}{100-t}\, dt = -3 \ln(100-t) = \ln(100-t)^{-3}$

Then $A = \frac{1}{(100-t)^{-3}} \left( \int (100-t)^{-3} dt + C \right)$

$\Rightarrow A = (100-t)^3 \left( \frac{1}{2}(100-t)^{-2} + C \right)$

$\Rightarrow A = \frac{1}{2}(100-t) + C(100-t)^3$

$0 = A(0) = \frac{1}{2}(100) + C(100)^3$

$-50 = C(1 \text{ million}) \Rightarrow C = -\frac{1}{20000}$

So $A(t) = \frac{1}{2}(100-t) - \frac{(100-t)^3}{20000}$

Ex - Consider a tank with 100 L of pure water. Suppose a brine solution containing 0.5 kg/L of salt is poured into the tank at a rate of 4 L/min. The tank is kept well-stirred, and the mixture flows out of the tank at a rate of 3 L/min

1) Write the DE for this scenario
2) Solve

1) $\frac{dA}{dt} = \left(\begin{matrix}\text{inflow} \\ \text{rate}\end{matrix}\right)\left(\begin{matrix}\text{inflow} \\ \text{concentration}\end{matrix}\right) - \left(\begin{matrix}\text{outflow} \\ \text{rate}\end{matrix}\right)\left(\begin{matrix}\text{outflow} \\ \text{conc.}\end{matrix}\right)$

$$\frac{dA}{dt} = 4(0.5) - 3\left(\frac{A}{100+t}\right), \quad A(0)=0$$

$$\frac{dA}{dt} = 2 - \frac{3A}{100+t}$$

$$\frac{dA}{dt} + \frac{3A}{100+t} = 2$$

$$P(t) = \frac{3}{100+t} \qquad Q(t) = 2$$

Since $\int P(t)\,dt = \int \frac{3}{100+t}\,dt = 3\ln(100+t)$,

the integrating factor is $u(t) = e^{\int P(t)\,dt} = (100+t)^3$

Then $$A = \frac{1}{(100+t)^3}\left(\int 2(100+t)^3\,dt + C\right)$$

$$A = \frac{1}{(100+t)^3}\left(\frac{1}{2}(100+t)^4 + C\right)$$

$$A = \frac{1}{2}(100+t) + \frac{C}{(100+t)^3}$$

$$0 = A(0) = \frac{1}{2}(100) + \frac{C}{100^3}$$

$$-50 = \frac{C}{1\text{ million}} \rightarrow C = -50\text{ million}$$

Thus $$A(t) = \frac{1}{2}(100+t) - \frac{50\text{ million}}{(100+t)^3}$$

Ex - Consider a 100 L tank of brine with a concentration of 0.5 kg/L of salt. Suppose pure water is poured into the tank at a rate of

5 L/min. The tank is kept well-stirred, and the mixture flows out of the tank at the same rate

1) Setup DE
2) Solve

1) $\frac{dA}{dt} = \begin{pmatrix}\text{inflow}\\ \text{rate}\end{pmatrix}\begin{pmatrix}\text{inflow}\\ \text{conc.}\end{pmatrix} - \begin{pmatrix}\text{outflow}\\ \text{rate}\end{pmatrix}\begin{pmatrix}\text{outflow}\\ \text{conc.}\end{pmatrix}$

$$\frac{dA}{dt} = 5(0) - 5\left(\frac{A}{100}\right)$$

$$\frac{dA}{dt} = -\frac{1}{20}A, \quad A(0) = 50$$

We saw last time that DEs of this form have general solution

$A(t) = Ce^{-\frac{1}{20}t}$, $\quad 50 = A(0) = C$

Thus $A(t) = 50e^{-\frac{1}{20}t}$