Compute the Laplace transform of

$$f(t) = \begin{cases} 2 & 0 \le t < 4 \\ 0 & 4 \le t < 9 \\ 5 & 9 \le t \end{cases}$$

Give the domain

$$L(f) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^4 2e^{-st}\,dt + \cancel{\int_4^9 0\,dt}^{\,0} + \int_9^\infty 5e^{-st}\,dt$$

$$= \left[-\frac{2}{s}e^{-st}\right]_0^4 + \lim_{N\to\infty}\int_9^N 5e^{-st}\,dt$$

$$= \left[-\frac{2}{s}e^{-4s} + \frac{2}{s}\right] + \lim_{N\to\infty}\left[-\frac{5}{s}e^{-st}\Big|_9^N\right]$$

$$= -\frac{2}{s}e^{-4s} + \frac{2}{s} + \lim_{N\to\infty}\left[-\frac{5}{s}e^{-sN} + \frac{5}{s}e^{-9s}\right]$$

$$= -\frac{2}{s}e^{-4s} + \frac{2}{s} + \frac{5}{s}e^{-9s}, \qquad s > 0$$

$y'' + 4y = -12\sec(2t)$

Use VoP b/c ~~[illegible]~~ right hand side is <u>not</u> e, sin, cos, polynomial.

Step 1: Find y_1, y_2 which solve $y'' + 4y = 0$

Aux eq is $r^2 + 4 = 0 \Rightarrow r = \pm 2i$

So homogeneous sol. is $y_h = c_1\cos(2t) + c_2\sin(2t)$

$y_1 = \cos(2t), \quad y_2 = \sin(2t)$

Step 2: Find $y_p = v_1 y_1 + v_2 y_2$ where

$$v_1 = \int \frac{-y_2(-12\sec(2t))}{y_1 y_2' - y_1' y_2}\,dt$$

$$v_2 = \int \frac{y_1(-12\sec(2t))}{y_1 y_2' - y_1' y_2}\,dt$$

$y_1 = \cos(2t) \Rightarrow y_1' = -2\sin(2t)$

$y_2 = \sin(2t) \Rightarrow y_2' = 2\cos(2t)$

$$y_1 y_2' - y_1' y_2 = 2\cos^2(2t) + 2\sin^2(2t) = 2$$

$$V_1 = \int \frac{-y_2(-12\sec(2t))}{2}\,dt = \int 6\sin(2t)\sec(2t)\,dt$$

$$= \int 6\frac{\sin(2t)}{\cos(2t)}\,dt$$

$$= -3\ln|\cos(2t)|$$

$$V_2 = \int \frac{y_1(-12\sec(2t))}{2}\,dt = \int -6\cos(2t)\sec(2t)\,dt$$

$$= \int -6\,dt$$

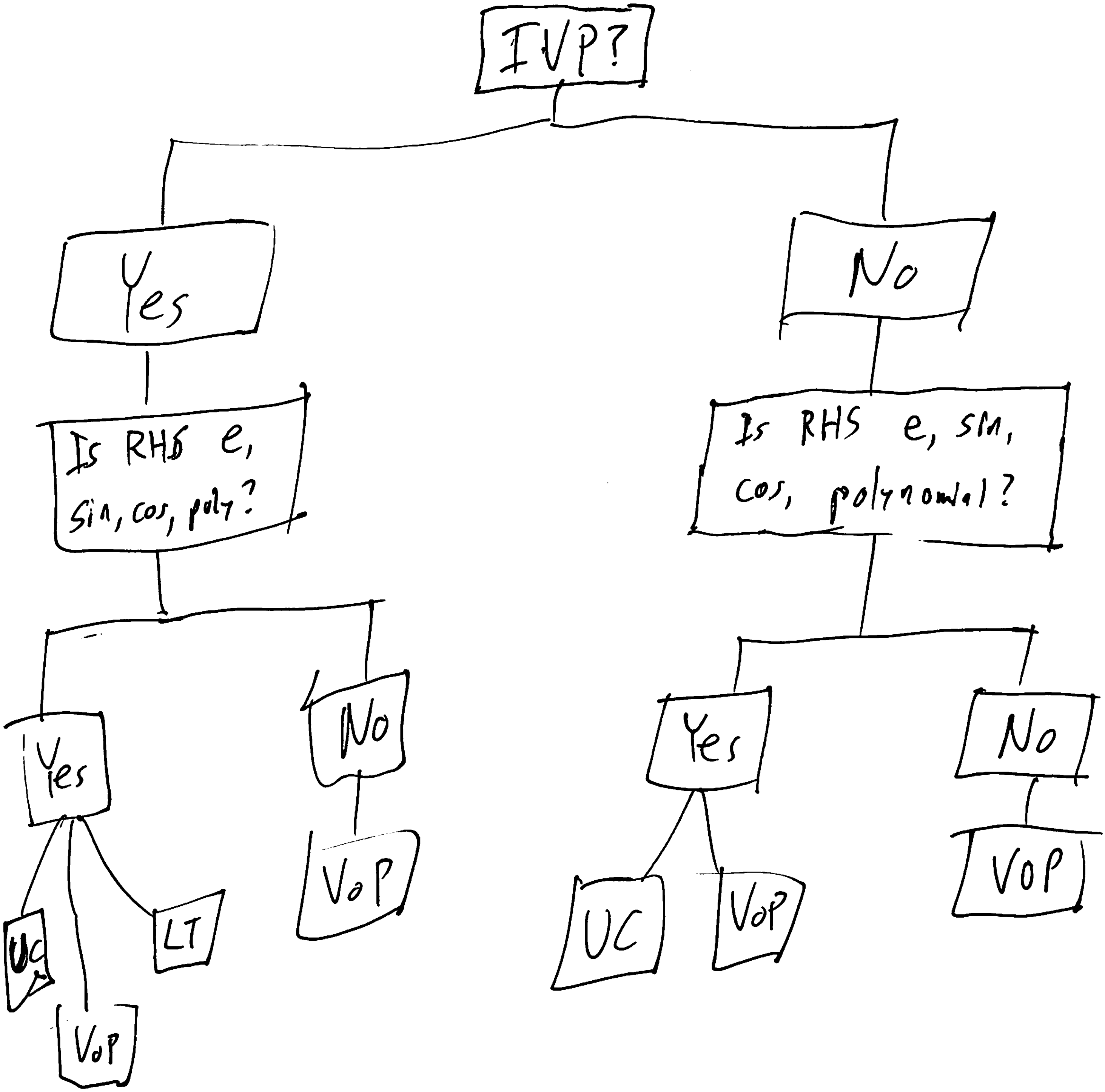
$$= -6t$$

So $y_p = v_1 y_1 + v_2 y_2 = -3\ln|\cos(2t)|\cos(2t) - 6t\sin(2t)$

Step 3! $y = y_h + y_p$

$$y = c_1\cos(2t) + c_2\sin(2t) - 3\ln|\cos(2t)|\cos(2t) - 6t\sin(2t)$$

Which method do I use?



$$y'' - 2y' + 10y = e^{7t}, \quad y(0) = 1, \quad y'(0) = 2$$

$$L(y'' - 2y' + 10y) = L(e^{7t})$$

$$L(y'') - 2L(y') + 10L(y) = \frac{1}{s-7}$$

$$s^2L(y) - sy(0) - y'(0) - 2\left(sL(y) - y(0)\right) + 10L(y) = \frac{1}{s-7}$$

$$s^2L(y) - s - 2 - 2sL(y) + 2 + 10L(y) = \frac{1}{s-7}$$

$$(s^2 - 2s + 10)L(y) = s + \frac{1}{s-7} = \frac{s^2 - 7s + 1}{s-7}$$

$$L(y) = \frac{s^2 - 7s + 1}{(s-7)(s^2 - 2s + 10)}$$

$$\frac{s^2 - 7s + 1}{(s-7)(s^2-2s+10)} = \frac{A}{s-7} + \frac{Bs+C}{s^2-2s+10}$$

$$s^2 - 7s + 1 = A(s^2 - 2s + 10) + (Bs + C)(s-7)$$

Plug in $s=7$ to get $1 = 45A$ so $A = \frac{1}{45}$

Plug in $s=0$ to get $1 = \frac{10}{45} - 7C$

$$1 - \frac{2}{9} = -7C$$

$$\frac{7}{9} = -7C \quad \text{so} \quad C = -\frac{1}{9}$$

Plug in $s=1$ to get $-5 = \frac{1}{5} + (B - \frac{1}{9})(-6)$

$$-5 = \frac{1}{5} - 6B + \frac{2}{3}$$

$$6B = \frac{75}{15} + \frac{3}{15} + \frac{10}{15} = \frac{88}{15}$$

$$B = \frac{88}{90} = \frac{44}{45}$$

Thus, $$L(y) = \frac{1/45}{s-7} + \frac{44/45\, s - \frac{1}{9}}{s^2 - 2s + 10}$$

$$= \frac{1/45}{s-7} + \frac{44/45\, s - \frac{1}{9}}{(s-1)^2 + 9}$$

Note: $$s^2 + bs + c = s^2 + bs + \frac{b^2}{4} + c - \frac{b^2}{4}$$

$$= \left(s + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}$$

$$L(y) = \frac{1/45}{s-7} + \frac{\frac{44}{45}(s-1) + \frac{39}{45}}{(s-1)^2 + 9}$$

$$= \frac{1/45}{s-7} + \frac{44}{45} \cdot \frac{s-1}{(s-1)^2+9} + \frac{13}{45} \cdot \frac{3}{(s-1)^2+9}$$

$$y = \frac{1}{45} L^{-1}\left(\frac{1}{s-7}\right) + \frac{44}{45} L^{-1}\left(\frac{s-1}{(s-1)^2+9}\right) + \frac{13}{45} L^{-1}\left(\frac{3}{(s-1)^2+9}\right)$$

$$y = \frac{1}{45} e^{7t} + \frac{44}{45} e^{t} \cos(3t) + \frac{13}{45} e^{t} \sin(3t)$$

$$y'' - 4y' + 3y = e^{t}$$

General solution using UC

Step 1: Find y_h general solution to $y'' - 4y' + 3y = 0$

Aux eq $\quad r^2 - 4r + 3 = 0 \Rightarrow (r-3)(r-1) = 0$

$\Rightarrow r = 3, r = 1$

So $\quad y_h = c_1 e^{3t} + c_2 e^{t}$

Step 2: Find y_p

Base case is $\quad y_p = Ae^{t}$

Aside: If RHS was te^t, we would start with $y_p = (At+B)e^t$

Since 1 is a root of the auxiliary equation, we multiply through by t ie $y_p = Ate^t$

$y_p' = Ate^t + Ae^t$

$y_p'' = Ate^t + 2Ae^t$

Plug into the DE $y'' - 4y' + 3y = e^t$ to get

$Ate^t + 2Ae^t - 4Ate^t - 4Ae^t + 3Ate^t = e^t$

$-2Ae^t = e^t$

$-2A = 1$

$A = -1/2$

Thus $y_p = -\frac{1}{2}te^t$

Step 3: $y = y_h + y_p = c_1e^{3t} + c_2e^t - \frac{1}{2}te^t$

$$x'' - 2x' + x = 24t^2e^t$$

Aux eq is $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$ double root

For y_p, start with $y_p = (At^2 + Bt + C)e^t$

Since $r=1$ is double root of aux eq., we multiply through by t^2

ie $y_p = t^2(At^2 + Bt + C)e^t = (At^4 + Bt^3 + Ct^2)e^t$

Undamped mass-spring system is

$my'' + ky = 0$

Solution is $y = C_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$

Natural frequency $= \dfrac{\sqrt{\frac{k}{m}}}{2\pi}$

period $= \dfrac{2\pi}{\sqrt{\frac{k}{m}}}$

Damped motion is $my'' + by' + Ky = 0$

Distinct real roots $\Rightarrow$ Overdamping

single repeated root $\Rightarrow$ critical damping

complex roots $\Rightarrow$ underdamping

$$\frac{3}{(2s+5)^3} = \text{[illegible]} + \cancel{\frac{B}{2s+5}}$$

Recall Inverse Laplace of $\frac{n!}{(s-a)^{n+1}}$ is $t^n e^{at}$

$$\frac{3}{(2s+5)^3} = \frac{3}{\left[2(s+\frac{5}{2})\right]^3} = \frac{3}{8(s+\frac{5}{2})^3} = \frac{3/8}{(s+5/2)^3}$$

$$= \frac{3}{16} \cdot \frac{2}{(s+\frac{5}{2})^3}$$

$$\Rightarrow L^{-1} = \frac{3}{16} L^{-1}\left(\frac{2}{(s+\frac{5}{2})^3}\right)$$

$$= \frac{3}{16} t^2 e^{-\frac{5}{2}t}$$