A <u>differential equation</u> is an equation involving derivatives

Ex- $\frac{dy}{dx} = 2x \Rightarrow y = x^2 + C$

Observe infinitely many solutions.
Can get a single solution if we specify an "initial condition"

In this equation, there are two variables. The variable whose derivative we are taking (in this case y) is the <u>dependent variable</u>. The variable that we are taking derivatives with respect to (in this case x), is the <u>independent variable</u>.

Example (compound Interest)

Recall that if P dollars is initially invested at interest rate r, compounded continuously, then the amount A in the account at time t is $A = Pe^{rt}$

Observe $\frac{dA}{dt} = rPe^{rt} = rA$

Hence, the differential equation modelling compound Interest is $\frac{dA}{dt} = rA$

Also infinitely many solutions (P is arbitrary)

Differential equations can involve higher-order derivatives as well.

Ex- Let s be the position of a projectile at time t. Then $\frac{d^2s}{dt^2}$ is the acceleration of the object.

If we assume constant downward acceleration due to gravity ($-32\ ft/sec^2$), then our DE is $\frac{d^2s}{dt^2} = -32$

$$\Rightarrow \quad \frac{ds}{dt} = -32t + c_1$$

$$\Rightarrow \quad s(t) = -16t^2 + c_1 t + c_2$$

The <u>order</u> of a DE is the order of the highest-order derivative in the equation.

Ex- $\frac{dy}{dx} = 2x$ and $\frac{dA}{dt} = rA$ both first order. $\frac{d^2s}{dt^2} = -32$ is second-order

If a DE has order n, then the general solution will have n arbitrary constants. We thus sometimes refer to the general solution as an n-parameter family of solutions.

Each DE thus far has involved ordinary derivatives with respect to a single independent variable. We call such equations <u>ordinary differential equations.</u>

A DE involving partial derivatives is called a <u>partial differential equation</u>

Example- The one-dimensional heat equation says that the temperature u over an interval is modeled by

$u_t = \alpha u_{xx}$ where t is time, x is position, and α is a constant.

An ordinary differential equation is <u>linear</u> if it is of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x).$$

That is, the dependent variable and its derivatives appear as additive combinations of their first powers.

Ex - $\frac{dy}{dx} = 2x$, $\underbrace{\frac{dA}{dt} = rA}_{\frac{dA}{dt} - rA = 0}$, $\frac{d^2s}{dt^2} = -32$

all linear

Ex - Determine if the following are linear or nonlinear

1) $e^x \frac{d^2y}{dx^2} - \cos(x)\frac{dy}{dx} + y = \ln(x)$

2) $\frac{dy}{dx} + y^3 = 4x^2$

3) $y\frac{dy}{dx} = \frac{7}{x}$

1) linear 2) nonlinear 3) nonlinear

An ODE is an equation involving a dependent variable and its derivatives with respect to an independent variable.

An <u>explicit solution</u> to the ODE is a function $y(x)$ that, when substituted for the dependent variable in the ODE, preserves equality.

Ex - 1) $x \frac{dy}{dx} = 2y$ $\quad y(x) = x^2$

2) $\frac{d^2y}{dx^2} + y = 0$ $\quad y(x) = 2\cos(x) - 3\sin(x)$

3) $\frac{dy}{dx} + 2y = 0$ $\quad y(x) = Ce^{-2x}$
(C a constant)

Show that these are solutions

1) $y = x^2 \Rightarrow \frac{dy}{dx} = 2x$

So LHS is $x(2x) = 2x^2$ $\quad =$

And RHS is $2(x^2) = 2x^2$

2) $y = 2\cos(x) - 3\sin(x) \Rightarrow \frac{dy}{dx} = -2\sin(x) - 3\cos(x)$

$\Rightarrow \frac{d^2y}{dx^2} = -2\cos(x) + 3\sin(x)$

So LHS is $(-2\cos(x) + 3\sin(x)) + (2\cos(x) - 3\sin(x))$
$= 0$

And RHS is 0 $=$

3) $y = Ce^{-2x} \Rightarrow \frac{dy}{dx} = -2Ce^{-2x}$

So LHS is $-2Ce^{-2x} + 2Ce^{-2x} = 0$

And RHS is 0 $=$

We can't always express solutions explicitly
i.e. "y in terms of x"

In these cases, we define solutions implicitly i.e. in the form $G(x,y) = 0$

Ex- Show $x + y + e^{xy} = 0$ satisfies

$$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0 \qquad *$$

$$\frac{d}{dx}(x + y + e^{xy}) = \frac{d}{dx}(0)$$

$$1 + \frac{dy}{dx} + e^{xy}\left(x\frac{dy}{dx} + y\right) = 0$$

$$1 + \frac{dy}{dx} + xe^{xy}\frac{dy}{dx} + ye^{xy} = 0$$

$$1 + (1 + xe^{xy})\frac{dy}{dx} + ye^{xy} = 0 \quad *$$

Ex - 1) Show $x^2 - \sin(x+y) = 1$ satisfies

$$\frac{dy}{dx} = 2x\sec(x+y) - 1 \quad ✓$$

2) Show $x^2 - y^2 = C$ satisfies

$$\frac{dy}{dx} = \frac{x}{y} \qquad (C \text{ a constant})$$

1) $$\frac{d}{dx}\left(x^2 - \sin(x+y)\right) = \frac{d}{dx}(1)$$

$$2x - \cos(x+y)\left(1 + \frac{dy}{dx}\right) = 0$$

$$2x = \cos(x+y)\left(1 + \frac{dy}{dx}\right)$$

$$1 + \frac{dy}{dx} = \frac{2x}{\cos(x+y)} = 2x\sec(x+y)$$

$$\frac{dy}{dx} = 2x\sec(x+y) - 1 \quad ✓$$

2) $$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(c)$$

$$2x - 2y\frac{dy}{dx} = 0$$

$$2x = 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x}{y} \quad \checkmark$$

Observe that arbitrary constants appear often when solving ODEs, implying infinitely many solutions.

We usually identify a single solution by specifying initial conditions.

Can we always do this?

Definition - Solving an <u>initial value problem</u> for an n-th order DE means finding a solution to the DE that satisfies n initial conditions on the dependent variable and its derivatives up to order $n-1$.

Ex- 1) $\frac{dy}{dx} = 2x, \quad y(3) = 10$

2) $\frac{d^2s}{dt^2} = -32$, $s(0)=1$, $\frac{ds}{dt}(0) = 7$

3) $\frac{dy}{dx} + 2y = 0$, $y(0) = 14$

1) General solution is $y = x^2 + C$

$10 = y(3) = 3^2 + C = 9 + C$

$\Rightarrow C = 1$ ie $y = x^2 + 1$

2) General solution is $s = -16t^2 + C_1 t + C_2$

$\Rightarrow \frac{ds}{dt} = -32t + C_1$

$7 = \frac{ds}{dt}(0) = -32(0) + C_1 = C_1$

$1 = s(0) = -16(0)^2 + C_1(0) + C_2 = C_2$

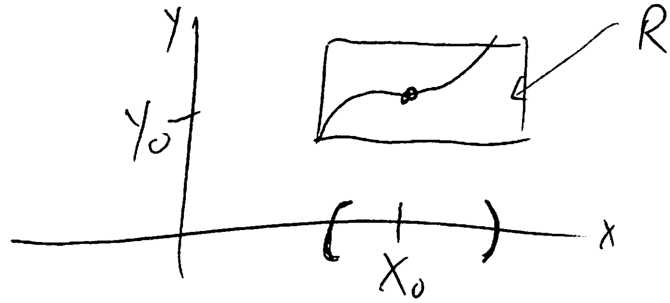
So $s(t) = -16t^2 + 7t + 1$

3) General solution is $y = Ce^{-2x}$

$14 = y(0) = Ce^{-2(0)} = C$

So $y = 14e^{-2x}$

Let $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ be a first-order initial value problem. If f and $\frac{\partial f}{\partial y}$ are both continuous in a region $R \subseteq \mathbb{R}^2$ containing (x_0, y_0), then the IVP has a unique solution in some interval around x_0.



Examples - 1) $\frac{dy}{dx} = y^4 - x^4, \quad y(0) = 7$

2) $3y\frac{dy}{dx} + 4x = 0, \quad y(2) = -3$

3) $y\frac{dy}{dx} = x, \quad y(1) = 0$

4) $\frac{dy}{dx} = xy^{2/3}, \quad y(5) = 0$

1) $f(x,y) = y^4 - x^4 \Rightarrow$ continuous at and around $(0,7)$

$\frac{\partial f}{\partial y} = 4y^3 \Rightarrow$ cont. at and around $(0,7)$

Thus, IVP has a unique solution around $x = 0$

2) $3y\frac{dy}{dx} + 4x = 0 \Rightarrow \frac{dy}{dx} = \frac{-4x}{3y}$

$f(x,y) = \frac{-4x}{3y}$; continuous at and around $(2,-3)$

$\frac{\partial f}{\partial y} = \frac{4x}{3y^2}$; cont. at and around $(2,-3)$

Thus, IVP has a unique solution around $x = 2$

3) $y\frac{dy}{dx} = x \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

$f(x,y) = \frac{x}{y}$; not continuous at $(1,0)$

Thus, there does not exist a unique solution

4) $f(x,y) = xy^{2/3}$; cont. at and around $(5,0)$

$\frac{\partial f}{\partial y} = \frac{2}{3}xy^{-1/3} = \frac{2x}{3y^{1/3}}$; not cont. at $(5,0)$

Thus, there does not exist a unique solution.

Once existence / uniqueness has been determined, we can often determine the largest interval for x on which a solution exists and is unique. This is done by determining the largest interval containing x_0 for which f is continuous

Ex. $\frac{dy}{dx} = \frac{y}{x-7}$, $y(0) = 4$

The largest interval for which solution is unique is $(-\infty, 7)$

Ex - $\frac{dy}{dx} = \frac{6y}{x^2-4}$ with

1) $y(0) = 3$
2) $y(5) = 1$
3) $y(-11) = 8$

f is continuous on $(-\infty, -2)$, $(-2, 2)$, $(2, \text{infinity})$

1) The largest interval containing $x=0$ is $\underline{(-2,2)}$

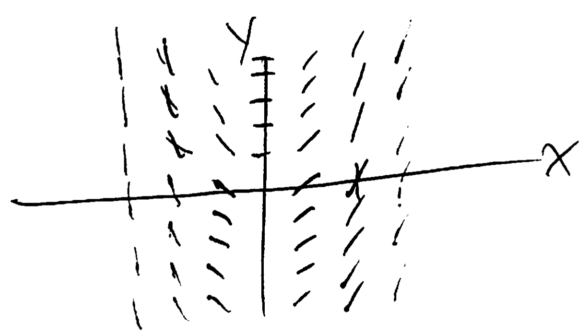
2) The largest interval containing $x=5$ is $(2,\infty)$

3) The largest interval containing $x=-11$ is $(-\infty,-2)$

Consider a DE $\frac{dy}{dx} = f(x,y)$. It follows that $f(x_0, y_0)$ is the slope of solutions y at the point (x_0, y_0).

A plot in the xy plane consisting of short line segments showing the slope at different points is called a <u>direction field</u> for the DE

Ex - $\frac{dy}{dx} = 2x$



1) $\frac{dy}{dx} = y$

2) $\frac{dy}{dx} = \frac{-x}{y}$