Recall that derivatives of single-variable functions are defined using limits.

We need to extend the concept of limits of multivariable functions to define derivatives of multivariable functions.

Suppose we want to compute $\lim_{(x,y)\to(2,-1)} e^{\frac{x^2}{4}+y}$ in the x-direction.

We can fix $y=-1$ and compute $\lim_{x\to 2} e^{\frac{x^2}{4}-1} = 1$

Likewise, to compute the limit in the y-direction, fix $x=2$ and compute $\lim_{y\to -1} e^{1+y} = 1$

To compute the derivative of $f(x,y)$ with respect to x, we can treat y as a constant and differentiate x terms as usual. This is called the partial derivative of f with respect to x, denoted f_x or $\frac{\partial f}{\partial x}$.

To compute the derivative of $f(x,y)$ with respect to y, we treat x as a constant and differentiate y terms as usual. This is the partial derivative of f with respect to y, denoted f_y or $\frac{\partial f}{\partial y}$.

Examples - 1) $f(x,y) = x^2 + y^2$

$f_x = 2x$ $\qquad$ $f_y = 2y$

2) $f(x,y) = x^2 y^2$

$f_x = 2xy^2$ $\qquad$ $f_y = 2yx^2$

3) $f(x,y) = x^3 + 6x^2y - 5xy^2 + 7xy - 2y^3 + 3x + 3y - 8$

$f_x = 3x^2 + 12xy - 5y^2 + 7y + 3$ | $f_y = 6x^2 - 10xy + 7x - 6y^2 + 3$

4) $f(x,y) = x^2 \sin(y)$

$f_x = 2x\sin(y)$ $\qquad$ $f_y = x^2\cos(y)$

5) $f(x,y) = \frac{x}{y}$

$f_x = \frac{1}{y}$ $\qquad f_y = \frac{-x}{y^2}$

6) $f(x,y) = y\ln(x)$

$f_x = \frac{y}{x}$ $\qquad f_y = \ln(x)$

7) $f(x,y) = y^3 e^x$

$f_x = y^3 e^x$ $\qquad f_y = 3y^2 e^x$

8) $f(x,y) = e^{x^2+y^2}$

$f_x = 2x e^{x^2+y^2}$ $\qquad f_y = 2y e^{x^2+y^2}$

9) $f(x,y) = \cos(3x+5y)$

$f_x = -3\sin(3x+5y)$ $\qquad f_y = -5\sin(3x+5y)$

For functions of 3, 4, etc. variables, the idea is the same.

Example- $f(x,y,z) = 3x^2 + 4xy^2 - 2xyz + z^2 - zy + 8$

$f_x = 6x + 4y^2 - 2yz$ $\qquad f_y = 8xy - 2xz - z$ $\qquad f_z = -2xy + 2z - y$

We can compute higher order (ie second, third, etc.)

derivatives of multivariable functions as well.

For a two variable function $f(x,y)$:

We can compute the partial derivative of f_x with respect to x, denoted f_{xx} or $\frac{\partial^2 f}{\partial x^2}$

We can compute the partial derivative of f_x with respect to y, denoted f_{xy} or $\frac{\partial^2 f}{\partial y \partial x}$

We can compute the partial derivative of f_y with respect to x, denoted f_{yx} or $\frac{\partial^2 f}{\partial x \partial y}$

We can compute the partial derivative of f_y with respect to y, denoted f_{yy} or $\frac{\partial^2 f}{\partial y^2}$

Example - $f(x,y) = x^3 + 6x^2y - 5xy^2 + 7xy - 2y^3 + 3x + 3y - 8$

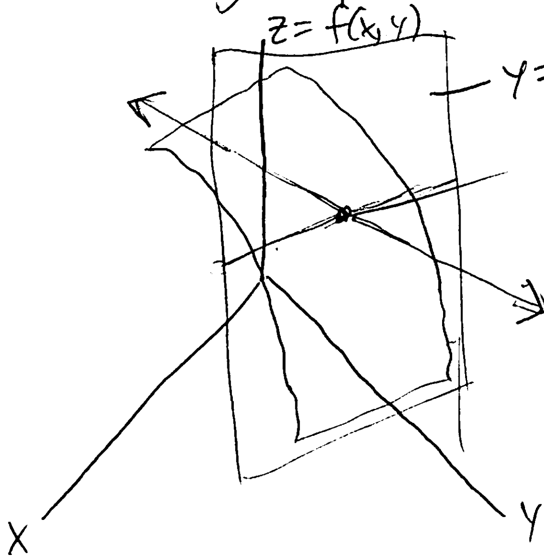
$$f_x = 3x^2 + 12xy - 5y^2 + 7y + 3$$

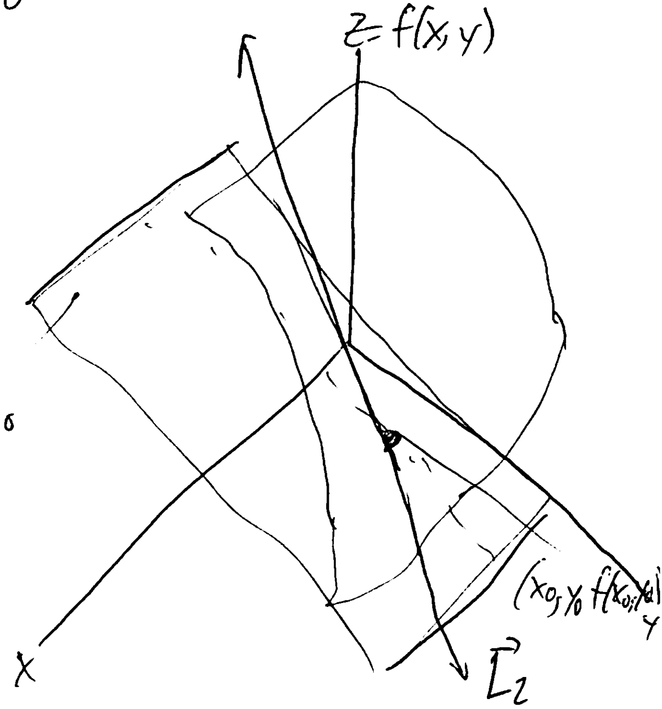
$$f_y = 6x^2 - 10xy + 7x - 6y^2 + 3$$

$$f_{xx} = 6x + 12y \qquad f_{xy} = 12x - 10y + 7$$

$$f_{yx} = 12x - 10y + 7 \qquad f_{yy} = -10x - 12y$$

Recall that for a single variable function $f(x)$, $f'(x_0)$ gives us the slope of <u>the</u> tangent line to the graph of f at X_0.





$\vec{L}_1$ has direction vector

$$\langle 1, 0, f_x(x_0, y_0) \rangle$$

$\vec{L}_2$ has direction vector $\langle 0, 1, f_y(x_0, y_0) \rangle$

Example - Find the two tangent lines to $f(x,y) = x^2 - 2xy + y^2$ at the point $(3,4)$

Point is $(3, 4, f(3,4)) = (3, 4, 1)$

$f_x = 2x - 2y \Rightarrow f_x(3,4) = -2$

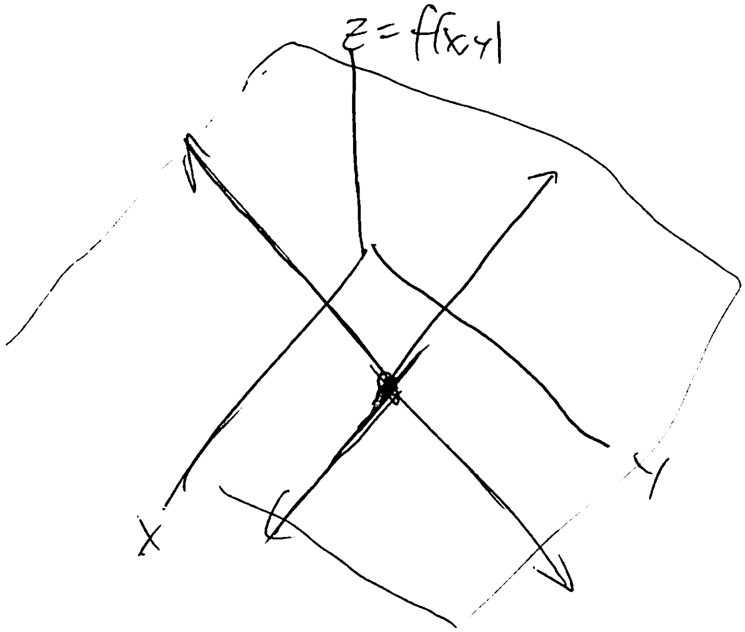
$f_y = -2x + 2y \Rightarrow f_y(3,4) = 2$

$\vec{L}_1$ has direction vector $\langle 1, 0, f_x(3,4) \rangle = \langle 1, 0, -2 \rangle$

$\vec{L}_2$ has direction vector $\langle 0, 1, f_y(3,4) \rangle = \langle 0, 1, 2 \rangle$

$\vec{L}_1 : \langle 3+t,\ 4,\ 1-2t \rangle$

$\vec{L}_2 : \langle 3,\ 4+t,\ 1+2t \rangle$



The <u>tangent plane</u> to the graph of $f(x,y)$ at the point $(x_0, y_0, f(x_0, y_0))$ is the plane containing this point and the two tangent lines to f at this point. It has normal vector $\vec{n} = \langle 1, 0, f_x(x_0,y_0) \rangle \times \langle 0, 1, f_y(x_0,y_0) \rangle$

Example. $f(x,y) = x^2 - 2xy + y^2$, $(x_0, y_0) = (3, 4)$

Point is $(3, 4, f(3,4)) = (3, 4, 1)$

$\vec{n} = \langle 1, 0, -2 \rangle \times \langle 0, 1, 2 \rangle = \langle 2, -2, 1 \rangle$

So plane equation is $2(x-3)-2(y-4)+(z-1)=0$

$\Rightarrow 2x-6-2y+8+z-1=0$

$\Rightarrow 2x-2y+z=-1$

The normal line to the graph of $f(x,y)$ at the point $(x_0, y_0, f(x_0, y_0))$ is the line through this point that is perpendicular to the tangent plane at that point.

The direction vector of the line is the normal vector of the tangent plane.

Example- $f(x,y) = x^2 - 2xy + y^2$ at $(3,4)$

Point is $(3,4,1)$. Direction vector is $\langle 2,-2,1 \rangle$

So normal line is $\langle 3+2t, 4-2t, 1+t \rangle$

Recall that the linearization (or tangent line approximation) of $f(x)$ at $x = x_0$ is

$L(x) = f(x_0) + f'(x_0)(x-x_0)$. It allows us to approximate $f(x)$ for values of x "near" x_0.

The linearization of $f(x,y)$ at (x_0, y_0) is

$$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example - Find the linearization of $f(x,y) = \sqrt{x^2 + y^2}$ at $(3,4)$. Then approximate $\sqrt{(3.1)^2 + (4.1)^2}$

$$f(3,4) = \sqrt{3^2 + 4^2} = 5$$

$$f_x = \frac{x}{\sqrt{x^2+y^2}} \Rightarrow f_x(3,4) = \frac{3}{\sqrt{3^2+4^2}} = \frac{3}{5}$$

$$f_y = \frac{y}{\sqrt{x^2+y^2}} \Rightarrow f_y(3,4) = \frac{4}{\sqrt{3^2+4^2}} = \frac{4}{5}$$

$$L(x,y) = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$$

$$\sqrt{(3.1)^2 + (4.1)^2} = f(3.1, 4.1) \approx L(3.1, 4.1)$$

$$= 5 + \frac{3}{5}(3.1 - 3) + \frac{4}{5}(4.1 - 4)$$

$$= 5 + \frac{3}{5}\left(\frac{1}{10}\right) + \frac{4}{5}\left(\frac{1}{10}\right)$$

$$= 5 + \frac{3}{50} + \frac{4}{50}$$

$$= 5 + \frac{7}{50}$$

$$= 5.14$$

The <u>gradient</u> of $f(x,y)$ is the vector $\nabla f = \langle f_x, f_y \rangle$

Example - $f = 6ye^{x^2} + y^3 \sin(x)$.

Find ∇f and $\nabla f(0,2)$

$f_x = 12xye^{x^2} + y^3 \cos(x)$

$f_y = 6e^{x^2} + 3y^2 \sin(x)$

$\nabla f = \langle 12xye^{x^2} + y^3\cos(x),\ 6e^{x^2} + 3y^2 \sin(x) \rangle$

$\nabla f(0,2) = \langle 8, 6 \rangle$

Likewise for functions of 3, 4, etc. variables

e.g for $f(x,y,z)$, $\nabla f = \langle f_x, f_y, f_z \rangle$

Gradient properties - For differentiable multivariable functions f and g, and a differentiable single-variable function h:

1) $\nabla(f \pm g) = \nabla f \pm \nabla g$

2) $\nabla(cf) = c\nabla f$ for all constants $c \in \mathbb{R}$

3) $\nabla(fg) = f\nabla g + g\nabla f$

4) $\nabla(h \circ f) = (h' \circ f)\nabla f$

Example - Verify 4) for $f(x,y) = x^2 + y^2$ and $h(w) = w^2$

$$h \circ f = (x^2 + y^2)^2$$

$$(h \circ f)_x = 2(x^2+y^2)(2x) = 4x(x^2+y^2)$$

$$(h \circ f)_y = 2(x^2+y^2)(2y) = 4y(x^2+y^2)$$

$$\nabla(h \circ f) = \langle 4x(x^2+y^2),\ 4y(x^2+y^2) \rangle$$

$$h'(w) = 2w \Rightarrow (h' \circ f) = 2(x^2+y^2)$$

$=$

$$\nabla f = \langle f_x, f_y \rangle = \langle 2x, 2y \rangle$$

$$\Rightarrow (h' \circ f)\nabla f = 2(x^2+y^2)\langle 2x, 2y \rangle$$

$$= \langle 4x(x^2+y^2),\ 4y(x^2+y^2) \rangle$$

The derivative of a multivariable function f in the direction of a vector $\vec{v}$ is called a directional derivative. It is computed as $D_{\vec{v}} f = \nabla f \cdot \frac{\vec{v}}{\|\vec{v}\|}$

Example - 1) $f(x,y) = 6ye^{x^2} + y^3 \sin(x)$, $\vec{v} = \langle 3, 4 \rangle$

Compute $D_{\vec{v}} f$ and $D_{\vec{v}} f(0, 2)$

2) For a general function $f(x,y)$, compute $D_{\langle 1,0 \rangle} f$ and $D_{\langle 0,1 \rangle} f$

1) $\nabla f = \langle 12xye^{x^2} + y^3\cos(x),\ 6e^{x^2} + 3y^2 \sin(x) \rangle$

$\|\vec{v}\| = 5 \Rightarrow \frac{\vec{v}}{\|\vec{v}\|} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

$D_{\vec{v}} f = \nabla f \cdot \frac{\vec{v}}{\|\vec{v}\|} = \frac{3}{5}(12xye^{x^2} + y^3\cos(x)) + \frac{4}{5}(6e^{x^2} + 3y^2 \sin(x))$

$D_{\vec{v}} f(0, 2) = \frac{3}{5}(8) + \frac{4}{5}(6) = \frac{48}{5}$

2) $\nabla f = \langle f_x, f_y \rangle$ $\quad \|\langle 1,0 \rangle\| = 1 = \|\langle 0,1 \rangle\|$

$D_{\langle 1,0 \rangle} f = \langle f_x, f_y \rangle \cdot \langle 1, 0 \rangle = f_x$

$D_{\langle 0,1 \rangle} f = \langle f_x, f_y \rangle \cdot \langle 0, 1 \rangle = f_y$

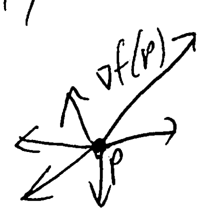
The directional derivative $D_{\vec{v}}f$ tells us the rate of change of f in the direction of $\vec{v}$.

Directional derivative properties: For differentiable multivariable functions f and g, a vector $\vec{v}$, and a differentiable single-variable function h:

1) $D_{\vec{v}}(f \pm g) = D_{\vec{v}}f \pm D_{\vec{v}}g$
2) $D_{\vec{v}}(cf) = c D_{\vec{v}}f$ for all constants $c \in \mathbb{R}$
3) $D_{\vec{v}}(fg) = f D_{\vec{v}}g + g D_{\vec{v}}f$
4) $D_{\vec{v}}(h \circ f) = (h' \circ f) D_{\vec{v}}f$

Let f be a differentiable multivariable function and P a point where $\nabla f(P) \neq \vec{0}$.

There is a unique gradient vector $\nabla f(P)$ at P, but infinitely many other direction vectors $\vec{v}$ at P.



The directional derivative (and hence the rate of change) of f at P varies with $\vec{v}$.

When is the rate of change greatest (i.e. greatest increase/decrease)?

When is the rate of change 0 (constant)?

Recall $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$ where θ is the angle between $\vec{a}$ and $\vec{b}$.

$$D_{\vec{v}} f(P) = \nabla f(P) \cdot \frac{\vec{v}}{\|\vec{v}\|} = \|\nabla f(P)\| \left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| \cos(\theta)$$
$$= \|\nabla f(P)\| \cos(\theta)$$

where θ is the angle between $\nabla f(P)$ and $\vec{v}$.

Since $\|\nabla f(P)\|$ is fixed, the directional derivative (and hence rate of change) of f at P is determined primarily by θ.

In particular, since $-1 \leq \cos(\theta) \leq 1$, we have
$$-\|\nabla f(P)\| \leq D_{\vec{v}} f(P) \leq \|\nabla f(P)\|$$

Thus, the greatest rate of ~~decrea~~ increase of f at P is $\|\nabla f(P)\|$.

This occurs when $\cos(\theta) = 1$ ie $\theta = 0$ ie. $\vec{v}$ points in the direction of $\nabla f(P)$

The greatest rate of decrease of f at P is $-\|\nabla f(P)\|$. This occurs when $\cos(\theta) = -1$ ie $\theta = \pi$ ie $\vec{v}$ points in the direction opposite of $\nabla f(P)$.

Furthermore, rate of change is 0 (constant) when $\cos(\theta) = 0$ ie $\theta = \frac{\pi}{2}$ ie $\vec{v}$ is orthogonal to $\nabla f(P)$.

Example- $f(x,y) = 6ye^{x^2} + y^3\sin(x)$. $P = (0,2)$.

1) Find the greatest rate of increase and decrease of f at P, and the directions at which these occur.

2) Find two directions where the rate of change of f at P is 0.

1) $\nabla f(P) = \langle 8, 6 \rangle \Rightarrow \|\nabla f(P)\| = 10$

So greatest rate of increase is 10, in the direction of $\nabla f(p)$ so direction is $\frac{\nabla f(p)}{\|\nabla f(p)\|}$

$= \frac{1}{10}\langle 8, 6\rangle = \langle \frac{4}{5}, \frac{3}{5}\rangle$.

And greatest rate of decrease is -10, in the direction opposite to $\nabla f(p)$ (ie $-\nabla f(p)$) So direction is $\frac{-\nabla f(p)}{\|\nabla f(p)\|} = \frac{1}{10}\langle -8, -6\rangle = \langle -\frac{4}{5}, -\frac{3}{5}\rangle$

2) We find vectors that are orthogonal to $\langle 8,6\rangle$, say $\langle -6, 8\rangle$ and $\langle 6, -8\rangle$ so directions are $\frac{\langle -6, 8\rangle}{\|\langle -6, 8\rangle\|} = \langle -\frac{3}{5}, \frac{4}{5}\rangle$ and

$$\frac{\langle 6, -8\rangle}{\|\langle 6, -8\rangle\|} = \langle \frac{3}{5}, \frac{-4}{5}\rangle$$

Let $f(x,y)$ be a differentiable function, and let $x(t)$ and $y(t)$ be single-variable functions of t. Then

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}(x(t), y(t))\frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t))\frac{dy}{dt}$$

$$= \nabla f(x(t), y(t)) \cdot \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle$$

Example - $f(x,y) = x^2y^2$, $x(t) = e^{3t}$, $y(t) = \sin(t)$.

$$\frac{\partial f}{\partial x} = 2xy^2 \Rightarrow \frac{\partial f}{\partial x}(x(t), y(t)) = 2e^{3t}\sin^2(t)$$

$$\frac{\partial f}{\partial y} = 2x^2y \Rightarrow \frac{\partial f}{\partial y}(x(t), y(t)) = 2e^{6t}\sin(t)$$

$$\frac{dx}{dt} = 3e^{3t} \quad \text{and} \quad \frac{dy}{dt} = \cos(t).$$

$$\frac{d}{dt}f = 2e^{3t}\sin^2(t)(3e^{3t}) + 2e^{6t}\sin(t)\cos(t)$$
$$= 6e^{6t}\sin^2(t) + 2e^{6t}\sin(t)\cos(t).$$

Likewise for functions of 3, 4, etc. variables

Let $f(x,y)$ be a differentiable function, and let $x(s,t)$ and $y(s,t)$ be differentiable functions of s and t. Then

$$\frac{\partial}{\partial s} f(x(s,t), y(s,t)) = \frac{\partial f}{\partial x}(x(s,t), y(s,t))\frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}(x(s,t), y(s,t))\frac{\partial y}{\partial s}(s,t)$$

$$\frac{\partial}{\partial t} f(x(s,t), y(s,t)) = \frac{\partial f}{\partial x}(x(s,t), y(s,t))\frac{\partial x}{\partial t}(s,t) +$$
$$\frac{\partial f}{\partial y}(x(s,t), y(s,t))\frac{\partial y}{\partial t}(s,t)$$

Example - $f(x,y) = x^2y^2$, $x(s,t) = se^t$, $y(s,t) = t\sin(s)$.

$$\frac{\partial f}{\partial x} = 2xy^2 \Rightarrow \frac{\partial f}{\partial x}(x(s,t), y(s,t)) = 2se^t t^2 \sin^2(s)$$

$$\frac{\partial f}{\partial y} = 2x^2y \Rightarrow \frac{\partial f}{\partial y}(x(s,t), y(s,t)) = 2s^2e^{2t} t\sin(s)$$

$$\frac{\partial x}{\partial s} = e^t \quad \text{and} \quad \frac{\partial y}{\partial s} = t\cos(s)$$

$$\frac{\partial x}{\partial t} = se^t \quad \text{and} \quad \frac{\partial y}{\partial t} = \sin(s)$$

$$\frac{\partial}{\partial s} f = 2se^t t^2 \sin^2(s) e^t + 2s^2e^{2t} t\sin(s)\, t\cos(s)$$

$$\frac{\partial}{\partial t} f = 2se^t t^2 \sin^2(s)\, se^t + 2s^2e^{2t} t\sin(s)\sin(s)$$

Likewise for functions of 3, 4, etc. variables

Recall implicit differentiation from single-variable calculus: Consider $x^2y^2 = 5$ where y is a function of x. Compute $\frac{dy}{dx}$

$x^2\left(2y\frac{dy}{dx}\right) + y^2(2x) = 0 \Rightarrow \frac{dy}{dx} = \frac{-2xy^2}{2x^2y} = \frac{-y}{x}$

We can do implicit differentiation in multivariable calculus as well, using partial derivatives.

Example - $xyz^2 + ye^z = x\sin(z)$ where z is a function of x and y.

Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

For x: $(xy)2z\frac{\partial z}{\partial x} + z^2y + ye^z\frac{\partial z}{\partial x} = x\cos(z)\frac{\partial z}{\partial x} + \sin(z)$

For y: $(xy)2z\frac{\partial z}{\partial y} + z^2x + ye^z\frac{\partial z}{\partial y} + e^z = x\cos(z)\frac{\partial z}{\partial y}$

Rearranging, we get

$$\frac{\partial z}{\partial x} = \frac{\sin(z) - z^2y}{2xyz + ye^z - x\cos(z)}$$

and

$$\frac{\partial z}{\partial y} = \frac{z^2x + e^z}{x\cos(z) - 2xyz - ye^z}$$