CS 294 Topics in Algorithmic Game Theory	**September 27, 2011**

Lecture 5

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1 Approximate Nash equilibria

1.1 Introduction

The solution concept of Nash equilibrium is vastly influential because intellectually it is useful and neat, and philosophically the fact that every game has a Nash equilibrium is reassuring. However, we have also seen that it is intractible unless **PPAD** $\subseteq$ **P**, and this certainly poses a problem to us.

Traditionally, there are three types of remedies to the hardness of a problem. The first is to solve interesting special cases. In fact, we have already touched some work in this domain through examples of zero-sum games and separable network games. In this section, we discuss the second remedy, which is to try to find an approximate solution to the problem. We start with the following definition.

Definition 1. A strategy profile $(x_1, \ldots, x_n)$ is an ϵ-approximate Nash equilibrium if for every player i and every action a in the support of x_i, $\mathrm{E}[u_i(a, x_{-i})] \geq \mathrm{E}[u_i(b, x_{-i})] - \epsilon$ for any action $b \in A_i$.

Certainly, for this definition to make sense, we have to assume that the utility function is normalized ($u_i(\vec{x}) \in [0, 1]$). Note that ϵ acts additively, in contrast to the common practice in the field of approximation algorithms where ϵ usually acts multiplicatively. In fact, it is known that the multiplicative version is **PPAD**-complete [1].

An unsatisfactory aspect of this concept is that unlike approximate solutions in most optimization problems, an approximate Nash equilibrium is hardly useful because once it is found, each player immediately knows how they can achieve better payoffs by deviating from it. Thus, an ϵ-approximate Nash equilibrium is not a compelling proposal unless ϵ is very small. Thus it becomes an important question whether there is an efficient scheme for approximating Nash equilibria. An optimization problem is said to have a PTAS (polynomial-time approximation scheme) if there exists an algorithm that computes the ϵ-approximate solution in time $O(n^{f(1/\epsilon)})$. It is said to have a FPTAS (fully polynomial-time approximation scheme) if there exists an algorithm with time complexity $O((1/\epsilon)^c n^d)$.

In 2006, Chen et al. [2] proved that a FPTAS for Nash equilibria is impossible unless **PPAD** $\subseteq$ **P**. Whether there exists a PTAS for this problem still remains to be a major open question. Currently, the smallest value of ϵ for which we have a polynomial algorithm is 0.3393 [3].

1.2 A quasi-polynomial time approximation scheme for Nash equilibrium

In this section, we introduce a quasi-polynomial time algorithm by Lipton, Markakis and Mehta [4] which computes an ϵ-approximate Nash equilibrium in time $O(n^{\log n/\epsilon^2})$. The algorithm employs the following simple idea.

Suppose (X, Y) is a Nash equilibrium of a two-player game. Assuming we know (X, Y), it is easy to sample each player's action using the distributions X and Y. Thus, we can repeat this sampling procedure many times to obtain a discrete approximation of the original distribution. Here, $t = 8 \log n/\epsilon^2$ samples should suffice for our purpose. Let $(\hat{X}, \hat{Y})$ be the approximate distribution we obtained by sampling. Then, the following lemma holds.

Lemma 1. $(\hat{X}, \hat{Y})$ is an ϵ-approximate Nash equilibrium with high probability.

Clearly, all probabilities in $\hat{X}$ and $\hat{Y}$ have a denominator t. Thus, even if we do not know what (X, Y) is, we can search exhaustively on all probability distributions with denominator t to find the ϵ-approximate Nash equilibrium in time $O(n^t) = O(n^{\log n/\epsilon^2})$. (Naively, it might appear that the time complexity should be $O(t^n)$, because each of the n probabilities can take values in $\{0, 1, \ldots, t\}$. However, we can think of it as distributing t objects into n bins, in which case the number of ways becomes $\binom{t+n-1}{n-1} \sim n^t$.) The final step is to prove the above lemma.

Proof. The proof consists of two parts. In the first part, we use a Chernoff-style argument to prove that $\hat{X}$ is close to X with high probability. In this lecture, we will omit this step and directly proceed to showing that if $\hat{X}$ is close to X, it must be an ϵ-approximate Nash equilibrium. For this, we first introduce the following definition.

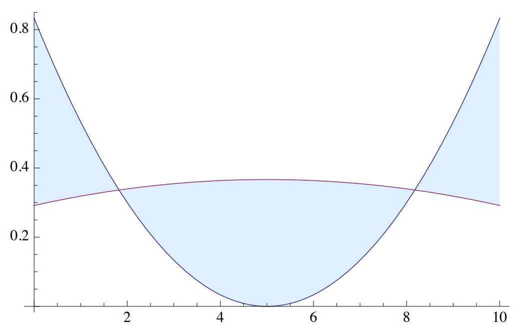

Figure 1: TV distance

Definition 2. The TV distance (total variation distance) between two (discrete) distributions D_1 and D_2 is defined as follows:

$$||D_1 - D_2||_{TV} = \sum_i |D_1(i) - D_2(i)|.$$

Now, suppose $(X_1, \ldots, X_m)$ and $(X'_1, \ldots, X'_m)$ are two strategy profiles (m is the number of players). What is the difference in the utilities between the two strategy profiles? Observe that for any player i,

$$\begin{aligned} |u_i(X) - u_i(X')| &\leq u_{max} \sum_{a \in A} |\Pr_X(a) - \Pr_{X'}(a)| \\ &= u_{max} ||X - X'||_{TV} \\ &\leq u_{max} \sum_i ||X_i, X'_i||_{TV}, \end{aligned}$$

where u_{max} is the biggest payoff that appears in the utility table. This means that if two profiles are pairwise close in the TV distance, the payoff of each player will be close. This completes our proof of the lemma. $\square$

1.3 Approximate Nash equilibria in anonymous games

Now we shall study the approximate Nash equilibria in anonymous games, which is a type of a succinct game. Anonymous games are games in which once you fix a player, the utility function becomes symmetric in other players' actions. In other words, this means that our payoff is completely determined by "how many" of the other players choose each action. In anonymous games, it is usually assumed that there are only two possible actions (e.g. do I take BART to San Francisco or drive?). If n is the number of players and s is the number of (pure) strategies ($n \gg s$), the representation size of an anonymous game is sn^s (using the same argument as in the time complexity analysis of the LMM algorithm), as opposed to s^n of an ordinary game.

How do we find approximate Nash equilibria in such games? We can use the following algorithm due to Daskalakis and Papadimitriou [5]. First, pick a positive integer k. As we did in the LMM algorithm, we will discretize probabilities to multiples of $1/k$, i.e., we will exhaustively search on all strategies that have probabilities that are multiples of $1/k$. Then, we can pick the one that is closest to the equilibrium. This algorithm has time complexity $O(n^{k+1})$, because we need to explore all possible ways of distributing n people into $k+1$ bins. In order to fix the value of k, we appeal to the following theorem.

Theorem 1. Let $X_1, \ldots, X_n$ be independent Bernoulli random variables with expectations $p_1, \ldots, p_n$. Then, there exist $q_1, \ldots, q_n$ which are multiples of $1/k$ such that if $Y_1, \ldots, Y_n$ are independent Bernoulli random variables with expectations $q_1, \ldots, q_n$,

$$||X - Y||_{TV} \leq O\left(1/\sqrt{k}\right),$$

where X and Y are distributions of $\sum_i X_i$ and $\sum_i Y_i$, respectively.

Thus, if we take $k = O(1/\epsilon^2)$, we obtain a PTAS with time complexity $n^{O(1/\epsilon^2)}$. In general, if $s > 2$, there exists a PTAS with time complexity $O(n^{f(s)/\epsilon^2})$. If $s = 2$, the time complexity can be improved to $(1/\epsilon)^{O(1/\epsilon^2)}\text{poly}(n)$. In fact, the exponent $O(1/\epsilon^2)$ can be further improved to $O(\log 1/\epsilon)$, which has the following optimality result [6].

Theorem 2. No oblivious algorithm for anonymous games can achieve time complexity less than $(1/\epsilon)^{O(\log 1/\epsilon)}\text{poly}(n)$. No oblivious algorithm for general games can achieve time complexity less than $O(n^{\log n/\epsilon^2})$.

An algorithm is said to be oblivious if it just samples a fixed distribution on pairs of mixed strategies, and the input is only used to determine whether the sampled strategies comprise an ϵ-approximate Nash equilibrium.

2 Correlated equilibria

2.1 Definition and example of Correlated equilibria

Consider the symmetric game refereed as the chicken game. In this game, the two players can be viewed as very competitive drivers speeding from different streets to an intersection. Each of them has two strategies: Stop or go. The table below shows the payoffs of both drivers in playing the game. [8]

	stop	go
stop	4,4	1,5
go	5,1	0,0

Table 1: The chicken game

There are two pure equilibria (One player stop and the other go) and the symmetric mixed equilibrium (1/2, 1/2). The respective pure strategy profiles of these three Nash equilibria are the following:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$$

But consider now the following distribution on the strategy profiles:

$$\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

It is not a Nash equilibrium because there is no two mixed strategies for the two players that can generate this distribution. However, this distribution is a rational outcome of the game. For instance, suppose a trusted third party picks randomly between (go-stop and stop-go) behaving like a traffic signal) and recommend players to follow the outcome picked. For example, if the third-party picks go-stop, he recommends go (Green) for Player 1 and stop (red) for Player 2. This distribution is self enforcing in the sense that if a player assume that all other players are going to abide to the trusted party, he would have no incentive to play another strategy than the one proposed by the third party. This distribution is an example of correlated equilibria as defined below:

Definition 3. *(Aumann, 1979)* A correlated equilibrium is a probability distribution $\{p_x\}$ on the space of strategy profiles that obey the following conditions: For each player i, and every two different strategies j, j' of i, conditioned on the event that a strategy profile with j as his strategy was drawn from this distribution, the expected utility of playing j is no smaller than that of playing j':

$$\sum_{s \in S_{-i}} (u_{sj} - u_{sj'}) p_{sj} \geq 0 \tag{1}$$

S_{-i} denotes the strategy profile of all players except for i. sj denotes the strategy profile in which player i plays j and the others play s. This inequality says that if a strategy profile is drawn from a distribution $\{p_x\}$ and each player is told, privately, his or her own component of the outcome, and if furthermore all players assume that the others will follow the recommendation, then the recommendation is self enforcing.

Notice that if p^i , $i = 1, \ldots, n$, is a set of mixed strategies of the players, and we consider the product distribution induced by it $p_x = \prod_i (p^i_{x_i})$, the above inequalities states that these mixed strategies constitute a mixed Nash equilibrium. Therefore, any Nash equilibrium is a correlated equilibrium and Nash equilibrium is a special case of correlated equilibrium in which p_s's are restricted to come from a product (uncorrelated) distribution.

The correlated equilibria inequalities for the driver game are:

$$(4-5)p_{11} + (1-0)p_{12} \geq 0$$

$$(5-4)p_{21} + (0-1)p_{22} \geq 0$$

$$(4-5)p_{11} + (1-0)p_{21} \geq 0$$

$$(5-4)p_{12} + (0-1)p_{22} \geq 0$$

We observe that correlated equilibrium can be solve in polynomial time using liner programming. We can find the correlated equilibria that maximize the sum of the payoffs of players by maximizing the objective function $8p_{11} + 6p_{12} + 6p_{21}$ given the above constraints. The resulting optimal distribution is:

$$\begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix}$$

2.2 Correlated Equilibria versus Nash Equilibria on polytope

The polytope defined by the correlated equlibria inequalities for the driven game in Figure 2. Every point in this polytope is a correlated equilibrium. The three Nash equilibria are located on some vertices of the polytope. The "traffic light" correlated equilibrium $C1$ and the optimum one $C2$ are also shown. [7]

Theorem 3. In any non-degenerate two-player game, the Nash equilibria are vertices of the polytope defined by the correlated equlibria inequalities.

Note that a least one vertex of the polytope is required to be a Nash equilibria. The theorem does not apply for three-players games that have irrational solutions.

2.3 Correlated Equilibria in Succinct Games

Theorem 4. (Papadimitriou, 2005) In any succinctly representable game of polynomial type for which the expected utility problem can be solved in polynomial time, the problem of finding a correlated equilibrium can be solved in polynomial time as well.

Therefore, there is a polynomial time algorithm for finding a correlated equilibrium for network, congestion, scheduling, symmetric, anonymous and other succinctly representable games.

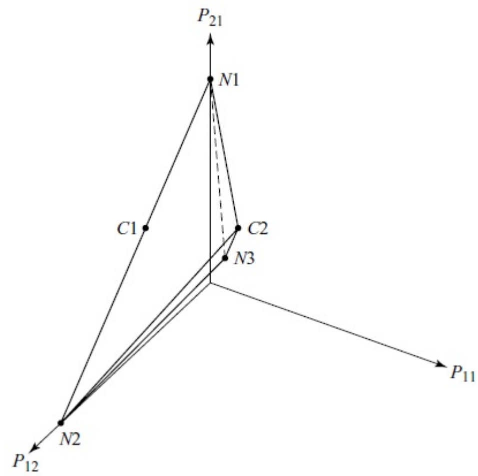

Figure 2: The three Nash equilibria (N1,N2, N3) and two correlated equilibria (C1, C2) of the drivers' game position on the polytope

2.4 The existence proof of correlated equilibria

This section present the proof of existence of correlated equilibria without having to refer to Nash equilibria. [7]

Theorem 5. Every game has a correlated equilibrium.

Proof: Consider the linear program (P)

$$max \sum_s x_s$$

$$Ux \geq 0$$

$$x \geq 0$$

The program is either trivial (with maximum zero) or unbounded. In the case where the solution is unbounded, the game has a correlated equilibrium. Hence, by duality, it is sufficient to prove that the dual of the problem is infeasible to prove that the game has a correlated equilibrium. Below are the dual constraints:

$$U^T y \leq 0$$

$$y \geq 0$$

Lemma 2. For every $y \geq 0$ there is a product distribution x such that $xU^T y = 0$.

Proving this Lemma 2 is sufficient enough to state that the dual problem is infeasible. For instance, for $xU^T y = 0$ to be true, at least one $U^T y_i$ should be greater of equal to zero. The proof of the lemma goes as follow: We look at the expression $xU^T y$ setting $x_s = x^1_{s_1} \cdot \cdots \cdot x^n_{s_n}$ This product is linear and homogeneous in the utilities. Consider therefore the coefficient in this expression of the utility u^p_{is} where $s \in S_{-p}$ and $i \in S_p$. This coefficient is

$$\left[\prod_{q\neq p} x^q_{s_q}\right] \cdot \left[x^p_i \sum_{j\in S_P} y^p_{ji} - \sum_{j\in S_p} x^p_j y^p_{ij}\right]$$

The second term of this expression is recognized as the equilibrium equation of the steady-state distribution x^p_i of a Markov chain y^p_{ij}. Therefore, by setting, for each player p, the x^p_i's to be the steady state distribution of the Markov chain defined by the normalized y^p_{ij}'s all second factors become zero and so does the expression xU^Ty. The dual variable y^p_{ij} may be thought of as the tendency of player p to switch from strategy i to strategy j. A final observation is that the mapping from y values to x values is universal for all games of the same type, since it does not depend on the u^p_s's.

2.5 Ellipsoid Against Hope - polynomial algorithm

This section presents a polynomial algorithm for finding a correlated equilibrium. The idea is to apply the ellipsoid algorithm to the dual (D) which is guaranteed to be infeasible. The Dual has a polynomial number of variables (about nm^2) and exponential number of constraints (about m^n). For the primal, the opposite holds. Hence, the ellipsoid algorithm is appropriate for the dual. At each step i we have a candidate solution y_i; we use the lemma to obtain x_i and the violated inequality $x_iU^Ty \leq -1$. The algorithm end up reporting infeasible but by then we will have a polynomial collection of product distributions. Hence, after the termination of the ellipsoid algorithm at the Lth step, we have L product distributions $x_1, \ldots, x_L$ such that for each $i \leq L$, $\left[x_iU^T\right] y \leq -1$ is violated by y_i. This means that $\left[XU^T\right] y \leq -1$ where X is the matrix whose rows are the x_i's is itself an infeasible linear program. Now solving for the primal with L non-zero variables $\left[UX^T\right] \alpha \leq 0, \alpha \leq 0$ gives a correlated equilibrium in polynomial time.

Jiang and Leyton-Brown have presented a variant of the Ellipsoid Against Hope algorithm that uses pure strategy profiles to compute correlated equilibria with polynomial sized supports. [9]

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