

# Primal and Dual Decomposition: Theory and Distributed Algorithms

Chee Wei Tan

CS 8292 : Advanced Topics in Convex Optimization and its Applications  
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# Outline

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- Distributed Algorithms
- Primal Decomposition
- Lagrange Dual Decomposition

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# Distributed Algorithms

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In previous lectures, we have studied optimization algorithms like

- Descent algorithm, e.g., Gradient method, Newton method
- Interior point algorithm
- Cutting plane algorithm
- Fixed-point iteration algorithm, e.g., power method

Distributed algorithms are preferred because:

- It's **scalable**
- It's **robust**
- Centralized command is **not** feasible or is too costly

## Key issues:

- Local computation vs. global communication
- Scope, scale, and physical meaning of communication overhead
- Theoretical issues: Convergence? Optimality? Speed?
- Practical issues: Robustness? Synchronization? Complexity? Stability?
- Problem separability structure for decomposition: vertical and horizontal

# Decomposition: LP Example

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LP with variables  $u, v$ :

$$\begin{array}{ll}\text{maximize} & c_1^T u + c_2^T v \\ \text{subject to} & A_1 u \preceq b_1 \\ & A_2 v \preceq b_2 \\ & F_1 u + F_2 v \preceq h\end{array}$$

Coupling constraint:  $F_1 u + F_2 v \preceq h$ . Otherwise, **separable** into two LP

# Primal Decomposition

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Introduce variable  $z$  and rewrite coupling constraint as

$$F_1 u \preceq z, \quad F_2 v \preceq h - z$$

LP **decomposed** into a **master problem** and **two subproblems**:

$$\text{minimize}_z \phi_1(z) + \phi_2(z)$$

where

$$\phi_1(z) = \inf_u \{c_1^T u \mid A_1 u \preceq b_1, F_1 u \preceq z\}$$

$$\phi_2(z) = \inf_v \{c_2^T v \mid A_2 v \preceq b_2, F_2 v \preceq h - z\}$$

**Subgradient** of function  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  at  $x$  is a vector  $g$  such that

$$f(y) \geq f(x) + g^T(y - x), \quad \forall y$$

# Primal Decomposition

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For each iteration  $t$ :

1. **Solve two separate LPs** to obtain optimal  $u(t), v(t)$  and associated dual variables  $\lambda_1(t), \lambda_2(t)$
2. **Subgradient update**:  $g(t) = -\lambda_1(t) + \lambda_2(t)$
3. **Master algorithm update**:  $z(t+1) = z(t) - \alpha(t)g(t)$  where  $\alpha(t) \geq 0$ ,  $\lim_{t \rightarrow \infty} \alpha_t = 0$  and  $\sum_{t=1}^{\infty} \alpha(t) = \infty$

**Interpretation:**

- $z$  fixes allocation of resources between two subproblems and master problem iteratively finds best **allocation** of resources
- More of each resource is allocated to the subproblem with larger Lagrange multiplier at each step

# Dual Decomposition

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Form partial Lagrangian:

$$\begin{aligned} L(u, v, \lambda) &= c_1^T u + x_2^T v + \lambda^T (F_1 u + F_2 v - h) \\ &= (F_1^T \lambda + c_1)^T u + (F_2^T \lambda + c_2)^T v - \lambda^T h \end{aligned}$$

Dual function:

$$\begin{aligned} q(\lambda) &= \inf_{u,v} \{L(u, v, \lambda) \mid A_1 u \preceq b_1, A_2 v \preceq b_2\} \\ &= -\lambda^T h + \inf_{u: A_1 u \preceq b_1} (F_1^T \lambda + c_1)^T u + \inf_{v: A_2 v \preceq b_2} (F_2^T \lambda + c_2)^T v \end{aligned}$$

Dual problem:

$$\begin{array}{ll} \text{maximize} & q(\lambda) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$



# Dual Decomposition

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Solve the following LP in  $u$ , with minimizer  $u^*(\lambda(t))$

$$\begin{array}{ll}\text{minimize} & (F_1^T \lambda(t) + c_1)^T u \\ \text{subject to} & A_1 u \preceq b_1\end{array}$$

Solve the following LP in  $v$ , with minimizer  $v^*(\lambda(t))$

$$\begin{array}{ll}\text{minimize} & (F_2^T \lambda(t) + c_2)^T v \\ \text{subject to} & A_2 v \preceq b_2\end{array}$$

Use the following subgradient (to  $-q$ ) to **update**  $\lambda$ :

$$g(t) = -F_1 u^*(\lambda(t)) - F_2 v^*(\lambda(t)) + h, \quad \lambda(t+1) = [\lambda(t) - \alpha(t)g(t)]^+$$

**Interpretation:**

Master algorithm adjusts **prices**  $\lambda$ , which regulates the separate solutions of two subproblems

# Summary

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- Decouple a coupling constraint: primal or dual decomposition
- Dual decomposition algorithm: a large-scale engineering application in the next lecture

## **Reading assignment:**

- Sections 3.4, 7.5 in D. P. Bertsekas and J. N. Tsitsiklis, Parallel and Distributed Computation: Numerical Methods, Athena Scientific 1999.