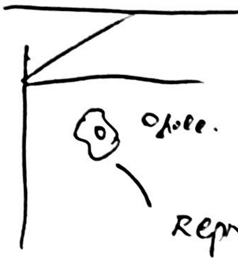
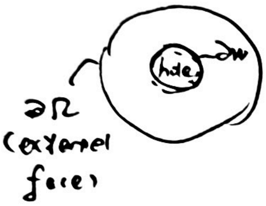
Derivation of Gurson model.

Definition.

I. Homogenization.



Representative volume element (RVE)



total domain Ω

void domain w (No traction on void surface)

$$\partial\Omega \cap \partial w = 0$$

porosity : $f = \dfrac{w}{\Omega}$

About stress and strain in RVE.

stress : $\underline{\sigma}$ in Ω stain : $\underline{d}$ $\underline{\varepsilon}$ in Ω

0 in w

Suppose some kind of velocity field exists in w, and C^1 consistent with $\underline{d}$.

Note. $\underline{d}$: rate of deformation (2nd symmetric tensor)

I.1 Kinematic Approach

① kinematic boundary condition ($\partial\Omega$)

$$\forall \hat{x} \in \partial\Omega, \quad v_i = D_{ij} x_j \quad \text{(Gurson, 1977)}$$

$\underline{D}$ boundary rate of deformation.

② Macroscopic stress

$$\Sigma_{ij} = \langle \sigma_{ij} \rangle_{\Omega} = \frac{1}{V_\Omega}\int_{\Omega} \sigma_{ij}\, dV$$

$$\text{or} = \frac{V_{\Omega\setminus w}}{V_\Omega}\frac{1}{V_{\Omega\setminus w}}\int_{\Omega\setminus w} \sigma_{ij}\, dV = (1-f)\langle \sigma_{ij} \rangle_{\Omega\setminus w}.$$

Note. $\underline{\sigma}$, $\underline{d}$ no need to follow constitutive relation.

2.

③ Intergral expression of D.

Green's Theorem + Boundary condition ($v_i = D_{ij} \cdot x_j$)

Prove that $\langle d_{ij} \rangle_\Omega = \frac{1}{V_\Omega} \int_\Omega d_{ij} \, dV$

$$= \frac{1}{V_\Omega} \int_\Omega \frac{1}{2} (v_{i,j} + v_{j,i}) \, dV$$

$$= \frac{1}{V_\Omega} \int_\Omega \frac{1}{2} [(D_{ik} x_k)_{,j} + (D_{jk} x_k)_{,i}] \, dV$$

$$= \cancel{\frac{1}{2V} \int (D_{ik} x_k)_{,j}}$$

$$= \frac{1}{2V_\Omega} \left\{ \int D_{ik} \delta_{kj} \cdot dV + \int D_{jk} \delta_{ik} \, dV \right\}$$

$$= \frac{1}{2V_\Omega} \cdot (D_{ij} + D_{ji}) \cdot \int dV = D_{ij}$$

$$\Rightarrow \underline{\underline{\langle d_{ij} \rangle_\Omega = D_{ij}}}$$ (Green theorem useless)

So, $D \Rightarrow$ Macroscopic rate of deformation.

④ Hill-Mandel Lemma.

$\hat{v}$ / kinematic admissable velocity field /

$\equiv \hat{v}_i = D_{ij} x_j \ (\partial\Omega)$

σ / statically admissable stress field /

$$\equiv \begin{cases} \sigma_{ij,j} = 0 & (\Omega \setminus w) \\ \sigma_{ij} = 0 & (\partial w) \end{cases}$$

$$\Rightarrow \underline{\underline{\langle \sigma_{ij} d_{ij} \rangle_\Omega = \Sigma_{ij} D_{ij}}}$$

Proof: $\langle \sigma_{ij} d_{ij} \rangle_\Omega = \frac{1}{V_\Omega}\int_\Omega \sigma_{ij} d_{ij}\, dV$ (definition)

$$= \frac{1}{V_\Omega}\int_{\partial\Omega} T_i v_i \, dS = \frac{1}{V_\Omega}\int_{\partial\Omega} \sigma_{ij} n_j v_i \, dS$$ (virtual work principle)

$$= \frac{1}{V_\Omega}\int_{\partial\Omega} \sigma_{ij} n_j D_{ik} x_k \, dS$$ (Boundary condition)

$$= \frac{1}{V_\Omega}\int_\Omega (\sigma_{ij} D_{ik} x_k)_{,j} \, dV$$ (Green's Theorem)

$$= \underbrace{\frac{1}{V_\Omega} D_{ik}\int_\Omega \sigma_{ij,j}\, x_k \, dV}_{0 \text{ (equilibrium)}} + \frac{1}{V_\Omega} D_{ik}\int \sigma_{ij}\, x_{k,j} \, dV$$

$$= \frac{1}{V_\Omega}\int D_{ik}\, \delta_{kj}\, \sigma_{ij} \, dV$$

$$= \Sigma_{ij} D_{ij}$$ ■

Also $\langle \sigma_{ij} d_{ij} \rangle_\Omega =$

$$= \frac{1}{V_\Omega}\int_\Omega \sigma_{ij} d_{ij} \, dV$$

$$= \frac{1}{V_\Omega}\int_{\Omega\backslash\omega} \sigma_{ij} d_{ij} \, dV$$

$$= (1-f)\, \frac{1}{V_{\Omega\backslash\omega}}\int_{\Omega\backslash\omega} \sigma_{ij} d_{ij} \, dV$$

$$= \underline{(1-f)\, \langle \sigma_{ij} d_{ij} \rangle_{\Omega\backslash\omega}} = \Sigma_{ij} D_{ij}$$

Suppose. 1. Matrix : incompressible ($tr(\underline{d}) = 0$), rigid-ideal plastic.

Mises yield.

Define. Plastic Dissipation

$$\pi(d) = \sup_{\sigma^* \in C} \sigma^*_{ij} d_{ij}$$

C is all possible stress inside yield convex surface.

Now consider.

A stress field $\underline{\sigma}$.

velocity field $\hat{v} \Rightarrow \underline{d}$

satisfy $\underline{\sigma} \in C$, $\langle \sigma_{ij} \rangle_\Omega = \Sigma_{ij}$

$\hat{v}_i = D_{ij} x_j |_{\partial\Omega}$, $\langle d_{ij} \rangle_\Omega = D_{ij}$

$$\Rightarrow \quad \Sigma_{ij} D_{ij} = \langle \sigma_{ij} d_{ij} \rangle_\Omega \leq \sup_{\sigma \in C} \langle \sigma_{ij} d_{ij} \rangle_\Omega = \langle \pi(\underline{d}) \rangle_\Omega$$

for $\forall \underline{d}$.

sup 最小上界

inf 最大下界

A Tighter upper bound.

$$\Sigma_{ij} D_{ij} \leq \Pi(\underline{D}) = \inf_{d \in K(D)} \langle \pi(\underline{d}) \rangle_\Omega$$

$K(D)$ kinematically admissible microscopic deformations

$$K(D) = \{ \underline{d} \mid \forall \hat{x} \in \Omega \backslash \omega,\ d_{kk} = 0 \text{ and } \exists \hat{v},\ \forall \hat{x} \in \Omega,\ d_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) \text{ and } \forall \hat{x} \in \partial\Omega,\ v_i = D_{ij} x_j \}$$

$\Pi(\underline{D})$ Macro plastic dissipation with $\underline{D}$

Define.

Set of Macroscopic stress $\underline{\Sigma}$, denoted $\mathscr{C}$

$$\mathscr{C} = \{ \underline{\Sigma} \mid \forall D,\ \underline{\Sigma} : \underline{D} \leq \pi(\underline{D}) \}$$

$fr(\mathscr{C})$: boundary of the set $\mathscr{C}$.

$$fr(\mathscr{C}) = \{ \underline{\Sigma} \mid \underline{\Sigma} : \underline{D} = \pi(\underline{D}) \}$$

suppose $\pi(\underline{D})$ 1 degree homogenous with D

$\cancel{\underline{\Sigma} : \underline{D} = \pi(\underline{D})}$

$$\pi(\underline{D}) = \frac{\partial \pi(\underline{D})}{\partial D_{ij}} D_{ij}, \quad \text{with } \underline{\Sigma} : \underline{D} = \pi(\underline{D})$$

$$\Rightarrow \frac{\partial \pi(\underline{D})}{\partial \underline{D}} = \underline{\Sigma} \quad (fr(\mathscr{C})) \quad \bigstar$$

$\Rightarrow$ $\mathscr{C}$ Elactic domain and yield surface

$fr(\mathscr{C})$ yield surface

It is possible to Elimination $\underline{D}$ from $\frac{\partial \pi(\underline{D})}{\partial \underline{D}} = \underline{\Sigma}$.

Because $\frac{\partial \pi(\underline{D})}{\partial \underline{D}}$ 0 degree homogenous

$$\Rightarrow \phi(\underline{\Sigma}) = 0$$

Micro Structure Evolution.

$\dot{V}_\Omega = \dot{V}_{\Omega|w} + \dot{V}_w$. and $\Omega|w$ incompressible

$\Rightarrow \dot{V}_\Omega = \dot{V}_w$

$$\dot{f} = \frac{d}{dt}\left(\frac{V_w}{V_\Omega}\right) = \frac{d}{dt}\left(\frac{V_\Omega - V_{\Omega|w}}{V_\Omega}\right)$$

$$= \frac{(\dot{V}_\Omega - \dot{V}_{\Omega|w})V_\Omega}{V_\Omega^2} - \frac{(V_\Omega - V_{\Omega|w})\dot{V}_\Omega}{V_\Omega^2}$$

$$= \frac{\dot{V}_\Omega V_\Omega}{V_\Omega^2} - \frac{f V_\Omega \dot{V}_\Omega}{V_\Omega^2} = (1-f)\frac{\dot{V}_\Omega}{V_\Omega}$$

$$= \underline{\underline{(1-f) D_{kk}}}$$

$$= \underline{\underline{(1-f) H \frac{\partial \Phi}{\partial \Sigma_m}}}$$

H. plastic Multiplier.

$\Sigma_m = \Sigma_{kk}/3$

Void Growth Models

$$\Pi(D) = \inf_{\underline{d} \in K(\Omega)} \langle \sup_{\underline{\sigma} \in C} \sigma_{ij} d_{ij} \rangle_\Omega$$

Among all modes of plastic deformation,

Smallest Macroscopic average dissipation over RVE define macroscopic yielding.

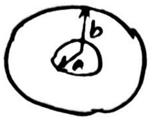
Then, how to choose B. D.

$\underline{\underline{\text{kinematically}}}$ ~~or~~ St. statie. V_s.
prefer

Because, $\mathcal{C}(\underline{\underline{D}} = D\dot{x}) \supset \mathcal{C}_{real} \supset \mathcal{C}_{\sigma n = \Sigma n}$.
$\vee$
upper bound

$\boxed{\text{Gurson Model.}}$

Basic ingredients

(i) RVE :  $f = \frac{a^3}{b^3}$

(ii) J_2 flow theory

(iii) Trial velocity field : $\left\{ \begin{array}{l} \text{incompressible velocity field} \\ + \\ \text{linear field (uniform deformation)} \end{array} \right.$

Microscopic yield criterion

$$\sigma_{eq} = \sqrt{\frac{3}{2}\sigma' : \sigma'}$$

$$\underline{d} = \frac{3}{2}\frac{d_{eq}}{\bar{\sigma}}\sigma' \qquad \sigma' : \text{deviatoric stress}$$

$$d_{eq} = \sqrt{\frac{2}{3}\underline{d} : \underline{d}} \qquad \bar{\sigma} : \text{yield stress}$$

$$\Rightarrow \quad \pi(\underline{d}, \hat{x}) = \left\{ \begin{array}{ll} \bar{\sigma}\, d_{eq}(\hat{x}) & \text{(Matrix)} \\ 0 & \text{(void)} \end{array} \right.$$

Velocity field is given by

$$\forall \hat{x} \in \Omega \backslash w, \quad v_i(\hat{x}) = A\, v_i^A(\hat{x}) + \beta_{ij} x_j$$

$$v_i^A(\hat{x}) = \frac{1}{r^2}\hat{e}_r$$

☆ $\boxed{\text{B.D. at } \hat{x} = b\,\hat{e}_r}$: $A = b^3 D_m \qquad (D_m = D_{kk}/3)$

$$\underline{\beta} = D'$$

$\hat{v}$ leads to deformation field d^A

$\Rightarrow$ velocity field fully determined (D is specified through B.D.)

$$\Pi(\underline{D}) = \langle \pi(\underline{d}) \rangle_\Omega = \frac{1}{V_\Omega} \int_\Omega \bar{\sigma}\, d_{eq}\, dV$$

$$= \frac{1}{V_\Omega} \int_{\Omega \setminus w} \bar{\sigma}\, d_{eq}\, dV \quad (\pi(d(\hat{x})) \text{ at void} = 0)$$

$$= \frac{V_{\Omega\setminus w}}{V_\Omega} \frac{1}{V_{\Omega \setminus w}} \int_{\Omega\setminus w} \bar{\sigma}\, d_{eq}\, dV$$

$$= (1-f)\, \bar{\sigma} \langle d_{eq} \rangle_{\Omega \setminus w}$$

or $$\langle \pi(d) \rangle_\Omega = \frac{1}{V_\Omega} \int_{\Omega\setminus w} \bar{\sigma}\, d_{eq}\, dV$$

$$= \frac{1}{V_\Omega} \bar{\sigma} \int_a^b \langle d_{eq} \rangle_{S(r)}\, S(r)\, dr$$

($S(r)$, surface area of r-sphere, $S(r) = 4\pi r^2$)

Cauchy - Schwartz inequality

$$\langle d_{eq} \rangle_S = \frac{1}{S} \int_S d_{eq}\, dS$$

$$= \frac{1}{S} \lim_{n\to\infty} \sum_1^n d_{eq} \cdot \Delta S$$

$$= \lim_{n\to\infty} \frac{1}{n \Delta S} \Sigma\, d_{eq} \cdot \Delta S$$

$$\leq \lim_{n\to\infty} \frac{1}{n\Delta S} \sqrt{\cancel{d_{eq}} \sum_1^n d_{eq}^2}\, \sqrt{n (\Delta S)^2}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sqrt{\frac{1}{\Delta S} \sum_1^n d_{eq}^2\, \Delta S}$$

$$= \lim_{n\to\infty} \sqrt{\frac{1}{n\Delta S} \sum_1^n d_{eq}^2\, \Delta S}$$

$$= \cancel{\lim_{n\to\infty}} \sqrt{\langle d_{eq}^2 \rangle_S}.$$

Substitute $\langle d_{eq} \rangle_s \le \sqrt{\langle d_{eq}^2 \rangle_s}$ into $\Pi(\underline{D}) = \frac{\bar{\sigma}}{V_\Omega} \int_a^b \langle d_{eq} \rangle_s S \, dr$

$$\Rightarrow \quad \Pi(\underline{D}) = \frac{\bar{\sigma}}{V_\Omega} \int_a^b \sqrt{\langle d_{eq}^2 \rangle_s} \cdot S \, dr$$

☆ & Rewriting d_{eq} in term of $\underline{d}^A$, β_{ij}

$$\underline{\langle d_{eq}^2 \rangle_s = A^2 \langle d_{eq}^{A^2} \rangle_s + \beta_{eq}^2 + \frac{4}{3} A \langle \underline{d}^A \rangle_s : \beta}$$

With $\underline{d}^A = \frac{b^3}{r^3} D_m [-2 \hat{e}_r \otimes \hat{e}_r + \hat{e}_\theta \otimes \hat{e}_\theta + \hat{e}_\varphi \otimes \hat{e}_\varphi]$

$$d_{eq}^{A^2} = 4 D_m^2 \frac{b^6}{r^6}, \quad \beta_{eq}^2 = D_{eq}^2$$

Note $\langle \hat{e}_i \otimes \hat{e}_i \rangle_s = \frac{1}{3} \Rightarrow \boxed{\underline{d}^A = 0}$.

$$\Pi(\underline{D}) = \frac{\bar{\sigma}}{V_\Omega} \int_a^b \sqrt{\langle d_{eq}^2 \rangle_s} \cdot S \, dr$$

$$= \frac{\bar{\sigma}}{V_\Omega} \int_a^b \sqrt{A^2 \langle d_{eq}^{A^2} \rangle_s + \beta_{eq}^2} \; S \, dr$$

$$= \frac{\bar{\sigma}}{b^3} \int_{\xi}^{\xi f} \sqrt{1 + u^2} \, \frac{du}{u^2},$$

$$u = \xi \frac{b^3}{r^3}, \quad \xi = \frac{2 D_m}{D_{eq}}$$

Elimination $\underline{D}$ in $\frac{\partial \Pi(\underline{D})}{\partial \underline{D}} = \underline{\Sigma} \quad (f(\xi))$

$\Rightarrow$ Gurson Model.

$$\Phi^{GTN}(\underline{\Sigma}; f^*) \equiv \frac{\Sigma_{eq}^2}{\bar{\sigma}^2} + 2 q_1 f^* \cosh\left(\frac{3}{2} q_2 \frac{\Sigma_m}{\bar{\sigma}}\right) - 1 - q_1^2 f^{*2}$$

Parameter.

q_1. q_2, f^*

q_1, q_2 allow more accurate observation.

f include void coalescence effect

$$f^* = \begin{cases} f & \text{if } f \le f_c \\ f_c + \left(\frac{1}{q_1} - f_c\right) \frac{f - f_c}{f_R - f_c} & \text{otherwise} \end{cases}$$

Rupture occurs when

$$f^* = \frac{1}{q_1} \quad (\text{equally } f = f_R.)$$

f_R. damage porosity

Void Evolution (~~which~~ contain Nucleation)

$$\dot{f}_{growth} = (1 - f) D_{kk}$$

$$\dot{f}_{Nucleation} = \cancel{A \dot{\bar{\varepsilon}}_p A_n}$$

$$A \dot{\bar{\varepsilon}} + B (\dot{\Sigma}_{eq} + c \dot{\Sigma}_m)$$

$A \dot{\bar{\varepsilon}}$ strain controlled Nucleation

$B(\dot{\Sigma}_{eq} + c \dot{\Sigma}_m)$ stress controlled Nucleation, >0

e.g. $A = \dfrac{f_N}{S_N \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\dfrac{\bar{\varepsilon} - \varepsilon_N}{S_N}\right)^2\right]$

$$\dot{f} = \dot{f}_{growth} + \dot{f}_{nucleation}.$$