Normal Form Games

Multiplayer game where all players take an action simultaneously

Definition – Normal form game

A normal form game is a tuple $G = (N, A, u)$

- N is a set of n number of players
- $A = A_1 \times \ldots \times A_n$ is an *action profile*, where A_i is the set of actions for player i. Thus, an action profile $a = (a_1, \ldots, a_n)$ describes the simultaneous moves by all players.
- $u : A \rightarrow \mathbb{R}^N$ is a reward function that returns an N-tuple specifying the payoff each player receives in state S. This is called the *utility* for an action.

- Each player receives a utility (aka reward) based on their action and every other players' actions
- Goal of each player is to maximise their own utility (Rational)
- Players do not care about other players' utilities (Self-Interested)

Can use a matrix/table to represent outcomes

Left number is utility for left player
Right number for Top player

		Prisoner B: admit	Prisoner B: deny
Prisoner A	admit	-2,-2	0,-4
	deny	-4, 0	-1,-1

- All players have perfect information

Strategies:

Pure Strategy: Always pick the same action every time

Mixed Strategy: Each action has a fixed % chance of being chosen

Strategy Dominance

Weakly Dominant Strategy

S_1 dominates S_2 if the utility received by S_1 against any other strategy is always greater than or equal to the utility received by S_2 against the same opponents.

Strongly dominant strategy

Same as weak version, but S_1 has to always have better utility than S_2 (never equal)

A strategy that dominates all other strategies is known as a dominant strategy

Even if the opponent knows our strategy, they would not gain any advantage if it is a dominant strategy

Solving Normal Form Games

Best Response: Strategy maximising utility assuming we know what strategy the opponent will use

Nash Equilibrium: When all players have the best response against all other players.

i.e. If a player know what all other players' strategies are, they still will not want to change their current strategy

Algorithm – Best response

Input: Normal form game $G = (N, A, u)$, agent i, and strategy profile s_{-i} for agents other than i

Output: Set of best responses

$best\_response = \emptyset$
$best\_response\_value = -\infty$
For each $s_i \in S_i$
    if $u(s_i, s_{-i}) > best\_response\_value$ then
        $best\_response = \{s_i\}$
    else if $u(s_i, s_{-i}) = best\_response\_value$ then
        $best\_response = best\_response \cup \{s_i\}$
return $best\_response$

Algorithm – Nash equilibria

Input: Normal form game $G = (N, A, u)$

Output: Set of Nash equilibria

$nash\_equilibria = \emptyset$
For each $s_1 \in A_1$
  For each $s_2 \in A_2$
    if $s_i \in BestResponse(i, s_j)$ and $s_j \in BestResponse(j, s_i)$ then
      $nash\_equilibria = nash\_equilibria \cup \{(s_i, s_j)\}$
return $nash\_equilibria$

To find if an outcome is a nash equilibrium:

Start from that state

Check if any player can improve by picking a different action

If no player can improve, then it is a nash equilibrium

```
                 Agent 2
               split    steal
                        |
         split   1, 1   | [0, 2]
Agent 1        ---------|---------
         steal  [2, 0]  | [0, 0]
                        |
```

Expected Utility

Average Utility received from playing a pure strategy given the opponents strategies

Indifference

If two pure strategies have the same expected utility

Mixed Strategy Equilibria

A mixed Strategy which makes all opponents indifferent to their pure strategies.

```
                          Player Odd
                        heads     tails
                                |
               heads    1,-1    | -1, 1
Player Even            ---------|---------
               tails   -1, 1    |  1,-1
                                |
```

Finding a mixed strategy equilibria.

(assuming 2 player 2action game)

		Player 2 C	Player 2 D
player 1	y A	e, s	g, h
	$1-y$ B	i, j	k, l

We want to find y: the probability of selecting action A

$1-y$ becomes the probability of selecting action B

$U(C) = U(D)$

$U(C) = y \times g + (1-y) \times j$

$U(D) = y \times h + (1-y) \times l$

Solve for y and $1-y$

		Adversary Terminal 1	Adversary Terminal 2
Defender	Terminal 1	5,-3	-1, 1
	Terminal 2	-5, 5	2,-1

$E[1] = E[2]$

$E[1] = -3 \times y + 5 \times (1-y) = -3y + 5 - 5y = -8y+5$ //

$E[2] = 1 \times y + (-1) \times (1-y) = y - 1 + y = 2y-1$

$-10y = -6$

$y = \frac{3}{5}$

			Column player $\frac{1}{8}$ L	$\frac{7}{8}$ R
Row player	$\frac{1}{2}$	T	320, 40	40,80
	$\frac{1}{2}$	B	40, 80	80,40

$E[L] = E[R]$

$E[L] = y \times 40 + (1-y) \times 80 = 40y + 80 - 80y = -40y + 80$ //

$E[R] = y \times 80 + (1-y) \times 40 = 80y + 40 - 40y = 40y + 40$

$80y = 40$

$y = \frac{1}{2}$

Game 2:

			Column player L	R
Row player	$\frac{1}{2}$	T	44, 40	40,80
	$\frac{1}{2}$	B	40, 80	80,40