MAT2040-T13 Homework 1

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Question 1.

1.1)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \tag{1}$$

$$B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \tag{2}$$

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ -1 & 1 & 2 & 2 \\ 1 & 1 & -1 & 1 \end{array}\right] \tag{3}$$

1.2)

$$[A|b] \xrightarrow[R_3 \to -R_1 + R_3]{R_2 \to R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 3 & 3 & 5 \\ 0 & -1 & -2 & -2 \end{array}\right] \xrightarrow{R_3 \to 3R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 3 & 3 & 5 \\ 0 & 0 & -3 & -1 \end{array}\right]$$
$$\xrightarrow{R_3 \to -\frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 3 & 3 & 5 \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right] \xrightarrow[R_2 \to -3R_3 + R_2]{R_1 \to -R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & \frac{8}{3} \\ 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right]$$
$$\xrightarrow[R_1 \to -2R_2 + R_1]{R_2 \to \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right] \tag{4}$$

$$S = \begin{bmatrix} 0 \\ \frac{4}{3} \\ \frac{1}{3} \end{bmatrix} \tag{5}$$

Question 2.

$$\left[\begin{array}{ccccc|c} 4 & 5 & 3 & 3 & 4 & -5 \\ 2 & 3 & 1 & 0 & 1 & -3 \\ 3 & 4 & 2 & 1 & 2 & -1 \end{array}\right] \xrightarrow[R_3 \to -\frac{3}{4}R_1 + R_3]{R_2 \to -\frac{1}{2}R_1 + R_2} \left[\begin{array}{ccccc|c} 4 & 5 & 3 & 3 & 4 & -5 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{5}{4} & -1 & \frac{11}{4} \end{array}\right]$$
$$\xrightarrow{\substack{R_2 \to 2R_2 \\ R_3 \to 4R_3 \\ R_3 \to R_3 - R_2}} \left[\begin{array}{ccccc|c} 4 & 5 & 3 & 3 & 4 & -5 \\ 0 & 1 & -1 & -3 & -2 & -1 \\ 0 & 0 & 0 & -2 & -2 & 12 \end{array}\right] \xrightarrow{\substack{R_3 \to -\frac{1}{2}R_3 \\ R_2 \to R_2 + 3R_3 \\ R_1 \to R_1 - 3R_3}} \left[\begin{array}{ccccc|c} 4 & 5 & 3 & 0 & 1 & 13 \\ 0 & 1 & -1 & 0 & 1 & -19 \\ 0 & 0 & 0 & 1 & 1 & -6 \end{array}\right]$$
$$\xrightarrow{\substack{R_1 \to R_1 - 5R_2 \\ R_1 \to \frac{1}{4}R_1}} \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 27 \\ 0 & 1 & -1 & 0 & 1 & -19 \\ 0 & 0 & 0 & 1 & 1 & -6 \end{array}\right] \tag{6}$$

$$S = \begin{bmatrix} -2x_3 + x_5 + 27 \\ x_3 - x_5 - 19 \\ x_3 \\ -x_5 - 6 \\ x_6 \end{bmatrix} \tag{7}$$

Question 3.

In this question, we could use partition matrix:

$$AB = \left[\begin{array}{cc|cc} 1 & 0 & a & b \\ 0 & 1 & c & d \end{array}\right] \left[\begin{array}{cc} -a & -b \\ -c & -d \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{8}$$

$$BA = \left[\begin{array}{cc} -a & -b \\ -c & -d \\ \hline 0 & 0 \\ -1 & 1 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 0 & a & b \\ 0 & 1 & c & d \end{array}\right] = \begin{bmatrix} -a & -b & -a^2 - bc & -ab - bd \\ -c & -d & -ac - cd & -bc - d^2 \\ 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix} \tag{9}$$

Question 4.

4.1)

$$A^2 - B^2$$
$$= A \cdot A + B \cdot B$$
$$= A^T \cdot A^T - B^T \cdot B^T$$
$$= (A \cdot A)^T - (B \cdot B)^T$$
$$= ((A^2) - (B^2))^T \tag{10}$$

, so it is symmetric.

4.2) Let LHS $= (A+B)(A-B)$ and RHS $= [(A+B)(A-B)]^T$, then

$$\text{RHS} = (A-B)^T(A+B)^T = (A^T - B^T)(A^T - B^T) = (A-B)(A+B) \neq \text{LHS} \tag{11}$$

, so it is not symmetric.

4.3) Let LHS $= ABA$ and RHS $= (ABA)^T$, then

$$\text{RHS} = (BA)^T A^T = A^T B^T A^T = ABA = \text{LHS} \tag{12}$$

, so it is symmetric.

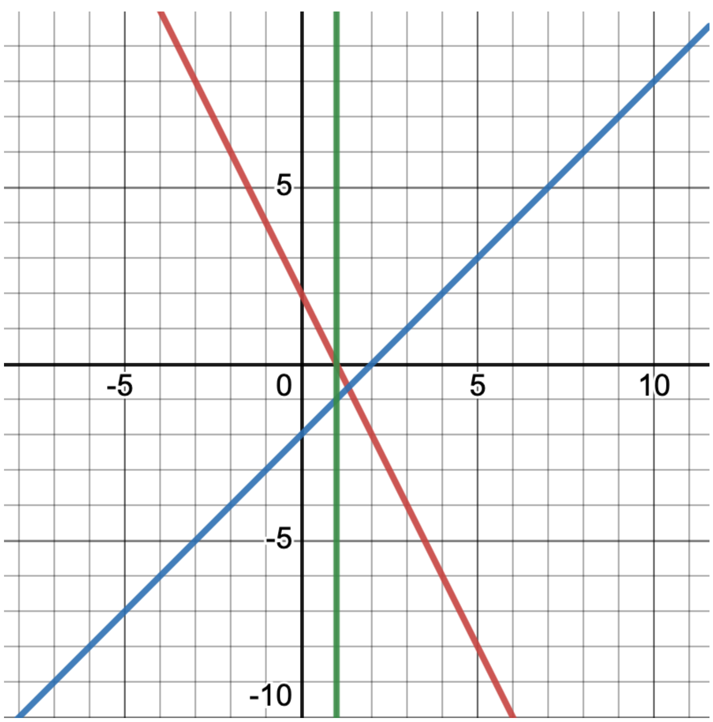
4.4) Let LHS $= ABAB$ and RHS $= (ABAB)^T$, then

$$\text{RHS} = (AB)^T(AB)^T = B^T A^T B^T A^T = BABA \neq \text{LHS} \tag{13}$$

, so it is not symmetric.

Question 5.

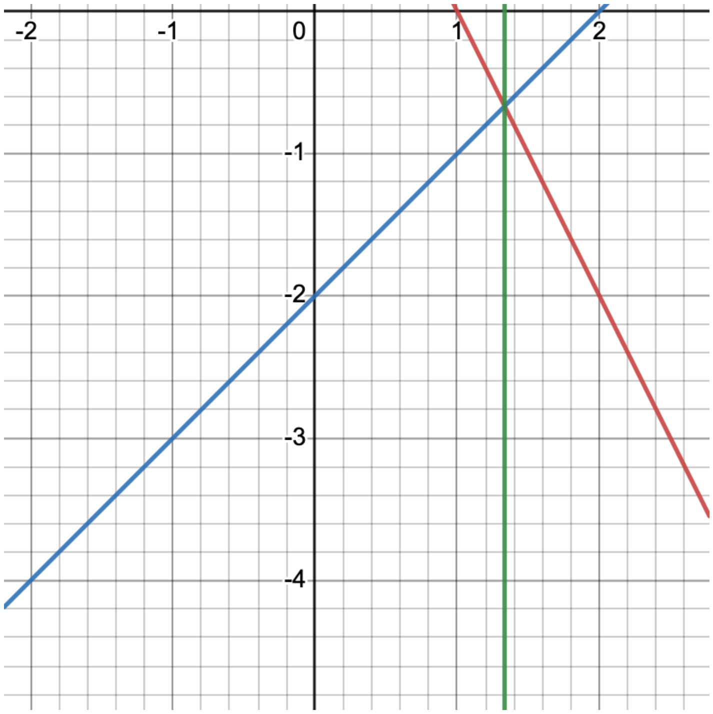
5.1)



No solution.

5.2)

$$\begin{cases} 2x + y & = 2 \\ x - y & = 2 \\ x & = \frac{4}{3} \end{cases} \tag{14}$$



Question 6.

$$\begin{bmatrix} \boxed{a} & 1 & 3 & r \\ a & a & 4 & s \\ a & a & 2a & t \end{bmatrix} \to \begin{bmatrix} \boxed{a} & 1 & 3 & r \\ 0 & a-1 & 1 & s-r \\ 0 & a-1 & 2a-3 & t-r \end{bmatrix} \tag{15}$$

when $a \neq 0$, the elimination fails to give the pivot.

$$\begin{bmatrix} \boxed{a} & 1 & 3 & r \\ 0 & a-1 & 1 & s-r \\ 0 & a-1 & 2a-3 & t-r \end{bmatrix} \to \begin{bmatrix} a & 1 & 3 & r \\ 0 & \boxed{a-1} & 1 & s-r \\ 0 & 0 & 2a-4 & t-s \end{bmatrix} \tag{16}$$

when $a \neq 1$, the elimination fails to give the pivot.

$$\begin{bmatrix} a & 1 & 3 & r \\ 0 & a-1 & 1 & s-r \\ 0 & 0 & \boxed{2a-4} & t-s \end{bmatrix} \tag{17}$$

when $a \neq 2$, the elimination fails to give the pivot.

Hence,

$$a = \begin{cases} 0 \\ 1 \\ 2 \end{cases} \tag{18}$$

Question 7.

By

$$ab^T = \begin{bmatrix} 39 & 99 & 420 & 207 \\ 45 & 33 & 782 & 199 \\ 100 & 80 & 49 & 73 \\ 59 & 18 & 75 & 46 \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 \end{bmatrix} \tag{19}$$

we could get

$$a^Tb = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 = 39 + 33 + 49 + 46 = 167 \tag{20}$$

Question 8.

$$\begin{aligned} A^3 = (ab^T)^3 = (\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \end{bmatrix})^3 &= \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 0 \end{bmatrix}^3 \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 6 & 6 & 0 \\ 9 & 9 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 9 & 0 \\ 18 & 18 & 0 \\ 27 & 27 & 0 \end{bmatrix} \end{aligned} \tag{21}$$

Question 9.

Given

$$(I - xy^T)^{-1} = (I + xy^T) \tag{22}$$

$$(I - xy^T)(I - xy^T)^{-1} = (I - xy^T)(I + xy^T) \tag{23}$$

$$(I - xy^T)(I - xy^T)^{-1} = I^2 - xy^T + xy^T + (xy^T)^2 \tag{24}$$

$$I = I + x(y^Tx)y^T \tag{25}$$

$$0 = x(y^Tx)y^T \tag{26}$$

if $y^Tx = 0$, then eq. (26) is true.

Question 10.

$$C^n = B^{-1}ABB^{-1}AB \dots B^{-1}ABB^{-1}AB = B^{-1} \underbrace{A \dots A}_{\text{n times}} B = B^{-1}A^nB \tag{27}$$