The Graetz Problem

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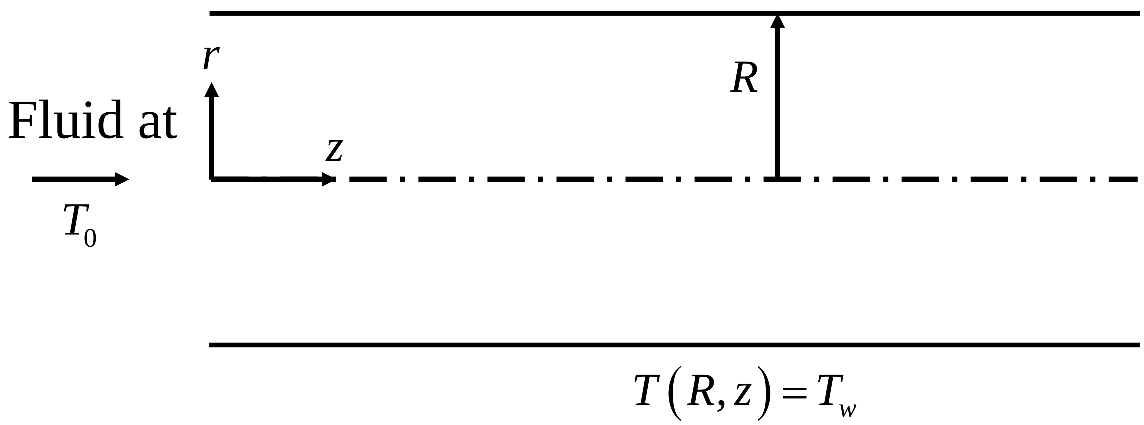
As a good model problem, we consider steady state heat transfer to fluid in steady flow through a tube. The fluid enters the tube at a temperature T_0 and encounters a wall temperature at T_w, which can be larger or smaller than T_0. A simple version of this problem was first analyzed by Graetz (1883). A sketch of the system is shown below.



Objective

To obtain the steady temperature distribution $T(r,z)$ in the fluid, and to calculate the rate of heat transfer from the wall to the fluid

Assumptions

1. Steady fully developed laminar flow; steady temperature field.

2. Constant physical properties ρ, μ, k, C_p -- This assumption also implies incompressible Newtonian flow.

3. Axisymmetric temperature field $\Rightarrow \dfrac{\partial T}{\partial \varphi} \equiv 0$, where we are using the symbol φ for the polar angle. This is because we want to use the symbol θ to represent dimensionless temperature later.

4. Negligible viscous dissipation

Velocity Field

Poiseuille Flow

$v_r = 0; \qquad v_\varphi = 0$

$v_z(r) = v_0\left(1-\frac{r^2}{R^2}\right)$ $\qquad$ v_0 : Maximum velocity existing at the centerline

Energy Equation

Subject to assumption (2), Equation (B.9.2) from Bird et al. (page 850) can be written as follows.

$$\rho C_p\left[\underbrace{\cancel{\frac{\partial T}{\partial t}}}_{\boxed{1}} + \underbrace{\cancel{v_r}}_{\boxed{v_r = 0}}\frac{\partial T}{\partial r} + \frac{\overbrace{\cancel{v_\varphi}}^{\boxed{v_\varphi = 0}}}{r}\frac{\partial T}{\partial \varphi} + v_z\frac{\partial T}{\partial z}\right]$$

$$= k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\overbrace{\cancel{\frac{\partial^2 T}{\partial \varphi^2}}}^{\boxed{3}} + \frac{\partial^2 T}{\partial z^2}\right] + \overbrace{\cancel{\mu\Phi_v}}^{\boxed{4}}$$

and therefore, simplified to

$$v_0\left(1-\frac{r^2}{R^2}\right)\frac{\partial T}{\partial z} = \alpha\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right]$$

where $\alpha = k/\left(\rho C_p\right)$ is the thermal diffusivity of the fluid.

Boundary Conditions

Inlet: $\qquad T(r,0) = T_0$

Wall: $\qquad T(R,z) = T_w$

Centerline: $\qquad T(0,z)$ is finite

or $\qquad \frac{\partial T}{\partial r}(0,z) = 0$

Because of the appearance of the axial conduction term in the governing differential equation, we should write another boundary condition in the $z-$coordinate. But actually, the inlet condition written above is incompatible with the inclusion of axial conduction in the problem, because conduction will lead to some of the information about the step change in wall temperature at the inlet to propagate backward. As we shall see shortly, we'll neglect axial conduction, which will obviate the need for writing a second condition in the $z-$coordinate.

Non-Dimensionalization

We shall use the following scheme for scaling (or non-dimensionalizing) the variables.

$$\theta = \frac{T - T_w}{T_0 - T_w}\ , \quad Y = \frac{r}{R}, \quad Z = \frac{z}{R\,Pe}, \quad \text{where the Péclet Number } Pe = \frac{Rv_0}{\alpha}.$$

This permits us to transform the governing differential equation and boundary conditions to the following form.

$$\left(1 - Y^2\right)\frac{\partial \theta}{\partial Z} = \frac{1}{Y}\frac{\partial}{\partial Y}\left(Y\frac{\partial \theta}{\partial Y}\right) + \frac{1}{Pe^2}\frac{\partial^2 \theta}{\partial Z^2}$$

$$\theta(Y,0) = 1$$

$$\theta(1,Z) = 0$$

$$\theta(0,Z) \text{ is finite}$$

$$\text{or} \qquad \frac{\partial \theta}{\partial Y}(0,Z) = 0$$

The Péclet Number

The Péclet number plays the same role in heat transport as the Reynolds number does in fluid mechanics. First, we note that the Péclet number is the product of the Reynolds and Prandtl numbers.

$$Pe = \frac{Rv_0}{\alpha} = \frac{Rv_0}{\nu} \times \frac{\nu}{\alpha} = \text{Re} \times \text{Pr}$$

The physical significance of the Péclet number can be inferred by recasting it slightly.

$$Pe = \frac{\rho v_0 C_p \Delta T}{k\dfrac{\Delta T}{R}} = \frac{\text{Rate of energy transport by convection}}{\text{Rate of energy transport by conduction}}$$

Note that the numerator represents the order of magnitude of the convective flux in the main flow direction, whereas the denominator stands for the order of magnitude of the conduction flux in the radial direction. If we wish to compare the rates of energy transport by these two mechanisms in the same direction, we can multiply the Péclet number by L/R where L is a characteristic length in the axial direction.

For large values of Pe, we can see that $\frac{1}{Pe^2} \ll 1$. Therefore, in the scaled energy equation, the term involving axial conduction can be safely neglected. Physically, there are two mechanisms for transporting energy in the axial direction, namely, convection and conduction. Because the Péclet number is large, we are able to neglect transport by conduction in comparison with transport by convection. On the other hand, in the radial direction, there is only a single mechanism for transport of energy, namely conduction. By performing calculations including conduction in the axial direction, it has been established that it is safe to neglect axial conduction for $Pe \geq 100$. To learn about how to include axial conduction, you can consult the articles by Davis (1973), Acrivos (1980), and Papoutsakis et al. (1980).

Let us make a sample calculation of the Péclet number for laminar flow heat transfer in a tube. The thermal diffusivity of common liquids is typically in the range $10^{-7} - 2\times10^{-7}\ \frac{m^2}{s}$, and we'll use the larger limit. Choose

$$R = 10\ mm, \quad v_0 = 0.05\ \frac{m}{s}, \quad \alpha = 2\times10^{-7}\ \frac{m^2}{s}$$

This yields, $Pe = 2,500$, which is much larger than 100. We can check to see if the flow is laminar by calculating the Reynolds number. If the fluid is water, $\nu \approx 10^{-6}\ \frac{m^2}{s}$, which yields a Prandtl number $\mathrm{Pr} = \frac{\nu}{\alpha} = 5$. Therefore, the Reynolds number is $\mathrm{Re} = 500$, which is comfortably in the laminar flow regime.

The final version of the scaled energy equation is

$$\left(1 - Y^2\right)\frac{\partial \theta}{\partial Z} = \frac{1}{Y}\frac{\partial}{\partial Y}\left(Y\frac{\partial \theta}{\partial Y}\right)$$

We can solve this equation by separation of variables, because the boundary conditions in the $Y-$coordinate are homogeneous. The method of separation of variables yields an infinite series solution for the scaled temperature field.

$$\theta\left(Y, Z\right) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 Z}\phi_n\left(Y\right)$$

In the above solution, the functions $\phi_n(Y)$ are the characteristic functions or eigenfunctions of a proper Sturm-Liouville system.

$$\frac{1}{Y}\frac{d}{dY}\left(Y\frac{d\phi}{dY}\right)+\lambda^2\left(1-Y^2\right)\phi=0$$

$$\frac{d\phi}{dY}(0)=0 \quad \text{or} \quad \phi(0) \ \text{ is finite}$$

$$\phi(1)=0$$

The above ordinary differential equation for $\phi(Y)$ can be solved by applying the following transformations to both the dependent and the independent variables (Lauwerier, 1951, Davis, 1973).

$$X=\lambda Y^2 \qquad\qquad W(X)=e^{\frac{X}{2}}\phi(Y)$$

This leads to the following differential equation for $W(X)$.

$$X\frac{d^2W}{dX^2}+(1-X)\frac{dW}{dX}+\left(\frac{\lambda}{4}-\frac{1}{2}\right)W=0$$

This is known as Kummer's equation. It has two linearly independent solutions, but only one is bounded at $X=0$. Because $\phi(0)$ must be bounded, we must require that $W(0)$ also remain bounded. This rules out the singular solution, leaving us with the regular solution

$$W(X)=c\,M\left(\frac{1}{2}-\frac{\lambda}{4},\ 1,\ X\right)$$

where c is an arbitrary multiplicative constant. The function $M(a,b,X)$ is the confluent hypergeometric function, or Kummer function, and is discussed in Chapter 13 of the "Handbook of Mathematical Functions" by M. Abramowitz and I. A. Stegun, It is an extension of the exponential function, and is written in the form of the following series.

$$M(a,b,X)=1+\frac{a}{b}X+\frac{a(a+1)}{b(b+1)}\frac{X^2}{2!}+\cdots$$
$$+\frac{a(a+1)\cdots(a+n-1)}{b(b+1)\cdots(b+n-1)}\frac{X^n}{n!}+\cdots$$

You can see that when $a=b$,

$$M(a,a,X)=e^X$$

Application of the boundary condition at the tube wall, $\phi(1)=0$, leads to the following transcendental equation for the eigenvalues.

$$M\left(\frac{1}{2}-\frac{\lambda}{4},\ 1,\ \lambda\right)=0$$

The above equation has infinitely many discrete solutions for λ, which we designate as λ_n, with n assuming positive integer values beginning from 1. Corresponding to each value λ_n, there is an eigenfunction $\phi_n(Y)$ given by

$$\phi_n(Y)=e^{-\frac{\lambda_n Y^2}{2}}W_n\left(\lambda_n Y^2\right)$$

The first few eigenvalues are reported in the table.

n	λ_n	λ_n^2
1	2.7044	7.3136
2	6.6790	44.609
3	10.673	113.92
4	14.671	215.24
5	18.670	348.57

Note that technically $\left\{\lambda_n^2\right\}$ is the set of eigenvalues, even though we use the term loosely to designate $\left\{\lambda_n\right\}$ as that set for convenience.

The most important property of a proper Sturm-Liouville system is that the eigenfunctions are orthogonal with respect to a weighting function that is specific to that system. In the present case, the orthogonality property of the eigenfunctions can be stated as follows.

$$\int_0^1 \phi_m(Y)\phi_n(Y)Y\left(1-Y^2\right)dY=0, \quad m\neq n$$

Using this orthogonality property, it is possible to obtain a result for the coefficients in the solution by separation variables.

$$A_n = \frac{\int_0^1 \phi_n(Y)Y\left(1-Y^2\right)dY}{\int_0^1 \phi_n^2(Y)Y\left(1-Y^2\right)dY}$$

The Heat Transfer Coefficient

The heat flux from the wall to the fluid, $q_w(z)$ is a function of axial position. It can be calculated directly by using the result

$$q_w(z) = k\frac{\partial T}{\partial r}(R,z)$$

but as we noted earlier, it is customary to define a heat transfer coefficient $h(z)$ via

$$q_w(z) = h(z)(T_w - T_b)$$

where the bulk or cup-mixing average temperature T_b is introduced. The way to experimentally determine the bulk average temperature is to collect the fluid coming out of the system at a given axial location, mix it completely, and measure its temperature. The mathematical definition of the bulk average temperature was given in an earlier section.

$$T_b = \frac{\int_0^R 2\pi r V(r) T(r,z)\, dr}{\int_0^R 2\pi r V(r)\, dr}$$

where the velocity field $V(r) = v_0\left(1 - r^2/R^2\right)$. You can see from the definition of the heat transfer coefficient that it is related to the temperature gradient at the tube wall in a simple manner.

$$h(z) = \frac{k\frac{\partial T}{\partial r}(R,z)}{(T_w - T_b)}$$

We can define a dimensionless heat transfer coefficient, which is known as the Nusselt number.

$$Nu(Z) = \frac{2hR}{k} = -2\;\frac{\frac{\partial \theta}{\partial Y}(1,Z)}{\theta_b(Z)}$$

where θ_b is the dimensionless bulk average temperature.

By substituting from the infinite series solution for both the numerator and the denominator, the Nusselt number can be written as follows.

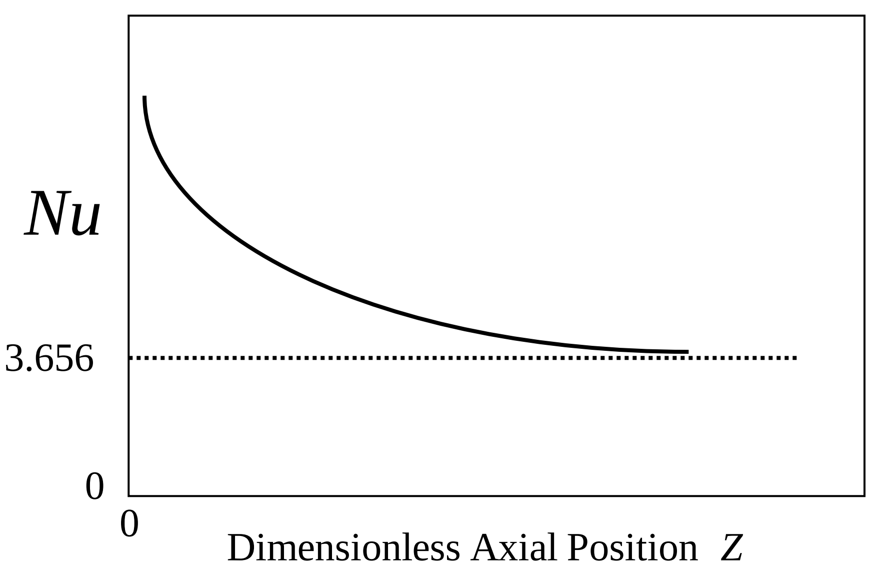
$$Nu(Z) = -2 \frac{\sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 Z} \frac{d\phi_n}{dY}(1)}{4 \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 Z} \int_0^1 Y\left(1-Y^2\right)\phi_n(Y)\,dY}$$

The denominator can be simplified by using the governing differential equation for $\phi_n(Y)$, along with the boundary conditions, to finally yield the following result.

$$Nu = \frac{\sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 Z} \frac{d\phi_n}{dY}(1)}{2 \sum_{n=1}^{\infty} A_n \frac{e^{-\lambda_n^2 Z}}{\lambda_n^2} \frac{d\phi_n}{dY}(1)}$$

We can see that for large Z, only the first term in the infinite series in the numerator, and likewise the first term in the infinite series in the denominator, is important. Therefore, as $Z \to \infty$, $Nu \to \frac{\lambda_1^2}{2} = 3.656$.

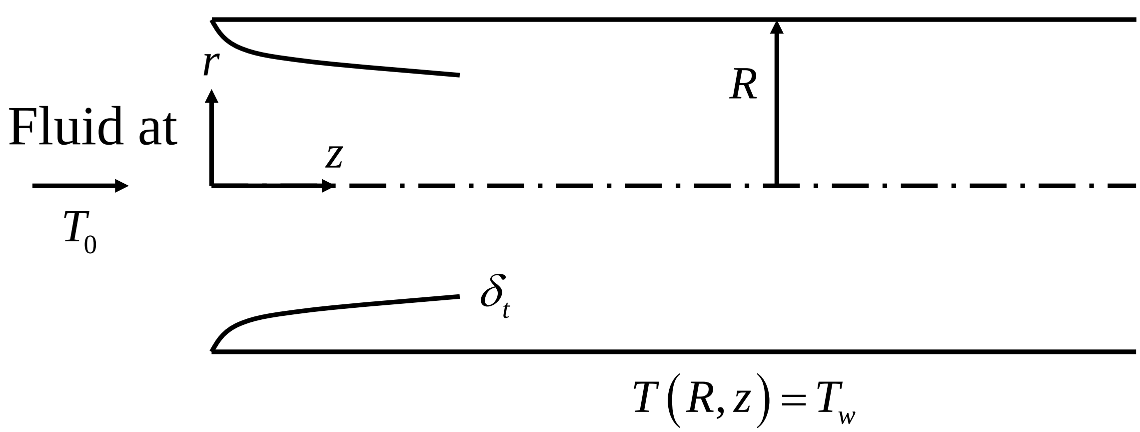
The sketch qualitatively illustrates the behavior of the Nusselt number as a function of dimensionless axial position.



A similar analysis is possible in the case of a uniform wall flux boundary condition. Extensions of the Graetz solution by separation of variables have been made in a variety of ways, accommodating non-Newtonian flow, turbulent flow, and other geometries besides a circular tube.

The Lévêque Approximation

The orthogonal function expansion solution obtained above is convergent at all values of the axial position, but convergence is very slow as the inlet is approached. The main reason for this is the assistance provided by $e^{-\lambda_n^2 Z}$ in accelerating convergence for sufficiently large values of Z. Lévêque (1928) considered the thermal entrance region in a tube and developed an alternative solution, which is useful precisely where the orthogonal function expansion converges too slowly.



We shall now construct the Lévêque solution which is built on the assumption that the thickness of the thermal boundary layer $\delta_t \ll R$. This assumption leads to the following simplifications.

1. Curvature effects can be neglected in the radial conduction term. This means that the derivative $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$ can be approximated by $\frac{1}{R}\frac{\partial}{\partial r}\left(R\frac{\partial T}{\partial r}\right) = \frac{\partial^2 T}{\partial r^2}$.

2. Because we are only interested in the velocity distribution within the thermal boundary layer, we expand the velocity field in a Taylor series in distance measured from the tube wall and retain the first non-zero term.

Defining $x = R - r$, we can rewrite the velocity distribution as

$$v_z(r) \; = \; v_0\left(1 - \frac{(R-x)^2}{R^2}\right) = v_0\left(2\frac{x}{R} - \frac{x^2}{R^2}\right) \approx 2v_0\frac{x}{R}$$

Recall that a power series obtained by any method is a Taylor series. The above approach is simpler than working out the derivatives of $v_z(r)$ in the $x-$coordinate, evaluating them at the wall, and constructing the Taylor series.

3. Because the conditions outside the thermal boundary layer are those in the fluid entering the tube, we shall use the boundary condition $T(x \to \infty) \to T_0$ instead of the centerline boundary condition employed in obtaining the Graetz solution.

Beginning with the simplified energy equation in which axial conduction has been neglected already, and invoking the above assumptions, we have the following governing equation for the temperature field.

$$2v_0 \frac{x}{R} \frac{\partial T}{\partial z} = \alpha \frac{\partial^2 T}{\partial x^2}$$

where the chain rule has been used to transform the second derivative in r to the second derivative in x.

The temperature field $T(x,z)$ satisfies the following boundary conditions.

$$T(x,0) = T_0$$
$$T(0,z) = T_w$$
$$T(\infty,z) = T_0$$

We shall work with a dimensionless version of these equations. For consistency, we scale the temperature and axial coordinate in the same manner as before.

$$\theta = \frac{T - T_w}{T_0 - T_w} \qquad\qquad Z = \frac{z}{R\,Pe}$$

We define a new scaled distance from the wall via $X = x/R$. The scaled governing equation and boundary conditions are given below.

$$2X \frac{\partial \theta}{\partial Z} = \frac{\partial^2 \theta}{\partial X^2}$$

$$\theta(X,0) = 1$$
$$\theta(0,Z) = 0$$
$$\theta(\infty,Z) = 1$$

The similarity of this governing equation and boundary conditions to those in the fluid mechanical problem in which we solved for the velocity distribution between two plates when one of them is held fixed and the other is moved suddenly is not a coincidence. For small values of time in the fluid mechanical problem, we replaced the boundary condition at the top plate with one at an infinite distance from the suddenly moved plate, and used the method of combination of variables to solve the equations. It would be worthwhile for you to go back and review the notes on "combination of variables" at this stage.

By invoking ideas very similar to those used in the fluid mechanical problem, we postulate that a similarity solution exists for the temperature field in the present problem. That is, we assume $\theta(X,Z)=F(\eta)$ where the similarity variable $\eta = X/\delta(Z)$. The variable $\delta(Z)$ represents the scaled thermal boundary layer thickness, and is unknown at this stage. We make the necessary transformations using the chain rule.

$$\frac{\partial \theta}{\partial Z} = \frac{\partial \eta}{\partial Z}\frac{dF}{d\eta} = \left(-\frac{X}{\delta^2}\frac{d\delta}{dZ}\right)\frac{dF}{d\eta} = -\frac{\eta}{\delta}\frac{d\delta}{dZ}\frac{dF}{d\eta}$$

$$\frac{\partial \theta}{\partial X} = \frac{\partial \eta}{\partial X}\frac{dF}{d\eta} = \frac{1}{\delta}\frac{dF}{d\eta}$$

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial}{\partial X}\left[\frac{1}{\delta(Z)}\frac{dF}{d\eta}\right] = \frac{1}{\delta}\frac{\partial}{\partial X}\left[\frac{dF}{d\eta}\right] = \frac{1}{\delta}\frac{\partial \eta}{\partial X}\frac{d}{d\eta}\left[\frac{dF}{d\eta}\right] = \frac{1}{\delta^2}\frac{d^2F}{d\eta^2}$$

Using these results, the partial differential equation for $\theta(X,Z)$ is transformed to an ordinary differential equation for $F(\eta)$.

$$\frac{d^2F}{d\eta^2} + 2\eta^2\left(\delta^2\frac{d\delta}{dZ}\right)\frac{dF}{d\eta} = 0$$

It is evident that the similarity hypothesis will fail unless the quantity inside the parentheses is required to be independent of Z, and therefore, a constant. For convenience, we set this constant to 3/2. Therefore, we have an ordinary differential equation for $F(\eta)$ and another for $\delta(Z)$.

$$\frac{d^2F}{d\eta^2} + 3\eta^2\frac{dF}{d\eta} = 0$$

$$\delta^2\frac{d\delta}{dZ} = \frac{3}{2}$$

To derive the boundary conditions on these functions, we must go to the boundary conditions on $\theta(X,Z)$. In a straightforward way, we see that $\theta(0,Z)=0$ yields $F(0)=0$, and $\theta(\infty,Z)=1$ leads to $F(\infty)=1$. The remaining (inlet) condition gives

$$\theta(X,0)=F\left(\frac{X}{\delta(0)}\right)=1$$

By choosing $\delta(0)=0$, this condition collapses into the condition $F(\infty)=1$ obtained already from the boundary condition on the scaled temperature field as $X\to\infty$. Summarizing the boundary conditions on $F(\eta)$ and $\delta(Z)$, we have

$F(0)=0,\quad F(\infty)=1$, and

$\delta(0)=0$

Integration yields the following solution for the scaled boundary layer thickness $\delta(Z)$.

$$\delta(Z)=\left(\frac{9}{2}Z\right)^{1/3}$$

The solution for $F(\eta)$ is

$$F(\eta) \;=\; \frac{\int_0^\eta e^{-\gamma^3}d\gamma}{\int_0^\infty e^{-\gamma^3}d\gamma} \;=\; \frac{1}{\Gamma(4/3)}\int_0^\eta e^{-\gamma^3}d\gamma$$

Here, $\Gamma(x)$ represents the Gamma function, discussed in the "Handbook of Mathematical Functions" by Abramowitz and Stegun. The numerical value of $\Gamma(4/3)\approx 0.89298$, so that we can write $1/\Gamma(4/3)\approx 1.1199$ or roughly 1.120.

Heat Transfer Coefficient

In the thermal entrance region, when the thermal boundary layer is thin, we can approximate the bulk average temperature T_b by the temperature of the fluid entering the tube T_0. Therefore, we define the heat transfer coefficient in this entrance region by

$$q_w = k\frac{\partial T}{\partial r}(R,z)=h(T_w - T_0)$$

Transforming to dimensionless variables, and defining a Nusselt number $Nu = 2hR/k$, we can write

$$Nu(Z) \; = \; 2\frac{\partial\theta}{\partial X}(0,Z) \; = \; \frac{2}{\delta(Z)}\frac{dF}{d\eta}(0)$$

By substituting for $\delta(Z)$ and $\frac{dF}{d\eta}(0)$, we obtain the following approximate result for the Nusselt number in the thermal entrance region.

$$Nu(Z) \; \approx \; 1.357 \; Pe^{1/3}\left(\frac{R}{z}\right)^{1/3}$$

Comparison with the exact solution shows this result is a good approximation in the range

$$\frac{Pe}{2500} \leq \left(\frac{z}{R}\right) \leq \frac{Pe}{50}$$

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