Non-Dimensionalizing

$$2\,u_m(1-r^2)\,\frac{\partial\theta}{\partial x} = \frac{\alpha}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial\theta}{\partial r}\right) + \alpha\,\frac{\partial^2\theta}{\partial x^2}$$

$$r^* = \frac{r}{r_0} \qquad x^* = \frac{1}{2\,u_m}\,\frac{\alpha}{r_0{}^2}\,x \overset{\text{same as}}{\rightarrow} x^+ = \frac{2}{\text{Re Pr}}\,\frac{x}{D} \rightarrow \frac{\text{conduction}}{\text{convection}}$$

$$(1 - r^*)\,\frac{\partial\theta}{\partial x^*} = \frac{1}{r^*}\,\frac{\partial}{\partial r^*}\left(r^*\,\frac{\partial\theta}{\partial r^*}\right) + \frac{\alpha^2}{4\,u_m{}^2\,r_0{}^2}\,\frac{\partial^2\theta}{\partial x^{*2}}$$

$$\frac{\alpha^2}{4\,u_m{}^2\,r_0{}^2} = \left(\frac{1}{\text{Re Pr}}\right)^2 = \beta$$

$$(1 - r^*)\,\frac{\partial\theta}{\partial x^*} = \frac{1}{r^*}\,\frac{\partial}{\partial r^*}\left(r^*\,\frac{\partial\theta}{\partial r^*}\right) + \beta\,\frac{\partial^2\theta}{\partial x^{*2}}$$

β for This Problem

$$\beta = \frac{\alpha^2}{4\,u_m{}^2\,r_0{}^2} = \left(\frac{1}{2\,u_m\,r_0}\,\frac{k}{\rho\,C_p}\right)^2$$

Given

$D = 0.02\,m$

$k = 0.6\,\frac{W}{m\,K}$

$\rho = 998\,\frac{\text{kg}}{m^3}$

$v = \frac{\mu}{\rho} = 10^{-6}\,\frac{m^2}{s}$

$C_p = 4182\,\frac{J}{\text{kg}\,K}$

$\dot{m} = \rho A\,u_m = 0.01\,\frac{\text{kg}}{s}$

$r_0 = 0.01\,m$

$$u_m = \frac{\dot{m}}{\rho A} = \frac{\dot{m}}{\rho\,\pi\,r_0{}^2}$$

$$\beta = \left(\frac{1}{2\,u_m\,r_0}\,\frac{k}{\rho\,C_p}\right)^2 = \left(\frac{1}{2\,r_0}\,\frac{k}{\rho\,C_p}\,\frac{\rho\,\pi\,r_0{}^2}{\dot{m}}\right)^2 = \left(\frac{1}{2}\,\frac{k}{C_p}\,\frac{\pi\,r_0}{\dot{m}}\right)^2$$

$$\beta = \left(\frac{\pi}{2}\,\frac{0.6}{4182}\,\frac{0.01}{0.01}\right)^2\left(\frac{W}{m\,K}\,\frac{\text{kg}\,K}{J}\,\frac{s}{\text{kg}}\,\frac{m}{1}\right)^2 \simeq 0.011\,\frac{1}{m} \simeq 5*10^{-8}$$

In[ ]:=
$$\left(\frac{\pi\,0.6}{2\times 4182}\right)^2$$

Out[ ]= 5.07895×10^{-8}

Nusselt No.

$q_{\text{out,wall}} = h\,(T_{\text{mean}} - T_{\text{wall}})$

$h = \frac{q_o}{T_m - T_w} = \frac{-k\left(\frac{\partial T}{\partial r}\right)_{r=r_0}}{T_m - T_W}$

$\text{Nu}_x = h_x \frac{D}{k} = \frac{-\left(\frac{\partial T}{\partial r}\right)_{r=r_0}}{T_m - T_W} D$

In terms of non-dimensional $\theta = \frac{T - T_w}{T_{\text{in}} - T_w}$ and $r^* = \frac{r}{r_0}$

$T = \theta(T_{\text{in}} - T_w) + T_w$

$T_m = \frac{1}{N}\sum T_i = \frac{1}{N}\sum[\theta_i(T_{\text{in}} - T_w) + T_w] = T_w + (T_{\text{in}} - T_w)\frac{1}{N}\sum\theta_i$

$T_m = \theta_m(T_{\text{in}} - T_w) + T_w$

$\frac{\partial T}{\partial r} = (T_{\text{in}} - T_w)\frac{\partial \theta}{\partial r}$

$\frac{\partial \theta}{\partial r} = \frac{\partial \theta}{\partial r^*}\frac{\partial r^*}{\partial r} = \frac{1}{r_0}\frac{\partial \theta}{\partial r^*}$

$\frac{\left(\frac{\partial T}{\partial r}\right)_{r=r_0}}{T_m - T_W} = \frac{(T_{\text{in}} - T_w)\left(\frac{\partial \theta}{\partial r}\right)_{r_0}}{\theta_m(T_{\text{in}} - T_w)} = \frac{\frac{1}{r_0}\left(\frac{\partial \theta}{\partial r^*}\right)_{r^*=1}}{\theta_m}$

$\text{Nu}_x = -D\frac{\left(\frac{\partial T}{\partial r}\right)_{r=r_0}}{T_m - T_W} = -D\frac{\frac{1}{r_0}\left(\frac{\partial \theta}{\partial r^*}\right)_{r_0}}{\theta_m} = -2\frac{\left(\frac{\partial \theta}{\partial r^*}\right)_{r^*=1}}{\theta_m}$

Approximate $\left(\frac{\partial \theta}{\partial r^*}\right)_{r=r_0}$ using finite difference using non-dimensional $\theta = \frac{T - T_w}{T_{\text{in}} - T_w}$

$\theta_1 = \theta(1 - \Delta r^*), \qquad \theta_2 = \theta(1 - 2\,\Delta r^*), \qquad \theta_0 = \theta(1) = 0$

$\theta_1 = -\Delta r^*\left(\frac{\partial \theta}{\partial r^*}\right)_0 + \frac{\Delta r^{*2}}{2}\left(\frac{\partial^2 \theta}{\partial r^{*2}}\right)_0 - \frac{\Delta r^{*3}}{6}\left(\frac{\partial^3 \theta}{\partial r^{*3}}\right)_0 + h.o.t$

$\theta_2 = -2\,\Delta r^*\left(\frac{\partial \theta}{\partial r^*}\right)_0 + 4\,\frac{\Delta r^{*2}}{2}\left(\frac{\partial^2 \theta}{\partial r^{*2}}\right)_0 - 8\,\frac{\Delta r^{*3}}{6}\left(\frac{\partial^3 \theta}{\partial r^{*3}}\right)_0 + h.o.t$

$\theta_3 = -3\,\Delta r^*\left(\frac{\partial \theta}{\partial r^*}\right)_0 + 9\,\frac{\Delta r^{*2}}{2}\left(\frac{\partial^2 \theta}{\partial r^{*2}}\right)_0 - 27\,\frac{\Delta r^{*3}}{6}\left(\frac{\partial^3 \theta}{\partial r^{*3}}\right)_0 + h.o.t$

$\theta_1 - \frac{1}{4}\theta_2 = \left(\frac{\partial \theta}{\partial r^*}\right)_0\left(-\Delta r^* + \frac{1}{2}\Delta r^*\right) + O(\Delta r^{*3})$

$\left(\frac{\partial \theta}{\partial r^*}\right)_0 = \frac{\frac{1}{4}\theta_2 - \theta_1}{\frac{1}{2}\Delta r^*} = \frac{2}{\Delta r^*}\left[\frac{1}{4}\theta(1 - 2\,\Delta r^*) - \theta(1 - \Delta r^*)\right] + O(\Delta r^{*2})$

$\left(\frac{\partial \theta}{\partial r^*}\right)_0 \simeq \frac{3}{\Delta r}\left(-\theta_1 + \frac{1}{2}\theta_2 - \frac{1}{9}\theta_3\right) + O(\Delta r^{*3})$

Fully Developed Condition

$x_f \simeq 0.05 * D * \text{Re Pr}$

Non-dimensional fully developed condition:

$x^* = \frac{2}{\text{Re Pr}} \frac{x}{D} \rightarrow x = \frac{D * \text{Re Pr}}{2} x^*$

So we have

$\frac{D * \text{Re Pr}}{2} x_f^* \simeq 0.05 * D * \text{Re Pr}$

$x_f^* \simeq 0.1$

Finite Difference

Interior Nodes

$$u_j\left(\frac{\theta-\theta_{i-1}}{\Delta x}\right) = \left(\frac{\theta_{j+1}+\theta_{j-1}-2\,\theta}{\Delta r^2}\right) + \frac{1}{r_j}\left(\frac{\theta_{j+1}-\theta_{j-1}}{2\,\Delta r}\right) + \beta\left(\frac{\theta_{i+1}+\theta_{i-1}-2\,\theta}{\Delta x^2}\right)$$

$$\left(\frac{1}{r_j\,2\,\Delta r}-\frac{1}{\Delta r^2}\right)\theta_{j-1} + \left(\frac{2}{\Delta r^2}+\frac{2\,\beta}{\Delta x^2}+\frac{u_j}{\Delta x}\right)\theta + \left(-\frac{1}{r_j\,2\,\Delta r}-\frac{1}{\Delta r^2}\right)\theta_{j+1} = \left(\frac{\beta}{\Delta x^2}+\frac{u_j}{\Delta x}\right)\theta_{i-1} + \frac{\beta}{\Delta x^2}\theta_{i+1}$$

$$A = \frac{1}{\Delta r^2} \qquad B = \frac{\beta}{\Delta x^2} \qquad C_j = \frac{1}{r_j\,2\,\Delta r} \qquad D_j = \frac{u_j}{\Delta x}$$

$$[C_j - A]\,\theta_{j-1} + [2\,A + 2\,B + D_j]\,\theta + [-C_j - A]\,\theta_{j+1} = [B + D_j]\,\theta_{i-1} + B\,\theta_{i+1}$$

Axis of Symmetry

$j = 0$

$$u_j\left(\frac{\theta-\theta_{i-1}}{\Delta x}\right) = 2\left(\frac{2\,\theta_{j+1}-2\,\theta}{\Delta r^2}\right) + \beta\left(\frac{\theta_{i+1}+\theta_{i-1}-2\,\theta}{\Delta x^2}\right)$$

$$\left(\frac{4}{\Delta r^2}+\frac{2\,\beta}{\Delta x^2}+\frac{u_j}{\Delta x}\right)\theta + 4\left(-\frac{1}{\Delta r^2}\right)\theta_{j+1} = \left(\frac{\beta}{\Delta x^2}+\frac{u_j}{\Delta x}\right)\theta_{i-1} + \frac{\beta}{\Delta x^2}\theta_{i+1}$$

$$A = \frac{1}{\Delta r^2} \qquad B = \frac{\beta}{\Delta x^2} \qquad C_j = \frac{1}{r_j\,2\,\Delta r} \qquad D_j = \frac{u_j}{\Delta x}$$

$$[4\,A + 2\,B + D_j]\,\theta + [-4\,A]\,\theta_{j+1} = [B + D_j]\,\theta_{i-1} + B\,\theta_{i+1}$$

Right Nodes

$i = N - 1$

$$u_j\left(\frac{\theta-\theta_{i-1}}{\Delta x}\right) = \left(\frac{\theta_{j+1}+\theta_{j-1}-2\,\theta}{\Delta r^2}\right) + \frac{1}{r_j}\left(\frac{\theta_{j+1}-\theta_{j-1}}{2\,\Delta r}\right) + \beta\left(\frac{2\,\theta_{i-1}-2\,\theta}{\Delta x^2}\right)$$

$$\left(\frac{1}{r_j\,2\,\Delta r}-\frac{1}{\Delta r^2}\right)\theta_{j-1} + \left(\frac{u_j}{\Delta x}+\frac{2}{\Delta r^2}+\frac{2\,\beta}{\Delta x^2}\right)\theta + \left(-\frac{1}{r_j\,2\,\Delta r}-\frac{1}{\Delta r^2}\right)\theta_{j+1} = \left(\frac{2\,\beta}{\Delta x^2}+\frac{u_j}{\Delta x}\right)\theta_{i-1}$$

$$A = \frac{1}{\Delta r^2} \qquad B = \frac{\beta}{\Delta x^2} \qquad C_j = \frac{1}{r_j\,2\,\Delta r} \qquad D_j = \frac{u_j}{\Delta x}$$

$$[C_j - A]\,\theta_{j-1} + [2\,A + 2\,B + D_j]\,\theta + [-C_j - A]\,\theta_{j+1} = 2[B + D_j]\,\theta_{i-1}$$

Axis Right Corner

$i = N - 1$

$j = 0$

$$u_j\left(\frac{\theta-\theta_{i-1}}{\Delta x}\right) = 4\left(\frac{\theta_{j+1}-\theta}{\Delta r^2}\right) + 2\,\beta\left(\frac{\theta_{i-1}-\theta}{\Delta x^2}\right)$$

$$\left(\frac{u_j}{\Delta x}+\frac{4}{\Delta r^2}+\frac{2\,\beta}{\Delta x^2}\right)\theta + 4\left(-\frac{1}{\Delta r^2}\right)\theta_{j+1} = \left(\frac{2\,\beta}{\Delta x^2}+\frac{u_j}{\Delta x}\right)\theta_{i-1}$$

$$A = \frac{1}{\Delta r^2} \qquad B = \frac{\beta}{\Delta x^2} \qquad C_j = \frac{1}{r_j\,2\,\Delta r} \qquad D_j = \frac{u_j}{\Delta x}$$

$$[4\,A + 2\,B + D_j]\,\theta + 4[-A]\,\theta_{j+1} = 2[B + D_j]\,\theta_{i-1}$$