Graetz Problem with Conduction:
Comparison of Nusselt Number from Simulation with Analytic
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November 27, 2018

To investigate the Graetz problem, two simulations were executed: one where axial conduction was neglected, and one where it was included. The resulting Nusselt No. as a function of non-dimensional axial distance x^* (Nu_x) was compared with that for the analytical solution, which neglects axial conduction. As the mesh is refined, the values for Nu_x without conduction approach the analytical solution for short axial distances, but converge to a slightly lower Nu_x in the thermally fully-developed region. When conduction is included, the values for Nu_x are smaller across the board.

Problem Setup and Theory

The Graetz problem considers fully developed fluid flow of constant temperature T_{in} in a circular pipe flowing into a region where the pipe walls are held at constant temperature T_w. The fluid is assumed to be incompressible, so that the flow decoupled from the energy equation. The governing equation for the flow at steady state is:

$$u_x(r)\,\frac{\partial T}{\partial x} = \frac{\alpha}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right) + \alpha\,\frac{\partial^2 T}{\partial x^2}$$

where the flow is in the axial direction x and depends only on radial distance r. The fully developed flow is given as:

$$u_x(r) = 2\,u_m(1-r^2).$$

Here u_m is the mean flow rate. It is beneficial to non-dimensionalize the above equation for the temperature. The following non-dimensional parameters are used:

$$r^* = \frac{r}{r_0} \qquad x^* = \frac{1}{2\,u_m}\,\frac{\alpha}{r_0^{\,2}}\,x \quad \theta = \frac{T-T_w}{T_{\mathrm{in}}-T_w},$$

where r_0 is the pipe radius and $\alpha = \frac{\text{conductivity}}{\text{density} * \text{heat capacity}} = \frac{k}{\rho\,C_p}$. Note that the non-dimensional temperature, θ, is 1 at the inlet and 0 at the wall. Note also that $x^* = \frac{1}{2\,u_m}\,\frac{\alpha}{r_0^{\,2}}\,x = \frac{2}{\mathrm{Re}\,\mathrm{Pr}}\,\frac{x}{D} = \frac{2}{\text{Reynolds No.} * \text{Prandlt No.}}\,\frac{\text{axial distance}}{\text{pip diameter}}$. Applying these relations, the governing equation becomes:

$$\left(1 - r^{*2}\right)\frac{\partial\theta}{\partial x^*} = \frac{1}{r^*}\,\frac{\partial}{\partial r^*}\left(r^*\,\frac{\partial\theta}{\partial r^*}\right) + \beta\,\frac{\partial^2\theta}{\partial x^{*2}}.$$

Here $\beta = \frac{\alpha^2}{4\,u_m^{\,2}\,r_0^{\,2}} = \left(\frac{1}{\mathrm{Re}\,\mathrm{Pr}}\right)^2$.

For this problem, the thermal properties of water at room-temperature were used to assign a value to β. A mass flow rate of $\dot{m} = 0.01\ \mathrm{kg}/s$ was also used. Using the mass flow rate to replace the mean velocity, and taking the conductivity k and heat capacity C_p of water to be 0.6 $W/m\cdot K$ and 4182 $J/m\cdot K$ respectively, we have

$$\beta = \frac{\pi}{4}\,\frac{k}{\dot{m}\,C_p} = \frac{\pi}{4}\,\frac{0.6}{0.01*4182}\,m^{-1} \simeq 0.011\,\frac{1}{m}.$$

Notice that this is a sufficiently small number that we might expect neglecting axial conduction to provide a reasonable approximation to the thermal distribution in the flow.

In this problem, we will be looking at the the local Nusselt No, Nu_x given by the relations:

$$q_{\text{out,wall}} = h_x(T_m - T_w) = -k\left(\frac{\partial T}{\partial r}\right)_{r=r_0}$$

$$\mathrm{Nu}_x = h_x\,\frac{D}{k} = \frac{-\left(\frac{\partial T}{\partial r}\right)_{r=r_0}}{T_m - T_W}\,D$$

Here the convective heat transfer coefficient h_x is used, and again, D is the pipe diameter, and T_m is the mean temperature at a given axial position. Converting this expression for Nu_x to one using our non-dimensional definitions we use $T_m = (T_{\text{in}} - T_w)\,\theta_m$ and $\frac{\partial T}{\partial r} = (T_{\text{in}} - T_w)\,\frac{1}{r_0}\,\frac{\partial \theta}{\partial r^*}$ to give:

$$\mathrm{Nu}_x = -2\,\frac{\left(\frac{\partial \theta}{\partial r}\right)_{r=1}}{\theta_m}$$

After running the simulation, $\left(\frac{\partial \theta}{\partial r^*}\right)_{r^*=1}$ is approximated using finite difference as

$$\left(\frac{\partial \theta}{\partial r^*}\right)_0 = \frac{2}{\Delta r^*}\left[\frac{1}{4}\,\theta(1 - 2\,\Delta r^*) - \theta(1 - \Delta r^*)\right] + O(\Delta r^{*2}).$$

A further refinement of the approximation can be made using the next farthest temperature node and is given as

$$\left(\frac{\partial \theta}{\partial r^*}\right)_0 = \frac{3}{\Delta r}\left(-\theta_1 + \frac{1}{2}\,\theta_2 - \frac{1}{9}\,\theta_3\right) + O(\Delta r^{*3}).$$

However, for the mesh refinements used in the simulations, the adjustment this better approximation makes was found to have negligible impact on the resulting Nu_x values.

To produce meaningful results, it was necessary to execute the simulation over a reasonable axial distance. This was taken to be 2 to 4 times larger than the approximate distance for which the thermal distribution is fully developed x_f, which was given as $x_f \simeq 0.05 * D * \mathrm{Re}\,\mathrm{Pr}$. Since, in terms of our non-dimensional variable x^*, we have $x = x^* * D *\ \mathrm{Re}\,\mathrm{Pr}/2$, the non-dimensional fully-developed distance is:

$$x_f{}^* \simeq 0.1$$

For this problem, some analytical values for Nu_{x^*} are given in the following table

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x^*	0.001	0.004	0.01	0.04	0.08	0.1	0.2
Nu_{x^*}	12.8	8.03	6.	4.17	3.77	3.71	3.66

Simulation Setup

Two finite-difference simulations were developed for the problem. Both simulations were written in c++ and use the GNU Scientific Library (GSL). Also, for both, the following boundary conditions were used:

$$\theta_{\text{in}} = \theta_{0,j} = 1 \qquad \theta_w = \theta_{i,M} = 0 \qquad \left(\frac{\partial \theta}{\partial r^*}\right)_{r^*=0} = 0$$

The first, which neglects conduction, uses the temperature distribution at the current nodes and a forward-difference formula to approximate the temperature distribution at the next set of nodes. The following finite-difference equations are used:

$$\text{Interior Nodes: } u_j\left(\frac{\theta_{i,j} - \theta_{i-1,j}}{\Delta x}\right) = \left(\frac{\theta_{i,j+1} + \theta_{i,j-1} - 2\,\theta_{i,j}}{\Delta r^2}\right) + \frac{1}{r_j}\left(\frac{\theta_{i,j+1} - \theta_{i,j-1}}{2\,\Delta r}\right)$$

$$\text{Node at Symmetry Axis: } u_j\left(\frac{\theta_{i,0} - \theta_{i-1,0}}{\Delta x}\right) = 4\left(\frac{\theta_{i,1} - \theta_{i,0}}{\Delta r^2}\right)$$

The finite-difference formulae are solved at each step using the sparse-matrix solver GMRES (Generalized Minimum Residual Method) provided by GSL. Note: $u_j = (1 - r_j^{*2})$.

The second simulation, which includes conduction, reads in the temperature distribution created by the first simulation and then iterates over the distribution column by column, once again employing the sparse-matrix solver to update each column, after which, it calculates the residual-norm for the nodes. In this way, *one iteration* of the simulation consists of a march through the columns using the sparse-matrix solver followed by a check of the residuals. The solver employs the following finite-difference equations:

Interior Nodes: $u_j\left(\frac{\theta_{i,j} - \theta_{i-1,j}}{\Delta x}\right) = \left(\frac{\theta_{i,j+1} + \theta_{i,j-1} - 2\,\theta_{i,j}}{\Delta r^2}\right) + \frac{1}{r_j}\left(\frac{\theta_{i,j+1} - \theta_{i,j-1}}{2\,\Delta r}\right) + \beta\left(\frac{\theta_{i+1,j} + \theta_{i-1,j} - 2\,\theta_{i,j}}{\Delta x^2}\right)$

Axis of Symmetry: $u_j\left(\frac{\theta_{i,0} - \theta_{i-1,0}}{\Delta x}\right) = 4\left(\frac{\theta_{i,1} - \theta_{i,0}}{\Delta r^2}\right) + \beta\left(\frac{\theta_{i+1,0} + \theta_{i-1,0} - 2\,\theta_{i,0}}{\Delta x^2}\right)$

Last Column: $u_j\left(\frac{\theta_{N,j} - \theta_{N-1,j}}{\Delta x}\right) = \left(\frac{\theta_{N,j+1} + \theta_{N,j-1} - 2\,\theta_{N,j}}{\Delta r^2}\right) + \frac{1}{r_j}\left(\frac{\theta_{N,j+1} - \theta_{N,j-1}}{2\,\Delta r}\right) + 2\,\beta\left(\frac{\theta_{N-1,j} - \theta_{N,j}}{\Delta x^2}\right)$

Last Column, Axis Node: $u_j\left(\frac{\theta_{N,0} - \theta_{N-1,0}}{\Delta x}\right) = 4\left(\frac{\theta_{N,1} - \theta_{N,0}}{\Delta r^2}\right)$

Results and Discussion

For the final simulation results, the mesh spanned r^* from 0 to 1 and x^* from 0 to 0.4 in steps of $dr^* = 0.002$ and $dx^* = 0.0002$. Though a finer mesh may have given better results, the computation time made finer meshes prohibitive. As well, 5,000 iterations were made for the iterative simulation, to come as close to convergence as was time-wise manageable. From the resulting non-dimensional temperature distributions, the following values for Nu_{x^*} were calculated:

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x^*	Analytic	Without Conduction	With Conduction
0.001	12.8	14.2	13.34
0.004	8.03	8.622	8.015
0.01	6.	6.334	5.941
0.04	4.17	4.151	4.072
0.08	3.77	3.642	3.581
0.1	3.71	3.569	3.508
0.2	3.66	3.504	3.444
∞/0.3	3.66	3.502	3.444

Table: Nu_{x^*} *for specific values of* x^*

For both simulations, calculated values for Nu_{x^*} follow the form of the analytic, no-conduction problem. However, the values are are initially higher, but then dip lower in the fully-developed region. The following plot shows the simulated temperature distributions.

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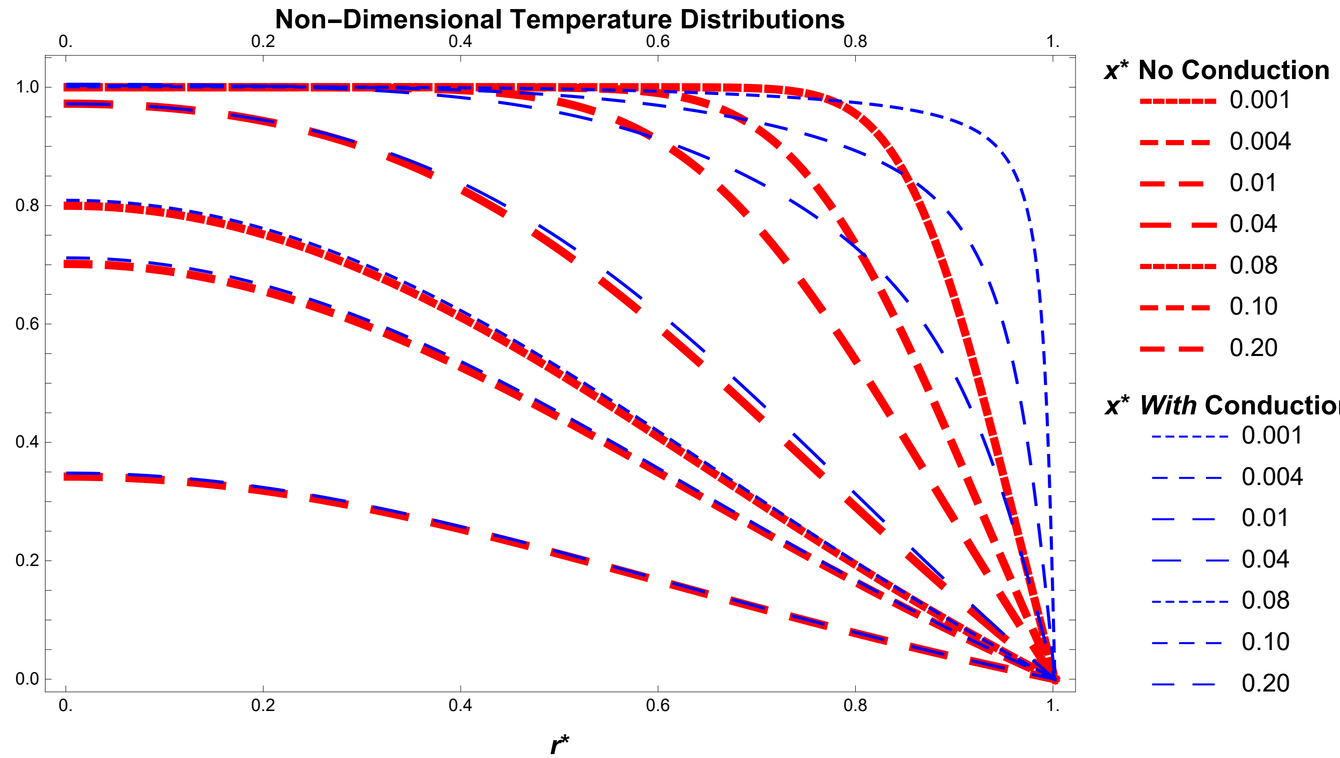

Figure: Temperature Distributions for values of x^*

The temperature distributions highlight the battle between the 2 terms in the equation for Nu_{x^*} which Notice that the temperatures are generally higher for the solution that includes axial conduction. This makes sense, since the presence of conduction allows for more heat to be drawn forward out of the initial incoming flow. This is especially true near the wall of the pipe where the slow flow velocity means that convection is minimum. In fact, even with quite a small value for β, conduction actually dominates the axial flow of heat for the region near the inlet and very near the pipe walls.

Considering the temperature plot for the simulations and the definition of Nu_{x^*}, is it notable that the temperature at the pipe wall near the inlet is steeper when axial conduction is included. However, the mean temperature is also higher, enough so that it has a greater effect on Nu_{x^*} than the steeper gradient. In the fully-developed region, the gradients at the wall are nearly identical, while the overall average temperature is higher, which also results in lower local Nusselt numbers.

Considering these effects of conduction very near the pipe wall, it comes to mind that it might be interesting to perform the same simulation, but allow for back-conduction. This could be accomplished by setting the boundary conditions such that the the heated wall boundary does not begin until some small positive distance into the simulation. Observing the plot, one might think to set that distance at $x^* = 0.04$. However, since the distance of consideration is so short, other effects, such as conduction in the pipe wall itself, may prove a higher correction than back-conduction in the fluid.

For both solutions, the Nu_{x^*} has basically converged by the position $x^* = 2\,x_f^* \simeq 0.2$. Note that, even though the local Nusselt number is smaller when axial conduction is included, the temperature gradients at the wall are nearly identical. This means that the heat load on the wall is nearly identical. According to the above plot, the gradients converge over very short distances ($x^* \leq 0.04$). When conduction is included, the heat load on the wall is only substantially different over a very short distances, which may have quite negligible effects in a real-world system.

Simulation Mesh, Iterations and Convergence

When running the simulations, different mesh sizes were attempted. In general, the returned solutions approached the analytical solution as the mesh density was increased.

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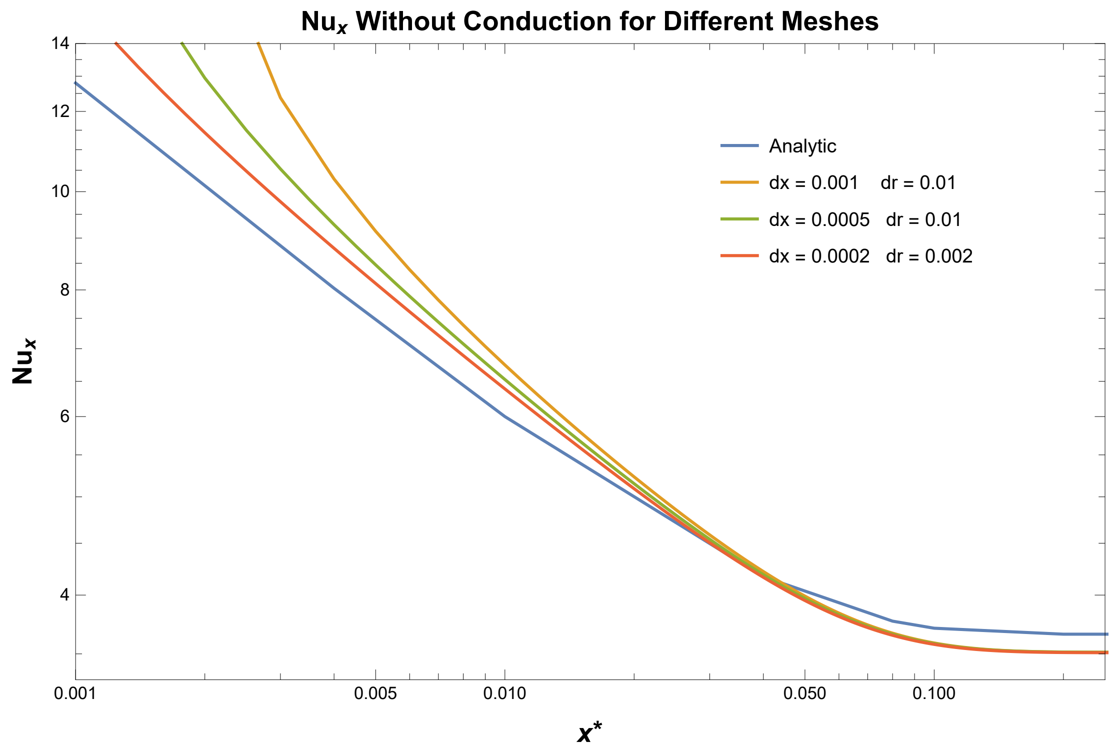

Figure: Values for Nu_x *calculated from simulations with different mesh densities. Note the log-log scale of the plot.*

Observing Figure above, we see that Nu_{x^*} for small values of x^* approaches the analytic solution as the mesh density increases. However, in the fully developed region, all the mesh densities converged to a value of approximately 3.50, which is smaller than the analytic value of 3.66. It is possible that this due to the type of solver used: one that marches along using forward difference. The mesh used for the final values where $dx^* = 0.0002$ and $dr^* = 0.002$ was chosen due to the longer computation time for denser meshes, which was mostly problematic in the iterative solver.

For the Iterative solver, convergence proved to be difficult, especially for very fine meshes, and deciding on convergence criterion was difficult. Due to computation time, the final iterative solver was executed 5,000 times, but is not believed to have converged.

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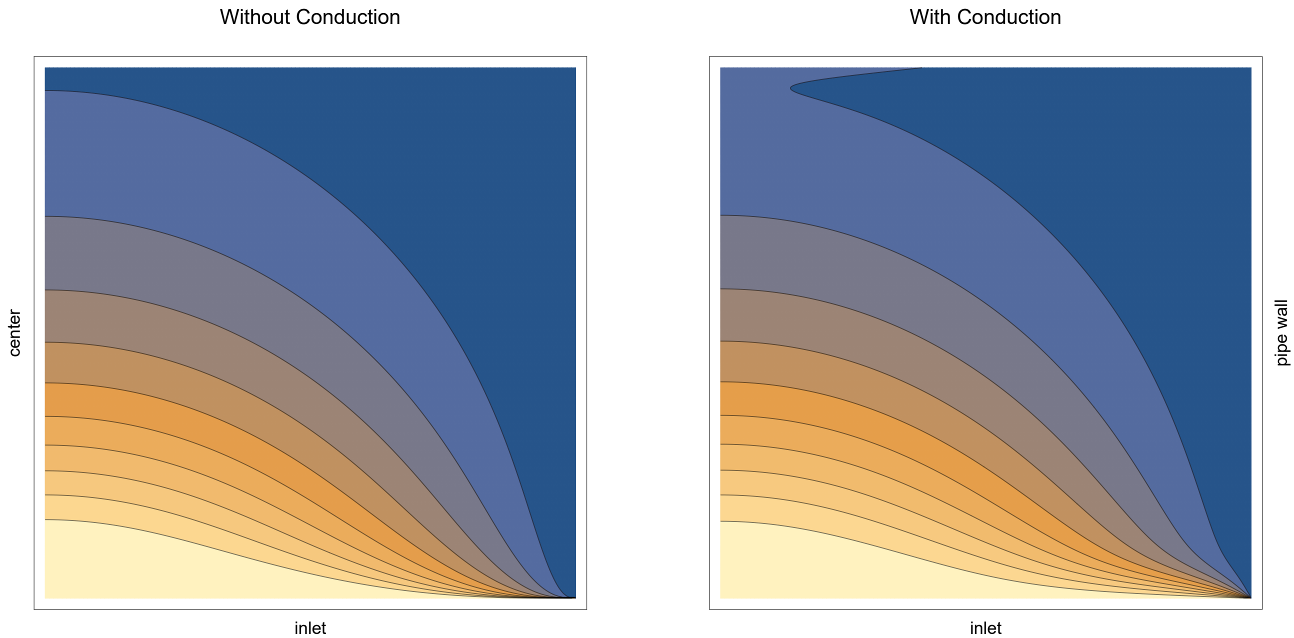

Figure: Qualitative Contour Plots of the Temperature Distribution Returned by the Solvers

Observing the figure above, notice the curvature in the lower right corner of the simulation where conduction was included. With coarser meshes (to reduce computation time) and more iterations, this curvature was observed to smooth out and “push” the contour lines farther out, meaning that higher temperatures should exist further downstream than is observed here. However, it likely that the temperature gradient near the wall would not increase by much over the current simulation. This would drive the local downstream Nusselt number down even further.

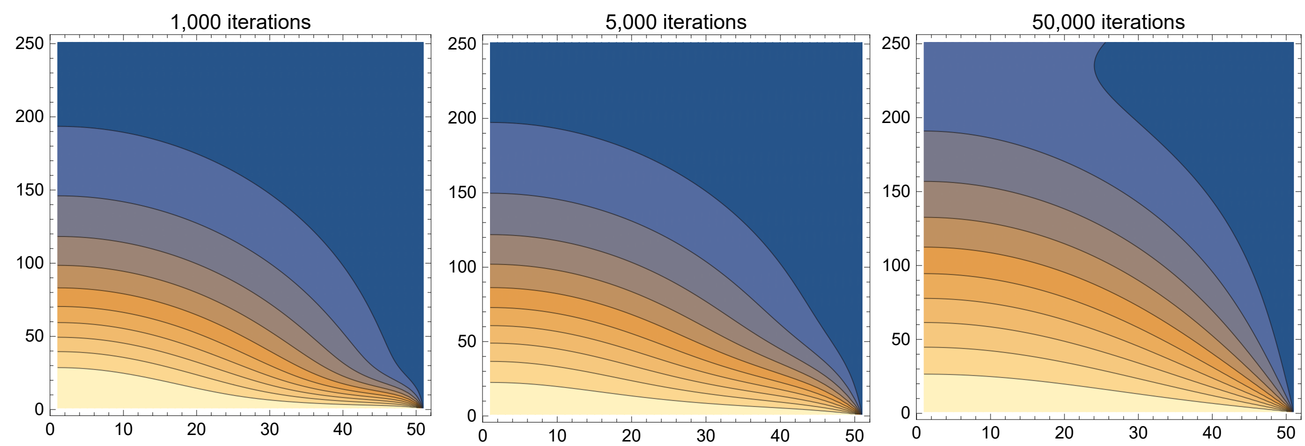

Figure: Shows the smoothing of the curvature near the inlet and the extension of the temperature field further into the pipe as more iterations are made .

Unfortunately, observing the residuals for the simulations involving further iterations can be misleading. Just because the curve appears flat or has dropped to a value much smaller than the initial value for the residuals.

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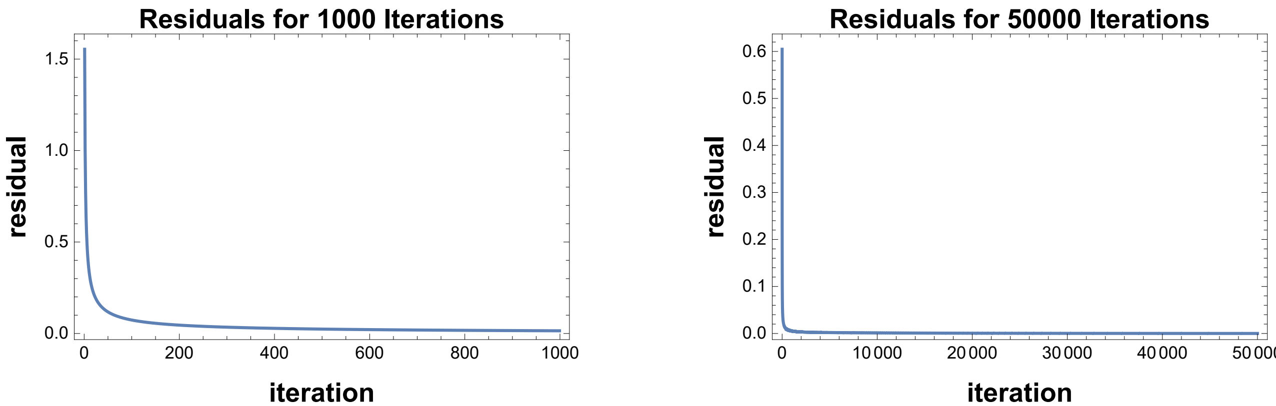

Figure: Residuals for the simulations in the above figure.

The slow convergence of the solver to the “true” temperature distribution in the pipe may be due to the calculation scheme. Perhaps a point-wise Gauss-Seidel method would work more efficiently, rather than using a sparse-matrix solver and running over column by column. Also, a better initial guess for the solution might prove very helpful. An axial conduction term could be included in the marching, forward-difference solver, which may provide a better initial guess for the iterative solver to start with. Finally, it is possible that making the mesh coarser for larger values of x^* where the gradients are smaller could also reduce computation time while minimally affecting the results.

Conclusions

Despite numerical difficulties, it is pretty evident that the addition of conduction when the product of the Reynolds and Prandtl number (β is small) does not greatly affect the heat load on the walls of the pipe except for a very small region near the entrance. For this reason, when making calculations for real-world applications, tabulated analytic results can often suffice. As far as the numerics are concerned, more work could be done to try and suss-out the issues and improve the results.