

Generative Adversarial Networks

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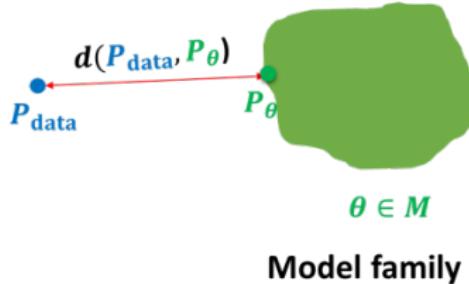
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Lecture 9

Recap



$$\mathbf{x}_i \sim P_{\text{data}} \\ i = 1, 2, \dots, n$$



- Model families
 - Autoregressive Models: $p_\theta(\mathbf{x}) = \prod_{i=1}^n p_\theta(x_i | \mathbf{x}_{<i})$
 - Variational Autoencoders: $p_\theta(\mathbf{x}) = \int p_\theta(\mathbf{x}, \mathbf{z}) d\mathbf{z}$
 - Normalizing Flow Models: $p_X(\mathbf{x}; \theta) = p_Z(\mathbf{f}_\theta^{-1}(\mathbf{x})) \left| \det \left(\frac{\partial \mathbf{f}_\theta^{-1}(\mathbf{x})}{\partial \mathbf{x}} \right) \right|$
- All the above families are based on maximizing likelihoods (or approximations)
- Is the likelihood a good indicator of the quality of samples generated by the model?

Towards likelihood-free learning

- **Case 1:** Optimal generative model will give best **sample quality** and highest test **log-likelihood**
- For imperfect models, achieving high log-likelihoods might not always imply good sample quality, and vice-versa (Theis et al., 2016)

Towards likelihood-free learning

- **Case 2:** Great test log-likelihoods, poor samples. E.g., For a discrete noise mixture model $p_\theta(\mathbf{x}) = 0.01p_{\text{data}}(\mathbf{x}) + 0.99p_{\text{noise}}(\mathbf{x})$
 - 99% of the samples are just noise
 - Taking logs, we get a lower bound

$$\begin{aligned}\log p_\theta(\mathbf{x}) &= \log[0.01p_{\text{data}}(\mathbf{x}) + 0.99p_{\text{noise}}(\mathbf{x})] \\ &\geq \log 0.01p_{\text{data}}(\mathbf{x}) = \log p_{\text{data}}(\mathbf{x}) - \log 100\end{aligned}$$

- For expected likelihoods, we know that
 - Lower bound

$$E_{p_{\text{data}}}[\log p_\theta(\mathbf{x})] \geq E_{p_{\text{data}}}[\log p_{\text{data}}(\mathbf{x})] - \log 100$$

- Upper bound (via non-negativity of KL)

$$E_{p_{\text{data}}}[\log p_{\text{data}}(\mathbf{x})] \geq E_{p_{\text{data}}}[\log p_\theta(\mathbf{x})]$$

- As we increase the dimension of \mathbf{x} , absolute value of $\log p_{\text{data}}(\mathbf{x})$ increases proportionally but $\log 100$ remains constant. Hence, $E_{p_{\text{data}}}[\log p_\theta(\mathbf{x})] \approx E_{p_{\text{data}}}[\log p_{\text{data}}(\mathbf{x})]$ in very high dimensions

Towards likelihood-free learning

- **Case 3:** Great samples, poor test log-likelihoods. E.g., Memorizing training set
 - Samples look exactly like the training set (cannot do better!)
 - Test set will have zero probability assigned (cannot do worse!)
- The above cases suggest that it might be useful to disentangle likelihoods and samples
- **Likelihood-free learning** consider objectives that do not depend directly on a likelihood function

Comparing distributions via samples



$$S_1 = \{\mathbf{x} \sim P\}$$



$$S_2 = \{\mathbf{x} \sim Q\}$$

Given a finite set of samples from two distributions $S_1 = \{\mathbf{x} \sim P\}$ and $S_2 = \{\mathbf{x} \sim Q\}$, how can we tell if these samples are from the same distribution? (i.e., $P = Q?$)

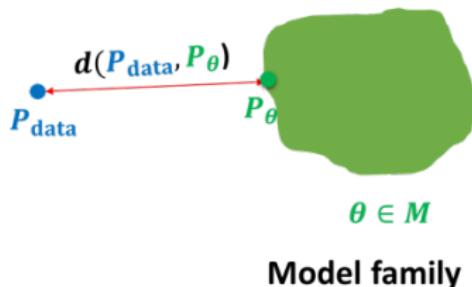
Two-sample tests

- Given $S_1 = \{\mathbf{x} \sim P\}$ and $S_2 = \{\mathbf{x} \sim Q\}$, a **two-sample test** considers the following hypotheses
 - Null hypothesis $H_0: P = Q$
 - Alternate hypothesis $H_1: P \neq Q$
- Test statistic T compares S_1 and S_2 e.g., difference in means, variances of the two sets of samples
- If T is less than a threshold α , then accept H_0 else reject it
- Key observation:** Test statistic is **likelihood-free** since it does not involve the densities P or Q (only samples)

Generative modeling and two-sample tests



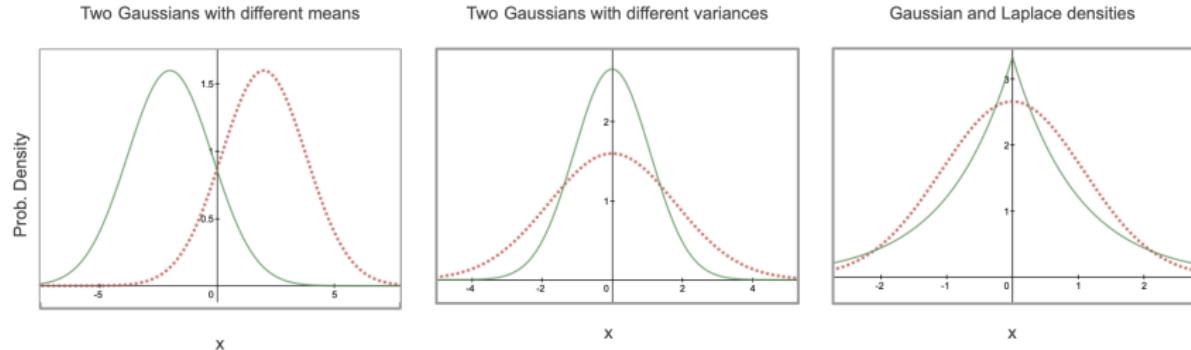
$$\mathbf{x}_i \sim P_{\text{data}} \\ i = 1, 2, \dots, n$$



- Apriori we assume direct access to $S_1 = \mathcal{D} = \{\mathbf{x} \sim p_{\text{data}}\}$
- In addition, we have a model distribution p_θ
- Assume that the model distribution permits efficient sampling (e.g., directed models). Let $S_2 = \{\mathbf{x} \sim p_\theta\}$
- **Alternate notion of distance between distributions:** Train the generative model to minimize a two-sample test objective between S_1 and S_2

Two-Sample Test via a Discriminator

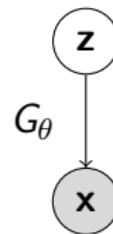
- Finding a two-sample test objective in high dimensions is hard



- In the generative model setup, we know that S_1 and S_2 come from different distributions p_{data} and p_{θ} respectively
- Key idea: Learn** a statistic that **maximizes** a suitable notion of distance between the two sets of samples S_1 and S_2

Generative Adversarial Networks

- A two player minimax game between a **generator** and a **discriminator**

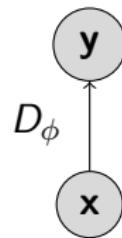


- **Generator**

- Directed, latent variable model with a deterministic mapping between z and x given by G_θ
- Minimizes a two-sample test objective (in support of the null hypothesis $p_{\text{data}} = p_\theta$)

Generative Adversarial Networks

- A two player minimax game between a generator and a discriminator



- **Discriminator**

- Any function (e.g., neural network) which tries to distinguish “real” samples from the dataset and “fake” samples generated from the model
- Maximizes the two-sample test objective (in support of the alternate hypothesis $p_{\text{data}} \neq p_\theta$)

Example of GAN objective

- **Training objective for discriminator:**

$$\max_D V(G, D) = E_{\mathbf{x} \sim p_{\text{data}}} [\log D(\mathbf{x})] + E_{\mathbf{x} \sim p_G} [\log(1 - D(\mathbf{x}))]$$

- For a fixed generator G , the discriminator is performing binary classification with the cross entropy objective
 - Assign probability 1 to true data points $\mathbf{x} \sim p_{\text{data}}$
 - Assign probability 0 to fake samples $\mathbf{x} \sim p_G$
- Optimal discriminator

$$D_G^*(\mathbf{x}) = \frac{p_{\text{data}}(\mathbf{x})}{p_{\text{data}}(\mathbf{x}) + p_G(\mathbf{x})}$$

Example of GAN objective

- **Training objective for generator:**

$$\min_G V(G, D) = E_{\mathbf{x} \sim p_{\text{data}}} [\log D(\mathbf{x})] + E_{\mathbf{x} \sim p_G} [\log(1 - D(\mathbf{x}))]$$

- For the optimal discriminator $D_G^*(\cdot)$, we have

$$\begin{aligned} & V(G, D_G^*(\mathbf{x})) \\ &= E_{\mathbf{x} \sim p_{\text{data}}} \left[\log \frac{p_{\text{data}}(\mathbf{x})}{p_{\text{data}}(\mathbf{x}) + p_G(\mathbf{x})} \right] + E_{\mathbf{x} \sim p_G} \left[\log \frac{p_G(\mathbf{x})}{p_{\text{data}}(\mathbf{x}) + p_G(\mathbf{x})} \right] \\ &= E_{\mathbf{x} \sim p_{\text{data}}} \left[\log \frac{p_{\text{data}}(\mathbf{x})}{\frac{p_{\text{data}}(\mathbf{x}) + p_G(\mathbf{x})}{2}} \right] + E_{\mathbf{x} \sim p_G} \left[\log \frac{p_G(\mathbf{x})}{\frac{p_{\text{data}}(\mathbf{x}) + p_G(\mathbf{x})}{2}} \right] - \log 4 \\ &= \underbrace{D_{KL} \left[p_{\text{data}}, \frac{p_{\text{data}} + p_G}{2} \right] + D_{KL} \left[p_G, \frac{p_{\text{data}} + p_G}{2} \right]}_{2 \times \text{Jenson-Shannon Divergence (JSD)}} - \log 4 \\ &= 2D_{JSD}[p_{\text{data}}, p_G] - \log 4 \end{aligned}$$

Jenson-Shannon Divergence

- Also called as the symmetric KL divergence

$$D_{JSD}[p, q] = \frac{1}{2} \left(D_{KL}\left[p, \frac{p+q}{2}\right] + D_{KL}\left[q, \frac{p+q}{2}\right] \right)$$

- Properties
 - $D_{JSD}[p, q] \geq 0$
 - $D_{JSD}[p, q] = 0$ iff $p = q$
 - $D_{JSD}[p, q] = D_{JSD}[q, p]$
 - $\sqrt{D_{JSD}[p, q]}$ satisfies triangle inequality \rightarrow Jenson-Shannon Distance
- Optimal generator for the JSD/Negative Cross Entropy GAN

$$p_G = p_{\text{data}}$$

- For the optimal discriminator $D_{G^*}^*(\cdot)$ and generator $G^*(\cdot)$, we have

$$V(G^*, D_{G^*}^*(x)) = -\log 4$$

The GAN training algorithm

- Sample minibatch of m training points $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(m)}$ from \mathcal{D}
- Sample minibatch of m noise vectors $\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \dots, \mathbf{z}^{(m)}$ from p_z
- Update the generator parameters θ by stochastic gradient **descent**

$$\nabla_{\theta} V(G_{\theta}, D_{\phi}) = \frac{1}{m} \nabla_{\theta} \sum_{i=1}^m \log(1 - D_{\phi}(G_{\theta}(\mathbf{z}^{(i)})))$$

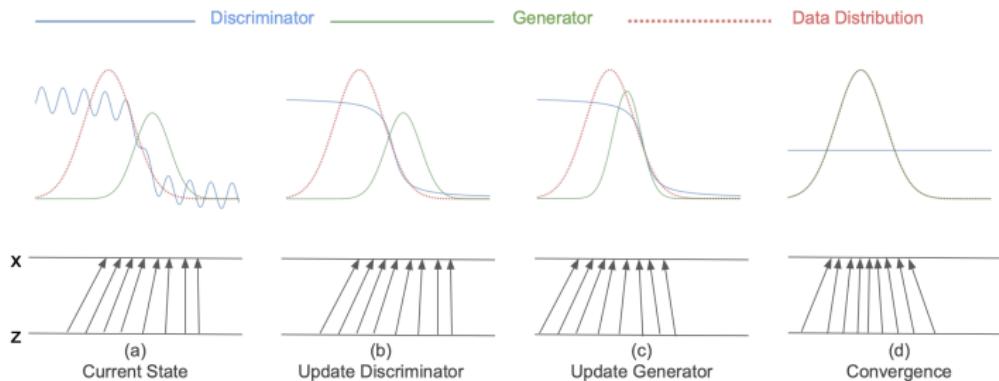
- Update the discriminator parameters ϕ by stochastic gradient **ascent**

$$\nabla_{\phi} V(G_{\theta}, D_{\phi}) = \frac{1}{m} \nabla_{\phi} \sum_{i=1}^m [\log D_{\phi}(\mathbf{x}^{(i)}) + \log(1 - D_{\phi}(G_{\theta}(\mathbf{z}^{(i)})))]$$

- Repeat for fixed number of epochs

Alternating optimization in GANs

$$\min_{\theta} \max_{\phi} V(G_{\theta}, D_{\phi}) = E_{\mathbf{x} \sim p_{\text{data}}} [\log D_{\phi}(\mathbf{x})] + E_{\mathbf{z} \sim p(\mathbf{z})} [\log(1 - D_{\phi}(G_{\theta}(\mathbf{z})))]$$



Frontiers in GAN research

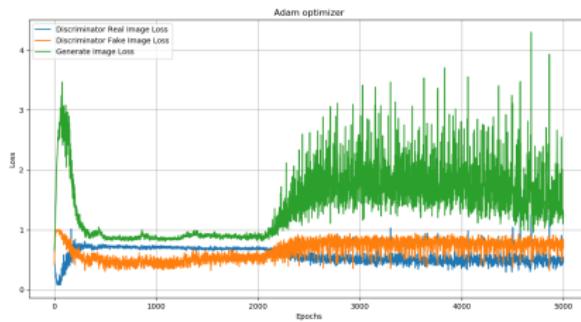


- GANs have been successfully applied to several domains and tasks
- However, working with GANs can be very challenging in practice
 - Unstable optimization
 - Mode collapse
 - Evaluation
- Many bag of tricks applied to train GANs successfully

Image Source: Ian Goodfellow. Samples from Goodfellow et al., 2014, Radford et al., 2015, Liu et al., 2016, Karras et al., 2017, Karras et al., 2018

Optimization challenges

- **Theorem (informal):** If the generator updates are made in function space and discriminator is optimal at every step, then the generator is guaranteed to converge to the data distribution
- **Unrealistic assumptions!**
- In practice, the generator and discriminator loss keeps oscillating during GAN training



Source: Mirantha Jayathilaka

- No robust stopping criteria in practice (unlike MLE)

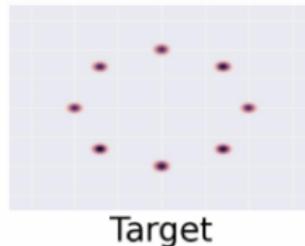
Mode Collapse

- GANs are notorious for suffering from **mode collapse**
- Intuitively, this refers to the phenomena where the generator of a GAN collapses to one or few samples (dubbed as “modes”)

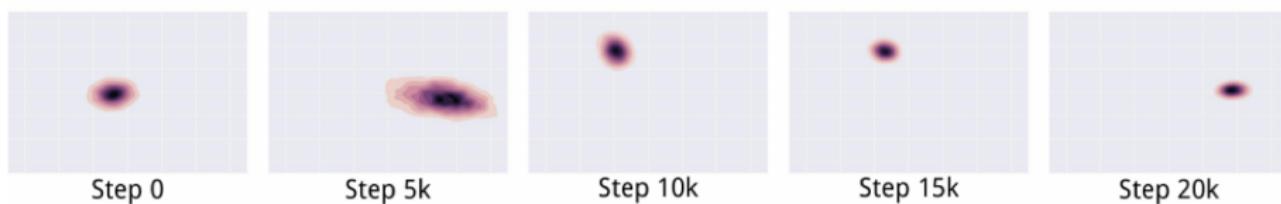


Arjovsky et al., 2017

Mode Collapse



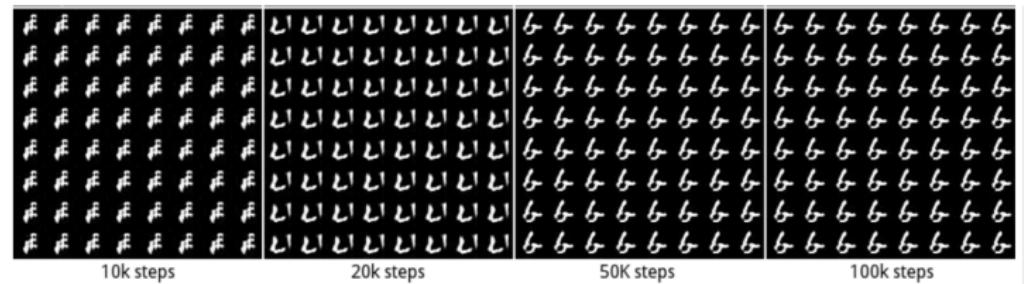
- True distribution is a mixture of Gaussians



Source: Metz et al., 2017

- The generator distribution keeps oscillating between different modes

Mode Collapse



Source: Metz et al., 2017

- Fixes to mode collapse are mostly empirically driven: alternate architectures, adding regularization terms, injecting small noise perturbations etc.
- <https://github.com/soumith/ganhacks>
How to Train a GAN? Tips and tricks to make GANs work by Soumith Chintala

Beauty lies in the eyes of the discriminator



Source: Robbie Barrat, Obvious

GAN generated art auctioned at Christie's.

Expected Price: \$7,000 – \$10,000

True Price: \$432,500

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Lecture 10

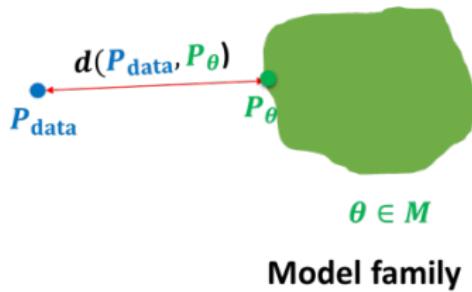
Selected GANs

- <https://github.com/hindupuravinash/the-gan-zoo>
The GAN Zoo: List of all named GANs
- Today
 - Rich class of likelihood-free objectives via f -GANs
 - Inferring latent representations via BiGAN
 - Application: Image-to-image translation via CycleGANs

Beyond KL and Jenson-Shannon Divergence



$$\mathbf{x}_i \sim P_{\text{data}} \\ i = 1, 2, \dots, n$$



What choices do we have for $d(\cdot)$?

- KL divergence: Autoregressive Models, Flow models
- (scaled and shifted) Jenson-Shannon divergence: original GAN objective

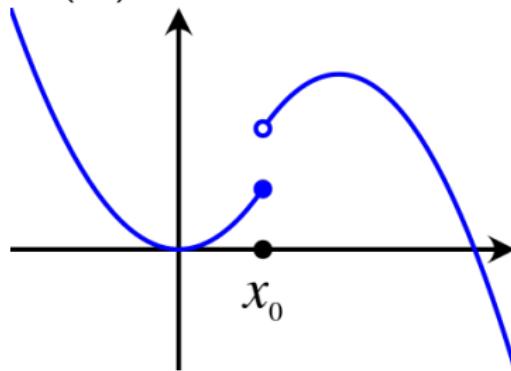
f divergences

- Given two densities p and q , the f -divergence is given by

$$D_f(p, q) = E_{\mathbf{x} \sim q} \left[f \left(\frac{p(\mathbf{x})}{q(\mathbf{x})} \right) \right]$$

where f is any convex, lower-semicontinuous function with $f(1) = 0$.

- Convex: Line joining any two points lies above the function
- Lower-semicontinuous: function value at any point \mathbf{x}_0 is close to $f(\mathbf{x}_0)$ or greater than $f(\mathbf{x}_0)$



- Example: KL divergence with $f(u) = u \log u$

f divergences

Many more f-divergences!

Name	$D_f(P\ Q)$	Generator $f(u)$
Total variation	$\frac{1}{2} \int p(x) - q(x) dx$	$\frac{1}{2} u - 1 $
Kullback-Leibler	$\int p(x) \log \frac{p(x)}{q(x)} dx$	$u \log u$
Reverse Kullback-Leibler	$\int q(x) \log \frac{q(x)}{p(x)} dx$	$-\log u$
Pearson χ^2	$\int \frac{(q(x)-p(x))^2}{p(x)} dx$	$(u - 1)^2$
Neyman χ^2	$\int \frac{(p(x)-q(x))^2}{q(x)} dx$	$\frac{(1-u)^2}{u}$
Squared Hellinger	$\int \left(\sqrt{p(x)} - \sqrt{q(x)} \right)^2 dx$	$(\sqrt{u} - 1)^2$
Jeffrey	$\int (p(x) - q(x)) \log \left(\frac{p(x)}{q(x)} \right) dx$	$(u - 1) \log u$
Jensen-Shannon	$\frac{1}{2} \int p(x) \log \frac{2p(x)}{p(x)+q(x)} + q(x) \log \frac{2q(x)}{p(x)+q(x)} dx$	$-(u + 1) \log \frac{1+u}{2} + u \log u$
Jensen-Shannon-weighted	$\int p(x)\pi \log \frac{p(x)}{\pi p(x)+(1-\pi)q(x)} + (1-\pi)q(x) \log \frac{q(x)}{\pi p(x)+(1-\pi)q(x)} dx$	$\pi u \log u - (1 - \pi + \pi u) \log(1 - \pi + \pi u)$
GAN	$\int p(x) \log \frac{2p(x)}{p(x)+q(x)} + q(x) \log \frac{2q(x)}{p(x)+q(x)} dx - \log(4)$	$u \log u - (u + 1) \log(u + 1)$
α -divergence ($\alpha \notin \{0, 1\}$)	$\frac{1}{\alpha(\alpha-1)} \int \left(p(x) \left[\left(\frac{q(x)}{p(x)} \right)^\alpha - 1 \right] - \alpha(q(x) - p(x)) \right) dx$	$\frac{1}{\alpha(\alpha-1)} (u^\alpha - 1 - \alpha(u - 1))$

Source: Nowozin et al., 2016

f -GAN: Variational Divergence Minimization

- To use f -divergences as a two-sample test objective for likelihood-free learning, we need to be able to estimate it only via samples
- Fenchel conjugate: For any function $f(\cdot)$, its convex conjugate is defined as

$$f^*(t) = \sup_{u \in \text{dom}_f} (ut - f(u))$$

- Duality: $f^{**} = f$. When $f(\cdot)$ is convex, lower semicontinuous, so is $f^*(\cdot)$

$$f(u) = \sup_{t \in \text{dom}_{f^*}} (tu - f^*(t))$$

f -GAN: Variational Divergence Minimization

- We can obtain a lower bound to any f -divergence via its Fenchel conjugate

$$\begin{aligned} D_f(p, q) &= E_{\mathbf{x} \sim q} \left[f \left(\frac{p(\mathbf{x})}{q(\mathbf{x})} \right) \right] \\ &= E_{\mathbf{x} \sim q} \left[\sup_{t \in \text{dom}_{f^*}} \left(t \frac{p(\mathbf{x})}{q(\mathbf{x})} - f^*(t) \right) \right] \\ &:= E_{\mathbf{x} \sim q} \left[T^*(x) \frac{p(\mathbf{x})}{q(\mathbf{x})} - f^*(T^*(x)) \right] \\ &= \int_{\mathcal{X}} [T^*(x)p(\mathbf{x}) - f^*(T^*(x))q(\mathbf{x})] d\mathbf{x} \\ &\geq \sup_{T \in \mathcal{T}} \int_{\mathcal{X}} (T(\mathbf{x})p(\mathbf{x}) - f^*(T(\mathbf{x}))q(\mathbf{x})) d\mathbf{x} \\ &= \sup_{T \in \mathcal{T}} (E_{\mathbf{x} \sim p} [T(\mathbf{x})] - E_{\mathbf{x} \sim q} [f^*(T(\mathbf{x}))]) \end{aligned}$$

where $\mathcal{T} : \mathcal{X} \mapsto \mathbb{R}$ is an arbitrary class of functions

- Note:** Lower bound is likelihood-free w.r.t. p and q

f -GAN: Variational Divergence Minimization

- Variational lower bound

$$D_f(p, q) \geq \sup_{T \in \mathcal{T}} (E_{\mathbf{x} \sim p} [T(\mathbf{x})] - E_{\mathbf{x} \sim q} [f^*(T(\mathbf{x}))])$$

- Choose any f -divergence
- Let $p = p_{\text{data}}$ and $q = p_G$
- Parameterize T by ϕ and G by θ
- Consider the following f -GAN objective

$$\min_{\theta} \max_{\phi} F(\theta, \phi) = E_{\mathbf{x} \sim p_{\text{data}}} [T_{\phi}(\mathbf{x})] - E_{\mathbf{x} \sim p_{G_{\theta}}} [f^*(T_{\phi}(\mathbf{x}))]$$

- Generator G_{θ} tries to minimize the divergence estimate and discriminator T_{ϕ} tries to tighten the lower bound

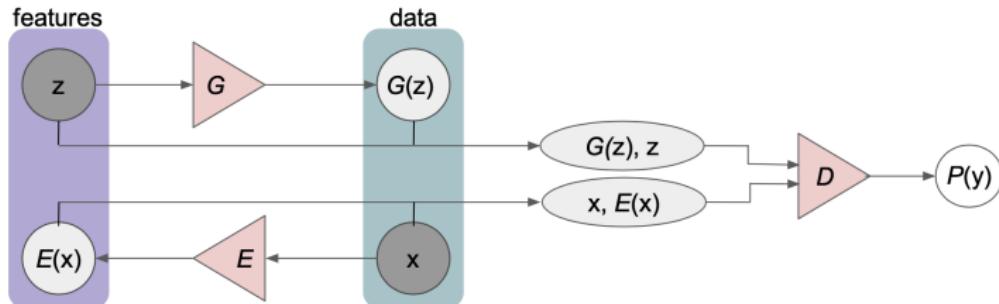
Inferring latent representations in GANs

- The generator of a GAN is typically a directed, latent variable model with latent variables \mathbf{z} and observed variables \mathbf{x} . How can we infer the latent feature representations in a GAN?
- Unlike a normalizing flow model, the mapping $G : \mathbf{z} \mapsto \mathbf{x}$ need not be invertible
- Unlike a variational autoencoder, there is no inference network $q(\cdot)$ which can learn a variational posterior over latent variables
- **Solution 1:** For any point \mathbf{x} , use the activations of the prefinal layer of a discriminator as a feature representation
- Intuition: Similar to supervised deep neural networks, the discriminator would have learned useful representations for \mathbf{x} while distinguishing real and fake \mathbf{x}

Inferring latent representations in GANs

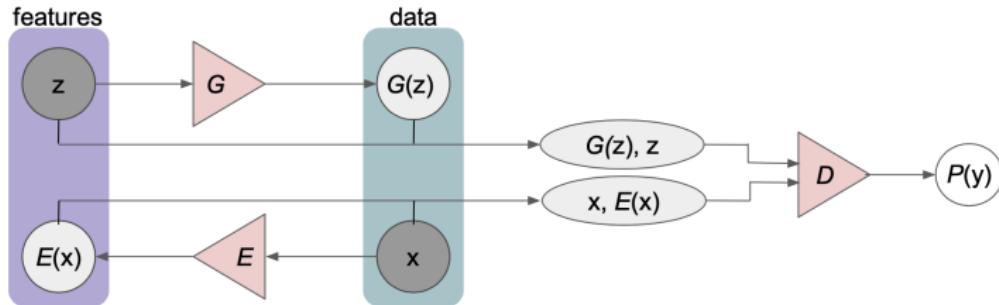
- If we want to directly infer the latent variables \mathbf{z} of the generator, we need a different learning algorithm
- A regular GAN optimizes a two-sample test objective that compares samples of \mathbf{x} from the generator and the data distribution
- **Solution 2:** To infer latent representations, we will compare samples of \mathbf{x}, \mathbf{z} from the joint distributions of observed and latent variables as per the model and the data distribution
- For any \mathbf{x} generated via the model, we have access to \mathbf{z} (sampled from a simple prior $p(\mathbf{z})$)
- For any \mathbf{x} from the data distribution, the \mathbf{z} is however unobserved (latent)

Bidirectional Generative Adversarial Networks (BiGAN)



- In a BiGAN, we have an encoder network E in addition to the generator network G
- The encoder network only observes $x \sim p_{\text{data}}(x)$ during training to learn a mapping $E : x \mapsto z$
- As before, the generator network only observes the samples from the prior $z \sim p(z)$ during training to learn a mapping $G : z \mapsto x$

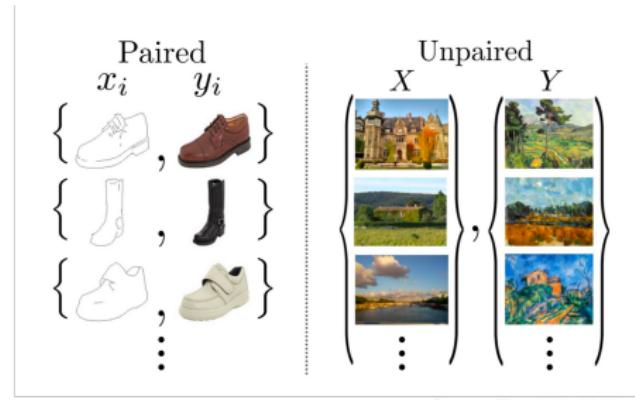
Bidirectional Generative Adversarial Networks (BiGAN)



- The discriminator D observes samples from the generative model $z, G(z)$ and the encoding distribution $E(x), x$
- The goal of the discriminator is to maximize the two-sample test objective between $z, G(z)$ and $E(x), x$
- After training is complete, new samples are generated via G and latent representations are inferred via E

Translating across domains

- Image-to-image translation: We are given images from two domains, \mathcal{X} and \mathcal{Y}
- Paired vs. unpaired examples

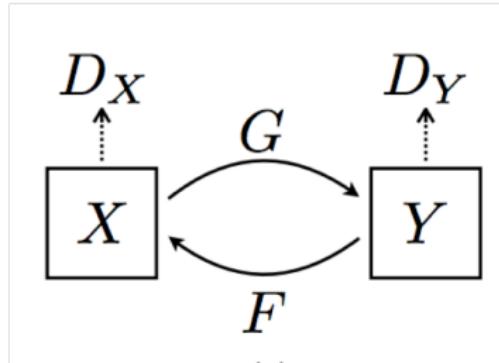


Source: Zhu et al., 2016

- Paired examples can be expensive to obtain. Can we translate from $\mathcal{X} \leftrightarrow \mathcal{Y}$ in an unsupervised manner?

CycleGAN: Adversarial training across two domains

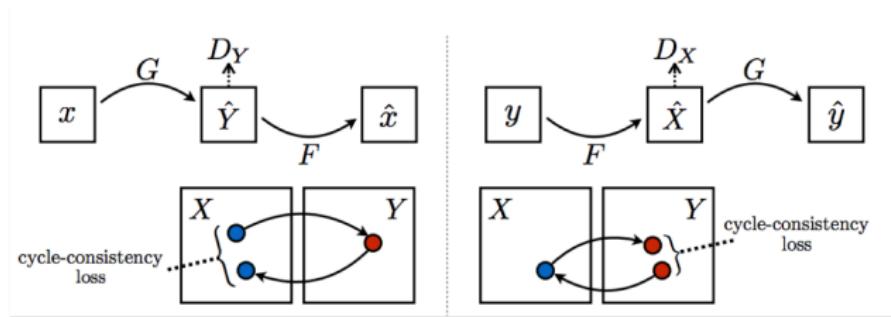
- To match the two distributions, we learn two parameterized conditional generative models $G : \mathcal{X} \leftrightarrow \mathcal{Y}$ and $F : \mathcal{Y} \leftrightarrow \mathcal{X}$
- G maps an element of \mathcal{X} to an element of \mathcal{Y} . A discriminator D_Y compares the observed dataset Y and the generated samples $\hat{Y} = G(X)$
- Similarly, F maps an element of \mathcal{Y} to an element of \mathcal{X} . A discriminator D_X compares the observed dataset X and the generated samples $\hat{X} = F(Y)$



Source: Zhu et al., 2016

CycleGAN: Cycle consistency across domains

- **Cycle consistency:** If we can go from X to \hat{Y} via G , then it should also be possible to go from \hat{Y} back to X via F
 - $F(G(X)) \approx X$
 - Similarly, vice versa: $G(F(Y)) \approx Y$

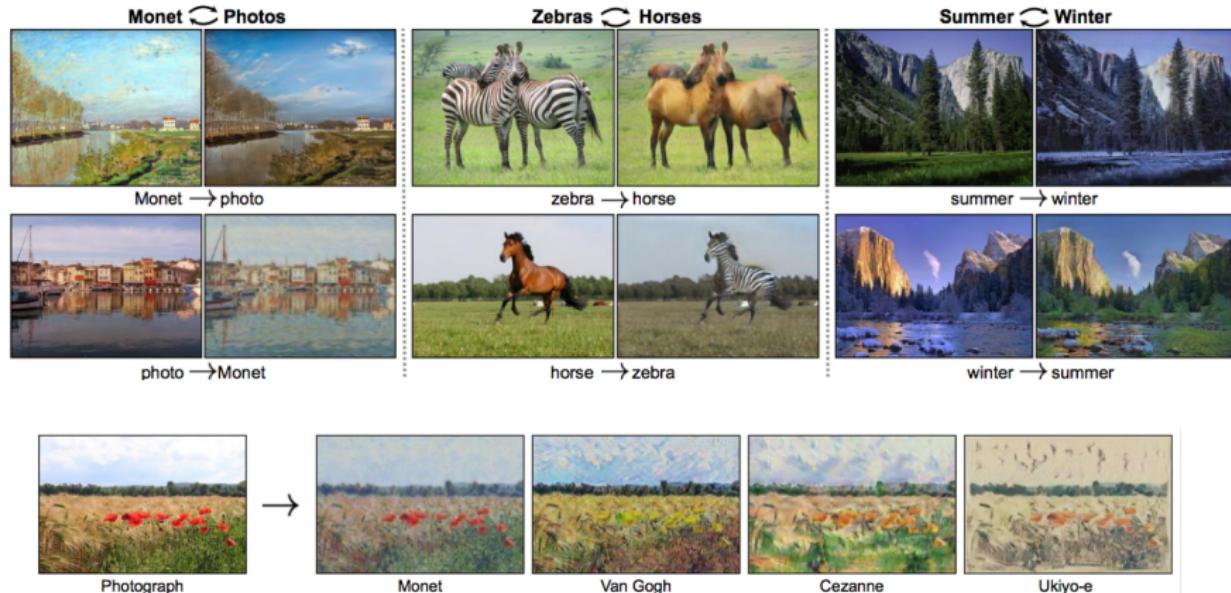


Source: Zhu et al., 2016

- Overall loss function

$$\begin{aligned} & \min_{F, G, D_X, D_Y} \mathcal{L}_{\text{GAN}}(G, D_Y, X, Y) + \mathcal{L}_{\text{GAN}}(F, D_X, Y, X) \\ & + \lambda \underbrace{\left(E_X[\|F(G(X)) - X\|_1] + E_Y[\|G(F(Y)) - Y\|_1] \right)}_{\text{cycle consistency}} \end{aligned}$$

CycleGAN in practice



Source: Zhu et al., 2016

AlignFlow

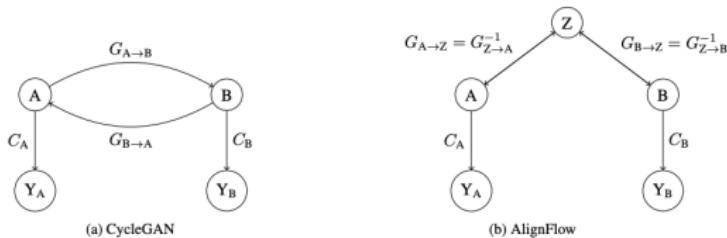


Figure 1: CycleGAN v.s. AlignFlow for unpaired cross-domain translation. Unlike CycleGAN, AlignFlow specifies a single invertible mapping $G_{A \rightarrow Z} \circ G_{B \rightarrow Z}^{-1}$ that is exactly cycle-consistent, represents a shared latent space Z between the two domains, and can be trained via both adversarial training and exact maximum likelihood estimation. Double-headed arrows denote invertible mappings. Y_A and Y_B are random variables denoting the output of the critics used for adversarial training.

- What if G is a flow model?
- No need to parameterize F separately! $F = G^{-1}$
- Can train via MLE and/or adversarial learning!
- Exactly cycle-consistent
 $F(G(X)) = X$
 $G(F(Y)) = Y$

Summary of Generative Adversarial Networks

- Key observation: Samples and likelihoods are not correlated in practice
- Two-sample test objectives allow for learning generative models only via samples (likelihood-free)
- Wide range of two-sample test objectives covering f -divergences (and more)
- Latent representations can be inferred via BiGAN
- Cycle-consistent domain translations via CycleGAN and AlignFlow