

# Statistical Models & Computing Methods

## Lecture 12: Variational EM



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- ▶ EM algorithm finds the MLE for latent variable model

$$\mathcal{L}(\theta) = \log p(x|\theta) = \log \sum_z p(x, z|\theta)$$

- ▶ EM update formula

$$\theta^{(t+1)} = \arg \max_{\theta} Q^{(t)}(\theta) = \arg \max_{\theta} \mathbb{E}_{p(z|x, \theta^{(t)})} \log p(x, z|\theta)$$

- ▶ EM requires the posterior  $p(z|x, \theta^{(t)})$  is known. What if  $p(z|x, \theta^{(t)})$  is unknown?
  - ▶ If somehow we can sample from  $p(z|x, \theta^{(t)})$ , we can use Monte Carlo estimates, that is Monte Carlo EM.
  - ▶ However, the associated computation may be expensive.



- Recall EM maximizes the lower bound

$$\mathcal{F}(q, \theta^{(t)}) = \mathbb{E}_{q(z)} \log \frac{p(x, z | \theta)}{q(z)} \leq \mathcal{L}(\theta), \quad \forall q(z)$$

- When the best  $q(z) = p(z|x, \theta^{(t)})$  is not available, we can use approximate  $q(z)$  instead.
- A widely used approximation is the mean-field approximation

$$q(z) = \prod_{i=1}^d q_i(z_i)$$



- In that case, the lower bound is

$$\begin{aligned}\mathcal{F}(q(z), \theta^{(t)}) &= \int \prod_{i=1}^d q_i(z_i) \log \frac{p(x, z | \theta^{(t)})}{\prod_{i=1}^d q_i(z_i)} dz_1 dz_2 \dots dz_d \\ &= \int \prod_{i=1}^d q_i(z_i) \log p(x, z | \theta^{(t)}) dz_1 dz_2 \dots dz_d \\ &\quad - \sum_{i=1}^d \int q_i(z_i) \log q_i(z_i) dz_i\end{aligned}$$

- Coordinate Ascent

$$q_i^{(t)}(z_i) \propto \exp \left( \mathbb{E}_{-q_i} \log p(x, z | \theta^{(t)}) \right), i = 1, \dots, d$$



- E-step. Run coordinate ascent several times to obtain good mean-field approximation

$$q^{(t)}(z) = \prod_{i=1}^d q_i^{(t)}(z_i)$$

compute the expected complete data log-likelihood

$$Q^{(t)}(\theta) = \mathbb{E}_{q^{(t)}(z)} \log p(x, z | \theta)$$

- M-step. Update  $\theta$  to maximize  $Q^{(t)}(\theta)$

$$\theta^{(t+1)} = \arg \max_{\theta} Q^{(t)}(\theta)$$



- Now let us consider Bayesian inference for latent variable models

$$p(z, \theta | x) \propto p(x, z | \theta) p(\theta)$$

- We can lower bound the marginal likelihood

$$\begin{aligned}\mathcal{L}(x) &= \log p(x) = \log \int p(x, z | \theta) p(\theta) dz d\theta \\ &= \log \int q(z, \theta) \frac{p(x, z | \theta) p(\theta)}{q(z, \theta)} dz d\theta \\ &\geq \int q(z, \theta) \log \frac{p(x, z | \theta) p(\theta)}{q(z, \theta)} dz d\theta \\ &= \mathcal{F}(q(z, \theta))\end{aligned}$$

- Maximizing this lower bound  $\mathcal{F}$  is equivalent to minimizing  $D_{\text{KL}}(q(z, \theta) \| p(z, \theta | x))$



- ▶ Again, we consider a simple factorized approximation
$$q(z, \theta) = q_z(z)q_\theta(\theta)$$

$$\begin{aligned}\mathcal{L}(x) &\geq \int q_z(z)q_\theta(\theta) \log \frac{p(x, z|\theta)p(\theta)}{q_z(z)q_\theta(\theta)} dz d\theta \\ &= \mathcal{F}(q_z(z), q_\theta(\theta))\end{aligned}$$

- ▶ Maximizing this lower bound  $\mathcal{F}$ , leads to **EM**-like iterative updates

$$\begin{aligned}q_z^{(t+1)}(z) &\propto \exp \left( \mathbb{E}_{q_\theta^{(t)}(\theta)} \log p(x, z|\theta) \right) \\ q_\theta^{(t+1)}(\theta) &\propto p(\theta) \cdot \exp \left( \mathbb{E}_{q_z^{(t+1)}(z)} \log p(x, z|\theta) \right)\end{aligned}$$



Let's focus on conjugate-exponential (CE) models, which satisfy

## Condition 1

The joint probability over variables is in the exponential family

$$p(x, z|\theta) = h(x, z) \exp(\phi(\theta) \cdot T(x, z) - A(\theta))$$

## Condition 2

The prior over parameters is conjugate to this joint probability

$$p(\theta|\eta, \nu) \propto \exp(\phi(\theta) \cdot \nu - \eta A(\theta))$$

Conjugate priors are computationally convenient and have an intuitive interpretation:

- ▶  $\eta$ : number of pseudo-observations
- ▶  $\nu$ : values of pseudo-observations





Now suppose we have an iid data set  $x = \{x_1, \dots, x_n\}$

► VB E-step.

$$\begin{aligned} q_z^{(t+1)}(z) &\propto \exp \left( \mathbb{E}_{q_\theta^{(t)}(\theta)} \log p(x, z | \theta) \right) \\ &\propto \prod_{i=1}^n h(x_i, z_i) \exp \left( \bar{\phi} \cdot T(x_i, z_i) \right) \end{aligned}$$

where  $\bar{\phi} = \mathbb{E}_{q_\theta^{(t)}}(\phi(\theta))$

► VB M-step

$$q_\theta^{(t+1)}(\theta) \propto \exp \left( \phi(\theta) \cdot \left( \nu + \sum_{i=1}^n \bar{T}(x_i, z_i) \right) - (\eta + n) A(\theta) \right)$$

where  $\bar{T}(x_i, z_i) = \mathbb{E}_{q_z^{(t+1)}}(T(x_i, z_i))$



## EM for MAP

- ▶ **Goal:** maximize  $p(x, \theta)$
- ▶ **E-step:** compute

$$q_z^{(t+1)}(z) = p(z|x, \theta^{(t)})$$

- ▶ **M-step:**

$$\theta^{(t+1)} = \arg \max_{\theta} Q^{(t)}(\theta)$$

$$Q^{(t)}(\theta) = \mathbb{E}_{q_z^{(t+1)}} \log p(x, z, \theta)$$

## Variational Bayesian EM

- ▶ **Goal:** lower bound  $p(x)$
- ▶ **VB E-step:** compute

$$q_z^{(t+1)}(z) = p(z|x, \bar{\phi})$$

- ▶ **VB M-step:**

$$q_{\theta}^{(t+1)}(\theta) \propto \exp \left( Q^{(t)}(\theta) \right)$$

$$Q^{(t)}(\theta) = \mathbb{E}_{q_z^{(t+1)}} \log p(x, z, \theta)$$



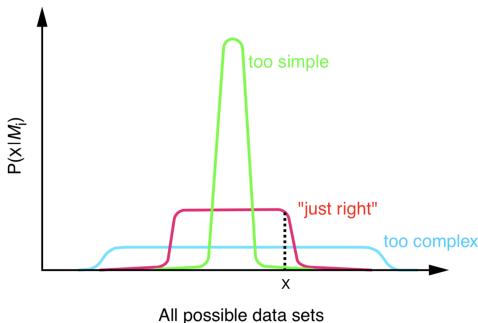
- ▶ Reduces to the EM algorithm if  $q_{\theta}(\theta) = \delta(\theta - \theta^*)$ .
- ▶  $\mathcal{F}$  increases monotonically, and incorporates the model complexity penalty.
- ▶ Analytical parameter distributions
- ▶ VB E-step has the same complexity as corresponding E step, and is almost identical except that it uses the expected natural parameters,  $\bar{\phi}$ .
- ▶ The lower bound given by VBEM can be used for model selection.

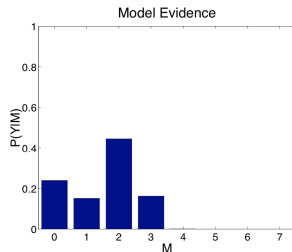
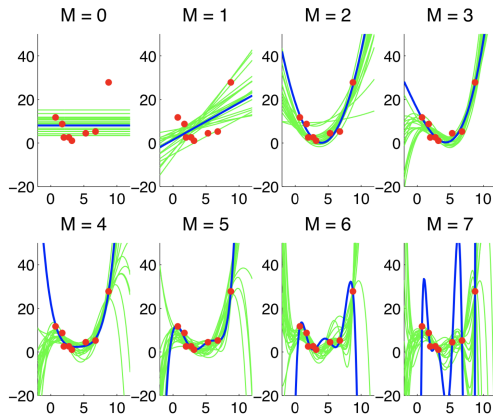


- In Bayesian model selection, we want to select the model class with the highest marginal likelihood (**evidence**)

$$p(x|m) = \int p(x|\theta, m)p(\theta|m)d\theta$$

- Occam's Razor





Adapted from Zoubin Ghahramani

- **Bayesian Information Criterion (BIC):**

$$\log p(x|m) \approx \log p(x|\hat{\theta}_{\text{MAP}}, m) - \frac{d}{2} \log n$$

- **Annealed Importance Sampling (AIS):**

$$Z_k = \int p(x|\theta, m)^{\tau_k} p(\theta|m) d\theta, \quad 0 = \tau_0 < \dots < \tau_K = 1$$

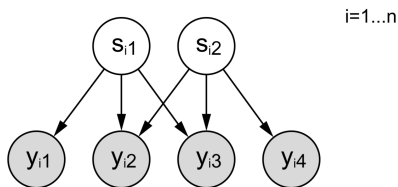
$$\log p(x|\theta) = Z_K = \prod_{k=0}^{K-1} \frac{Z_{k+1}}{Z_k}$$

where  $\frac{Z_{k+1}}{Z_k}$  can be estimated via importance sampling.

- **Variational Bayesian EM (VB):** use VBEM lower bound estimate

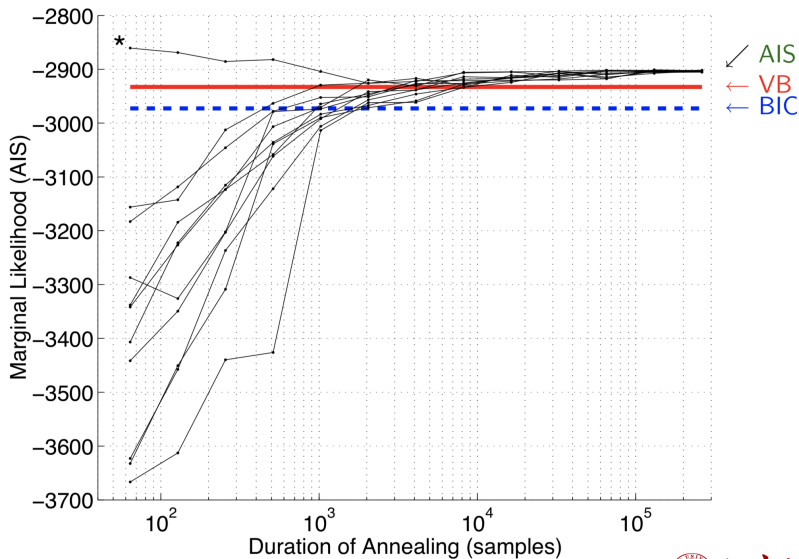


- ▶ A simple bipartite graphical model: **two** binary hidden variables, and **four** five-valued discrete observed variables



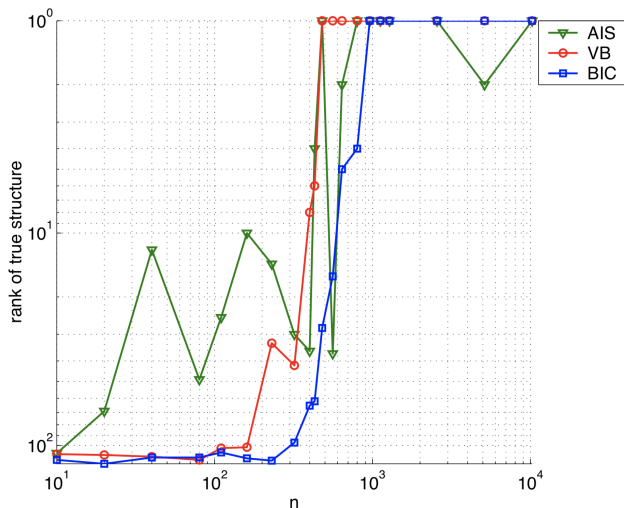
- ▶ Experiment: there are 136 distinct structures with 2 latent variables as potential parents of 4 conditionally independent observed variables
- ▶ Score each structure with 3 methods: **BIC**, **VB** and the gold standard **AIS**.







VB score finds correct structure earlier, and more reliably



- ▶ Schwarz, G. (1978). Estimating the dimension of a model. *The Annals of Statistics*, 6:461–464.
- ▶ Neal, R. M. (2001). Annealed importance sampling. *Statistics and Computing*, 11:125–139.
- ▶ M. J. Beal and Z. Ghahramani, “The variational bayesian em algorithm for in- complete data: With application to scoring graphical model structures”, *Bayesian statistics*, vol. 7, J. Bernardo, M. Bayarri, J. Berger, A. Dawid, D Heckerman, A. Smith, M West, et al., Eds., pp. 453–464, 2003.

