

# Supplementary Material: Projection-Free Bandit Convex Optimization over Strongly Convex Sets

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## A Proof of Theorem 1

We start this proof by introducing standard definitions for smooth functions and strongly convex functions[5].

**Definition 1.** Let  $f(\mathbf{x}) : \mathcal{K} \rightarrow \mathbb{R}$  be a function over  $\mathcal{K}$ . It is called  $\beta$ -smooth over  $\mathcal{K}$  if for all  $\mathbf{x}, \mathbf{y} \in \mathcal{K}$

$$f(\mathbf{y}) \leq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{\beta}{2} \|\mathbf{y} - \mathbf{x}\|_2^2.$$

**Definition 2.** Let  $f(\mathbf{x}) : \mathcal{K} \rightarrow \mathbb{R}$  be a function over  $\mathcal{K}$ . It is called  $\alpha$ -strongly convex over  $\mathcal{K}$  if for all  $\mathbf{x}, \mathbf{y} \in \mathcal{K}$

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{\alpha}{2} \|\mathbf{y} - \mathbf{x}\|_2^2.$$

Then, let  $\mathbf{x}^* \in \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}} \sum_{t=1}^T f_t(\mathbf{x})$  and  $\mathbf{y}^* = (1 - \delta/r)\mathbf{x}^*$ . It is easy to verify that

$$\begin{aligned} \operatorname{Regret}(T) &= \sum_{t=1}^T f_t(\mathbf{y}_t + \delta \mathbf{u}_t) - \sum_{t=1}^T f_t(\mathbf{x}^*) \\ &\leq \sum_{t=1}^T \left( f_t(\mathbf{y}_t) + \delta G \|\mathbf{u}_t\|_2 - f_t(\mathbf{y}^*) + G \left\| \frac{\delta}{r} \mathbf{x}^* \right\|_2 \right) \\ &\leq \sum_{t=1}^T (f_t(\mathbf{y}_t) - f_t(\mathbf{y}^*)) + \delta GT + \frac{\delta GRT}{r} \end{aligned} \quad (11)$$

where the first inequality is due to Assumption 2, and the last inequality is due to  $\|\mathbf{u}_t\|_2 = 1$  and Assumption 1.

To further bound the right side of (11), we define the  $\delta$ -smooth version of  $f_t(\mathbf{x})$  as

$$\hat{f}_{t,\delta}(\mathbf{x}) = \mathbb{E}_{\mathbf{u} \sim \mathcal{B}^d} [f_t(\mathbf{x} + \delta \mathbf{u})]$$

and introduce the following lemma.

**Lemma 4 (Lemma 2.6 in Hazan[20]).** *Let  $f(\mathbf{x}) : \mathbb{R}^d \rightarrow \mathbb{R}$  be  $\alpha$ -strongly convex and  $G$ -Lipschitz over a convex and compact set  $\mathcal{K} \subseteq \mathbb{R}^d$ . Then, its  $\delta$ -smooth version defined in (5) has the following properties:*

- $\widehat{f}_\delta(\mathbf{x})$  is  $\alpha$ -strongly convex over  $\mathcal{K}_\delta$ ;
- $|\widehat{f}_\delta(\mathbf{x}) - f(\mathbf{x})| \leq \delta G$  for any  $\mathbf{x} \in \mathcal{K}_\delta$ ;
- $\widehat{f}_\delta(\mathbf{x})$  is  $G$ -Lipschitz over  $\mathcal{K}_\delta$ .

By applying Lemma 4, we have

$$\begin{aligned}
& \sum_{t=1}^T f_t(\mathbf{y}_t) - \sum_{t=1}^T f_t(\mathbf{y}^*) \\
& \leq \sum_{t=1}^T (\widehat{f}_{t,\delta}(\mathbf{y}_t) + \delta G) - \sum_{t=1}^T (\widehat{f}_{t,\delta}(\mathbf{y}^*) - \delta G) \\
& \leq \sum_{t=1}^T \widehat{f}_{t,\delta}(\mathbf{y}_t) - \sum_{t=1}^T \widehat{f}_{t,\delta}(\mathbf{y}^*) + 2\delta GT.
\end{aligned} \tag{12}$$

Then, by utilizing the convexity of  $\widehat{f}_{t,\delta}(\cdot)$ , we have

$$\begin{aligned}
& \sum_{t=1}^T \widehat{f}_{t,\delta}(\mathbf{y}_t) - \sum_{t=1}^T \widehat{f}_{t,\delta}(\mathbf{y}^*) \\
& \leq \sum_{t=1}^T \langle \nabla \widehat{f}_{t,\delta}(\mathbf{y}_t), \mathbf{y}_t - \mathbf{y}^* \rangle \\
& \leq \sum_{t=1}^T \langle \nabla \widehat{f}_{t,\delta}(\mathbf{y}_t), \mathbf{y}_t^* - \mathbf{y}^* \rangle + \sum_{t=1}^T \langle \nabla \widehat{f}_{t,\delta}(\mathbf{y}_t), \mathbf{y}_t - \mathbf{y}_t^* \rangle \\
& \leq \sum_{t=1}^T \langle \nabla \widehat{f}_{t,\delta}(\mathbf{y}_t), \mathbf{y}_t^* - \mathbf{y}^* \rangle + \sum_{t=1}^T G \|\mathbf{y}_t - \mathbf{y}_t^*\|_2.
\end{aligned} \tag{13}$$

Combining (11), (12) and (13), we have

$$\begin{aligned}
& \mathbb{E}[\text{Regret}(T)] \\
& \leq \mathbb{E} \left[ \sum_{t=1}^T \langle \nabla \widehat{f}_{t,\delta}(\mathbf{y}_t), \mathbf{y}_t^* - \mathbf{y}^* \rangle \right] + \mathbb{E} \left[ \sum_{t=1}^T G \|\mathbf{y}_t - \mathbf{y}_t^*\|_2 \right] \\
& \quad + 3\delta GT + \frac{\delta GRT}{r} \\
& = \mathbb{E} \left[ \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{y}_t^* - \mathbf{y}^* \rangle \right] + \mathbb{E} \left[ \sum_{t=1}^T G \|\mathbf{y}_t - \mathbf{y}_t^*\|_2 \right] \\
& \quad + 3\delta GT + \frac{\delta GRT}{r}
\end{aligned} \tag{14}$$

where the last equality is due to Lemma 1.

To bound the term  $\|\mathbf{y}_t - \mathbf{y}_t^*\|_2$  in (14), we notice that as proved by [16], if a function  $f(\mathbf{x}) : \mathcal{K} \rightarrow \mathbb{R}$  is  $\alpha$ -strongly convex, it holds that

$$\frac{\alpha}{2} \|\mathbf{x} - \mathbf{x}^*\|_2^2 \leq f(\mathbf{x}) - f(\mathbf{x}^*) \quad (15)$$

for  $\mathbf{x}^* = \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}} f(\mathbf{x})$  and any  $\mathbf{x} \in \mathcal{K}$ . It is easy to verify that  $F_{t-1}(\mathbf{y}) = \eta \sum_{\tau=1}^{t-1} \langle \mathbf{g}_\tau, \mathbf{y} \rangle + \|\mathbf{y} - \mathbf{y}_1\|_2^2$  is 2-strongly convex. Therefore, according to (15), we have

$$\|\mathbf{y}_t - \mathbf{y}_t^*\|_2 \leq \sqrt{F_{t-1}(\mathbf{y}_t) - F_{t-1}(\mathbf{y}_t^*)} \leq \frac{\sqrt{C}}{(t+2)^{1/4}} \quad (16)$$

where the last inequality is due to Lemma 3.

Moreover, to bound the term  $\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{y}_t^* - \mathbf{y}^* \rangle$ , we introduce the following lemma.

**Lemma 5 (Lemma 2.3 in Shalev-Shwartz [30]).** *Let  $\{\ell_t(\mathbf{x})\}_{t=1}^T$  be a sequence of loss functions over a convex set  $\mathcal{K}'$  and let  $\mathbf{x}_t^* = \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}'} \sum_{\tau=1}^{t-1} \ell_\tau(\mathbf{x}) + \mathcal{R}(\mathbf{x})$  for any  $t \in [T+1]$ , where  $\mathcal{R}(\mathbf{x})$  is a strongly convex function over  $\mathcal{K}'$ . Then, for any  $\mathbf{x} \in \mathcal{K}'$ , it holds that*

$$\sum_{t=1}^T \ell_t(\mathbf{x}_t^*) - \sum_{t=1}^T \ell_t(\mathbf{x}) \leq \mathcal{R}(\mathbf{x}) - \mathcal{R}(\mathbf{x}_1^*) + \sum_{t=1}^T (\ell_t(\mathbf{x}_t^*) - \ell_t(\mathbf{x}_{t+1}^*)).$$

By applying Lemma 5 with  $\ell_t(\mathbf{x}) = \langle \mathbf{g}_t, \mathbf{x} \rangle$ ,  $\mathcal{R}(\mathbf{x}) = \|\mathbf{x} - \mathbf{y}_1\|_2^2 / \eta$ , and  $\mathcal{K}' = \mathcal{K}_\delta$ , we have

$$\begin{aligned} \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{y}_t^* - \mathbf{y}^* \rangle &\leq \frac{\|\mathbf{y}^* - \mathbf{y}_1\|_2^2}{\eta} - \frac{\|\mathbf{y}_1^* - \mathbf{y}_1\|_2^2}{\eta} + \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{y}_t^* - \mathbf{y}_{t+1}^* \rangle \\ &\leq \frac{4R^2}{\eta} + \sum_{t=1}^T \|\mathbf{g}_t\|_2 \|\mathbf{y}_t^* - \mathbf{y}_{t+1}^*\|_2. \end{aligned} \quad (17)$$

Since  $F_t(\mathbf{x})$  is 2-strongly convex and 2-smooth over  $\mathcal{K}_\delta$  for  $t = 0, \dots, T$ . Then, from (15), for any  $t \in [T]$ , we have

$$\begin{aligned} \|\mathbf{y}_t^* - \mathbf{y}_{t+1}^*\|_2^2 &\leq F_t(\mathbf{y}_t^*) - F_t(\mathbf{y}_{t+1}^*) \\ &= F_{t-1}(\mathbf{y}_t^*) - F_{t-1}(\mathbf{y}_{t+1}^*) + \langle \eta \mathbf{g}_t, \mathbf{y}_t^* - \mathbf{y}_{t+1}^* \rangle \\ &\leq \eta \|\mathbf{g}_t\|_2 \|\mathbf{y}_t^* - \mathbf{y}_{t+1}^*\|_2 \end{aligned}$$

which implies that

$$\|\mathbf{y}_t^* - \mathbf{y}_{t+1}^*\|_2 \leq \eta \|\mathbf{g}_t\|_2 \leq \frac{\eta d M}{\delta}. \quad (18)$$

Therefore, substituting (18) into (17), we have

$$\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{y}_t^* - \mathbf{y}^* \rangle \leq \frac{4R^2}{\eta} + \sum_{t=1}^T \eta \|\mathbf{g}_t\|_2^2 \leq \frac{4R^2}{\eta} + \frac{\eta d^2 M^2 T}{\delta^2}. \quad (19)$$

By combining (14) with (16) and (19), we finally have

$$\begin{aligned}\mathbb{E}[\text{Regret}(T)] &\leq \frac{4R^2}{\eta} + \frac{\eta d^2 M^2 T}{\delta^2} + \sum_{t=1}^T \frac{\sqrt{C}G}{(t+2)^{1/4}} + 3\delta GT + \frac{\delta GRT}{r} \\ &\leq \frac{4RdM(T+2)^{3/4}}{c} + \frac{RdMT^{3/4}}{c} + \frac{4\sqrt{C}G(T+2)^{3/4}}{3} \\ &\quad + 3cGT^{3/4} + \frac{cGRT^{3/4}}{r}\end{aligned}\quad (20)$$

where the second inequality is derived by substituting the value of  $\eta$  and  $\delta$ .

## B Proof of Theorem 2

By combining (11), (12) and (13), we first have

$$\text{Regret}(T) \leq \sum_{t=1}^T \langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_t), \mathbf{y}_t^* - \mathbf{y}^* \rangle + \sum_{t=1}^T G \|\mathbf{y}_t - \mathbf{y}_t^*\|_2 + 3\delta GT + \frac{\delta GRT}{r}. \quad (21)$$

To bound  $\sum_{t=1}^T \langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_t), \mathbf{y}_t^* - \mathbf{y}^* \rangle$ , we introduce the following lemma.

**Lemma 6.** *Let  $\mathbf{y}_t^* = \arg\min_{\mathbf{y} \in \mathcal{K}_\delta} F_{t-1}(\mathbf{y})$ ,  $\mathbf{y}^* = (1 - \delta/r)\mathbf{x}^*$ , where  $\mathbf{x}^* \in \arg\min_{\mathbf{x} \in \mathcal{K}} \sum_{t=1}^T f_t(\mathbf{x})$ . Under Assumption 1, 2 and 3, with probability at least  $1 - \gamma$ , Algorithm 1 ensures*

$$\sum_{t=1}^T \langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_t), \mathbf{y}_t^* - \mathbf{y}^* \rangle \leq 2R \left( G + \frac{dM}{\delta} \right) \sqrt{2T \ln \frac{1}{\gamma}} + \frac{4R^2}{\eta} + \frac{\eta d^2 M^2 T}{\delta^2}.$$

By combining (16), (21), and Lemma 6, with probability at least  $1 - \gamma$ , we have

$$\begin{aligned}\text{Regret}(T) &\leq 2R \left( G + \frac{dM}{\delta} \right) \sqrt{2T \ln \frac{1}{\gamma}} + \frac{4R^2}{\eta} + \frac{\eta d^2 M^2 T}{\delta^2} \\ &\quad + \sum_{t=1}^T \frac{G\sqrt{C}}{(t+2)^{1/4}} + 3\delta GT + \frac{\delta GRT}{r}.\end{aligned}$$

By substituting  $\eta = \frac{cR}{dM(T+2)^{3/4}}$  and  $\delta = cT^{1/4}$  into the above inequality, we have

$$\begin{aligned}\text{Regret}(T) &\leq 2RG \sqrt{2 \ln \frac{1}{\gamma}} T^{1/2} + \frac{2RdM}{c} \sqrt{2 \ln \frac{1}{\gamma}} T^{3/4} \\ &\quad + \frac{4RdM(T+2)^{3/4}}{c} + \frac{4\sqrt{C}G(T+2)^{3/4}}{3} \\ &\quad + \frac{RdMT^{3/4}}{c} + 3cGT^{3/4} + \frac{cGRT^{3/4}}{r}.\end{aligned}\quad (22)$$

## C Proof of Lemma 2

Let  $\mathbf{x}, \mathbf{y}$  be any vectors in  $\mathcal{K}_\delta$ . According to the definition of  $\mathcal{K}_\delta$ , we must have

$$\frac{\mathbf{x}}{1 - (\delta/r)} \in \mathcal{K}, \quad \frac{\mathbf{y}}{1 - (\delta/r)} \in \mathcal{K}.$$

Then, since  $\mathcal{K}$  is  $\alpha_K$ -strongly convex with respect to a norm  $\|\cdot\|$ , according to Definition 1, for any  $\gamma \in [0, 1]$  and  $\mathbf{z} \in \mathbb{R}^d$  such that  $\|\mathbf{z}\| = 1$ , it holds that

$$\gamma \frac{\mathbf{x}}{1 - (\delta/r)} + (1 - \gamma) \frac{\mathbf{y}}{1 - (\delta/r)} + \gamma(1 - \gamma) \frac{\alpha_K}{2} \left\| \frac{\mathbf{x} - \mathbf{y}}{1 - (\delta/r)} \right\|^2 \mathbf{z} \in \mathcal{K}.$$

By using the definition of  $\mathcal{K}_\delta$  again, for any  $\gamma \in [0, 1]$  and  $\mathbf{z} \in \mathbb{R}^d$  such that  $\|\mathbf{z}\| = 1$ , it holds that

$$\gamma \mathbf{x} + (1 - \gamma) \mathbf{y} + \gamma(1 - \gamma) \frac{\alpha_K}{2(1 - (\delta/r))} \|\mathbf{x} - \mathbf{y}\|^2 \mathbf{z} \in \mathcal{K}_\delta.$$

Finally, combining the above result with Definition 1,  $\mathcal{K}_\delta$  is  $\frac{\alpha_K}{1 - (\delta/r)}$ -strongly convex with respect to the norm  $\|\cdot\|$ .

## D Proof of Lemma 3

For brevity, we define  $h_t = F_{t-1}(\mathbf{y}_t) - F_{t-1}(\mathbf{y}_t^*)$  for  $t = 1, \dots, T$  and  $h_t(\mathbf{y}_{t-1}) = F_{t-1}(\mathbf{y}_{t-1}) - F_{t-1}(\mathbf{y}_t^*)$  for  $t = 2, \dots, T$ .

For  $t = 1$ , since  $\mathbf{y}_1 = \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}_\delta} \|\mathbf{x} - \mathbf{y}_1\|_2^2$ , we have

$$h_1 = F_0(\mathbf{y}_1) - F_0(\mathbf{y}_1^*) = 0 \leq \frac{C}{\sqrt{1+2}} = \epsilon_1. \quad (23)$$

For any  $T + 1 \geq t \geq 2$ , assuming  $h_{t-1} \leq \epsilon_{t-1}$ , we first note that

$$\begin{aligned} h_t(\mathbf{y}_{t-1}) &= F_{t-1}(\mathbf{y}_{t-1}) - F_{t-1}(\mathbf{y}_t^*) \\ &= F_{t-2}(\mathbf{y}_{t-1}) - F_{t-2}(\mathbf{y}_t^*) + \langle \eta \mathbf{g}_{t-1}, \mathbf{y}_{t-1} - \mathbf{y}_t^* \rangle \\ &\leq F_{t-2}(\mathbf{y}_{t-1}) - F_{t-2}(\mathbf{y}_{t-1}^*) + \langle \eta \mathbf{g}_{t-1}, \mathbf{y}_{t-1} - \mathbf{y}_t^* \rangle \\ &\leq \epsilon_{t-1} + \eta \|\mathbf{g}_{t-1}\|_2 \|\mathbf{y}_{t-1} - \mathbf{y}_t^*\|_2 \\ &\leq \epsilon_{t-1} + \eta \|\mathbf{g}_{t-1}\|_2 \|\mathbf{y}_{t-1} - \mathbf{y}_{t-1}^*\|_2 + \eta \|\mathbf{g}_{t-1}\|_2 \|\mathbf{y}_{t-1}^* - \mathbf{y}_t^*\|_2 \\ &\leq \epsilon_{t-1} + \frac{\eta \sqrt{\epsilon_{t-1}} d M}{\delta} + \frac{\eta^2 d^2 M^2}{\delta^2} \end{aligned} \quad (24)$$

where the first inequality is due to  $\mathbf{y}_{t-1}^* = \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}_\delta} F_{t-2}(\mathbf{x})$  and the last inequality is due to  $\|\mathbf{y}_{t-1} - \mathbf{y}_{t-1}^*\|_2 \leq \sqrt{F_{t-2}(\mathbf{y}_{t-1}) - F_{t-2}(\mathbf{y}_{t-1}^*)}$  and (18).

Then, by substituting  $\eta = \frac{cR}{dM(T+2)^{3/4}}$  and  $\delta = cT^{-1/4}$  into (24), we have

$$\begin{aligned} h_t(\mathbf{y}_{t-1}) &\leq \epsilon_{t-1} + \frac{R\sqrt{C}}{\sqrt{T+2}(t+1)^{1/4}} + \frac{R^2}{(T+2)} \\ &\leq \epsilon_{t-1} + \frac{\epsilon_{t-1}}{4(t+1)^{1/4}} + \frac{\epsilon_{t-1}}{16(t+1)^{1/4}} \\ &\leq \left(1 + \frac{1}{2(t+1)^{1/4}}\right) \epsilon_{t-1} \end{aligned} \quad (25)$$

where the second inequality is due to  $T \geq t-1$  and  $R \leq \frac{\sqrt{C}}{4}$ .

Then, to bound  $h_t = F_{t-1}(\mathbf{y}_t) - F_{t-1}(\mathbf{y}_t^*)$  by  $\epsilon_t$ , we further introduce the following lemma.

**Lemma 7 (Derived from Lemma 1 of Garber and Hazan[15]).** *Let  $f(\mathbf{x}) : \mathcal{K}' \rightarrow \mathbb{R}$  be a convex and  $\beta_f$ -smooth function, where  $\mathcal{K}'$  is  $\alpha_{K'}$ -strongly convex with respect to the  $\ell_2$  norm. Moreover, let  $\mathbf{x}_{\text{in}} \in \mathcal{K}$  and  $\mathbf{x}_{\text{out}} = \mathbf{x}_{\text{in}} + \sigma'(\mathbf{v} - \mathbf{x}_{\text{in}})$ , where  $\mathbf{v} \in \arg\min_{\mathbf{x} \in \mathcal{K}'} \langle \nabla f(\mathbf{x}_{\text{in}}), \mathbf{x} \rangle$  and  $\sigma' = \arg\min_{\sigma \in [0,1]} \langle \sigma(\mathbf{v} - \mathbf{x}_{\text{in}}), \nabla f(\mathbf{x}_{\text{in}}) \rangle + \frac{\sigma^2 \beta_f}{2} \|\mathbf{v} - \mathbf{x}_{\text{in}}\|_2^2$ . For any  $\mathbf{x}^* \in \arg\min_{\mathbf{x} \in \mathcal{K}} f(\mathbf{x})$ , we have*

$$f(\mathbf{x}_{\text{out}}) - f(\mathbf{x}^*) \leq (f(\mathbf{x}_{\text{in}}) - f(\mathbf{x}^*)) \max\left(\frac{1}{2}, 1 - \frac{\alpha_{K'} \|\nabla f(\mathbf{x}_{\text{in}})\|_2}{8\beta_f}\right).$$

We note that  $F_{t-1}(\mathbf{x})$  is 2-smooth over  $\mathcal{K}_\delta$  for any  $t \in [T+1]$  and  $\mathcal{K}_\delta$  is  $\frac{\alpha_K}{1-(\delta/r)}$ -strongly convex with respect to the  $\ell_2$  norm. By applying Lemma 7 with  $f(\mathbf{x}) = F_{t-1}(\mathbf{x})$ ,  $\mathcal{K}' = \mathcal{K}_\delta$ ,  $\alpha_{K'} = \frac{\alpha_K}{1-(\delta/r)}$ , and  $\mathbf{x}_{\text{in}} = \mathbf{y}_{t-1}$ , for any  $t \in [T+1]$ , we have  $\mathbf{x}_{\text{out}} = \mathbf{y}_t$  and

$$h_t \leq h_t(\mathbf{y}_{t-1}) \max\left(\frac{1}{2}, 1 - \frac{\alpha_{K'} \|\nabla F_{t-1}(\mathbf{y}_{t-1})\|_2}{16}\right). \quad (26)$$

Because of (25), (26), and  $1 + \frac{1}{2(t+1)^{1/4}} \leq \frac{3}{2}$ , if  $\frac{1}{2} \leq \frac{\alpha_{K'} \|\nabla F_{t-1}(\mathbf{y}_{t-1})\|_2}{16}$ , it is easy to verify that

$$h_t \leq \frac{3}{4} \epsilon_{t-1} = \frac{3}{4} \frac{C}{\sqrt{t+1}} = \frac{C}{\sqrt{t+2}} \frac{3\sqrt{t+2}}{4\sqrt{t+1}} \leq \frac{C}{\sqrt{t+2}} = \epsilon_t \quad (27)$$

where the last inequality is due to  $\frac{3\sqrt{t+2}}{4\sqrt{t+1}} \leq 1$  for any  $t \geq 2$ .

Then, if  $\frac{1}{2} > \frac{\alpha_{K'} \|\nabla F_{t-1}(\mathbf{y}_{t-1})\|_2}{16}$ , there exist two cases. First, if  $h_t(\mathbf{y}_{t-1}) \leq \frac{3C}{4\sqrt{t+1}}$ , it is easy to verify that

$$h_t \leq h_t(\mathbf{y}_{t-1}) \leq \frac{3C}{4\sqrt{t+1}} \leq \epsilon_t \quad (28)$$

where the last inequality has been proved in (27).

Second, if  $h_t(\mathbf{y}_{t-1}) > \frac{3C}{4\sqrt{t+1}}$ , we have

$$\begin{aligned} h_t &\leq h_t(\mathbf{y}_{t-1}) \left( 1 - \frac{\alpha_{K'} \|\nabla F_{t-1}(\mathbf{y}_{t-1})\|_2}{16} \right) \\ &\leq \epsilon_{t-1} \left( 1 + \frac{1}{2(t+1)^{1/4}} \right) \left( 1 - \frac{\alpha_{K'} \|\nabla F_{t-1}(\mathbf{y}_{t-1})\|_2}{16} \right) \end{aligned}$$

where the last inequality is due to (25). Then, we notice that as proved by Garber and Hazan[16], if a function  $f(\mathbf{x}) : \mathcal{K} \rightarrow \mathbb{R}$  is  $\alpha$ -strongly convex, it holds that

$$\|\nabla f(\mathbf{x})\|_2 \geq \sqrt{\frac{\alpha}{2}} \sqrt{f(\mathbf{x}) - f(\mathbf{x}^*)} \quad (29)$$

for  $\mathbf{x}^* = \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}} f(\mathbf{x})$  and any  $\mathbf{x} \in \mathcal{K}$ . Because of (29) and the fact that  $F_{t-1}(\mathbf{y})$  is 2-strongly convex, we further have

$$\begin{aligned} h_t &\leq \epsilon_{t-1} \left( 1 + \frac{1}{2(t+1)^{1/4}} \right) \left( 1 - \frac{\alpha_{K'} \sqrt{h_t(\mathbf{y}_{t-1})}}{16} \right) \\ &\leq \epsilon_t \frac{\sqrt{t+2}}{\sqrt{t+1}} \left( 1 + \frac{1}{2(t+1)^{1/4}} \right) \left( 1 - \frac{\alpha_K \sqrt{3C}}{32(t+1)^{1/4}} \right) \end{aligned}$$

where the last inequality is due to  $\alpha_{K'} = \frac{\alpha_K}{1-(\delta/r)} \geq \alpha_K$ .

Since  $\frac{\sqrt{t+2}}{\sqrt{t+1}} \leq 1 + \frac{1}{\sqrt{t+1}}$  for any  $t \geq 0$ , it is easy to verify that

$$\frac{\sqrt{t+2}}{\sqrt{t+1}} \left( 1 + \frac{1}{2(t+1)^{1/4}} \right) \leq 1 + \frac{2}{(t+1)^{1/4}}$$

which further implies that

$$\begin{aligned} h_t &\leq \epsilon_t \left( 1 + \frac{2}{(t+1)^{1/4}} \right) \left( 1 - \frac{\alpha_K \sqrt{3C}}{32(t+1)^{1/4}} \right) \\ &\leq \epsilon_t \left( 1 + \frac{2}{(t+1)^{1/4}} \right) \left( 1 - \frac{2}{(t+1)^{1/4}} \right) \\ &\leq \epsilon_t \end{aligned} \quad (30)$$

where the second inequality is due to  $\alpha_K \sqrt{3C} \geq 64$ .

By combining (23), (27), (28) and (30), we complete the proof.

## E Proof of Lemma 6

We introduce the classical Azuma's inequality [4] for martingales in the following lemma.

**Lemma 8.** Suppose  $D_1, \dots, D_r$  is a martingale difference sequence and

$$|D_j| \leq c_j$$

almost surely. Then, we have

$$\Pr \left( \sum_{j=1}^r D_j \geq \Delta \right) \leq \exp \left( \frac{-\Delta^2}{2 \sum_{j=1}^r c_j^2} \right).$$

To apply Lemma 8, we define

$$D_t = \left( \nabla \hat{f}_{t,\delta}(\mathbf{y}_t) - \mathbf{g}_t(\mathbf{x}_t) \right) (\mathbf{y}_t^* - \mathbf{y}^*) \quad (31)$$

where  $\mathbf{y}_t^* = \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}_\delta} f_t(\mathbf{x})$ ,  $\mathbf{y}^* = (1 - \delta/r)\mathbf{x}^*$  and  $\mathbf{x}^* \in \operatorname{argmin}_{\mathbf{x} \in \mathcal{K}} \sum_{t=1}^T f_t(\mathbf{x})$ . According to Algorithm 1 and Lemma 1, we have

$$\mathbb{E}[D_t \mid \mathbf{y}_t, \mathbf{y}_t^*] = 0$$

which further implies that  $D_1, \dots, D_T$  is a martingale difference sequence with

$$\begin{aligned} |D_t| &= \left| \left( \nabla \hat{f}_{t,\delta}(\mathbf{y}_t) - \mathbf{g}_t(\mathbf{x}_t) \right) (\mathbf{y}_t^* - \mathbf{y}^*) \right| \\ &\leq \left\| \nabla \hat{f}_{t,\delta}(\mathbf{y}_t) - \mathbf{g}_t(\mathbf{x}_t) \right\|_2 \|\mathbf{y}_t^* - \mathbf{y}^*\|_2 \\ &\leq 2R \left( \left\| \nabla \hat{f}_{t,\delta}(\mathbf{y}_t) \right\|_2 + \|\mathbf{g}_t(\mathbf{x}_t)\|_2 \right) \\ &\leq 2R \left( G + \frac{dM}{\delta} \right) \end{aligned}$$

where the last inequality is due to Lemma 4 and Assumption 3.

Then, applying Lemma 8 with  $\Delta = 2R(G + \frac{dM}{\delta})\sqrt{2T \ln \frac{1}{\gamma}}$ , with probability at least  $1 - \gamma$ , we have

$$\sum_{t=1}^T D_t \leq \Delta = 2R(G + \frac{dM}{\delta})\sqrt{2T \ln \frac{1}{\gamma}}. \quad (32)$$

Additionally, combining (31), we further have

$$\sum_{t=1}^T \langle \nabla \hat{f}_{t,\delta}(\mathbf{y}_t), \mathbf{y}_t^* - \mathbf{y}^* \rangle = \sum_{t=1}^T D_t + \sum_{t=1}^T \mathbf{g}_t(\mathbf{x}_t)^\top (\mathbf{y}_t^* - \mathbf{y}^*). \quad (33)$$

By combining (19), (32) and (33), we complete the proof.