informs
http://pubsonline.informs.org/journal/ijds

ANY INFORMS JOURNAL
Vol. 00, No. 0, Xxxxx 0000, pp. 000–000
ISSN 2694-4022, EISSN 2694-4030

**Submitted to *Any INFORMS Journal***

# A descent-direction-constrained method for multiobjective optimization

Anonymous author
Anonymous anonymous@informs.org

Anonymous author
Anonymous anonymous@informs.org



**Abstract.** This paper presents a descent-direction-constrained method without a priori parameters (e.g., weighting factors) for locating Pareto optimal solutions of multiobjective optimization problems by iteratively searching descent directions for a fixed objective with consideration to the others and evaluating their step lengths. The descent direction is determined through a quadratic subproblem constrained by the approximations of all but the fixed objective function of the original multiobjective problem. In the nonconvex case, both linear and quadratic approximations are considered with a penalty added to the objective function of the subproblem. In the convex case, a multiobjective subproblem composed of quadratic approximations of original objective functions, a generalization of Newton-type method, is proposed to analyze improvement upon traditional methods. Under certain differentiability assumptions, the algorithm is shown to converge to a point satisfying the proposed first-order necessary conditions for Pareto optimality, simultaneously avoiding weakly Pareto solutions and superlinearly and quadratically to the Pareto points with further sufficient conditions satisfied under strict convex assumptions. Finally, the numerical tests are conducted to demonstrate superiority and effectiveness of our method.

**Key words:** Multiobjective optimization; descent-direction-constrained method; Newton-type method;scalarization

## 1. Introduction

In the realm of optimization, particularly in the context of complex decision-making scenarios, multiobjective optimization stands out as a powerful tool. Unlike traditional single-objective optimization, where a single goal is pursued, multiobjective optimization deals with the simultaneous optimization of multiple conflicting objectives. This approach is crucial in situations where no single solution can satisfy all objectives simultaneously, leading to the concept of $Pareto\ optimal\ solutions$. Pareto optimal solutions represent the optimal trade-offs among conflicting objectives, offering decision-makers a range of optimal choices rather than a single solution. These solutions play a pivotal role in various fields, including economics (Keller et al. 2009), policy design (Papalexopulos 2023) chemical and biological engineering (Logist et al. 2009, 2010, Fu et al. 2023), electric power distribution (Ganguly et al. 2013), mineral processing (Lu et al. 2018), and medical (Rosenthal and Markus 2017), among others.

In recent 20 years, significant progress has been made in developing algorithms for solving multiobjective optimization problems, where the main stream algorithms can be classified into two general categories: heuristic/evolutionary algorithms (Deb et al. 2002, Cai et al. 2018, Pal and Charkhgard 2019, Pang et al. 2023) and deterministic algorithms (Dennis and Schnabel 1983, Das and Dennis 1998, Miettinen 1999, Messac at al. 2003, Eichfelder 2006, Burachik et al. 2014, 2017, Prudente et al. 2022, Nie and Yang 2024). Heuristic or evolutionary algorithms, such as NSGA-II proposed by Deb et al. (2002), do not require gradient information and are less likely to get stuck in local optima. They typically use techniques like retaining and eliminating individuals based on evaluation indexes (Li et al. 2018, Shen et al. 2019, Ma et al. 2019). However, these algorithms will generate different solutions in different runs, and the convergence of evolutionary algorithms is not always guaranteed.

On the other hand, deterministic algorithms, relying on gradient-based methods and scalarization techniques, generate more stable solutions with provable convergence properties. In this paper, we primarily focus on discussing deterministic algorithms due to their stability and convergence guarantees. The scalarization (Miettinen 1999) is one of the most famous deterministic techniques. It converts the multiobjective problem into a series of single-objective problems by scalarization techniques, such as the weighted sum or norm functions. The popular methods include the $\varepsilon$-constraint method (Eichfelder 2006), NBI (Das and Dennis 1998), NNC (Messac at al. 2003, Burachik et al. 2014). Then we solve them serially by the mature single objective optimization methods such as SQP, trust-region method (Nocedal and Wright 1999), et al. Typically, these methods need a priori information, such as the choice of the parameters, which brings a barrier to decision makers. Based on a critical condition (Fliege et al. 2000), Fliege takes the lead to propose an a priori parameter-free method (Fliege et al. 2009) that is an extension of Newton's method for multiobjective optimization problems. This method is proved to display a quadratic rate of convergence under some differential assumptions. Fliege et al. (2009) gives a necessary condition for Pareto optimality and defines the descent direction mathematically for multiobjective optimization, laying a great foundation for the independent theory of multiobjective optimization instead of relying on single-objective optimization methods. Many successful generalizations of this algorithm have been developed in recent years, for exmaple, Carrizo et al. (2016) improves an algorithm for nonconvex cases.

In this study, we undertake an exaplanation of Fliege et al. (2009) through the lens of a novel multiobjective subproblem framework. Subsequently, we introduce an innovative method constrained by descent directions, designed to modify said descent trajectories. First, we put forth a novel indispensable criterion for Pareto optimality, adept at circumventing the inclusion of weakly Pareto optimal solutions. Next, we present a quadratic subproblem formulation wherein the primary objective entails an approximation of a predetermined objective function within the original multiobjective context. This formulation is subject to constraints derived from linear and quadratic approximations of the remaining objective functions. Then, we introduce a multiobjective subproblem aimed at elucidating the connections to Fliege et al. (2009) employing the $\varepsilon$-constraint method. Finally, we establish the convergence of the solution sequence towards Pareto optimal solutions, contingent upon specific continuity assumptions. Additionally, we rigorously demonstrate both superlinear and quadratic convergence properties, particularly within the context of strictly convex scenarios.

The paper is structured as follows: Section 2 provides a review of fundamental definitions and preliminary outcomes. Section 3 presents a quadratic subproblem aimed at ascertaining the optimal search direction, alongside a multiobjective subproblem designed to investigate the application of Newton-type methodologies within the convex domain. This section further proposes a descent-direction-constrained algorithm. Section 4 delves into a comprehensive examination of the convergence properties inherent in the proposed algorithm. Section 5 undertakes a comparative analysis, employing various cases to substantiate the efficacy and superiority of the proposed algorithm versus Fliege et al. (2009). Lastly, Section 6 furnishes a conclusive summary and outlines prospective avenues for future research endeavors.

## 2. Preliminary Results and Analysis

Multiobjective optimization is to minimize a vector of objective functions simultaneously, and can be formulated as:

$$\min_{x\in\mathcal{X}} \; F(x) = [f_1(x), \cdots, f_m(x)]^{\mathrm{T}}, \tag{MOP}$$

where $x$ denotes an $n$-dimension variable, and the feasible set $\mathcal{X} \subset \mathbb{R}^n$ is supposed to be an open set, $F : \mathcal{X} \to \mathbb{R}^m$ denotes a vector-valued objective function. The objective functions $f_1, \cdots, f_m$ are supposed to be twice continuously differentiable. This paper mainly considers MOPs with box constraints.

Some general assumptions should be noted to avoid the trivial cases,

**Assumption 1.** *There is an attainable infimum of each objective function denoted as* $\inf_{x\in\mathcal{X}} f_i(x)$.

**Assumption 2.** *There does not exists a solution that can optimize all the objective functions simultaneously.*

And the Pareto optimality is defined as follow:

**Definition 1.** *A feasible solution $x^* \in \mathcal{X}$ is called (globally) Pareto optimal if $\nexists\, x \in \mathcal{X}$ and $x \neq x^*$ such that $f_i(x) \le f_i(x^*)$ for all $i = 1, \cdots, m$, and $f_j(x) < f_j(x^*)$ for at least one $j$. If there exists a neighbour $\mathcal{N}(x^*)$ of $x^*$ such that this property holds in $\mathcal{N}(x^*)$, then $x^*$ is called locally Pareto optimal. Furthermore, $x^*$ is weakly Pareto optimal if $\nexists\, x \in \mathcal{X}$ such that $f_i(x) < f_i(x^*)$ for all $i = 1, \cdots, m$.*

Generally, all the Pareto optimal solutions are called *Pareto solutions*, and the set of all Pareto solutions is called *Pareto optimal set*, which appears as the *Pareto front* in the objective space. Then, given two feasible points $x_1,\ x_2$, we have the definition of Pareto dominance through strict partial order induced by the cone:

$$\mathbb{R}^m_{++} = \{x \in \mathbb{R}^m : x > 0\},\ \mathbb{R}^m_{+} = \{x \in \mathbb{R}^m : x \geq 0\}, \tag{1}$$

and the dominance is defined by:

$$F(x_1) \prec F(x_2) \Leftrightarrow F(x_2) - F(x_1) \in \mathbb{R}^m_{++};$$
$$F(x_1) \preceq F(x_2) \Leftrightarrow F(x_2) - F(x_1) \in \mathbb{R}^m_{+} \backslash \{0\}.$$

We call $x_1$ dominates $x_2$ if and only if $F(x_1) \prec F(x_2)$. And we call $x_1$ weakly dominates $x_2$ if and only if $F(x_1) \preceq F(x_2)$. So $x^*$ is Pareto optimal if and only if there does not exist an $x \in \mathcal{X}$ such that it weakly dominates $x^*$. Throughout this paper, unless extra notation, we will assume that the objective vecotr is twice continuously differentiable on $\mathcal{X}$, that is, $F \in \mathcal{C}^2(\mathcal{X}, \mathbb{R}^m)$, and let $\mathcal{J}(F(x))$ denote the Jacobian matrix of $F(x)$, and we define that $\|\cdot\|$ denotes 2-Norm. A necessary condition for Pareto optimality is defined that a point $x^* \in \mathcal{X}$ is critical (Fliege et al. 2009) for $F(x)$ if

$$\mathcal{J}(F(x^*)) \cap (-\mathbb{R}^m_{++}) = \emptyset. \tag{2}$$

However, this condition is relatively stringent and particularly applicable to strongly convex cases, as it necessitates the absence of any potential descent direction for all objectives. Even in some convex cases, adhering to this condition may result in weak Pareto optimality, as there could be a direction that allows for the descent of some objectives while maintaining the non-increase of others. So, for the first time, we define $x^*$ is *strictly critical* for $F(x)$ if

$$\mathcal{J}(F(x^*)) \cap (-\mathbb{R}^m_{+} \backslash \{0\}) = \emptyset, \tag{3}$$

which means if $x^*$ is strictly critical, for all $s \in \mathbb{R}^n$ there exists $j_0$ such that

$$\nabla f_{j_0}(x^*)^{\mathrm{T}} s > 0,$$

or $\nabla f_i(x^*)^{\mathrm{T}} s = 0,\ \forall i = 1, \cdots, m$. If $x \in \mathcal{X}$ is not strictly critical, then there exists $s \in \mathbb{R}^n$ such that $\nabla f_i(x)^{\mathrm{T}} s \leq 0$ for all $i = 1, \cdots, m$, and $\nabla f_j(x)^{\mathrm{T}} s < 0$ for some $j = 1, \cdots, m$. Then it holds that

$$\lim_{t \to 0} \frac{f_i(x + \alpha s) - f_i(x)}{\alpha} = \nabla f_i(x)^{\mathrm{T}} s \leq 0, \tag{4}$$

for all $i = 1, \cdots, m$, and strictly holds for some $j = 1, \cdots, m$. So $s$ is a descent direction for $F$ at $x$, namely, there exists $\alpha_0 > 0$ such that

$$F(x + \alpha s) \preceq F(x) \text{ for all } \alpha \in (0, \alpha_0]. \tag{5}$$

The relationship between the properties of being critical, strictly critical, and Pareto optimal was introduced in the following theorem.

**Theorem 1.** *Herein, $F$ is assumed to be continuously differentiable on $\mathbb{R}^n$, i.e., $F \in \mathcal{C}^1(\mathcal{X}, \mathbb{R}^m)$, then*

1. *If $x^*$ is strictly critical, then it is critical.*
2. *If $x^*$ is locally Pareto optimal, then $x^*$ is a strictly critical solution for $F$.*
3. *Suppose that $\mathcal{X}$ is convex and $F$ is $\mathbb{R}^m$-convex, if $x^* \in \mathcal{X}$ is strictly critical for $F$, then $x^*$ is Pareto optimal.*

**Proof:** We first prove item 1. If $x^*$ is strictly critical, then $\mathcal{J}(F(x^*)) \cap (-\mathbb{R}^m_+ \backslash \{0\}) = \emptyset$. Because $-\mathbb{R}^m_{++} \subset -\mathbb{R}^m_+ \backslash \{0\}$, we have $\mathcal{J}(F(x^*)) \cap (-\mathbb{R}^m_{++}) = \emptyset$. So, item 1 holds.

To prove item 2. Assume that $x^*$ is locally Pareto optimal, if it not strictly critical, then there exist an $s \in \mathbb{R}^n$ and $\alpha_0 > 0$ such that $F(x + \alpha s) \preceq F(x)$, which is a contradiction to locally Pareto optimality of $x^*$; so item 2 is true.

To prove item 3, take any $x \in U$, since it is strictly critical, then for all $s \in \mathbb{R}^n$ there exists an index $j_0$ such that $\nabla f_{j_0}(x^*)^{\mathrm{T}} s > 0$. Since $F$ is convex, then

$$f_{j_0}(x_1) \geq f_{j_0}(x^*) + \nabla f_{j_0}(x^*)^{\mathrm{T}}(x_1 - x^*) > f_{j_0}(x^*)$$

or $\nabla f_i(x^*)^{\mathrm{T}} s = 0$ for all $i = 1, \cdots, m$, which imply that $x^*$ is Pareto optimal. So, item 3 holds. □

The weighting method is one of the most common traditional scalarization techniques because of its simple structure and provable convergence to Pareto solutions for convex MOPs (Miettinen 1999). However, some inevitable drawbacks in nonconvex cases are pointed out in Das and Dennis (1997), i.e., failure in capturing nonconvex parts of Pareto front, nonuniform spread of Pareto points using uniform spread of the parameter. In Fliege et al. (2000), a steepest descent method is proposed to give a descent direction which is also a feasible direction for constrained cases. In single-objective optimization problems, the Newton's method minimizes a subproblem of the quadratic approximation of the objective function. A Newton-type method for MOP introduced in Fliege et al. (2009) minimizes the maximal descent of quadratic approximations of the objectives. Thus, we will obtain a descent direction for all objective functions from the subproblem. Following that, Fliege et al. (2009) proposes a Newton-type descent direction $s_F$ for MOP is further defined as the solution of

$$\min_{s \in \mathbb{R}^n} \max_{i=1,\ldots,m} \nabla f_i(x)^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_i(x) s, \tag{6}$$

and the optimal value of problem (6) is denoted as:

$$\theta_F(x) = \min_{s \in \mathbb{R}^n} \max_{i=1,\ldots,m} \nabla f_i(x)^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_i(x) s. \tag{7}$$

Since (6) is not differentiable, a min-max problem is solved in reality.

This method firstly attempts to establish a characteristic methodology for multiobjective optimization instead of relying on single-objective optimization methods. It is a successful method without a priori information being required, and many results are derived from it in recent years (Carrizo et al. 2016, Prudente et al. 2022). Moreover, this type of method can be applied to nonconvex-constrained problems via quasi-Newton method and SQP-type methods (Prudente et al. 2022, Fliege et al. 2016)).

However, Fliege's method is a particular case of the weighting method with fixed parameters to some extent. Therefore, this paper will analyze the drawbacks of this method via a generalized multiobjective subproblem and propose a novel approach for multiobjective problems.

In MOP, what really matters for a noncritical point to search a Pareto solution is the searching direction. Evidently, the descent direction for all the objective functions, such as the solution of problem (6), is definitely an ideal search direction for solving the multiobjective problem. Because it has achieved that all the objective functions have a desirable descent, and leads to a $\mathbb{R}^m_+$-decreasing converging sequence. However, according to the critical condition, the limitation of this method is that the descent direction at the terminating iteration is constrained in cone $\mathbb{R}^m_{++}$. The feasible region is actually smaller for a direction $s$ than it should be and will lead to weakly Pareto optimal points. Therefore, in the next section, we propose a novel Newton-type method to handle this issue, based on the strict critical condition for Pareto optimality.

# 3. Descent-direction-constrained method

The main theoretical results of this paper are presented in this section and the next. In this section, first, we propose a descent-direction-constrained subproblem constructed by the approximations of origin objective functions. The subproblem is considered in two cases according to the positivity of the Hessian matrices of these objective functions. Then, we analyze the relationship with Fliege's method through a multiobjective subproblem. At last, we give the algorithm for locating the Pareto optimal points.

For $x \in \mathcal{X}$, we define the direction $s_d(x)$ at $x$, as the optimal solution of a descent-direction-constrained quadratic program problem,

$$\begin{aligned} \min_{s\in\mathbb{R}^n} \quad & \nabla f_d(x)^{\mathrm{T}} s + H_{d,x}(s) \\ \text{s.t.} \quad & \nabla f_i(x)^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_i(x) s \le 0, \\ & \nabla f_i(x)^{\mathrm{T}} s \le 0,\ i = 1, \cdots, m,\ i \ne d \end{aligned} \tag{8}$$

where $H_{d,x}(s)$ is the term determined before solving and is defined as

$$H_{d,x}(s) = \begin{cases} \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_d(x) s, & \nabla^2 f_d \text{ is positive,} \\ \frac{1}{2} s^{\mathrm{T}} s, & \text{else.} \end{cases}$$

Specifically, $f_d$ is called the descent function in this paper. Problem (8) is constructed by a quadratic objective function with convex (linear and quadratic) inequality constraints. It has a convex feasible region and is a convex problem. The feasible region can be described as

$$\mathcal{S} = \{ s \mid \nabla f_i(x)^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_i(x) s \le 0, \nabla f_i(x)^{\mathrm{T}} s \le 0,\ i = 1, \cdots, m,\ i \ne d, \}$$

Since $\mathcal{S}$ is compact, and if $\mathcal{S}$ has more than one elements except $s=0$ which is always feasible, it is easy to find an interior point for the linear constraints. Even, when $\nabla^2 f_i(x) > 0$, we can delete the linear constraints. Suppose there is not an interior point. So there exists an $s' \in \mathbb{R}^n$ such that

$$\nabla f_i(x)^{\mathrm{T}} s' + \frac{1}{2} s'^{\mathrm{T}} \nabla^2 f_i(x) s' = 0$$

Then, for any $\alpha > 0$, let $s = \alpha s'$, $s \in \mathbb{R}^n$, we have

$$\begin{aligned} &\nabla f_i(x)^{\mathrm{T}} s' + \frac{1}{2} s'^{\mathrm{T}} \nabla^2 f_i(x) s' \\ =&\nabla f_i(x)^{\mathrm{T}} \alpha s' + \frac{1}{2} \alpha s'^{\mathrm{T}} \nabla^2 f_i(x) \alpha s' \\ =&\alpha(\nabla f_i(x)^{\mathrm{T}} s' + \frac{\alpha}{2} s'^{\mathrm{T}} \nabla^2 f_i(x) s'). \end{aligned}$$

There exists a small enough $\alpha$ such that the right-hand side is negative. We conclude that problem (8) has a slater point for any noncritical point.

Now, we divide problem (8) into two cases, considering different circumstances of the objective function.
*Case I.* $\nabla^2 f_d$ is nonpositive, then the objective function of the subproblem is:

$$\nabla f_d(x)^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} s \tag{9}$$

As for this case, a penalty $s^{\mathrm{T}} s$ (same as $\|s\|_2^2$) is utilized, for particular consideration of the case when $\nabla^2 f_i(x)$ ( $i = 1, \cdots, m,\ i \neq d$) are nonpositive. Problem (8) becomes a *quadratic program* (QP):

$$\begin{aligned} \min_{s \in \mathbb{R}^n} \quad & \nabla f_d(x)^{\mathrm{T}} s + \frac{1}{2}\|s\|_2^2 \\ \text{s.t.} \quad & \nabla f_i(x)^{\mathrm{T}} s \leq 0, \\ & i = 1, \cdots, m,\ i \neq d. \end{aligned} \tag{10}$$

Since none of the Hessian matrices is included, it is predictable that problem (10) represents the worst case of the convergent rate. It can be regarded as a formulation of another steepest descent method for MOP, akin to Fliege et al. (2000). $\frac{1}{2}\|s\|_2^2$ denotes a simultaneous minimization of the norm of the searching direction $s$ and can be converted into an inequality constraint $\|s\|_2^2 \leq 1$. This restriction is necessary as the feasible region restricted a group of linear inequality constraints may not be closed, i.e., the minimum of this problem may be unobtainable. That is why we divide $H_{d,x}(s)$ into two cases herein.
*Case II.* $\nabla^2 f_d$ is positive, then we get the objective function as

$$\nabla f_d(x)^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_d(x) s \tag{11}$$

In this case, $H_{d,x}(s) = \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_d(x) s$, we have an ideal circumstance with all the Hessian matrices being positive. Problem (8) becomes a *quadratically constrained quadratic program* (QCQP),

$$\begin{aligned} \min_{s \in \mathbb{R}^n} \quad & \nabla f_d(x)^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_d(x) s \\ \text{s.t.} \quad & \nabla f_i(x)^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_i(x) s \leq 0, \\ & i = 1, \cdots, m,\ i \neq d. \end{aligned} \tag{12}$$

Problem (12) is a formulation of Newton's method for MOP through scalarization, akin to Fliege et al. (2009). In this case, $\nabla^2 f_i(x)$ is positive for $i = 1, \cdots, m$. Then we propose the following multiobjective subproblem where each objective function is a quadratic approximation of original objective function to obtain all the possible descent directions:

$$\min_{s\in\mathbb{R}^n} \left\{ \begin{array}{c} \nabla f_1(x)^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_1(x) s \\ \cdots \\ \nabla f_m(x)^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_m(x) s \end{array} \right\}. \tag{MOSP}$$

Since all the objective functions and the feasible region are convex, MOSP is a convex problem by definition 2.1.3 in Miettinen (1999). This problem that includes all the possible descent and non-descent directions can be regarded as a generalization of the subproblem in Fliege et al. (2009).

**Remark 1.** *It is clear that there are many algorithms that can solve the convex subproblem MOSP, and different methods will obtain different descent directions. The method in Fliege et al. (2009) can be regarded as the adoption of the Tchebycheff method (Miettinen 1999) for solving the multiobjective subproblem MOSP to obtain a minmax scalarization problem. This method does not require parameters like the weighting coefficients, but it is the same as setting the weighting vector as* $w = 1^{\mathrm{T}}$.

We can easily find that problem (12) is exactly the form of the $\varepsilon$-constraint method implemented in MOSP with parameters $\varepsilon_i$ to be $0$. Before using the $\varepsilon$-constraint method, single objective optimization problems are generally solved for individual minimums to obtain the upper and lower bounds of the parameters, and a sequence of parameters can be chosen from the region restricted by these bounds.

We consider the subproblem of $d$th individual objective solved by Newton's method:

$$\min_{s\in\mathbb{R}^n} \quad \nabla f_d(x)^{\mathrm{T}} s + H_{d,x}(s) \tag{IOP}$$

In *Case II*, by choosing proper parameters, we generally obtain the optimal solution $x^*$ such that $\nabla f_i(x^*)^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_i(x^*) s = \varepsilon_i$ for $i = 1, \cdots, m, i \neq d$. However, we basically desire one feasible descent direction, so it is not that appropriate to proceed as the scalarization methods. Herein we set $\varepsilon_i = 0$, so we propose a parameter-free subproblem (12) to obtain the descent direction. It is different from the traditional $\varepsilon$-constraint method as its constraints may be inactive at the optimal point.

Compared to problem (12), IOP lacks $m-1$ inequality constraints converted from the other objectives. It is evident that the solutions to these two problems are the same if the constraints are inactive. Now we need to analyze the case that they are different, i.e., the quadratic constraints are active. Let $\lambda_i$ be the multiplier of the inequality constraint converted from $i$th objective function. Then we consider:

*(a)* If all the inequality constraints are nonactive, i.e., $\lambda_i = 0 \ \forall \ i = 1, \cdots, m, \ i \neq d$, problem (12) has the same solution as IOP.

*(b)* If some inequality constraints are active, i.e., $\lambda_i \neq 0$ for some $i = 1, \cdots, m, \ i \neq d$, it implies that the solutions of these two problems are different.

As for (a), it is necessary to alter the descent objective to avoid obtaining the same direction as IOP. Otherwise, many initial points will converge to several individual minimums. So, we should analyze the multipliers after solving problem (12), to judge (a) and (b). For (a), we should alter the different descent objective, and for (b), we proceed to next iteration. This process is equally important for problem (8).

**Lemma 1.** *The solution of problem (12) is a Pareto optimal solution of MOSP.*

**Proof:** Since problem (12) is a strongly convex problem, then its solution is unique without further checking (page 87, Miettinen 1999) if there is a solution of (12), and there does not exists weakly Pareto optimal solutions. So, there must exists a unique solution for problem (12), and this solution is a Pareto optimal solution of MOSP. □

Define $\theta_d(x)$ as the optimal value of problem (8) with $d$th objective being the descent direction

$$\begin{aligned} \theta_d(x) = \inf_{s\in\mathbb{R}^n} \quad & \nabla f_d(x)^{\mathrm{T}} s + H_{d,x}(s) \\ \text{s.t.} \quad & \nabla f_i(x)^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_i(x) s \le 0, \\ & \nabla f_i(x)^{\mathrm{T}} s \le 0,\ i = 1, \cdots, m,\ i \ne d, \end{aligned} \tag{13}$$

and $s_d(x)$ is

$$\begin{aligned} s_d(x) = \arg\min_{s\in\mathbb{R}^n} \quad & \nabla f_d(x)^{\mathrm{T}} s + H_{d,x}(s) \\ \text{s.t.} \quad & \nabla f_i(x)^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_i(x) s \le 0, \\ & \nabla f_i(x)^{\mathrm{T}} s \le 0,\ i = 1, \cdots, m,\ i \ne d, \end{aligned} \tag{14}$$

Let $x^* \in \mathbb{R}^n$ be a Pareto optimal solution of MOP, then all the inequality constraints of problem (12) at $x^*$ are active. Otherwise, there exist some $j$ such that $\nabla f_j(x^*)^{\mathrm{T}} s_d(x^*) + \max\{\frac{1}{2} s_d(x^*)^{\mathrm{T}} \nabla^2 f_j(x^*) s_d(x^*), 0\} < 0$, then $\nabla f_j(x^*)^{\mathrm{T}} s_d(x^*) < 0$, which is a contradiction to the strict criticality.

The Lagrangian of problem (8) is

$$L(s,\lambda) = \nabla f_d(x)^{\mathrm{T}} s + H_{d,x}(s) + \sum_{\substack{i=1\\ i\ne d}}^{m} \lambda_i^1 (\nabla f_i(x)^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_i(x) s) + \sum_{\substack{i=1\\ i\ne d}}^{m} \lambda_i^2 \nabla f_i(x)^{\mathrm{T}} s,$$

and the Karush-Kuhn-Tucker (KKT) conditions are calculated:

$$\nabla f_d(x) + \nabla_s H_{d,x}(s) + \sum_{\substack{i=1\\ i\ne d}}^{m} \lambda_i^1 (\nabla f_i(x) + \nabla^2 f_i(x) s) + \sum_{\substack{i=1\\ i\ne d}}^{m} \lambda_i^2 \nabla f_i(x) = 0, \tag{15}$$

$$\lambda_i^1 (\nabla f_i(x)^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_i(x) s) = 0, \quad \lambda_i^2 \nabla f_i(x)^{\mathrm{T}} s = 0, \quad \lambda_i^1, \lambda_i^2 \ge 0. \tag{16}$$

$$\nabla f_i(x)^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} \nabla^2 f_i(x) s \le 0, \nabla f_i(x)^{\mathrm{T}} s \le 0,\ 1 \le i \le m, i \ne d, \tag{17}$$

where $\lambda = (\lambda^1, \lambda^2)$, $\lambda_i = (\lambda_i^1, \lambda_i^2)$, and

$$\lambda^1 = (\lambda_1^1, \cdots, \lambda_{d-1}^1, \lambda_{d+1}^1, \cdots, \lambda_m^1) \in \mathbb{R}_+^{m-1},$$
$$\lambda^2 = (\lambda_1^2, \cdots, \lambda_{d-1}^2, \lambda_{d+1}^2, \cdots, \lambda_m^2) \in \mathbb{R}_+^{m-1},$$

are the Lagrange multiplier vectors for quadratic and linear constraints respectively. Then we can obtain:

$$s_d(x) = -\left(\nabla_s^2 H_{d,x}(s) + \sum_{\substack{i=1\\ i\neq d}}^{m} \lambda_i^1 \nabla^2 f_i(x)\right)^{-1} \left(\nabla f_d(x) + \sum_{\substack{i=1\\ i\neq d}}^{m} \lambda_i^1 \nabla f_i(x) + \sum_{\substack{i=1\\ i\neq d}}^{m} \lambda_i^2 \nabla f_i(x)\right), \tag{18}$$

It is obvious that, in *Case II*, problem (12), $s_d(x)$ is also a Newton-type direction implicitly induced by weighting the all but one objective function by the a priori multipliers. Since problem (8) is convex and has a slater point,there is no duality gap, then we have:

$$\theta_d(x) = \inf_{s\in\mathbb{R}^n} \max_{\lambda\geq 0}\ L(s,\lambda) = \sup_{\lambda\geq 0} \min_{s\in\mathbb{R}^n}\ L(s,\lambda). \tag{19}$$

Now we can conclude the following property of problem (8).

**Lemma 2.** *Let $\theta_d(x)$ and $s_d(x)$ be as above, then we have:*

1. *For any $x \in \mathcal{X}$ and any $d = 1, \cdots, m$, $\theta_d(x) \leq 0$.*
2. *For a fixed index d, the function $s_d : \mathcal{X} \to \mathbb{R}^n$ is bounded on compact sets and $\theta_d : \mathcal{X} \to \mathbb{R}_-$ is continuous in $\mathcal{X}$ respectively in Case I and Case II.*
3. *The following statements are equivalents*

*(a) The point $x$ is not strictly critical,*

*(b) $\theta_d(x) < 0$ for at least one d, $1 \leq d \leq m$,*

*(c) $s_d(x) \neq 0$.*

**Proof:** We first prove item 1. For any $x \in \mathcal{X}$ and any $d = 1, \cdots, m$, $s = 0$ is always a feasible solution. And note that, $\theta_d(x)$ is the minimal value of (8), we have $\theta_d(x) \leq \nabla f_d(x)^{\mathrm{T}} 0 + \frac{1}{2} 0^{\mathrm{T}} H_d(x) 0$ for $d = 1, \cdots, m$, so $\theta_d(x) \leq 0$, item 1 holds.

Secondly we prove item 2. Let $\mathcal{W} \in \mathcal{X}$ be a compact set. According to item 1, for any $y \in \mathcal{W}$, we have

$$H_{d,y}(s_d(y)) \leq -\nabla f_d(y)^{\mathrm{T}} s_d(y), \text{ for all } d = 1, \cdots, m.$$

Since all $f_d$ are twice continuously differentiable everywhere, and from the definition of $H_{d,x}(s)$, we have

$$H_{d,x}(s) \geq \frac{1}{2}\|s\|_2^2,$$

then it is uniformly bounded, and let $K_d = \|\nabla f_d(y)\|$ , so for any $d = 1, \cdots, m$

$$\min_{x\in\mathcal{W}, \|u\|=1} H_{d,x}(u) = \frac{1}{2}.$$

Then for $y \in \mathcal{W}$, we have

$$\frac{1}{2}\|s_d(y)\|^2 \leq \|\nabla f_d(x)\|\|s_d(y)\| \leq K_d\|s_d(y)\|.$$

So $\|s_d(y)\| \leq 2K_d$ holds for all $y \in \mathcal{W}$ and any $d = 1, \cdots, m$, i.e., $s_d$ is bounded on compact sets.

For $x \in \mathcal{W}$ and given $d$, define $\varphi_{d,x} : \mathcal{W} \to \mathbb{R}$ and $\varphi_{i,x} : \mathcal{W} \to \mathbb{R}$

$$\begin{aligned}
&\varphi_{d,x}(z) = \begin{cases} \nabla f_d(z)^{\mathrm{T}} s_d(x) + \frac{1}{2} s_d(x)^{\mathrm{T}} s_d(x), & Case I \\ \nabla f_d(z)^{\mathrm{T}} s_d(x) + \frac{1}{2} s_d(x)^{\mathrm{T}} \nabla^2 f_d(z) s_d(x), & Case II \end{cases} \\
&\varphi_{i,x}^1(z) = \nabla f_i(z)^{\mathrm{T}} s_d(x) + \frac{1}{2} s_d(x)^{\mathrm{T}} \nabla^2 f_i(z) s_d(x), \\
&\varphi_{i,x}^2(z) = \nabla f_i(z)^{\mathrm{T}} s_d(x).
\end{aligned}$$

Evidently, $\varphi_{d,x}$ is continuous respectively in *Case I* and *Case II* (It is fixed before solving the problem), besides $\varphi_{i,x}^1$ and $\varphi_{i,x}^2$ are continuous for $x \in \mathcal{W}$. Because $\lambda_i^2 \varphi_{i,x}^2(z) = 0$, then the family

$$\Phi_x(z) = \max_{\lambda_i \geq 0} \varphi_{d,x}(z) + \sum_{\substack{i=1 \\ i \neq d}}^{m} \lambda_i^1 \varphi_{i,x}^1(z) + \sum_{\substack{i=1 \\ i \neq d}}^{m} \lambda_i^2 \varphi_{i,x}^2(z)$$

is also continuous (the Lagrange multipliers are defined non-negative and continuous).

Take $\epsilon > 0$, there exists $\delta > 0$ such that $\forall z_1,\ z_2 \in \mathcal{W}$

$$\|z_1 - z_2\| < \delta \Rightarrow |\Phi_x(z_1) - \Phi_x(z_2)| < \epsilon \quad \forall x \in \mathcal{W}.$$

Hence for $\|z_1 - z_2\| < \delta$, we have

$$\begin{aligned}
\theta_d(z_1) &\leq \max_{\lambda_i \geq 0}\ \varphi_{d,z_2}(z_1) + \sum_{\substack{i=1 \\ i \neq d}}^{m} \lambda_i^1 \varphi_{i,z_2}^1(z_1) + \sum_{\substack{i=1 \\ i \neq d}}^{m} \lambda_i^2 \varphi_{i,z_2}^2(z_1) \\
&= \Phi_{z_2}(z_1) \leq \Phi_{z_2}(z_1) + |\Phi_{z_2}(z_2) - \Phi_{z_2}(z_1)| < \theta_d(z_2) + \epsilon
\end{aligned}$$

Interchanging $z_1$ and $z_2$, we can easily obtain $\theta_d(z_2) < \theta_d(z_1) + \epsilon$. So we can conclude that $|\theta_d(z_1) - \theta_d(z_2)| < \epsilon$, i.e., $\theta_d(x)$ is continuous in $\mathcal{X}$ for any $d = 1, \cdots, m$, respectively in *Case I* and *Case II*.

At last, we prove item 3.

$(a) \Rightarrow (b)$ Suppose that $(a)$ holds, so $\mathcal{J}(F(x^*)) \cap (-\mathbb{R}_+^m \backslash \{0\}) \neq \emptyset$, then for a fixed $d$, there exists $s' \in \mathcal{S}$ such that $\nabla f_i(x)^{\mathrm{T}} s' \leq 0$ for all $i = 1, \cdots, m$ and $\nabla f_j(x)^{\mathrm{T}} s' < 0$ for some $j = 1, \cdots, m$. Then, take $d = j$ as the index with $\nabla f_d(x)^{\mathrm{T}} s' < 0$, so for all $\alpha > 0$, we have:

$$\begin{aligned}
\theta_d(x) &\leq \max_{\lambda \geq 0} L(\alpha s', \lambda) \\
&\leq \nabla f_d(x)^{\mathrm{T}} \alpha s' + \frac{1}{2} \alpha s'^{\mathrm{T}} \nabla^2 f_d(x) \alpha s' \\
&= \alpha(\nabla f_d(x)^{\mathrm{T}} s' + \frac{\alpha}{2} s'^{\mathrm{T}} \nabla^2 f_d(x) s')
\end{aligned}$$

and

$$\begin{cases} \theta_d(x) \leq \alpha(\nabla f_d(x)^{\mathrm{T}} s' + \frac{\alpha}{2} s'^{\mathrm{T}} \nabla^2 f_d(x) s'), & \nabla^2 f_d(x) \text{ is positive} \\ \theta_d(x) \leq \alpha(\nabla f_d(x)^{\mathrm{T}} s' + \frac{\alpha}{2} s'^{\mathrm{T}} s'), & \text{else} \end{cases} \tag{20}$$

There must exists a small enough $\alpha$ such that the right-hand side is negative, so $(b)$ holds.

$(b) \Rightarrow (a)$ If $\theta_d(x) < 0$, then we have

$$\nabla f_d(x)^{\mathrm{T}} s_d(x) < \nabla f_d(x)^{\mathrm{T}} s_d(x) + H_{d,x}(s_d(x)) < 0$$

where $s_d(x) \neq 0$ otherwise $\theta_d = 0$. And we also have

$$\nabla f_i(x)^{\mathrm{T}} s_d(x) \leq 0 \quad \forall i = 1, \cdots, m, \ i \neq d.$$

So, $\mathcal{J}(F(x^*)) \cap (-\mathbb{R}^m_+ \backslash \{0\}) \neq \emptyset$, $(a)$ holds.

$(b) \Rightarrow (c)$ If $\theta_d(x) < 0$, suppose that $s_d(x) = 0$, since $s_d(x)$ is the optimal solution of (8), then we have $\theta_d(x) = 0$, which is a contradiction. So $(c)$ holds for $d = 1, \cdots, m$.

$(c) \Rightarrow (a)$. Because $H_{d,x}(s)$ is positively defined, then for any given $d \in \{1, \cdots, m\}$ with $i = 1, \cdots, m$, $i \neq d$, we have:

$$\nabla f_d(x)^{\mathrm{T}} s_d(x) < \nabla f_d(x)^{\mathrm{T}} s_d(x) + H_{d,x}(s_d(x)) = \theta_d(x) \leq 0$$

and all the constraints ($i = 1, \cdots, m$) are satisfied:

$$\nabla f_i(x)^{\mathrm{T}} s_d(x) < \nabla f_i(x)^{\mathrm{T}} s_d(x) + \frac{1}{2} s_d(x)^{\mathrm{T}} \nabla^2 f_i(x)) s_d(x) \leq 0$$

Therefore, $x$ is not critical, nor strictly critical, $(a)$ holds. Item 3 holds. □

It should be emphasized that $s_d(x) \neq 0 \Rightarrow \theta_d(x) < 0$ for one $d$ may not hold in nonconvex cases, as is shown by Example 2 in Carrizo et al. (2016).

**Remark 2.** *Fliege et al. (2009) utilizes the Armijo conditions to achieve a shared descent among all objective functions. However, this goal becomes challenging to attain in our method, particularly when errors are present in the Hessian matrices and gradients, often due to the backtracking method. Therefore, this paper relaxes the requirement of a common descent and instead focuses on achieving at least one descent.*

Considering that linear constraints will lead to a non-full Newton step, we give the Armijo-type condition for the optimal step length.

**Corollary 1.** *If $\bar{x} \in U$ is noncritical point for $F$, then for any $0 < \sigma < 1$ there exist $r > 0$ such that $B[\bar{x}, r] \in U$ and for all $x \in B[\bar{x}, r]$, $t \in (0, 1]$, and for $d$,*

$$x + t s_d(x) \in U \ \textit{and} \ f_d(x + t s_d(x)) \leq f_d(x) + \sigma t \theta_d(x)$$

**Proof:** As $\bar{x}$ is noncritical in both cases, by the item 3 of Lemma 2, $\theta_d(\bar{x}) < 0$. Using the continuity of $\theta_d$, there exists $r > 0$ such that $B[\bar{x}, r] \in U$ and $\theta_d(x) \le \theta_d(\bar{x})/2$, $\forall x \in B[\bar{x}, r]$. In particular, any $x \in B[\bar{x}, r]$ is noncritical. Using the item 2 of Lemma 2, $s$ is bounded in $B[\bar{x}, r]$. So, there exists $0 < t_1 \le 1$ such that $x + ts_d(x) \in U$ for $x \in B[\bar{x}, r]$ and $t \in [0, t_1]$. Therefore, for $x \in B[\bar{x}, r]$, $t \in [0, t_1]$, and for all $1 \le d \le m$,

$$f_d(x + ts_d(x)) = f_d(x) + t\nabla f_d(x)^{\mathrm{T}} s_d(x) + o_d(ts_d(x), x)$$

where

$$\lim_{t \to 0+} \frac{o_d(ts_d(x), x)}{t\|s_d(x)\|} = 0,$$

and this limit is uniform for $x \in B[\bar{x}, r]$. Using once again the fact that $s$ is bounded on $B[\bar{x}, r]$, we have that $\lim_{t\to 0+} o_d(ts_d(x), x)/t = 0$ uniformly for $x \in B[\bar{x}, r]$. Since

$$\nabla f_d(x)^{\mathrm{T}} s_d(x) \le \nabla f_d(x)^{\mathrm{T}} s_d(x) + \frac{1}{2} s_d(x)^{\mathrm{T}} \nabla^2 f_d(x) s_d(x) = \theta_d(x),$$

the following holds for $t \in [0, t_1]$

$$\begin{aligned} f_d(x + ts_d(x)) &\le f_d(x) + t\theta_d(x) + o_d(ts_d(x), x) \\ &= f_d(x) + t\sigma_1\theta_d(x) + t[(1-\sigma_1)\theta_d(x) + \frac{o_d(ts_d(x), x)}{t}]. \end{aligned}$$

Since $\theta_d(x) \le \theta_d(\bar{x})/2 < 0$, for $t \in (0, t_1]$ small enough, the last term at the right-hand side of the above equations is not positive. The corollary is proved. □

We use the above Armijo-type rule which is applicable in both *Case I* and *Case II* to obtain the step length. Then a priori algorithm that alternates the descent direction can be given as Algorithm 1. If $x_k$ is not stationary for $F$, we have $\theta_{d_k}(x_k) < 0$, so from *(d)* of the Main loop we have $f_{d_k}(x_{k+1}) < f_{d_k}(x_k)$.

It should be noted that Algorithm 1 may fall into one particular individual minimum repeatedly as the constraints of (8) may be totally inactive, so we adopt some techniques to handle this issue. For the first iteration, if $all(\lambda = 0)$ is `true`, we change the descent function until there is at least one active constraint or there is no more choice.

## 4. Convergence Analysis

This section analyzes the convergence of Algorithm 1, under Assumptions 1 and 2. First of all, We prove the general convergence of Algorithm 1. For stopping criteria in Algorithm 1, we give a necessary condition that makes $\theta_d(x) = 0$ hold and the point strictly critical.

**Lemma 3.** *For a point $\bar{x}$, solving problem (8) with $f_{d^1}$ being the descent function, if $\theta_{d^1}(\bar{x}) = 0$ with another $d^2$th constraint being inactive, we have $\theta_{d^2}(\bar{x}) < 0$ for problem (8) with $f_{d^2}$ being the descent function.*

**Algorithm 1** Descent-direction-constrained Method

**Initialization**

Choose $x_0 \in \mathcal{X}$, $0 < \sigma < 1$, set $k := 0$ and define $J = \{1/2^n \mid n = 0, 1, \cdots\}$, index set $\mathcal{I} = \{1, \cdots, m\}$.

**Main loop**

(a) Choose one index $d \in \mathcal{I}$.

(b) Take $f_d(x)$ as the descent function and solve problem (8) to obtain $\theta_d(x_k)$ and $s_d(x_k)$ as in (13) and (14), and the Lagrange multiplier vector $\lambda$ of the constraints.

(c) If $\theta_{d_k}(x_k) = 0$, set $\mathcal{I} = \mathcal{I} \backslash \{d_k\}$, turn to *(f)*. Otherwise, proceed to *(d)*.

(d) Choose $t_k$ as the largest $t \in J$ such that

$$x_k + t s_{d_k}(x_k) \in \mathcal{X},$$
$$f_{d_k}(x_k + t s(x_k)) \leq f_{d_k}(x_k) + \sigma t_k \theta_{d_k}(x_k),$$

(e) Define $x_{k+1} = x_k + t_k s_{d_k}(x_k)$ and set $k := k + 1$. Turn to *(b)*.

(f) If $\mathcal{I} = \emptyset$, stop. Otherwise, proceed to *(a)*.

**Proof:** For the point $\bar{x}$, since $\theta_{d^1}(\bar{x}) = 0$ and the $d^2$th constraint is inactive, so

$$\nabla f_{d^2}(\bar{x})^{\mathrm{T}} s_{d^1}(\bar{x}) \leq \nabla f_{d^2}(\bar{x})^{\mathrm{T}} s_{d^1}(\bar{x}) + \max\{\frac{1}{2} s_{d^1}(\bar{x})^{\mathrm{T}} \nabla f_{d^2}(\bar{x}) s_{d^1}(\bar{x}), 0\} < 0,$$
$$\nabla f_i(\bar{x})^{\mathrm{T}} s_{d^1}(\bar{x}) \leq 0, \ \forall i = 1, \cdots, m, \ i \neq d^1,$$

we know that $\bar{x}$ is not strictly critical. Then, take $f_{d^2}$ as the descent function to solve problem (8), $s_{d^1}(\bar{x})$ can be a feasible solution such that

$$\theta_{d^2}(\bar{x}) \leq \nabla f_{d^2}(\bar{x})^{\mathrm{T}} s_{d^2}(\bar{x}) + \max\{\frac{1}{2} s_{d^1}(\bar{x})^{\mathrm{T}} \nabla f_{d^2}(\bar{x}) s_{d^1}(\bar{x}), 0\} < 0,$$

so the lemma holds. □

**Corollary 2.** *Every accumulation point of the sequence $\{x_k\}$ produced by Algorithm 1 is a Pareto optimal point. If the function $F$ has bounded level sets in the sense that the set $\mathcal{A} = \{x \in \mathbb{R}^n \mid F(x) \preceq F(x_1)\}$ is bounded, then the sequence $\{x_k\}_k$ stays bounded and has at least one accumulation point, and it is strictly critical.*

**Proof:** Since the level sets of $f_i$ $(i = 1, \cdots, m)$, are bounded, $\mathcal{A}$ is bounded. Let $x^*$ be an accumulation point of the sequence $\{x_k\}_k$. According to Lemma 2, we have that $\theta_d(x^*) = 0$ for all $d = 1, \cdots, m$, and because $\theta_d(x) < 0$ $(x \neq x^*)$ being lower bounded by single objective subproblems, $f_d$ is monotonically decreasing. Herein, by Lemma 3, when $\theta_d(x) < 0$ for at least one $d$, the algorithm will iterate the descent direction until the equality holds for all objectives. Then, we have

$$\lim_{k \to \infty} f_{d_k}(x_k) = f_d(x^*).$$

So,

$$\lim_{k\to\infty} |f_{d_{k+1}}(x_{k+1}) - f_{d_k}(x_k)| = 0.$$

Besides, from the Armijo-type condition, we have

$$f_{d_k}(x_k) - f_{d_{k+1}}(x_{k+1}) \geq -\sigma t_k \theta_{d_k}(x_k) \geq 0,$$

then

$$\lim_{k\to\infty} t_k \theta_{d_k}(x_k) \ = 0.$$

As $t_k > 0$ and then we have

$$\lim_{k\to\infty} \theta_{d_k}(x_k) = 0.$$

Because $\theta_d$ is continuous in both cases, for whatever case, we can conclude that $\theta_d(x^*) = 0$, then the point $x^*$ is a strictly critical point. Therefore, $\{x_k\}_k$ stops at a strictly critical point or converges to the individual minimum of $f_d$ which is also a locally Pareto optimal point. □

### 4.1. Convex case

Now we analyze the convergence of convex case, we are going to give sufficient conditions for local superlinear convergence based on some similar auxiliary techniques in Fliege et al. (2009). Some necessary technical results are presented at first, then we prove that the algorithm superlinearly converges to the Paret optimal solution, and further has a $q$-convergence under some continuous assumptions.

Before analyzing the convergence of Algorithm 1, we need some auxiliary technical results

**Lemma 4.** *Suppose that $V \subset \mathcal{X}$ is a convex set. Let $\epsilon, \delta > 0$ and $x, y \in V$ with $\|y - x\| < \delta$, it holds*

$$\|\nabla^2 f_i(y) - \nabla^2 f_i(x)\| < \epsilon \quad i = 1 \cdots, m,$$

*Then for any $x, y \in V$ such that $\|y - x\| < \delta$ we have that*

$$\|\nabla f_i(y) - [\nabla f_i(x) + \nabla^2 f_i(x)(y - x)]\| < \epsilon \|y - x\|$$

*and*

$$|f_i(y) - [f_i(x) + \nabla f_i(x)^{\mathrm{T}}(y - x) + \frac{1}{2}(y - x)^{\mathrm{T}} \nabla^2 f_i(x)(y - x)]| < \frac{\epsilon}{2}\|y - x\|^2$$

*hold for $i = 1, \cdots, l$.*

*If $\nabla^2 f_i(x)$ is Lipschitz continuous with constant $L$ for $i = 1, \cdots, m$, then*

$$\|\nabla f_i(y) - [\nabla f_i(x) + \nabla^2 f_i(x)(y - x)]\| < \frac{L}{2}\|y - x\|^2$$

*holds for all $i = 1, \cdots, m$.* □

**Proof:** It is easy applying Lemmas 4.1.9 and 4.1.12 from Dennis and Schnabel (1983).

Then, we construct the upper and lower bounds for $\theta_d(x)$.

**Lemma 5.** *Take $x \in \mathcal{X}$, take $a > 0$, if $\nabla^2 f_i(x) \geq aI$, $i = 1, \cdots, m$, then it holds*

$$\frac{|\theta_d(x)|}{1+\sum_{\substack{i=1\\i\neq d}}^m \lambda_i^1(x)} \geq \frac{a}{2}\|s_d(x)\|^2 \tag{21}$$

**Proof:** For the left inequality, recall that $\lambda_i(x)$ is the Lagrange multiplier for the $i$th constraint converted from the objective function, then define

$$D(x) := \frac{\nabla^2 f_d(x) + \sum_{\substack{i=1\\i\neq d}}^m \lambda_i^1 \nabla^2 f_i(x)}{1+\sum_{\substack{i=1\\i\neq d}}^m \lambda_i^1(x)},$$
$$v(x) := \frac{\nabla f_d(x) + \sum_{\substack{i=1\\i\neq d}}^m \lambda_i^1(x) \nabla f_i(x)}{1+\sum_{\substack{i=1\\i\neq d}}^m \lambda_i^1(x)}.$$

From the definitions of $H_{d,x}(s)$, we naturally have $D(x) \geq aI$, and $s_d(x) = -D(x)^{-1}v(x)$, and

$$\begin{aligned}\frac{\theta_d(x)}{1+\sum_{\substack{i=1\\i\neq d}}^m \lambda_i^1(x)} &= v^{\mathrm{T}} s_d(x) + \frac{1}{2} s_d(x)^{\mathrm{T}} D(x) s_d(x)\\ &= (-D(x)s_d(x))^{\mathrm{T}} s_d(x) + \frac{1}{2} s_d(x)^{\mathrm{T}} D(x) s_d(x)\\ &= -\frac{1}{2} s_d(x)^{\mathrm{T}} D(x) s_d(x)\end{aligned}$$

By Lemma 2, $\theta_d(x) \leq 0$, the inequality (21) holds. □

**Lemma 6.** *Take $x \in \mathcal{X}$, take $a > 0$, if $\nabla^2 f_i(x) \geq aI$ ($i = 1, \cdots, m$), then for all $\lambda_i^1 \geq 0$ ($i \neq d$), it holds*

$$\frac{|\theta_d(x)|}{1+\sum_{\substack{i=1\\i\neq d}}^m \lambda_i^1(x)} \leq \frac{\|\nabla f_d(x) + \sum_{\substack{i=1\\i\neq d}}^m \lambda_i^1 \nabla f_i(x)\|^2}{2a(1+\sum_{\substack{i=1\\i\neq d}}^m \lambda_i^1)^2} \tag{22}$$

**Proof:** For the right inequality, define

$$w := \frac{\nabla f_d(x) + \sum_{\substack{i=1\\i\neq d}}^m \lambda_i^1 \nabla f_i(x)}{1+\sum_{\substack{i=1\\i\neq d}}^m \lambda_i^1}$$
$$D(x) := \frac{\nabla^2 f_d(x) + \sum_{\substack{i=1\\i\neq d}}^m \lambda_i^1 \nabla^2 f_i(x)}{1+\sum_{\substack{i=1\\i\neq d}}^m \lambda_i^1},$$

we have $D(x) \geq aI$ for all $\lambda_i^1$, and

$$\begin{aligned}\frac{\theta_d(x)}{1+\sum_{\substack{i=1\\i\neq d}}^m \lambda_i^1(x)} &\geq \inf_{s\in\mathbb{R}^n} \frac{L(s,\lambda)}{1+\sum_{\substack{i=1\\i\neq d}}^m \lambda_i^1}\\ &= \inf_{s\in\mathbb{R}^n} \left(w^{\mathrm{T}} s + \frac{1}{2} s^{\mathrm{T}} D(x) s\right)\\ &\geq \inf_{s\in\mathbb{R}^n} \left(w^{\mathrm{T}} s + \frac{a}{2}\|s\|^2\right)\\ &= -\frac{\|w\|^2}{2a},\end{aligned} \tag{23}$$

where the minimum is achieved at $s$ such that $w+as=0$. As $\theta_d(x)\le 0$, then we can obtain exactly the inequality (22). □

Then we can estimate the the norm of the descent direction.

**Lemma 7.** *Take $\hat{x}\in\mathcal{X}$ and define $\hat{x}_+=\hat{x}+s(\hat{x})$ . If $a,r,\delta,\psi,\epsilon>0$ are such that:*

1. $aI\le\nabla^2 f_i(x)\le bI$ $(a<b,\ i=1,\cdots,m)$, *for* $x\in B[\hat{x},r]$,
2. $\|\nabla^2 f_i(x)-\nabla^2 f_i(y)\|\le\epsilon$ *for all* $x,y\in B[\hat{x},r]$ *with* $\|x-y\|<\delta$,
3. $B[\hat{x},r]\subset\mathcal{X}$,
4. $\|s_d(\hat{x})\|\le\min\{\delta,r\}$ *for any* $d\in\mathcal{I}$

*we have*

$$\|s_{d_+}(\hat{x}_+)\|\le\frac{\epsilon}{a}\|s_d(\hat{x})\|, \tag{24}$$

*and if it holds additionally that $t=1$ and*

$$\|\nabla^2 f_j(x)-\nabla^2 f_j(y)\|\le L\|x-y\|,\ \forall i=1,\cdots,m \tag{25}$$

*we further have*

$$\|s_{d_+}(\hat{x}_+)\|\le\frac{L}{2a}\|s_d(\hat{x})\|^2. \tag{26}$$

**Proof:** Let $\hat{\lambda}_i=\lambda_i(\hat{x})$ be the KKT multipliers of problem (8) at $x=\hat{x}$, as in (15)-(17). Define

$$\hat{v}_+=\frac{\nabla f_d(\hat{x}_+)+\sum_{\substack{i=1\\i\neq d_+}}^{m}\hat{\lambda}_i^1\nabla f_i(\hat{x}_+)}{(1+\sum_{\substack{i=1\\i\neq d_+}}^{l}\hat{\lambda}_i^1)}. \tag{27}$$

Note that $d_+$ may be different from $d$. By the KKT conditions and Lemma 6, for $x=\hat{x}$, we have:

$$\frac{\theta_{d+}(\hat{x}_+)}{1+\sum_{\substack{i=1\\i\neq d}}^{m}\lambda_i^1(\hat{x}_+)}\le\frac{\|\hat{v}_+\|^2}{2a}. \tag{28}$$

To estimate $\|\hat{v}_+\|$ of *Case II*, define for $x\in\mathcal{X}$

$$G(x)=\frac{f_d(x)+\sum_{\substack{i=1\\i\neq d}}^{m}\hat{\lambda}_i^1 f_i(x)}{1+\sum_{\substack{i=1\\i\neq d}}^{m}\hat{\lambda}_i^1}, \tag{29}$$

Then we have

$$\nabla G(x)=\frac{\nabla f_d(x)+\sum_{\substack{i=1\\i\neq d}}^{m}\hat{\lambda}_i^1\nabla f_i(x)}{1+\sum_{\substack{i=1\\i\neq d}}^{m}\hat{\lambda}_i^1},\quad \nabla^2 G(x)=\frac{\nabla^2 f_d(x)+\sum_{\substack{i=1\\i\neq d}}^{m}\hat{\lambda}_i^1\nabla^2 f_i(x)}{1+\sum_{\substack{i=1\\i\neq d}}^{m}\hat{\lambda}_i^1} \tag{30}$$

We can find that

$$\hat{v}_+=\nabla G(\hat{x}_+)=\nabla G(\hat{x}+ts(\hat{x})), \tag{31}$$

and

$$s_d(\hat{x}) = -\nabla^2 G(\hat{x})^{-1} \nabla G(\hat{x}). \tag{32}$$

Additionally, since $\nabla^2 G(x)$ is the convex combination of $\nabla^2 f_d$ and $\nabla^2 f_i$, for whatever descent function $f_d(x)$, we have $\nabla^2 G(x) \geq aI$

$$\|\nabla^2 G(y) - \nabla^2 G(x)\| \leq \epsilon, \quad \forall x, y \in B[\hat{x}, r] \text{ with } \|y - x\| \leq \delta$$

We apply Lemma 4, yielding

$$\|\nabla G(\hat{x} + t s_d(\hat{x})) - \nabla G(\hat{x}) - \nabla^2 G(\hat{x}) t s_d(\hat{x})\| \leq \epsilon \|s_d(\hat{x})\|$$

where (32) is the same as $\nabla G(\hat{x})$, and we get

$$\|\hat{v}_+\| \leq \epsilon \|s_d(\hat{x})\| \tag{33}$$

which, combined with (28) leads to:

$$\frac{\theta_{d_+}(\hat{x}_+)}{1 + \sum_{\substack{i=1 \\ i \neq d}}^{m} \lambda_i^1(\hat{x}_+)} \leq \frac{\epsilon^2}{2a} \|s_d(\hat{x})\|^2$$

Then, using the left inequality of Lemma 5, we can obtain:

$$\|s_{d_+}(\hat{x}_+)\| \leq \frac{\epsilon}{a} \|s_d(\hat{x})\|.$$

which is equal to (24).

And if $\nabla^2 f_i(x)$ are Lipschitz continuous with constant $L$ in $B[\hat{x}, r]$, $\nabla^2 G_i(x)$ is also Lipschitz continuous. So, by Lemma 4 and (28), we get

$$\|\hat{v}_+\| \leq \frac{L}{2} \|s_d(\hat{x})\|^2, \tag{34}$$

and combine with Lemma 5 and use the above inequality and item 1 of Lemma 2, we get:

$$\frac{a}{2} \|s_{d_+}(\hat{x}_+)\|^2 \leq \frac{1}{2a} \left(\frac{L}{2} \|s_d(\hat{x})\|^2\right)^2,$$

so

$$\|s_{d_+}(\hat{x}_+)\| \leq \frac{L}{2a} \|s_d(\hat{x})\|^2, \tag{35}$$

So we obtain (26). $\square$

### 4.2. Superlinear convergence

From above statements, we are going to give sufficient conditions for local superlinear convergence in this subsection and quadratic convergence in the next subsection.

**Theorem 2.** *Denote by $\{x_k\}_k$ a sequence generated by the Algorithm 1. Suppose that $V \in \mathcal{X}$, $d \in \mathcal{I}$, and $a, b, r, \delta, \epsilon > 0$*

*(a). $aI \leq \nabla^2 f_i(x) \leq bI$ $(a \leq b)$ in the sequence generated by Algorithm 1.*

*(b). $\|\nabla^2 f_i(x) - \nabla^2 f_i(y)\| \leq \epsilon$ for all $x, y \in V$ with $\|x - y\| \leq \delta$,*

*(c). $\epsilon/a < 1 - \sigma$*

*(d). $B[x_0, r] \in V$,*

*(e). $\|s_d(x_0)\| \leq \min\{\delta, r(1 - \epsilon/a)\}$.*

*Then for all $k$ and any fixed $d \in \mathcal{I}$, we have:*

1. $\|x_k - x_0\| \leq \|s_{d_k}(x_0)\| \frac{1-(\epsilon/a)^k}{1-\epsilon/a}$
2. $\|s_d(x_k)\| \leq \|s_d(x_0)\| (\epsilon/a)^k$
3. $t_k = 1$
4. $\|s_{d_{k+l}}(x_{k+1})\| \leq \|s_{d_k}(x_k)\| (\epsilon/a)$

*and, the sequence $\{x_k\}_k$ converges to some locally Pareto optimal point $x^* \in \mathbb{R}^n$ with*

$$\|x^* - x_0\| \leq \frac{\|s_d(x_0)\|}{1 - \epsilon/a} \leq r \tag{36}$$

*The convergence rate of $\{x_k\}_k$ is superlinear.*

**Proof:** First, we show that if items 1 and 2 hold for some $k$, then items 3 and 4 also hold for that $k$.

From the triangle inequality, item 1, and item 2, we have

$$\begin{aligned} \|x_{k+1} - x_0\| &= \|x_k + s_{d_k}(x_k) - x_0\| \\ &\leq \|s_{d_0}(x_0)\| \frac{1 - (\epsilon/a)^{k+1}}{1 - (\epsilon/a)} \end{aligned}$$

Hence, from assumption *(e)*, item 2, and assumption *(c)*, we get

$$\|x_{k+1} - x_0\| \leq r \ \& \ \|x_k - x_0\| \leq r \Rightarrow x_k, \ x_k + s_{d_k}(x_k) \in B[x_0, r] \tag{37}$$

and on the other hand, we get $\|x_{k+1} - x_k\| \leq \delta$. Besides, by (15)-(17), we have

$$\theta_d(x) = \nabla f_d(x)^{\mathrm{T}} s_d(x) + \frac{1}{2} s_d(x)^{\mathrm{T}} \nabla^2 f_d(x) s_d(x)$$

and then

$$\begin{aligned} &f_{d_k}(x_k + s_{d_k}(x_k)) \\ \leq &f_{d_k}(x_k) + \nabla f_{d_k}(x_k)^{\mathrm{T}} s_{d_k}(x_k) + \frac{1}{2} s_{d_k}(x_k)^{\mathrm{T}} \nabla^2 f_d(x) s_{d_k}(x) + \frac{\epsilon}{2} \|s_{d_k}(x_k)\|^2 \end{aligned}$$

$$
\begin{aligned}
&= f_{d_k}(x_k) + \theta_{d_k}(x_k) + \frac{\epsilon}{2}\|s_{d_k}(x_k)\|^2 \\
&= f_{d_k}(x_k) + \sigma\theta_{d_k}(x_k) + (1-\sigma)\theta_{d_k}(x_k) + \frac{\epsilon}{2}\|s_{d_k}(x_k)\|^2 \\
&= f_{d_k}(x_k) + \sigma\theta_{d_k}(x_k) + \frac{1}{2}\big(\epsilon - a(1-\sigma)(1+\sum_{\substack{i=1\\ i\neq d_k}}^{m}\lambda_i(x_k))\big)\|s_{d_k}(x_k)\|^2
\end{aligned}
$$

where by assumption *(c)*

$$
\epsilon - a(1-\sigma)(1+\sum_{\substack{i=1\\ i\neq d_k}}^{m}\lambda_i(x_k)) < \epsilon - a(1-\sigma) < 0.
$$

So

$$
f_{d_k}(x_k + s_{d_k}(x_k)) \le f_{d_k}(x_k) + \sigma\theta_{d_k}(x_k) \tag{38}
$$

Then we prove that items 1 and 2 hold for all $k$. For $k=0$, using assumption *(c)*, we see that they hold trivially. If items 1 and 2 hold for some $k$, then, as we already saw, items 3 and 4 also hold for such $k$, and this implies that $x_{k+1} = x_k + s_{d_k}(x_k)$, so item 1 is true. Item 2 follows from item 4.

Because items 1 and 2 hold for all $k$, items 3 and 4 also hold for all $k$. Therefore,

$$
\sum_{k=0}^{\infty}\|x_{k+1}-x_k\| = \sum_{k=0}^{\infty}\|s_{d_k}(x_k)\| \le \|s_{d_0}(x_0)\|\sum_{k=0}^{\infty}(\epsilon/a)^k < \infty, \tag{39}
$$

where the last inequality follows from assumption *(c)*. Thus, $\{x_k\}$ is a Cauchy sequence, and there exists $x^* \in \mathcal{X}$, such that

$$
\lim_{k\to\infty} x_k = x^*. \tag{40}
$$

Moreover, $\{\|s_{d_k}(x_k)\|\}$ converges to 0, so from assumption *(a)* and Lemma 2, $\theta_{d_k}(x_k) \to 0$ for $k \to \infty$. Therefore, combining (40) with the continuity of $\theta_d$ (item 3 of Lemma 2), we see that $\theta_d(x^*) = 0$. So, from item 2 of Lemma 2, $x^*$ is stationary for $F$ and we can conclude that $x^*$ is locally Pareto optimal. The first inequality in (36) is obtained from item 1 by taking the limit $k \to \infty$. As for the second one, it suffices to use assumption *(e)*.

To prove superlinear convergence, define

$$
\tau_k = \|s_{d_0}(x_0)\|\frac{(\epsilon/a)^k}{1-\epsilon/a}, \quad \delta_k = \|s_{d_0}(x_0)\|(\epsilon/a)^k, \quad k = 0,1,\cdots.
$$

Using the triagle inequality, item 1, assumptions *(e)* and *(d)*, we conclude that

$$
B[x_k, r_k] \subset B[x_0, r] \subset V. \tag{41}
$$

Take any $\tau > 0$ and define

$$
\hat{\epsilon} := \min\{a\frac{\tau}{1+2\tau}, \epsilon\}. \tag{42}
$$

For $k$ large enough

$$\|\nabla^2 f_i(x) - \nabla^2 f_i(y)\| \le \hat{\epsilon} \quad \forall x, y \in B[x_k, \tau_k], \|x-y\| \le \delta_k. \tag{43}$$

Hence, assumptions *(a)-(e)* are satisfied for $\hat{\epsilon}, \hat{r} = r_k, \hat{\delta} = \delta_k$, and $\hat{x}_0 = x_k$. Indeed, *(a)* and *(c)* follow from the fact that *(a)* and *(c)* hold for $\epsilon, a$ and $\sigma$; *(b)* and *(d)* are just (41) and (43) and *(e)* follows from the definitions of $\hat{\delta}, \hat{r}$, and $\hat{\epsilon}$.

Using (36) with $\hat{x}_0$ and $\hat{\epsilon}_0$ respectively, we get

$$\|x^* - x_k\| \le \|s_{d_k}(x_k)\| \frac{1}{1-\hat{\epsilon}/a}. \tag{44}$$

As this inequality also holds for $k+1$,

$$\|x^* - x_{k+1}\| \le \|s_{d_k}(x_{k+1})\| \frac{1}{1-\hat{\epsilon}/a} \le \|s_{d_{k+1}}(x_{k+1})\| \frac{\hat{\epsilon}/a}{1-\hat{\epsilon}/a}, \tag{45}$$

where the last inequality follows from item *4*, with $\hat{x}_0$ and $\hat{\epsilon}$.

Using the triangle inequality, the definition of $x_{k+1}$ and (45), we get

$$\begin{aligned} \|x^* - x_k\| &\ge \|x_{k+1} - x_k\| - \|x^* - x_{k+1}\| \\ &\ge \|s_{d_k}(x_k)\| - \|s_{d_k}(x_k)\| \frac{\hat{\epsilon}/a}{1-\hat{\epsilon}/a} \\ &= \|s_{d_k}(x_k)\| \frac{1-2\hat{\epsilon}/a}{1-\hat{\epsilon}/a}. \end{aligned} \tag{46}$$

In view of the definition of $\hat{\epsilon}$, we have $1 - 2\hat{\epsilon}/a > 0$, and from (45) and (46) we arrive at

$$\|x^* - x_{k+1}\| \le \|x^* - x_k\| \frac{\hat{\epsilon}/a}{1-2\hat{\epsilon}/a}, \tag{47}$$

which combined with the definition of $\hat{\epsilon}$, yield

$$\|x^* - x_{k+1}\| \le \tau \|x^* - x_k\|. \tag{48}$$

As $\tau > 0$ is arbitrary, we conclude that $\{x_k\}$ converges $q$-superlinear to $x^*$. □

Besides the considerable descent of $f_d$ in Theorem 2, we further have a acceptable descent of all the objectives under these assumptions. We define

$$G(x) = \frac{f_d(x) + \sum_{\substack{i=1 \\ i \neq d}}^m \lambda_i^1(x) f_i(x)}{1 + \sum_{\substack{i=1 \\ i \neq d}}^m \lambda_i^1(x)},$$

since $G(x)$ is a convex combination of $f_d(x)$ and $f_i(x)$, so it inherits the uniform continuity and

$$\nabla^2 G(x) = \frac{\nabla^2 f_d(x) + \sum_{\substack{i=1 \\ i \neq d}}^m \lambda_i^1(x) \nabla^2 f_i(x)}{1 + \sum_{\substack{i=1 \\ i \neq d}}^m \lambda_i^1(x)},$$

Then, using assumption *(b)* and Lemma 4, we take and get

$$\begin{aligned}
&G(x_k)+\nabla G(x_k)^{\mathrm{T}}s_{d_k}(x_k)+\frac{1}{2}s_{d_k}(x_k)^{\mathrm{T}}\nabla^2 G(x_k)s_{d_k}(x_k)+\frac{\epsilon}{2}\|s_{d_k}(x_k)\|^2\\
=&G(x_k)+\frac{\theta_{d_k}(x_k)}{1+\sum_{\substack{i=1\\ i\neq d_k}}^{m}\lambda_i^1(x_k)}+\frac{\epsilon}{2}\|s_{d_k}(x_k)\|^2\\
=&G(x_k)+\sigma\frac{\theta_{d_k}(x_k)}{1+\sum_{\substack{i=1\\ i\neq d_k}}^{m}\lambda_i^1(x_k)}+(1-\sigma)\frac{\theta_{d_k}(x_k)}{1+\sum_{\substack{i=1\\ i\neq d_k}}^{m}\lambda_i^1(x_k)}+\frac{\epsilon}{2}\|s_{d_k}(x_k)\|^2.
\end{aligned}$$

From Lemma 2 and Lemma 5, $\theta_d(x_k)\leq 0$, using assumptions *(a)* and *(c)*, we obtain

$$(1-\sigma)\frac{\theta_{d_k}(x_k)}{1+\sum_{\substack{i=1\\ i\neq d_k}}^{m}\lambda_i^1(x_k)}+\frac{\epsilon}{2}\|s_{d_k}(x_k)\|^2\leq\big(\epsilon-a(1-\sigma)\big)\frac{\|s_{d_k}(x_k)\|^2}{2}\leq 0.$$

Therefore, we have

$$G(x_k+s_{d_k}(x_k))\leq G(x_k)+\sigma\theta_{d_k}(x_k). \tag{49}$$

which denotes the considerable descent of the convex combination of all the objectives. This, to some extent, implies that Algorithm 1 has the same goal to optimize all the objectives as Fliege et al. (2009) in convex cases. The difference is that Algorithm 1 concentrates on one objective, while Fliege's method distributes the descent value to every objective.

Note that, the above results hold under no regularity assumptions of the convexity of the objective functions, so we consider the algorithm converges to some locally Pareto optimal point, with the whole sequence in that vicinity.

**Corollary 3.** *Suppose that $x^*$ is a locally Pareto optimal point, then there exists $\rho>0$ such that for any $x_0\in B[x^*,\rho]$, a set $V\subseteq\mathbb{R}^n$ and values $a,b,r,\delta,\epsilon$ exist and all hypotheses of Theorem 2 are satisfied. Additionally, let $\{x_k\}$ be the sequence generated by Algorithm 1 from $x_0$, $\{x_k\}$ converges to $x^*$ superlinearly.*

**Proof:** Take $R>0$ such that $B[x^*,R]\in U$ and define $V=B[x^*,R]$. There exists $a>0$ such that $\nabla^2 f_d(x)\geq aI$ for all $x\in V$, which is assumption (a) of Theorem 2. Take $\epsilon>0$ such that assumption (c) is satisfied; i.e., $\epsilon\leq a(1-\sigma)$. There exists $\delta>0$ such that

$$\|\nabla^2 f_i(x)-\nabla^2 f_i(y)\|\leq\epsilon,\ i=1,\cdots,m,\ \forall x,y\in V,\ \|x-y\|<\delta, \tag{50}$$

that is, assumption (b) holds.

Use item 1 of Theorem 1 to conclude that $x^*$ is strictly critical for $F$. According to item 2 of Lemma 2, $\theta_d$ is continuous in *Case II* and $\theta_d(x^*)=0$, there exists a $\rho$ with $0<\rho\leq R/2$ such that, for any $x\in B[x^*,\rho]$,

$$|\theta_d(x)|\leq\frac{a}{2}[\min\{\delta,(R/2)(1-\epsilon/a)\}]^2$$

Now take $r = R/2$. Suppose that $x_0 \in B[x^*, \rho]$. Note that $B[x_0, \rho] \subset B[x^*, \rho + r] \subset V$, which is assumption (d). Since $x = x_0$ satisfies the above inequality, we use Lemma 5 to conclude that assumption (e) is also satisfied.

Since the optimal value of corresponding single objective subproblem is always the lower bound of $\theta_d(x)$, and based on Assumption 1, this value is attainable, then $\{\theta_{d_k}(x_k)\}$ is at least bounded by the attainable optimal values of all the single objective subproblems. And by Lemma 2, item 3, and the Armijo-type condition, we know that $\{F(x_k)\}$ is bounded, the sequence $\{x_k\}$ is bounded. Hence from step *(d)* of Algorithm 1, we conclude that

$$\lim_{k\to\infty} t_k \theta_{d_k}(x_k) = 0$$

And by Theorem 2, the sequence superlinearly converges $\{x_k\}$ $x_k \to x^*$. □

Note that, by Theorem 2, the sequence $\{x_k\}$ generated by Algorithm 1 has accumulation point $x^*$, and it is a strictly critical point. So for large enough iterations, $x_k$ is in the convergence region of $x^*$, then the sequence converges superlinearly to $x^*$.

### 4.3. Quadratic convergence

With a more strict continuous assumption of the Hessian matrices, quadratic convergence is obtainable. Now, we prove the local quadratic converging rate of Algorithm 1.

**Theorem 3.** *Suppose that, in addition to all assumptions of Theorem 2, we have that $\nabla^2 f_i$ is Lipschitz continuous on $V$ with Lipschitz constant $L$. Define*

$$\tau_k = \frac{L}{2a} \|s_{d_k}(x_k)\|, \tag{51}$$

*and take $\zeta \in (0, 1/2)$. Then there exists $k_0$ such that for all $k \geq k_0$ it holds that $\tau_k < \zeta$ and*

$$\|x^* - x_{k+1}\| \leq \frac{L}{a} \cdot \frac{1-\tau_k}{(1-2\tau_k)^2} \|x^* - x_k\|^2 \leq \frac{L}{a} \cdot \frac{1-\zeta}{(1-2\zeta)^2} \|x^* - x_k\|^2, \tag{52}$$

*where $x^*$ is the Pareto optimal limit of $\{x_k\}$. The sequence $\{x_k\}_k$ converges q-quadratically to $x^*$.*

**Proof:** Due to item 2 of Theorem 2, it is clear that $\tau_k < \zeta$, for $k$ large enough, there is $k \geq 0$. Using Lemma 5, we conclude that

$$\|s_{d_{k+1}}(x_{k+1})\| \leq \frac{L}{2a} \|s_{d_k}(x_k)\|^2. \tag{53}$$

Let $i > k \geq k_0$. Then, from the triangle inequality and the fact that we are taking full Newton steps, we get

$$\|x_i - x_{k+1}\| \leq \sum_{j=k+2}^{i} \|x_j - x_{j-1}\| = \sum_{j=k+2}^{i} \|s_{d_{j-1}}(x_{j-1})\|. \tag{54}$$

So, letting $i \to \infty$ and using the convergence result of Theorem 2, we obtain

$$\|x^* - x_{k+1}\| \le \sum_{j=k+1}^{\infty} \|s_j(x_j)\|. \tag{55}$$

Then, combing (53) and (55) and using that $\tau_k = \frac{L}{2a}\|s_{d_k}(x_k)\| < 1/2$, we get

$$\begin{aligned}
\|x^* - x_{k+1}\| &\le \frac{L}{2a}\|s_{d_k}(x_k)\|^2 + (\frac{L}{2a})^3\|s_{d_k}(x_k)\|^4 + (\frac{L}{2a})^7\|s_{d_k}(x_k)\|^8 + \cdots \\
&= \frac{L}{2a}\|s_{d_k}(x_k)\|^2(1 + \tau_k^2 + \tau_k^6 + \cdots) \\
&\le \frac{L}{2a}\|s_{d_k}(x_k)\|^2 \sum_{j=0}^{\infty} \tau_k^j \\
&= \frac{L}{2a}\|s_{d_k}(x_k)\|^2 \frac{1}{1-\tau_k}.
\end{aligned} \tag{56}$$

From the triangle inequality, we get

$$\begin{aligned}
\|x^* - x_k\| &\ge \|x_{k+1} - x_k\| - \|x^* - x_{k+1}\| \\
&\ge \|s_{d_k}(x_k)\| - \frac{L}{2a}\|s_{d_k}(x_k)\| \cdot \frac{\|s_{d_k}(x_k)\|}{1-\tau_k} \\
&= \|s_{d_k}(x_k)\|^2 \frac{1-2\tau_k}{1-\tau_k},
\end{aligned} \tag{57}$$

where the last equality is a consequence of the definition of $\tau_k$. Now, from (56) and (57) we obtain

$$\begin{aligned}
\|x^* - x_{k+1}\| &\le \frac{L}{2a}\left[\|x^* - x_k\| \frac{1-\tau_k}{1-2\tau_k}\right]^2 \frac{1}{1-\tau_k} \\
&= \frac{L}{2a}\|x^* - x_k\|^2 \frac{(1-\tau_k)}{(1-2\tau_k)^2},
\end{aligned} \tag{58}$$

and (52) follows trivially. So, Algorithm 1 has the local $q$-quadratic convergence. □

From Corollaries 2 and 3 and Theorem 3, we can easily conclude that under the Lipschitz continuous assumptions of the Hessian matrices of all objective functions, the sequence generated by the algorithm converges quadratically to the Pareto optimal point from a point in the certain vicinity.

## 5. Numerical Results

This section implies the proposed algorithm to some bench mark problems from the literatures: Huband et al. (2006), Laumanns et al. (2002), Fliege et al. (2009), Hillermeier (2001). Besides, a comparison to the method in Fliege et al. (2009) is given to verify the correctness and advantages of Algorithm 1.

Considering the numerical error, we set the tolerance for stoping criterion and judging criterion in the step (c) of Algorithm 1 as

$$|\theta_d(x)| < tol, \ tol > 0,$$

**Table 1** **Main data of the comparison between Fliege's method Fliege et al. (2009) and Algorithm 1**

| Name | n | Lb | Ub | Source | Fliege's method | | Algorithm 1 | |
|---|---|---|---|---|---|---|---|---|
| | | | | | Iteration | NDP | Iteration | NDP |
| BK1 | 2 | $-[5,5]$ | $[10,10]$ | Huband et al. (2006) | 2.00 | 100% | 2.00 | 100% |
| JOSa | 5 | $-[2,\cdots,2]$ | $[2,\cdots,2]$ | Huband et al. (2006) | 2.00 | 100% | 2.00 | 100% |
| JOSb | 10 | $-[2,\cdots,2]$ | $[2,\cdots,2]$ | Huband et al. (2006) | 2.00 | 100% | 2.00 | 100% |
| JOSc | 50 | $-[2,\cdots,2]$ | $[2,\cdots,2]$ | Huband et al. (2006) | 2.00 | 100% | 2.00 | 100% |
| JOSd | 100 | $-[2,\cdots,2]$ | $[2,\cdots,2]$ | Huband et al. (2006) | 2.00 | 100% | 2.00 | 100% |
| ZDT1a | 10 | $[1,\cdots,1]\times 10^{-2}$ | $[1,\cdots,1]$ | Huband et al. (2006) | 2.38 | 4% | 2.02 | 94% |
| ZDT1b | 10 | $[1,\cdots,1]\times 10^{-4}$ | $[1,\cdots,1]$ | Huband et al. (2006) | 8.41 | 2% | 2.02 | 100% |
| ZDT1c | 10 | $[1,\cdots,1]\times 10^{-6}$ | $[1,\cdots,1]$ | Huband et al. (2006) | 12.65 | 5% | 2.02 | 100% |
| ZDT1d | 30 | $[1,\cdots,1]\times 10^{-6}$ | $[1,\cdots,1]$ | Huband et al. (2006) | 14.12 | 5% | 5.33 | 100% |
| ZDT1e | 50 | $[1,\cdots,1]\times 10^{-6}$ | $[1,\cdots,1]$ | Huband et al. (2006) | 12.55 | 7% | 10.71 | 90% |
| ZDT1f | 100 | $[1,\cdots,1]\times 10^{-6}$ | $[1,\cdots,1]$ | Huband et al. (2006) | 11.56 | 3% | 21.67 | 91% |
| VFM1 | 2 | $-[2,2]$ | $[2,2]$ | Huband et al. (2006) | 1.95 | 100% | 1.95 | 100% |
| LTDZ1 | 3 | $[0,\cdots,0]$ | $[1,\cdots,1]$ | Laumanns et al. (2002) | 2.00 | 100% | 2.00 | 100% |
| FDSa | 5 | $-[2,\cdots,2]$ | $[2,\cdots,2]$ | Fliege et al. (2009) | 5.11 | 100% | 5.97 | 100% |
| FDSb | 10 | $-[2,\cdots,2]$ | $[2,\cdots,2]$ | Fliege et al. (2009) | 5.72 | 100% | 5.76 | 100% |
| FDSc | 20 | $-[2,\cdots,2]$ | $[2,\cdots,2]$ | Fliege et al. (2009) | 6.07 | 100% | 28.85 | 100% |
| Hil | 2 | $[0,0]$ | $[5,5]$ | Hillermeier (2001) | 2.88 | 47% | 15.57 | 91% |
| IMl | 2 | $[1,4]$ | $[1,2]$ | Huband et al. (2006) | 2.11 | 73% | 2.52 | 100% |
| ZDT3a | 10 | $[1,\cdots,1]/100$ | $[1,\cdots,1]$ | Huband et al. (2006) | 4.16 | 33% | 2.72 | 83% |
| ZDT3b | 30 | $[1,\cdots,1]/100$ | $[1,\cdots,1]$ | Huband et al. (2006) | 3.96 | 36% | 5.68 | 83% |
| ZDT3c | 50 | $[1,\cdots,1]/100$ | $[1,\cdots,1]$ | Huband et al. (2006) | 4.51 | 35% | 9.83 | 84% |
| ZDT3d | 100 | $[1,\cdots,1]/100$ | $[1,\cdots,1]$ | Huband et al. (2006) | 4.30 | 35% | 18.72 | 82% |

where the prescribed parameter $tol$ is small enough. Box constraints can be handled by the same means as Fliege et al. (2009), i.e., adding the constraint: $L-x\leq s\leq U-x$ to problem (8) as a pair of box constraints.

All the tests are executed within Matlab (R2020a). The quadratic subproblem has been solved by the NLP solver `fmincon` of Matlab, setting `algorithm` as `sqp`. The corresponding parameters of the NLP solver are also set `Optimality Tolerance` $=1\times 10^{-12}$ and `Step Tolerance` $=1\times 10^{-12}$. In all the tests, we set the terminate tolerance $tol=1\times 10^{-8}$. We set $\sigma=0.1$ for the line search, and $500$ as the max number of iterations.

Note that we do not set a small value of $\zeta$, such as $0.1, 0.01$, because we desire a relatively strong influence of the converted constraints such that the algorithm is more likely to converge to the middle part of the Pareto front instead of the endpoints, avoiding repeating solutions as possible.

The details of the test problems are listed in Table 1. For each problem, we compute a population of individuals parallelly from 100 starting points, through the `parpool` toolbox. Two indexes, average iterations and the percent of non-dominated points in numerical solutions (NDP, *the non-dominated percent*), are evaluated. The first one illustrates the converge efficiency while the second index represents the accuracy of the numerical solutions. The main data of the numerical results are shown in Table 1 as well.

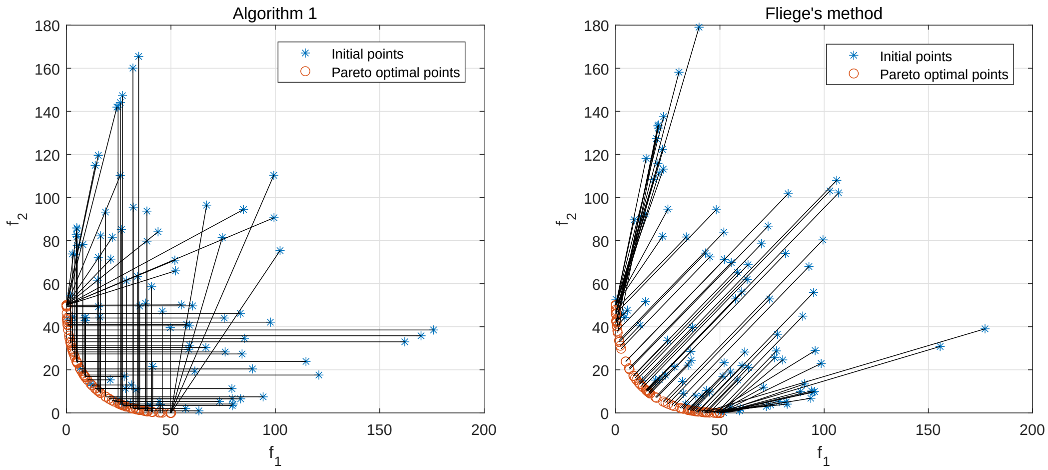


**Figure 1** Comparison of optimization trajectories between Algorithm 1 and Fliege's method of Prob. BK1.

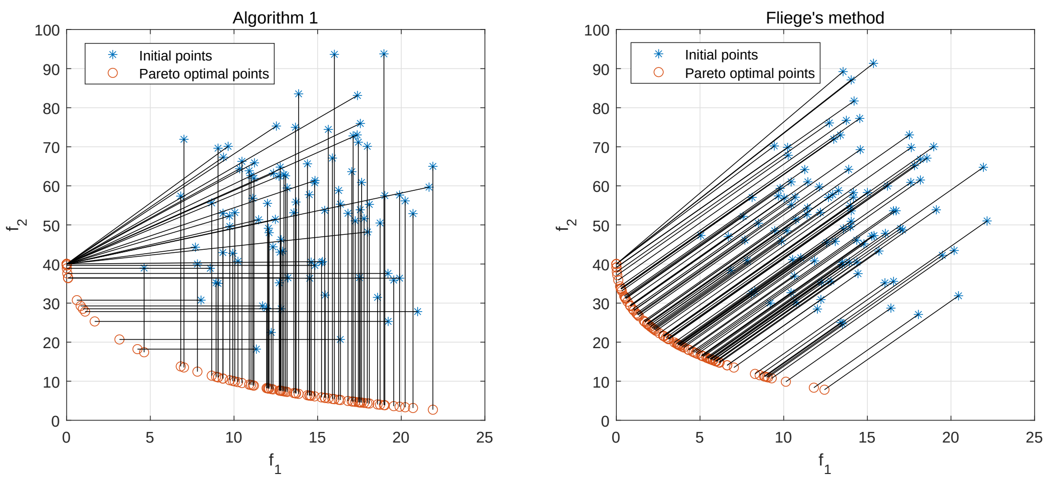


**Figure 2** Comparison of optimization trajectories between Algorithm 1 and Fliege's method of Prob. JOS with 10-dimension variables.

Prob. BK1 is a simple quadratic bi-objective problem with symmetrical definition domain $x_1, x_2 \in [-5, 10]$ corresponding to the objective functions. The trajectories of two algorithms are clearly presented in Fig. 1. In terms of a quadratic problem, many points generated by Algorithm 1 search along the vertical or horizontal directions, ensuring the descent of one objetive and keeping the other non-increasing. As mentioned in the former part, Fliege's method, to some extent, can be regarded as Tchebycheff method with the parameters set by $w = 1^{\mathrm{T}}$. So the trajectory of this method generally has an approximate slope of $45°$. Besides, the rest of trajectories direct to the individual minimums, as well as Algorithm 1, because the converted constraints are inactive, subproblem degenerating to the individual optimization problem.

Another bi-objective quadratic problem is JOS. The central axis of its feasible region is not the axis of symmetry of two objective functions. Besides, the dimension of the variable $n$ can be changed, so we choose $5, 10, 50, 100$ to analyze the performance of two algorithms. As is shown in Table 1, the average iterations

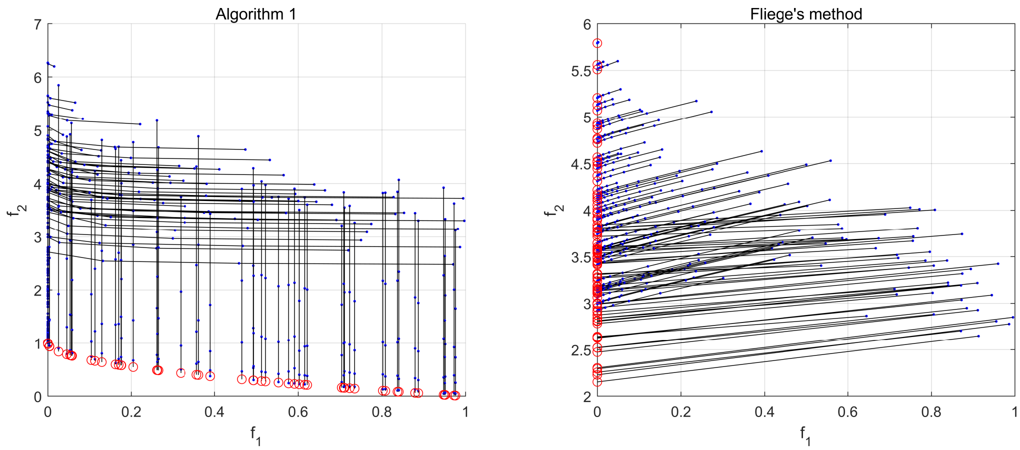


**Figure 3 Comparison of optimization trajectories between Algorithm 1 method and Fliege's method of Prob. ZDT1 with 200-dimension variables.**

of both algorithms grow along with the increase of the variable dimensions, but the growth is not that sharp as the objective functions are strongly convex and both algorithms have the $q$-quadratic convergence. The growth is likely aroused by the numerical error of the larger scale of Hessian matrices in solving the subproblems. It should be emphasized that right bound of the feasible region is exactly the optimal solution of the second objective function, so in Fig. 2, we can see there are not repeating solutions in the vicinity of the individual minimum of the second objective function, which is obviously different from Prob. BK1.

Prob. ZDT1 is a well-known benchmark problem and is mainly for evaluating the performance when there are weakly Pareto solutions. It is a non-quadratic bi-objective problem, where one objective function is linear, and the other is convex but relatively complex with changeable variable dimensions. The numerical difficulty of these two objective functions has a disparity. The numerical complexity of $f_2$ increases sharply with the dimension $n$, i.e., the scale of the Hessian matrices and the gradients will be larger and less accurate. Besides, there are many weakly Pareto optimal points when $f_1(x)$ gets its individual minimum.

Fig. 3 illustrates that Algorithm 1 converges to the Pareto optimal points instead of weakly Pareto optimal points, while all the numerical solutions of Fliege's method are weakly Pareto optimal. According to the stop criterion of Fliege's method, it stops if $\theta_d(x) = 0$, but $|\theta_d(x)| < tol$ is actually used in numerical studies. This example is a special circumstance that $f_1$ reaches its individual minimum, while $f_2$ does not. Then, $x_1 \to 0$ leads to extremely weak relationship between $f_2$ and $x_1$, as well as $f_1$. As a result, for the non-Pareto optimal solution with $x_1 \to 0$ (actually is weakly Pareto optimal solution), there exists a direction $s$ such that $f_1(x_{k+1}) - f_1(x_k) \to 0$, then, $\theta_{d_k}(x_k) \to 0$. So Flieges's method is more likely to be stuck in the weakly Pareto solution, as it is shown in Fig. 3. In contrast, Algorithm 1 will change the descent function at the weakly Pareto points, and it further converges to the Pareto optimal solution. In addition, the lower bound of this problem is designed as 0. However, we use a small positive constant for numerical stability, which also affects the accuracy and efficiency due to utilizing a tolerance of $\theta_d$, as shown in Table 1.

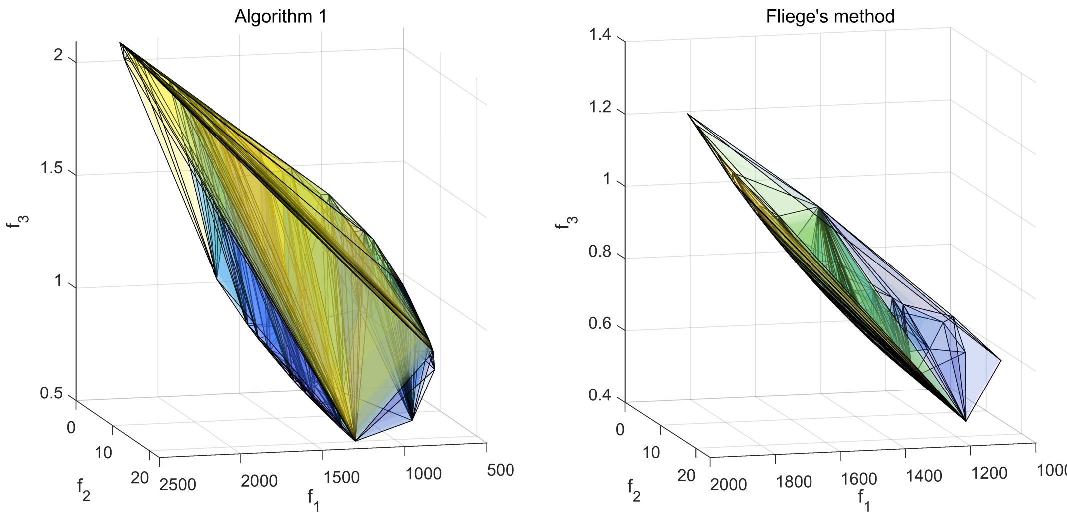


**Figure 4** **Comparison of Delaunay triangulations between Algorithm 1 and Fliege's method of Prob. FDS with 10-dimension variables.**

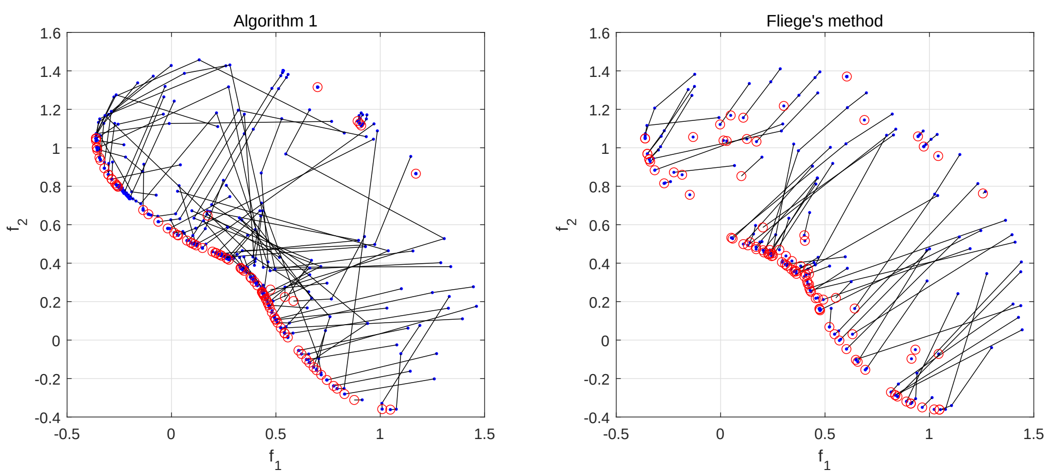


**Figure 5** **The value space of Pro. Hil for 100 random starting points, solving by Algorithm 1 and Fliege's method. Circled points indicate the numerical locally Pareto optimal points.**

Prob FDS is proposed in Fliege et al. (2009). It is a well-designed convex problem with three objectives whose numerical difficulty is sharply increasing along with the variable dimension $n$. From the data in Table 1, we can figure out a more significant increase in Algorithm 1 on the average iterations with the growing variable dimension $n$. This can be analyzed in Fig. 4 which presents the Delaunay triangulations generated by the two algorithms and is a visualization of the image of the numerical optimal points. Both methods have a considerable performance when the dimension is not too high. However, the Pareto front generated by Algorithm 1 covers a larger space, which indicates better performance, and this is because Fliege's method converges to a portion of Pareto points (vicinity to the individual minimums of some objectives).

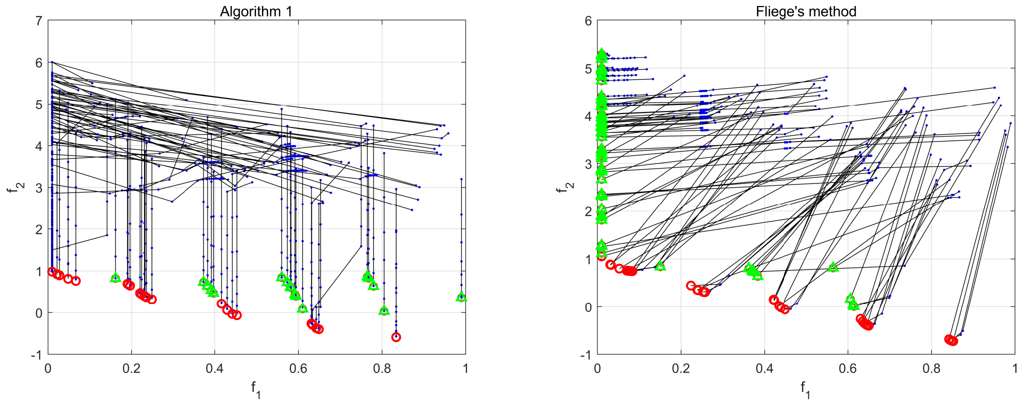


**Figure 6 Comparison of Pareto fronts between Algorithm 1 and Fliege's method of Prob. ZDT3 (n=50). Circled points indicate the numerical locally Pareto optimal points, and green triangled points indicate the dominated points, including locally but not globally Pareto optimal points and weakly Pareto points (in Fliege's method).**

Next, we consider nonconvex cases, to test global convergence of Algorithm 1. Prob. Hil (see Page 109, Hillermeier 2001) is a typical nonconvex, albeit smooth problem with many local minimums. Fliege's method gets stuck in locally Pareto optimal points, roughly generating an outline. As we can see in Fig. 5, even though several points are local Pareto optimal, a relatively complete outline is obtained by Algorithm 1, showing better performance than Fliege's method.

Prob. ZDT3 is a famous benchmark nonconvex problem with a discontinuous Pareto front. Algorithm 1 also performs considerably for local Pareto optimal points, while Fliege's method gets stuck in weakly Pareto optimal points as shown in Fig. 6. Algorithm 1 may be stuck in local Pareto points, but not in weakly Pareto points like Fliege's method. On the other hand, in Prob. ZDT1 and Prob. ZDT3, Algorithm 1 has a growth on iterations with a larger dimension of variables, particularly in the 100-dimension case, which lies in jumping out from the weak Pareto optimal points.

## 6. Conclusions

This paper proposes a descent-direction-constrained method for multiobjective problems based on a new strictly criteria condition for Pareto optimality. A multiobjective subproblem is introduced as a generalization to explain the relationships between Fliege et al. (2009) and traditional scalarization methods, analyzing the origin of the descent-direction-constrained subproblem, which adds the linear constraints considered in Carrizo et al. (2016). In the proposed algorithm, an Armijo-type condition is given to align with the optimality condition. The superlinear convergence of the proposed algorithm is provable under the assumptions of twice differentiability and local strong convexity of the objective functions. Furthermore, the solution sequence converges quadratically to the local Pareto optimal point if all the Hessian matrices of the objective functions are Lipschitz continuous.

The numerical results demonstrate the superiority of our method, particularly in terms of weakly Pareto optimal points, when compared to Fliege's method. However, challenges persist in dealing with high-dimensional problems due to the larger dimension of the variables and the complexity of objective functions. Some extra steps are needed to cover more areas of the Pareto front, for example, the vicinity portions of every individual minimum, and to navigate away from weak Pareto optimal points in some cases.

Our future plan entails the implementation of various existing multiobjective optimization methods, combining them to address large-scale nonconvex multiobjective optimization problems. Furthermore, we aim to explore practical optimal conditions, such as KKT conditions for single-objective optimization, for constrained multiobjective problems in our forthcoming research. Lastly, we intend to apply our theoretical methods to real-world industrial dynamic problems (considering the ordinary differential equations), contributing to the enhancement of relevant industries.

## References

Keller, L.-R., J. Simon, Y. Wang (2009) Multiple-objective decision analysis involving multiple stakeholders. *Decision technologies and applications*. Informs. 139–155.

Papalexopoulos, T., J. Alcorn, D. Bertsimas, R. Goff, D. Stewart, N. Trichakis (2023). Reshaping national organ allocation policy. *Operations Research*. Available at: `https://pubsonline.informs.org/doi/epdf/10.1287/opre.2022.0035`

Logist, F., P.-M.-M. Van Erdeghem, J.F. Van Impe (2009) Efficient deterministic multiple objective optimal control of (bio) chemical processes. *Chemical Engineering Science*, 64(11):2527–2538.

Logist, F., B. Houska, M. Diehl, J.-F. Van Impe (2010) Fast Pareto set generation for nonlinear optimal control problems with multiple objectives. *Structural and Multidisciplinary Optimization*, 42(4):591–603.

Fu, J., C. Zou, M. Zhang, X. Lu, Y. Li (2023) Multiobjective dynamic optimization of nonlinear systems with path constraints. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 53(3):1530–1542.

Ganguly, S., N.-C. Sahoo, D. Das (2013) Multi-objective planning of electrical distribution systems using dynamic programming. *International Journal of Electrical Power & Energy Systems*, 46:65–78.

Lu, X., B. Kiumarsi, T. Chai, Y. Jiang, and F. Lewis (2018) Operational control of mineral grinding processes using adaptive dynamic programming and reference governor. *IEEE Transactions on Industrial Informatics*, 15(4):2210–2221.

Susanne R. and M. Borschbach (2017) ”Design perspectives of an evolutionary process for multi-objective molecular optimization.” International Conference on Evolutionary Multi-Criterion Optimization. Cham: Springer International Publishing, 2017.

Deb, K., S. Agrawal, A. Pratap, T. Meyarivan (2002) A fast elitist non-dominated sorting genetic algorithm for multi-objective optimization: NSGA-II. *IEEE Transactions on Evolutionary Computation*, 6(2):182–197.

Cai, X., H. Sun, Q. Zhang, Y. Huang (2018) A grid weighted sum Pareto local search for combinatorial multi and many-objective optimization. *IEEE transactions on Cybernetics*, 49(9):3586-3598.

Pal, A. and H. Charkhgard (2019) A feasibility pump and local search based heuristic for bi-objective pure integer linear programming. *INFORMS Journal on Computing*. 31(1): 115-133.

Pang, L.-M., H. Ishibuchi, L. He, K. Shang, & L. Chen (2023) Hypervolume-based cooperative coevolution with two reference points for multi-objective optimization. *IEEE Transactions on Evolutionary Computation*. Available at: `https://ieeexplore.ieee.org/abstract/document/10155313`.

Dennis, J.-E., R.-B. Schnabel (1983) *Numerical Methods for Unconstrained Optimization and Nonlinear Systems*. Prentice-Hall, Englewood Cliffs, New Jersey.

Das, I. and J.-E. Dennis (1998) Normal-boundary intersection: A new method for generating the Pareto surface in nonlinear multicriteria optimization problems. *SIAM journal on optimization*, 8(2):86–98.

Miettinen, K. (1999) *Nonlinear multiobjective optimization*. Springer, Boston.

Messac, A., A. Ismail-Yahaya, C.-A. Mattson, (2003) The normalized normal constraint method for generating the Pareto frontier. *Structural and multidisciplinary optimization*, 25(2):86–98.

Eichfelder, G. (2006) $\varepsilon$-constraint method with adaptive parameter control and an application to intensity-modulated radiotherapy. *Multicriteria decision making and fuzzy systems, theory, methods and applications.* Shaker, Aachen, 25–42.

Burachik,R.-S., C.-Y. Kaya, M.-M. Rizvi (2104) A new scalarization technique to approximate Pareto fronts of problems with disconnected feasible sets. *Journal of Optimization Theory and Applications*, 162(2):428–446.

Burachik, R.-S., C.-Y. Kaya, M.-M. Rizvi (2017) A new scalarization technique and new algorithms to generate Pareto fronts. *SIAM Journal on Optimization*, 27(2):1010–1034.

Prudente, L.-F., D.-R. Souza (2022) A quasi-Newton method with Wolfe line searches for multiobjective optimization. Journal of Optimization Theory and Applications, 194(3):1107–1140.

Nie J. and Z. Yang (2024) The multi-objective polynomial optimization[J]. *Mathematics of Operations Research.* Available at: `https://pubsonline.informs.org/doi/epdf/10.1287/moor.2023.0200`.

Li, K., R. Chen, G. Fu, X. Yao (2018) Two-archive evolutionary algorithm for constrained multiobjective optimization. *IEEE Transactions on Evolutionary Computation*, 23(2):303–315.

Shen, X., D. Zou, N. Duan, Q. Zhang (2019) An efficient fitness-based differential evolution algorithm and a constraint handling technique for dynamic economic emission dispatch. *Energy*, 186:115801.

Ma, Z., Y. Wang, W. Song (2019) A new fitness function with two rankings for evolutionary constrained multiobjective optimization. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 51(8):5005–5016.

Nocedal, J., and S.-J. Wright (1999) *Numerical optimization*. Springer, New York.

Fliege, J., and B.-F. Svaiter (2000) Steepest descent methods for multicriteria optimization. *Mathematical methods of operations research*, 51:479–494.

Fliege, J., L.-M.-G. Drummond, B.-F. Svaiter (2009) Newton's method for multiobjective optimization. *SIAM Journal on Optimization*, 20(2):602–626.

Carrizo, G.-A., P.-A. Lotito, M.-C. Maciel (2016) Trust region globalization strategy for the nonconvex unconstrained multiobjective optimization problem. *Mathematical Programming*, 159(1a2):339–369.

Das, I., and J.-E. Dennis (1997) A closer look at drawbacks of minimizing weighted sums of objectives for Pareto set generation in multicriteria optimization problems. *Structural optimization*, 14:63–69.

Fliege, J., A. Vaz, F. Ismael (2016) A method for constrained multiobjective optimization based on SQP techniques. *SIAM Journal on Optimization*, 26(4):2091–2119.

Huband, S., P. Hingston, L. Barone, L. While (2006) A review of multiobjective test problems and a scalable test problem toolkit. *IEEE Transactions on Evolutionary Computation.* 10(5):477–506.

Laumanns, M., L. Thiele, K. Deb, E. Zitzler (2002) Combining convergence and diversity in evolutionary multiobjective optimization. *Evolutionary computation*, 10(3):263–282.

Hillermeier, C. (2001) *Nonlinear multiobjective optimization: a generalized homotopy approach*. Springer Science and Business Media.