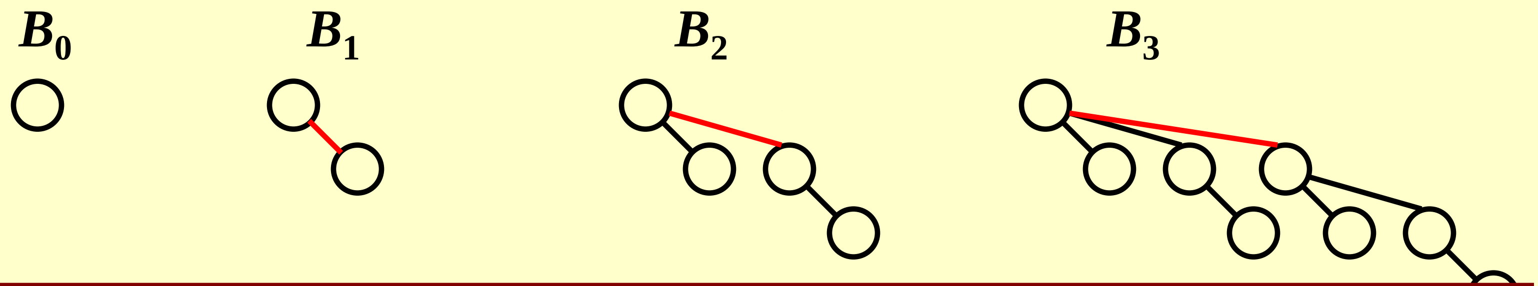
Binomial Queue

☞ Structure:

A binomial queue is not **a** heap-ordered tree, but rather a **collection** of heap-ordered trees, known as a **forest**. Each heap-ordered tree is a **binomial tree**.

A binomial tree of height **0** is a one-node tree.
A binomial tree, B_k, of height k is formed by attaching a binomial tree, B_{k-1}, to the root of another binomial tree, B_{k-1}.



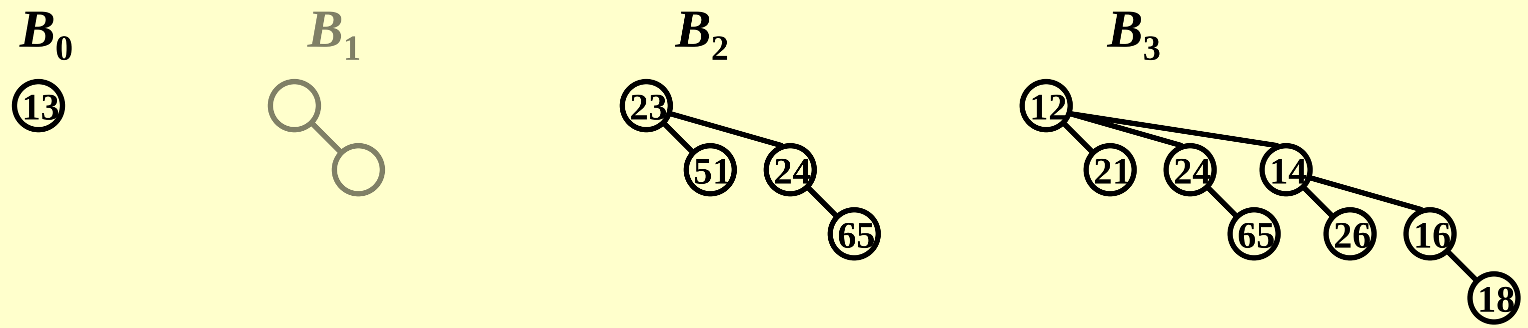
Observation: B_k consists of a root with ___ k ___ children, which are ___ $B_0, B_1, \ldots, B_{k-1}$ ___. B_k has exactly ___ 2^k ___ nodes. The number of nodes at depth d is ___ $\binom{k}{d}$ ___.

B_k structure + heap order + one binomial tree for each height

➡ **A priority queue of any size can be uniquely represented by a collection of binomial trees.**

【Example】Represent a priority queue of size 13 by a collection of binomial trees.

Solution: $13 = 2^0 + 0\times2^1 + 2^2 + 2^3 = 1101_2$



☞ Operations:

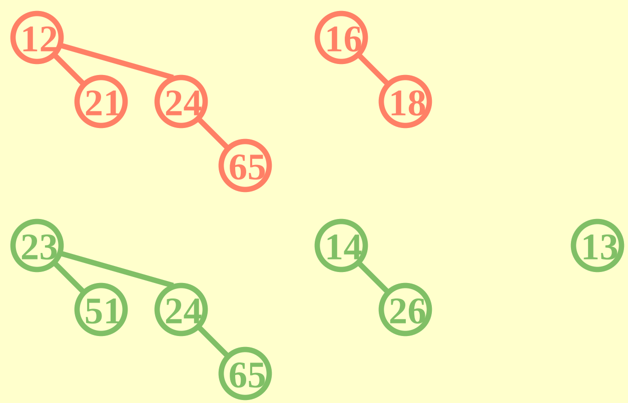
⌖ **FindMin**: The minimum key is in one of the **roots**.
There are at most $\lceil \log N \rceil$ roots, hence $T_p = O(\log N)$.

> **Note:** We can remember the minimum and update whenever it is changed. Then this operation will take **O(1)**.

⌖ **Merge**:

H_1

H_2



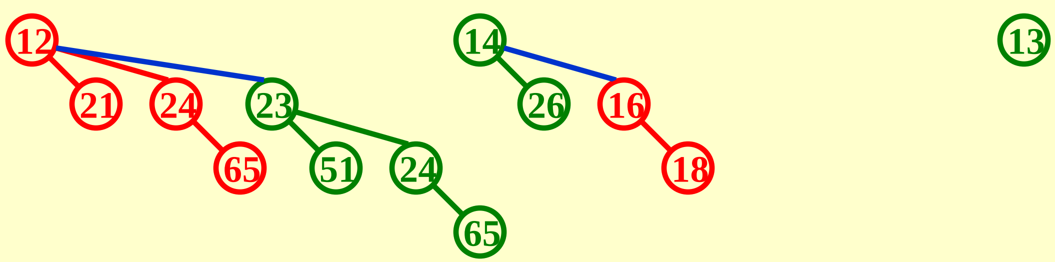
$$1\ 1\ 0_2 = 6$$

$$+\ 1\ 1\ 1_2 = 7$$

c c

$$1\ 1\ 0\ 1$$

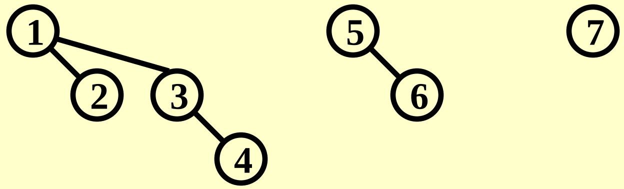
H_1



$T_p = O(\log N)$

⊕ **Insert**: a special case for merging.

【Example】Insert 1, 2, 3, 4, 5, 6, 7 into an initially empty queue.



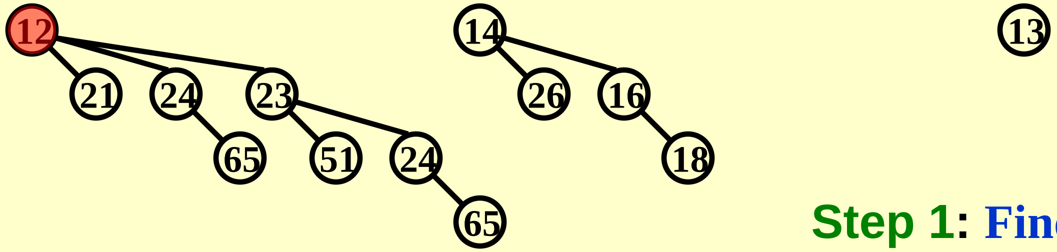
Note:

If the smallest nonexistent binomial tree is B_i, then $T_p = Const \cdot (i + 1)$.

Performing N Inserts on an initially empty binomial queue will take $O(N)$ worst-case time. Hence the average time is constant.

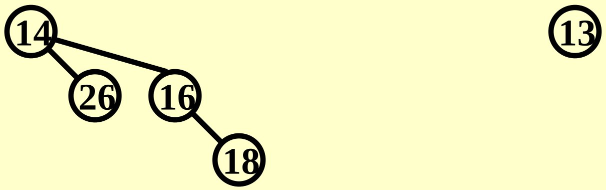
⌖ **DeleteMin (H):**

H



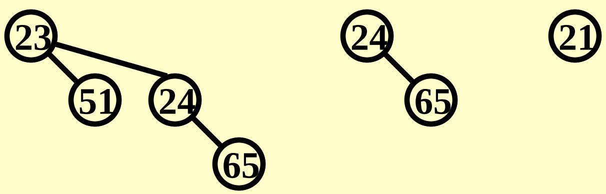
Step 1: FindMin in B_k
/* O(log *N*) */

H'



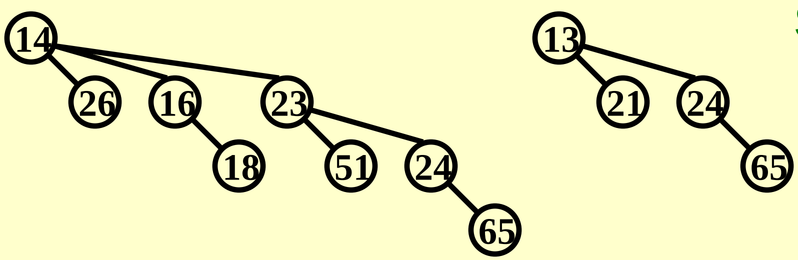
Step 2: Remove B_k from H
/* O(1) */

H"



Step 3: Remove root from B_k
/* O(log *N*) */

H

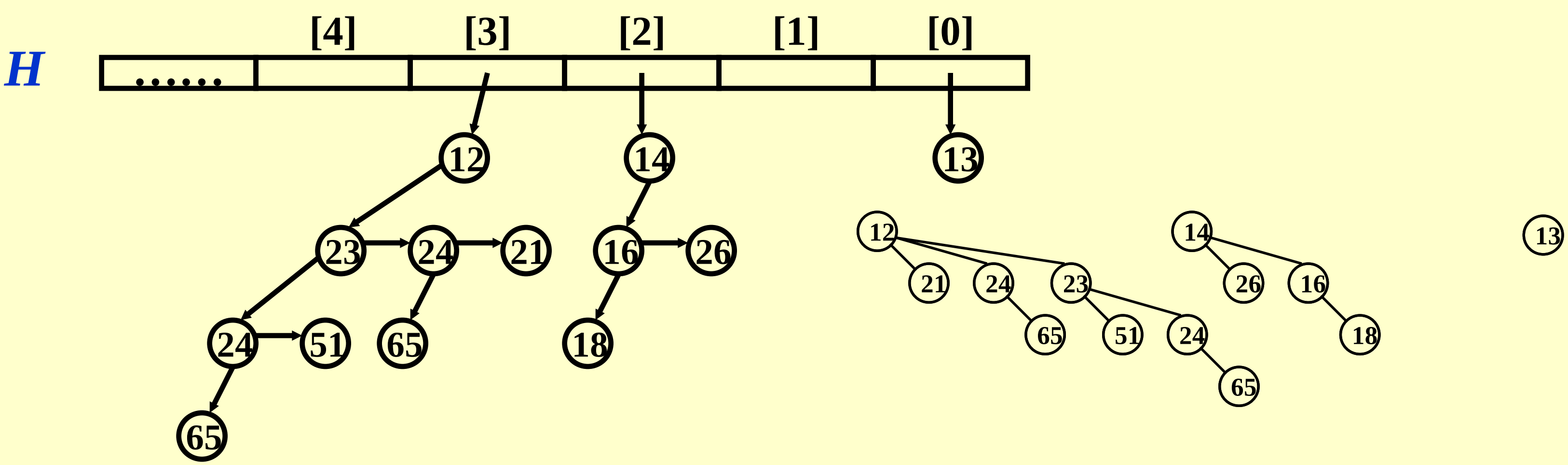


Step 4: Merge (*H'*, *H"*)
/* O(log *N*) */

☞ Implementation:

Binomial queue = array of binomial trees

Operation	Property	Solution
DeleteMin	**Find all the subtrees quickly**	**Left-child-next-sibling with linked lists**
Merge	**The children are ordered by their sizes**	**The new tree will be the largest. Hence maintain the subtrees in decreasing sizes**



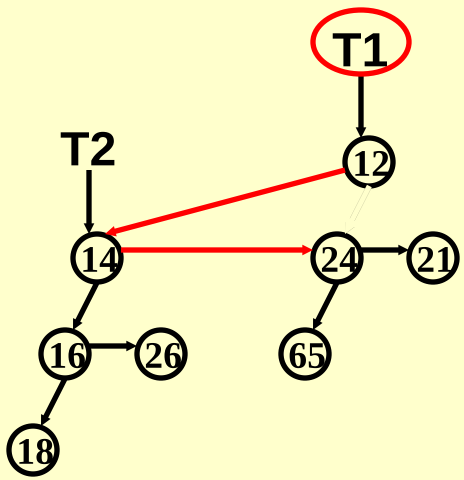
```
typedef struct BinNode *Position;
typedef struct Collection *BinQueue;
typedef struct BinNode *BinTree;  /* missing from p.176 */

struct BinNode
{
    ElementType     Element;
    Position        LeftChild;
    Position        NextSibling;
} ;

struct Collection
{
    int             CurrentSize;  /* total number of nodes */
    BinTree         TheTrees[ MaxTrees ];
} ;
```

```
BinTree
CombineTrees( BinTree T1, BinTree T2 )
{  /* merge equal-sized T1 and T2 */
    if ( T1->Element > T2->Element )
            /* attach the larger one to the smaller one */
            return CombineTrees( T2, T1 );
    /* insert T2 to the front of the children list of T1 */
    T2->NextSibling = T1->LeftChild;
    T1->LeftChild = T2;
    return T1;
}
```

$T_p = O(1)$



```
BinQueue  Merge( BinQueue H1, BinQueue H2 )
{   BinTree T1, T2, Carry = NULL;
    int i, j;
    if ( H1->CurrentSize + H2-> CurrentSize > Capacity )  ErrorMessage();
    H1->CurrentSize += H2-> CurrentSize;
    for ( i=0, j=1; j<= H1->CurrentSize; i++, j*=2 ) {
        T1 = H1->TheTrees[i]; T2 = H2->TheTrees[i]; /*current trees */
        switch( 4*!!Carry + 2*!!T2 + !!T1 ) {   /* assign each digit to a tree */
            case 0: /* 000 */
            case 1: /* 001 */  break;
            case 2: /* 010 */  H1->TheTrees[i] = T2; H2->TheTrees[i] = NULL; break;
            case 4: /* 100 */  H1->TheTrees[i] = Carry; Carry = NULL; break;
            case 3: /* 011 */  Carry = CombineTrees( T1, T2 );
                               H1->TheTrees[i] = H2->TheTrees[i] = NULL; break;
            case 5: /* 101 */  Carry = CombineTrees( T1, Carry );
                               H1->TheTrees[i] = NULL; break;
            case 6: /* 110 */  Carry = CombineTrees( T2, Carry );
                               H2->TheTrees[i] = NULL; break;
            case 7: /* 111 */  H1->TheTrees[i] = Carry;
                               Carry = CombineTrees( T1, T2 );
                               H2->TheTrees[i] = NULL; break;
        } /* end switch */
    } /* end for-loop */
    return H1;
}
```

Carry	T2	T1

```
ElementType  DeleteMin( BinQueue H )
{    BinQueue DeletedQueue;
     Position DeletedTree, OldRoot;
     ElementType MinItem = Infinity;  /* the minimum item to be returned */
     int i, j, MinTree; /* MinTree is the index of the tree with the minimum item */

     if ( IsEmpty( H ) )  {  PrintErrorMessage();  return –Infinity; }

     for ( i = 0; i < MaxTrees; i++) {  /* Step 1: find the minimum item */
        if( H->TheTrees[i] && H->TheTrees[i]->Element < MinItem ) {
                MinItem = H->TheTrees[i]->Element;  MinTree = i;    } /* end if */
     } /* end for-i-loop */
     DeletedTree = H->TheTrees[ MinTree ];
     H->TheTrees[ MinTree ] = NULL;   /* Step 2: remove the MinTree from H => H’ */
     OldRoot = DeletedTree;   /* Step 3.1: remove the root */
     DeletedTree = DeletedTree->LeftChild;   free(OldRoot);
     DeletedQueue = Initialize();   /* Step 3.2: create H” */
     DeletedQueue->CurrentSize = ( 1<<MinTree ) – 1;  /* 2^MinTree – 1 */
     for ( j = MinTree – 1; j >= 0; j – – ) {
        DeletedQueue->TheTrees[j] = DeletedTree;
        DeletedTree = DeletedTree->NextSibling;
        DeletedQueue->TheTrees[j]->NextSibling = NULL;
     } /* end for-j-loop */
     H->CurrentSize  – = DeletedQueue->CurrentSize + 1;
     H = Merge( H, DeletedQueue ); /* Step 4: merge H’ and H” */
     return MinItem;
}
```

【Claim】A binomial queue of N elements can be built by N successive insertions in O(N) time.

Proof 1 (Aggregate):

$\boxed{+1}$ B_0 /*step = 1 */

$\boxed{+0}$ B_1 /*step = 1, link = 1 */

$\boxed{+1}$ B_1 B_0 /*step = 1*/

$\boxed{-1}$ B_2 /*step = 1, link = 2 */

$\boxed{+1}$ B_2 B_0 /*step = 1*/

B_2 B_1 /*step = 1, link = 1*/

B_2 B_1 B_0 /*step = 1*/

$\boxed{-2}$ B_3 /*step = 1, link = 3*/

B_3 B_0 /*step = 1*/

… … …

Total steps = N

Total links =

$$N\left(\frac{1}{4}+2\times\frac{1}{8}+3\times\frac{1}{16}+\ldots\right)$$

$$=O(N)$$

Expensive insertions remove trees, while cheap ones create trees.

Proof 2: **An insertion that costs c units results in a net increase of $2 - c$ trees in the forest.**

C_i ::= cost of the *i*th insertion

Φ_i ::= number of trees *after* the *i*th insertion ($\Phi_0 = 0$)

$C_i + (\Phi_i - \Phi_{i-1}) = 2$ **for all $i = 1, 2, \ldots, N$**

Add all these equations up $\Rightarrow \sum_{i=1}^{N} C_i + \Phi_N - \Phi_0 = 2N$

$$\sum_{i=1}^{N} C_i = 2N - \Phi_N \le 2N = O(N)$$

∎

$T_{worst} = O(\log N)$, **but** $T_{amortized} = 2$

Reference:

Data Structure and Algorithm Analysis in C **(2nd Edition)：** **Ch.5，p.170-180； Ch.11，p.430-435**；***M.A.Weiss著、陈越改编，人民邮件出版社，2005***

Introduction to Algorithms, 3rd Edition: Ch.19, p. 505-530； ***Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein. The MIT Press. 2009***