Fundamentals of Applied Operations Research

Guochuan Zhang

zgc@zju.edu.cn

Research Group on Discrete Optimization and Algorithms

Zhejiang University

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Warm-Up

Spanning Tree

- Given an undirected graph $G = (V, E)$, find out as many edge disjoint spanning trees as possible.
- You may compute spanning trees one by one until none exists. Show it works or present a counter-example.
- * Can you do better if G is a complete graph?

Shortest Path

- Given an undirected graph $G = (V, E)$, each edge e_i is associated with two positive parameters b_i and c_i.
- A path p is evaluated by a pair (B_p, C_p) as well, where $B_p = \min_{e_i \in p} b_i$, and $C_p = \sum_{e_i \in p} c_i$.
- Find an s-t path p with (B_p, C_p), where no path is lexicographically better than p (larger B_p, smaller C_p).

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Roadmap

Operations Research - the Science of Better

- Explore the methodology for solving a great many of optimization problems with limited resources / information

Core - Optimization

- Problems
- Methods
- Culture

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A List of Topics

- Linear Programming
- Nonlinear Programming
- Integer Programming
- Combinatorial Optimization (approximation/online algorithms)
- Game Theory (optimization with interaction)

Main Issues

- Design and analysis of algorithms/mechanisms

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Major Conferences/Journals

Theory of Algorithms

- FOCS, STOC, SODA, ICALP, EC, ESA, STACS
- SIAM J Computing, ACM T Algorithms, J Computer and System Sciences, Information and Computation, Algorithmica

Operations Research

- IPCO
- Mathematics of Operations Research, Mathematical Programming, Operations Research, INFORMS J on Computing, Naval Research Logistics, IIE Transactions

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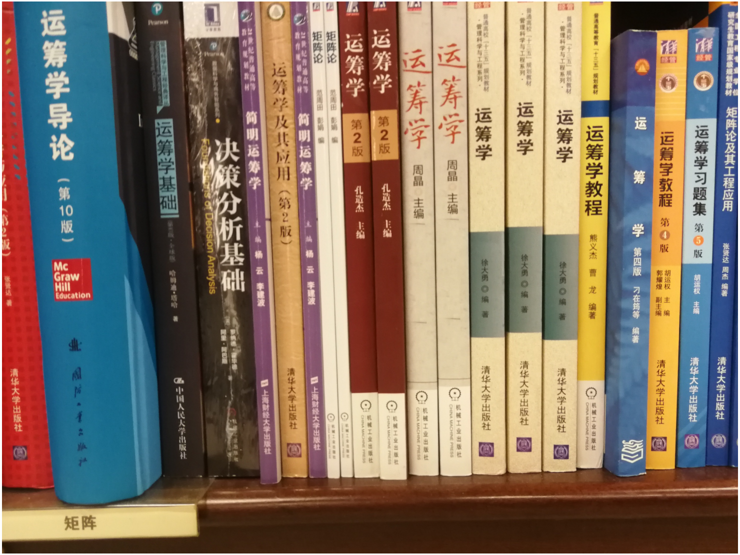
Reading Materials



Reading Materials

Books

- Any textbook on Operations Research or Optimization
- Any book on Combinatorial Optimization
  Recommended "Combinatorial Optimization - Algorithms and Complexity, Papademitriou and Steiglitz"
- Any book on Algorithms
  Recommended "Algorithm Design, Tardos and Kleinberg"
- Any book on Game Theory
  Recommended "Algorithmic Game Theory, Nisan, Roughgarden, Tardos, and Vazirani"

Who Are You?

Incentives

- Show great interests
- Have strong math background
- Enjoy finding out the truth
- Get credits (Sure, only if the above are satisfied)

Requirements

- 100% attendance (except for emergency)
- 100% attention
- Being active

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Grading Mechanism

In Class

- In-class discussions (quizzes)
- Final in-class exercise

After Class

- Individual homework
- Team work (paper-reading and presentations)

Final

- Writing?
- Oral?

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- Writing?
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Highlights

We are concerned about Problems but not a single Instance

We are doing Re-search but not simple Searches

We are working on Programming but not Coding

We are not only doing something correct but also showing the Correctness

We do prefer Intelligence

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Computability

Limits to Computers

- Computers can only carry out algorithms: precise and universally understood sequences of instructions that solve any instances of rigorously defined computational problems
- Are there well-defined mathematical problems for which there are no algorithms? YES! (Alan Turing)
- Undecidable problems do exist, say the Halting problem: given a computer program with its input, will it ever halt?

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Tractability

Time Bounds

- Away from the Turing's time in 1930s, computers nowadays deal with decidable problems. In principal, these problems admit an algorithm for solving every instance
- A new challenge is the running time of an algorithm, namely, the algorithm efficiency

Example

- TSP (the Travelling Salesman Problem): finding a shortest tour (a cycle), visiting each vertex exactly once, on a given weighted complete graph
- The number of possible tours is $(n-1)!/2$

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Input Size

- Basically, length of the sequence to encode the instance, the number of symbols in the sequence
- Testing a prime number: check if a given integer is prime?
- Size of a graph

Analysis of Algorithms (I)

- Deriving bounds for the time requirement of an algorithm
- Using the notations of O, Ω and Θ

Polynomial Time Algorithms

- Those running polynomially in the input size

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P vs NP

Decision Problems

- Decide if there is a solution (Yes/No questions)

Optimization Problems

- Determine an optimal solution

Class P

- A problem with a polynomial time algorithm solving all its instances (solved polynomially by a Turing machine)

Class NP

- If x is a Yes instance of the problem, there exists a certificate for x, whose validity can be checked in polynomial time (solved in polynomial time by a non-deterministic Turing machine)

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The Most Challenging Problem: $P = NP$?

Reduction

- A reduction f from problems B to A: given any instance I of B, $f(I)$ is an instance of A. I is Yes iff $f(I)$ is Yes

NP-Complete

- Problem A is one of the hardest problems in NP
- For any problem $B \in NP$, there is a polynomial time reduction from B to A
- A is NP-complete

NP-Hard

- For any problem $B \in NP$, there is a polynomial time reduction from B to A
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- Sounds impossible, as you have to show all problems can be polynomially reduced to a specific problem
- SAT, done by Cook in 1971

How to Find the next NPC Problems?

- Repeat Cook's work? Not necessarily
- Polynomial time reduction is transitive
- Show SAT can be reduced in polynomial time to the problem you expect
- Karp proved 21 NPC problems in 1972

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The Most Challenging Problem Becomes "Easier"

To Show $P = NP$

- Simply choose a suitable NPC problem and show a polynomial time algorithm

To Show $P \neq NP$

- Simply choose a suitable NPC problem and show it can not be solved in polynomial time

A "Common" Sense

- Most likely $P \neq NP$

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Solving Combinatorial Optimization Problems

Combinatorial Optimization Problem

$$\min f(x) \quad s.t.\ x \in \Omega$$

where Ω is a finite set

Our Concerns

- Efficiency: How fast to obtain a solution?
- Effectiveness: How good the solution is?

Tradeoff

Running Times versus Performance Bounds

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Distinguishing Problems

Easy Problems

Those admit a polynomial time algorithm, such as the minimum spanning tree problem, and the matching problem

Hard Problems

Those problems that can only be solved exponentially under the assumption $P \neq NP$

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Research Topics

Complexity

Show a problem is in P by providing a polynomial time algorithm, or prove it is hard under some known assumptions (e.g. $P \neq NP$)

Algorithm Design

- Exact algorithms for easy problems with very low running times
- Exact algorithms for hard problems, that run efficiently in practice
- Approximation algorithms for hard problems, that have good performance bounds
- (Meta-)heuristics for any problem, that work well for some real-world instances

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Highlighted Topics

Best with a Low Cost

Only exact solution is wanted, but with a reasonable running time

Cheap with a High Quality

Only (sub-, sup-) linear time is allowed, but with an acceptable performance bound

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Coming Back to Hard Problems

Approximation Algorithms

- Constant factor approximation algorithm
- PTAS
- FPTAS
- Absolute approximation algorithm

Hardness

- No polynomial time algorithms
- No FPTAS
- No PTAS
- No constant ratio

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Lecture 1

Optimization Problems

Introduction

Basic Models

- A number of variables (continuous or discrete)
- A feasible set, usually represented by a set of constraints on variables
- An objective function to be optimized

Math Formulation

$$\begin{array}{rl} \min\ (\max) & f(x) \\ s.t. & x \in \Omega \end{array}$$

Feasible Set (I)

$$\Omega : \{h_i(x) = 0,\ i = 1, 2, \ldots, m,\ g_j(x) \leq 0,\ j = 1, 2, \ldots, l\}$$

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Examples

Linear Programming

- Let m and n be positive integers, $b \in Z^m$ and $c \in Z^n$, and A be an $m \times n$ matrix with elements $a_{ij} \in Z$. Then an LP instance is defined as
  $\Omega = \{x : x \in R^n, Ax = b, x \geq 0\}$ and $f = c^T x$

TSP

- Given an integer $n > 0$, and the distance matrix $[d_{ij}]$ between every pair of n points, a tour is a closed path visiting every point exactly once
- $\Omega = \{\text{all cyclic permutation } \pi \text{ on n points}\}$
- The cost function is $f(\pi) = \sum_{j=1}^{n} d_{\pi_j \pi_{j+1}}$, where $\pi_{n+1} = \pi_1$

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Neighborhoods

Definition

Given an optimization problem with instances (Ω, f), a neighborhood is a mapping

$$N : \Omega \longrightarrow 2^{\Omega}$$

Examples

- If $\Omega = R^n$, the set of points within a fixed Euclidean distance gives a natural neighborhood
- In the TSP, we define a neighborhood 2-change as $N_2(\pi) = \{\tau \in \Omega :$ τ can be obtained from π by relink four points$\}$
- How about MST?

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Local and Global Optima

Definitions

- A feasible solution is local optimal with respect to a neighborhood N, if its value is the best among all points in N
- A feasible solution is globally optimal if its value is the best among all points in Ω
- A neighborhood N is exact if its local optimal solution is also global optimal

Examples

- TSP: N_2 is not exact, while N_n is
- MST?

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Convex Sets and Functions

Convex Set

- A convex combination of two points $x, y \in R^n$:
  $z = \lambda x + (1 - \lambda) y$, where $0 \leq \lambda \leq 1$
- A set S is convex if it contains all convex combinations of pairs of points $x, y \in S$
- The intersection of any number of convex sets is convex

Examples

- R^n, $\emptyset$, any interval in R
- $\{x : Ax = b, x \geq 0\}$

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Convex Functions

- Let S be a convex set in R^n (usually $S = R^n$)
- The function $f : S \longrightarrow R$ is convex in S if for any two points $x, y \in S$, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$, where $0 \leq \lambda \leq 1$
- For any $t \in R$, $S_t = \{x : f(x) \leq t, x \in S\}$ is convex
- f is concave if $-f$ is convex

Examples

- A linear function is convex and concave in any convex set S

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Convex Programming

Definition

- Minimization of a convex function on a convex set: f is convex and Ω is convex
- Usually Ω is defined by $\{x : g_i(x) \leq 0, i = 1, 2, \ldots, m\}$, where $g_i(x)$ is convex

A Smart Property

- The neighborhood $N_\epsilon(x) = \{y \in \Omega : \text{ and } ||x - y|| \leq \epsilon\}$ is exact for any $\epsilon > 0$
- Local optima are global as well (with respect to the Euclidean distance neighborhood)

Linear Programming

- LP is a special convex programming problem

Convex Programming

Definition

- Minimization of a convex function on a convex set: f is convex and Ω is convex
- Usually Ω is defined by $\{x : g_i(x) \leq 0, i = 1, 2, \ldots, m\}$, where $g_i(x)$ is convex

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Let us focus on LP first

Linear Programming Models

Example 1

- There are two products jointly produced by three firms

Firms	Product 1	Product 2	Resources
A	1	0	100
B	0	2	200
C	1	1	150

- Single values of the two products are 1 and 2, respectively
- Make a plan to maximize the total value of products

LP Formulation

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ s.t. \quad & x_1 \leq 100 \\ & 2x_2 \leq 200 \\ & x_1 + x_2 \leq 150 \\ & x_1, x_2 \geq 0 \end{aligned}$$

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Linear Programming Models

Extend to A General Problem

- There are n products jointly produced by m firms
- The j-th product has a value c_j
- The j-th product requires a_{ij} units of resources from the i-th firm
- The i-th firm has a resource amounting to b_i
- Maximize the total value

LP Formulation

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, 2, \ldots, m \\
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Linear Programming Models

A Simplified Formulation

- $$\begin{array}{ll} \max & c^T x \\ s.t. & Ax \leq b \\ & x \geq 0 \end{array}$$
- $c^T = (c_1, c_2, \ldots, c_n)$, $b = (b_1, b_2, \ldots, b_m)^T$, $A = (a_{ij})_{m \times n}$

Example 2

- The final of EURO Cup is coming soon. Fans are ready for bidding which team will be the champion
- There are n teams in the final
- There are m bids, each of an n-dimensional vector. Namely, bid $b_i = (a_{i1}, \ldots, a_{in})$, where a_{ij} is 1 if bid i supposes team j is the champion, $a_{ij} = 0$, otherwise
- Each bidder i would like to pay π_i for each bet, and he can buy at most q_i bets
- If a bid consists of a champion team (as the game is over), the bidder wins w for each bet
- The dealer decides if accepts the bids and if yes how many bets, so that his benefit is maximized

Linear Programming Models

Formulation

- Let x_i be the number of bets offered to the bidder i.
  $0 \leq x_i \leq q_i$.
- The objective function to maximize is
  $\sum_{i=1}^{m} \pi_i x_i - \max_{1 \leq j \leq n} \sum_{i=1}^{m} a_{ij} x_i w$

Linear Model

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{m} \pi_i x_i - y \\
& y \geq \sum_{i=1}^{m} a_{ij} x_i w, \quad j = 1, 2, \ldots, n \\
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