# Leftist Heaps

# and Skew Heaps

# Leftist Heaps

Target : **Speed up merging in O(*N*).**

**Heap: Structure Property + Order Property**

**Have to copy one array into another ➡ Θ(*N*)**

**Use pointers** **Slow down all the operations**

## Leftist Heap:

**Order Property – the same**

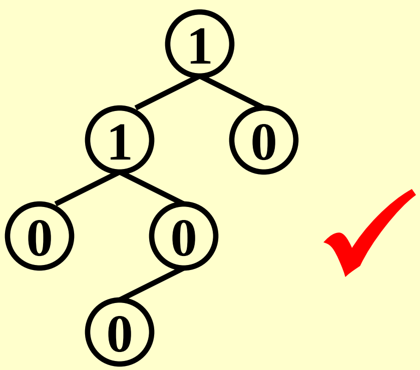
**Structure Property – binary tree, but *unbalanced***

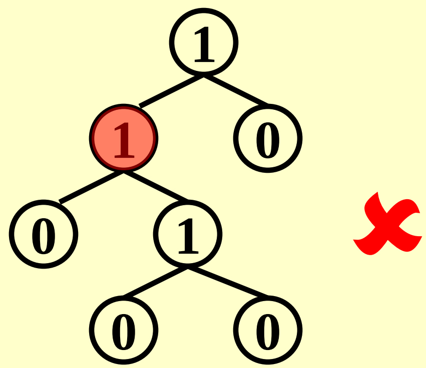
【Definition】The null path length, Npl(X), of any node X is the length of the shortest path from X to a node without two children. Define Npl(NULL) = –1.

**Note:**

$$\mathrm{Npl}(X) = \min\ \{\ \mathrm{Npl}(C) + 1 \text{ for all } C \text{ as children of } X\ \}$$

【Definition】The leftist heap property is that for every node X in the heap, the null path length of the left child is at least as large as that of the right child.





The tree is biased to get deep toward the *left*.

**【Theorem】A leftist tree with r nodes on the right path must have at least $2^r - 1$ nodes.**

**Proof: By induction on p. 162.**

**Note: The leftist tree of *N* nodes has a right path containing at most $\lfloor \log(N+1) \rfloor$ nodes.**

☝ **We can perform all the work on the *right* path, which is guaranteed to be short.**

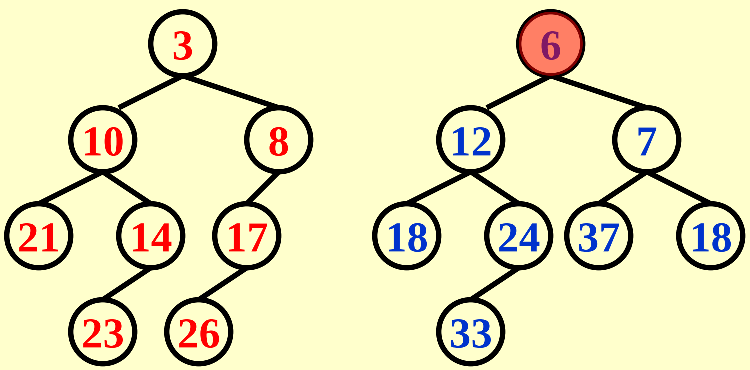
**Trouble makers: Insert and Merge**

**Note: Insertion is merely a special case of merging.**

## ☞ Declaration:

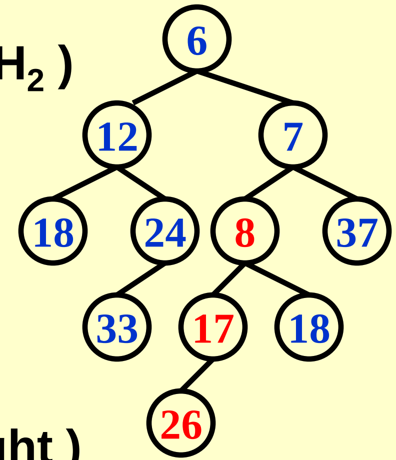
```
struct TreeNode
{
    ElementType      Element;
    PriorityQueue    Left;
    PriorityQueue    Right;
    int              Npl;
} ;
```

## ☞ Merge (recursive version):

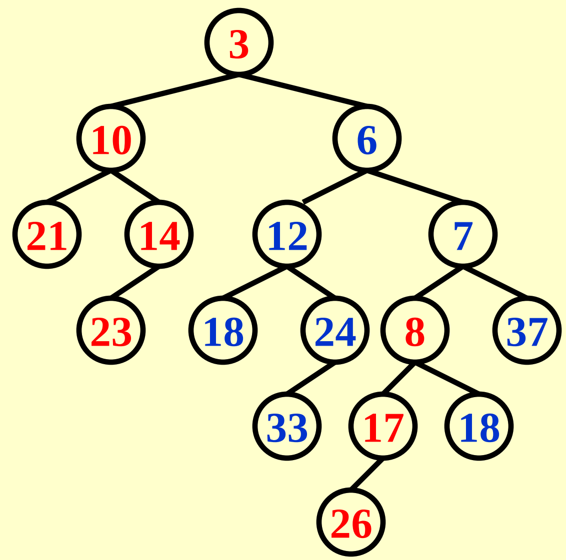



*$H_1$*          *$H_2$*

***Step 1***:
Merge( $H_1$->Right, $H_2$ )




***Step 2***:
Attach( $H_2$, $H_1$->Right )




***Step 3***:
Swap($H_1$->Right, $H_1$->Left )
if necessary

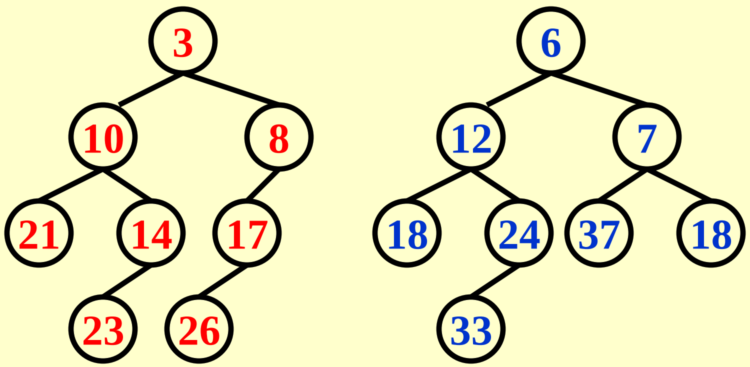
```
PriorityQueue  Merge ( PriorityQueue H1, PriorityQueue H2 )
{
    if ( H1 == NULL )   return H2;
    if ( H2 == NULL )   return H1;
    if ( H1->Element < H2->Element )  return Merge1( H1, H2 );
    else return Merge1( H2, H1 );
}
```

```
static PriorityQueue
Merge1( PriorityQueue H1, PriorityQueue H2 )
{
    if ( H1->Left == NULL )     /* single node */
            H1->Left = H2;      /* H1->Right is already NULL
                                   and H1->Npl is already 0 */
    else {
            H1->Right = Merge( H1->Right, H2 );     /* Step 1 & 2 */
            if ( H1->Left->Npl < H1->Right->Npl )
                    SwapChildren( H1 );             /* Step 3 */
            H1->Npl = H1->Right->Npl + 1;
    } /* end else */
    return H1;
}
```

$T_p = O(\log N)$

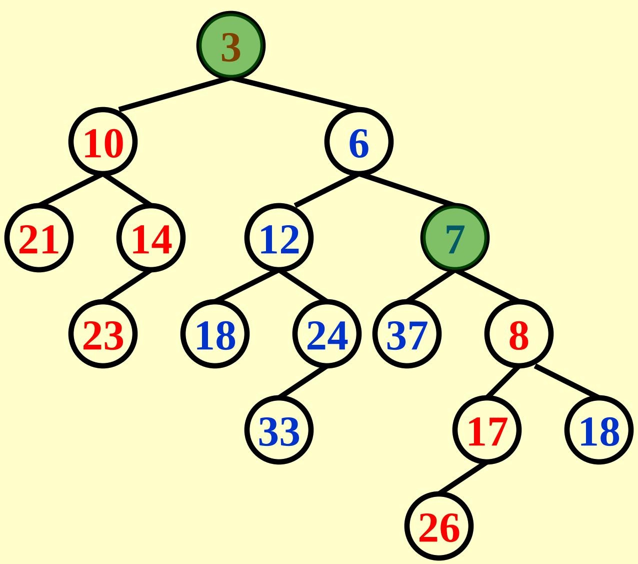
What if *Npl* is NOT updated?

## ☞ Merge (iterative version):

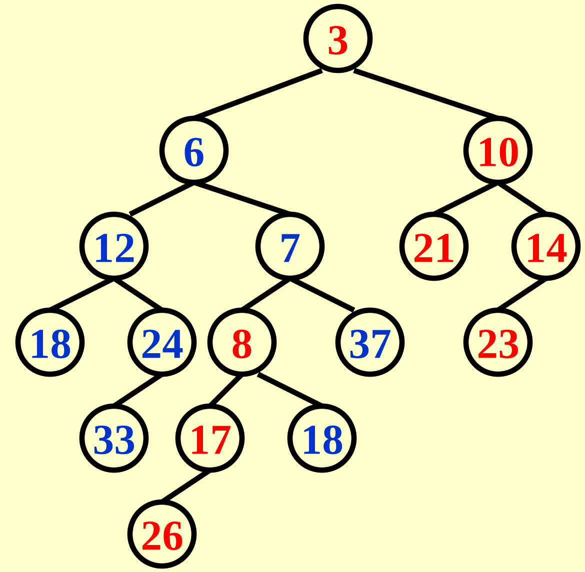



$H_1$ $H_2$

***Step 1***: **Sort the right paths without changing their left children**




***Step 2***: **Swap children if necessary**



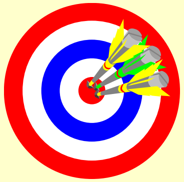

## ☞ DeleteMin:

***Step 1***: **Delete the root**

***Step 2***: **Merge the two subtrees**

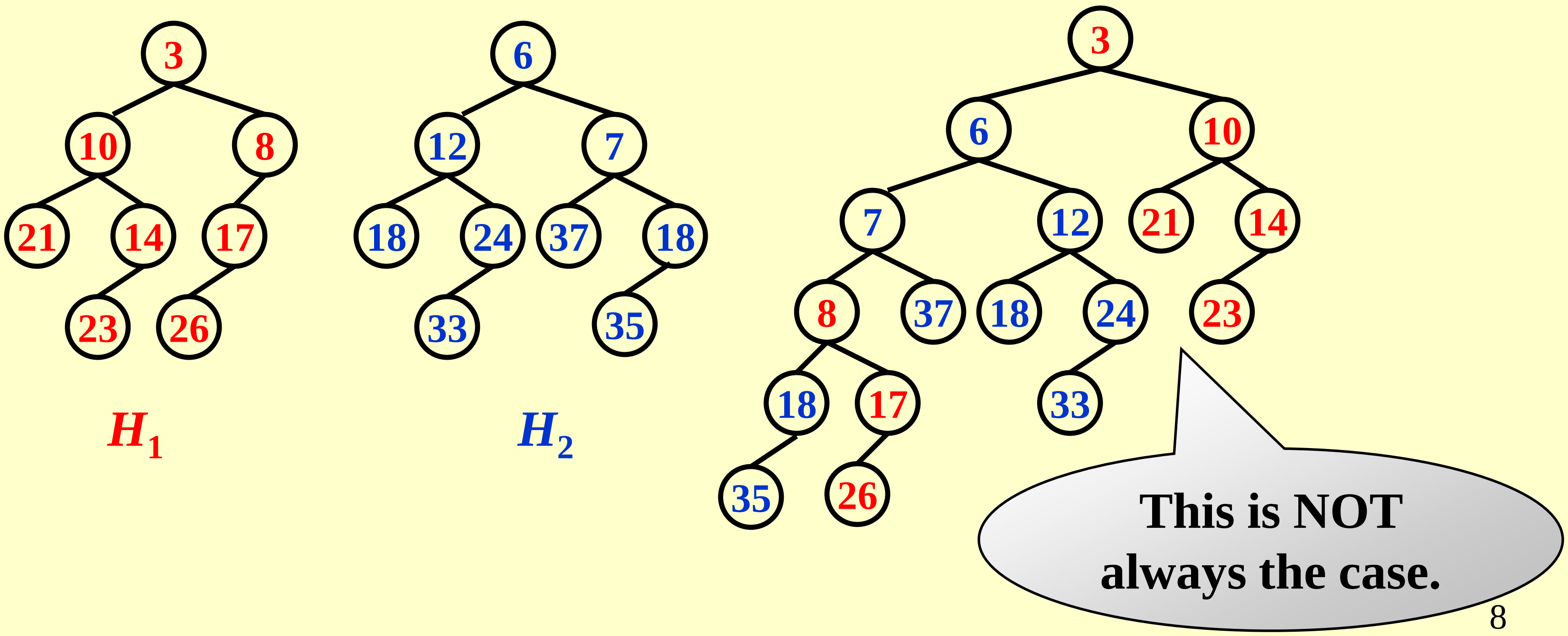
$T_p = O(\log N)$

# Skew Heaps -- a simple version of the leftist heaps

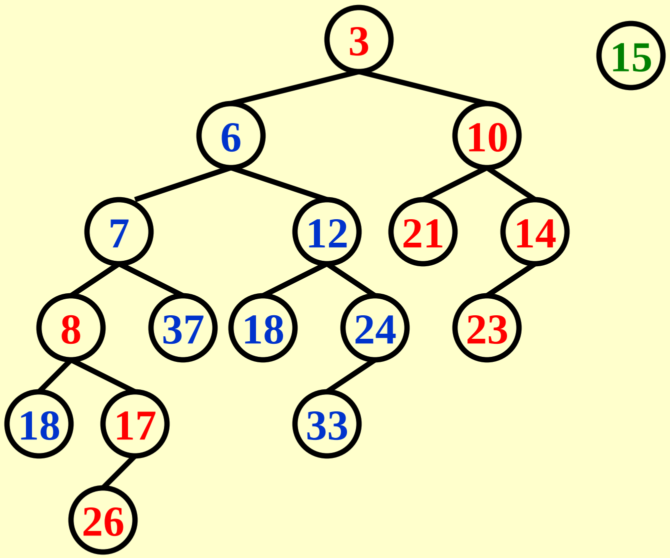


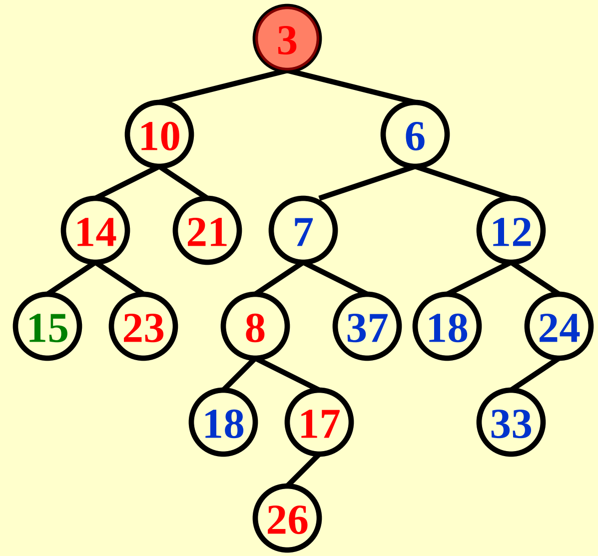
Target : **Any $M$ consecutive operations take at most $O(M \log N)$ time.**

☞ **Merge: Always swap the left and right children except that the largest of all the nodes on the right paths does not have its children swapped. No Npl.**

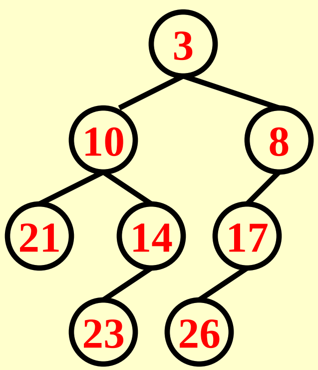


# 【Example】 Insert 15

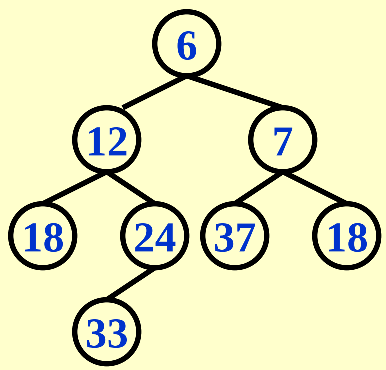


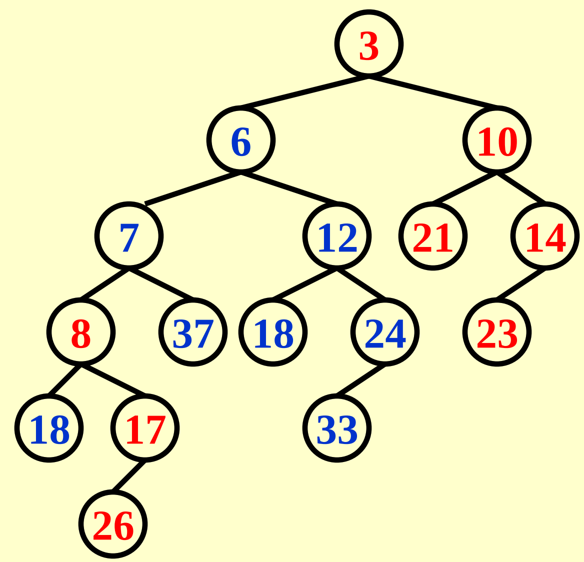

☞ **Merge (iterative version):**




$H_1$




$H_2$



**Note:**

☞ **Skew heaps have the advantage that no extra space is required to maintain path lengths and no tests are required to determine when to swap children.**

☞ **It is an open problem to determine precisely the expected right path length of both leftist and skew heaps.**

# Amortized Analysis for Skew Heaps

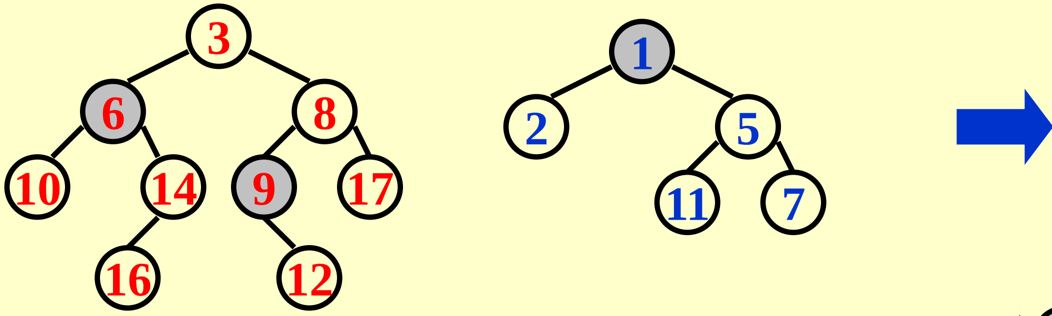
**Insert & Delete are just Merge**

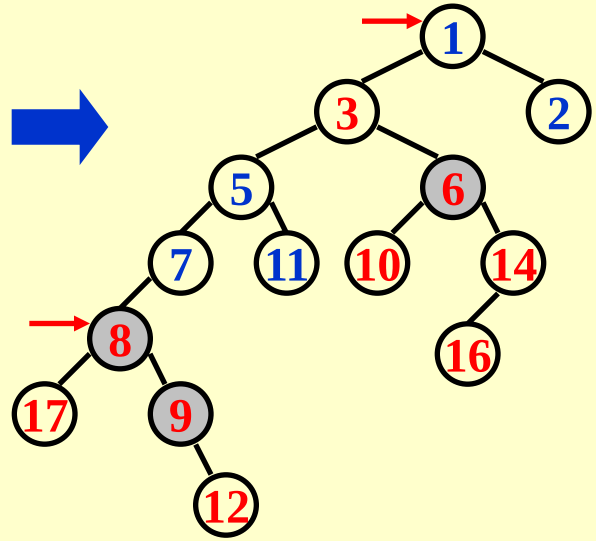
$T_{amortized} = O(\log N)$ ?

$D_i =$ **the root of the resulting tree**

$\Phi( D_i ) =$ **number of *heavy* nodes**

**【Definition】A node *p* is *heavy* if the number of descendants of *p*'s right subtree is at least half of the number of descendants of *p*, and *light* otherwise. Note that the number of descendants of a node includes the node itself.**




**The only nodes whose heavy/light status can change are nodes that are initially on the right path.**

$H_i : l_i + h_i \;\; ( i = 1, 2 )$ ⇒ $T_{worst} = l_1 + h_1 + l_2 + h_2$

Along the right path

**Before merge:** $\Phi_i = h_1 + h_2 + h$ $\qquad T_{amortized} = T_{worst} + \Phi_{i+1} - \Phi_i$

**After merge:** $\Phi_{i+1} \le l_1 + l_2 + h$ $\qquad \le 2\,(l_1 + l_2)$

$l = O(\log N)$ ⇒ $T_{amortized} = O(\log N)$

# Reference:

**Data Structure and Algorithm Analysis in C (2nd Edition)：Ch.5，p.161-169； Ch.11，p.435-437；*M.A.Weiss著、陈越改编，人民邮件出版社，2005***