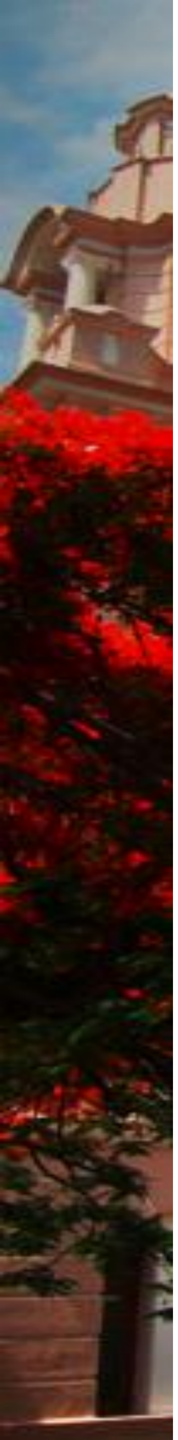
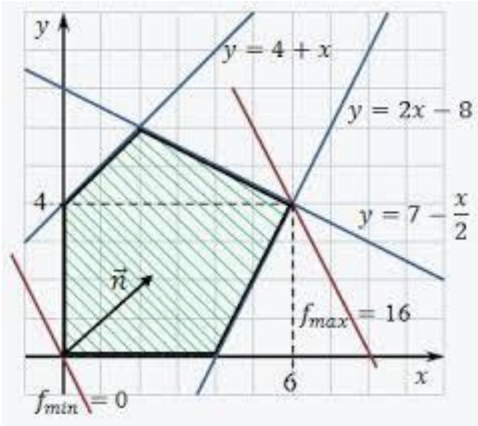
MSBA7003 Quantitative Analysis Methods

05 Linear Programming I



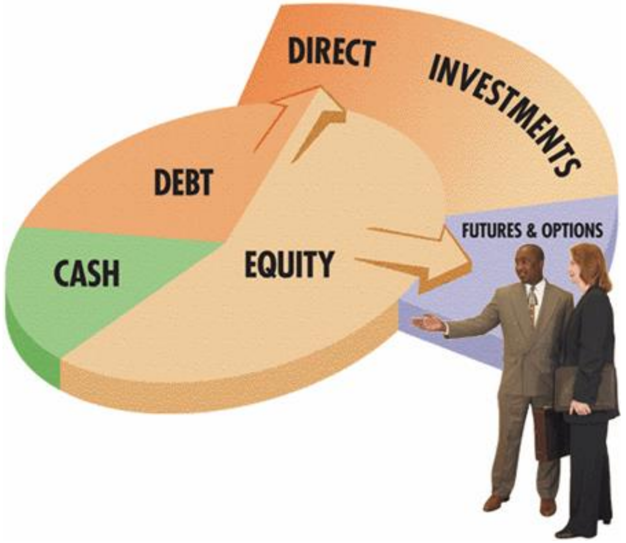
Agenda

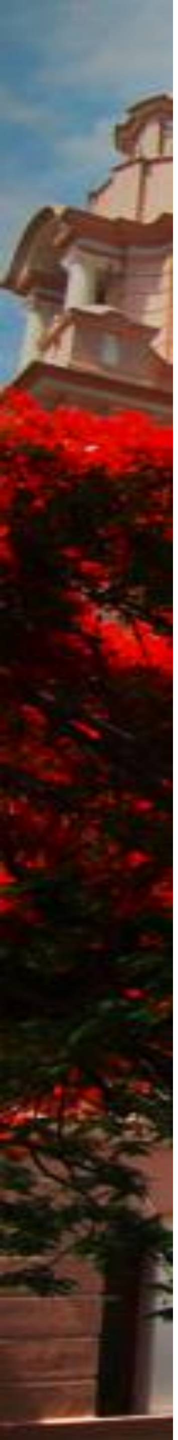
- Formulating a Linear Program
- Graphic Solution Methods
- Sensitivity Analysis
- Solving an LP with Excel Solver
- Solving an LP with Python and CPLEX
- Case: Harrah's Cherokee
- Dealing with Nonlinear functions



Introduction

- Many management decisions involve making the most effective use of limited resources.
  - E.g., space, money, labor, time, machinery, and raw materials
  - There are multiple decisions to make, and they are related by limited resources





Introduction

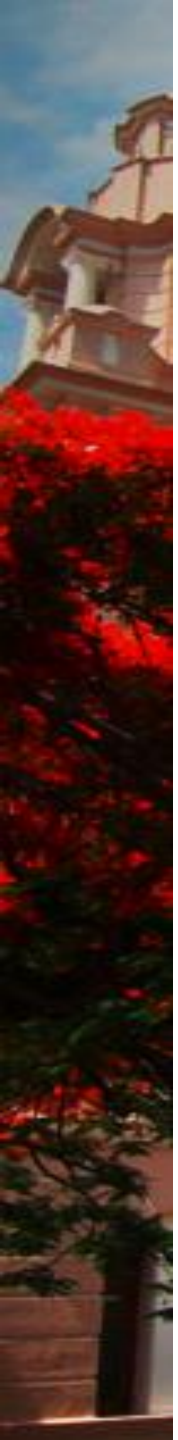
- Linear programming (LP) is a widely used mathematical modeling technique for resource allocation. It has been applied in many areas over the past 70 years

- A typical LP has the following properties
  - Maximizing or minimizing one objective
  - Alternative courses of action
  - One or more constraints
  - Objective function and constraints are linear
  - Parameters (not the decision environment) are certain
  - Nonnegative variables

Flair Furniture Company

- Flair Furniture Company needs to produce 100 tables and 100 chairs each week for contracted clients. Due do limited production capacity in the short term, the company needs to outsource the excessive demand to other firms. The profit margin for an outsourced unit is $10. The details for self-produced products are given in the table below. The company wants to determine the best combination of tables and chairs to produce to reach the maximum profit (A product mix problem).

	HOURS REQUIRED TO PRODUCE 1 UNIT		
DEPARTMENT	(*T*) TABLES	(*C*) CHAIRS	AVAILABLE HOURS THIS WEEK
Carpentry	4	3	240
Painting and varnishing	2	1	100
Profit per unit	$80	$60	



Formulating the LP problem

- The decision variables are

  T = number of tables to be produced per week

  C = number of chairs to be produced per week

- The objective is

  Maximize profit $= \$80T + \$60C + 10(100 - T + 100 - C)$

- The constraints are

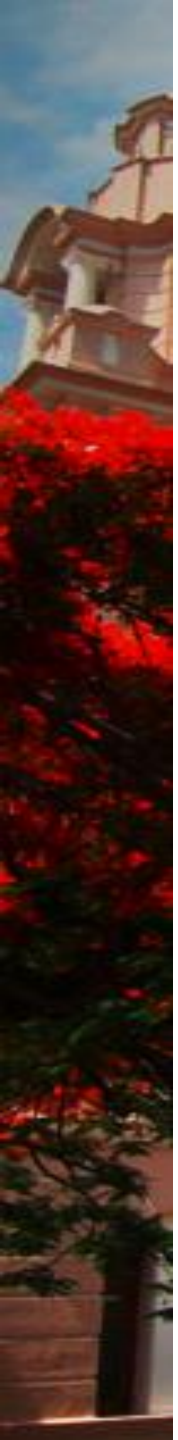
  1. The carpentry hours used cannot exceed 240 hours per week:

     $4T + 3C \leq 240$

  2. The painting and varnishing hours used cannot exceed 100 hours per week:

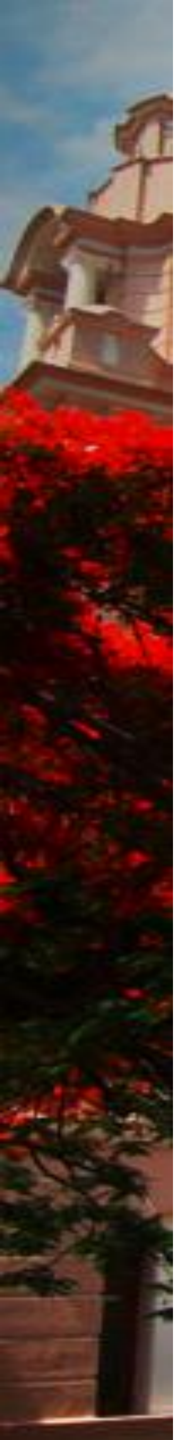
     $2T + 1C \leq 100$

  3. Sales constraints: $T, C \leq 100$
  4. Numbers are non-negative: $T, C \geq 0$



How to Find LP Solutions

- There are a number of ways to solve an LP
- Computer Software (e.g., Excel Solver, CPLEX solver)
- Graphical solution method
  - Isoprofit line solution method
  - Corner point solution method
- Easiest way to solve small LP problems is graphical
  - Provides valuable insight into how problems are solved in general
  - Only works when there are just two decision variables
  - Not possible to plot a solution for more than two variables

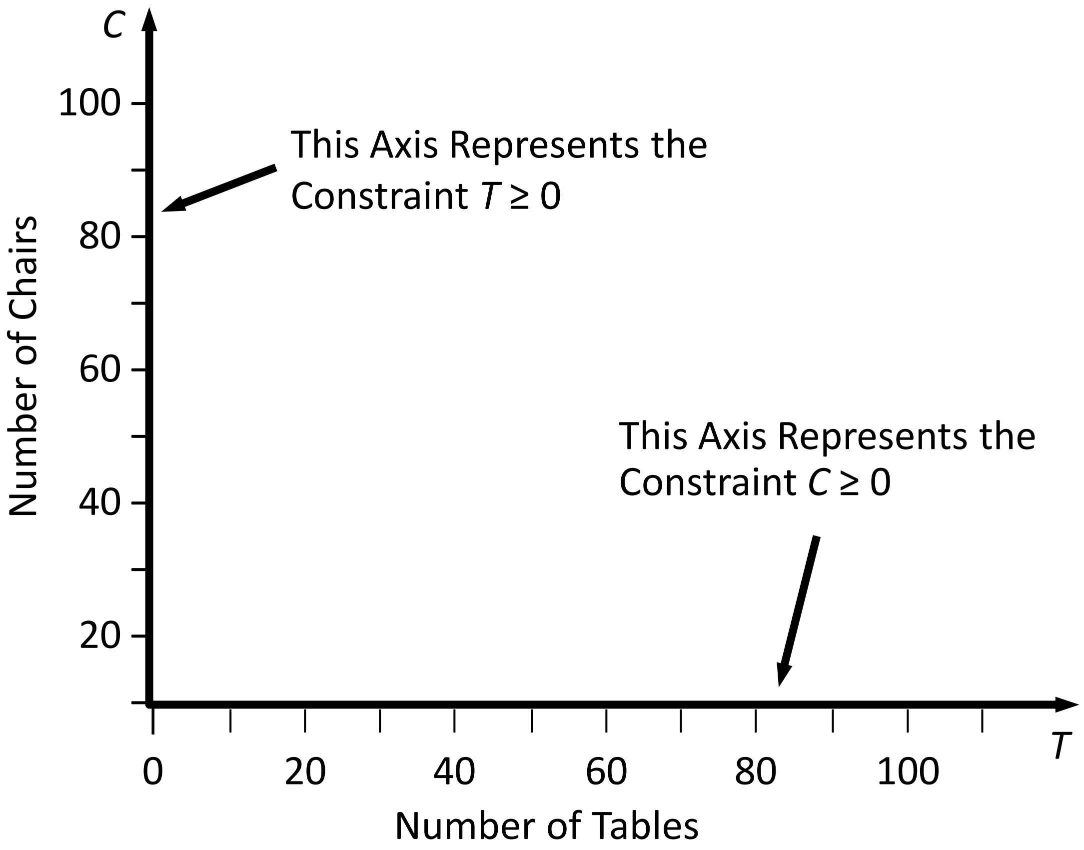


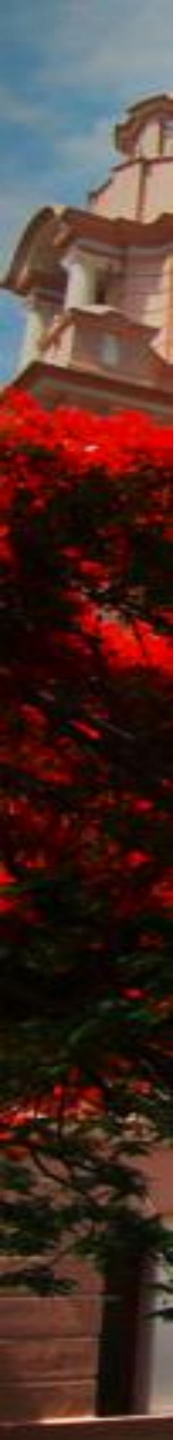
Graphical Solution Method

- The first step is to identify a set or region of feasible solutions by plotting each constraint on a graph and finding the common area.
  - Graph the equality portion of the constraint
    - Solve for the axis intercepts and draw the line
  - Identify the side that satisfies the constraint

- Start with the non-negativity constraints
  - Non-negativity constraints mean that we are always working in the first (or northeast) quadrant of a graph

Graphical Solution Method





Graphical Solution Method

- The carpentry hour constraint: $4T + 3C \leq 240$.
- When Flair produces no tables, the carpentry constraint is:

$$4(0) + 3C = 240$$
$$3C = 240$$
$$C = 80$$

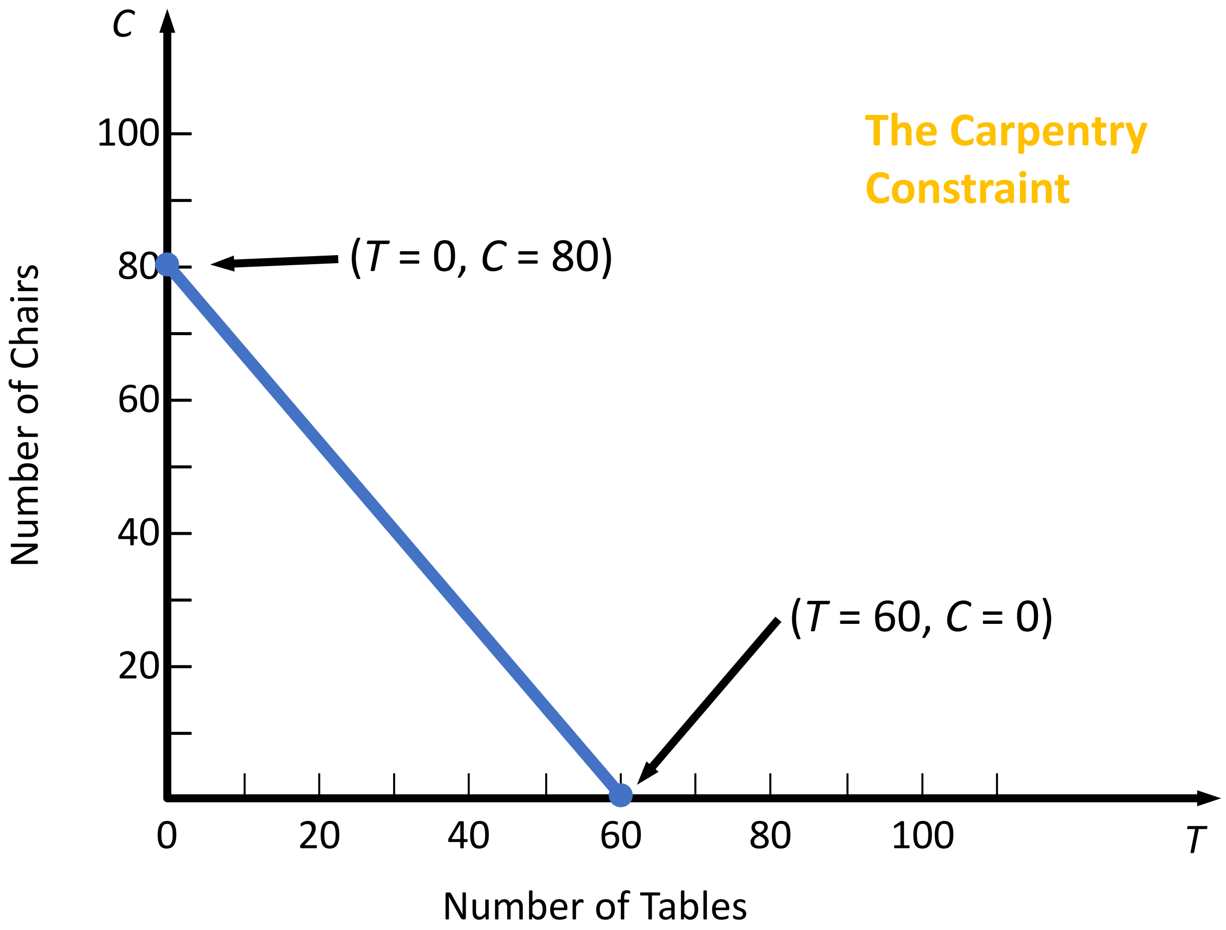
- Similarly for no chairs:

$$4T + 3(0) = 240$$
$$4T = 240$$
$$T = 60$$

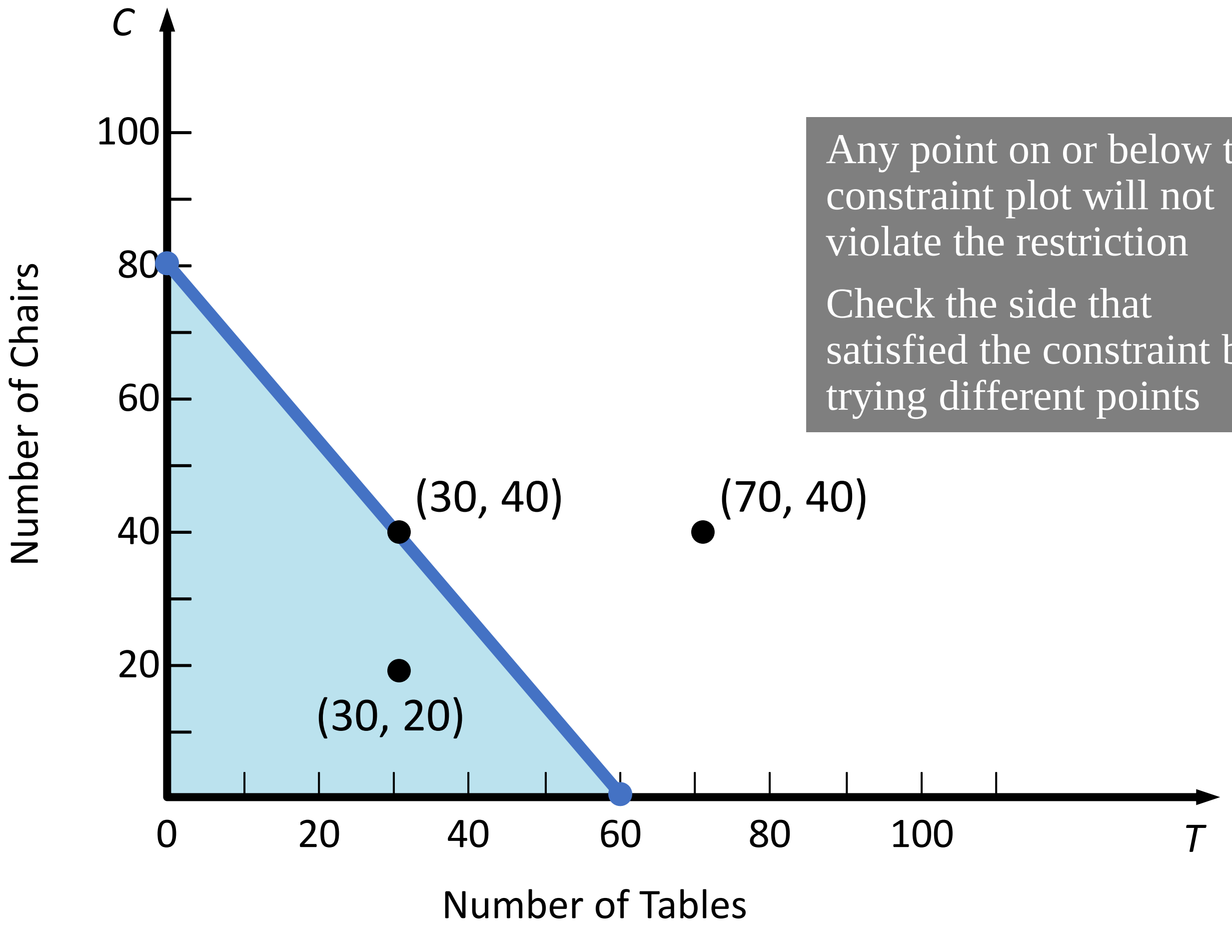
The equality part is a linear function:

$$C = \frac{1}{3}(240 - 4T)$$

Graphical Solution Method



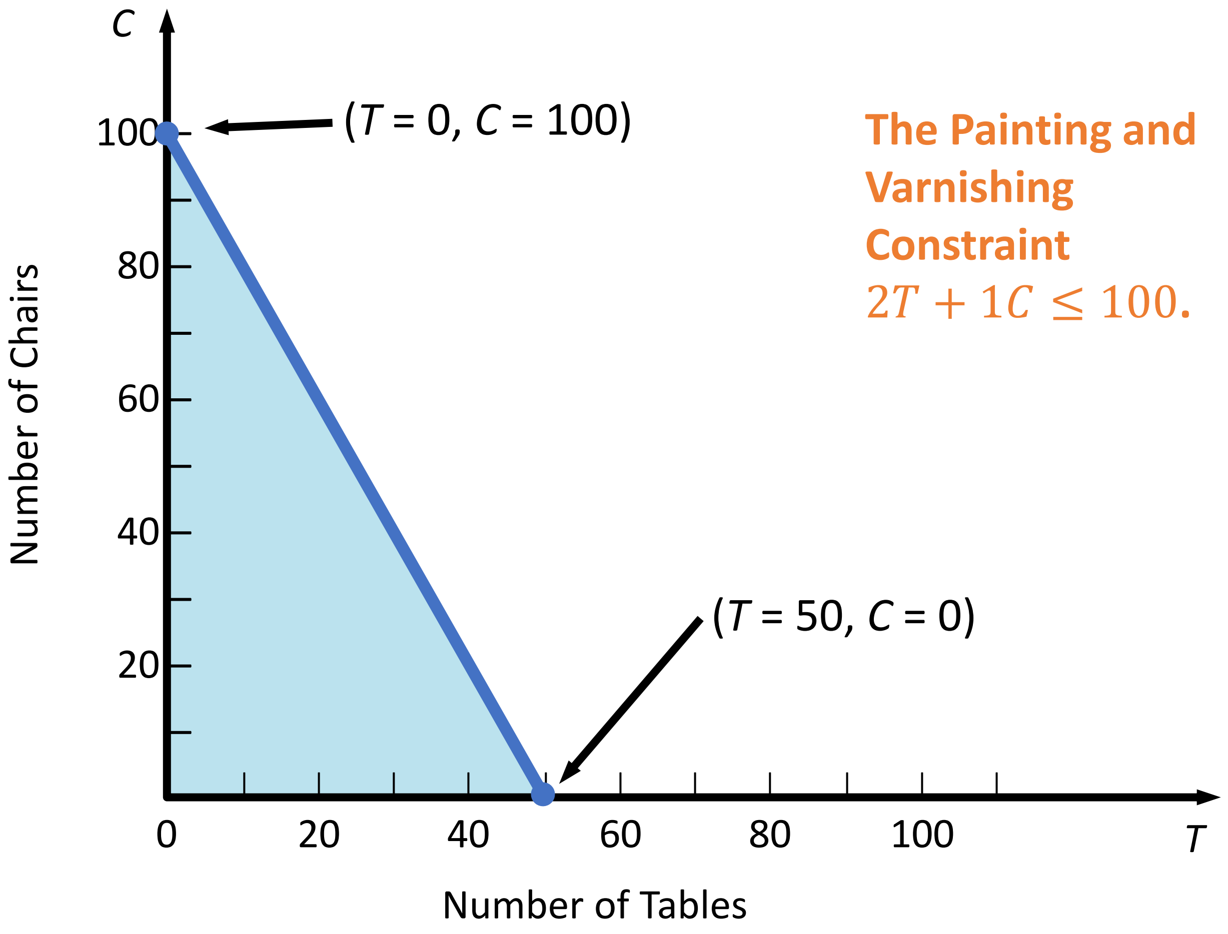
Graphical Solution Method



Any point on or below the constraint plot will not violate the restriction

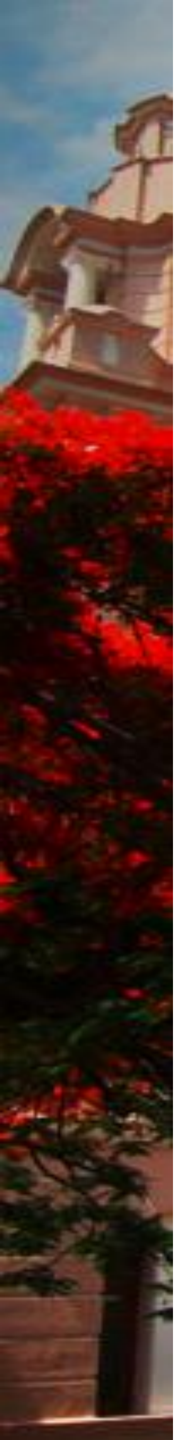
Check the side that satisfied the constraint by trying different points

Graphical Solution Method



The Painting and Varnishing Constraint

$2T + 1C \leq 100.$

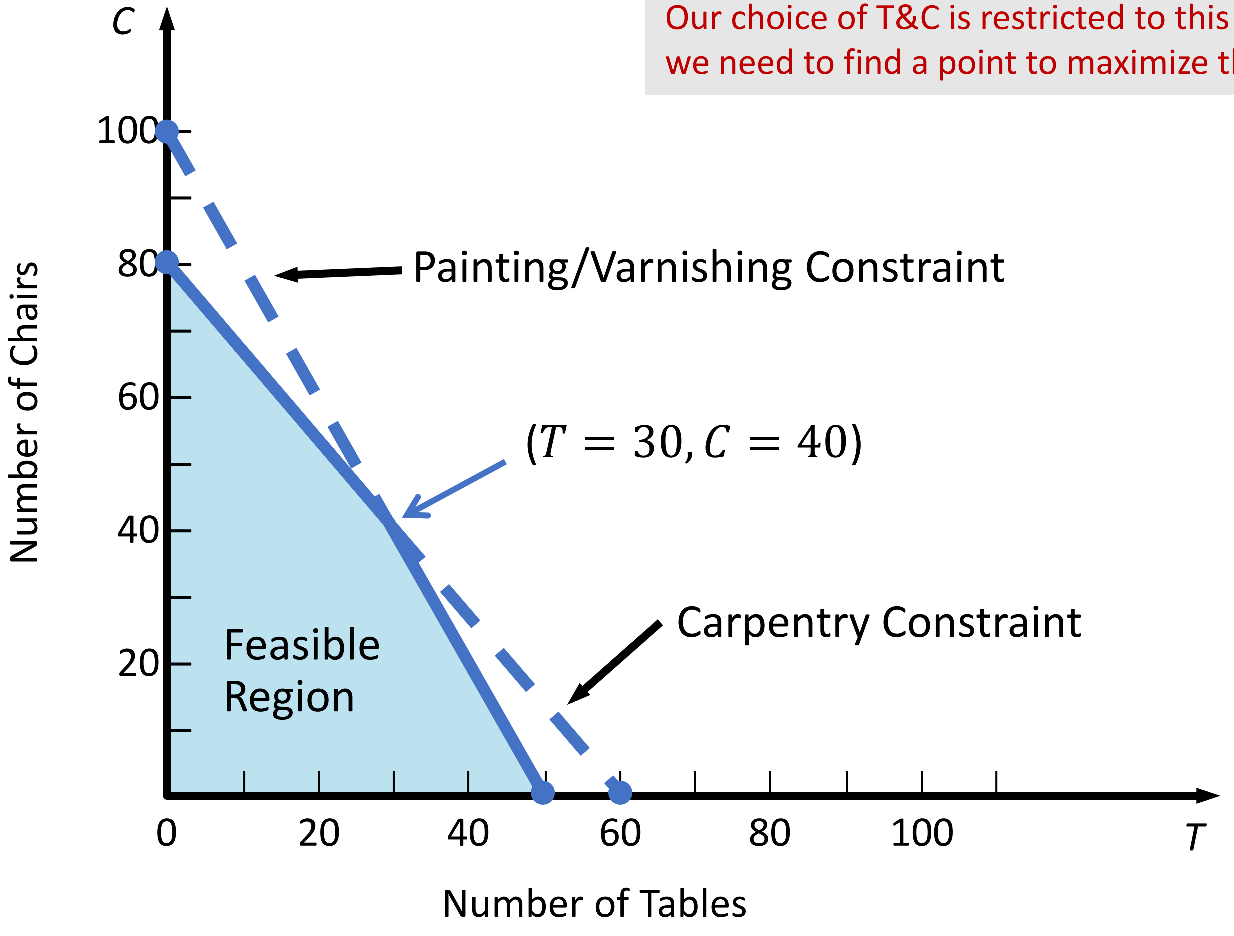


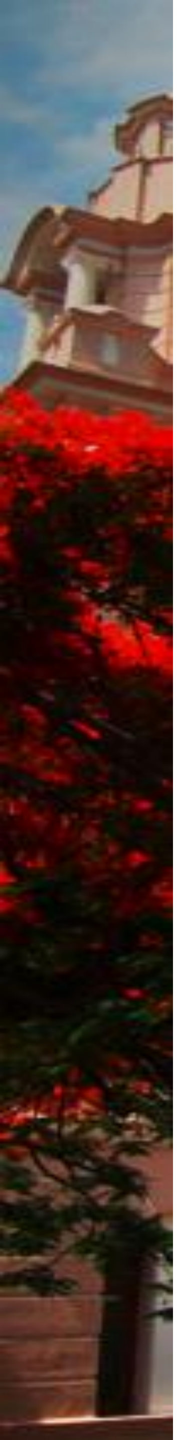
Graphical Solution Method

- To produce tables and chairs, both departments must be used, so a solution should satisfy both constraints, in addition to the non-negativity constraints.
- The feasible region is where all constraints are satisfied
  - Any point inside this region is a feasible solution
  - Any point outside the region is an infeasible solution
- Note that the sales constraints are not relevant (or redundant), because adding them does not change the feasible region.

Graphical Solution Method

Our choice of T&C is restricted to this area, and we need to find a point to maximize the profit.

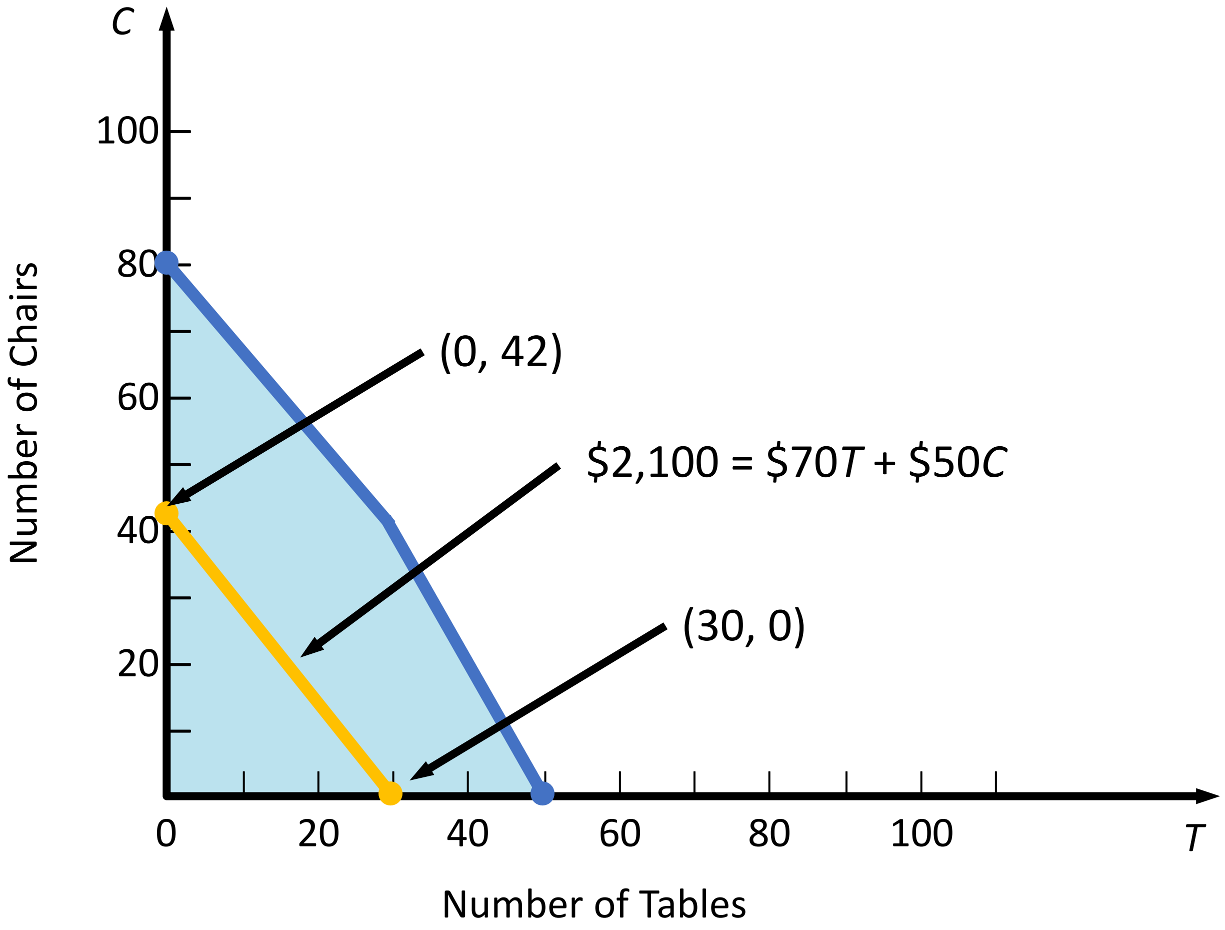




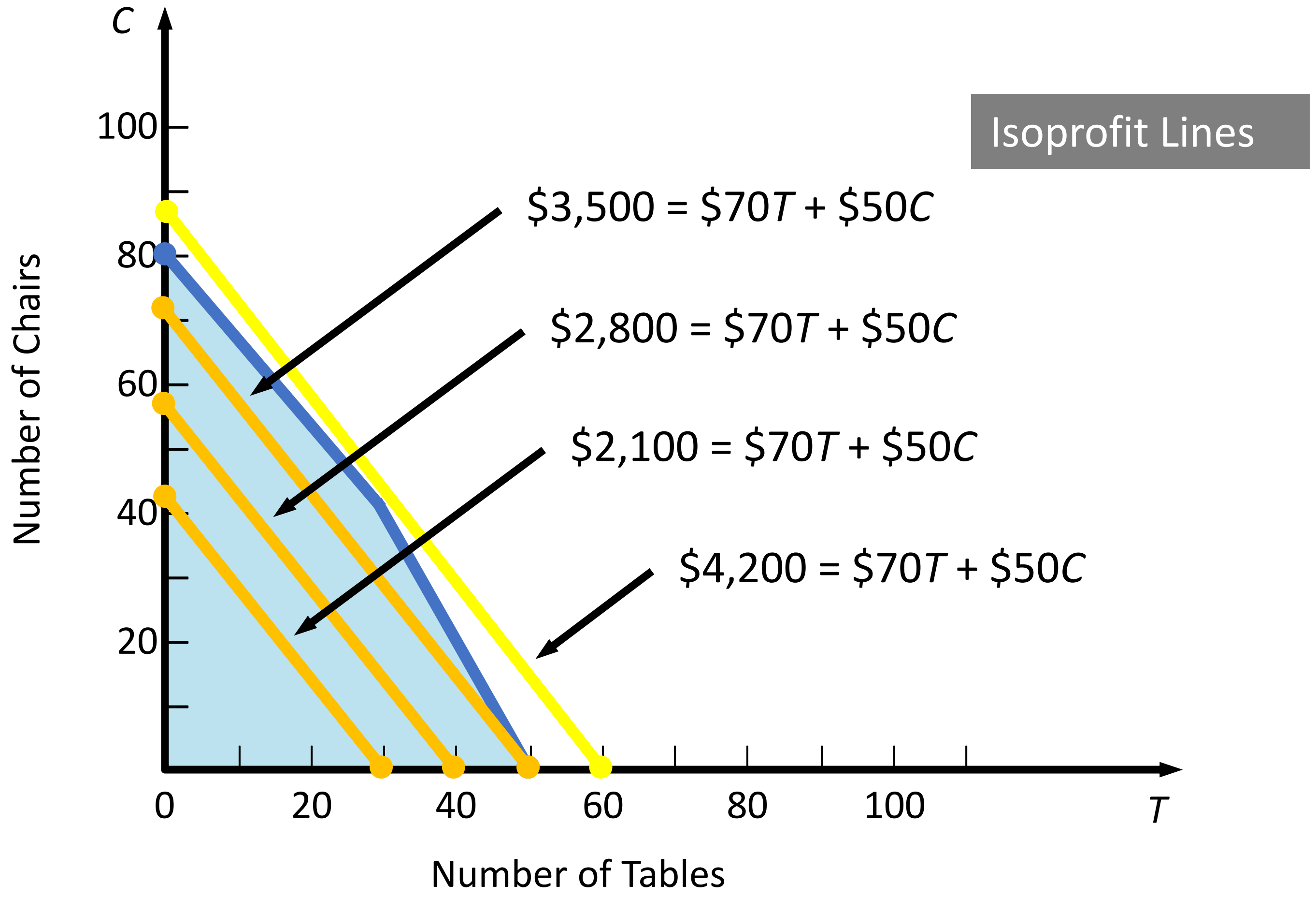
Graphical Solution Method

- Isoprofit line solution method
- How to describe profit in the graph?
- Recall that extra profit $= \$70T + \$50C$
- What does extra profit = $2,100 mean?
- It can be achieved by producing 30 tables or 42 chairs or some combinations of tables and chairs, which is a straight-line segment.

Graphical Solution Method

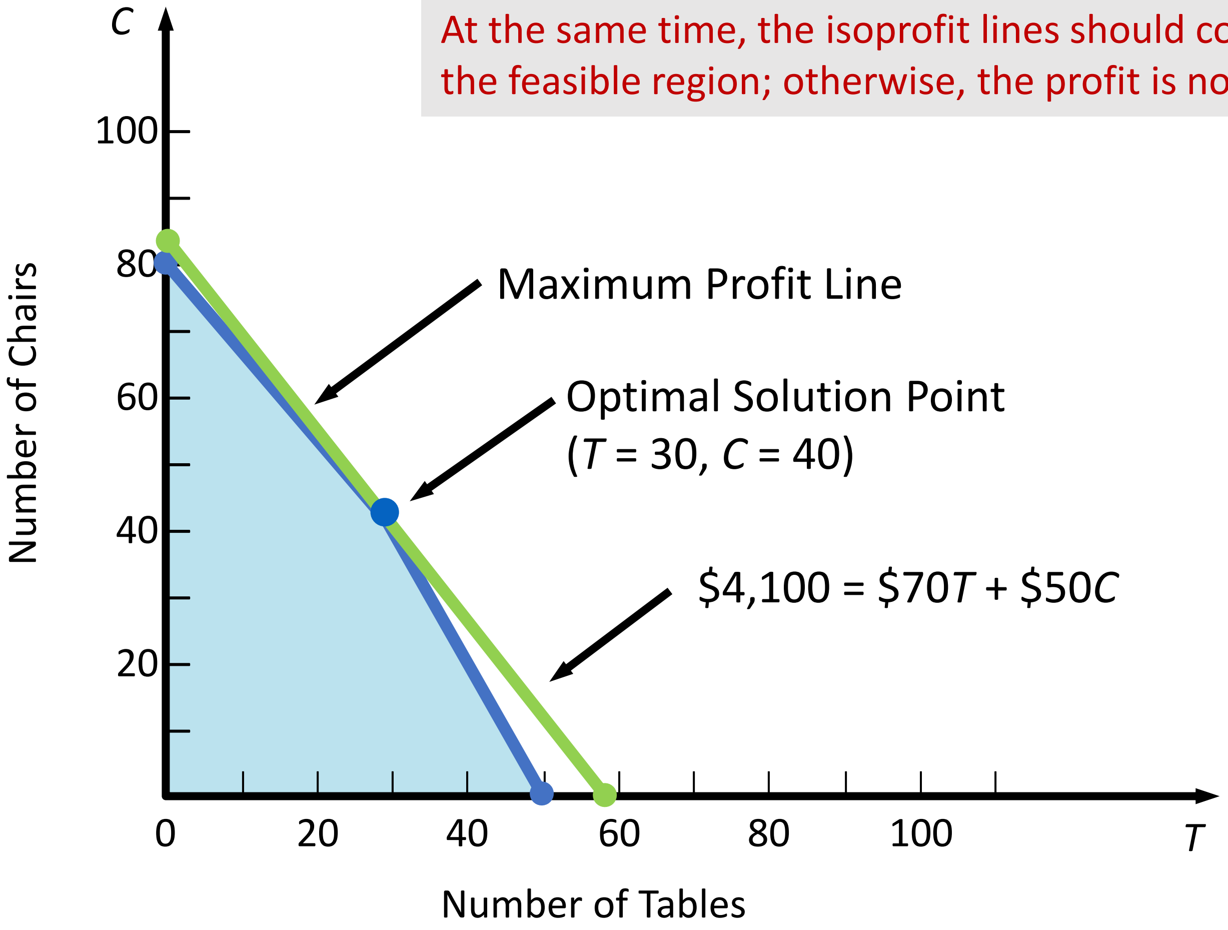


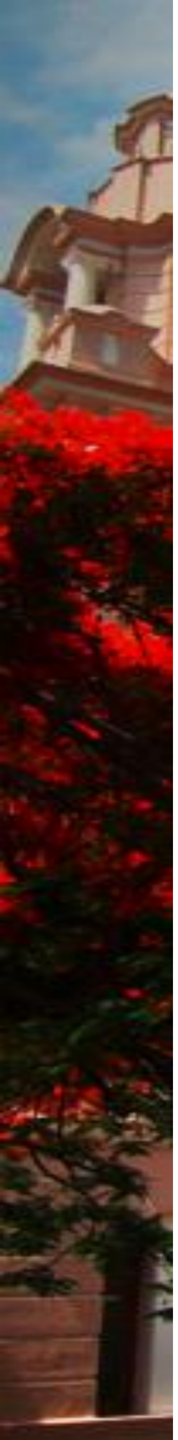
Graphical Solution Method



Graphical Solution Method

Profit maximization = go up as much as possible.
At the same time, the isoprofit lines should contain points in the feasible region; otherwise, the profit is not achievable.

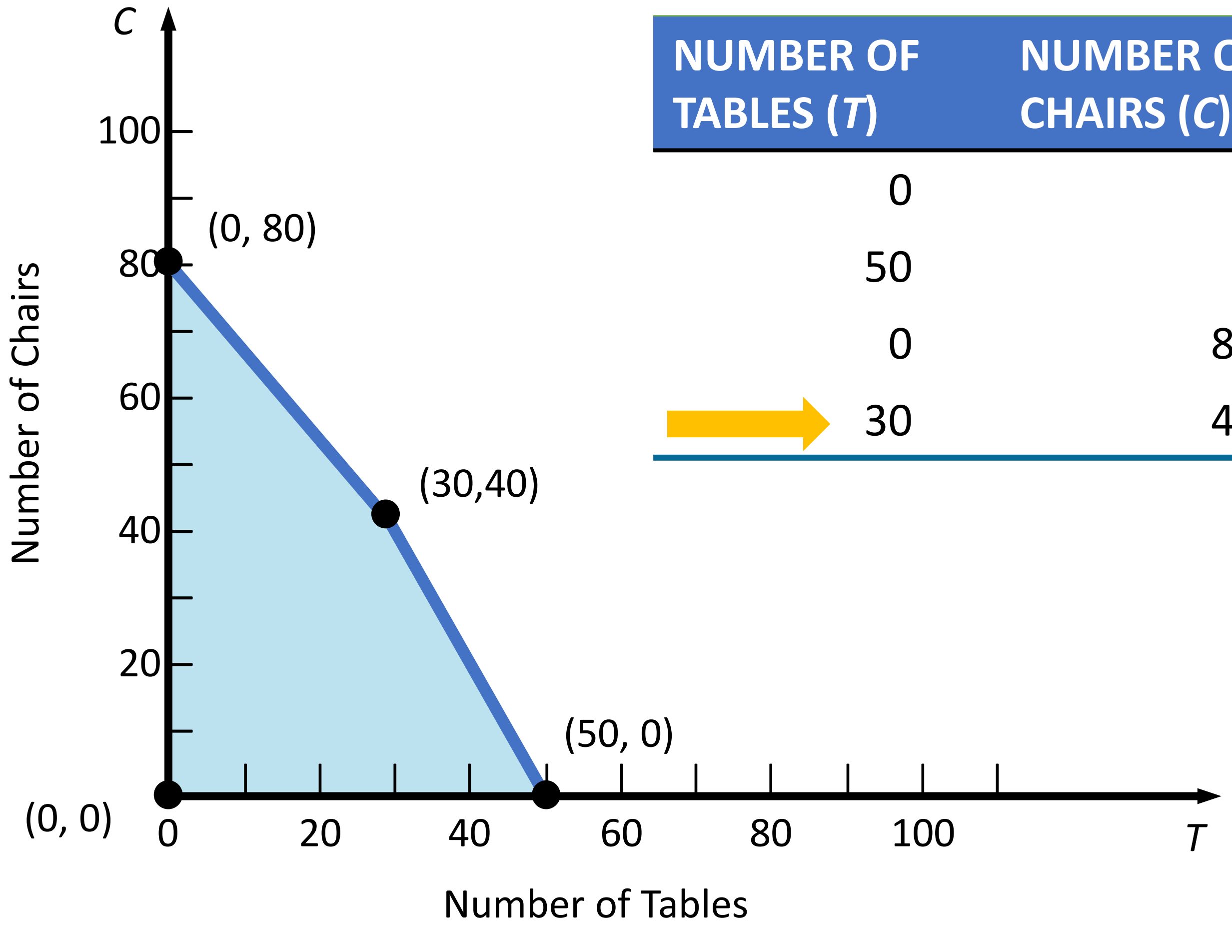




Graphical Solution Method

- Observation: the optimal solution must be achieved by one of the corner points (or vertexes).
- A second approach is corner point solution method: to compare profits achieved by different corner points.
- Remember: This is useful when there are only two decision variables. There will be too many corner points when there are too many variables or constraints.

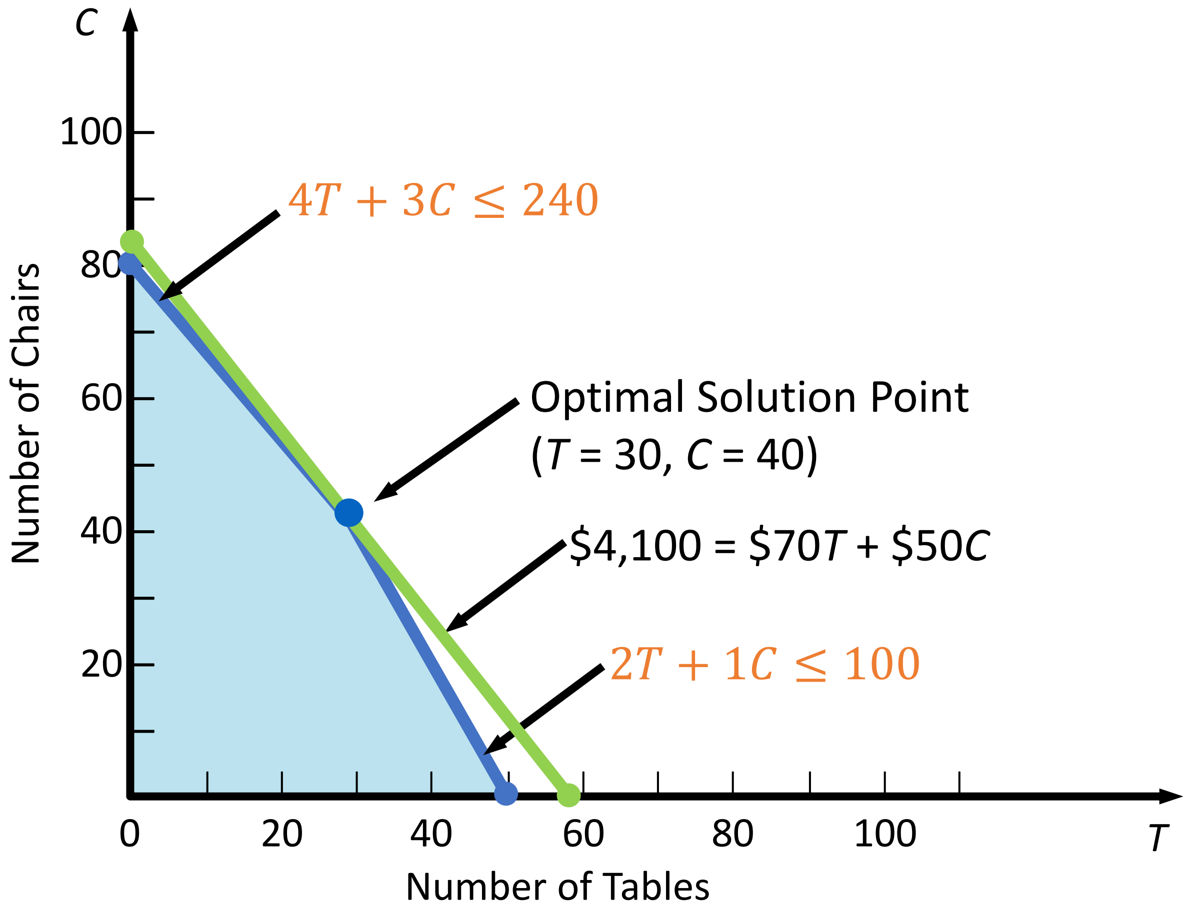
Graphical Solution Method

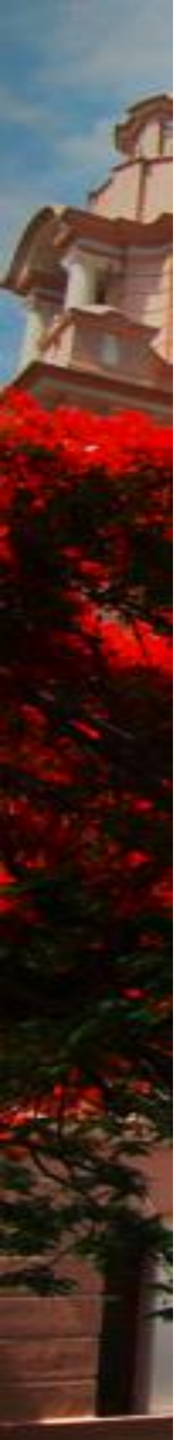


NUMBER OF TABLES (*T*)	NUMBER OF CHAIRS (*C*)	PROFIT = \$70*T* + \$50*C*
0	0	\$0
50	0	\$3,500
0	80	\$4,000
30	40	\$4,100

Sensitivity Analysis

- Note: we say a constraint is <u>binding</u> if the optimal solution lies on the boundary of this constraint.
- Changes in objective function coefficients
  - The iso-profit line rotates
- Changes in the technological coefficients
  - The boundary of a constraint rotates
- Changes in the resources or right-hand-side values
  - The boundary of a constraint is shifted parallelly

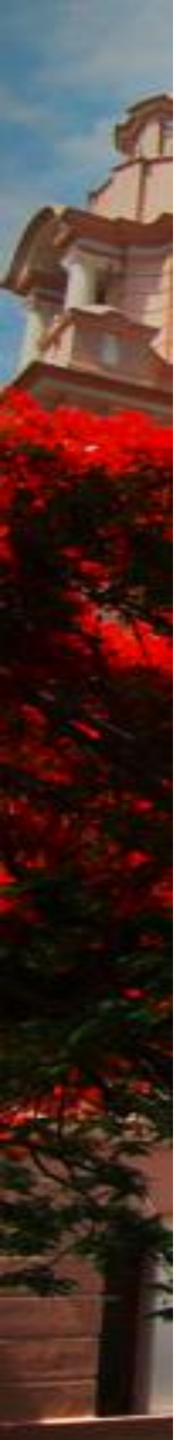




Shadow Price

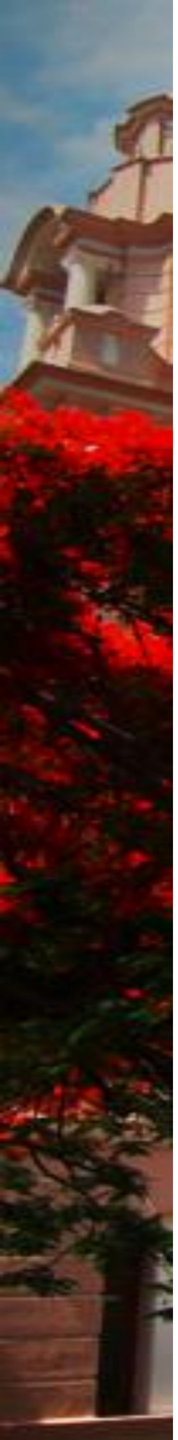
- Shadow price of a resource: If we increase the amount of resource (the right-hand-side constant) by one, the marginal impact on the optimal objective value.

- What if Flair has one more carpentry hour?
  - The maximum extra profit under the original condition: \$4,100
  - The maximum extra profit under the new condition: \$4,115
  - The weekly profit (objective) is increased by \$15
- If Flair outsources its carpentry hours, then the minimum price to charge is \$15.
- The shadow price of carpentry hours is \$15.



In-Class Exercise

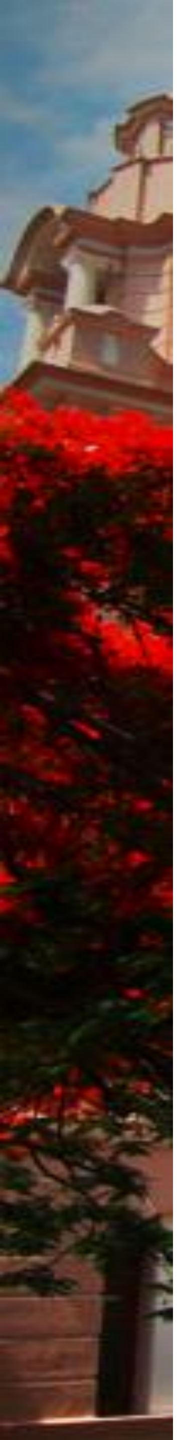
- If the profit of a chair is the same as a table, what is the shadow price of painting and varnishing hours?



Four Special Cases in LP

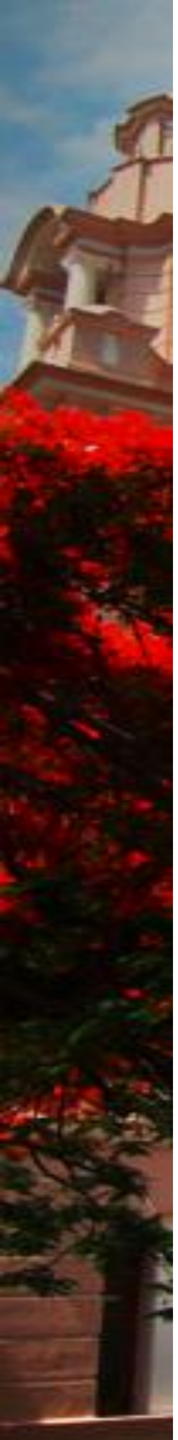
If you have no feasible solution or unbounded solution, in most cases your model is wrong.

- No feasible solution
  - Check the correctness of constraints
- Unboundedness
  - Check the choice of minimization or maximization
- Redundancy
  - The feasible region will not change if the constraint is dropped
  - Redundant constraints can be dropped without affecting the optimal solution
  - The resource corresponding to a redundant constraint has zero shadow price
- Multiple optimal solutions



In-class Exercise

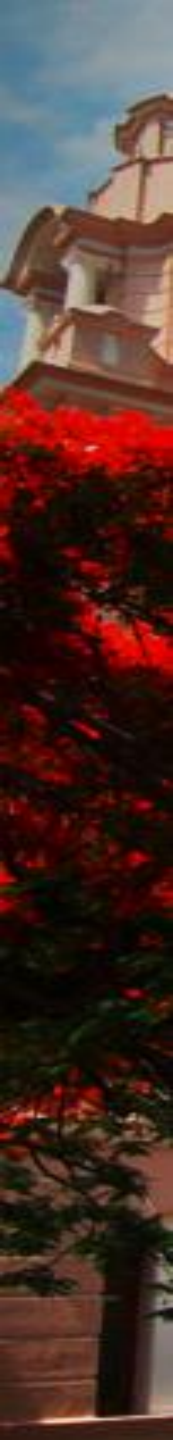
- Solve the following problem, and answer true/false questions:
- Max $6X + 3Y$
- $2X + Y \leq 10$
- $3X + 7Y \leq 42$
- $X \leq 6$
- $Y \leq 4$
- $X, Y \geq 0$
- Statement 1: There are two redundant constraints.
- Statement 2: There are more than one optimal solution.



Excel Solver

- Software solves LP with the same logic. The standard method is called the simplex method. Computer finds a corner point first and check the optimality. If optimal, stop; otherwise, move along the edge that increases the profit the most to the next corner point.
- You will solve almost all your LP problems in Excel!
- Step 1: Setup the framework and enter the data in Excel.

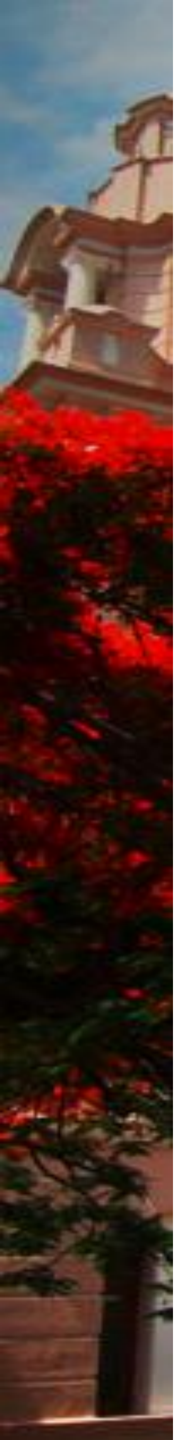
	A	B	C	D	E	F	G	H
1	**Flair Furniture**							
2								
3	Variables	T (Tables)	C (Chairs)		Total Profit			
4	Units Produced	0	0		0			
5	Profit per unit	70	50		Hours Used		Constraint	
6	Carpentry hours	4	3		0	≤	240	
7	P&V hours	2	1		0	≤	100	
8								



Excel Solver

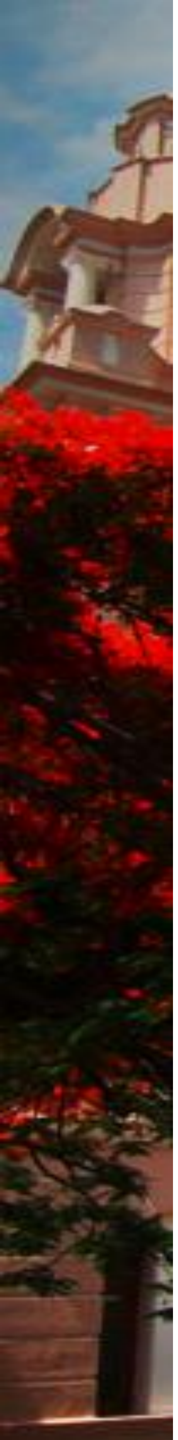
- Step 2: Write the formulas for the objective function and the constraints.

	A	B	C	D	E	F	G
1	**Flair Furniture**						
2							
3	Variables	T (Tables)	C (Chairs)		Total Profit		
4	Units Produced	0	0		=SUMPRODUCT(B4:C4,B5:C5)		
5	Profit per unit	70	50		Hours Used		Constraint
6	Carpentry hours	4	3		=SUMPRODUCT(B6:C6,B4:C4)	≤	240
7	P&V hours	2	1		=SUMPRODUCT(B7:C7,B4:C4)	≤	100



Excel Solver

- Step 3: Using Solver.
  - *Set Objective*: the pink cell
  - *By Changing Variable Cells*: the yellow cells
  - Click *Max* or *Min* for the problem.
  - Check the box for *Make Unconstrained Variables Non-Negative*.
  - Click *Select a Solving Method* and select *Simplex LP*.
  - Click *Add* to add the constraints.
  - In the *Cell Reference* box, input the golden cells.
  - Select <= or other appropriate relationships.
  - In the *Constraint* box, input the cells on the right-hand side.
  - Click *Solve*. In the *Solver Results* dialog box, select the reports you need and click OK.



Excel Solver: Sensitivity Report

Units Produced

Coefficient may change by these amounts and the optimal solution will not change.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B4	Units Produced T (Tables)	30	0	70	30	3.333333333
C4	Units Produced C (Chairs)	40	0	50	2.5	15

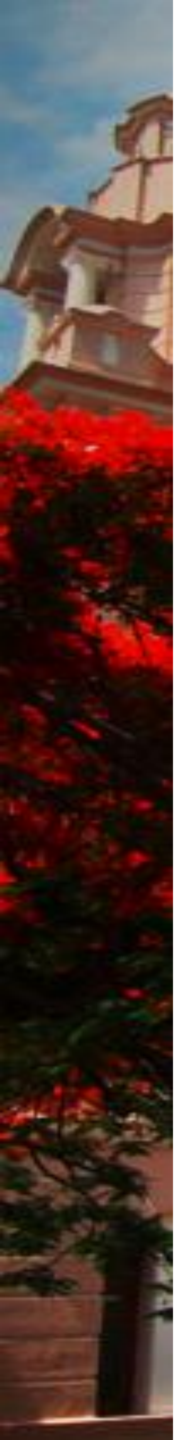
Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E6	Carpentry hours Hours Used	240	15	240	60	40
E7	P&V hours Hours Used	100	5	100	20	20

Resource may change by these amounts and the shadow price will not change.

Hours Used

If we add one more unit to the resource, the marginal impact on the optimal profit



Python + PuLP

```python
Products = ['Table','Chair']
Profits = {'Table':70,'Chair':50}
CarpentryHours = {'Table':4,'Chair':3}
PnVHours = {'Table':2,'Chair':1}
Resources = ['Carpentry','PnV']
ResHours = {'Carpentry':240,'PnV':100}

from pulp import *

prob = LpProblem("Flair_Furniture",LpMaximize)
production = LpVariable.dicts("Quantity",Products,lowBound=0)
prob += lpSum([production[i]*Profits[i] for i in Products]), "Total_Profit"
prob += lpSum([production[i]*CarpentryHours[i]
            for i in Products]) <= ResHours['Carpentry']
prob += lpSum([production[i]*PnVHours[i]
            for i in Products]) <= ResHours['PnV']
prob.solve()
print("The total profit =", value(prob.objective))
```

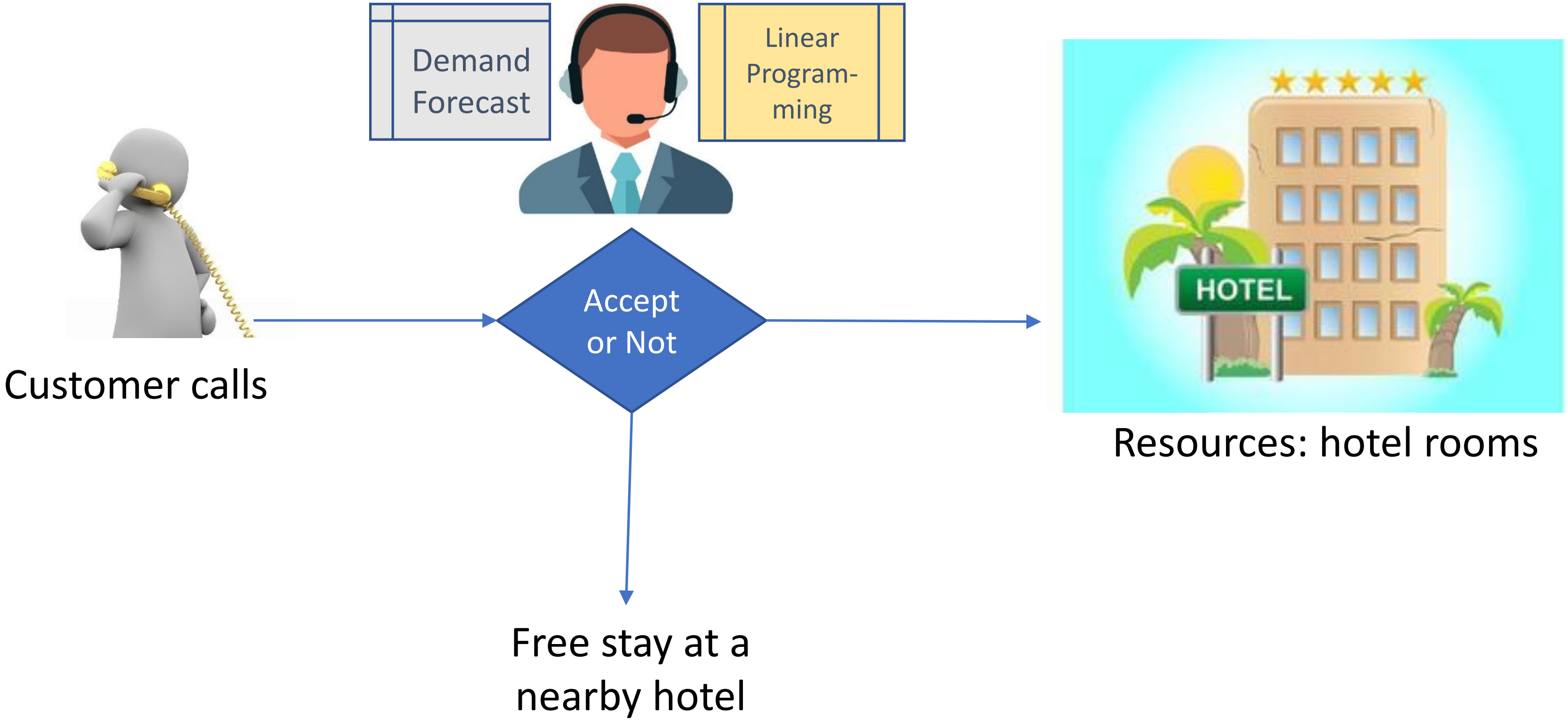
Variable names
Coefficients
Constraint names
Right-hand-side values

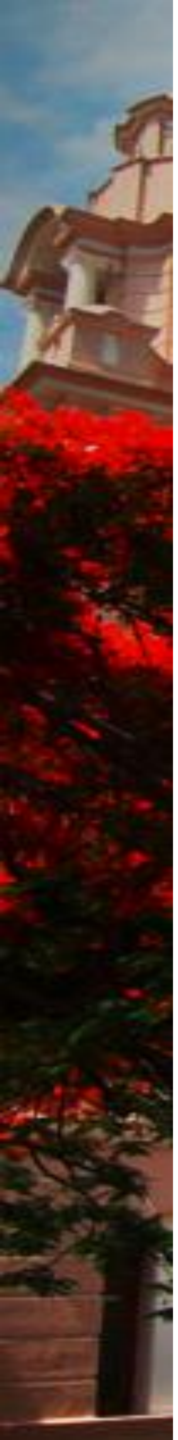
Python + CPLEX / DOCPLEX

```
import cplex
prob = cplex.Cplex()
prob.objective.set_sense(prob.objective.sense.maximize)
prob.variables.add(names=["nTable", "nChair"],
            obj=[70, 50])
prob.linear_constraints.add(names=["Carpentry_Constraint","P&V_Constraint"],
                rhs=[240,100],
                senses="LL")
rows=[0,0,1,1]
cols=[0,1,0,1]
vals=[4,3,2,1]
prob.linear_constraints.set_coefficients(zip(rows,cols,vals))
prob.solve()
```

```
from docplex.mp.model import Model

Problem = Model(name='FlairFurniture')

x1=Problem.continuous_var(lb=0,name='nTable')
x2=Problem.continuous_var(lb=0,name='nChair')

Problem.maximize(70*x1+50*x2)

Problem.add_constraint(4*x1+3*x2<=240)
Problem.add_constraint(2*x1+x2<=100)

Solution=Problem.solve()
Solution.display()
```

Harrah's Cherokee

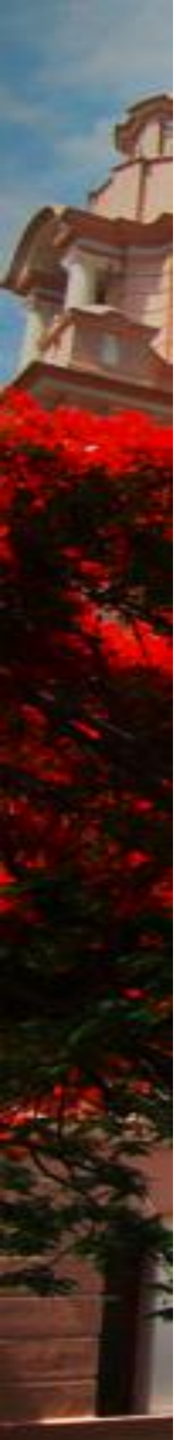




Harrah's Cherokee

- Simplified model:
- The focal customer stays for one night.
- Customers are divided into K groups; there are in total N rooms.
- r_i: the average nightly profit contribution of a customer in group i.
- d_i: the projected demand from group i.
- X_i: the number of room allocated to group i.
- Constraints: 1) the allocated number cannot exceed the projected demand for each group; 2) the allocated number cannot be negative; 3) the total allocated number cannot exceed the total number of rooms.

$$\begin{aligned}
\max \quad & \text{Total Revenue} = \sum_{i=1}^{K} r_i X_i \\
\text{s.t.} \quad & X_i \le d_i, \quad i = 1, \ldots, K, \\
& X_i \ge 0, \quad i = 1, \ldots, K, \\
& \sum_{i=1}^{K} X_i \le \text{Capacity}.
\end{aligned}$$

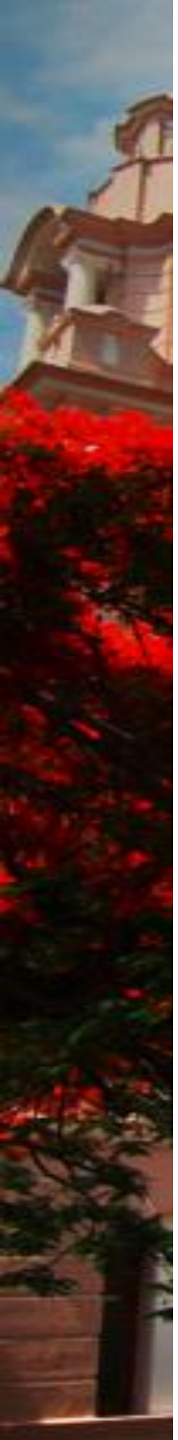


Harrah's Cherokee

- <u>Assume the focal customer books for 2 days</u>
- Parameters added:
- d_i^{one}: projected demand from group i on the first focal day
- d_i^{two}: projected demand from group i on the second focal day
- Decision variables:
- X_i: no. of rooms allocated to group i on the first focal day
- Y_i: no. of rooms allocated to group i on the second focal day

- n: the total no. of rooms
- Objective: MAX Two-Day Profit $= r_1(X_1 + Y_1) + \cdots + r_K(X_K + Y_K)$
- Constraint 1: $X_1 + \cdots + X_K \leq n$,
- Constraint 2: $Y_1 + \cdots + Y_K \leq n$,
- Constraint 3: $X_i \leq d_i^{one}, \quad i = 1, \ldots, K$,
- Constraint 4: $Y_i \leq d_i^{two}, \quad i = 1, \ldots, K$,
- Constraint 5: $X_i, Y_i \geq 0, \quad i = 1, \ldots, K$,
- Constraint 6: $n \leq N$

The shadow price of this constraint will determine whether to accept the booking



Harrah's Cherokee

	A	B	C	D	E	F	G	H	I	J
1	Extended Model: The focal customer is booking for two days									
2			Day 1	Day 1	Day 2	Day 2		Profit		
3	Group	Exp. Prof.	Proj. Dem.	Allocation	Proj. Dem.	Allocation		Day 1	Day 2	Total Profit
4	0	1000	119	119	98	98		372600	339700	712300
5	1	800	128	128	114	114				
6	2	600	126	126	110	110		Calculate the shadow price:		
7	3	400	150	150	102	102		No. Rooms	576	575
8	4	300	155	52	135	135		Total Prof.	712800	712300
9	5	200	168	0	152	16		Shadow Price	500	
10	6	100	144	0	124	0			(Total value for 2 days)	
11	7	50	103	0	110	0				
12	8	0	92	0	89	0				
13	9	0	45	0	33	0				
14			Total	575		575		Rooms for Allocation		575
15				<=		<=				<=
16		Rooms for Allocation		575		575		Total No. Rooms		575

Dealing with Nonlinear Functions

- Sometimes, nonlinear objective functions and constraint functions can be reformulated as linear ones in an LP.

Maximize $\min\{X, Y\}$

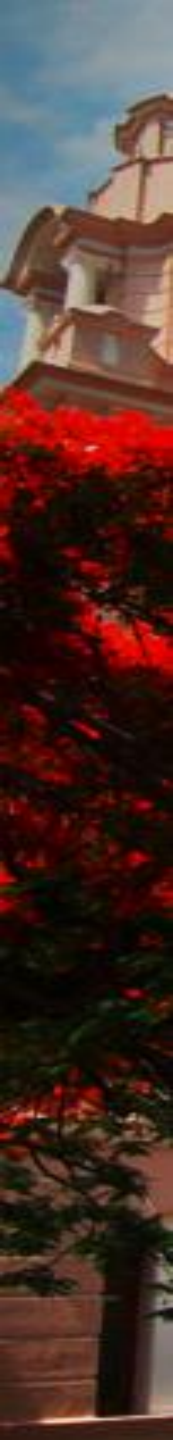
$\Downarrow$

Maximize Z
Subject to: $Z \leq X$ and $Z \leq Y$

Maximize $2X + 3Y$
Subject to: $2X + Y \leq \min\{X, Y - X\}$

$\Downarrow$

Maximize $2X + 3Y$
Subject to:
$2X + Y \leq X$ and $2X + Y \leq Y - X$



In-Class Exercise

- The Hong Kong Family Office is instructed by a client to invest \$250,000 among the following five asset classes. The market has three possible scenarios and the asset classes will have different returns.
- How to allocate the money such that worst case return is maximized?

Investment Return (%)	Class 1	Class 2	Class 3	Class 4	Class 5
Scenario 1	5	7	5	8	10
Scenario 2	5	6	3	12	15
Scenario 3	5	8	8	4	-5