MSBA7003 Quantitative Analysis Methods

Assignment 3 (Due October 2 at 23:55; Please submit with the template)

Q1.

Jimmy, an aircraft engineer, is designing the passenger cabin for a new aircraft model. This model will be used by an airline for a specific route. Jimmy needs to decide on the number of seats for both business class and economy class. There must be 4 seats per row in business class; the seat length is 1.2 length units; the seat weight is 2.2 weight units. In economy class, there must be 6 seats per row; the seat length is 1 unit; the seat weight is 1 weight unit. The total cabin length is 32 length units, and the seat weight limit is 200 weight units. Additionally, due to limited demand, business class cannot have more than 6 rows. Jimmy's optimization goal is to maximize the expected profit of the aircraft model for its entire lifecycle on this route. It is known that the profit margin of business class seats is twice that of economy class seats.

Which of the following statement(s) is(are) true?

A) In the optimal design plan, there should be 4 rows for business class and 27 rows for economy class.

B) In the optimal design plan, there should be 5 rows for business class and 26 rows for economy class.

C) In the following problem formulation given by GPT4, constraints 1 – 3 are all wrong.

GPT4 (Tokens used 750)

To solve this problem using linear programming, we first need to define the decision variables, constraints, and the objective function.

Let:
x = number of rows of business class seats
y = number of rows of economy class seats

Constraints:

1. Business class seat length constraint: 1.2 * 4x ≤ 32
2. Economy class seat length constraint: 1 * 6y ≤ 32
3. Total length constraint: 4.8x + 6y ≤ 32
4. Weight constraint: 2.2 * 4x + 1 * 6y ≤ 200
5. Demand constraint: x ≤ 6
6. Non-negativity constraints: x ≥ 0, y ≥ 0

Objective function:
Maximize profit: P = 2 * (4x) + 1 * (6y), where the profit from a business class seat is twice that of an economy class seat.

Now, we can solve this linear programming problem using a solver or any appropriate method.

D) The total weight constraint is not binding.

E) None of the above.

Notes: let x, y denote the # of rows of business class, economy class, respectively. We have the following formulation:

$$\max \quad (2*4)x + 6y$$
$$1.2x + y \leq 32$$
$$(2.2*4)x + 6y \leq 200$$
$$x \leq 6$$
$$x, y \geq 0$$

We use excel to build and solve the model as follows.

	# of business	# of economy			
Decision Variables	5	26			196
coeff.	8	6			
Constratints	1.2	1	32	<=	32
	8.8	6	200	<=	200
	1		5	<=	6

Hence, B and C are True.

(Please refer to "Assignment_3_Solutions")

Q2.

The Heinlein and Krampf Brokerage firm is instructed by a client to invest $250,000 among five possible asset classes as shown in the table below. The return of each security is uncertain and depends on the market scenario. The client listed the following guidelines:

(1) Investment Grade Bonds should constitute at least 20% of the invested value;
(2) At least 40% of the invested value should be placed in a combination of international equity and S&P 500;
(3) No more than 80% of the amount invested in bonds should be placed in hedge funds;
(4) No short selling, and it is okay to hold some cash.

Projected Return (%)	in Scenario 1	In Scenario 2
Investment Grade Bonds	1.1	-2.3
Commodities	-5.0	6.8
Hedge Funds	-1.7	4.9
International Equity	-0.3	8.4
S&P 500	-7.5	11.8

The objective is to maximize the worse-case return. Which of the following is(are) true?

A) It is optimal to hold some cash.

B) In the optimal solution, the investment in S&P 500 is zero.

C) In the optimal solution, the worst-case expected return is $1,350.

D) In the optimal solution, the return for an additional invested dollar is $0.0054.

E) None of the above.

Note: Define $\boldsymbol{x_i}$ as the amount of money allocated to class i. Denote as $\boldsymbol{r_{ij}}$ as the return rate of class i in scenario j.

The worst-case payoff is $\mathbf{min}\{\sum x_i r_{i1}, \sum x_i r_{i2}\}$. Hence, the LP can be formulated as follows

max t

$$
\begin{aligned}
\text{s.t.}\quad & t \le 1.1x_1 - 5.0x_2 - 1.7x_3 - 0.3x_4 - 7.5x_5 \\
& t \le -2.3x_1 + 6.8x_2 + 4.9x_3 + 8.4x_4 + 11.8x_5 \\
& x_1 \ge 20\%(x_1 + x_2 + x_3 + x_4 + x_5) \\
& x_4 + x_5 \ge 40\%(x_1 + x_2 + x_3 + x_4 + x_5) \\
& x_3 \le 80\% x_1 \\
& x_1 + x_2 + x_3 + x_4 + x_5 \le 250{,}000 \\
& x_1, x_2, x_3, x_4, x_5 \ge 0
\end{aligned}
$$

We use excel to build and solve the model as follows.

Investment	Investment Grade Bonds	Commodities	Hedge Funds	International Equity	S&P 500			
Scer.1-Projected Return (%)	0.011	-0.05	-0.017	-0.003	-0.075	1350		
Scer.2-Projected Return (%)	-0.023	0.068	0.049	0.084	0.118	4950		
Amount	150000	0	0	100000	0			
Total Return	1350							
Constraints								
	1	1	1	1	1	250000	<=	250000
	1					150000	>=	50000
				1	1	100000	>=	100000
			1			0	<=	120000

According to the optimal solutions, B) and C) is True. And according to the answer report, the constraint of G9 is binding, so it is optimal to invest all money; A) is False.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B6	Total Return Investment Grade Bonds	1350	0	1	1E+30	1
B4	Amount Investment Grade Bonds	150000	0	0	1E+30	0.009
C4	Amount Commodities	0	-0.061	0	0.061	1E+30
D4	Amount Hedge Funds	0	-0.028	0	0.028	1E+30
E4	Amount International Equity	100000	0	0	0.014	0.0135
F4	Amount S&P 500	0	-0.072	0	0.072	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B6	Total Return Investment Grade Bonds	1350	1	0	3600	1350
B6	Total Return Investment Grade Bonds	1350	0	0	1E+30	3600
G10		150000	0	0	100000	1E+30
G11		100000	-0.014	0	96428.57143	29752.06612
G12		0	0	0	1E+30	120000
G9		250000	0.0054	250000	1E+30	250000

Then from the sensitivity report, we can obtain that the Shadow Price for the constraints G9 (total investment) is equal to 0.0054; hence, D) is True.

(Please refer to "Assignment_3_Solutions")

Q3.

The Salem Board of Education wants to evaluate the efficiency of the town's four elementary schools. The three outputs of these schools are: (1) average reading score, (2) average mathematics score, and (3) average self-esteem score. The three inputs to these schools are: (1) average educational level of mothers (defined by highest grade completed: 12 = high school graduate, 16 = college graduate, and so on), (2) number of parent visits to school (per child), and (3) teacher-to-student ratio. The relevant information for the four schools is given in the table below.

School	Input 1	Input 2	Input 3	Output 1	Output 1	Output 3
A	14.5	3	0.25	3.5	2.7	3
B	13	2	0.13	3.3	2.5	2.4
C	15.5	4	0.28	3.8	3	3.3
D	16.2	3	0.33	4	3.8	4

Which of the following statement(s) is(are) true?

A) If we assume constant returns to scale, school both A and C are inefficient.

B) If we assume constant returns to scale, school Both B and D are efficient.

C) If we assume non-constant returns to scale, school B is efficient.

D) If we assume non-constant returns to scale, school C is efficient.

E) None of the above.

Note:

We take school A as an example. For schools B, C, and D, the solving process is similar.

We use Excel to build and solve the optimization problems for different schools under different assumptions. The results are as follows.

(Please refer to "Assignment_3_Solutions")

School	Input 1	Input 2	Input 3	Output 1	Output 1	Output 3
A	14.5	3	0.25	3.5	2.7	3
B	13	2	0.13	3.3	2.5	2.4
C	15.5	4	0.28	3.8	3	3.3
D	16.2	3	0.33	4	3.8	4

Assuming constant returns to scale

Evaluating A

	Input 1	Input 2	Input 3	Output 1	Output 1	Output 3
Weight	0.06896552	0	0	0.251341	0	0.02796935
	TOV		TIV			Efficiency
A	0.96360153	<=	1	fixed as 1		96%
B	0.89655172	<=	0.89655172			100%
C	1.04739464	<=	1.06896552			98%
D	1.11724138	<=	1.11724138			100%

Evaluating B

	Input 1	Input 2	Input 3	Output 1	Output 1	Output 3
Weight	0.06062271	0	1.63003663	0	0.4	0
	TOV		TIV			Efficiency
A	1.08	<=	1.28653846			84%
B	1	<=	1	fixed as 1		100%
C	1.2	<=	1.39606227			86%
D	1.52	<=	1.52			100%

Evaluating C

	Input 1	Input 2	Input 3	Output 1	Output 1	Output 3
Weight	0.06451613	0	0	0.23512545	0	0.02616487
	TOV		TIV			Efficiency
A	0.90143369	<=	0.93548387			96%
B	0.83870968	<=	0.83870968			100%
C	0.97982079	<=	1	fixed as 1		98%
D	1.04516129	<=	1.04516129			100%

Evaluating D

	Input 1	Input 2	Input 3	Output 1	Output 1	Output 3
Weight	0.0617284	0	0	0	0.26315789	0
	TOV		TIV			Efficiency
A	0.71052632	<=	0.89506173			79%
B	0.65789474	<=	0.80246914			82%
C	0.78947368	<=	0.95679012			83%
D	1	<=	1	fixed as 1		100%

Assuming non-constant returns to scale

Evaluating A							Weight
A	14.5	3	0.25	3.5	2.7	3	0
B	13	2	0.13	3.3	2.5	2.4	0.53125
C	15.5	4	0.28	3.8	3	3.3	0
D	16.2	3	0.33	4	3.8	4	0.46875
Virtual	14.5	2.46875	0.22375	3.62813	3.10938	3.15	Factor k
		k-multiple output		3.62813	2.79884	3.10982	1.03661
Evaluating B							Weight
A	14.5	3	0.25	3.5	2.7	3	0
B	13	2	0.13	3.3	2.5	2.4	1
C	15.5	4	0.28	3.8	3	3.3	0
D	16.2	3	0.33	4	3.8	4	0
Virtual	13	2	0.13	3.3	2.5	2.4	Factor k
		k-multiple output		3.3	2.5	2.4	1
Evaluating C							Weight
A	14.5	3	0.25	3.5	2.7	3	0
B	13	2	0.13	3.3	2.5	2.4	0.25
C	15.5	4	0.28	3.8	3	3.3	0
D	16.2	3	0.33	4	3.8	4	0.75
Virtual	15.4	2.75	0.28	3.825	3.475	3.6	Factor k
		k-multiple output		3.825	3.01974	3.32171	1.00658
Evaluating D							Weight
A	14.5	3	0.25	3.5	2.7	3	0
B	13	2	0.13	3.3	2.5	2.4	0
C	15.5	4	0.28	3.8	3	3.3	0
D	16.2	3	0.33	4	3.8	4	1
Virtual	16.2	3	0.33	4	3.8	4	Factor k
		k-multiple output		4	3.8	4	1

We can find that school B and school D are efficient under both assumptions and school A and school C are inefficient under both assumptions(constant returns to scale and non-constant returns to scale).

Therefore, options A), B), and C) are correct.

The formulation of this problem is provided below for your reference.

Let X_1, X_2, X_3, Y_1, Y_2, and Y_3 denote the coefficients of Input 1, Input 2, Input 3, Output 1, Output 2, and Output 3, respectively. Let W_A, W_B, W_C, and W_D denote the weights of school A, B, C, and D, respectively. $X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$; $Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$; $I_A = \begin{bmatrix} 14.5 \\ 3 \\ 0.25 \end{bmatrix}$; $I_B = \begin{bmatrix} 13 \\ 2 \\ 0.13 \end{bmatrix}$; $I_C = \begin{bmatrix} 15.5 \\ 4 \\ 0.28 \end{bmatrix}$; $I_D = \begin{bmatrix} 16.2 \\ 3 \\ 0.33 \end{bmatrix}$; $O_A = \begin{bmatrix} 3.5 \\ 2.7 \\ 3 \end{bmatrix}$; $O_B = \begin{bmatrix} 3.3 \\ 2.5 \\ 2.4 \end{bmatrix}$; $O_C = \begin{bmatrix} 3.8 \\ 3 \\ 3.3 \end{bmatrix}$; $O_D = \begin{bmatrix} 4 \\ 3.8 \\ 4 \end{bmatrix}$; $W = \begin{bmatrix} W_A \\ W_B \\ W_C \\ W_D \end{bmatrix}$; $I_1 = \begin{bmatrix} 14.5 \\ 13 \\ 15.5 \\ 16.2 \end{bmatrix}$; $I_2 = \begin{bmatrix} 3 \\ 2 \\ 3 \\ 3 \end{bmatrix}$; $I_3 = \begin{bmatrix} 0.25 \\ 0.13 \\ 0.28 \\ 0.33 \end{bmatrix}$; $O_1 = \begin{bmatrix} 3.5 \\ 3.3 \\ 3.8 \\ 4 \end{bmatrix}$; $O_2 = \begin{bmatrix} 2.7 \\ 2.5 \\ 3 \\ 3.8 \end{bmatrix}$; $O_3 = \begin{bmatrix} 3 \\ 2.4 \\ 3.3 \\ 4 \end{bmatrix}$.

If we assume constant returns to scale, we build the following model to check whether school A is efficient or not.

Targets:

$$\max_{X,Y} O_A{}'Y$$

Decision variables: X_1, X_2, X_3, Y_1, Y_2, and Y_3.

Constraints:

$$\begin{gathered} I_A{}'X = 1, \\ O_A'Y \le I_A'X, \\ O_B'Y \le I_B'X, \\ O_C'Y \le I_C'X, \\ O_D'Y \le I_D'X, \\ X, Y \ge 0. \end{gathered}$$

Let Y^* denote the optimal weights. If $O_A'Y^* = 1$, school A is efficient; otherwise, it is not efficient.

If we assume non-constant returns to scale, we build the following model to check whether school A is efficient or not.

Targets:

$$\max_{W,K} K$$

Decision variables: W_A, W_B, W_C, W_D, and K.

Constraints:

$$W_A + W_B + W_C + W_D = 1,$$
$$I_1'W \le 14.5,$$
$$I_2'W \le 3,$$
$$I_3'W \le 0.25,$$
$$O_1'W \ge 3.5K,$$
$$O_2'W \ge 2.7K,$$
$$O_3'W \ge 3K,$$
$$W \ge 0.$$

Under the optimal solution, if $K = 1$, school A is efficient; otherwise, it is not efficient.