Poisson regression, LDA and QDA

MSBA7002 Business Statistics Tutorial 3

ZHAO Yuwei

The University of Hong Kong

zhaoyw16@hku.hk

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Poisson regression

Suppose a random variable Y takes on nonnegative discrete values 0,1,2.... If Y follows a Poisson distribution with mean $\lambda(x_1, ..., x_p)$, the density function of Y is

$$Pr(Y = k) = \frac{exp(-\lambda(x_1, ..., x_p))\lambda(x_1, ..., x_p)^k}{k!} \quad \text{for k=0,1,...} \tag{1}$$

Poisson regression

For a Poisson regression, we assume that

$$log(\lambda(x_1, ..., x_p)) = \alpha + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_p x_p \tag{2}$$

because $\lambda(x_1, ..., x_p)$ is the mean of Y and has to be positive.

Poisson regression

Assume that for observation i, we have $(x_{i,1}, ..., x_{i,p})$ and y_i, the likelihood function is

$$L = \Pi_{i=1}^{n} \frac{exp(-\lambda(x_{i,1}, ..., x_{i,p}))\lambda(x_{i,1}, ..., x_{i,p})^{y_i}}{y_i!}. \tag{3}$$

In Poisson regression, we seek to maximize the likelihood function.

Poisson regression

Regression model	Linear	logistic	Poisson
Distribution	Gaussian	binomial	Poisson
Link function	$\mu = x'\beta$	$log(\frac{p}{1-p}) = x'\beta$	$log(\lambda) = x'\beta$

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Linear Discriminant Analysis

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Linear Discriminant Analysis

Discriminat Analysis

Step 1. Estimate the probability distribution for each class

$$\hat{f}_k(\mathbf{X}) = \hat{f}_k(X_1, \ldots, X_p), \quad k = 1, 2, \ldots, K$$

Step 2. Calculate the conditional probability of each observation for each class

$$\{\hat{p}(\mathbf{x_i}|\mathbf{y} = \mathbf{1}), \ldots, \hat{p}(\mathbf{x_i}|\mathbf{y} = \mathbf{K})\}, \quad i = 1, 2, \ldots, N$$

Step 3. Calculate the (posterior) probability of each observation for each class

$$\{\hat{p}(\mathbf{y} = \mathbf{1}|\mathbf{x_i}), \ldots, \hat{p}(\mathbf{y} = \mathbf{K}|\mathbf{x_i})\}, \quad i = 1, \ldots, N$$

Step 4. Predict the class based on estimated probability

$$k_i = \arg \max_{y \in \{1,\ldots,K\}} \{\hat{p}(\mathbf{y} = \mathbf{1}|\mathbf{x_i}), \ldots, \hat{p}(\mathbf{y} = \mathbf{K}|\mathbf{x_i})\}, \quad i = 1, \ldots, N$$

Linear Discriminant Analysis

Discriminat Analysis

Step 1

Estimate the probability distribution for each class

$$\hat{f}_k(\mathbf{X}) = \hat{f}_k(X_1, \ldots, X_p), \quad k = 1, 2, \ldots, K$$

- We need to assume the distribution of $(X_1, \ldots, X_p)$ for each group
  - Multivariate normal distribution

    LDA: assume the same covariance matrix for each group

    QDA: assume a different covariance matrix for each group
  - Other distributions

Linear Discriminant Analysis

Discriminat Analysis

Step 2

Calculate the conditional probability

$$\{\hat{p}(\mathbf{x_i}|\mathbf{y}=\mathbf{1}),\ldots,\hat{p}(\mathbf{x_i}|\mathbf{y}=\mathbf{K})\},\quad i=1,2,\ldots,N$$

- Suppose obsevation i belong to a given class
- if observation i belongs to class $k(k=1,2,\ldots,K)$,

$$\hat{p}(\mathbf{x_i}|y=k)=\hat{f}_1(\mathbf{x}_i)=\hat{f}_k(x_{i1},\ldots,x_{ip})$$

Linear Discriminant Analysis

Discriminat Analysis

Step 3

Calculate the (posterior) probability of each observation

$$\{\hat{p}(\mathbf{y}=\mathbf{1}|\mathbf{x_i}),\dots,\hat{p}(\mathbf{y}=\mathbf{K}|\mathbf{x_i})\},\quad i=1,\dots,N$$

- it is a Bayesian method
- we need to add prior probability of class $\{\pi_1,\pi_2,\dots,\pi_K\}$

$$\hat{p}(y=1|\mathbf{x_i})=\frac{\pi_1\hat{p}(\mathbf{x_i}|y=1)}{\sum_{k=1}^K\pi_k\hat{p}(\mathbf{x_i}|y=k)}$$

$$\vdots$$

$$\hat{p}(y=K|\mathbf{x_i})=\frac{\pi_K\hat{p}(\mathbf{x_i}|y=K)}{\sum_{k=1}^K\pi_k\hat{p}(\mathbf{x_i}|y=k)}$$

Linear Discriminant Analysis

Discriminat Analysis

Step 3

Calculate the (posterior) probability of each observation

$$\{\hat{p}(\mathbf{y} = \mathbf{1}|\mathbf{x_i}), \ldots, \hat{p}(\mathbf{y} = \mathbf{K}|\mathbf{x_i})\}, \quad i = 1, \ldots, N$$

- Prior Probability of Class $\{\pi_1, \pi_2, \ldots, \pi_K\}$
  - Prior information
  - Without any prior information
    - i Use proportation of each class as the prior

$$\pi_k = \frac{n_k}{\sum_{j=1}^{K} n_j}$$

    - ii Use equal prior probability

$$\pi_1 = \pi_2 = \ldots = \pi_K = 1/K$$

    - iii Other mehods

Linear Discriminant Analysis

Discriminat Analysis

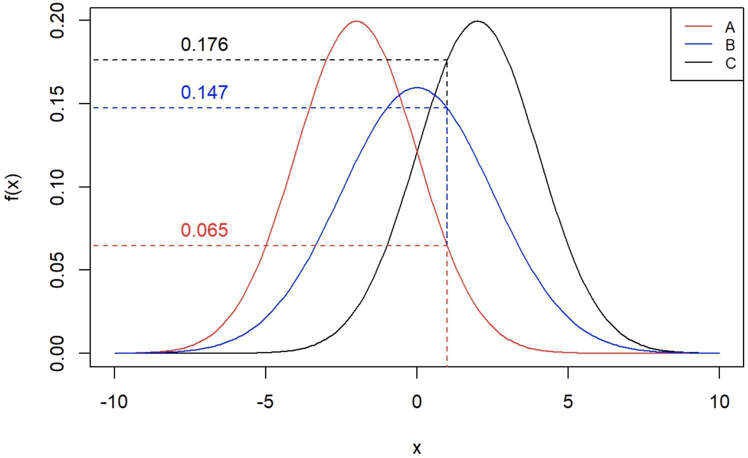
Step 4

Predict the class based on estimated probability

$$k_i = \arg\max_{k\in\{1,\ldots,K\}} \{\hat{p}(\mathbf{k}=\mathbf{1}|\mathbf{x_i}),\ldots,\hat{p}(\mathbf{k}=\mathbf{K}|\mathbf{x_i})\}, \quad i=1,\ldots,N$$

Linear Discriminant Analysis

Discriminat Analysis



Linear Discriminant Analysis

Discriminat Analysis

y	π_k	$p(\mathbf{x}\|y=k)$	$\pi_k p(\mathbf{x}\|y=k)$	$\frac{\pi_k p(\mathbf{x}\|y=k)}{\sum_{j=1}^{K} \pi_k p(\mathbf{x}\|y=j)}$	$\hat{y}$
A	0.6	0.065	0.039	0.3764479	✓
B	0.2	0.147	0.0294	0.2837838	
C	0.2	0.176	0.0352	0.3397683	
Σ			0.1036	1.0	

Linear Discriminant Analysis

Linear Discriminat Analysis

Three Hypotheses

i. $X_1, \ldots, X_p$ follow Mulitivariate Guassian Distribution
ii. Homoscedasticity assumption:
The class covariance matrices are identical

$$\Sigma_1 = \Sigma_2 = \ldots = \Sigma_K$$

Linear Discriminant Analysis

Linear Discriminat Analysis

Two features

- Easy to estimate probability function

$$f_k(x_1, \ldots, x_p) = \prod_{j=1}^{p} f_k(x_j)$$

$$f_k(x_j) = \frac{1}{\sqrt{2\pi}\sigma_j} e^{-\frac{1}{2}(\frac{x_j - \mu_{jk}}{\sigma_j})^2}$$

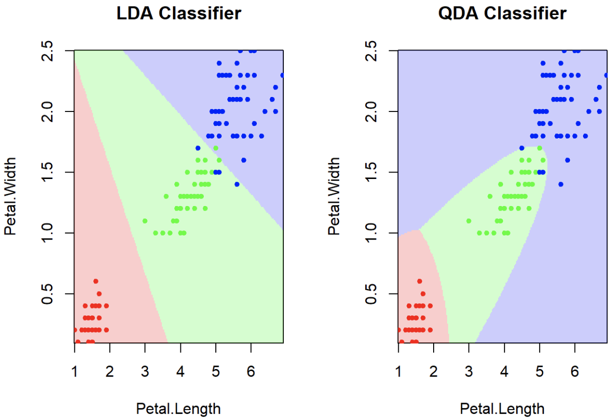
$$\ln(f_k(x_j)) = -\frac{1}{2}[(\frac{x_j - \mu_{jk}}{\sigma_j})^2 + \ln(2\pi) + \ln(\sigma_j^2))]$$

$$\frac{\partial f_k}{\partial \mu_{jk}} = 0, \quad \frac{\partial f_k}{\partial \sigma_j^2} = 0$$

Linear Discriminant Analysis

Linear Discriminat Analysis

ii Provide a linear Bayes Decision Boundary



Linear Discriminant Analysis

Linear Discriminat Analysis

ii Provide a linear Bayes Decision Boundary

- The rotation is meanful !
- Let Y "supervise" the rotation
  $\Rightarrow$ **Discriminant Variables**