MSBA 7004
Operations Analytics

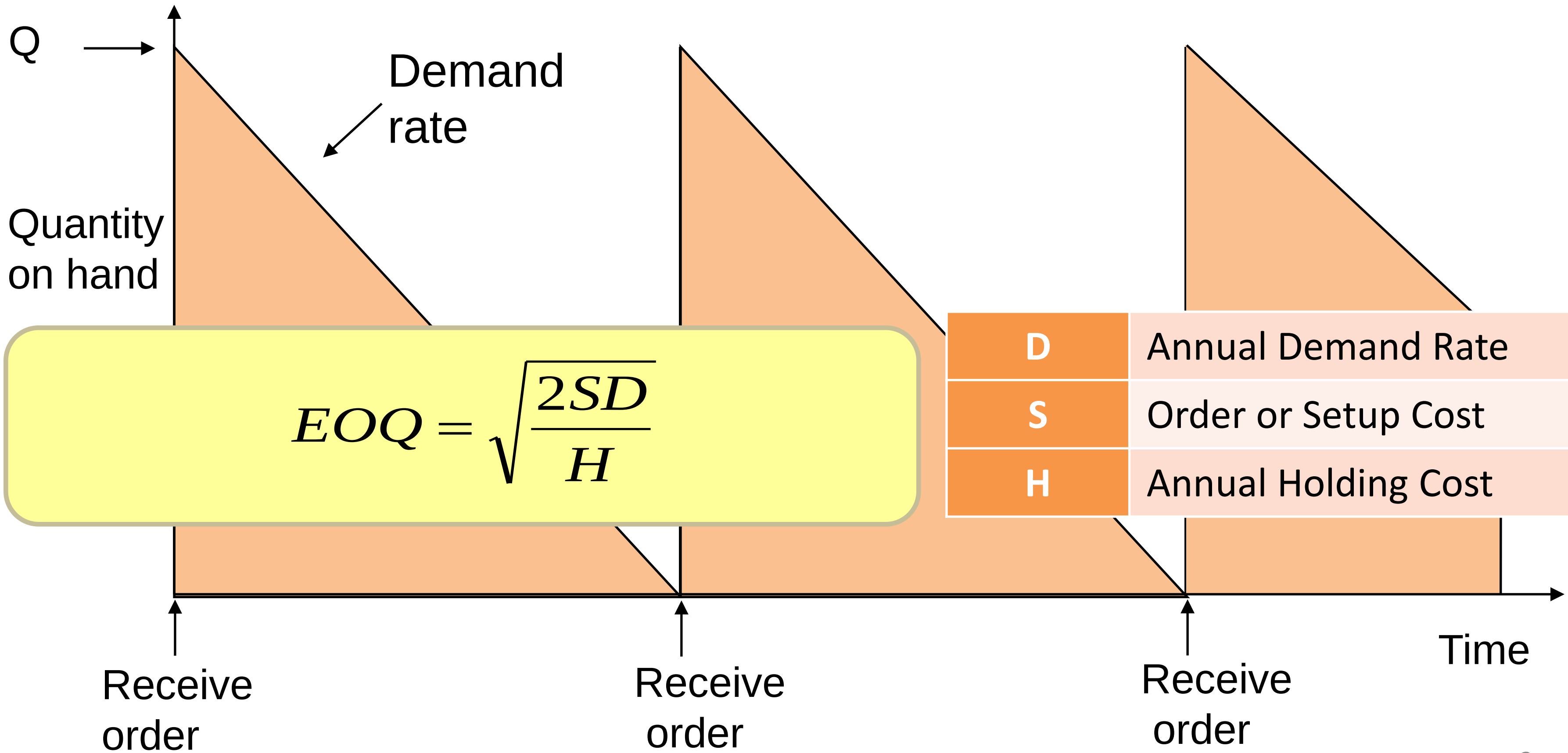
Class 7-1: Inventory Analysis (II)
Lead Time and Demand Uncertainty,
Continuous vs. Periodic Review Models
2023

Learning Objectives

- Understand the impact of lead time and demand uncertainty in inventory models
- Understand continuous review (or fixed-order-quantity) and periodic review (or fixed-time-period) models

Cycle Stocks: Tradeoff between fixed costs and holding costs

Profile of Inventory Level over Time

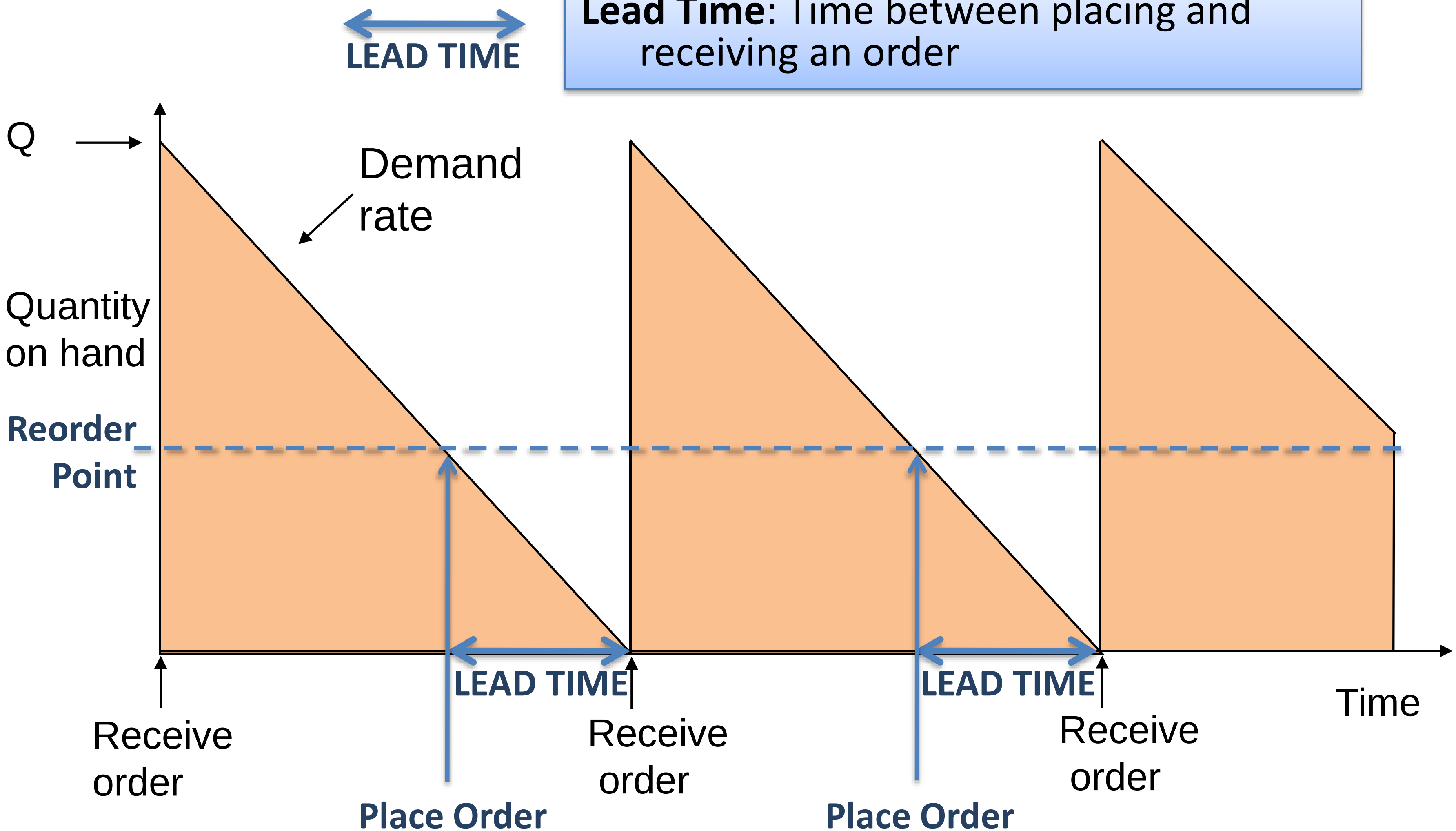


$$EOQ = \sqrt{\frac{2SD}{H}}$$

D	Annual Demand Rate
S	Order or Setup Cost
H	Annual Holding Cost

EOQ Model: With a “Lead Time”

Lead Time: Time between placing and receiving an order



Q-r (EO$\underline{Q}$, $\underline{R}$OP) Model: EOQ with Reorder Point

$$EOQ = \sqrt{\frac{2SD}{H}}$$

D	(Annual) Demand Rate
S	Order or Setup Cost
H	(Annual) Holding Cost

Reorder Point: ROP = D * LT

D	Demand Rate
LT	Lead Time

Valid provided LT < Cycle Time
We need a reorder point due to the lead time

Practice Problem: QMH (1)

Queen Marry Hospital consumes 100 boxes of bandages per week. The price of bandages is $70 per box. The hospital operates 52 weeks per year. The cost of processing an order is $60, and the cost of holding one box for a year is 15% of the value of the material. What is the EOQ and the corresponding total cost?

If the lead-time is one-half week, what should be the reorder point?

D	100*52 per year
S	$60
C	$70
i	15%

H	i*C = (0.15)(70)=10.50

$$TC(Q) = \frac{Q}{2}H + \frac{D}{Q}S$$

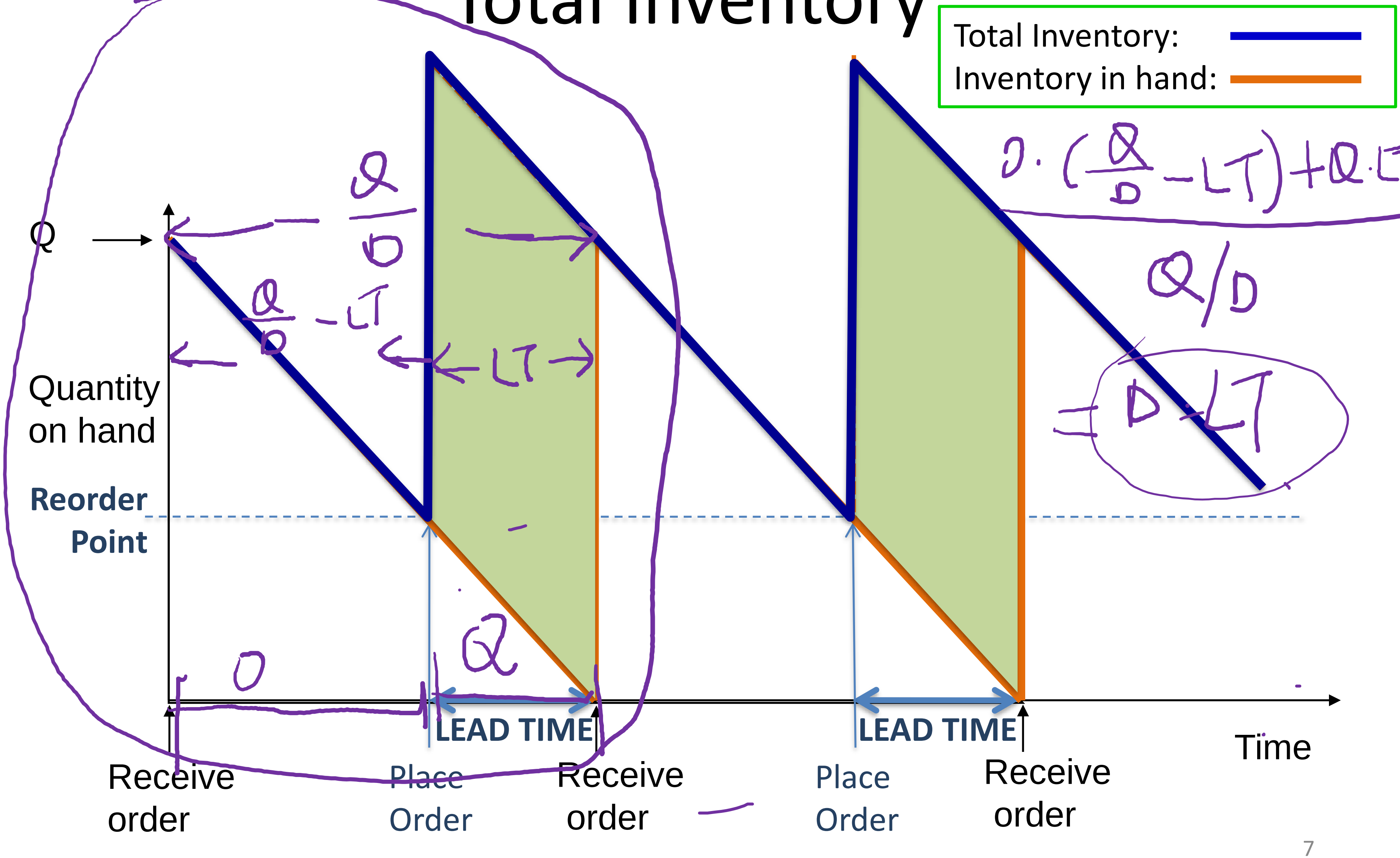
$$EOQ = \sqrt{\frac{2SD}{H}} = 244$$

$$TC(244) = \frac{244}{2}H + \frac{D}{244}S = 2459.7$$

$$ROP = D \cdot LT$$

$$= (100 * 52) \cdot (0.5/52) = 50$$

Cycle Stock, Pipeline Inventory, and Total Inventory



Accounting for *Pipeline* Inventory

- Pipeline inventory is the inventory "on route"

Average Pipeline Inventory

$$D \cdot LT$$

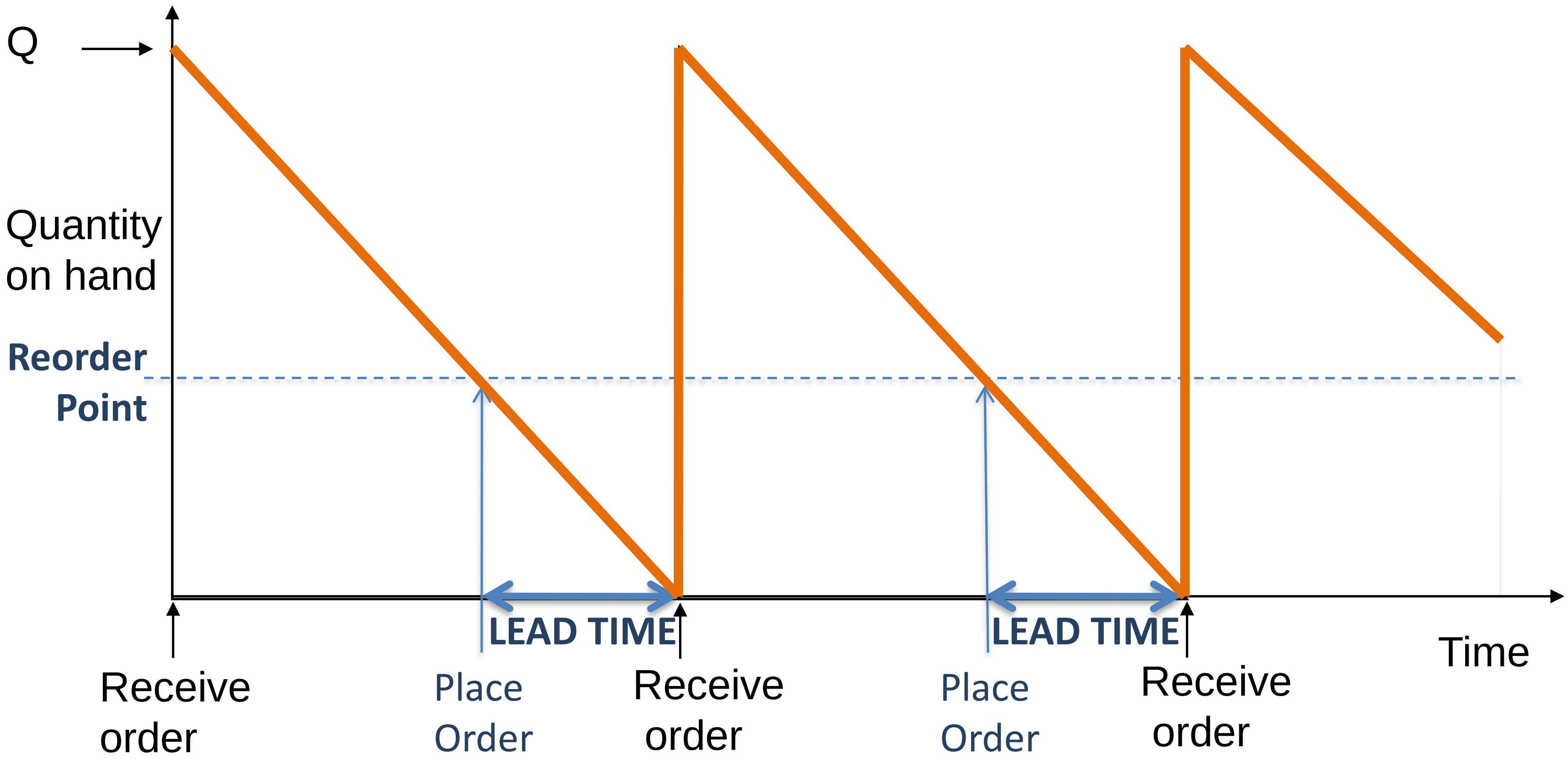
In each inventory cycle of duration Q/D, you have pipeline inventory of amount Q only in the time window of duration LT. In other periods, pipeline inventory is 0. Your average pipeline inventory: $\frac{\mathrm{LT}*\mathrm{Q}+\left(\frac{Q}{D}-LT\right)*0}{Q/D}$=D*LT

- **<u>If</u>** the buyer pays for inventory when it is *shipped* from the seller, then the buyer needs to consider the cost of the pipeline inventory

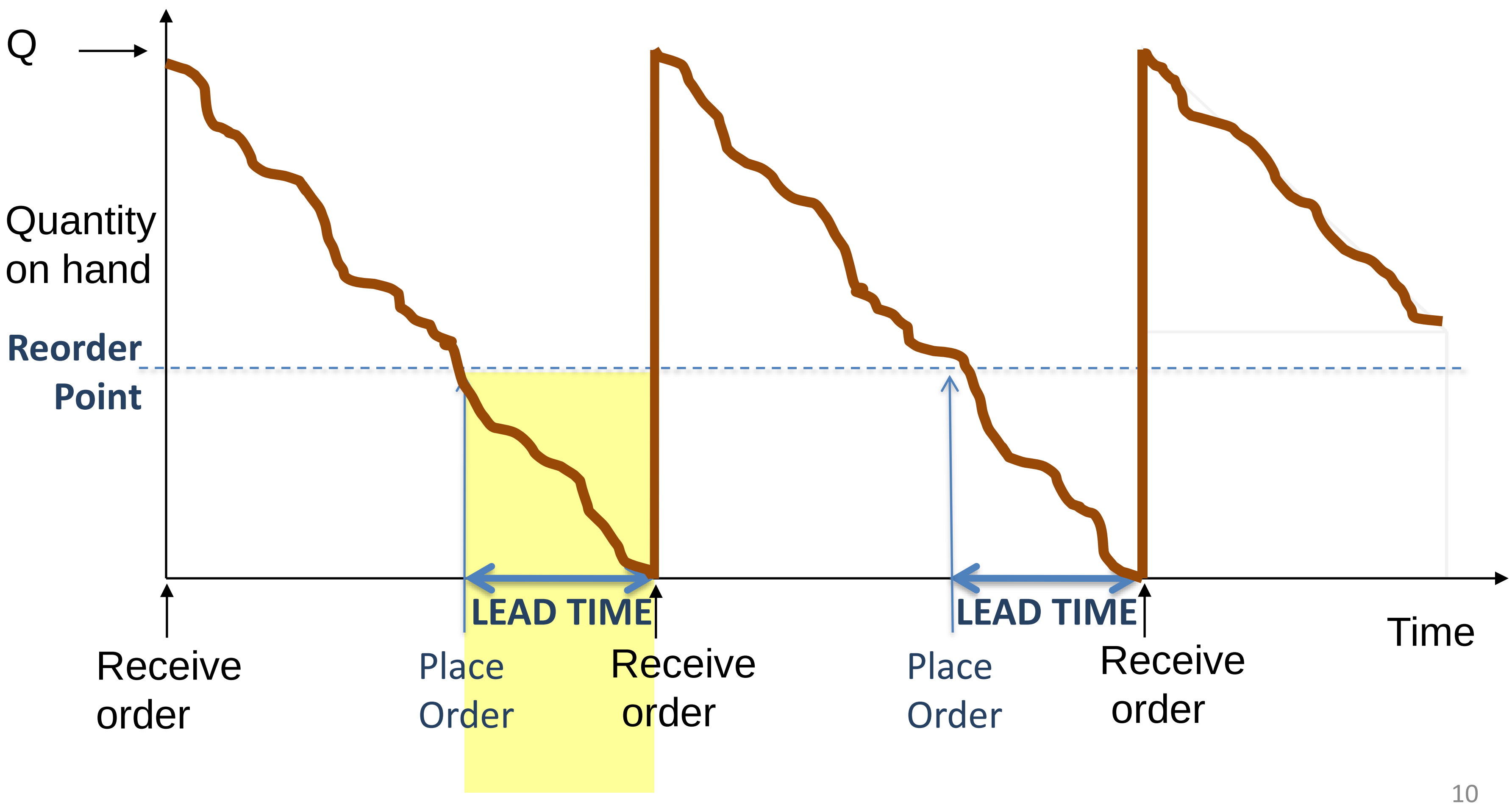
Average Total Inventory = **Average Cycle Stock (Q/2)** + **Average Pipeline Inventory (D ·LT)**

Q-r Model with Constant Demand

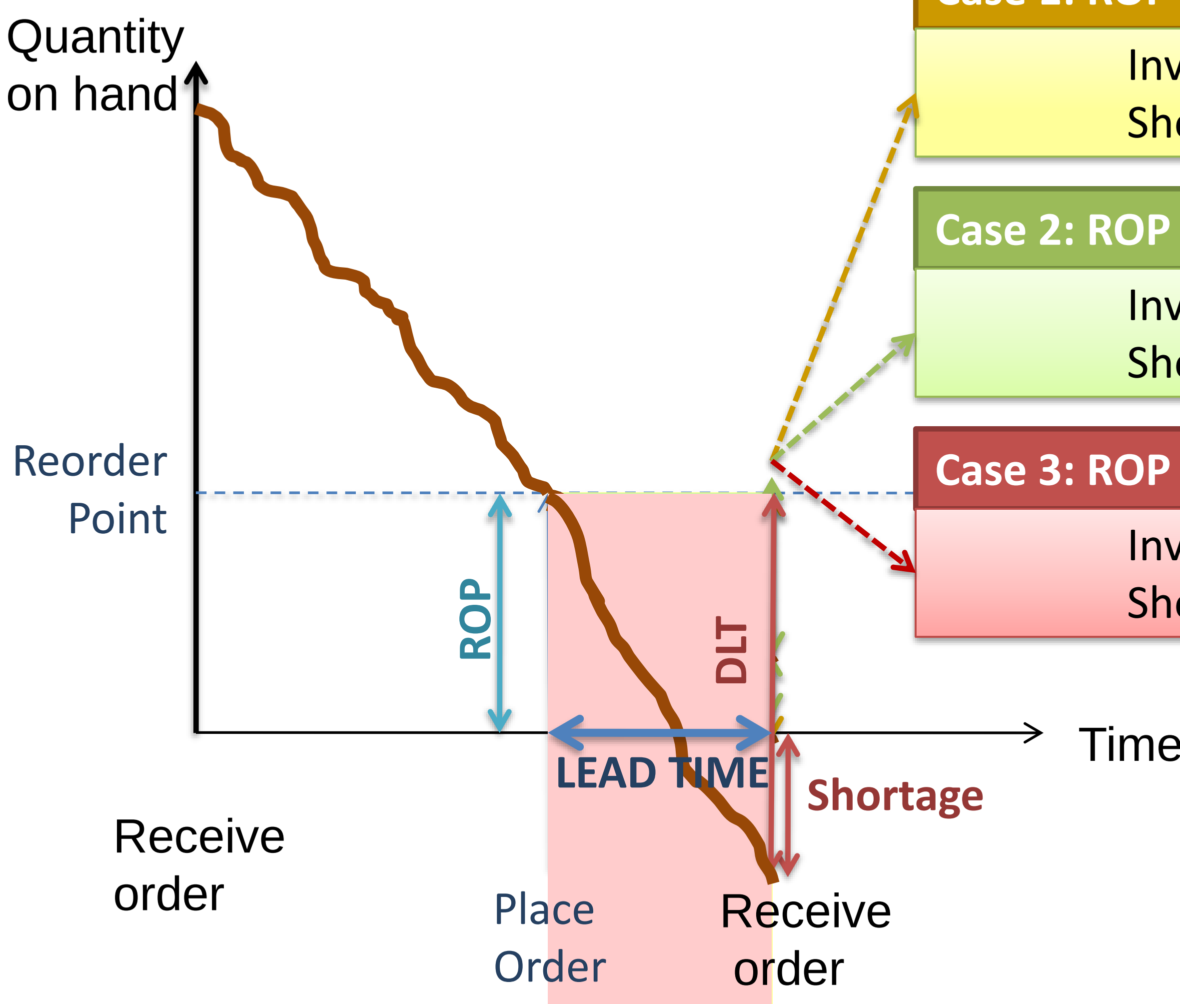
So far, we assumed demand is **constant** …



… But what if demand is uncertain?



Demand During Lead Time: Uncertain



Case 1: ROP = DLT

Inventory = 0
Shortage = 0

Case 2: ROP > DLT

Inventory = ROP – DLT
Shortage = 0

Case 3: ROP < DLT

Inventory = 0
Shortage = DLT - ROP

DLT = Demand during the Lead Time

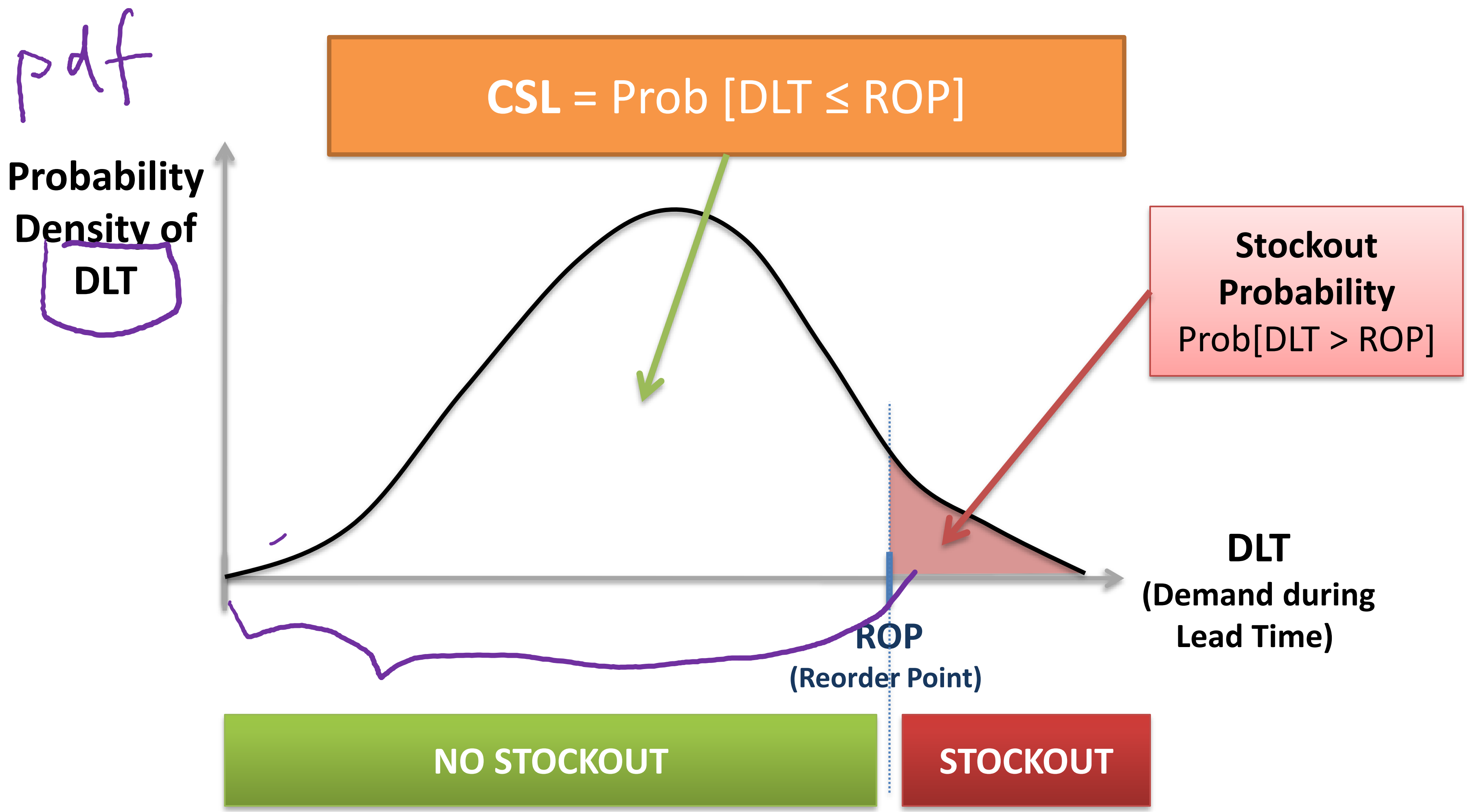
(Cycle) Service Level

- Shortage occurs if demand during lead time (DLT) exceeds reorder point (ROP)
- **Cycle Service Level** (CSL), or **Service Level** (SL)
  - Measure of reliability of the system
  - Probability of ***not*** stocking out in a cycle

CSL = Prob [DLT ≤ ROP]

- Desired service level helps us determine ROP
- Service level is a managerial decision
  - How often do you not want to stock out

How to Determine Reorder Point (ROP)?

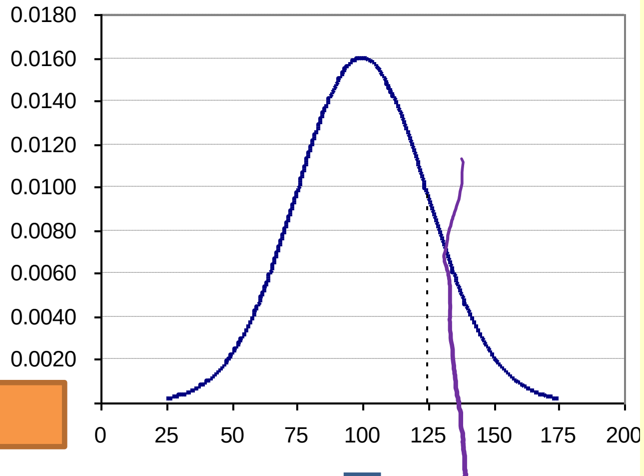


Normal Distribution Tutorial (1): Rescaling into Z-score

Start with
μ= 100,
σ= 25.

$D \sim N(\mu, \sigma^2)$

S = 125



P(D ≤ 125)

=

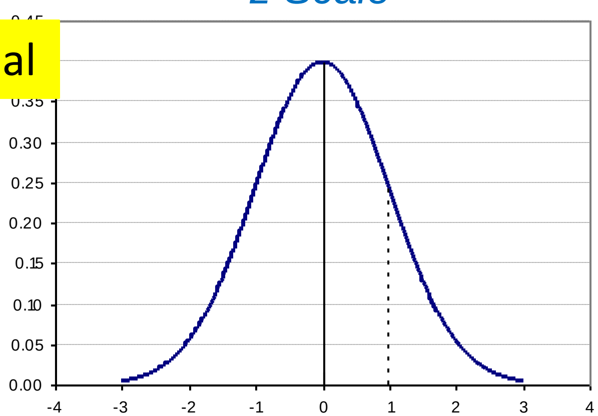
P(X ≤ 1)

z-Scale

Standard normal

$X \sim N(0,1)$

$$z = \frac{Q-\mu}{\sigma} = \frac{125-100}{25} = 1$$



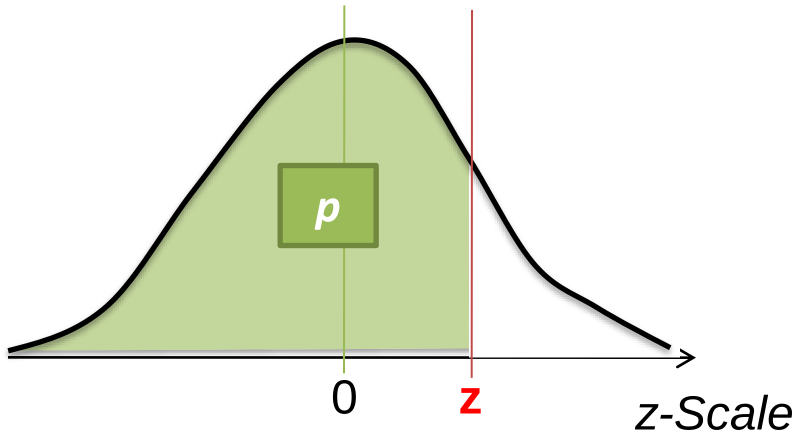
- Let S be the order quantity, and (μ, σ) the parameters of the normal demand distribution
- *Prob*{demand is *S* or lower} = *Prob*{the outcome of a standard normal is *z* or lower}, where

$$z = \frac{S-\mu}{\sigma} \quad \text{or} \quad S = \mu + z \times \sigma$$

- Look up *Prob*{the outcome of a standard normal is *z* or lower} in the Standard Normal Distribution Function Table, or NORMSDIST (excel), norm.cdf (scipy)

Normal Distribution Tutorial (2): Relationship between P and Z (using excel)

Consider the Standard Normal Distribution (in z-scale).



Given z, compute probability p using **NORMSDIST**(z), norm.cdf(z).

Given p, compute z using **NORMSINV**(p), norm.ppf(p).

Adding Normal Distributions

Summing k distributions

If $$X_i \sim N(m_i, (\sigma_i)^2)$$

then $$\sum_{i=1}^{k} X_i \sim N\left(\sum_{i=1}^{k} m_i, \sum_{i=1}^{k} (\sigma_i)^2\right)$$

Summing k *identical* distributions

If $$X_i \sim N(m, \sigma^2)$$

then $$\sum_{i=1}^{k} X_i \sim N(km, (\sqrt{k}\sigma)^2)$$

Normal Distribution Tutorial (3): Relationship between P and Z (using Z-score chart)

For a z-score of 1.13 what is the service level?

z	0	0.01	0.02	0.03	0.04	0.05
0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894
1	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943

ROP and Normal Demand

Suppose weekly demand has a normal distribution: $N(m, \sigma^2)$

Suppose lead time is *k* weeks.

Then, DLT has normal distribution $N(m_{LT}, \sigma_{LT}{}^2)$

with mean $m_{LT} = k \cdot m$ and standard deviation $\sigma_{LT} = \sqrt{k} \cdot \sigma$

Given ROP, CSL is

$P[DLT \leq ROP] = P[N(0,1) \leq z] = normsdist\ (z)$ where $z = \dfrac{ROP - m_{LT}}{\sigma_{LT}}$

Given CSL, *ROP* satisfying $\mathrm{CSL} = P[DLT \leq \mathrm{ROP}]$ is

$\mathrm{ROP} = m_{LT} + z \cdot \sigma_{LT}$ where z=*normsinv*(CSL)

Safety Stock (SS)

Safety Stock and Normal Distribution

- Safety stock is the extra inventory beyond the expected demand during LT (D*LT)
- When demand has (unpredictable) uncertainty, safety stock is held to cushion against uncertainties
- Safety stock is based on $SS = z \cdot \sigma_{LT} = z \cdot \sqrt{k} \cdot \sigma$
  - Desired service level (z)
  - Length of the lead time (k)
  - Demand variability (σ)

Given CSL, *SS* achieving this CSL (under Normal demand) is

$SS = z \cdot \sigma_{LT}$ where z=*normsinv*(CSL)

Summary: Q-r model with SS (due to demand uncertainty)

- To compute Q: Use the EOQ Formula $EOQ = \sqrt{\frac{2SD}{H}}$
- ROP = $m_{LT} + SS$

$SS = z\sigma_{LT}$

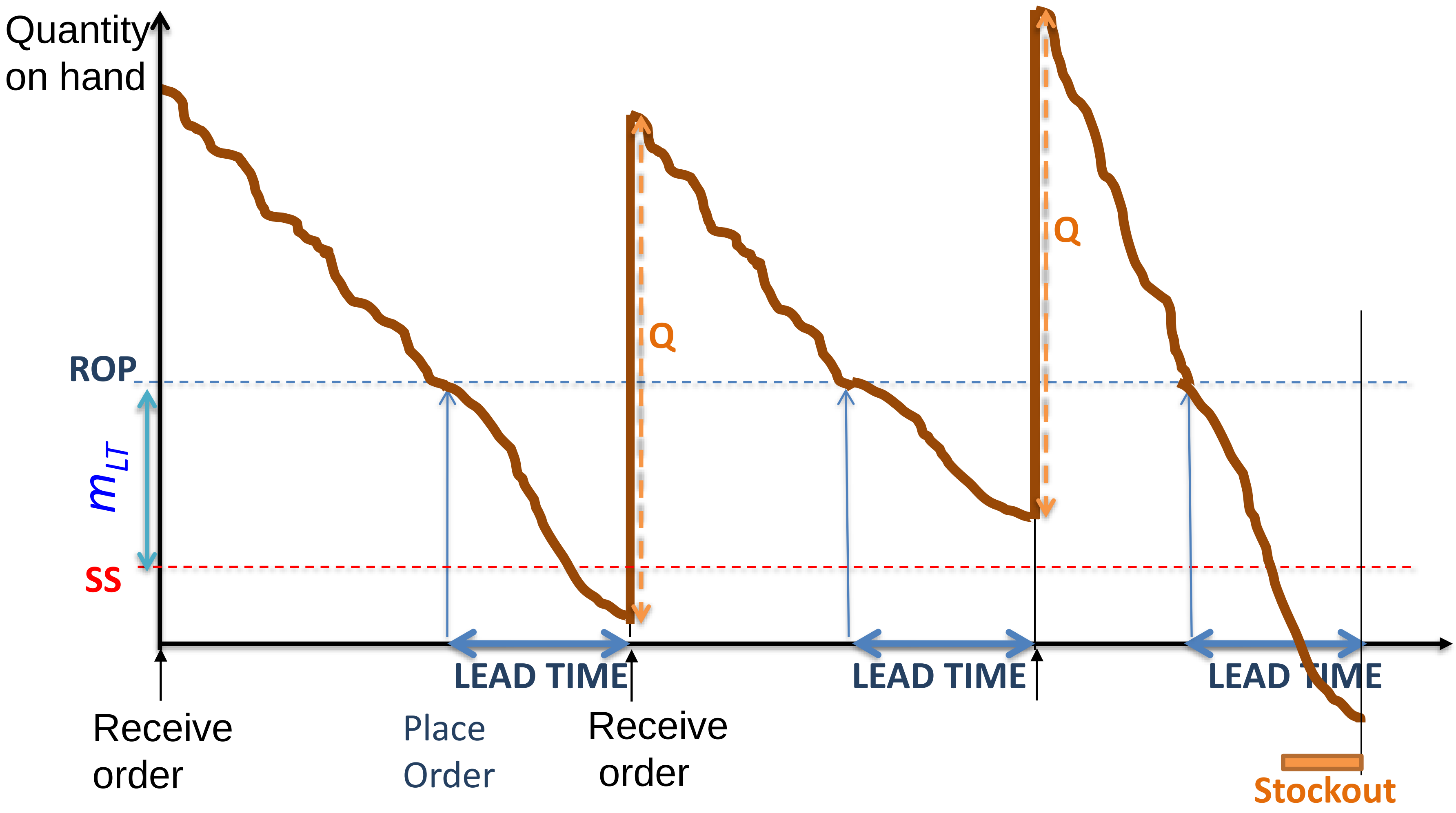
$\sigma_{LT} = \sqrt{k}\sigma$

k: Lead time in “unit time”

σ: STD of demand per “time unit”

Mean demand during lead time
(mean demand per “time unit”)*(lead time in “unit time”)
=D*k

ROP and SS


Quantity on hand
ROP
m_{LT}
SS
Q
Q
LEAD TIME
LEAD TIME
LEAD TIME
Receive order
Place Order
Receive order
Stockout

Average Inventory

If the firm pays for inventory "in stock"	If the firm pays for inventory "in stock" and "on route"
Average Inventory = Q/2 + SS	Average Inventory = Q/2 + SS + m_{LT}
	Pipeline Inventory

↑ "Default" case in this course
(when there is no other explanation)

Practice Problem: QMH (2)

Demand is normally distributed with mean 100 boxes of bandages per week and a standard deviation 20 boxes. The lead time is one-half week.

What safety stock is necessary if the hospital uses a 97% service level?

What should be the reorder point?

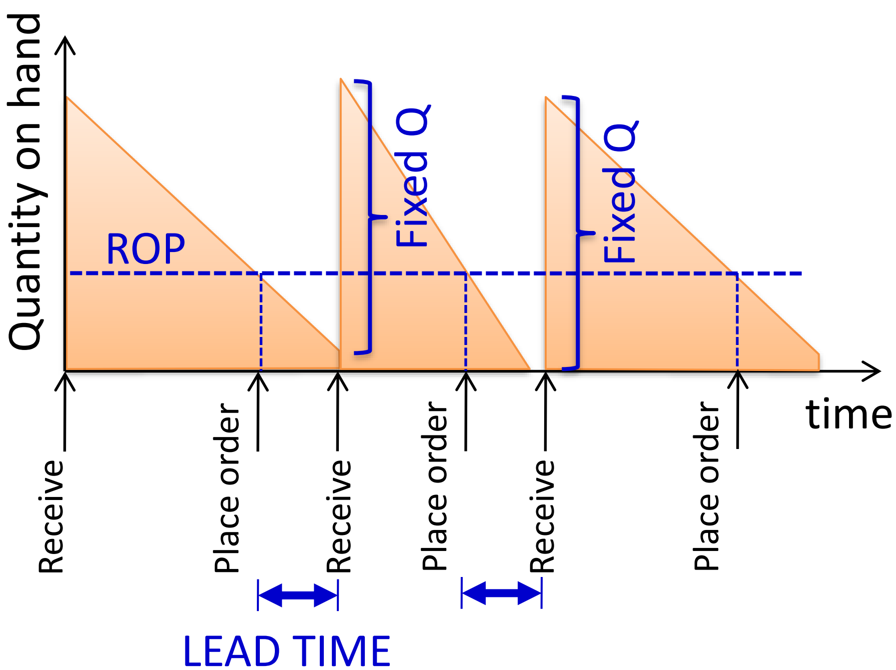
D	100 per week (average)
σ	20 for weekly demand
k	0.5 week
SL	97%

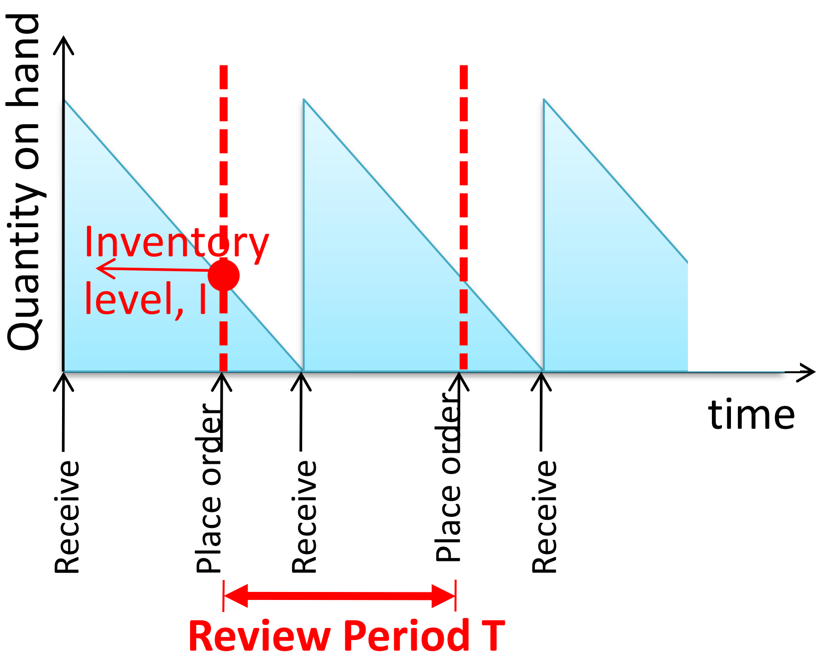
Continuous vs. Periodic Review

Continuous Review Systems	Periodic Review Systems
Event-triggered Example: every time the inventory falls below a reorder point **R**, order a fixed quantity **Q**	Time-triggered Inventory accounted at particular pre-determined times and ordering decision made accordingly An order quantity that usually depends on the current inventory-on-hand is ordered
Also called **fixed-order-quantity models or (Q,r) policy**	Also called **fixed-time-period models or (R,T) policy**

Continuous vs. Periodic Review

Fixed-Order-Quantity Model (Continuous Review)	Fixed-Time-Period Model (Periodic Review)





Order Size for Fixed-Time-Period Model

Assume Normal demand uncertainty

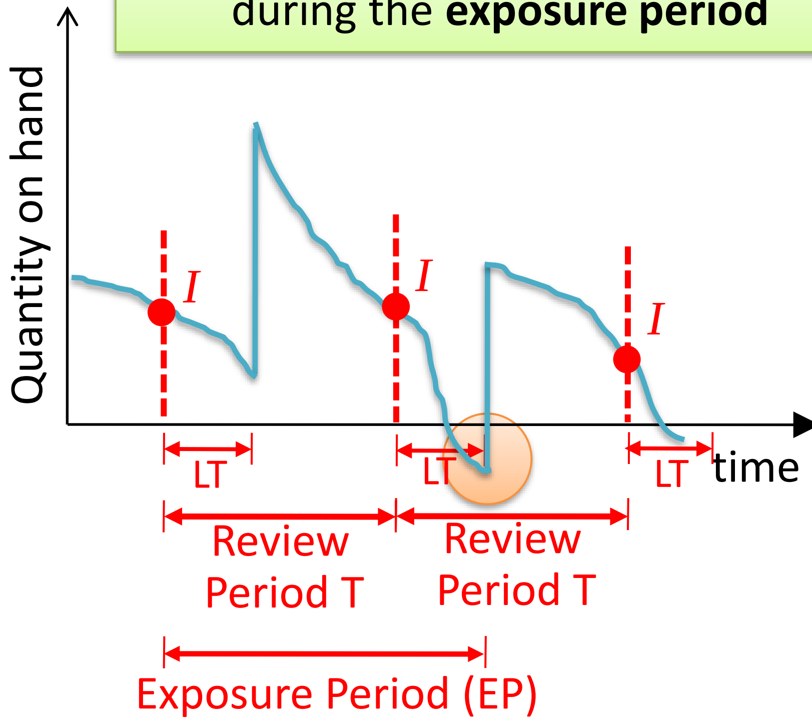
Target Level or "Inventory Position"
= (LT + T) * D + $z\sigma_{LT+T}$

Order Quantity
= Target Level - I

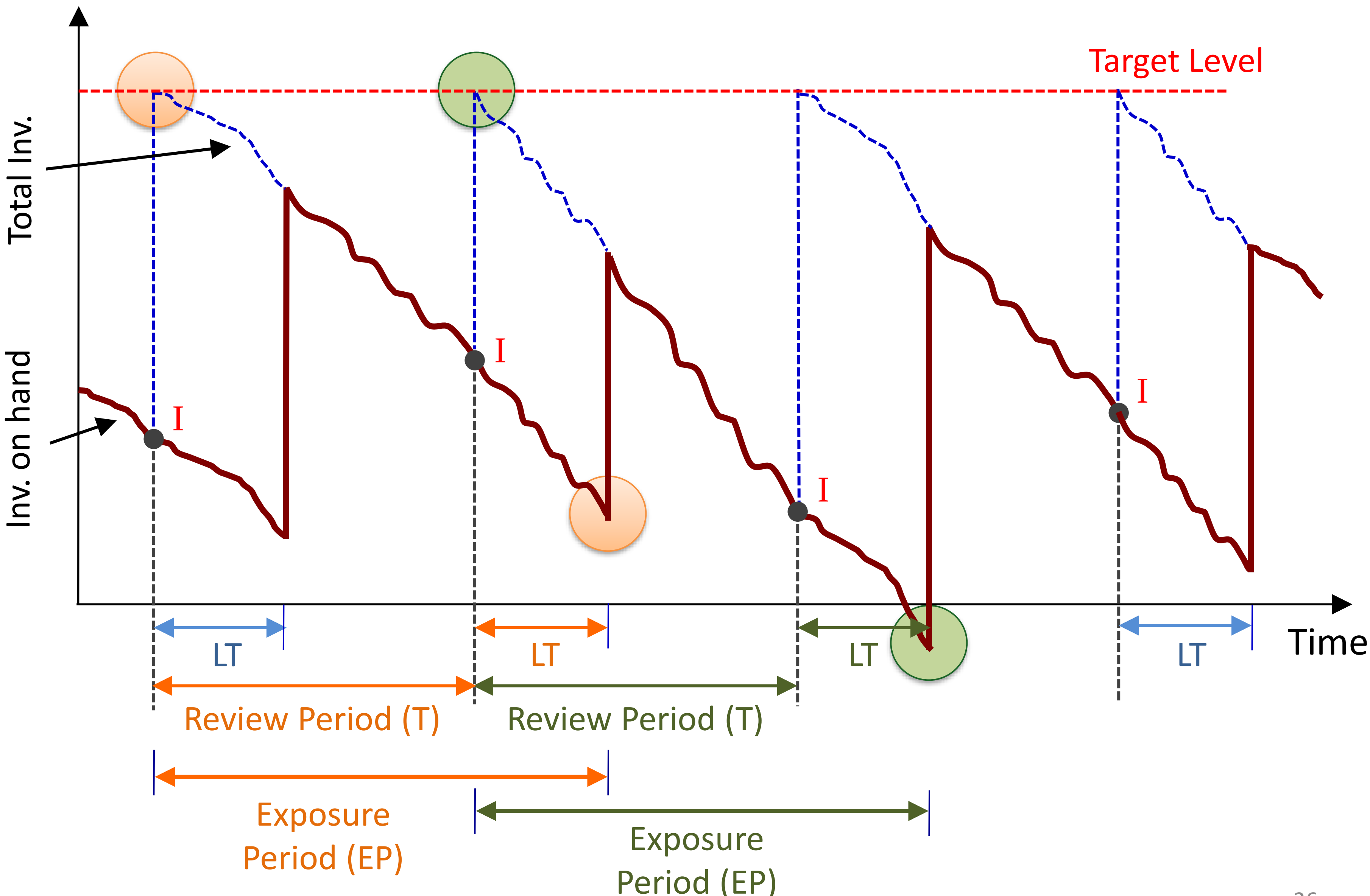
I	Current inventory level
T	Review Period (time between reviews)
EP	**Exposure Period** (= LT+T)

Given Cycle service level (CSL), we can find z=normsinv(CSL)

σ_{LT+T} Standard deviation of demand during the **exposure period**



Fixed-Time-Period Model: Illustration



Example: Fixed-Time-Period Model

Average daily demand for a product is 20 units.

The review period is 30 days and lead time is 10 days.

Policy: Not stocking out 96% of the time.

At the beginning of the review period, there are 200 units in inventory.

The daily demand standard deviation is 4 units.

How many units should be ordered?

D	20 units per day
σ	4 for daily demand
SL	96%
T	30 days
LT	10 days
I	200 units

$$z = NORMSINV(SL) = 1.75$$

$$\sigma_{LT+T} = sqrt(LT+T)*\sigma = 25.3$$

Target: $(LT + T) * D + z\sigma_{LT+T}$
$= (40)(20) + (1.75)*(25.3)$
$= 844.30$

Order Quantity:
$(LT + T) * D + z\sigma_{LT+T} - I$
$= 844.30 – 200 = 644.30$

Average Inventory Comparison

	Fixed-Order-Quantity Model	Fixed-Time-Period Model
Average Pipeline Inventory	LT * D	LT * D
Average Cycle Stock	Q/2 (where Q=EOQ)	Q/2 (where Q=T*D)
Safety Stock	$z\sigma_{LT}$	$z\sigma_{LT+T}$

LT: Lead Time

T: Review Period (Time between Reviews)

Total Cost Analysis

- Total Supply Chain Cost

  =Inventory Cost (Holding + Order) + Transportation Cost

- We can analyze annual supply chain cost

  Annual Inventory Cost + Annual Transportation Cost

- Can also look at supply chain cost per unit

  (Annual Inventory Cost + Annual Transportation Cost) divided by (Annual Sales)

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Operations Analytics

Class 7-2: Inventory Analysis (III)
Demand Uncertainty, Newsvendor Problem
2023

Learning Objectives

- Understand more impact of demand uncertainty
  - On exact profit
- Understand the trade-off between *stock-out* (excess demand) and *excess inventory* (excess supply)
  - Newsvendor Model

True or False?

The motivation of the EOQ model is to match the demand with the right quantity of supply.

How much inventory should you hold?

- Trade-off #1: How much to order each time?

Inventory ordering costs
(Economies of scale)

Inventory holding costs

Economic Order Quantity (EOQ) Model

- Trade-off #2: How much to store each time?

Cost of running out

Cost of having excess inventory

Newsvendor Model

What is the Best Service Level (SL*) ?

- Trade-off:

	Inventory Holding Cost (Overage Cost)	Loss of Revenue & Goodwill (Underage Cost)
High Service Level	High	Low
Low Service Level	Low	High

Newsvendor Model: SL* Driven Decision

- A newsvendor stocks newspapers to sell that day
- Trade-offs:
  - If stocks <u>too few</u> newspapers, misses potential sales.
  - If stocks <u>too many</u> newspapers, money wasted on unsold newspapers.

How many newspapers should be stocked?

Newsvendor-type Problems: Not just Newspapers

- Fashion goods with short product life-cycles
- “Perishable” items (e.g., food, newspapers)
- Seasonal products
- Short-run capacity planning
- Revenue management
  for airlines/hotels

An Oil Rig Example

- Problem: Decide a suitable stock level for a perishable product which has historically random demand
- This product is used on an oil rig
  - The rig is re-supplied weekly from stocks on-shore
  - At the beginning of each week, a planner determines how much stock of the product is to be placed on the rig
  - When a shortage occurs, an emergency shipment is made
- How should we approach making this stocking decision?
  - Define an objective
  - Identify what data are required
  - Measure consequences of decision

Oil Rig Example: Cost Data

- What are the cost implications of our decision?

Cost of having too <u>much</u> stock	$\$C_o$ = \$1000 per unit
Cost of having too <u>little</u> stock (Emergency resupply cost)	$\$C_u$ = \$9000 per unit

- Decision Structure

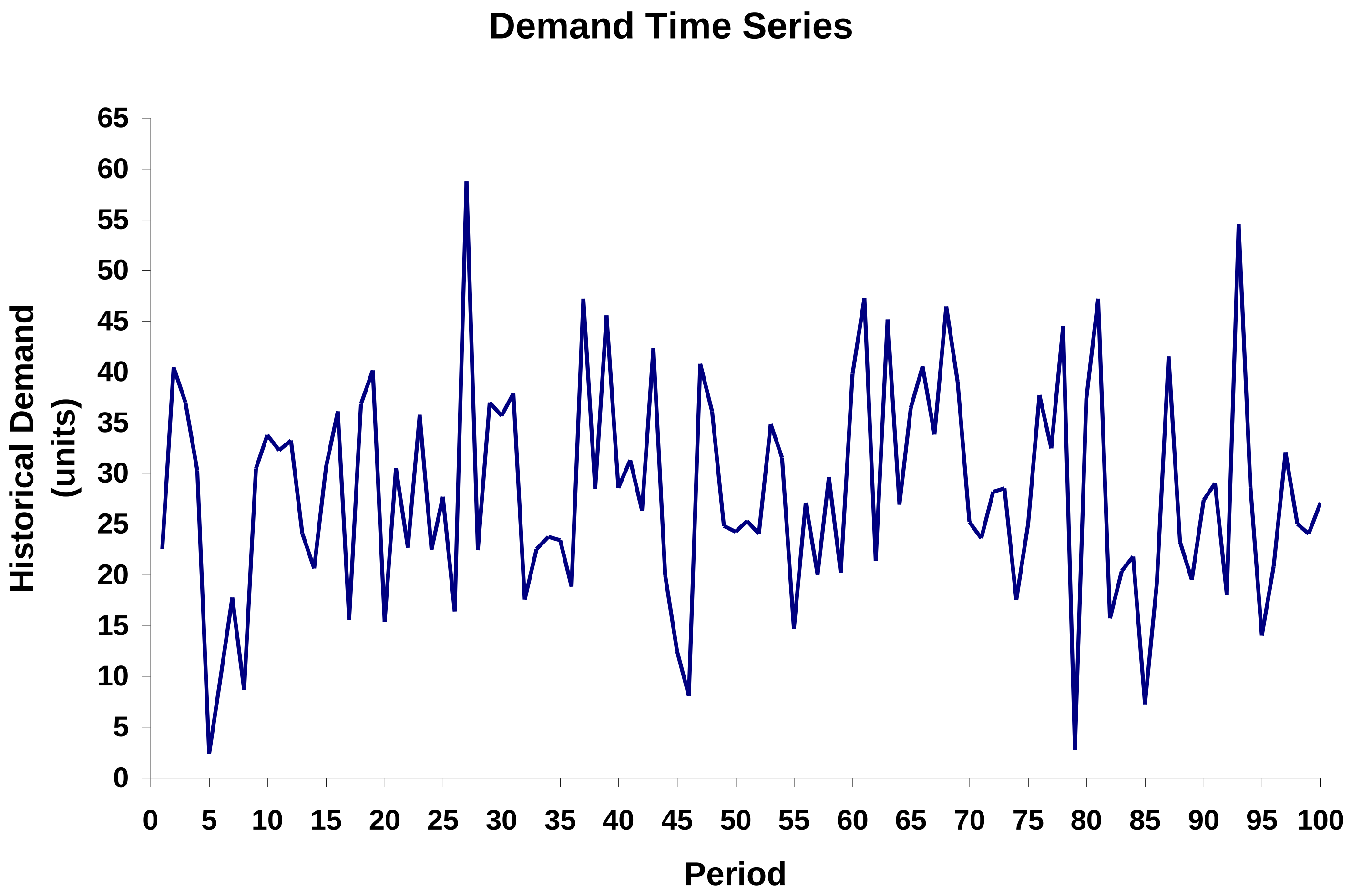
D	Customer demand that will occur in a week (random variable)
S	Our decision of how much to stock each week

Representing the Decision Problem

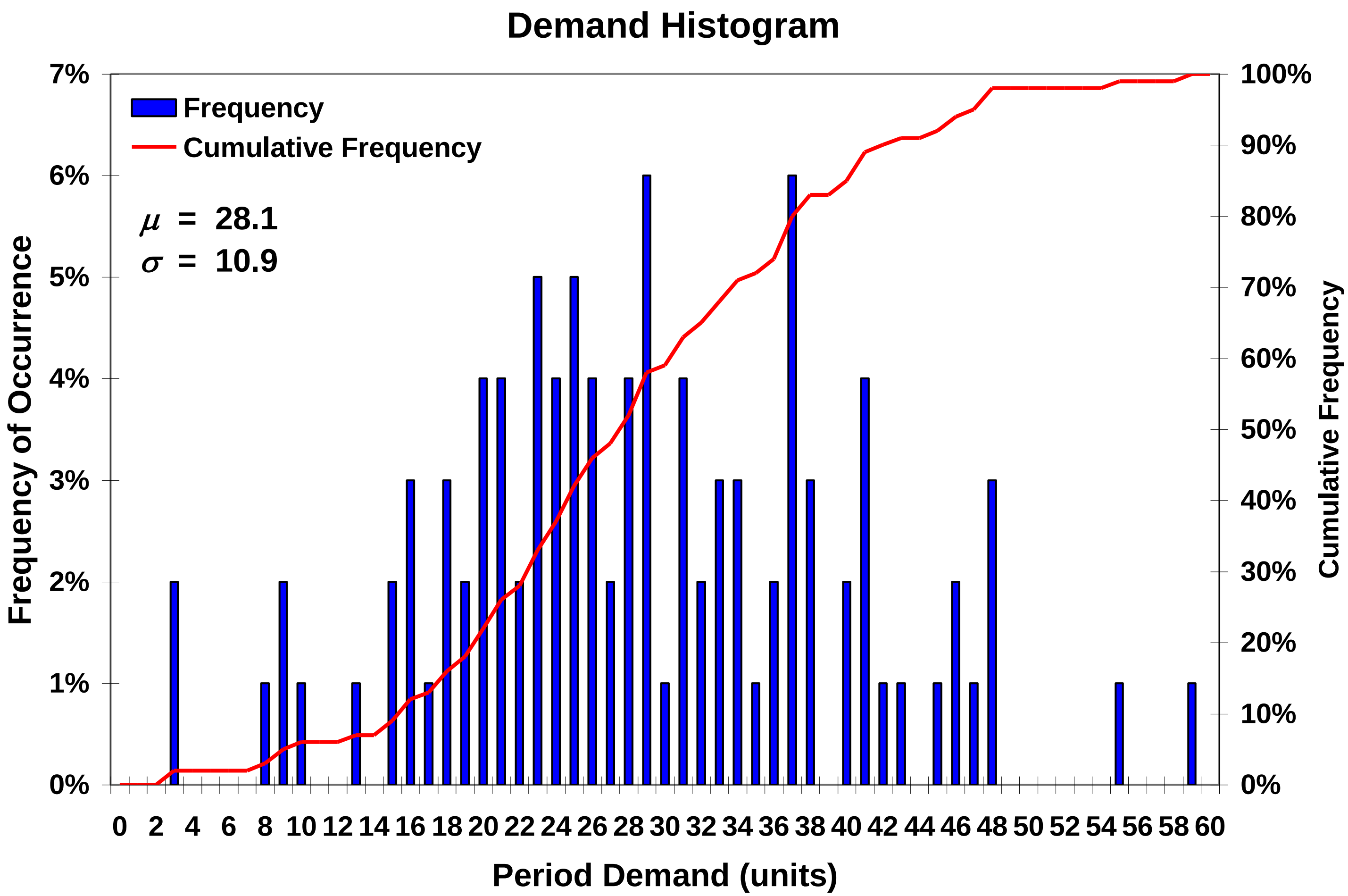
What is the cost if the demand (D) and the supply (S) were known? For example, if we had stocked 30 units each period for the previous 100 weeks.

- What would the *costs* have been each week?
  - What if demand exceeds supply that week?
  - What if supply exceeds demand that week?
- What is the total cost per week?
  - Holding and shortage costs
- What *fraction of the time (out of 100 cycles)* would we have had enough stock to satisfy demand?
  - Definition of SL in EOQ setting
- What *fraction of the time (out of 100 days)* would we have had enough stock to satisfy demand?
  - Definition of SL in Newsvendor setting

Oil Rig Example: Historical Demand



Oil Rig Example: Demand Distribution



Parameters In the Newsvendor Model

- Uncertain demand D
- Costs:

C_o	Overage Cost per Unit
C_u	Underage Cost per Unit

- *Decision*: Stocking quantity S

If Demand Were Known

- What is the cost if the demand (D) and the supply (S) were known?
  - What if demand exceeds supply?
  - What if supply exceeds demand?

- What is the total cost per week?
  - Overstock and shortage costs

Cost For the First Six Weeks, If S=30

Demand D	Stock S	Underage Cost	Overage Cost	Total Cost
23	30			
40	30			
30	30			
16	30			
2	30			
11	30			

Unit: thousand

C_o = \$1 per unit (excess inventory, has to be reprocessed)

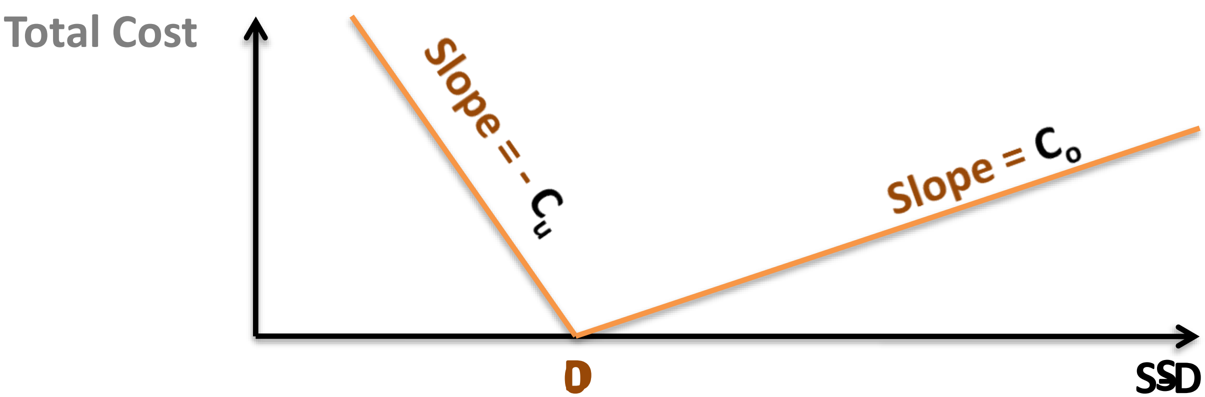
C_u = \$9 per unit (insufficient inventory, emergent call for resupply)

Do you think S=30 is good? Shall increase or decrease?

Cost Function (Demand Known)

- If D were known, the cost function would be

$$
\begin{aligned}
\mathbf{C(S, D) = Cost\ (S, D)} \\
&= C_o \cdot \mathbf{max\{0, S\text{-}D\}} + C_u \cdot \mathbf{max\{0, D\text{-}S\}} \\
&= C_o \cdot \underbrace{\mathbf{[S\text{-}D]^+}}_{\text{Had too much stock}} + C_u \cdot \underbrace{\mathbf{[D\text{-}S]^+}}_{\text{Had too Little stock}}
\end{aligned}
$$



There exists a fundamental economic tradeoff

Building the Cost Model

- Recall that the cost of (S) units is

Overage	$\$C_o$ = \$1000 per unit when supply > demand
Underage	$\$C_u$ = \$9000 per unit when supply < demand

- P(D): probability that demand is equal to D
- The cost when demand equals D is given by the relationship:

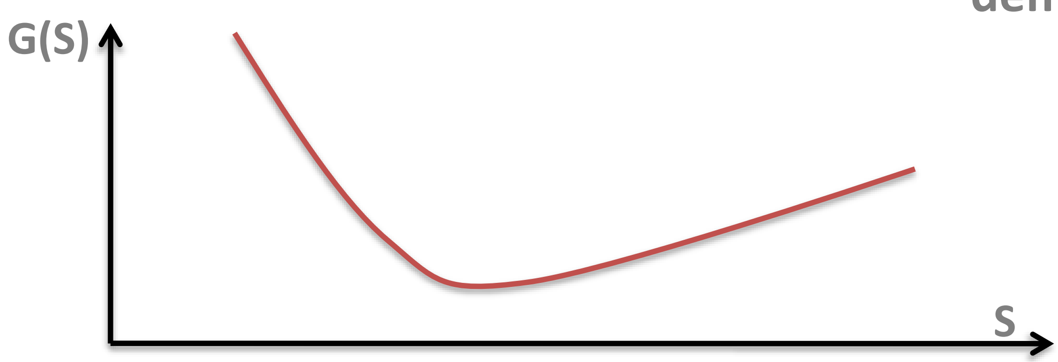
$$\mathbf{C(S, D)} = C_o \cdot \mathbf{[S\text{-}D]^+} + C_u \cdot \mathbf{[D\text{-}S]^+}$$

- The expected cost is a weighted average of all possible demands and costs

Expected Cost Function (Demand Uncertain)

- Expected Overage and Underage Cost Given Demand Distribution D and Stock S

$$G(S) = \overbrace{E[C(S,D)]}^{\text{Expected Cost}} = \sum_{\substack{\text{All possible}\\ \text{demand values}}} \overbrace{C(S,D)}^{\substack{\text{Cost when stock equals S}\\ \text{and demand equals D}}} \cdot \underbrace{P\{D\}}_{\substack{\text{Probability that}\\ \text{demand equals D}}}$$



The Newsvendor Model

- Uncertain demand D
- Costs:

C_o	Overage Cost per Unit
C_u	Underage Cost per Unit

- *Decision*: Stocking quantity S

Determining the Optimal Stock Level

Minimize Expected Overage and Underage Cost

$$\min_{S} \quad G(S) = E[C(S,D)] = C_o \underbrace{E[S-D]^+}_{\text{Expected overage}} + C_u \underbrace{E[D-S]^+}_{\text{Expected shortage}}$$

Solution (Newsvendor Solution):

- Find the value of S (denoted S*) such that the probability of meeting all the demand is $C_u/(C_o + C_u)$

That is, $\mathbf{P[D \le S^*]} = \underbrace{C_u/(C_o + C_u)}$

"Critical Ratio/Fractile", or "**Newsvendor Ratio/Fractile**"

Newsvendor Solution: Explanation

Marginal Analysis: Suppose you stock S units

Marginal Overage Cost with S		Marginal Underage Cost with S	
Probability of S being "over"	**$P\{D \le S\}$**	Probability of S being "under"	**$1 - P\{D \le S\}$**
Marginal cost of over-stocking	**$P\{D \le S\} * C_o$**	Marginal cost of under-stocking	**$\{1-P\{D \le S\}\} * C_u$**

To find the *optimal* stocking level:

$P\{D \le S\} * C_o$ = **Marginal Cost of Over-Stocking** = **Marginal Cost of Under-Stocking** = $\{1-P\{D \le S\}\} * C_u$

$$P\{D \le S\} = C_u / (C_u + C_o)$$

The problem is to find such S based on the demand distribution (CDF)!