MSBA 7004
Operations Analytics

Class 8-1: Inventory Analysis (III)
Demand Uncertainty, Newsvendor Problem
2023

Learning Objectives

- Understand more impact of demand uncertainty
  - On exact profit
- Understand the trade-off between *stock-out* (excess demand) and *excess inventory* (excess supply)
  - Newsvendor Model

True or False?

The motivation of the EOQ model is to match the demand with the right quantity of supply.

How much inventory should you hold?

- Trade-off #1: How much to order each time?

Inventory ordering costs
(Economies of scale)

Inventory holding costs

Economic Order Quantity (EOQ) Model

- Trade-off #2: How much to store each time?

Cost of running out

Cost of having excess inventory

Newsvendor Model

What is the Best Service Level (SL*) ?

- Trade-off:

	Inventory Holding Cost (Overage Cost)	Loss of Revenue & Goodwill (Underage Cost)
High Service Level	High	Low
Low Service Level	Low	High

Newsvendor Model: SL* Driven Decision

- A newsvendor stocks newspapers to sell that day
- Trade-offs:
  - If stocks <u>too few</u> newspapers, misses potential sales.
  - If stocks <u>too many</u> newspapers, money wasted on unsold newspapers.

How many newspapers should be stocked?

Newsvendor-type Problems: Not just Newspapers

- Fashion goods with short product life-cycles
- “Perishable” items (e.g., food, newspapers)
- Seasonal products
- Short-run capacity planning
- Revenue management for airlines/hotels

An Oil Rig Example

- Problem: Decide a suitable stock level for a perishable product which has historically random demand
- This product is used on an oil rig
  - The rig is re-supplied weekly from stocks on-shore
  - At the beginning of each week, a planner determines how much stock of the product is to be placed on the rig
  - When a shortage occurs, an emergency shipment is made
- How should we approach making this stocking decision?
  - Define an objective
  - Identify what data are required
  - Measure consequences of decision

Oil Rig Example: Cost Data

- What are the cost implications of our decision?

Cost of having too <u>much</u> stock	$\$C_o$ = \$1000 per unit
Cost of having too <u>little</u> stock (Emergency resupply cost)	$\$C_u$ = \$9000 per unit

- Decision Structure

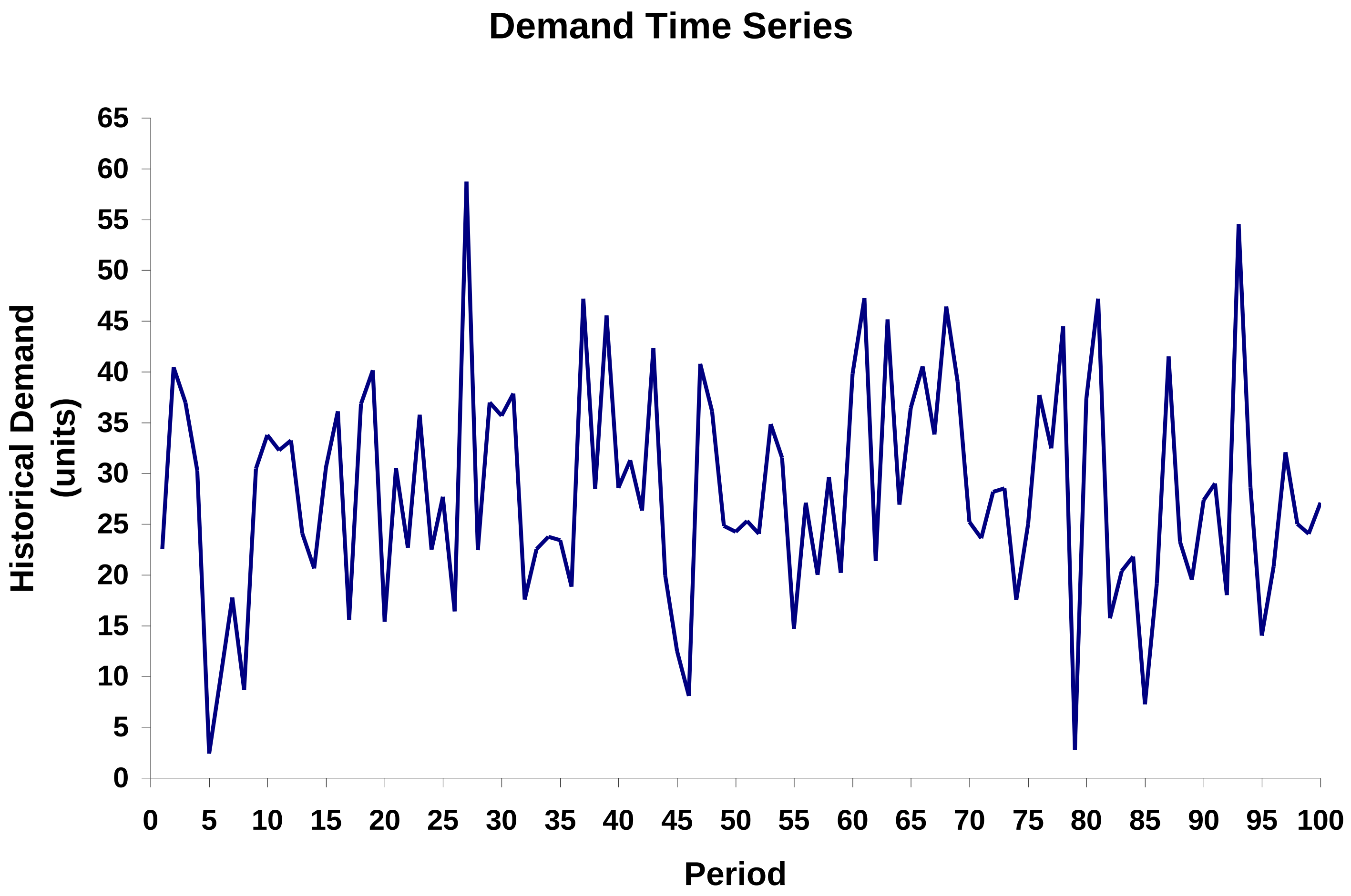
D	Customer demand that will occur in a week (random variable)
S	Our decision of how much to stock each week

Representing the Decision Problem

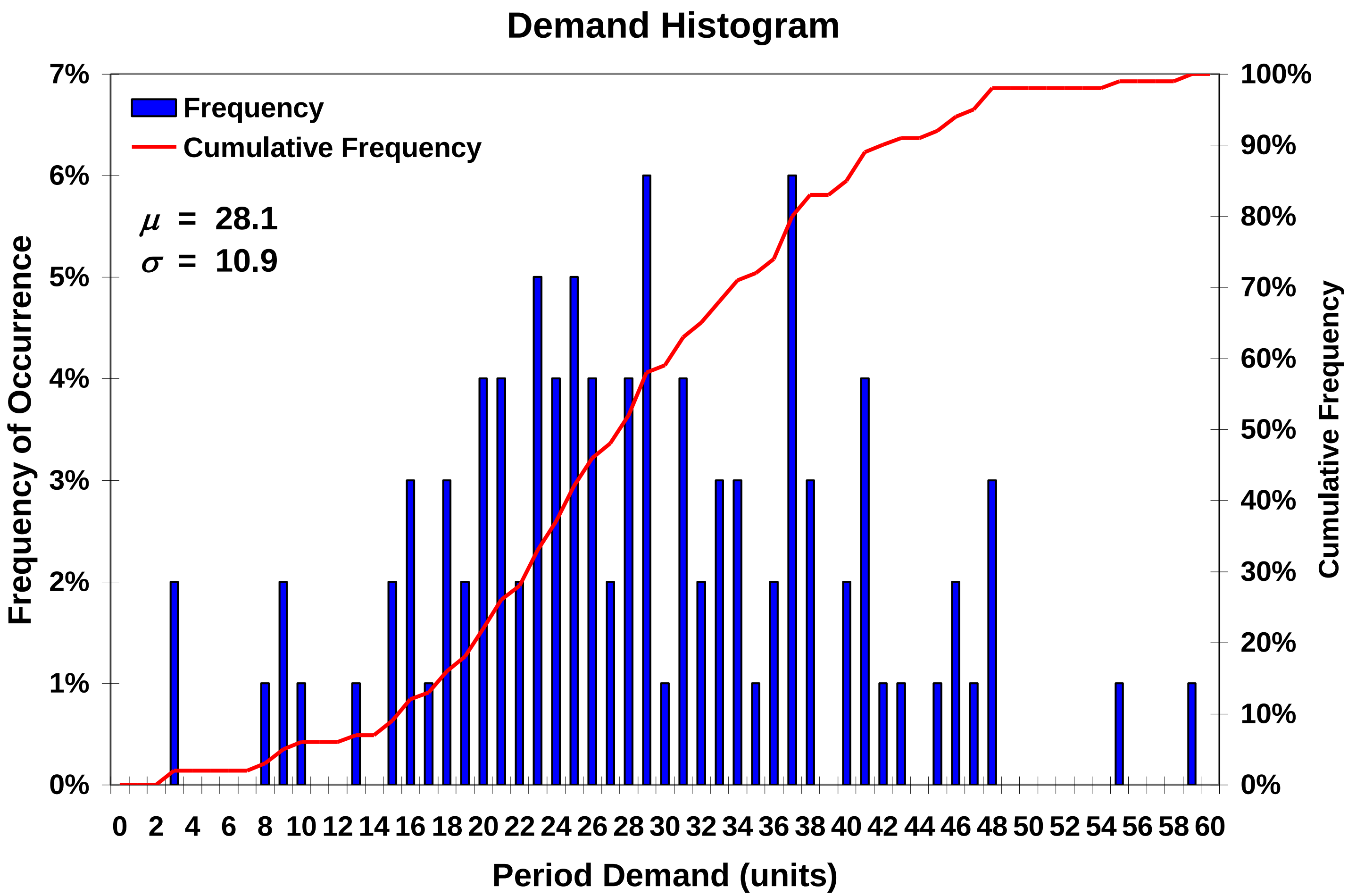
What is the cost if the demand (D) and the supply (S) were known? For example, if we had stocked 30 units each period for the previous 100 weeks.

- What would the *costs* have been each week?
  - What if demand exceeds supply that week?
  - What if supply exceeds demand that week?
- What is the total cost per week?
  - Holding and shortage costs
- What *fraction of the time (out of 100 cycles)* would we have had enough stock to satisfy demand?
  - Definition of SL in EOQ setting
- What *fraction of the time (out of 100 days)* would we have had enough stock to satisfy demand?
  - Definition of SL in Newsvendor setting

Oil Rig Example: Historical Demand



Oil Rig Example: Demand Distribution



Parameters In the Newsvendor Model

- Uncertain demand D
- Costs:

c_o	Overage Cost per Unit
c_u	Underage Cost per Unit

- *Decision*: Stocking quantity S

If Demand Were Known

- What is the cost if the demand (D) and the supply (S) were known?
  - What if demand exceeds supply?
  - What if supply exceeds demand?

- What is the total cost per week?
  - Overstock and shortage costs

Cost For the First Six Weeks, If S=30

Demand D	Stock S	Underage Cost	Overage Cost	Total Cost
23	30	0	7	7
40	30	90	0	90
30	30	0	0	0
16	30	0	14	14
2	30	0	28	28
11	30	0	19	19

Unit: thousand

C_o = \$1 per unit (excess inventory, has to be reprocessed)

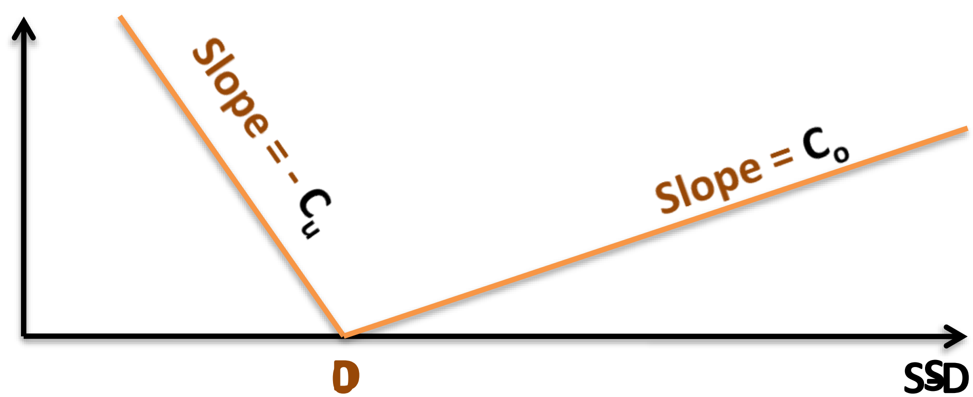
C_u = \$9 per unit (insufficient inventory, emergent call for resupply)

Do you think S=30 is good? Shall increase or decrease?

Cost Function (Demand Known)

- If D were known, the cost function would be

$$C(S, D) = \text{Cost}\,(S, D)$$
$$= C_o \cdot \max\{0, S\text{-}D\} + C_u \cdot \max\{0, D\text{-}S\}$$
$$= C_o \cdot \underbrace{[S\text{-}D]^+}_{\text{Had too much stock}} + C_u \cdot \underbrace{[D\text{-}S]^+}_{\text{Had too Little stock}}$$



There exists a fundamental economic tradeoff

Building the Cost Model

- Recall that the cost of (S) units is

Overage	$\$C_o$ = \$1000 per unit when supply > demand
Underage	$\$C_u$ = \$9000 per unit when supply < demand

- P(D): probability that demand is equal to D
- The cost when demand equals D is given by the relationship:

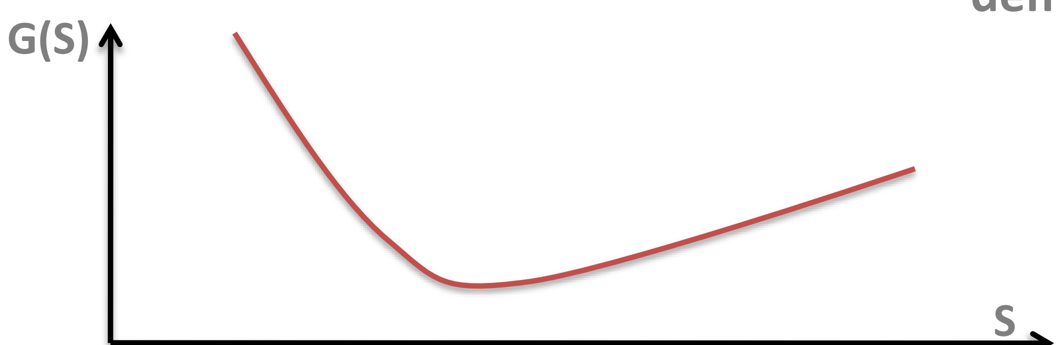
$$\mathbf{C(S, D)} = C_o \cdot \mathbf{[S\text{-}D]^+} + C_u \cdot \mathbf{[D\text{-}S]^+}$$

- The expected cost is a weighted average of all possible demands and costs

Expected Cost Function (Demand Uncertain)

- Expected Overage and Underage Cost Given Demand Distribution D and Stock S

$$G(S) = \overbrace{E[C(S,D)]}^{\text{Expected Cost}} = \sum_{\substack{\text{All possible} \\ \text{demand values}}} \overbrace{C(S,D)}^{\substack{\text{Cost when stock equals S} \\ \text{and demand equals D}}} \cdot \underbrace{P\{D\}}_{\substack{\text{Probability that} \\ \text{demand equals D}}}$$



The Newsvendor Model

- Uncertain demand D
- Costs:

C_o	Overage Cost per Unit
C_u	Underage Cost per Unit

- *Decision*: Stocking quantity S

Determining the Optimal Stock Level

Minimize Expected Overage and Underage Cost

$$\min_{S} \quad G(S) = E[C(S,D)] = C_o \underbrace{E[S-D]^+}_{\text{Expected overage}} + C_u \underbrace{E[D-S]^+}_{\text{Expected shortage}}$$

Solution (Newsvendor Solution):

- Find the value of S (denoted S*) such that the probability of meeting all the demand is $C_u/(C_o + C_u)$

That is, $P[D \le S^*] = \underbrace{C_u/(C_o + C_u)}$

"Critical Ratio/Fractile", or "**Newsvendor Ratio/Fractile**"

Newsvendor Solution: Explanation

Marginal Analysis: Suppose you stock S units

Marginal Overage Cost with S		Marginal Underage Cost with S	
Probability of S being "over"	**P{D ≤ S}**	Probability of S being "under"	**1 - P{D ≤ S}**
Marginal cost of over-stocking	**P{D ≤ S} * C_o**	Marginal cost of under-stocking	**{1-P{D ≤ S}} * C_u**

To find the *optimal* stocking level:

P{D ≤ S} * C_o = **Marginal Cost of Over-Stocking** = **Marginal Cost of Under-Stocking** = **{1-P{D ≤ S}} * C_u**

$$P\{D \leq S\} = C_u / (C_u + C_o)$$

The problem is to find such S based on the demand distribution (CDF)!

How to Interpret C_u and C_o?

C_u , C_o are the unit opportunity costs.

You should compute C_u:

Suppose the actual demand D is larger than my current stock S. If I could return to the past and order one unit more, then it increases my profit (or reduces my cost) by C_u.

You should compute C_o:

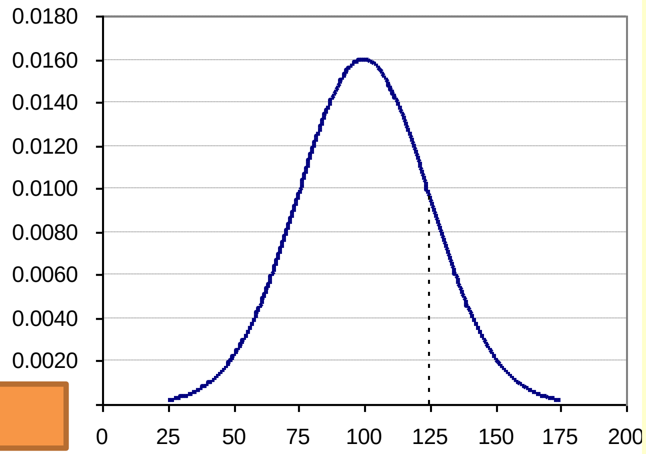
Suppose the actual demand D is smaller than my current stock S. I could return to the past and order one unit less, then it increases my profit (or reduces my cost) by C_o.

Solving S*: Demand with Normal Distribution (1)

Start with
$\mu = 100$,
$\sigma = 25$.

$D \sim N(\mu, \sigma^2)$

S = 125



P(D ≤ 125)

=

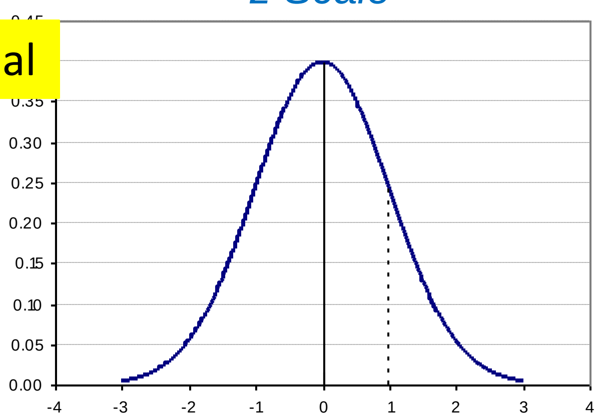
P(X ≤ 1)

z-Scale

Standard normal

$X \sim N(0,1)$

$$z = \frac{Q - \mu}{\sigma} = \frac{125 - 100}{25} = 1$$



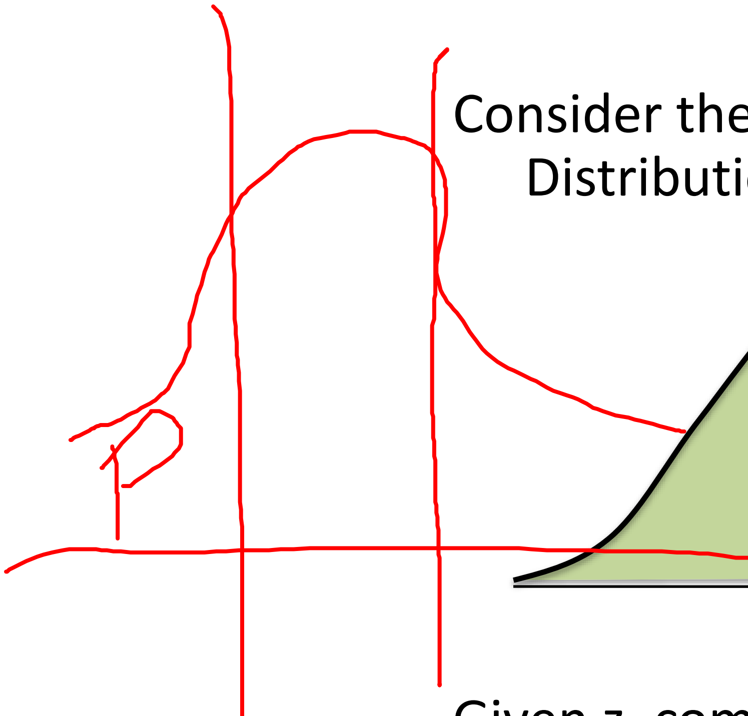
Problem: How to find S such that *Prob*$\{D \leq S\}$ =critical ratio(known)?

- Let *S* be the order quantity, and (μ, σ) the parameters of the normal demand distribution
- *Prob*$\{D \leq S\}$ = *Prob*{a standard normal random variable $X \leq z$}, where

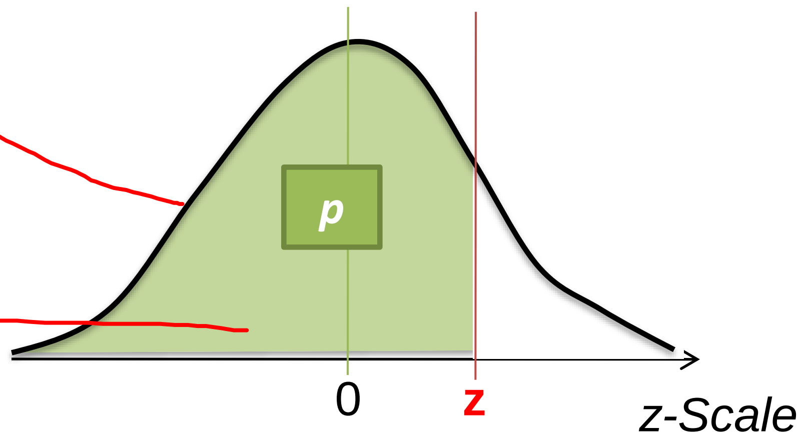
$$z = \frac{S - \mu}{\sigma} \quad \text{or} \quad S = \mu + z \times \sigma$$

- Given the critical ratio, look up *Prob*{the outcome of a standard normal is *z* or lower} in the Standard Normal Distribution Function Table, excel NORM.S.INV function, python norm.ppf(p) or any statistical software, then solve z and S

Solving S*: Demand with Normal Distribution (2)



Consider the Standard Normal Distribution (in z-scale).



Given z, compute probability p using Excel NORM.S.DIST(z) or python norm.cdf(z).

Given p, compute z using Excel NORM.S.INV(p) or python norm.ppf(p).

Example: given C_u = 87, C_o =13, then the critical ratio

$$P(D \leq S^*) = \frac{C_u}{C_o + C_u} = 0.87, \text{ how to find S?}$$

z	0	0.01	0.02	0.03	0.04	0.05
0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894
1	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943

$$P(D \leq S^*) = P(X \leq z^*) = \frac{C_u}{C_o + Cu} = 0.87$$

If P<0.5, then Z takes a negative value

z^*=1.13, S= $\mu + z^* \sigma = 100 + 1.13 * 25 = 128.5$

Example 1: Newsvendor Model with Normal Demand

For the academic year 2020-2021, demand for HKU T-shirts is normally distributed with mean 1000 and standard deviation 200.

Cost of shirts is $10.

Selling price is $15.

Unsold shirts can be sold off at $8 in the summer of 2021.

How many shirts should the HKU bookstore buy for the 2020-2021 academic year?

Mean demand	1000
STD of demand	200

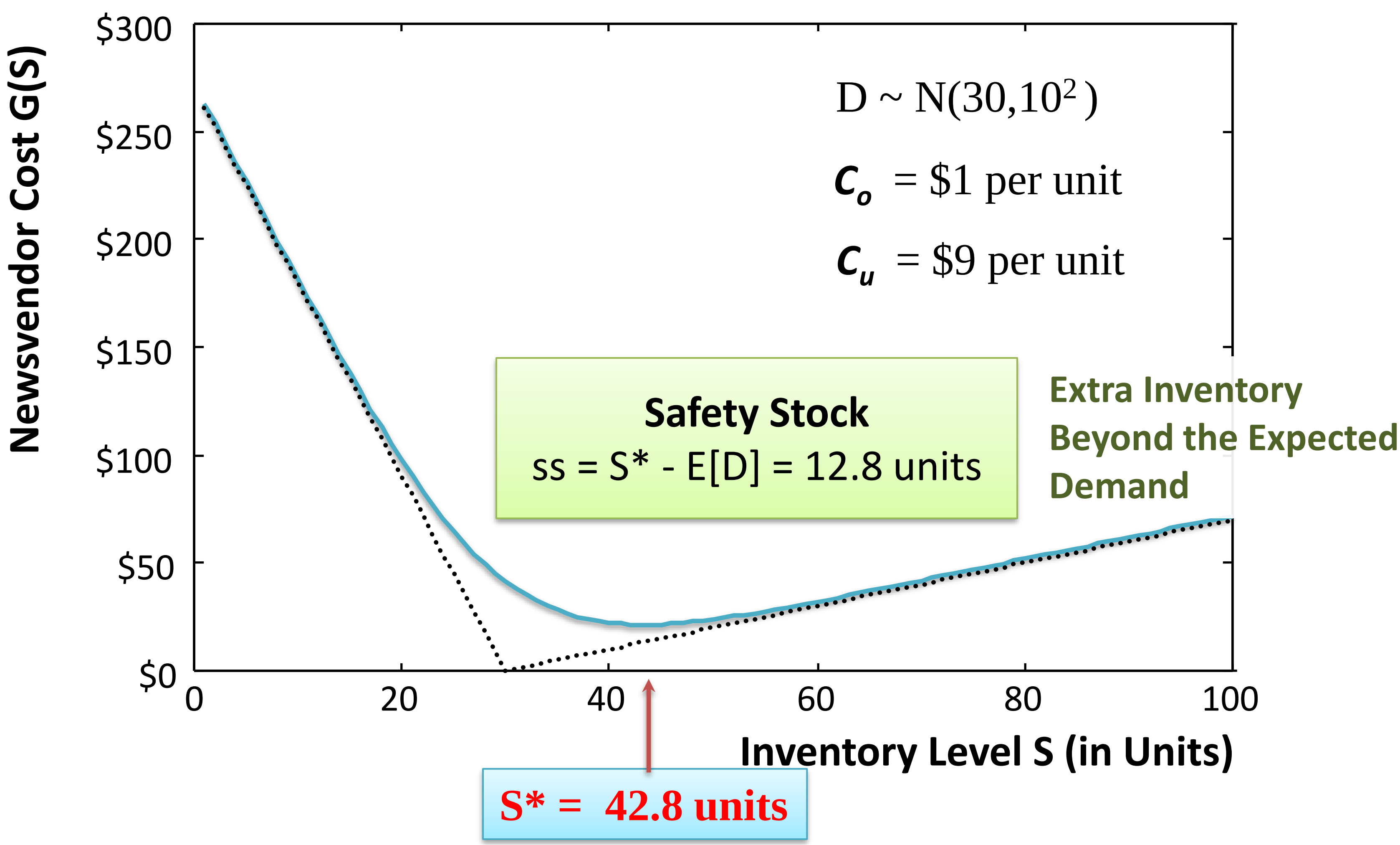
C_o	10 – 8 = 2
C_u	15 – 10 = 5

$$p = \frac{C_u}{C_u + C_o} = \frac{5}{5+2} = \frac{5}{7} = 71.4\%$$

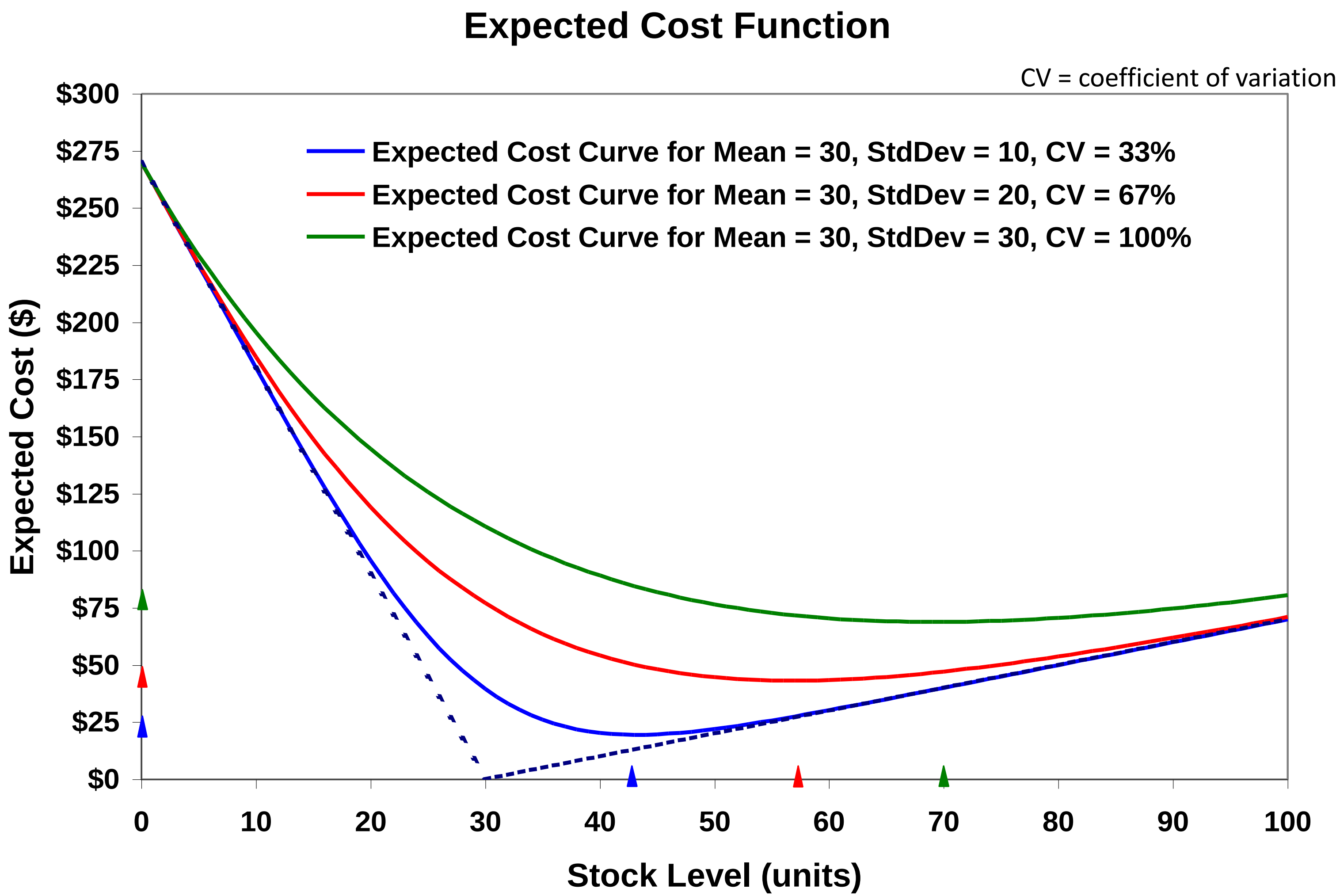
$$z = \mathrm{NORMSINV}(p) = 0.57$$

$$S = 1000 + 200 \cdot z = 1113.2$$

Example 2: Newsvendor Model with Normal Demand



Impact of Demand Variability



4.5G* Service Plan

Recontracting offers are up for grabs! >>

4.5G Service Plan | 4.5G Multi-User Service Plan

Monthly SIM only plan fee(1)(3) $158	Monthly SIM only plan fee(1)(3) $198	Monthly SIM only plan fee(1)(3) $298
Monthly handset plan fee(2)(3) $298	Monthly handset plan fee(2)(3) $408	Monthly handset plan fee(2)(3) $548
Local data usage(6) 5GB	Local data usage(6) 6GB+2GB	Local data usage(6) 6GB+2GB
Voice call mins(8)(9)(10) Unlimited	Voice call mins(8)(9)(10) Unlimited	Voice call mins(8)(9)(10) Unlimited
-	-	Mainland China & Macau roaming data usage 2GB(24)
Local Mobile data Top-up Plan(19) Top-up Data Package HK$28/200MB or HK$50/GB -	Local Mobile data Top-up Plan(19) Top-up Data Package HK$28/200MB or HK$50/GB -	Local Mobile data Top-up Plan(19) Top-up Data Package HK$28/200MB or HK$50/GB Capacity Data Package(22) HK$58/month

Example: Cell Phone Data Plan

- Monthly plan charge: $150 fixed charge plus 5 cents/MB
- Over-the-plan charge: 35 cents/MB
- Monthly usage: N(2000MB, 400^2 MB^2)
- Suppose you can pre-select any amount of data for your plan. What is the amount of MBs you should select for your plan?

Mean demand	2000
STD of demand	400

C_o	5
C_u	35 – 5 = 30

$$p = \frac{C_u}{C_u + C_o} = \frac{30}{30+5} = \frac{6}{7} = 85.7\%$$

$$z = \text{NORMSINV}(p) = 1.06$$

$$S = 2000 + 400 \cdot z = 2427$$

Application: Revenue Management

- Revenue Management: Technique to maximize revenue by matching fixed supply with uncertain demand
- Use it when …

Fixed inventory/capacity Expensive or impossible to expand	Inventory/Capacity **committed** to a customer before all demand is known
Different customer segments Can differentiate or price-discriminate among customers	**Same Unit of inventory/capacity** can satisfy different customer segments

15:05 HKG — 4h 10m — 19:15 SIN

CX635

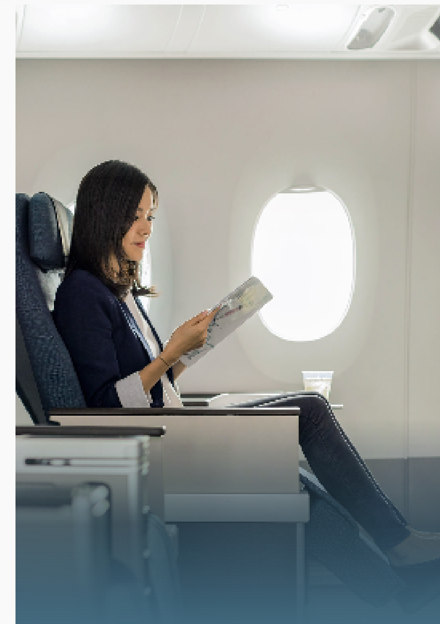
View details

From HKD3,800

From HKD5,940

View fare

	Light	Essential	Flex
Seat	Reserve seat **For a fee**	Regular seat **Free of charge**	Extra legroom seat **Free of charge** Preferred seat **Free of charge** Regular seat **Free of charge**
Baggage	Cabin baggage **1pc, 7kg each** Check-in baggage **1 pc , 23kg each**	Cabin baggage **1pc, 7kg each** Check-in baggage **2 pcs , 23kg each**	Cabin baggage **1pc, 7kg each** Check-in baggage **2 pcs , 23kg each**
Loyalty	Status Points **8** Asia Miles **800** Upgrade with Asia Miles -	Status Points **10** Asia Miles **1000** Upgrade with Asia Miles **Eligible**	Status Points **20** Asia Miles **2000** Upgrade with Asia Miles **Eligible**
Flexibility	Flight change fee HKD**1,100** + Potential fare difference Cancellation fee HKD**1,100** No show fee HKD**1,300** Standby for earlier flight -	Flight change fee HKD**900** + Potential fare difference Cancellation fee HKD**900** No show fee HKD**1,300** Standby for earlier flight -	Flight change fee **Free of charge** + Potential fare difference Cancellation fee HKD**900** No show fee HKD**1,300** Standby for earlier flight **Available**
	HKD**3,800**	HKD**4,250**	HKD**4,750**
	Select Light	Select Essential	Select Flex



Upgrade to Premium Economy

Get more personal space to stay relaxed and productive at work with Premium Economy Class

View fares

Please view fare condition explanation to find the detailed explanation of each fare condition.

Revenue Management: Example

- HKG-SIN Flight: 200 seats (coach seats only)
- Two customer types

Leisure Traveler	Business Traveler
Advance booking	Late booking
Discount fare ($750)	Full fare ($2500)

Assumption

The number of business travelers is normally distributed with mean 75 and standard deviation 25.

C_o	750
C_u	2500-750 = 1750

Set aside:
75 + normsinv(1750/2500)*25 = 88.1
number of seats for business travelers

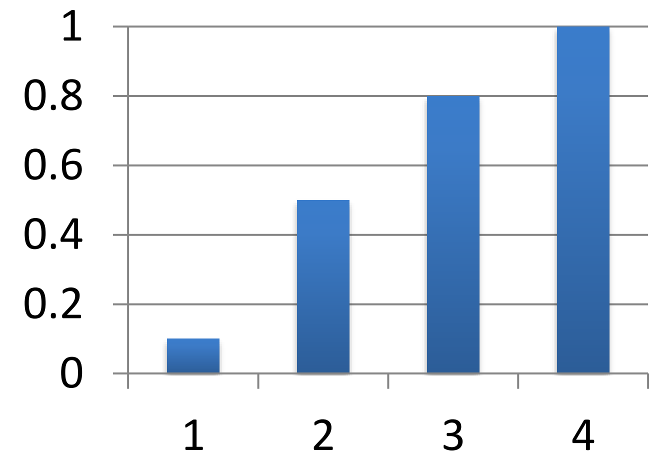
Demand with Discrete Distribution (1)

$$\min_{S} \quad G(S) = E[C(S,D)] = C_o E[S-D]^+ + C_u E[D-S]^+$$

G(S) is the expected overage and underage cost if we order S units.

- Suppose C_u=3 and C_o =1

d	Prob[D=d]	CDF(d)
1	0.1	0.1
2	0.4	0.5
3	0.3	0.8
4	0.2	1.0

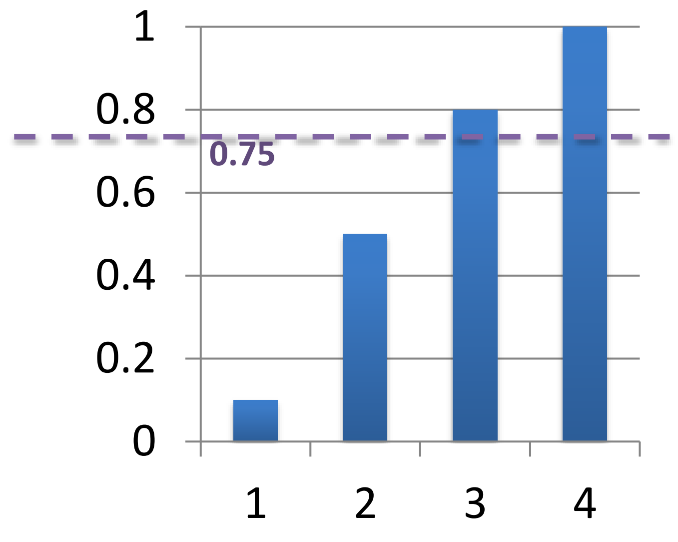


- What is the optimal stocking quantity S*?
- What is the optimal expected cost G(S*)?

Demand with Discrete Distribution (2)

$$\min_{S} \quad G(S) = E[C(S,D)] = C_o E[S-D]^+ + C_u E[D-S]^+$$

S	Expected overage $E[S\text{-}D]^+$	Expected underage $E[D\text{-}S]^+$	G(S)
1	(0.1)*0 + (0.4)*0 + (0.3)*0 + (0.2)*0 = 0	(0.1)*0 + (0.4)*1 + (0.3)*2 + (0.2)*3 = 1.6	1*(0) + 3*(1.6) = 4.8
2	(0.1)*1 + (0.4)*0 + (0.3)*0 + (0.2)*0 = 0.1	(0.1)*0 + (0.4)*0 + (0.3)*1 + (0.2)*2 = 0.7	1*(0.1) + 3*(0.7) = 2.2
3	(0.1)*2 + (0.4)*1 + (0.3)*0 + (0.2)*0 = 0.6	(0.1)*0 + (0.4)*0 + (0.3)*0 + (0.2)*1 = 0.2	1*(0.6) + 3*(0.2) = 1.2
4	(0.1)*3 + (0.4)*2 + (0.3)*1 + (0.2)*0 = 1.4	(0.1)*0 + (0.4)*0 + (0.3)*0 + (0.2)*0 = 0	1*(1.4) + 3*(0) = 1.4



Solution: **S^* = smallest S such that P[D ≤ S]** >= $C_u/(C_o + C_u)$

Risk Pooling: Example

Four markets

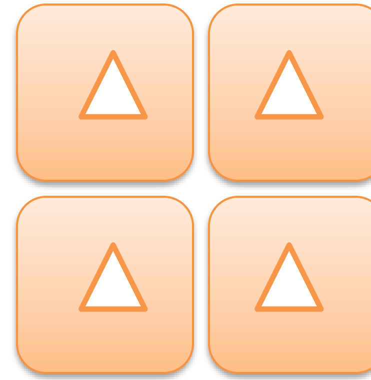
One warehouse for each market

Each warehouse experiences demand $N(\mu, \sigma^2)$ with mean μ=100 and std σ=50.

Assume ***service level*** (probability of not stocking out) of 90%.

What is the safety stock at each warehouse?

What is the total safety stock at the four warehouses combined?



z = NORMSINV(0.90) = 1.28

Inventory level:

$S = \mu + \sigma * z = 164.1$

Safety Stock:

$ss = S - \mu = 64.1$

Total Safety Stock:

4 * ss = 256.3

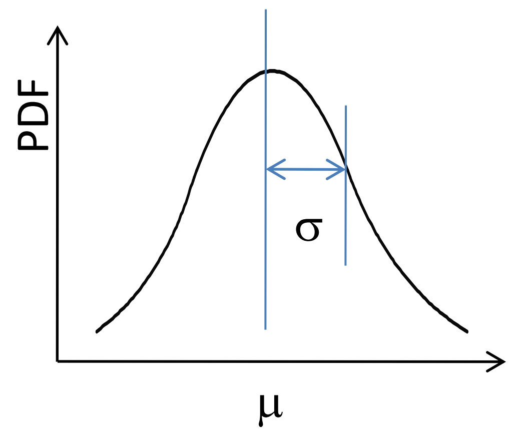
If you decide to serve all four markets from a central warehouse, what is the total safety stock?

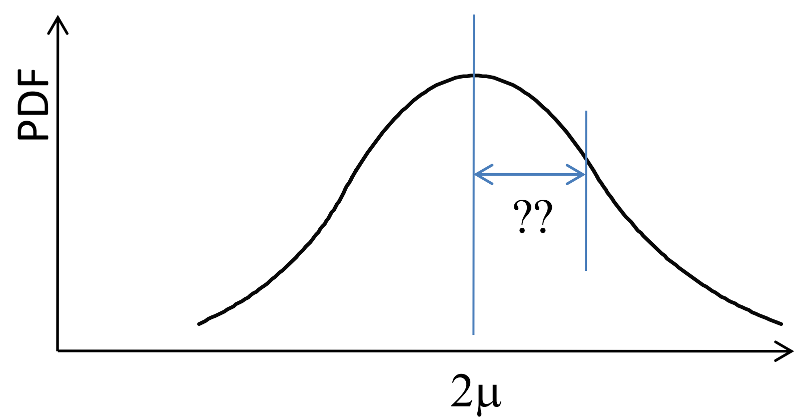
Risk Pooling (or Demand Aggregation)

- Independent demand streams impose greater variability when compared to a “pooled” demand stream
- Approach: Adding independent random variables
- Example Applications
  - Component commonality in product design
  - Portfolio effects in finance
  - Safety stock

Adding Normal Distributions

- Daily demand at one warehouse: $N(100, 50^2)$
- Suppose you combine two identical warehouses into one. What is the mean and standard deviation at the combined warehouses?



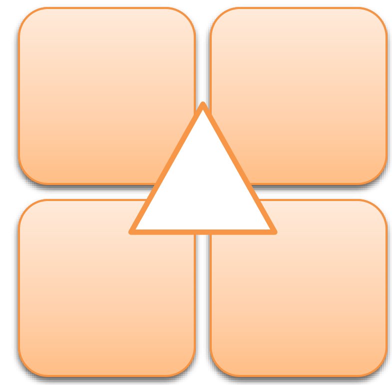


Risk Pooling: Example (Revisited)

Four markets with Normal demand $N(\mu, \sigma^2)$ with mean μ=100 and std σ=50.

With four separate warehouses …

Total Safety Stock (Before):
256.3



If you decide to serve all four markets from a central warehouse, what is the total safety stock?

With the central warehouse, the aggregate demand is normally distributed with mean 4μ=400 and std $\sqrt{4}\sigma$=2σ=100.

What is the safety stock now?

z = NORMSINV(0.90) = 1.28

Inventory level:
S = $4\mu + (2\sigma) * z$ = 528.2

New Safety Stock:
ss = S - 4μ = 128.2

Risk Pooling (Another Example)

Computer A
With Hard Drive X

$N(100, 30^2)$

Computer B
With Hard Drive Y

$N(300, 40^2)$

Order lead time = 1 day
97.8% Critical fractile

z = NORMSINV(0.978) = 2.0

Inventory Level
A: 100+(2.0)(30) = 160
B: 300+(2.0)(40) = 380

Total Inventory: 160 + 380 = 540

What if I use the same hard drive for both products?

Common Hard Drive

Normal with
Mean = ???
STD = ???

Inventory Level
400+(2.0)(50) = 500

Centralization/Pooling: Benefits

- Centralization reduces safety stocks (pooling)
  - Can offer better service for the same inventory investment, or same service with smaller inventory investment
- Some methods to achieve pooling efficiencies
  - Physical centralization
  - Information centralization
  - Commonality
  - Postponement / Late customization
- Cost saving are proportional to the square root of the number of locations pooled (square root rule)

Newsvendor Summary

- Newsvendor model balances lost opportunity cost and over-stocked waste cost
  - Stocking decision is driven by demand uncertainty and over-stocking and under-stocking cost structure
  - Safety stock is used to cope with demand uncertainty
- Many revenue management techniques use newsvendor principles

Useful contents from textbook

Exhibit 14.1

A PROCESS FOR EVALUATING THE PROBABILITY DEMAND IS EITHER LESS THAN OR EQUAL TO *Q* (WHICH IS *F*(*Q*)) OR MORE THAN *Q* (WHICH IS 1 − *F*(*Q*))

If the demand forecast is a normal distribution with mean μ and standard deviation σ, then follow steps A and B:

A. Evaluate the z-statistic that corresponds to Q:

$$z = \frac{Q - \mu}{\sigma}$$

B. The probability demand is less than or equal to Q is $\Phi(z)$. With Excel $\Phi(z)$ can be evaluated with the function Normsdist(z); otherwise, look up $\Phi(z)$ in the Standard Normal Distribution Function Table in Appendix B. If you want the probability demand is greater than Q, then your answer is $1 - \Phi(z)$.

If the demand forecast is a discrete distribution function table, then look up $F(Q)$, which is the probability demand is less than or equal to Q. If you want the probability demand is greater than Q, then the answer is $1 - F(Q)$.

Useful contents from textbook

Exhibit 14.2

A PROCEDURE TO FIND THE ORDER QUANTITY THAT MAXIMIZES EXPECTED PROFIT IN THE NEWSVENDOR MODEL

Step 1 Evaluate the critical ratio: $\frac{C_u}{C_o+C_u}$ In the case of the Hammer 3/2, the underage cost is C_u = Price – Cost and the overage cost is C_o = Cost – Salvage value.

Step 2 If the demand forecast is a normal distribution with mean μ and standard deviation σ, then follow steps A and B:

A. Find the optimal order quantity if demand had a standard normal distribution. One method to achieve this is to find the z value in the Standard Normal Distribution Function Table such that

$$\Phi\left(z\right) = \frac{C_u}{C_o + C_u}$$

(If the critical ratio value does not exist in the table, then find the two z values that it falls between. For example, the critical ratio 0.80 falls between z = 0.84 and z = 0.85. Then choose the larger of those two z values.) A second method is to use the Excel function Normsinv: z = Normsinv(Critical ratio).

B. Convert z into the order quantity that maximizes expected profit, Q: $Q = \mu + z \times \sigma$

Practice Problems

- Office Hours: 9:30-11:30 AM
- Breakout room B104-1