Homework for Linear Algebra
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Exercise 1. (i)

- S' is a basis for span$(S) \Rightarrow S'$ is maximally linearly independent:
  From the definition of basis, S' is linearly independent.S' satisfies the definition of maximally linearly independent. For any $\mathbf{v} \in span(S) \backslash S'$, it can be represented by a linear combination of vectors in S' ,so $S' \cup \mathbf{v}$ is linear dependent.

- S' is maximally linearly independent $\Rightarrow S'$ is a basis for span(S):
  From the definition of maximally linearly independent, for any $\mathbf{v} \in S \backslash S'$,it can be repensented by a linear combination of vectors in S'.
  For any vector in $span(S)$,it can be repensented by vectors in S ,and inductively, be repensented by vectors in S'. Also S' is linear independent, so it satisfies a basis for $span(S)$.

(ii)

- S is a basis for $V \Rightarrow S$ is maximally linearly independent in V:
  Since S is a basis, it is linear independent and spans V.
  So if it is not maximally linear independent, there exists $\mathbf{v} \in V \setminus S$ that $S \cup \mathbf{v}$ is linear independent.
  This contradicts that S spans V.

- S is maximally linearly independent $\Rightarrow S$ is a basis for V:
  If S isn't a basis for V, there exists $\mathbf{v} \in V$that $\mathbf{v} \notin span(S)$.
  So $S \cup \mathbf{v}$ remains linear independent.Contradict the definition.

(iii)Assume S' is a subset of S and is always linear independent. To build a S', we inductively add vectors in S one by one.Until no more vectors can be added.Since S is a finite set ,the process will always have a end.
So for any $\mathbf{v} \in S \setminus S'$, $S' \cup \mathbf{v}$ will lead to linear dependent.
Now S' satisfies the definition of maximally linear independent.

Exercise 2. First,consider the situation that any of $\mathbf{v}, \mathbf{w} = \mathbf{0}$.
Obviously, (i) or (ii) will hold in this situation.
Under the condition that none of $\mathbf{v}, \mathbf{w} = \mathbf{0}$.Since $\mathbf{u}_1 \cdots \mathbf{u}_n, \mathbf{v}, \mathbf{w}$ are linearly dependent, there exists

$$c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n + a\mathbf{v} + b\mathbf{w} = \mathbf{0}$$

(Assume not all $c1 \cdots cn, a, b$ are zero.)
Since $\mathbf{u}_1 \cdots \mathbf{u}_n$ are linearly dependent ,the only way that

$$c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n = \mathbf{0}$$

is all the coefficients are zero. So a,b cannot all be zero. So we have:
If $b = 0, a \neq 0$, $c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n = -a\mathbf{v}$.(i) holds.
If $a = 0, b \neq 0$, $c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n = -b\mathbf{w}$.(ii) holds.
If $a \neq 0, b \neq 0$, $c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n = -b\mathbf{w} - \mathbf{v}$.(iii) holds.

Exercise 3. $\mathbf{v}_3$

Exercise 4. If V has finite many vectors and they are all linearly independent.

Exercise 5. (1,1,0,1),(10,7,2,3),(0,0,1,0),(0,0,0,1)

Exercise 6.

- $\mathbf{W} \subsetneq \mathbf{V} \Rightarrow dim(\mathbf{W}) < dim(\mathbf{V})$
  Assume $dim(\mathbf{W}) = dim(\mathbf{V})$.
  Since $\mathbf{W} \subsetneq \mathbf{V}$, there exists $\mathbf{v} \in \mathbf{V} \setminus \mathbf{W}$ and cannot be represented by the basis of $\mathbf{W}$.So $dim(\mathbf{W} \cup \mathbf{v}) = dim(\mathbf{W}) + 1 > \mathbf{V}$
  Since $\mathbf{v} \cup \mathbf{W} \in \mathbf{V}$, $dim(\mathbf{v} \cup \mathbf{W}) \leq dim(\mathbf{V})$. Contradict!
- $\mathbf{W} \subsetneq \mathbf{V} \Leftarrow dim(\mathbf{W}) < dim(\mathbf{V})$
  Assume $\mathbf{W} = \mathbf{V}$, from the definition of basis ,if $dim(\mathbf{W}) < dim(\mathbf{V})$, then the current basis of V doesn't hold.Contradict!

Exercise 7. (i)Since all the column vectors of A $\in R^m$, so

$$dim(C(A)) <= dim(R^m) = m.$$

So the basis of $C(A)$ has at most m vectors. Since we have n column vectors, they are linearly dependent.
(ii)For A $\begin{bmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_n\end{bmatrix}$, we have the linear combination:

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

And since A is linearly dependent ,there exists nonzero solution to :

$$c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$$

So if $A\mathbf{x} = \mathbf{b}$ holds, $A\begin{bmatrix}x_1 + k * c_1 \\ \vdots \\ x_n + k * c_n\end{bmatrix} = \mathbf{b}$ $(k \in Z)$ holds too ,it has infinite many solutions.
If $dim(C(A)) < m$,which means $span(C(A)) \subsetneq R^m$,there exists $\mathbf{v} \in R^m \setminus span(C(A))$ that has no solution.
(iii)

- column rank of A is m $\Rightarrow A\mathbf{x} = \mathbf{b}$ has infinite many solutions
  From previous provements,we know that if $dim(C(A)) = m = R^m$, $span(C(A)) = R^m$, and if the equation has a solution, it will have infinite many solutions.

- column rank of A is m $\Leftarrow A\mathbf{x} = \mathbf{b}$ has infinite many solutions
  If there always exists a solution, that means $R^m \subseteq span(C(A))$. Since the column vectors $\in R^m$ So $span(C(A)) = R^m$.So the column rank of A is m.

Exercise 8. Write

$$A = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{bmatrix}$$

As row vectors in R^n.So does

$$I = \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix}$$

. Assume there exist B thats holds the equation, then

$$\mathbf{e}_1 \cdots \mathbf{e}_n \in span(\{\mathbf{a}_1 \cdots \mathbf{a}_m\})$$

So it implies

$$R^n = span(\{\mathbf{e}_1 \cdots \mathbf{e}_n\}) \subseteq span(span(\{\mathbf{a}_1 \cdots \mathbf{a}_m\})) = span(\{\mathbf{a}_1 \cdots \mathbf{a}_m\}) \subseteq R^n$$

So the row rank of A is n.

If the row rank of A is less than n.That means $span(\{\mathbf{a}_1 \cdots \mathbf{a}_m\}) \subsetneq R^n$. So there must exist a $\mathbf{e}_i \in R^n \setminus R(A)$ that can not be reresented by linear combinations of the row vectors. Unless $dim(span(\{\mathbf{a}_1 \cdots \mathbf{a}_m\})) = dim(R^n)$.So the row rank must be n.