Homework for Linear Algebra
September 29, 2024

Chengyu Zhang

Exercise 1. Total m*n matrices.

$$\left.\underbrace{\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix} \cdots \begin{bmatrix}0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix}}_{\text{n}}\quad\right\}\text{m}$$

Exercise 2. The given vector space has one basis element. Let it be $\mathbf{e}$
So for any $x \in V$, we can write x as a "scalar multiple" of $\mathbf{e}$. That is, there exists a scalar α such that $x = \alpha \otimes \mathbf{e} = \mathbf{e}^{\alpha}$.

Exercise 3. Yes. Define a vector space

$$\mathbf{V} = \{0, 1\}$$

with

$$x \oplus x' = (x + x')mod2, x \otimes x' = (x * x')mod2.$$

It only has two vectors(0),(1).

Exercise 4. $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4), (\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5), (\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_6), (\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6)$,those groups of vectors are each linear dependent.So the largest possible number is 3.

Exercise 5. (i)

$$\left\{\begin{array}{c} 2x_1 + 3x_2 + 4x_3 + 5x_4 + 1x_5 = 0 \\ x_2 + x_4 = 0 \end{array}\right. \rightarrow \left\{\begin{array}{c} 2x_1 + 3x_3 + 2x_4 + 1x_5 = 0 \\ x_2 + x_4 = 0 \end{array}\right.$$

$$\mathbf{N}(A) = span\left\{\begin{bmatrix}-2\\0\\1\\0\\0\end{bmatrix} \begin{bmatrix}-2\\-1\\0\\2\\0\end{bmatrix} \begin{bmatrix}-1\\0\\0\\0\\2\end{bmatrix}\right\}, \mathbf{C}(A) = span\left\{\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}3\\1\end{bmatrix}\right\}$$

(ii)

$$\mathbf{N}(A) = span\left\{\begin{bmatrix}1\\0\\0\\2\end{bmatrix} \begin{bmatrix}0\\0\\2\\1\end{bmatrix} \begin{bmatrix}0\\2\\0\\-1\end{bmatrix}\right\}, \mathbf{C}(A) = span\left\{\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\0\\2\\1\end{bmatrix} \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}\right\}$$

The null spaces and column spaces are written as a span of their basis. So the vectors above are their basis.

Exercise 6.

$$\begin{cases} x_1+(1-k)x_2+(1-k)x_3+(1-k)x_4=0 \\ x_2+(1-k)x_3+(1-k)x_4=0 \\ x_3+(1-k)x_4=0 \\ -kx_1-kx_2-kx_3+(1-k)x_4=0 \end{cases} \rightarrow \begin{cases} x_1-kx_2=0 \\ x_2-kx_3=0 \\ x_3+(1-k)x_4=0 \\ -kx_1-kx_2-kx_3+(1-k)x_4=0 \end{cases}$$

$$\rightarrow (k^3+k^2+k-1)(1-k)x_4=0$$

If $k=1$,the equation system will have infinite many solutions.So the vectors are linear dependent.

When $k \neq 1$ the vectors are linear independent.

Exercise 7. (i)Assume $\mathbf{v} \in span(S_1)$,$\mathbf{v}$ can be expressed by linear combination of vectors in span(S1). Since $S_1 \subseteq S_2$, $\mathbf{v} \in S_2$ .So $\mathbf{v}$ can be written in a linear combination form of vectors in $span(S_2)$, which is just how $span(S_2)$ is defined.
So every vector in $span(S_1)$ is in $span(S_2)$. $span(S_1) \subseteq span(S_2)$.

(ii)If span(S_1) $\neq$ span(span(S_2)), it means that span(S_1) isn't the SMALLEST subspace of V containing S_1. Contradict!

(iii)If $S_1 \subseteq$ span(S_2) , so span(S_1) $\subseteq S_1 \subseteq$ span(S_2)

Exercise 8. (i)S_1 can be represented by $span(S_1)$.If $span(S_1) \subseteq span(S_2)$. Every $\mathbf{v} \in S_1$ can be represented by $span(S_2)$.So S_1 can be represented by S_2, if and only if $span(S_1) \subseteq span(S_2)$.

(ii)If S_1 can be represented by S_2 ,then all the $v \in span(S_1) \subseteq S_1$ can also be represented by $span(S_2)$.Because all $\mathbf{v} \in S$ can be represented by $span(S_1)$, so it can be represented by $span(S_2)$
So S can be represented by S$_2$.

(iii) If $\mathbf{v}$ can be represented by $S \setminus \mathbf{v}$,then $\mathbf{v} \notin span(S)$. Because any vector in span(S) cannot be represented by other vectors in span(S).So we have

$$span(S \setminus \mathbf{v}) = span(S)$$