Homework for Linear Algebra
December 26, 2024

Chengyu Zhang

Exercise 1.

1.1 For any u,since the second vector is zero vector

$$\langle \mathbf{u}, \mathbf{0} \rangle = \langle \mathbf{u}, 0\mathbf{0} \rangle = 0\langle \mathbf{u}, \mathbf{0} \rangle = 0$$

1.2 If $\mathbf{v} \neq 0$, for any nonzero v, we have

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{u}||||\mathbf{v}||} = 0$$

But since the choose of $\mathbf{u}$ is arbitrary, we can always find $\mathbf{u}$ that the cos of angle of u and v is not zero. Contradict. So v must be zero.

1.3

$$\langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle Rightarrow \langle \mathbf{u} - \mathbf{v}, \mathbf{w} \rangle = 0$$

From what we have known in 1.2, $\mathbf{u} - \mathbf{v} = 0$ so $\mathbf{u} = \mathbf{v}$

Exercise 2

2.1 Since

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{u}||||\mathbf{v}||}$$

we have

$$\cos^2\theta||\mathbf{u}||^2||\mathbf{v}||^2 = \langle \mathbf{u}, \mathbf{v} \rangle^2$$

And from what we known

$$\langle \mathbf{u}, \mathbf{u} \rangle = ||\mathbf{u}||^2, \langle \mathbf{v}, \mathbf{v} \rangle = ||\mathbf{v}||^2$$

So when the equation

$$\langle \mathbf{u}, \mathbf{v} \rangle^2 = ||\mathbf{u}||^2||\mathbf{v}||^2$$

holds, we have $\cos^2\theta = 1$, which equivlant to u and v are linear dependent.

2.2

- $$||\mathbf{u} + \mathbf{v}|| = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle$$
  Since $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, also $\langle \mathbf{u}, \mathbf{u} \rangle = ||\mathbf{u}||^2$, we have $||\mathbf{u} + \mathbf{v}|| = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$
- From the same equation above, we have $\langle \mathbf{u}, \mathbf{v} \rangle = 0$,so we know $\mathbf{u} \perp \mathbf{v}$.

Exercise 3.

1. Since the fraction is curculant symmetric, when we change the position iof $\mathbf{v}$ and $\mathbf{u}$, the result wont change.

2. Assume $\mathbf{x}' = [x_1', x_2']^T$
$$\langle \mathbf{x}+\mathbf{x}', \mathbf{y}\rangle = (x_1+x_1')y_1 - (x_2+x_2')y_1 - (x_1+x_1')y_2 + 3(x_2+x_2')y_2$$
$$= (x_1y_1 - x_2y_1 - x_1y_2 + 3x_2y_2) + (x_1'y_1 - x_2'y_1 - x_1'y_2 + 3x_2'y_2) = \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{x}', \mathbf{y}\rangle$$

3.
$$\langle c\mathbf{x}, \mathbf{y}\rangle = (cx_1y_1 - cx_2y_1 - cx_1y_2 + 3cx_2y_2) = c(x_1y_1 - x_2y_1 - x_1y_2 + 3x_2y_2) = c\langle \mathbf{x}, \mathbf{y}\rangle$$

4.
$$\langle \mathbf{x}, \mathbf{x}\rangle = x_1^2 - x_1x_2 - x_1x_2 + 3x_2^2 = (x_1 - x_2)^2 + 2x_2^2 \geq 0$$
And only when $x_1 = x_2 = 0$, which x=0 , the fraction equals zero.

So the function sufficients a innerproduct.

Exercise 4.

1. From the form of equation, we have $\langle \mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{y}, \mathbf{x}\rangle$
2. Assume a function $f'[0,1] \to \mathbb{R}$ Use the property of intergral we have
$$\langle f+f', g\rangle = \int_0^1 (f(x)+f'(x))g(x)\,dx = \int_0^1 f(x)g(x)\,dx + \int_0^1 f'(x)g(x)\,dx = \langle f, g\rangle + \langle f', g\rangle$$

3.
$$\langle cf, g\rangle = \int_0^1 cf(x)g(x)\,dx = c\int_0^1 f(x)g(x)\,dx = c\langle f, g\rangle$$

4.
$$\langle f, f\rangle = \int_0^1 f(x)^2\,dx \geq 0$$
If and only if $f \equiv 0$, the fraction equals zero.

So it is an innerproduct.

For the equation
$$c_1f_1 + c_2f_2 + c_3f_3 = 0 \Rightarrow c_1 + c_2(x-1) + c_3(x-1)^2 = 0$$
Since the choose of x is arbitrary, the only way to hold the equation is $c_1 = c_2 = c_3 = 0$, so the three functions are linearly independent.

From what we have learnt $dim(span\{f_1, f_2, f_3\}) = 3$ since $||f_1|| = \sqrt{\langle f_1, f_1\rangle} = 1$, we choose it as the first unitg_1. Using generalized Gram-Schmidt,
$$g_2 = f_2 - \frac{\langle g_1, f_2\rangle}{\langle g_1, g_1\rangle} g_1 = x - \frac{1}{2}$$

$$g_3 = f_3 - \frac{\langle g_1, f_3\rangle}{\langle g_1, g_1\rangle} g_1 - \frac{\langle g_2, f_3\rangle}{\langle g_2, g_2\rangle} g_2 = x^2 - x + \frac{5}{6}$$

orthornormalize them, then $g_1 = 1, g_2 = 2\sqrt{3}(x - \frac{1}{2}), g_3 = \frac{2\sqrt{5}}{3}(x^2 - x + \frac{5}{6})$ And g_1, g_2, g_3 consists a orthornormal basis of $span\{f_1, f_2, f_3\}$.