Homework for Linear Algebra September 20,2024

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Exercise 1. $1.(i)\rightarrow(ii)$ Using the definition of f from (i):

$$f(x_1 + x_2) = a(x_1 + x_2) = ax_1 + ax_2 = f(x_1) + f(x_2)$$
$$f(cx) = acx = cf(x)$$

 $2.(ii)\rightarrow(i)$ When x=0, using additivity:

$$f(0) = 2f(0) = 0$$

When $x \neq 0$, using multiplication:

$$f(x) = f(x * 1) = x * f(1)$$

So, when a=f(1), we have f(x)=ax

Exercise 2 (a)line (b)plane (c)all of R^3

Exercise 3. c = 3 and d = 9 (3,3,6) isn't on the plane consisting of \boldsymbol{v} and \boldsymbol{w} .

Exercise 4. Taking a small scale as an example, assume

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

(i)
$$A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} = B+A$$

(ii)

$$c(A+B) = \begin{bmatrix} c(a_{11}+b_{11}) & c(a_{12}+b_{12}) \\ c(a_{21}+b_{21}) & c(a_{22}+b_{22}) \end{bmatrix} = \begin{bmatrix} ca_{11}+cb_{11} & ca_{12}+cb_{12} \\ ca_{21}+cb_{21} & ca_{22}+cb_{22} \end{bmatrix} = cA+cB$$

(iii)

$$A + (B + C) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} + c_{11} & b_{12} + c_{12} \\ b_{21} + c_{21} & b_{22} + c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} + c_{11} & a_{12} + b_{12} + c_{12} \\ a_{21} + b_{21} + c_{21} & a_{22} + b_{22} + c_{22} \end{bmatrix}$$

$$(A+B)+C = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = A + (B+C)$$

$$A(B+C) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} + c_{11} & b_{12} + c_{12} \\ b_{21} + c_{21} & b_{22} + c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(b_{11} + c_{11}) + a_{12}(b_{21} + c_{21}) & a_{11}(b_{12} + c_{12}) + a_{12}(b_{22} + c_{22}) \\ a_{21}(b_{11} + c_{11}) + a_{22}(b_{21} + c_{21}) & a_{21}(b_{12} + c_{12}) + a_{12}(b_{22} + c_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} + \begin{bmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{bmatrix} = AB + AC$$

$$(v)$$

$$(A+B)C = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11} + b_{11})c_{11} + (a_{12} + b_{12})c_{21} & (a_{11} + b_{11})c_{12} + (a_{12} + b_{12})c_{22} \\ (a_{21} + b_{21})c_{11} + (a_{22} + b_{22})c_{21} & (a_{21} + b_{21})c_{12} + (a_{22} + b_{22})c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{bmatrix} + \begin{bmatrix} b_{11}c_{11} + b_{12}c_{21} & b_{11}c_{12} + b_{12}c_{22} \\ b_{21}c_{11} + b_{22}c_{21} & b_{21}c_{12} + b_{22}c_{22} \end{bmatrix} = AC + BC$$

$$(vi)$$

$$A(BC) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11}c_{11} + b_{12}c_{21} & b_{11}c_{12} + b_{12}c_{22} \\ b_{21}c_{11} + b_{22}c_{21} & b_{21}c_{12} + b_{22}c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(b_{11}c_{11} + b_{12}c_{21}) + a_{12}(b_{21}c_{11} + b_{22}c_{21} & a_{11}(b_{11}c_{12} + b_{12}c_{22} \\ b_{21}c_{11} + b_{22}c_{21} & a_{21}(b_{11}c_{12} + b_{22}c_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(b_{11}c_{11} + b_{12}c_{21}) + a_{12}(b_{21}c_{11} + b_{22}c_{21}) & a_{11}(b_{11}c_{12} + b_{12}c_{22} + a_{22}(b_{21}c_{12} + b_{22}c_{22}) \\ a_{21}(b_{11}c_{11} + b_{12}c_{21}) + a_{22}(b_{21}c_{11} + b_{22}c_{21}) & a_{21}(b_{11}c_{12} + b_{12}c_{22}) + a_{22}(b_{21}c_{12} + b_{22}c_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(b_{11}c_{11} + b_{12}c_{21}) + a_{12}(b_{21}c_{11} + b_{22}c_{21}) & a_{11}(b_{11}c_{12} + b_{12}c_{22}) + a_{22}(b_{21}c_{12} + b_{22}c_{22}) \\ a_{21}(b_{11}c_{$$

For larger scale, maybe it's better to focus on only one element of the matrix.

Exercise 5.

$$A^2 = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$A^3 = A^2 * A = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 30 & 10 \\ 10 & -30 \end{bmatrix}$$

$$A^4 = A^3 * A = \begin{bmatrix} 30 & 10 \\ 10 & -30 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$$
...

Following the pattern $A^{2n} = \begin{bmatrix} 10^n & 0 \\ 0 & 10^n \end{bmatrix} (n \in N)$

$$A^{50} = \begin{bmatrix} 10^{25} & 0 \\ 0 & 10^{25} \end{bmatrix}, A^{51} = \begin{bmatrix} 3 * 10^{25} & 10^{25} \\ 10^{25} & -3 * 10^{25} \end{bmatrix}$$