Homework for Linear Algebra
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Exercise 1. Let A be an $n \times n$ matrix. Prove the equivalence between:

(i) There is a B with $AB = I$.

(ii) There is a C with $CA = I$.

Hints: We suggest two approaches. The second is by Jiabao Lin.

1. It was shown in the class that the following are all equivalent.
   - A is invertible.
   - A has n pivots.
   - There is a B with $AB = I$.

   Now consider A^T.

2. Using transpose, it suffices (why?) to show that (i) implies (ii). So assume that there is a B with $AB = I$. Prove:
   - For any $\boldsymbol{b} \in \mathbb{R}^n$
   $$A\boldsymbol{x} = \boldsymbol{b}$$
   has a unique solution.
   - For any $n \times n$ matrix M
   $$AC = M$$
   has a unique solution C.
   - $ABA = A = AI$.

Exercise 2. Give an $m \times n$ matrix A such that:

- There is a B with $AB = I$.
- There is *no* C with $CA = I$.

Exercise 3. Find all solutions of the following systems of linear equations, where x_1, x_2, x_3, x_4 are variables and λ is a parameter.

$$(1) \quad \begin{cases} 2x_1 - x_2 + x_3 + x_4 & = 1, \\ x_1 + 2x_2 - x_3 + 4x_4 & = 2, \\ x_1 + 7x_2 - 4x_3 + 11x_4 & = \lambda. \end{cases}$$

$$(2) \quad \begin{cases} \lambda x_1 + x_2 + x_3 + x_4 & = 1 \\ x_1 + \lambda x_2 + x_3 + x_4 & = \lambda \\ x_1 + x_2 + \lambda x_3 + x_4 & = \lambda^2 \\ x_1 + x_2 + x_3 + \lambda x_4 & = \lambda^3 \end{cases}$$

Hint 1: The number of solutions of the system of linear equations may be different when λ varies. You need to figure out when there is a unique solution, when there are no solutions and when there are infinitely many solutions. If there are some solutions, write down them.

Hint 2: To find all solutions when the coefficient matrix A has less than n pivots, keep free variables "free" as variables and express pivots by numbers and these free variables. Finally, let the tuple of free variables $(x_{i_1}, x_{i_2}, ..., x_{i_k})$ range over $\mathbb{R}^k$ and you get all solutions.

Hint 3: For (2), the solutions may be in a complex form. You may find it useful to let $b = 1 + \lambda + \lambda^2 + \lambda^3$ and let $c = b/(\lambda + 3)$. You can express your solution using numbers, λ, b and c.

Exercise 4. Use Gaussian elimination to calculate an upper triangular system $U\boldsymbol{x} = \boldsymbol{c}$ for the linear system $A\boldsymbol{x} = \boldsymbol{b}$. Write down the elementary matrix in each step and point out the failures you meet.

(i) $A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

(ii) $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ and $\boldsymbol{b} = \boldsymbol{0}$.

(iii) $A = \begin{bmatrix} 0 & 1 & 2 \\ 7 & 8 & 3 \\ 6 & 5 & 4 \end{bmatrix}$ and $\boldsymbol{b} = \boldsymbol{0}$.

Exercise 5. Use Gauss-Jordan to calculate the inverse of the following matrices.

(i) $\begin{bmatrix} 16 & 15 & 14 & 13 \\ 5 & 4 & 3 & 12 \\ 6 & 1 & 2 & 11 \\ 7 & 8 & 9 & 10 \end{bmatrix}$. (ii) $\begin{bmatrix} 1 & 1 & 10 & 1 \\ 1 & 10 & 1 & 1 \\ 1 & 1 & 1 & 10 \\ 10 & 1 & 1 & 1 \end{bmatrix}$. (iii) $\boldsymbol{a}\boldsymbol{a}^T - I_n$ with $\boldsymbol{a} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$.