Homework for Linear Algebra
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Exercise 1.

- Assume the result of left side $[\mathbf{u}_1, \cdots, \mathbf{u}_m]$

$$\mathbf{u}_i = \sum_{k=1}^{n} (A + A')(k,i)\mathbf{v}_i = \sum_{k=1}^{n} A(k,i)\mathbf{v}_i + \sum_{k=1}^{n} A'(k,i)\mathbf{v}i$$

  So the equation holds.

- Let $C = AB$, $C_{ij} = \sum_{q=1}^{m} A(i,q)B(q,j)$. Assume the result of left side $[\mathbf{u}_1, \cdots, \mathbf{u}_l]$.

$$\mathbf{u}_i = \sum_{k=1}^{n}\sum_{q=1}^{m} A(k,q)B(q,i)\mathbf{v}_k$$

  Consider the right side

$$\mathbf{u}'_i = \sum_{q=1}^{m}(\sum_{k=1}^{n} A(k,i)\mathbf{v}_k)B(q,i) = \mathbf{u}_i$$

  So the equation holds.

Exercise 2.

(i)We have $(x, y) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$

$$\begin{cases} x = c_1 + 2c_2 \\ y = c_1 + 3c_2 \end{cases}$$

The coordinate vector $(3x - 2y, y - x)$.

(ii)

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} M = \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix}$$

$$M = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$$

Exercise 3.

(i)

$$T(S(\mathbf{u}_1 + \mathbf{u}_2)) = T(S(\mathbf{u}_1) + S(\mathbf{u}_2)) = T(S(\mathbf{u}_1)) + T(S(\mathbf{u}_2))$$

$$T(S(c\mathbf{u})) = T(cS(\mathbf{u})) = cTS(\mathbf{u})$$

So TS is a linear transformation.

(ii) Assume the coordinate vector of $\mathbf{u}$ is $(c_1, \cdots, c_n)$. From definition, we can represent the coordinate vector of $S(\mathbf{u})$ with respect to $\overline{\mathbf{w}}$

$$A_S \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

So the coordinate vector of $T(S(\mathbf{u}))$ will be

$$A_T(A_S \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}) = (A_T A_S) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

So $A_{TS} = A_T A_S$.

Exercise 4.

(i)

- $$(T+T')(\mathbf{v}_1+\mathbf{v}_2) = T(\mathbf{v}_1+\mathbf{v}_2)+T'(\mathbf{v}_1+\mathbf{v}_2) = T(\mathbf{v}_1)+T'(\mathbf{v}_1)+T(\mathbf{v}_2)+T'(\mathbf{v}_2) = (T+T')(\mathbf{v}_1)+(T+T')(\mathbf{v}_2)$$
  $$(T+T')(c\mathbf{v}) = T(c\mathbf{v}) + T'(c\mathbf{v}) = c(T+T')(\mathbf{v})$$
- $$cT(\mathbf{v}_1 + \mathbf{v}_2) = cT(\mathbf{v}_1) + cT(\mathbf{v}_2)$$
  $$cT(a\mathbf{v}) = caT(\mathbf{v})$$

(ii)

1. $T + T'(\mathbf{v}) = T(\mathbf{v}) + T'(\mathbf{v}) = (T' + T)(\mathbf{v})$
2. $T_1 + (T_2 + T_3) = T_1 + T_2 + T_3 = (T_1 + T_2) + T_3$
3. LET $T_0(\mathbf{v}) \equiv 0$, $T_0 + T = T$.
4. For every T, we can have T' that for every $\mathbf{v}$, $T(\mathbf{v}) = -T'(\mathbf{v})$. So $T+T' = 0$
5. Let c=1, $1 \cdot T = T$.
6. From we have proved in (i), $c_1(c_2T) = (c_1c_2)T$.
7. Obvious $c(T + T') = cT + cT'$.
8. Also $(c_1 + c_2)T = c_1T + c_2T$.

(iii)The dimension is m*n.
Proof:Consider transforming T into the corresponding matrix A_T. We have proved in class that this transformation is a bijection. So each T corresponds to a unique m*n matrix A_T. Since the vector space $M_{m*n}\{\mathbb{R}\}$ has dimension m*n, so do **T(V,W)**.