Homework for Linear Algebra
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Exercise 1. (i) ⇒ (ii)
If exists a B with AB = I, that means A has n pivots.
So there exists a medthod of Gauss-Jordan process that produces

$$D \cdots E \cdots P \cdots E \cdots A = I$$

Let $C = D \cdots E \cdots P \cdots E \cdots$, then we have $CA = I$.
(ii) ⇒ (i)
Since $CA = I$ and $(CA)^T = A^T C^T$, we have $A^T C^T = I^T = I$.
From previous provement, we know there exists $DA^T = I$
So $(DA^T)^T = AD^T = I$, let $D^T = B$, AB = I exists.

Exercise 2.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Exercise 3. (i)

$$A\mathbf{x} = \mathbf{b} \Rightarrow \begin{bmatrix} 1 & 0 & 1/5 & 7/5 \\ 0 & 1 & -3/5 & 7/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4/5 \\ 3/5 \\ \lambda - 5 \end{bmatrix}$$

If $\lambda = 5$,it has infinite many solutions,if $\lambda \neq 5$,it has no solution.
(ii)

$$(\lambda+3)(x_1+x_2+x_3+x_4) = (1+\lambda+\lambda^2+\lambda^3) \Rightarrow (x_1+x_2+x_3+x_4) = \frac{b}{c}(c \neq 0)$$

If $\lambda = -3$ or $\lambda = 1$ it has infinite many solutions.
Else we have $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{\frac{b}{c}-1}{\lambda-1} \\ \frac{\frac{b}{c}-\lambda}{\lambda-1} \\ \frac{\frac{b}{c}-\lambda^2}{\lambda-1} \\ \frac{\frac{b}{c}-\lambda^3}{\lambda-1} \end{bmatrix}$

Exercise 4. (i)

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Failure :Temporary failure for a zero pivot.Permanent failure with no solution.(A pivot is missing and 0x = 1)
(ii)

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Failure :Permanent failure with infinite many solution.(A pivot is missing and $0x = 0$)

(iii)

$$P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6/7 & 0 & 1 \end{bmatrix} E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 13/7 & 1 \end{bmatrix}$$

Failure:Temporary failure for a zero pivot.

Exercise 5. (i)

$$\begin{bmatrix} 16 & 15 & 14 & 13 & 1 & 0 & 0 & 0 \\ 5 & 4 & 3 & 12 & 0 & 1 & 0 & 0 \\ 6 & 1 & 2 & 11 & 0 & 0 & 1 & 0 \\ 7 & 8 & 9 & 10 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 83/690 & -2/15 & 1/6 & -62/345 \\ 0 & 1 & 0 & 0 & 34/345 & 11/30 & -1/3 & -139/690 \\ 0 & 0 & 1 & 0 & -17/138 & -1/3 & 1/6 & 26/69 \\ 0 & 0 & 0 & 1 & -6/115 & 1/10 & 0 & 11/230 \end{bmatrix}$$

(ii)

$$\begin{bmatrix} 1 & 1 & 10 & 1 & 1 & 0 & 0 & 0 \\ 1 & 10 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 10 & 0 & 0 & 1 & 0 \\ 10 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -1/117 & -1/117 & -1/117 & 4/39 \\ 0 & 1 & 0 & 0 & -1/117 & 4/39 & -1/117 & -1/117 \\ 0 & 0 & 1 & 0 & 4/39 & -1/117 & -1/117 & -1/117 \\ 0 & 0 & 0 & 1 & -1/117 & -1/117 & 4/39 & -1/117 \end{bmatrix}$$

(iii)

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & \ddots & \ddots & 1 & 0 & \ddots & \ddots & 0 \\ 1 & \ddots & \ddots & 1 & 0 & \ddots & \ddots & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -2/3 & 1/3 & 1/3 & 1/3 \\ 0 & \ddots & \ddots & 0 & 1/3 & \ddots & \ddots & 1/3 \\ 0 & \ddots & \ddots & 0 & 1/3 & \ddots & \ddots & 1/3 \\ 0 & 0 & 0 & 1 & 1/3 & 1/3 & 1/3 & -2/3 \end{bmatrix}$$

The process are way too long to print.Only show the results.