Homework for Linear Algebra
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Exercise 1.

We have known the equation $det(A) = det(A^T)$. And

$$det(-A) = \sum_{\sigma \in Perm(n)} (-1)^{\tau(\sigma)}(-a_{\sigma(1)1}) \cdots (-a_{\sigma(n)n}) = \sum_{\sigma \in Perm(n)} (-1)^{\tau(\sigma)}(-1)^n(a_{\sigma(1)1}) \cdots (a_{\sigma(n)n})$$

If n is odd, $det(-A) = -det(A) = det(A)$. So $det(A) = 0$. If n is even, $det(-A) = det(A)$, not true.

Exercise 2.

Do elementary row operation to the matrix

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & c^2-a^2-\frac{c-a}{b-a}(b^2-a^2) \end{bmatrix} = \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-a)(c-b) \end{bmatrix}$$

$$det(A) = (b-a)(c-a)(c-b)$$

Exercise 3.

Calculate the matrix's determinant by big formula

$$det(A) = \sum_{\sigma \in Perm(n)} (-1)^{\tau(\sigma)} a_{\sigma(1)1} \cdots a_{\sigma(n)n} = \sum_{\sigma \in Perm(n)} (-1)^{\tau(\sigma)}$$

Since A has two same columns, det(A)=0, there's det(A)=0. Since

$$(-1)^{\tau(\sigma)} = \begin{cases} -1(\sigma \text{ is odd }) \\ 1(\sigma \text{ is even }) \end{cases}$$

So there're equal number of odd and even permuntations to maintain the det=0.

Exercise 4.

First prove for any column vector of A and row column of A*

$$\mathbf{a}_i * (\mathbf{a}^*)_j^T \begin{cases} 0(i \neq j) \\ det(A)(i = j) \end{cases}$$

When $i = j$, it is the form of cofactor formula.

When $i \neq j$, it equals the cofactor formula of a matrix that copies the jth column to ith column. And from definition, we know its determinant equals 0.

Exercise 5.

Use the equation proved in Exercise 4. Apply cofactor formula to calulate.

$$A^{-1} = \frac{A^*}{det(A)}$$

1.
$$A^* = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}, det(A) = 1 * (-2) = -2, A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

2.
$$A^* = \begin{bmatrix} -2 & 17 \\ -3 & 19 \end{bmatrix}, det(A) = 19 * \frac{13}{19} = 13, A^{-1} = \begin{bmatrix} -\frac{2}{13} & \frac{17}{13} \\ -\frac{3}{13} & \frac{19}{13} \end{bmatrix}$$

3.
$$A^* = \begin{bmatrix} 5 & -3 \\ 0 & 1 \end{bmatrix}, det(A) = 1 * 5 = 5, A^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{3}{5} & \frac{1}{5} \end{bmatrix}$$

4.
$$A^* = \begin{bmatrix} -1 & -1 & 2 \\ -2 & 1 & -2 \\ 2 & -1 & -1 \end{bmatrix}, det(A) = 1*(-1)*3 = -3, A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$