Homework for Linear Algebra
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Exercise 1.

$$\mathbf{q}_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \mathbf{q}_2 = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix}$$

Exercise 2.
2.1

$$\mathbf{q}_1 = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} = (\frac{1}{10}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{7}{10})$$

$$\mathbf{q}_2 = \mathbf{a}_2 - \frac{\|\mathbf{a}_2\| \cos\theta_1}{\|\mathbf{q}_1\|}\mathbf{q}_1 = (-\frac{7}{10}, \frac{3}{10}, \frac{2}{5}, -\frac{1}{2}, \frac{1}{10})$$

2.2 The projection vector of (1,0,0,0,0) is

$$\mathbf{p} = (\frac{1}{2}, -\frac{9}{50}, -\frac{6}{25}, \frac{2}{5}, 0)$$

Exercise 3.

3.1 From the second equation we know

$$\Delta(\mathbf{a}_1, \cdots, c\mathbf{a}_j, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_n) = 0$$

Combine that with the first equation

$$\Delta(\mathbf{a}_1, \cdots, \mathbf{a}_i + c\mathbf{a}_j, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_n) = \Delta(\mathbf{a}_1, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_n) + \Delta(\mathbf{a}_1, \cdots, c\mathbf{a}_j, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_n)$$
$$= \Delta(\mathbf{a}_1, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_n)$$

3.2 Add column j to column i

$$\Delta(\mathbf{a}_1, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_n) = \Delta(\mathbf{a}_1, \cdots, \mathbf{a}_i + \mathbf{a}_j, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_n)$$

Substract column i from column j

$$\Delta(\mathbf{a}_1, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_n) = \Delta(\mathbf{a}_1, \cdots, \mathbf{a}_i + \mathbf{a}_j, \cdots, -\mathbf{a}_i, \cdots, \mathbf{a}_n)$$

Add column j to column i

$$\Delta(\mathbf{a}_1, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_n) = \Delta(\mathbf{a}_1, \cdots, \mathbf{a}_j, \cdots, -\mathbf{a}_i, \cdots, \mathbf{a}_n)$$

Multiply -1 to column j

$$-\Delta(\mathbf{a}_1, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_n) = \Delta(\mathbf{a}_1, \cdots, \mathbf{a}_j, \cdots, \mathbf{a}_i, \cdots, \mathbf{a}_n)$$

Exercise 4.

An available sequence: Each time picks the first element, swap it with the next element if this element's $\sigma(i)$ is bigger than $\sigma(j)$ of the next element. Repeat n-1 times.
In this method, everytime we do a swap will erase a pair of inversion. So the exact exchangenumber will be $\tau(\sigma)$.

Exercise 5.

5.1 Since $\sigma_1\sigma_2 \in Perm(n)$ and there're only two differences between $\sigma_1(i)$ and $\sigma_2(i)$. So there must exist i_1, i_2 that satisfy

$$\sigma_1(i_1) = \sigma_2(i_2), \sigma_1(i_2) = \sigma_2(i_1)$$

To create precisely 2 differences.

5.2 Assume $i_1 < i_2$ and see the swap as a series of swaps on adjacent elements. Since every element is unique , doing the swap will create a -1 or 1 change on $\tau(\sigma)$. Finishing the swap of $\sigma_1(i_1)$ and $\sigma_1(i_2)$ will need

$$(i_2 - i_1 - 1) + (i_2 - i_1) = 2(i_2 - i_1) - 1$$

and this will create an **odd** change on $\tau(\sigma)$. So $\tau(\sigma_1)$ and $\tau(\sigma_2)$ will have different parity.

Exercise 6.

6.1 An available solution: Rewrite σ_1 by the form of $\sigma_2(i)$.For example

$$\sigma_1 = 13425, \sigma_2 = 12345 \Rightarrow \sigma_1 = \sigma_2(1)\sigma_2(3)\sigma_2(4)\sigma_2(3)\sigma_2(5)$$

Similar to **Exercise 4.**. There always exists a sequence of swapping to change $\sigma_2(1)\sigma_2(3)\sigma_2(4)\sigma_2(3)\sigma_2(5)$ to $\sigma_2(1)\sigma_2(2)\sigma_2(3)\sigma_2(4)\sigma_2(5)$ So the statement holds.

6.2

First rewrite the permutations the same way as **6.1**. So we can see the sequence σ_1 to σ_2 as a permutation transforming to the basic permutation.
In other words, the parity of σ_1 to σ_2 equals σ_1' to σ_2' which $\sigma_2' = 1234 \cdots n$.

From **5.2** we know that any 2-element swap will change the parity of $\tau(\sigma)$. Assume $\alpha(\sigma)$ as the length of the sequence transforming arbitrary σ into basic permutation $1234 \cdots n$. Now we prove that the parity of $\alpha(\sigma)$ is unique for every σ.

Since the basic permutation has 0 inversions, regard it has a even $\tau(\sigma)$. So for any permutation σ, if it has an odd $\tau(\sigma)$, it will need odd number of swaps to transform into basic permutation. Same for the even ones.

In this way we prove that the parity of sequence length σ_1 to σ_2 will not change. And by the provement above, we can know that the parity of σ_1 to σ_2 equals the parity of first transforing σ_1 to the basic permutation (in the form of σ_1' to σ_2') then transforming σ_2 to the basic form.
So the parity of σ_1 to σ_2 = the parity of $\tau(\sigma_1) + \tau(\sigma_2)$.

Exercise 7. Assume adding the jth column onto the ith column.

$$det(A') = \sum_{\sigma \in Perm(n)} (-1)^{\tau(\sigma)} \mathbf{a}_{\tau(1)1} \cdots (\mathbf{a}_{\tau(i)i} + \mathbf{a}_{\tau(j)j}) \cdots \mathbf{a}_{\tau(j)j} \cdots \mathbf{a}_{\tau(n)n}$$

$$= \sum_{\sigma \in Perm(n)} (-1)^{\tau(\sigma)} \mathbf{a}_{\tau(1)1} \cdots \mathbf{a}_{\tau(i)i} \cdots \mathbf{a}_{\tau(j)j} \cdots \mathbf{a}_{\tau(n)n}$$

$$+ \sum_{\sigma \in Perm(n)} (-1)^{\tau(\sigma)} \mathbf{a}_{\tau(1)1} \cdots \mathbf{a}_{\tau(j)j} \cdots \mathbf{a}_{\tau(j)j} \cdots \mathbf{a}_{\tau(n)n}$$

From **Exercise 3.** we know that if A has two same columns det(A)=0. So

$$det(A') = \sum_{\sigma \in Perm(n)} (-1)^{\tau(\sigma)} \mathbf{a}_{\tau(1)1} \cdots \mathbf{a}_{\tau(i)i} \cdots \mathbf{a}_{\tau(j)j} \cdots \mathbf{a}_{\tau(n)n} = det(A)$$