Homework for Linear Algebra
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Exercise 1. Write B as a matrix of column vectors.

$$AB = \begin{bmatrix} A\mathbf{b}_1 & \dots & A\mathbf{b}_q \end{bmatrix}$$

For any c_{il} in the product AB , it belongs to A $\mathbf{b}_l$. And

$$\mathbf{c}_l = \mathbf{a}_1 b_{1l} + \ldots + \mathbf{a}_p b_{pl}, c_{il} = a_{i1} b_{1l} + \ldots + a_{ip} b_{pl} = \sum_{k=1}^{p} a_{ik} b_{kl}$$

Exercise 2. (i) There is only one entry in every $\mathbf{p}_i$ equals 1 , and all the 1 entries are in different rows. So each pair of $\mathbf{p}_i \cdot \mathbf{p}_j$ equals 0.

(ii) Each $\| \mathbf{p}_i \| = 1$, so each of the vector is a unit matrix.

(iii) To prove $P^{-1} = P^T$, we need to prove $P^T P = I$.

$$P^T P = \begin{bmatrix} \mathbf{p}_1^T \\ \vdots \\ \mathbf{p}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 & \dots & \mathbf{p}_n \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1^T \mathbf{p}_1 & \dots & \mathbf{p}_1^T \mathbf{p}_n \\ \vdots & \ddots & \vdots \\ \mathbf{p}_n^T \mathbf{p}_1 & \dots & \mathbf{p}_n^T \mathbf{p}_n \end{bmatrix}$$

Using (i) and (ii), we know that $\mathbf{p}_1^T = \mathbf{p}_1$, and $\mathbf{p}_1^2 = 1$, So $P^T P = I$.

Exercise 3. (i) Since $A^{-1} = A^T$, we have $A^{-}1\mathbf{a}_1 = A^T \mathbf{a}_1$.

$$A^{-1}\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n^T \mathbf{a}_1 \end{bmatrix}$$

Since $\mathbf{a}_i^T = \mathbf{a}_i$, we have for any $i \neq j$, $\mathbf{a}_i \mathbf{a}_j = 0$, so each pair of $\mathbf{a}_i$ and $\mathbf{a}_j$ are perpendicular to each other.

(ii) Using the same method as (i), we can prove that $\| \mathbf{a}_i^2 \| = \| \mathbf{a}_i \| = 1$. So it is a unit vector.

Exercise 4. Since $BA = AC = I$, we have

$$BAA^{-1} = IA^{-1} \Rightarrow B = A^{-1}, A^{-1}AC = A^{-1}I \Rightarrow C = A^{-1}.$$

So $A^{-1} = B = C$.

Exercise 5. (i)

$$(0+1)\mathbf{v} = 1\mathbf{v} \to 0\mathbf{v} + 1\mathbf{v} = 1\mathbf{v} + \mathbf{0} \to 0\mathbf{v} = \mathbf{0}$$

(ii)

$$0\mathbf{v} = \mathbf{0} \to (-1+1)\mathbf{v} = \mathbf{0} \to (-1)\mathbf{v} + \mathbf{v} = \mathbf{0} \to (-1)\mathbf{v} = -\mathbf{v}$$

(iii)

$$(-1)\mathbf{v} = -\mathbf{v}, (-1)\mathbf{w} = -\mathbf{w} \to -(\mathbf{v}+\mathbf{w}) = (-\mathbf{v}) + (-\mathbf{w})$$

(iv)

$$c\mathbf{0} = \sum_{1}^{c} \mathbf{0}, \mathbf{0} + \mathbf{0} = \mathbf{0} \to c\mathbf{0} = \mathbf{0}$$

(v)

$$(-1)\mathbf{v} = -\mathbf{v} \to c(-\mathbf{v}) = (-c)\mathbf{v}$$

$$\sum_{1}^{c} \mathbf{v} = c\mathbf{v} \to -\sum_{1}^{c} \mathbf{v} = -(c\mathbf{v}) \to (-c)\mathbf{v} = -(c\mathbf{v})$$