Homework for Linear Algebra
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Exercise 1.

1.1

$$rank(A) + dim(N(A)) = m, rank(BA) + dim(N(BA)) = m$$

Since each row vector in BA can be seen as a linear combination of row vectors of A. So

$$dim(C((BA)^T)) \leq dim(C(A^T)) \Rightarrow rank(BA) \leq rank(A)$$

So

$$dim(N(A)) \leq dim(N(BA)) \Rightarrow N(A) \subseteq N(BA)$$

1.2 *First we reduce A into its reduced row echelon form R with $r = rank(A)$ pivots.*

$$A \Rightarrow \begin{bmatrix} \mathbf{0} & \mathbf{U} \end{bmatrix} (\mathbf{U} \textit{ is an n*r upper-trangular matrix })$$

Since $rank(B) = n$, each column vector of B is linearly independent to other. So the ith column vector of BU is a linear combination of $\mathbf{b}_1 \cdots \mathbf{b}_i$ with the ith coefficient $\neq 0$. So the column vectors are independent to others.

$$BA = \begin{bmatrix} \mathbf{0} & BU \end{bmatrix}$$

So $dim(C(BA)) = dim(C(BU)) = dim(C(U)) = r$, $rank(BA) = rank(A)$. From the equations in 1.1, we can conclude

$$N(A) = N(BA)$$

The converse is't true. Let $A = \mathbf{0}$. Obviously we can not conclude $rank(B) = n$.

Exercise 2.

$$A\mathbf{x} = 0 \Rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

Apply Gauss-Jordan to the augmented matrix, we get

$$\begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So we get a patricular solution to $A\mathbf{x} = \mathbf{b}$.

$$\mathbf{x}_p = \begin{cases} x_1 = 4 \\ x_2 = -1 \\ x_3 = 0 \\ x_4 = 0 \end{cases}$$

And we can get the special solutuons of N(A)

$$\mathbf{s}_1 = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 6 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

So the complete solution is

$$\mathbf{x} = \mathbf{x}_p + c_1\mathbf{s}_1 + c_2\mathbf{s}_2$$

Exercise 3. *Assume we have a linear combination*

$$c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n = 0$$

Multiply $\mathbf{v}_1$ *to the equation. Since* $\mathbf{v}_i \perp \mathbf{v}_j$, *we have*

$$\mathbf{v}_1(c_1\mathbf{a}_1 + \cdots c_n\mathbf{v}_n) = 0 * \mathbf{v}_1 \Rightarrow c_1\mathbf{v}_1^2 = 0$$

Since $\mathbf{v}_1^2 \neq 0$, $c_1 = 0$. *In the same way we conclude* $c_1 = \cdots = c_n = 0$, *so* $\mathbf{v}_1 \cdots \mathbf{v}_n$ *are linearly independent.*

Exercise 4.

4.1 *Since a non-zero vector can not be prependicular to itself, there won't exist a vector* $\mathbf{v}$ *expect* $\mathbf{0}$ *that exists in both sets.*
Otherwise it means that the vector is prependicular to itself.

4.2 *We have*

$$\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{w}_1 - \mathbf{w}_2$$

Since V, W *are subspaces, we have*

$$\mathbf{v}_1 - \mathbf{v}_2 \in V, \mathbf{w}_1 - \mathbf{w}_2 \in W$$

From what we proved in 4.1, we know

$$\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{w}_1 - \mathbf{w}_2 = \mathbf{0}$$

So

$$\mathbf{v}_1 = \mathbf{v}_2, \mathbf{w}_1 = \mathbf{w}_2$$

4.3 *(i)* $\mathbf{0}$ *is in* $V + W$.
(ii)

$$\mathbf{v}_1, \mathbf{v}_2 \in V \Rightarrow \mathbf{v}_1 + \mathbf{v}_2 \in V, \mathbf{w}_1, \mathbf{w}_2 \in W \Rightarrow \mathbf{w}_1 + \mathbf{w}_2 \in W$$
$$\Rightarrow \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{w}_1 + \mathbf{w}_2 \in V + W$$

(iii)

$$\mathbf{v} \in V \Rightarrow c\mathbf{v} \in V, \mathbf{w} \in W \Rightarrow c\mathbf{w} \in W$$
$$\Rightarrow c(\mathbf{v} + \mathbf{w}) \in V + W$$

Since V, W *are subspaces of* $R^n, soV + Wisasubspaceof R^n$.

4.4 *Assume* $\mathbf{v}_1 \cdots \mathbf{v}_r$ *is a basis for V, and* $\mathbf{w}_1 \cdots \mathbf{w}_s$ *is a basis for W.*
For any $\mathbf{v} \in V$*, it can be represented by a linear combination of the basis, so does* $\mathbf{w} \in W$*.*
So for any $\mathbf{v} + \mathbf{w} \in V + W$ *,it can be represented by*

$$c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r + c_{r+1}\mathbf{w}_1 + \cdots + c_{r+s}\mathbf{w}_s$$

From 4.2 we know the way to represent this $\mathbf{v} + \mathbf{w}$ *vector is unique.*
And obvoiusly each $\mathbf{v}_i$ *and* $mathbfw_i$ *are linearly independent. So they form a basis of V+W*