Homework for Linear Algebra
October 24, 2024

Chengyu Zhang

Exercise 1.

1.1

$$\mathbf{v}\mathbf{w} = \mathbf{v}^T\mathbf{w}$$

*See the result as a 1*1 matrix.*

$$\mathbf{v}^T\mathbf{w} = (\mathbf{v}^T\mathbf{w})^T = \mathbf{w}^T\mathbf{v} = \mathbf{w}\mathbf{v}$$

1.2

$$(\mathbf{u}+\mathbf{v})\mathbf{w} = \sum_{i=1}^{m}(u_i+v_i)w_i = \sum_{i=1}^{m}u_iw_i + \sum_{i=1}^{m}v_iw_i = \mathbf{u}\mathbf{w}+\mathbf{v}\mathbf{w}$$

1.3

$$c\mathbf{u}\cdot\mathbf{v} = \sum_{i=1}^{m}(cu_i)v_i = \sum_{i=1}^{m}c(u_iv_i) = c(\mathbf{u}\mathbf{v})$$

1.4

$$\mathbf{u}\cdot\mathbf{u} = \sum_{i=1}^{m}u_i^2 \geq 0$$

When all $u_i = 0$, which means $\mathbf{u} = \mathbf{0}$, $\mathbf{u}\cdot\mathbf{u} = 0$

Exercise 2.

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\cdot\mathbf{u}\cdot\mathbf{v}$$

$$\Rightarrow$$

$$\mathbf{u}\cdot\mathbf{v} = 0 \Rightarrow \|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

$$\Leftarrow$$

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \Rightarrow \mathbf{u}\cdot\mathbf{v} = 0 \Rightarrow \mathbf{u}\perp\mathbf{v}$$

Exercise 3.

3.1 *Since $rank(A) = n$ and A has n columns, we can know that the column vectors of A are linearly independent.*
So $dim(C(A)) = dim(V) = n$, and there will be a matrix A whose column vectors comsist a basis of V.

3.2 *Name a basis of* V $\{\mathbf{a}_1 \cdots \mathbf{a}_n\}$*.We need to find*

$$\mathbf{p} = \hat{x}_1\mathbf{a}_1 + \cdots + \hat{x}_n\mathbf{a}_n$$

Let

$$\hat{\mathbf{x}} = (\hat{x}_1, \cdots \hat{x}_n), A = \begin{bmatrix} \mathbf{a}_1 & \cdots \mathbf{a}_n \end{bmatrix}$$

Then $\mathbf{p} = A\hat{\mathbf{x}}$

$$\mathbf{e} \perp V \Rightarrow \mathbf{v} \perp \{\mathbf{a}_1 \cdots \mathbf{a}_n\} \Rightarrow \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} \cdot \mathbf{e} = \mathbf{0}$$

$$A^T\mathbf{e} = A^T(\mathbf{v} - A\hat{\mathbf{x}}) = \mathbf{0} \Rightarrow A^T\mathbf{v} = A^TA\hat{\mathbf{x}} \Rightarrow (A^TA)^{-1}A^T\mathbf{v}\hat{\mathbf{x}}$$

So there exists a unique way to get the $\hat{\mathbf{x}}$*, and the* $\mathbf{e}$ *is unique.*

3.3

$$\|\mathbf{v} - \mathbf{u}\| = \|\mathbf{v} - \mathbf{p} + \mathbf{p} - \mathbf{u}\|$$

$$\|\mathbf{v} - \mathbf{p} + \mathbf{p} - \mathbf{u}\|^2 = (\mathbf{v} - \mathbf{p})^2 + (\mathbf{p} - \mathbf{w})^2 + 2 * \mathbf{e}(\mathbf{p} - \mathbf{u})$$

Since

$$\mathbf{p}, \mathbf{u} \in V \mathbf{e} \perp V$$

$$\Rightarrow \mathbf{e} \perp \mathbf{p}, \mathbf{e} \perp \mathbf{u}$$

So

$$\|\mathbf{v} - \mathbf{u}\|^2 - \|\mathbf{e}\|^2 = (\mathbf{p} - \mathbf{w})^2 \geq 0$$

When $\mathbf{p} = \mathbf{w}$, we have $\|\mathbf{v} - \mathbf{u}\| = \|(\mathbf{e})\|$

3.4 *From the definition the dist is the shortest vector prependicular to* V*,and the* $\mathbf{e}$ *in 3.2 satisfies.So*

$$dist(\mathbf{v}, V) = min\|\mathbf{v} - \mathbf{u}\| = \|\mathbf{e}\|$$

Exercise 4. *Using*

$$\mathbf{p} = \frac{\|\mathbf{b}\|cos\theta}{\|\mathbf{a}\|}\mathbf{a}$$

$$\mathbf{p}_1 = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{3} \\ \frac{5}{3} \end{bmatrix}, \mathbf{e}_1 = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\mathbf{p}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 5. *Using*

$$\mathbf{p} = A(A^TA)^{-1}A^T\mathbf{b}$$

$$(A^TA)^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{p_1} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} 3/2 & -1 \\ -1 & 1 \end{bmatrix}, A(A^TA)^{-1}A^T = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p_2} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$