Homework for Linear Algebra
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Exercise 1. 1.(i)→(ii) Using the definition of f from (i):

$$f(x_1 + x_2) = a(x_1 + x_2) = ax_1 + ax_2 = f(x_1) + f(x_2)$$

$$f(cx) = acx = cf(x)$$

2.(ii)→(i) When $x = 0$, using additivity :

$$f(0) = 2f(0) = 0$$

When $x \neq 0$, using multiplication :

$$f(x) = f(x * 1) = x * f(1)$$

So,when a=f(1), we have $f(x) = ax$

Exercise 2 (a)line (b)plane (c)all of R^3

Exercise 3. $c = 3$ and $d = 9$
(3,3,6) isn't on the plane consisting of $\boldsymbol{v}$ and $\boldsymbol{w}$.

Exercise 4. Taking a small scale as an example,assume

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

(i)

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} = B + A$$

(ii)

$$c(A+B) = \begin{bmatrix} c(a_{11} + b_{11}) & c(a_{12} + b_{12}) \\ c(a_{21} + b_{21}) & c(a_{22} + b_{22}) \end{bmatrix} = \begin{bmatrix} ca_{11} + cb_{11} & ca_{12} + cb_{12} \\ ca_{21} + cb_{21} & ca_{22} + cb_{22} \end{bmatrix} = cA+cB$$

(iii)

$$A+(B+C) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} + c_{11} & b_{12} + c_{12} \\ b_{21} + c_{21} & b_{22} + c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} + c_{11} & a_{12} + b_{12} + c_{12} \\ a_{21} + b_{21} + c_{21} & a_{22} + b_{22} + c_{22} \end{bmatrix}$$

$$(A + B) + C = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = A + (B + C)$$

(iv)

$$A(B+C) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(b_{11}+c_{11})+a_{12}(b_{21}+c_{21}) & a_{11}(b_{12}+c_{12})+a_{12}(b_{22}+c_{22}) \\ a_{21}(b_{11}+c_{11})+a_{22}(b_{21}+c_{21}) & a_{21}(b_{12}+c_{12})+a_{22}(b_{22}+c_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix} + \begin{bmatrix} a_{11}c_{11}+a_{12}c_{21} & a_{11}c_{12}+a_{12}c_{22} \\ a_{21}c_{11}+a_{22}c_{21} & a_{21}c_{12}+a_{22}c_{22} \end{bmatrix} = AB+AC$$

(v)

$$(A+B)C = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}+b_{11})c_{11}+(a_{12}+b_{12})c_{21} & (a_{11}+b_{11})c_{12}+(a_{12}+b_{12})c_{22} \\ (a_{21}+b_{21})c_{11}+(a_{22}+b_{22})c_{21} & (a_{21}+b_{21})c_{12}+(a_{22}+b_{22})c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}c_{11}+a_{12}c_{21} & a_{11}c_{12}+a_{12}c_{22} \\ a_{21}c_{11}+a_{22}c_{21} & a_{21}c_{12}+a_{22}c_{22} \end{bmatrix} + \begin{bmatrix} b_{11}c_{11}+b_{12}c_{21} & b_{11}c_{12}+b_{12}c_{22} \\ b_{21}c_{11}+b_{22}c_{21} & b_{21}c_{12}+b_{22}c_{22} \end{bmatrix} = AC+BC$$

(vi)

$$A(BC) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11}c_{11}+b_{12}c_{21} & b_{11}c_{12}+b_{12}c_{22} \\ b_{21}c_{11}+b_{22}c_{21} & b_{21}c_{12}+b_{22}c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(b_{11}c_{11}+b_{12}c_{21})+a_{12}(b_{21}c_{11}+b_{22}c_{21}) & a_{11}(b_{11}c_{12}+b_{12}c_{22})+a_{12}(b_{21}c_{12}+b_{22}c_{22}) \\ a_{21}(b_{11}c_{11}+b_{12}c_{21})+a_{22}(b_{21}c_{11}+b_{22}c_{21}) & a_{21}(b_{11}c_{12}+b_{12}c_{22})+a_{22}(b_{21}c_{12}+b_{22}c_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}b_{11}+a_{12}b_{21})c_{11}+(a_{11}b_{12}+a_{12}b_{22})c_{21} & (a_{11}b_{11}+a_{12}b_{21})c_{12}+(a_{11}b_{12}+a_{12}b_{22})c_{22} \\ (a_{21}b_{11}+a_{22}b_{21})c_{11}+(a_{21}b_{12}+a_{22}b_{22})c_{21} & (a_{21}b_{11}+a_{22}b_{21})c_{12}+(a_{21}b_{12}+a_{22}b_{22})c_{22} \end{bmatrix}$$

$= (AB)C$

For larger scale,maybe it's better to focus on only one element of the matrix.

Exercise 5.

$$A^2 = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$A^3 = A^2 * A = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 30 & 10 \\ 10 & -30 \end{bmatrix}$$

$$A^4 = A^3 * A = \begin{bmatrix} 30 & 10 \\ 10 & -30 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$$

$$\cdots$$

Following the pattern $A^{2n} = \begin{bmatrix} 10^n & 0 \\ 0 & 10^n \end{bmatrix} (n \in N)$

$$A^{50} = \begin{bmatrix} 10^{25} & 0 \\ 0 & 10^{25} \end{bmatrix}, A^{51} = \begin{bmatrix} 3*10^{25} & 10^{25} \\ 10^{25} & -3*10^{25} \end{bmatrix}$$