

# Homework for Linear Algebra

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### Exercise 1. (i)

- $S'$  is a basis for  $\text{span}(S) \Rightarrow S'$  is maximally linearly independent:  
From the definition of basis,  $S'$  is linearly independent.  $S'$  satisfies the definition of maximally linearly independent. For any  $\mathbf{v} \in \text{span}(S) \setminus S'$ , it can be represented by a linear combination of vectors in  $S'$ , so  $S' \cup \mathbf{v}$  is linear dependent.
- $S'$  is maximally linearly independent  $\Rightarrow S'$  is a basis for  $\text{span}(S)$ :  
From the definition of maximally linearly independent, for any  $\mathbf{v} \in S \setminus S'$ , it can be represented by a linear combination of vectors in  $S'$ .  
For any vector in  $\text{span}(S)$ , it can be represented by vectors in  $S$ , and inductively, be represented by vectors in  $S'$ . Also  $S'$  is linear independent, so it satisfies a basis for  $\text{span}(S)$ .

(ii)

- $S$  is a basis for  $V \Rightarrow S$  is maximally linearly independent in  $V$ :  
Since  $S$  is a basis, it is linear independent and spans  $V$ .  
So if it is not maximally linear independent, there exists  $\mathbf{v} \in V \setminus S$  that  $S \cup \mathbf{v}$  is linear independent.  
This contradicts that  $S$  spans  $V$ .
- $S$  is maximally linearly independent  $\Rightarrow S$  is a basis for  $V$ :  
If  $S$  isn't a basis for  $V$ , there exists  $\mathbf{v} \in V$  that  $\mathbf{v} \notin \text{span}(S)$ .  
So  $S \cup \mathbf{v}$  remains linear independent. Contradict the definition.

(iii) Assume  $S'$  is a subset of  $S$  and is always linear independent. To build a  $S'$ , we inductively add vectors in  $S$  one by one. Until no more vectors can be added. Since  $S$  is a finite set, the process will always have a end.  
So for any  $\mathbf{v} \in S \setminus S'$ ,  $S' \cup \mathbf{v}$  will lead to linear dependent.  
Now  $S'$  satisfies the definition of maximally linear independent.

### Exercise 2. First, consider the situation that any of $\mathbf{v}, \mathbf{w} = \mathbf{0}$ .

Obviously, (i) or (ii) will hold in this situation.

Under the condition that none of  $\mathbf{v}, \mathbf{w} = \mathbf{0}$ . Since  $\mathbf{u}_1 \cdots \mathbf{u}_n, \mathbf{v}, \mathbf{w}$  are linearly dependent, there exists

$$c_1 \mathbf{u}_1 + \cdots + c_n \mathbf{u}_n + a\mathbf{v} + b\mathbf{w} = \mathbf{0}$$

(Assume not all  $c_1 \cdots c_n, a, b$  are zero.)

Since  $\mathbf{u}_1 \cdots \mathbf{u}_n$  are linearly dependent, the only way that

$$c_1 \mathbf{u}_1 + \cdots + c_n \mathbf{u}_n = \mathbf{0}$$

is all the coefficients are zero. So a,b cannot all be zero. So we have:

If  $b = 0, a \neq 0, c_1 \mathbf{u}_1 + \cdots + c_n \mathbf{u}_n = -a\mathbf{v}$ .(i) holds.

If  $a = 0, b \neq 0, c_1 \mathbf{u}_1 + \cdots + c_n \mathbf{u}_n = -b\mathbf{w}$ .(ii) holds.

If  $a \neq 0, b \neq 0, c_1 \mathbf{u}_1 + \cdots + c_n \mathbf{u}_n = -b\mathbf{w} - a\mathbf{v}$ .(iii) holds.

**Exercise 3.**  $\mathbf{v}_3$

**Exercise 4.** Only if  $V = Z$ , or the vectors consist the basis can be transformed by linear combination.

**Exercise 5.**  $(1,1,0,1), (10,7,2,3), (0,0,1,0), (0,0,0,1)$

**Exercise 6.**

- $\mathbf{W} \subsetneq \mathbf{V} \Rightarrow \dim(\mathbf{W}) < \dim(\mathbf{V})$   
Assume  $\dim(\mathbf{W}) = \dim(\mathbf{V})$ .  
Since  $\mathbf{W} \subsetneq \mathbf{V}$ , there exists  $\mathbf{v} \in \mathbf{V} \setminus \mathbf{W}$  and cannot be represented by the basis of  $\mathbf{W}$ . So  $\dim(\mathbf{W} \cup \mathbf{v}) = \dim(\mathbf{W}) + 1 > \dim(\mathbf{V})$   
Since  $\mathbf{v} \cup \mathbf{W} \in \mathbf{V}$ ,  $\dim(\mathbf{v} \cup \mathbf{W}) \leq \dim(\mathbf{V})$ . Contradict!
- $\mathbf{W} \subsetneq \mathbf{V} \Leftarrow \dim(\mathbf{W}) < \dim(\mathbf{V})$   
Assume  $\mathbf{W} = \mathbf{V}$ , from the definition of basis, if  $\dim(\mathbf{W}) < \dim(\mathbf{V})$ , then the current basis of  $\mathbf{V}$  doesn't hold. Contradict!

**Exercise 7.** (i) Since all the column vectors of  $A \in R^m$ , so

$$\dim(C(A)) \leq \dim(R^m) = m.$$

So the basis of  $C(A)$  has at most m vectors. Since we have n column vectors, they are linearly dependent.

(ii) For  $A [\mathbf{a}_1 \cdots \mathbf{a}_n]$ , we have the linear combination:

$$x_1 \mathbf{a}_1 + \cdots + x_n \mathbf{a}_n = \mathbf{b}$$

And since  $A$  is linearly dependent, there exists nonzero solution to :

$$c_1 \mathbf{a}_1 + \cdots + c_n \mathbf{a}_n = \mathbf{0}$$

So if  $A\mathbf{x} = \mathbf{b}$  holds,  $A \begin{bmatrix} x_1 + k * c_1 \\ \vdots \\ x_n + k * c_n \end{bmatrix} = \mathbf{b} \ (k \in Z)$  holds too, it has infinite many

solutions.

If  $\dim(C(A)) < m$ , which means  $\text{span}(C(A)) \subsetneq R^m$ , there exists  $\mathbf{v} \in R^m \setminus \text{span}(C(A))$  that has no solution.

(iii)

- column rank of A is  $m \Rightarrow A\mathbf{x} = \mathbf{b}$  has infinite many solutions  
From previous provements, we know that if  $\dim(C(A)) = m = R^m$ ,  $\text{span}(C(A)) = R^m$ , and if the equation has a solution, it will have infinite many solutions.
- column rank of A is  $m \Leftarrow A\mathbf{x} = \mathbf{b}$  has infinite many solutions  
If there always exists a solution, that means  $R^m \subseteq \text{span}(C(A))$ . Since the column vectors  $\in R^m$  So  $\text{span}(C(A)) = R^m$ . So the column rank of A is  $m$ .

**Exercise 8.** Write

$$A = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{bmatrix}$$

As row vectors in  $R^n$ . So does

$$I = \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{bmatrix}$$

. Assume there exist B that holds the equation, then

$$\mathbf{e}_1 \cdots \mathbf{e}_n \in \text{span}(\{\mathbf{a}_1 \cdots \mathbf{a}_m\})$$

So it implies

$$R^n = \text{span}(\{\mathbf{e}_1 \cdots \mathbf{e}_n\}) \subseteq \text{span}(\text{span}(\{\mathbf{a}_1 \cdots \mathbf{a}_m\})) = \text{span}(\{\mathbf{a}_1 \cdots \mathbf{a}_m\}) \subseteq R^n$$

So the row rank of A is  $n$ .

If the row rank of A is less than  $n$ . That means  $\text{span}(\{\mathbf{a}_1 \cdots \mathbf{a}_m\}) \subsetneq R^n$ . So there must exist a  $\mathbf{e}_i \in R^n \setminus \text{span}(\{\mathbf{a}_1 \cdots \mathbf{a}_m\})$  that can not be represented by linear combinations of the row vectors. Unless  $\dim(\text{span}(\{\mathbf{a}_1 \cdots \mathbf{a}_m\})) = \dim(R^n)$ . So the row rank must be  $n$ .