Homework for Linear Algebra
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Exercise 1.

$$A^+ = [1]\left[\tfrac{1}{5}, 0\right]\begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix} = \begin{bmatrix} \frac{3}{25} & \frac{4}{25} \end{bmatrix}$$

$$A^+A = \begin{bmatrix} 1 \end{bmatrix}, AA^+ = \begin{bmatrix} \frac{9}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{16}{25} \end{bmatrix}$$

$$\mathbf{x}_1^+ = A^+\mathbf{b}_1 = 1, \mathbf{x}_2^+ = A^+\mathbf{b}_2 = 0$$

Exercise 2.

From the difinition of $\mathbf{u}$,we have $\mathbf{u}_i^T\mathbf{u}_j = 1$ when i=j, and if $i \neq j$, it equals zero. So

$$A^+A = \sum_{i\in r} \frac{\mathbf{v}_i\mathbf{u}_i^T}{\sigma_i} * \sum_{i\in r} \sigma_i\mathbf{u}_i\mathbf{v}_i^T = \sum_{i\in r} \mathbf{v}_i\mathbf{v}_i^T$$

So does for $\mathbf{v}_i$, so

$$(A^+A)^2 = \sum_{i\in r} \mathbf{v}_i\mathbf{v}_i^T * \sum_{i\in r} \mathbf{v}_i\mathbf{v}_i^T = \sum_{i\in r} \mathbf{v}_i\mathbf{v}_i^T = A^+A$$

For "projection", we have known that $\mathbf{v}_1 \cdots \mathbf{v}_r$ consists a basis for $C(A^T)$, also we can see from the equation that each column vector of A^TA is a linear combination of $\mathbf{v}_1 \cdots \mathbf{v}_r$. We prove that A^+A is exactly the projection matrix onto the vector space $C(A^T)$. That is to say, for any $\mathbf{b}$ its projection vector on $C(A^T)$ is $A^+A\mathbf{b}$. Proof: Also use the property of $\mathbf{v}_i$,

$$A(\mathbf{b} - A^+A\mathbf{b}) = \sum_{i\in r} \sigma_i\mathbf{u}_i\mathbf{v}_i^T\mathbf{b} - \sum_{i\in r} \sigma_i\mathbf{u}_i\mathbf{v}_i^T * \sum_{i\in r} \mathbf{v}_i\mathbf{v}_i^T\mathbf{b} = \sum_{i\in r} \sigma_i\mathbf{u}_i\mathbf{v}_i^T\mathbf{b} - \sum_{i\in r} \sigma_i\mathbf{u}_i\mathbf{v}_i^T\mathbf{b} = 0$$

That is to say, the error vector $\mathbf{b} - A^+A\mathbf{b}$ is perpendicular to all the column vectors of A, which is to say it's perpendicular to $C(A^T)$.

Exercise 3.

For the matrix $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$, since each column vector of it is orthornormal,we have $A^{-1} = A^T$. So

$$AA^T = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} = \mathbf{v}_1\mathbf{v}_1^T + \cdots + \mathbf{v}_n\mathbf{v}_n^T = AA^{-1} = I_n$$

Exercise 4.

4.1

Since $A^+ = V\Sigma^+U^T = \sum_{i\in r} \frac{\mathbf{v}_i\mathbf{u}_i^T}{\sigma_i}$, and from definition, each $\mathbf{v}_i$ is linearly independent to others. So we can see each column of A^+ as a linear combination of $\mathbf{v}_1 \cdots \mathbf{v}_r$,which means $rank(A^+) = rank(A) = r$.

4.2

If A is invertible, that means in SVD, the matrix Σ will have the whole n singular values. From the property of orthornormal matrix, we have

$$A^+A = V\Sigma^+U^TU\Sigma V^T = V\Sigma^+\Sigma V^T = VIV^T = I$$

Also $AA^+ = I$, which means $A^+ = A^{-1}$.

Exercise 5.

5.1 Assume $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$, where $\mathbf{v}_1, \cdots, \mathbf{v}_n$ consists a orthonormal basis for $\mathbb{R}^n$, and the first r vector of it is exactly the column vectors in V of A's SVD. From what we known in class, it's possible to build a basis like this. So

$$\frac{||A\mathbf{x}||}{||\mathbf{x}||} = \frac{||(\sum_{i\in r}\sigma_i\mathbf{u}_i\mathbf{v}_i^T)(\sum_{i\in n}c_i\mathbf{v}_i)||}{\sqrt{\sum_{i\in n}c_i^2}} = \frac{\sqrt{\sum_{i\in r}\sigma_i^2c_i^2}}{\sqrt{\sum_{i\in n}c_i^2}}$$

Also

$$\sum_{i\in r}\sigma_i^2c_i^2 = \sum_{i\in r}\sigma_i^2c_i^2 + \sum_{i=r+1}^{n} 0 * c_i^2 \leq \sigma_1^2\sum_{i=1}^{n}c_i^2$$

So

$$\max\frac{||A\mathbf{x}||}{||\mathbf{x}||} = \sigma_1$$

holds.

5.2 First we prove the fact that there exist a nonzero $\mathbf{x} \in V_{k+1} = span(\{\mathbf{v}_1 \cdots \mathbf{v}_{k+1}\}) \cap R(B)$

Proof: Since $dim(V_{k+1}) = k+1 > rank(B)$, there must exist a vector in $\mathbf{v}_1, \cdots, \mathbf{v}_{k+1}$ that in $V_{k+1} \setminus C(B^T)$, so the error vector of its projection (which is also in V_{k+1} since it can be represented by the vectors consist the basis of $C(B^T)$ and the unique vector). This vector sufficients $\mathbf{x}$ since it is perpendicular to all the row vectors of B. Now we have $\mathbf{x} = \sum_{i=1}^{k+1} c_i\mathbf{v}_i$, then we calculate the fraction.

$$||(A-B)\mathbf{x}|| = ||A\mathbf{x}|| = \sqrt{\sum_{i=1}^{k+1}c_i^2\sigma_i^2} \geq \sqrt{\sigma_{k+1}\sum_{i=1}^{k+1}c_i^2}$$

So $\max\frac{||(A-B)\mathbf{x}||}{||\mathbf{x}||} \geq \sigma_{k+1}$.

5.3 the equality holds when $c_2 = \cdots = c_n = 0$ which means $\mathbf{x}$ is linearly dependent to $\mathbf{v}_1$.