Modelling of Ice Melting Process

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1. Problem Definition

1)Restatement of the task

Considering a system shown in the *Figure 1*, which includes a container, a bottle of water and a floating ice tube. Try to model the ice melting process.

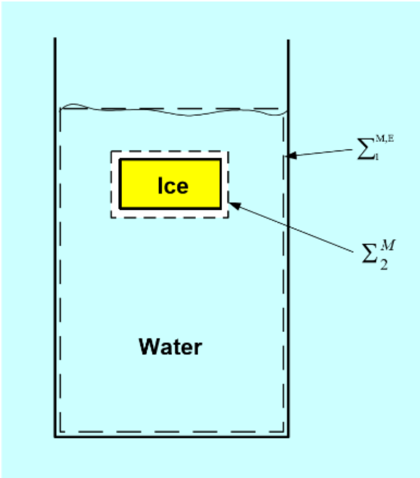

Figure 1

2) Underlying assumptions

a. During the process, the external environment stays stable, which means variables like air temperature T_a and air pressure $P_0 = 1atm$.

b. During the process, we assume the relationship $V_{water} >> V_{ice}$ always exists. Thus, we could ignore the changing volume of water.

c. To simplify the process, we choose an ice ball as our modelling object (as the parameters of sphere are isotropic in space), and heat transfer processes are isotropic under the same circumstance.

d. During the process, we assume the ice ball stays spherical, which means we ignore the shape difference brought by the heat conduction characteristics difference of air and water.

e. Different materials are perfectly contacted, which means the temperature changes continuously at the interface of different materials.

f. The inner temperature of the ice ball varies linearly with radius.

g. Heat flows distribute uniformly inside the sphere, in other words, whenever heat flows injected in, temperature changes the same for every point in the sphere.

h. As both temperature and temperature differences are small, we ignore radiation between materials, only conduction and convection exist during the process.

i. Temperature changes with both space and time, thus, the system is a distributed and dynamic system

3) Intended goals or use

a. To describe the ice melting process precisely

b. To predict the ice melting process or other similar processes.

c. Control design

d. $\pm 10\%$ accuracy

4) Anticipated inputs/disturbance

Inputs: Initial volume of ice V_i, initial temperature of ice T_i, the temperature of external environment T_a

Output: The radius of the ice ball $r(t)$

2. Controlling Factors

a. No chemical reactions involved

b. Mass diffusion obeys mass transfer law

c. Heat Conduction obeys heat transfer law

d. No evaporation included

e. Liquid stays still

f. Perfect contact

3. Problem Data

R_0: Bottom radius of the container

H_0: Height of the container

V_{w0}: Initial volume of water

V_e: volume of ice exposed to the air

T_{a0}: Initial temperature of air

T_{i0}: Initial temperature of ice

T_{w0}: Initial temperature of water

t_i: ice temperature during the process

t_w: average water temperature during the process

t_{co}: outer wall temperature of the container during the process

t_{ci}: inner wall temperature of the container during the process

S_{ai}: The contact area between air and ice

S_{aw}: The contact area between air and water

S_{iw}: The contact area between ice and water

λ_a: Thermal conductivity of water, we choose $\lambda_a = 2.55\times10^{-2} W/(m\cdot K)$

λ_i: Thermal conductivity of ice, we choose $\lambda_i = 2.24 W/(m\cdot K)$

λ_w: Thermal conductivity of water, we choose $\lambda_w = 0.599 W/(m\cdot K)$

ρ_w: The density of water, we choose $\rho_w = 1.0\times10^3 kg/m^3$

ρ_i: The density of ice, we choose $\rho_i = 0.9\times10^3 kg/m^3$

C_w: Heat capacity of water, we choose $C_w = 4.18\times10^3 J/kg\cdot℃$

C_i : Heat capacity of ice, we choose $C_i = 2.1\times10^3\, J/kg\cdot℃$

γ_i : latent heat of liquefaction of ice, we choose $\gamma_i = 3.35\times10^5\, J/kg$

α_w :Convection coefficient of water, we choose $\alpha_w = 1.0\times10^3\, J/kg$

α_a :Convection coefficient of air, we choose $\alpha_a = 3.5\times10^3\, J/kg$

d : The thickness of the container wall

r_i : ice radius during the process

4. Model Construction

Summary: As the whole process is quite sophisticate, we divide the whole process into 5 smaller part, showing as follows:

① Energy conservation: Regarding the container, water and ice as a whole system, develop the law of energy conservation:

$$(Input\ energy) - (Output\ energy) = (Energy\ change\ of\ the\ system)$$

This process involves two parts. One is the heat conduction between air and container wall, air and water, air and ice. The other one is radiation of air, water, container wall and ice.

② Heat conduction in the container wall

③ Heat conduction in the water

④ Heat conduction between water and ice

⑤ Heat absorption and melting process

Modelling:

① Energy conservation

Firstly, for the flank of the container, we assume the point which is kR_0 distant from the center axis of the cylinder has the same temperature as the surrounding environment. According to our experience, we pick $k = \inf$. And according to underlying assumption d, we could figure out that the point close to the container wall has the same temperature as the container wall. List the heat conduction equation between air and the flank as follows:

$$Q_1 = q_1 A = 2\pi r H_0 \lambda_a \frac{dt}{dr} \Rightarrow \int_{t_{c0}}^{T_A} dt = \int_{R_0}^{kR_0} \frac{Q_1}{2\pi\lambda_a H_0}\cdot\frac{dr}{r}$$

Solve the ODE, we could get the heat flow from the air to the flank:

$$Q_1 = \frac{2\pi\lambda_a H_0 (T_a - t_{c0})}{\ln k} \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

Secondly, for the bottom of the container, we also assume the point which is kR_0 distant from the bottom of the cylinder has the same temperature as the surrounding environment and we pick $k=2$. We use average temperature gradient to calculate the heat flow from air to the bottom.

$$Q_2 = \frac{\pi R_0^{\,2}(T_a - t_{co})}{(k-1)R_0} \cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$$

Thirdly, according to underlying assumption b that $V_{water} >> V_{ice}$, we could regard the area of water surface as a constant. Similarly, we are supposed to calculate the heat flow form air to the water surface.

$$Q_3 = \frac{\pi R_0^{\,2}(T_a - t_w)}{(k-1)R_0} \cdots\cdots\cdots\cdots\cdots\cdots\cdots (3)$$

Fourthly, considering the heat conduction between ice and air. To do this , we have to calculate the area exposed to the air.

As shown in *Figure 2*, according to Archimedes principle, we are supposed to know that the volume of ice exposed to the air:

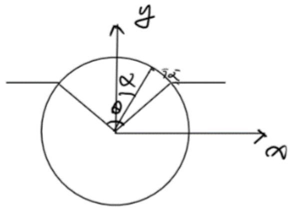

Figure 2

$$V_e = \frac{\rho_w - \rho_i}{\rho_w} \cdot V_i = 0.1V_i$$

Calculate the value of θ by using volume integral:

$$V_e = \int_0^{\frac{\theta}{2}} 2\pi r_i \sin\alpha r_i \cos\alpha r_i (1-\cos\alpha) d\alpha = 2\pi r_i^3 (\frac{1}{2}\sin^2\frac{\theta}{2} + \frac{1}{3}\cos^3\frac{\theta}{2} - \frac{1}{3}) = \frac{1}{10}\cdot\frac{4}{3}\pi r_i^3$$
$$\Rightarrow \frac{1}{2}\sin^2\frac{\theta}{2} + \frac{1}{3}\cos^3\frac{\theta}{2} = 0.4$$

Solve the equation, we could get $\theta = 110.9^o$

Then we are supposed wo calculate the contact area between ice and air S_{ai} and contact area between ice and water S_{iw} as follows:

$$S_{ai} = \int_0^{\frac{\theta}{2}} 2\pi r_i^2 \sin\alpha d\alpha = 0.86586\pi r_i^2, S_{iw} = 4\pi r_i^2 - \int_0^{\frac{\theta}{2}} 2\pi r_i^2 \sin\alpha d\alpha = 3.13414\pi r_i^2$$

To simplify the modelling process, we regard S_{ai} as a two dimension circle surface, we can calculate the heat flow form air to the exposed area of ice.

$$Q_4 = \frac{0.86586\pi r_i^2 (T_a - t_w)}{(k-1)R_0} \cdots\cdots\cdots\cdots\cdots\cdots\cdots (4)$$

② Heat conduction in the container wall

The temperature of the outer wall of the container t_{co} is higher than that of the inner wall t_{ci} , and as the container s a cylinder, we could construct the heat flow formula as follows.

$$Q_5 = \frac{2\pi\lambda_c H_0 (t_{c0} - t_{ci})}{\ln\frac{r_2}{r_1}} \cdots\cdots\cdots\cdots\cdots\cdots (5)$$

(λ_c : Thermal conductivity of the container; r_1 :Radius of inner wall; r_2 :Radius of the outer wall)

According to our experiment results, we find that the wall of container is quite thin, the temperatures of outer and inner wall are approximately the same. So, to simplify the modelling process, **we ignore the heat conduction in the container wall**. Thus, we have:

$$t_{co} \approx t_{ci}$$

③ Heat conduction in water

According to our underlying assumptions, as materials are perfect contacting with each other, so there's no temperature changes continuously at the interface of different materials.

To construct the model, we regard the process as shown on the right. For water outside the semi-circle ①, it is quite far away from the ice ball, so we assume that water temperature in this area stays stable with the value same as t_{ci}. We can easily construct the following equation:

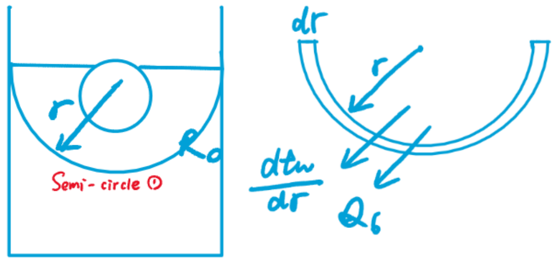

Figure 3

$$2\pi r^2 \lambda_w \frac{dt_w}{dr} = Q_6 \Rightarrow t_w(R_0) - t_w(r_i) = \frac{Q_6}{2\pi}\left(\frac{1}{r_i} - \frac{1}{R_0}\right)$$

Give boundary condition, we have: $t_w(R_0) = t_{co}, t_w(r_i) = 0℃$

Solve the ODE, we have the following conclusion:

$$Q_6 = \frac{2\pi \lambda_w t_{ci} R_0 r_i}{R_0 - r_i} \cdots\cdots\cdots\cdots\cdots\cdots (6)$$

④ Heat conduction between water and ice

Considering the above equations (1) to (6), we find undefined variables are $t_{co} \approx t_{ci}, t_w$.

Pick algebraic average value of water temperature as t_w, we have: $t_w = \frac{t_w(R_0) + t_w(r_i)}{2} = \frac{t_{co}}{2}$

According to our underlying assumption f and g, temperature inside the ice ball changes linearly with radius, which means $t_i(r) = \frac{r}{r_i} T_i$, and temperature changes the same for every point inside the ice ball, we can calculate the average temperature of the ice ball: $\bar{t_i} = \frac{r_i}{2R} T_i$.

Similar solution to this process, we have heat conduction between water and ice is calculate as follow:

$$Q_7 = \alpha_w S_w \left(t_w - \frac{r_i}{2R} T_1\right) = 3.13414\pi r^2 \alpha_w \left(t_w - \frac{r_i}{2R} T_1\right) \cdots\cdots (7)$$

⑤ Heat absorption and melting process

Referring to *Figure 4*, when state changes from A to B, ice ball radius change from r to $r - dr$.

During the process, the changing of radius has the corresponding energy consumption which are used for both ice-ball temperature increase and ice-melting.

Ice ball temperature increase consumes energy:

$$Q_{r1}=\frac{4}{3}\pi r_i^3\rho_i C_i\frac{dt_i}{dt}=\frac{4}{3R_i}\pi r_i^3\rho_i C_i T_i\frac{dr}{dt}$$

Melting process consumes energy: $Q_{r2}=-4\pi r_i^2\rho_i\gamma_i\frac{dr}{dt}$

Q_4 and Q_7 are energies influx into the ice ball, applying energy conservation equation:

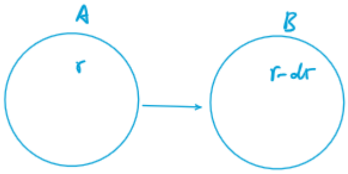

Figure 4

$$Q_4+Q_7=Q_{r1}+Q_{r2}$$

$$\Rightarrow 0.86586\pi r_i^2\alpha_a(T_a-t_i)+3.13414\pi r_i^2\alpha_w\cdot(t_w-\frac{r}{2R}T_i)=\frac{4}{3R_i}\pi r_i^3\rho_i C_i T_i\frac{dr}{dt}-4\pi r_i^2\rho_i\gamma_i\frac{dr}{dt}\cdots\cdots(8)$$

⑥ Solution to water temperature

Above all, we have already complete the major solution of this process. However, we find t_w undefined, so we need additional equations to meet the degree of freedom of this model. Q_1,Q_2,Q_3,Q_7 are energy that water exchanges with other materials. Applying energy conservation equation:

$$Q_1+Q_2+Q_3-Q_7=\frac{dW_w}{dt}=C_w\rho_w V_w\frac{dt_w}{dt}$$

$$\Rightarrow\frac{dt_w}{dt}=\frac{1}{C_w\rho_w V_w}(2\pi R_0H_0\alpha_a(T_a-t_{co})+Q_2=\pi R_0^2\alpha_a(T_a-t_{co})+\pi R_0^2\alpha_a(T_a-t_w)-3.13414\pi r_i^2\alpha_w\cdot(t_w-\frac{r_i}{2R}T_i))\cdots\cdots(9)$$

Up to now, we transfer the question into solving the above differential ordinary equations (8) and (9). And we have already complete the model construction step.

5. Model Solution

Firstly, we apply **DOF analysis** to verify whether the current model is solvable, showing as follows:

a. Model:

Ordinary differential equations:

$$\frac{dr_i}{dt}=\frac{0.86586\pi r_i^2\alpha_a(T_a-t_i)+3.13414\pi r_i^2\alpha_w\cdot(t_w-\frac{r}{2R}T_i)}{\frac{4}{3R_i}\pi r_i^3\rho_i C_i T_i-4\pi r_i^2\rho_i\gamma_i}$$

$$\frac{dt_w}{dt}=\frac{1}{C_w\rho_w V_w}(2\pi R_0H_0\alpha_a(T_a-t_{co})+\pi R_0^2\alpha_a(T_a-t_{co})+\pi R_0^2\alpha_a(T_a-t_w)-3.13414\pi r_i^2\alpha_w\cdot(t_w-\frac{r_i}{2R}T_i))$$

$Initial\ \ condition: t_w(0)=23.6℃,\ \ r_i(0)=30.1mm$

Algebraic equations:

$$S_{iw}=3.13414\pi r_i^2$$

$$S_{ai}=0.86586\pi r_i^2$$

$$S_{aw}=\pi R_0^2-S_{ai}$$

$$t_{ci} = t_{co}$$

$$t_{co} = 2t_w$$

b. DOF analysis:
State variable: r_i, t_w
Algebraic variables: $t_i, t_{co}, t_{ci}, S_{ai}, S_{aw}, S_{iw}$
Parameters: $R_0, H_0, V_{w0}, V_e, T_{a0}, T_{i0}, T_{w0}, d$
Constants: $\lambda_a, \lambda_i, \lambda_w, \rho_w, \rho_i, C_w, C_i, \gamma_i, \alpha_w, \alpha_a$
Equations
$N_{DF} = N_U - N_E = 0$

Thus, we've verified that the model is well posed such that the "excess" variables and degree of freedom are satisfied. Also, the current model avoids certain numerical problems such as high index systems.

6. Model Verification

① **Experiment**

In order to verify the model, we carry out the experiment in order to provide enough data for verification. Experiment steps are shown as follows:

a. experimental equipment

Figure 5

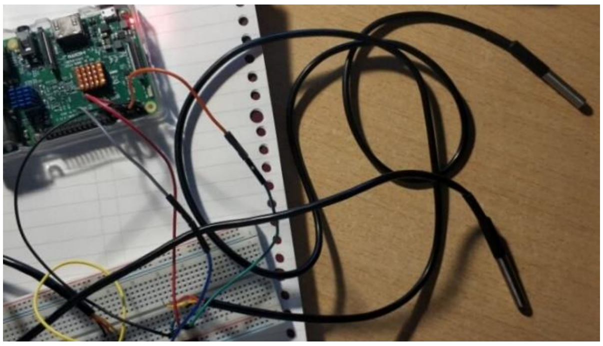

Figure 6

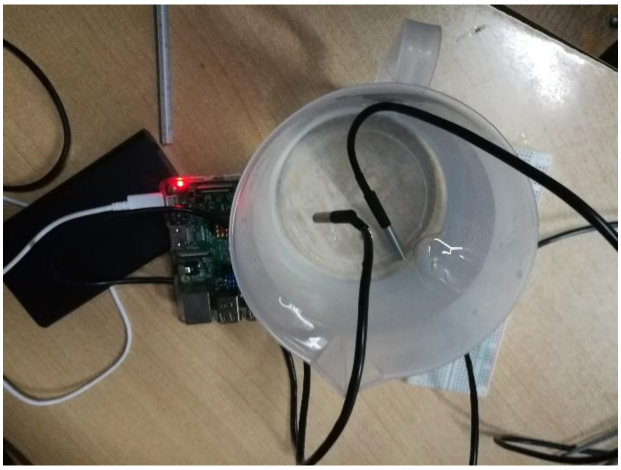

Figure 7

Figure 5 shows the container we use during the experiment, *Figure 6* shows the temperature sensor we use during the experiment, and *Figure 7* is an image during the experiment. The corresponding parameters of these experimental equipment are shown in the following table:

Table 1:Parameters of the container

Radius(mm)	Hight(mm)	Wall thickness(mm)
160	260	2

Table 2:Parameters of the ice ball

Radius(mm)	Initial Temperature(℃)
31	-11.4

Table 3:Other parameters

Environment temperature(℃)	25.0
Initial water temperature(℃)	23.6
Initial water volume(L)	17.3
Ice melting time(s)	820

Table 4:Water temperature

Time(s)	Water temperature (℃)
0	23.6
70	23.6
152	23.3
221	23.1
300	22.9
360	22.9
431	22.7
516	22.7
576	22.7
670	22.6
740	22.5
780	22.5
820	22.5

Structed programming approach & Modular code

Applying MATLAB to solve the model numerically, MATLAB codes are showing as follows:

Main function

```
% Main function
clc
clear
format short;
Tw=23.6; %Initial water temperature
R0=0.031; %Initial radius
[x,y]=ode45('eq1',[0,1000],[Tw,R0]); %call subfunction 'eq1' to solve ODEs
n=length(y);
for i=1:n %Restrict the range of radius(r>0)
    if y(i,2)<0
        y(i,2)=0;
    end
end
figure(1);
plot(x,y(:,1),'b-o');
hold on;
a=[0,70,152,221,300,360,431,516,576,670,740,780,820]; %Set experiment
data(time)
b=[23.6,23.6,23.3,23.1,22.9,22.9,22.7,22.7,22.7,22.6,22.5,22.5,22.5]; %Set experiment
data(water temperature)
[xData, yData] = prepareCurveData( a, b ); %Fit experiment data
% Set up fittype and options.
ft = fittype( 'smoothingspline' );
opts = fitoptions( 'Method', 'SmoothingSpline' );
opts.SmoothingParam = 3.56776105295262e-07;
[fitresult, gof] = fit( xData, yData, ft, opts ); % Fit model to data.
h = plot( fitresult,'r'); %Plot fit result
xlabel 'Time(s)';
ylabel 'Temperature(℃)';
title 'Ice melting process (T~t)'; %plot water temperature figure
figure(2);
plot(x,1000*y(:,2),'r-o');
xlabel 'Time(s)';
ylabel 'Radius(mm)';
title 'Ice melting process (r~t)'; %plot radius figure
```

ODE construction subfunction

```
% eq1 function, to establish the corresponding ODEs
function dy=eq1(~,y)
% Define constants (SI)
Ui=900; %Density of ice
Li=3.35*10^5; %Latent heat of liquefaction of ice
Ci=2.1*10^3; %Heat capacity of ice
Uw=1000; %Density of water
Aw=600; %Convection coefficient of water
Cw=4.2*10^3; %Heat capacity of water
Aa=15; %Convection coefficient of water
R=0.031; %Initial radius
Ti=-11.4; %Initial ice temperature
H0=0.26; %Height of the container
R0=0.08; %Bottom radius of the container
Ta=25; %Initial air temperature
Vw=17.3*10^-3; %Initial water temperature

k1=2*pi*Aa*H0*R0;
k2=pi*R0^2*Aa;
k3=-3.13414*pi*Aw;
k4=0.86586*pi*Aa;
k5=3.13414*pi*Aw;
k6=-4*pi*Ui*Ci*Ti/(3*R);
k7=-4*pi*Ui*Li; %k1~k7 are define constants to simplify the following ODEs

% Define ODEs
dy=zeros(2,1);
dy(1)=(k1*(Ta-2*y(1))+k2*(Ta-2*y(1))+k2*(Ta-y(1))+k3*y(2)^2*(y(1)-
y(2)*Ti/(2*R)))/(Cw*Uw*Vw);
dy(2)=(k4*y(2)^2*(Ta-y(2)*Ti/(2*R))+k5*y(2)^2*(y(1)-
y(2)*Ti/(2*R)))/(k6*y(2)^3+k7*y(2)^2);
```

Run the above code under our experiment condition, we could approach the following result:

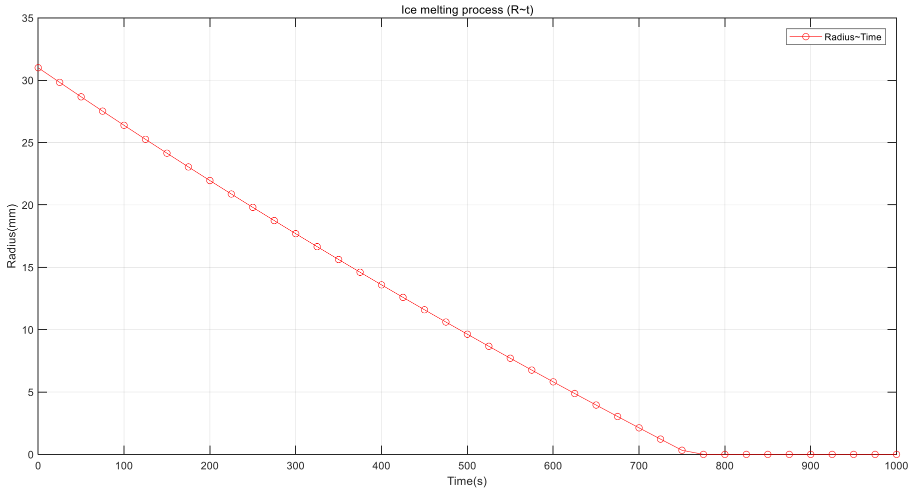

Figure 8: Ice-ball radius(mm)~Time(s)

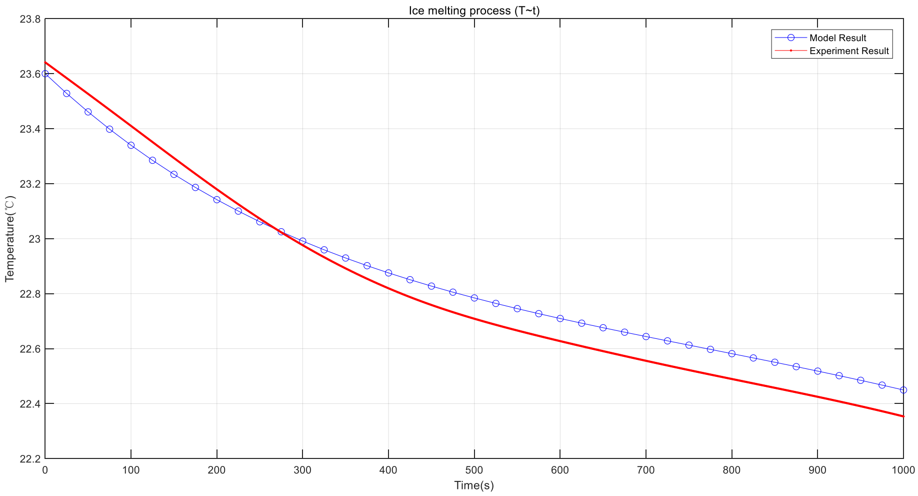

Figure 9: Water temperature(℃)~Time(s)

Afterwards, we examine our model result with our experiment result. According to *Figure 8*, we could easily find the estimate melting time $t = 775s$, which has the corresponding relative error $\varepsilon_r = 5.49\%$. According to *Figure 9*, we could find that the model curve are approximately the same with the experiment curve, with the maximum absolute error $\Delta T = 0.1℃$, that's quite accurate!

Above all, for melting time, our model have a relative error $\varepsilon_r = 5.49\% < 10\%$, which is acceptable and satisfies our intended goal ($\pm 10\%$ accuracy). For water temperature, we have a maximum absolute error $\Delta T = 0.1℃$, which is quite accurate.

PS: As ice-ball radius is hard to measure during the process, thus, we only examine the dynamic process of water temperature. However, as water temperature and radius are coupled variables, when one variable is correct, so does the other one.

7. Model Calibration & Validation

To calibrate the validity of the model, we examine whether the model behavior is logically correct when apply changes on single variable one by one, results are as follows:

To start with, we change **the value of water temperature**, we pick the following situations:

$$T_w = 15℃, 20℃, 25℃, 30℃, 35℃, 40℃$$

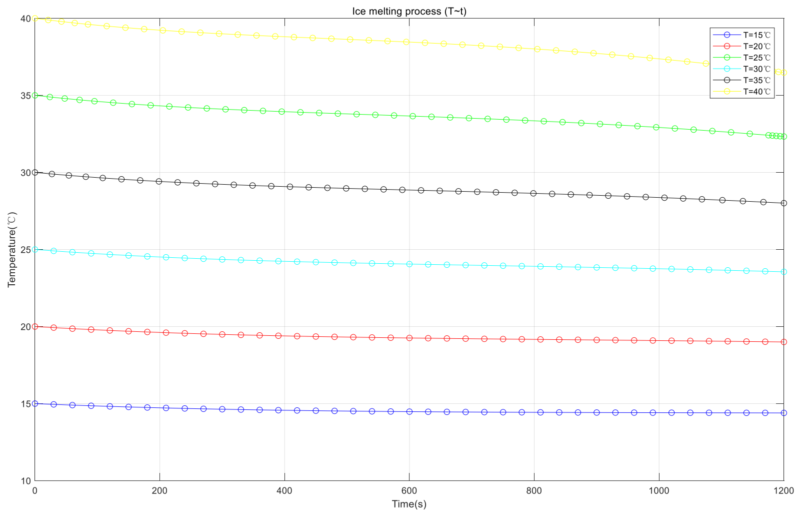

Figure 10: Water temperature(℃)~Time(s)

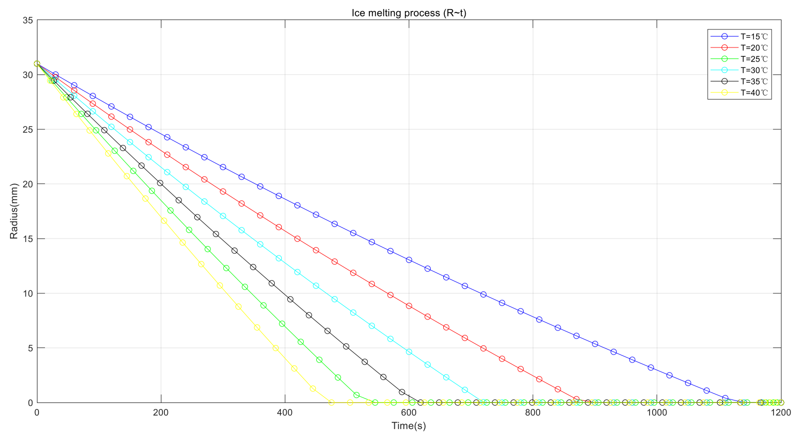

Figure 11:Radius(㎜)~Time(s)

From the *Figure 10* and *Figure 11*, we can easily find that with the increase of water temperature, firstly, water temperature changes more fiercely, which conforms to the reality as higher water temperature means higher temperature variation and more rapid changes.

Secondly, with the increase of water temperature, higher steady state water temperature is achieved, which conforms to the reality that higher initial value leads to higher final value.

Last but not least, with the increase of water temperature, ice-melting process is faster, which conforms to the reality that higher water temperature means higher water difference which leads to more fierce heat conduction. The melting rate could be represented by the slope of curves in *Figure 11*

In conclusion:

$$T_w \uparrow \Rightarrow changing\ rate\ of\ T_w \nearrow,\ final\ T_w \uparrow, ice-melting\ \ process \uparrow, melting\ time \downarrow$$

Then, we change the **initial radius of the ice ball**, we pick the following situations:

$$R = 10mm, 15mm, 20mm, 25mm, 30mm, 35mm$$

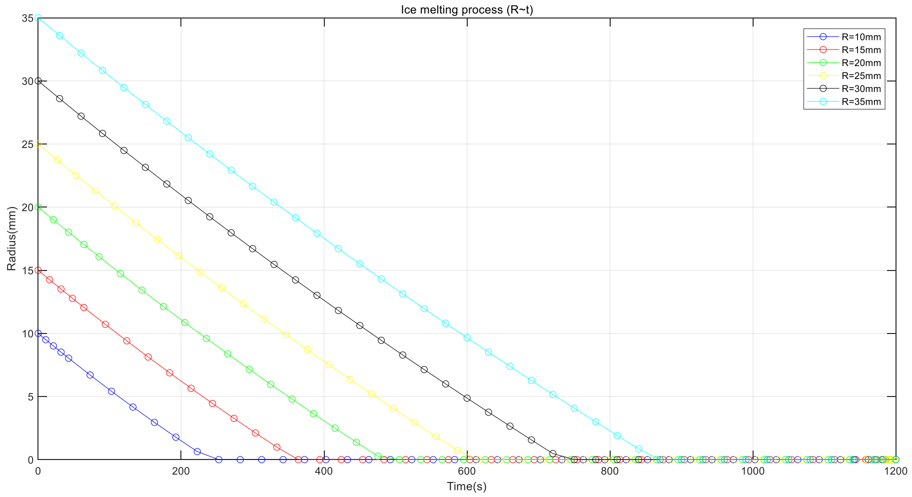

Figure 12: Water temperature(℃)~Time(s)

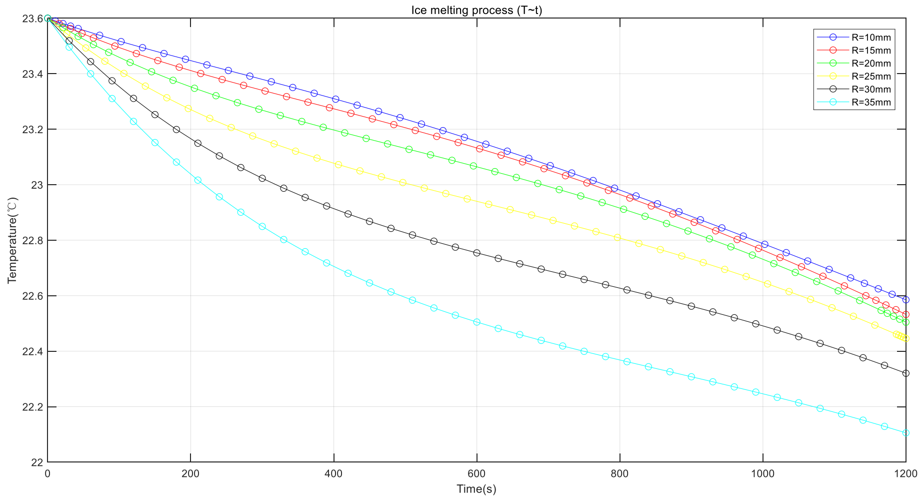

Figure 13:Radius(㎜)~Time(s)

From the *Figure 12* and *Figure 13*, we can easily find that with the increase of initial radius, firstly, water temperature changes more fiercely, which conforms to the reality as bigger ice ball means more fierce energy exchange during the process, thus, water temperature decreases faster.

Secondly, with the increase of radius, lower steady state water temperature is achieved, which conforms to the reality as bigger ice ball means more energy exchange during the process, thus, water temperature decreases more.

Last but not least, with the increase of radius, ice-melting process is faster, but needs more time to melt, which conforms to the reality that higher water temperature means higher water difference which leads to more fierce heat conduction.

In conclusion:

$$R\uparrow \Rightarrow changing\ rate\ of\ T_w \rightarrow, final\ T_w \downarrow, ice-melting\ \ process\uparrow, melting\ time\uparrow$$

Finally, we change **the value of air temperature**, we pick the following situations:

$$T_a = 10℃, 15℃, 20℃, 25℃, 30℃, 35℃$$

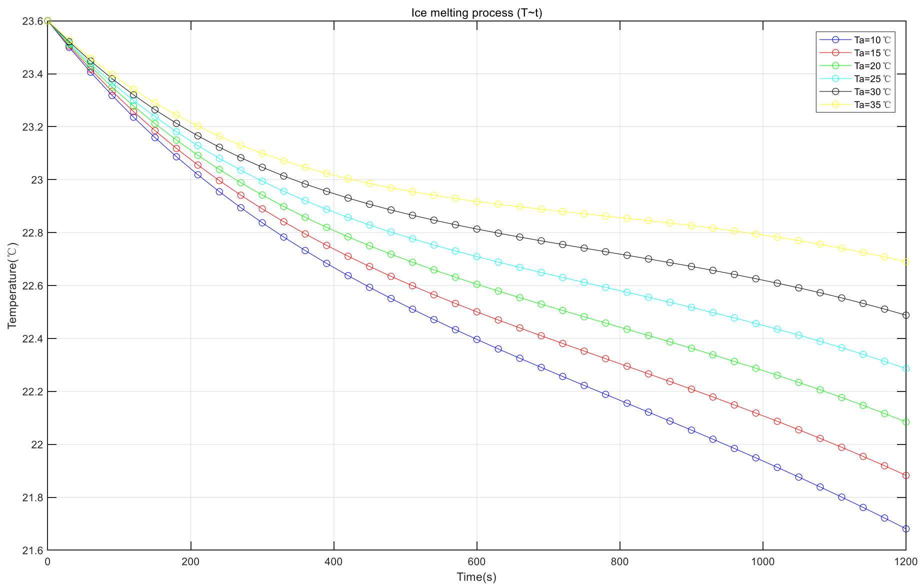

Figure 14: Water temperature(℃)~Time(s)

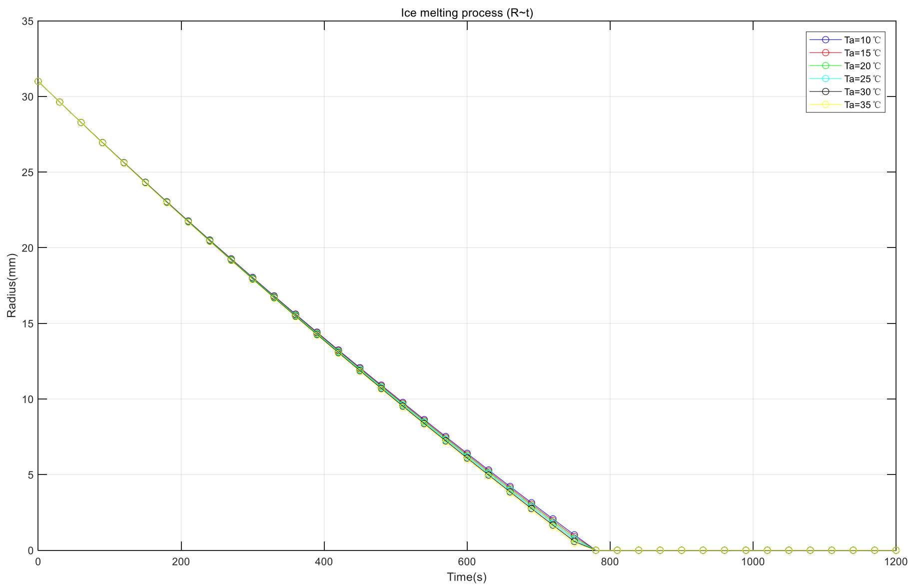

Figure 15:Radius(㎜)~Time(s)

From the *Figure 14* and *Figure 15*, we can easily find that with the increase of air temperature, firstly, water temperature changes more slowly, which conforms to the reality as higher air temperature means energy input to water during the process, thus, water temperature decreases slower.

Secondly, with the increase of water temperature, higher steady state water temperature is achieved, which conforms to the reality as higher air temperature means more energy input during the process, thus, water temperature decreases less.

Last but not least, air temperature has little influence on ice-melting process, which could be easily obtained from Figure 15 that curves are approximately the same.

In conclusion:

$$T_a \uparrow \Rightarrow changing\ rate\ of\ T_w \downarrow,\ final\ T_w \uparrow, ice-melting\ \ process —, melting\ time \searrow$$

PS: Changing rate of T_w is reflected by the slope of T~t curves.

Final T_w is reflected by the value of T_w when $R = 0mm$.

Ice-melting process is reflected by the slope of R~t curves.

Melting time is reflected by the value of t when $R = 0mm$.

From the above results, not only do we find our model is successfully adapted to our experiment data, but also do we find it logically correct for further validation, thus, the model is great and acceptable!