Dalhousie University

ECED 4301

Electromagnetic Waves and Propagation (Lecture Notes)

by

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Electrical and Computer Engineering

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Chapter 0

Vector Analysis

1. **Definition of a vector:**

<u>DEF:</u> A vector is said to be a quantity which has a **magnitude** and a **direction**.

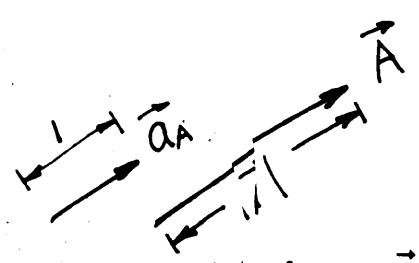
e.g. A vector $\vec{A}$ can be written as

$$\vec{A} = A\,\vec{a}_A$$

where $A=|\vec{A}|$ (≥ 0) is the magnitude of $\vec{A}$, and $\vec{a}_A$ specifies the direction of $\vec{A}$ and is a dimensionless vector having a unity magnitude. From the above equation:

$$\vec{a}_A = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A}$$

The vector $\vec{A}$ can be represented graphically by a directed straight-line segment of length $|\vec{A}| = A$ with its arrowhead pointing in the direction of $\vec{a}_A$, as shown in the figure below.



Graphical representatoı of vector $\vec{A}$

<u>OBS:</u> (1) Two vectors are equal if and only they have the same magnitude and the same direction, even though they mye displaced in space.

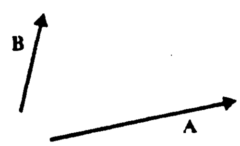
(2)If $\vec{A}$ is a vector, $-\vec{A}$ is another vector ich has the same magnitude as $\vec{A}$ but points to the opposite direction of $\vec{A}$.

2. **Vector addition and subtraction**

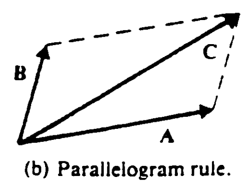
1. Vector addition

Consider two vectors, $\vec{A}$ and $\vec{B}$. Then, the sum of $\vec{A}$ and $\vec{B}$, $\vec{C}=\vec{A}+\vec{B}$, is another vector in the same plane. $\vec{C}$ can be obtained graphically in two ways:

A. By the parallelogram:



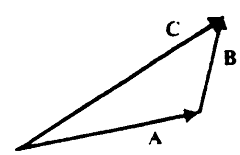
(a) Two vectors, $\vec{A}$ and $\vec{B}$



(b) Parallelogram rule

B. By the head-to-tail rule:

(a) Two vectors, $\vec{A}$ and $\vec{B}$



(b) Head-to-tail rule

Commutative Law:

$$\vec{C}=\vec{A}+\vec{B}=\vec{B}+\vec{A}$$

If the above equation exists, it is sometimes said that the vector $\vec{C}$ can be decomposed as (the sum of) two components, $\vec{A}$ and $\vec{B}$.

Associative Law: $\vec{A}+(\vec{B}+\vec{C}) = (\vec{A}+\vec{B})+\vec{C}$

2. Vector subtraction

Subtraction of two vectors, $\vec{\mathrm{A}}$ and $\vec{\mathrm{B}}$, can be obtained in terms of vector addition in the following way:

$$\vec{\mathrm{C}} = \vec{\mathrm{A}} - \vec{\mathrm{B}} = \vec{\mathrm{A}} + (-\vec{\mathrm{B}})$$

Note: It is meaningless to add or subtract a scalar from a vector, or to add or subtract a vector from a scalar !

Ex: Use the head-to-tail rule to find $\vec{\mathrm{C}}$ with $\vec{\mathrm{C}}=\vec{\mathrm{A}}+\vec{\mathrm{A}}$ $(=2\vec{\mathrm{A}})$.

3. Vector Multiplication

1) Multiplication of a vector $\vec{\mathrm{A}}$ by a scalar

DEF:

$$k\vec{\mathrm{A}} \triangleq k\,\vec{a}_A A = \mathrm{sign}(k)\,\vec{\mathrm{a}}_A\,|k|\,A$$

Note: Multiplication of a vector $\vec{\mathrm{A}}$ by a scalar k is a vector.

OBS: if k is a scalar and $\vec{\mathrm{A}}$ is a vector, $\vec{\mathrm{C}}=k\vec{\mathrm{A}}$ is another vector whose magnitude is k times of magnitude of $\vec{\mathrm{A}}$. $\vec{\mathrm{C}}$ will points to the same direction as $\vec{\mathrm{A}}$ if k is positive and points to the opposite direction of $\vec{\mathrm{A}}$ if k is a negative value.

It is not difficult to prove from the definition that

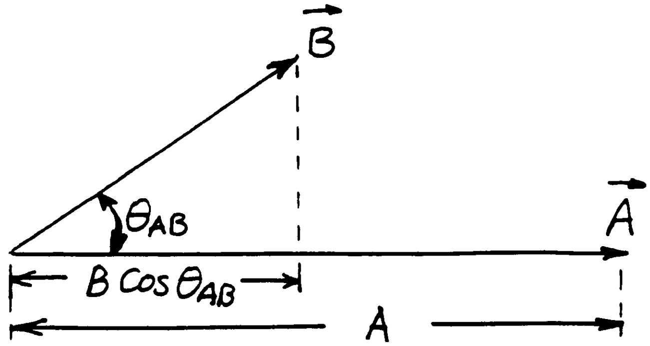
$$(k_1+k_2)\,\vec{\mathrm{A}} = k_1\,\vec{\mathrm{A}} + k_2\,\vec{\mathrm{A}}$$

$$k\,(\vec{\mathrm{A}} + \vec{\mathrm{B}}) = k\,\vec{\mathrm{A}} + k\,\vec{\mathrm{B}}$$

2) Scalar or Dot product

DEF:

$$\vec{\mathrm{A}} \bullet \vec{\mathrm{B}} \triangleq AB\cos\theta_{AB}$$



Geometrical illustration of the dot product of $\vec{A} \bullet \vec{B}$

Note: the dot product of two vectors is a scalar.

It is not difficult to prove from the definition that

$$\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$$

$$\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$$

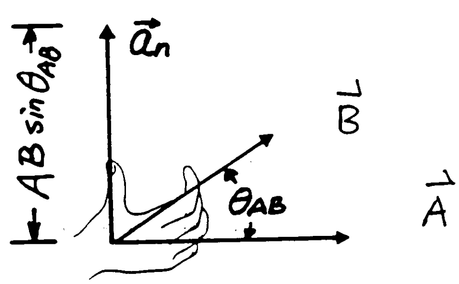
$$\vec{A} \bullet \vec{A} = |\vec{A}|^2 = A^2$$

Ex: Use vectors to prove the law of cosine for a triangle (P15, textbook)

3) Vector or Cross Product

DEF:

$$\vec{A} \times \vec{B} \triangleq \vec{a}_n AB \sin\theta_{AB}$$



Geometrical illustration of the cross product: $\vec{A} \times \vec{B}$

Here $\vec{a}_n$ is a unit vector normal (perpendicular) to the plane containing $\vec{A}$ and $\vec{B}$. The direction of $\vec{a}_n$ follows that of the thumb of *a right hand* when the fingers rotate from $\vec{A}$ to $\vec{B}$ through the angle θ_{AB} (*the right hand rule*).

Note: The vector and cross product of two vectors is a vector.

It is not difficult to prove from the definition that

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

4. Orthogonal Coordinate System

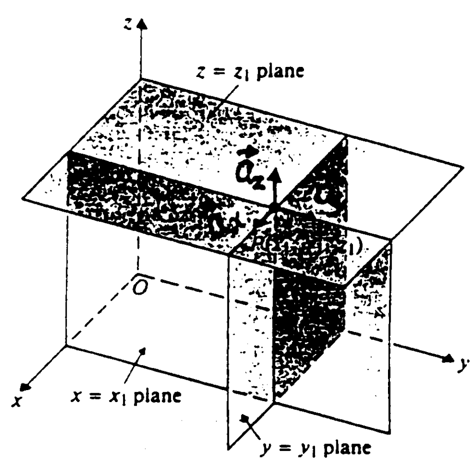
An orthogonal coordinate system can be used to help define a vector in space mathematically. An orthogonal system consist of families of surfaces: u_1=constant, u_2=constant, and u_3=constant, which are mutually perpendicular to each other. Then, any function of position, say f can be expressed as the function of u_1, u_2 and u_3.

e.g. : A function f can be written as $f=f(x, y, z)$ in a Cartesian or rectangular coordinate system. Here, $u_1=x$=constant, $u_2=y$=constant and $u_3=z$=constant are mutually perpendicular to each other (see the following figure).

Note: The u's need not all be the lengths, and some may be angles or others.

Most common and useful orthogonal coordinate systems are
1) Cartesian or rectangular coordinates
2) Cylindrical coordinates
3) Spherical coordinates

1) Cartesian or rectangular coordinates:



Cartesian coordinates (a) three mutually perpendicular planes (b) Intersection of the three planes in (a) specifies the location a point P

$\vec{a_x}$, $\vec{a_y}$ and $\vec{a_z}$ are the normal unit vectors which are perpendicular to the planes x=constant, y=constant and z=constant, respectively. They are mutually perpendicular to each other and are called the *base vectors* (the three coordinate directions).

Obviously,

$\vec{a_x} \times \vec{a_y} = \vec{a_z}$, $\vec{a_y} \times \vec{a_z} = \vec{a_x}$, and $\vec{a_z} \times \vec{a_x} = \vec{a_y}$

$\vec{a_x} \bullet \vec{a_y} = \vec{a_y} \bullet \vec{a_z} = \vec{a_z} \bullet \vec{a_x} = 0$

$\vec{a_x} \bullet \vec{a_x} = \vec{a_y} \bullet \vec{a_y} = \vec{a_z} \bullet \vec{a_z} = 1$

In the Cartesian coordinates, a vector $\vec{\mathrm{A}}$ can be written as

$$\vec{\mathrm{A}} = \vec{a_x} A_x + \vec{a_y} A_y + \vec{a_z} A_z$$

In other words, $\vec{\mathrm{A}}$ can be considered as the superposition of three vectors in the three coordinate directions and with magnitude A_x, A_y and A_z, respectively (this can be seen from the last figure). It is sometimes said that A_x, A_y and A_z are x, y and z components of $\vec{\mathrm{A}}$.

A vector differential length can be written as:

$$d\vec{l} = \vec{a_x}\, dx + \vec{a_y}\, dy + \vec{a_z}\, dz$$

A differential volume can be written as:

$$dv = dx\, dy\, dz$$

Once a vector can be expressed in terms of its x, y and z components, vector calculations can then be carried out in the following way, based on the rules governing the vectorial multiplication.

Suppose k is a scalar, $\vec{\mathrm{A}} = \vec{a_x} A_x + \vec{a_y} A_y + \vec{a_z} A_z$ and $\vec{\mathrm{B}} = \vec{a_x} B_x + \vec{a_y} B_y + \vec{a_z} B_z$. Then,

$$k\vec{\mathrm{A}} = \vec{a_x}\, kA_x + \vec{a_y}\, kA_y + \vec{a_z}\, kA_z$$

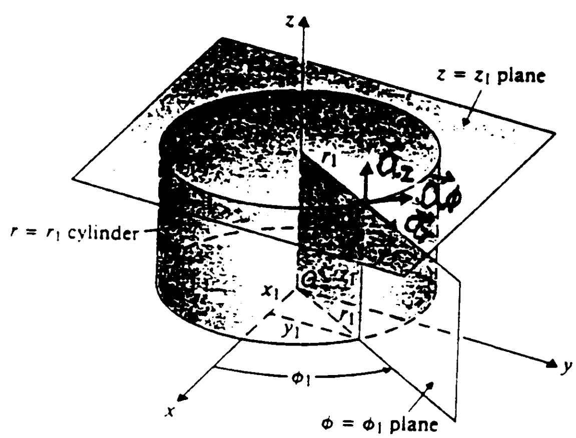
$$\vec{\mathrm{A}} \bullet \vec{\mathrm{B}} = A_x B_x + A_y B_y + A_z B_z$$

$$\vec{A}\times\vec{B} = \det\begin{bmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

$$= \vec{a}_x\,(A_y\,B_z - A_z\,B_y) + \vec{a}_y\,(A_z\,B_x - A_x\,B_z)$$
$$+ \vec{a}_z\,(A_x\,B_y - A_y\,B_x)$$

Ex: Given $\vec{A} = -\,\vec{a}_x + \vec{a}_y\,2 - \vec{a}_z\,2$. Find a) its magnitude b) the unit vector $\vec{a}_A$ in the direction of $\vec{A}$, c) the angle that $\vec{A}$ makes with the z-axis.
(Ans: 3, $-\,\vec{a}_x\,\frac{1}{3} + \vec{a}_y\,\frac{2}{3} - \vec{a}_z\,\frac{2}{3}$, 131.8^o)

2) Cylindrical coordinates:



Cylindrical coordinates (a) three mutually perpendicular planes (b) Intersection of the three planes in (a) specifies the location a point P

$\vec{a_r}$. $\vec{a_\phi}$ and $\vec{a_z}$ are the normal unit vectors which are perpendicular to the planes r=constant, ϕ=constant and z=constant, respectively. They are mutually perpendicular to each other and are called the *base vectors* (the three coordinate directions).

Obviously,

$\vec{a_r} \times \vec{a_\phi} = \vec{a_z}$, $\vec{a_\phi} \times \vec{a_z} = \vec{a_r}$, and $\vec{a_z} \times \vec{a_r} = \vec{a_\phi}$

$\vec{a_r} \bullet \vec{a_\phi} = \vec{a_\phi} \bullet \vec{a_z} = \vec{a_z} \bullet \vec{a_r} = 0$

$\vec{a_r} \bullet \vec{a_r} = \vec{a_\phi} \bullet \vec{a_\phi} = \vec{a_z} \bullet \vec{a_z} = 1$

In the Cylindrical coordinates, a vector $\vec{A}$ can be written as

$$\vec{A} = \vec{a_r} A_r + \vec{a_\phi} A_\phi + \vec{a_z} A_z$$

In other words, $\vec{A}$ can be considered as the superposition of three vectors in the three coordinate directions and with magnitude A_r, A_ϕ and A_z, respectively (this can be seen from the last figure). It is sometimes said that A_r, A_ϕ and A_z are r, ϕ and z components of $\vec{A}$.

A vector differential length can be written as:

$$d\vec{l} = \vec{a_r}\, dr + \vec{a_\phi}\, r\, d\phi + \vec{a_z}\, dz$$

A differential volume can be written as:

$$dv = r\, d\phi\, dr\, dz$$

Once a vector can be expressed in terms of its r, ϕ and z components, vector calculations can then be carried out in the following way, based on the rules governing the vectorial multiplication.

Suppose k is a scalar, $\vec{A} = \vec{a_r} A_r + \vec{a_\phi} A_\phi + \vec{a_z} A_z$ and $\vec{B} = \vec{a_r} B_r + \vec{a_\phi} B_\phi + \vec{a_z} B_z$. Then,

$$k\vec{A} = \vec{a_r}\, kA_r + \vec{a_\phi}\, kA_\phi + \vec{a_z}\, kA_z$$

$$\vec{A} \bullet \vec{B} = A_r B_r + A_\phi B_\phi + A_z B_z$$

$\vec{a}_R \bullet \vec{a}_\theta = \vec{a}_\theta \bullet \vec{a}_\phi = \vec{a}_\phi \bullet \vec{a}_R = 0$

$\vec{a}_R \bullet \vec{a}_R = \vec{a}_\theta \bullet \vec{a}_\theta = \vec{a}_\phi \bullet \vec{a}_\phi = 1$

In the Spherical coordinates, a vector $\vec{A}$ can be written as

$$\vec{A} = \vec{a}_R A_R + \vec{a}_\phi A_\phi + \vec{a}_\theta A_\theta$$

In other words, $\vec{A}$ can be considered as the superposition of three vectors in the three coordinate directions and with magnitude A_R, A_θ and A_ϕ, respectively (this can be seen from the last figure). It is sometimes said that A_R, A_θ and A_ϕ are R, θ and ϕ components of $\vec{A}$.

A vector differential length can be written as:

$$d\vec{l} = \vec{a}_R \, dR + \vec{a}_\theta \, R \, d\theta + \vec{a}_\phi \, R \sin\theta \, d\phi$$

A differential volume can be written as:

$$dv = R^2 \sin\theta \, dR \, d\theta \, d\phi$$

Once a vector can be expressed in terms of its R, θ and ϕ components, vector calculations can then be carried out in the following way, based on the rules governing the vectorial multiplication.

Suppose k is a scalar, $\vec{A} = \vec{a}_R A_R + \vec{a}_\theta A_\theta + \vec{a}_\phi A_\phi$ and $\vec{B} = \vec{a}_R B_R + \vec{a}_\theta B_\theta + \vec{a}_\phi B_\phi$. Then,

$$k\vec{A} = \vec{a}_R \, kA_R + \vec{a}_\theta \, kA_\theta + \vec{a}_\phi \, kA_\phi$$

$$\vec{A} \bullet \vec{B} = A_R B_R + A_\theta B_\theta + A_\phi B_\phi$$

$$\vec{A} \times \vec{B} = \det \begin{bmatrix} \vec{a}_R & \vec{a}_\theta & \vec{a}_\phi \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{bmatrix}$$

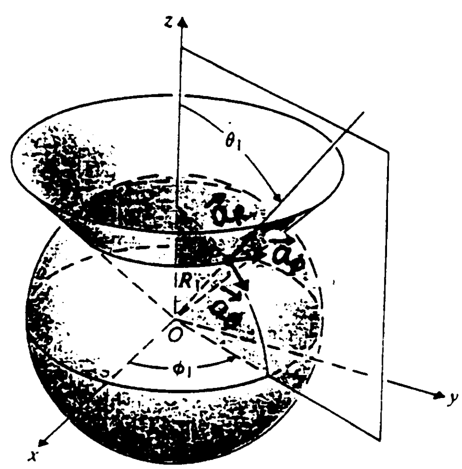
$$\vec{A} \times \vec{B} = \det \begin{bmatrix} \vec{a}_r & \vec{a}_\phi & \vec{a}_z \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{bmatrix}$$

$$= \vec{a}_r \, (A_\phi \, B_z - A_z \, B_\phi) + \vec{a}_\phi \, (A_z \, B_r - A_r \, B_z) + \vec{a}_z \, (A_r \, B_\phi - A_\phi \, B_r)$$

Ex: Assume a vector field expressed in cylindrical coordinates $\vec{A}_P = \vec{a}_r \, 3 \cos\phi - \vec{a}_\phi \, 2r + \vec{a}_z \, z$. Find a) $\vec{A}_P$ at P(4,60°,5), b) $\vec{A}_P$ at P(4,60°,5) in Cartesian coordinates, c) location of P in Cartesian coordinates.

Ans: $\vec{A}_P = \vec{a}_r \, \frac{3}{2} - \vec{a}_\phi \, 8 + \vec{a}_z \, 5$, $\vec{a}_x \, 7.68 - \vec{a}_y \, 2.7 + \vec{a}_z \, 5$, $(2, 2\sqrt{3}, 5)$.

3) Spherical coordinates:



Spherical coordinates (a) three mutually perpendicular planes (b) Intersection of the three planes in (a) specifies the location a point P

$\vec{a}_R$, $\vec{a}_\theta$ and $\vec{a}_\phi$ are the normal unit vectors which are perpendicular to the planes R=constant, θ=constant, and ϕ=constant respectively. They are mutually perpendicular to each other and are called the ***base vectors*** (the three coordinate directions).

Obviously,

$$\vec{a}_R \times \vec{a}_\theta = \vec{a}_\phi \, , \quad \vec{a}_\theta \times \vec{a}_\phi = \vec{a}_R \, , \text{ and } \vec{a}_\phi \times \vec{a}_R = \vec{a}_\theta$$

$$= \vec{a}_R\,(A_\theta\,B_\phi - A_\phi\,B_\theta) + \vec{a}_\theta\,(A_\phi\,B_R - A_R\,B_\phi)$$
$$+ \vec{a}_\phi\,(A_R\,B_\theta - A_\theta\,B_R)$$

Ex: Express $\vec{a}_z$ in the spherical coordinates.

5. Vector functions – vector fields

Sometimes, a vector is a function of position and time. That is, its magnitude and direction vary with position and time. In other words, the three coordinate components are functions of position and time. Mathematically, it can be written as:

$$\vec{A}=\vec{A}(x,y,z,t)=\vec{a}_x\,A_x(x,y,z,t) + \vec{a}_y\,A_y(x,y,z,t) + \vec{a}_z\,A_z(x,y,z,t)$$

or,

$$\vec{A}=\vec{A}(r,\phi,z,t)=\vec{a}_r\,A_r(r,\phi,z,t) + \vec{a}_\phi\,A_\phi(r,\phi,z,t) + \vec{a}_z\,A_z(r,\phi,z,t)$$

or,

$$\vec{A}=\vec{A}(R,\theta,\phi,t)=\vec{a}_R\,A_R(R,\theta,\phi,t) + \vec{a}_\phi\,A_\phi(R,\theta,\phi,t) + \vec{a}_\theta\,A_\theta(R,\theta,\phi,t)$$

6. Gradient of a Scalar Field

Consider a scalar field defined by a scalar function of space coordinates $V(u_1,\,u_2,\,u_3)$, where $u_1,\,u_2,\,u_3$ are the three space (orthogonal) coordinates. Then, we define *the vector that represents both the magnitude and the direction of the* ***maximum*** *space rate of increase of the scalar* $V(u_1,\,u_2,\,u_3)$ as the gradient of scalar $V(u_1,\,u_2,\,u_3)$. We write

$$\textbf{grad}\ V \equiv \nabla V \triangleq \vec{a}_n \frac{dV}{dn}$$

Here $\vec{a}_n$ is the unit vector pointing to the direction of maximum space rate of increase of ∇V. $\frac{dV}{dn}$ is the maximum rate of increase of V in $\vec{a}_n$ direction – (maximum) spatial directional derivative of V.

Note the gradient of a scalar is a vector.

The directional derivative along the other direction dl or $\vec{a}_l$ can be found as:

$$\frac{dV}{dl} = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}\cos\alpha = \frac{dV}{dn}\vec{a}_n \bullet \vec{a}_l = (\nabla V) \bullet \vec{a}_l$$

It represents that the rate of changes of V along $\vec{a}_l$ direction.

It can be shown that

$$\nabla V = \vec{a}_x \frac{\partial V}{\partial x} + \vec{a}_y \frac{\partial V}{\partial y} + \vec{a}_z \frac{\partial V}{\partial z} \qquad \text{(in the Cartesian coordinate)}$$

$$= \vec{a}_r \frac{\partial V}{\partial r} + \vec{a}_\phi \frac{1}{r} \frac{\partial V}{\partial \phi} + \vec{a}_z \frac{\partial V}{\partial z} \qquad \text{(in the cylindrical coordinate)}$$

$$= \vec{a}_R \frac{\partial V}{\partial R} + \vec{a}_\theta \frac{1}{R} \frac{\partial V}{d\theta} + \vec{a}_\phi \frac{1}{R \sin\theta} \frac{\partial V}{\partial \phi} \text{ (in the spherical coordinate)}$$

Ex: $V = V_0\, e^{-x} \sin\frac{\pi y}{4}$. Find $\vec{E} = -\nabla V$. (P45, textbook)

7. **Divergence of a Vector Field**

Consider a vector field defined by a vector function of space coordinates $\vec{A}(u_1, u_2, u_3)$, where u_1, u_2, u_3 are the three space (orthogonal) coordinates. Then, we define *the divergence of* $\vec{A}$ *at a point, abbreviated div*$\vec{A}$, *as the net outward flux of* $\vec{A}$ *per unit volume while the volume about the point tends to zero*:

$$\textbf{\textit{div}}\, \vec{A} \equiv \nabla \bullet \vec{A} \triangleq \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \bullet d\vec{s}}{\Delta v}$$

The numerator is a surface integral. The small circle on the integral sign indicates that the integral is to be carried out over the entire surface S enclosing a volume. In the integrand, the vector differential surface element $d\vec{s} = \vec{a}_n\, ds$ has a magnitude ds and a direction denoted by by the unit normal vector $\vec{a}_n$ pointing *outward* from the enclosing volume. Note the divergence of a vector is a scalar.

It can be shown that

$$\nabla \bullet \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad \text{(in the Cartesian coordinate)}$$

$$= \frac{1}{r} \frac{\partial (r A_r)}{\partial r} + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \qquad \text{(in the cylindrical coordinate)}$$

$$= \frac{1}{R^2}\frac{\partial(R^2 A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial(A_\theta \sin\theta)}{d\theta}$$
$$+ \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi} \qquad \text{(in the spherical coordinate)}$$

Ex: $\vec{B} = \vec{a}_\phi \frac{k}{r}$, show $\nabla \bullet \vec{B} = 0$

8. Curl of a Vector Field

Consider a vector field defined by a vector function of space coordinates $\vec{A}(u_1, u_2, u_3)$, where u_1, u_2, u_3 are the three space (orthogonal) coordinates. Then, we define *the curl of* $\vec{A}$ *at a point, abbreviated curl* $\vec{A}$, *as a vector whose magnitude is the maximum net circulation of* $\vec{A}$ *per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum:*

$$\textbf{\textit{curl}}\,\vec{A} \equiv \nabla\times\vec{A} \triangleq \lim_{\triangle S\to 0}\frac{\left(\vec{a}_n \oint_C \vec{A}\bullet\vec{dl}\right)_{\max}}{\triangle S}$$

The numerator is a line integral. The small circle on the integral sign indicates that the integral is to be carried out over a close path C enclosing the surface $\triangle S$. *In the integrand, the vector differential line element* $\vec{dl} = \vec{a}_l\, dl$ has a magnitude dl and a direction denoted by the unit normal vector $\vec{a}_l$ tangential to C. Note the curl of a vector is a vector.

It can be shown that

$$\nabla\times\vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \qquad \text{(in the Cartesian coordinate)}$$

$$=\frac{1}{r}\begin{vmatrix} \vec{a}_r & r\vec{a}_\phi & \vec{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} \quad \text{(in the cylindrical coordinate)}$$

$$=\frac{1}{R^2\sin\theta}\begin{vmatrix} \vec{a}_R & \vec{a}_\theta R & \vec{a}_\phi R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix} \quad \text{(in the spherical coordinate)}$$

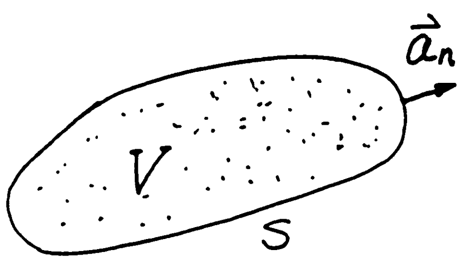
Ex: Show $\nabla\times\vec{A}=0$ if $\vec{A}=\vec{a}_\phi\frac{k}{r}$ (P57, textbook)

9. Two Important Theorems

1) Gaussian Theory – Divergence Theorem

The volume (V) integral of the divergence of a vector field equals to the total outward flux of vector through the surface (S) that bounds the volume; that is,

$$\int_V \nabla\bullet\vec{A}\,dv = \oint_S \vec{A}\bullet d\vec{s}$$



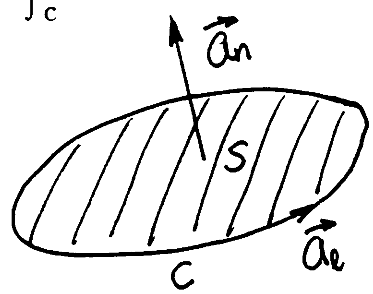
$d\vec{s}=\vec{a}_n ds$

Ex: $\vec{A}=\vec{a}_x x^2+\vec{a}_y xy+\vec{a}_z yz$. Find a) $\int_V \nabla\bullet\vec{A}\,dv$ and b) $\oint_S \vec{A}\bullet d\vec{s}$ where V is a unit cube (P52, textbook).

2) Stokes's Theory

The surface (S) integral of the curl of a vector field over an open surface is equal to the closed line (c) integral of the vector along the contour bounding the surface: that is,

$$\int_s (\nabla \times \vec{A}) \bullet d\vec{s} = \oint_c \vec{A} \bullet d\vec{l}$$



$$d\vec{S} = \vec{a}_n\, dS$$

$$d\vec{\ell} = \vec{a}_\ell\, d\ell$$

Ex: $\vec{F} = \vec{a_x}\, xy \; - \; \vec{a_y}\, 2x$. Find a) $\int_s (\nabla \times \vec{A}) \bullet d\vec{s}$ b) $\oint_c \vec{A} \bullet d\vec{l}$

10. Some Useful Vector Identities:

$$\vec{A} \bullet \vec{B} \times \vec{C} = \vec{B} \bullet \vec{C} \times \vec{A} = \vec{C} \bullet \vec{A} \times \vec{B}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \bullet \vec{C}) - \vec{C}(\vec{A} \bullet \vec{B})$$

$$\nabla(\psi V) = \psi\, \nabla V + V\, \nabla\psi$$

$$\nabla \bullet (\psi\vec{A}) = \psi\, \nabla \bullet \vec{A} + \vec{A} \bullet \nabla\psi$$

$$\nabla \times (\psi\vec{A}) = \psi\, \nabla \times \vec{A} + \nabla\psi \times \vec{A}$$

$$\nabla \bullet (\vec{A} \times \vec{B}) = \vec{B} \bullet (\nabla \times \vec{A}) - \vec{A} \bullet (\nabla \times \vec{B})$$

$$\nabla \bullet \nabla V = \nabla^2 V$$

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \bullet \vec{A}) - \nabla^2 \vec{A}$$

$$\nabla \times \nabla V = 0$$

$$\nabla \bullet (\nabla \times \vec{A}) = 0$$

11. Vector Phasor:

1) Definition of a Complex Number

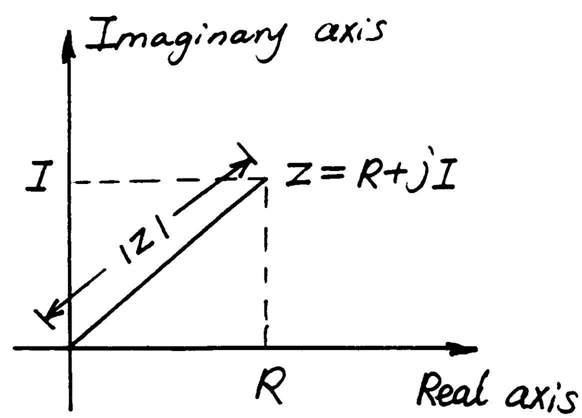
A complex number is defined as a number which has a real part and imaginary part. It can be written as

$$Z=R+jI$$

where $j \triangleq \sqrt{-1}$ or $j^2=j \cdot j= -1$, R is the value of real part and I is the value of imaginary part.

Graphically, it can be represented by a section of straight line which makes an angle φ with the real axis (as shown in the left figure). Note: a complex number is a scalar and does not have directions.

The length of the line is described as the magnitude $|Z|$ of a complex number while φ is the phase of the complex number denoted as $\arg(Z)$.



It is not difficult to see

$$|Z|=\sqrt{R^2+I^2}$$

$$\varphi = \arg(Z)=\tan^{-1}(\frac{I}{R})$$

Since

$$e^{j\varphi} = \cos\varphi+j\sin\varphi \text{ -----Euler's Equation}$$

then

$$Z=|Z|\, e^{j\varphi} \text{ ----- phasor}$$

2) Addition, Subtraction, Multiplication and Division of Complex Numbers

Suppose we have two complex numbers,

$$Z_1=R_1+jI_1=|Z_1|\, e^{j\varphi_1}$$

and

$$Z_2=R_2+jI_2=|Z_2|\, e^{j\varphi_2}$$

Then we have

$$Z_1+Z_2 = Z_2 + Z_1= (R_1+R_2) +j(I_1+I_2)$$

$$Z_1-Z_2 = -Z_2 + Z_1= (R_1-R_2) +j(I_1-I_2)$$

$$Z_1 \cdot Z_2 = Z_2 \cdot Z_1 = R_1 \cdot R_2 - I_1 \cdot I_2 + j(R_1 I_2 + R_2 I_1) = |Z_1||Z_2|\, e^{j(\varphi_1 + \varphi_2)}$$

$$\frac{Z_1}{Z_2} = \frac{R_1 \cdot R_2 + I_1 \cdot I_2 + j(R_2 I_1 - R_1 I_2)}{R_2^{\,2} + I_2^{\,2}} = \frac{|Z_1|}{|Z_2|}\, e^{j(\varphi_1 - \varphi_2)}$$

3) A vector phasor

A vector phasor is defined as a vector whose three components are complex numbers. The rules governing vector calculus are all valid for vector phasors with the addition of complex complex number arithmetic calculations as indicated above.

SUMMARY

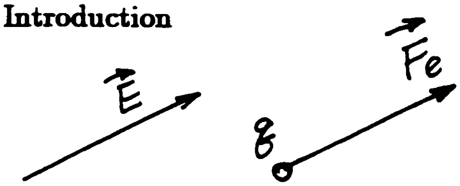
Vector analysis is an essential mathematical tool in electromagnetics. It provides a concise means for representing and expressing the relations of various quantities in the electromagnetic model. In this chapter we

- reviewed the basic rules of vector addition and subtraction, and of products of vectors,
- explained the properties of Cartesian, cylindrical, and spherical coordinate systems,
- introduced the differential del (∇) operator, and defined the gradient of a scalar field, and the divergence and the curl of a vector field,
- presented the divergence theorem that transforms the volume integral of the divergence of a vector field to a closed surface integral of the vector field, and vice versa,
- presented the Stokes's theorem that transforms the surface integral of the curl of a vector field to a closed line integral of the vector field, and vice versa,
- introduced two important null identities in vector field

Chapter 1

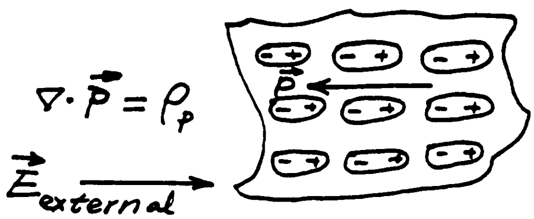
Time-Varying Electromagnetic Fields

1 – 1 Introduction



If a charge q is placed in an electric field, it will experience an electric force $\mathbf{F}_e$ on it.

DEF: Electric field intensity $\mathbf{E}$ (Volt/m) = $\mathbf{F}_e\,/q$ (electric force per unit charge).
OBS: $\mathbf{E}$ is a vector and can be a function of position and time. e.g. $\mathbf{E}(x,\ y,\ z,\ t)$



Polarization $\mathbf{P}(\mathrm{C/m^2})$ in a dielectric medium causes a change of electric field intensity within the medium.

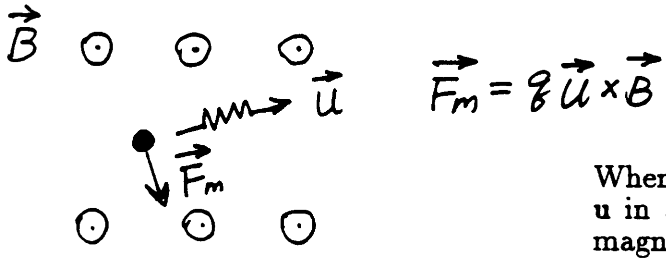
DEF: Electric flux density $\mathbf{D}\ (\mathrm{C/m^2}) \overset{\text{any medium}}{=} \epsilon_0\,\mathbf{E} + \mathbf{P} \overset{\text{linear media}}{=} \epsilon\mathbf{E} = \epsilon_0\,\epsilon_r\,\mathbf{E}$,
$\epsilon(\mathrm{F/m}) = \epsilon_0\,\epsilon_r$ -----permittivity of the medium
$\epsilon_0(\mathrm{F/m})$-----permittivity of the vacuum,
ϵ_r------relative permittivity (or dielectric constant).

OBS: $\mathbf{D}$ is a vector and can be a function of position and time. e.g. $\mathbf{D}(x,\ y,\ z,\ t)$

Dielectric Constants and Dielectric Strengths of Some Common Materials

Material	Dielectric Constant ϵ_r	Dielectric Strength (V/m)
Air (atmospheric pressure)	1.0	3×10^6
Mineral oil	2.3	15×10^6
Paper	2–4	15×10^6
Polystyrene	2.6	20×10^6
Rubber	2.3–4.0	25×10^6
Glass	4–10	30×10^6
Mica	6.0	200×10^6

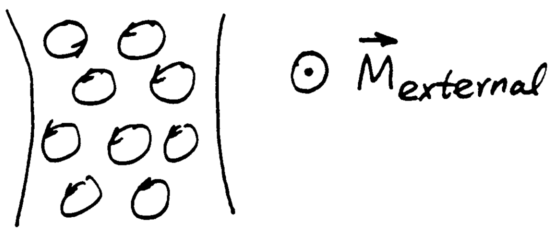
Ex: If a sinusoidal voltage $V(t)=\cos(\omega t+\varphi)$ (V) is applied on a parallel-plate capacitor, what is the electric field intensity between the two plates (a) when the medium is air (b) the medium is polystyrene?



When a charge q is moving at a velocity of $\mathbf{u}$ in a magnetic filed, it will experience a magnetic force $\mathbf{F}_m$.

DEF: Magnetic flux density $\mathbf{B}$: $\mathbf{F}_m = q\ \mathbf{u}\times\mathbf{B}$(T) (a measure of magnetic force per moving charge.

OBS: $\mathbf{B}$ is a vector and can be a function of position and time. e.g. $\mathbf{B}(x, y, z, t)$



Magnetization $\mathbf{M}$(T) in a magnetic medium causes a change of magnetic flux density within the medium.

DEF: Magnetic filed intensity $\mathbf{H}$ (A/m) $\overset{\text{any medium}}{=} \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \overset{\text{linear media}}{=} \frac{\mathbf{B}}{\mu} = \frac{\mathbf{B}}{\mu_0\,\mu_r}$,

μ(H/m)$=\mu_0\mu_r$ -----permeability of the medium,
μ_0(H/m)-----permeability of the vacuum,
μ_r------relative permeability (or dielectric constant).

OBS: $\mathbf{H}$ is a vector and can be a function of position and time. e.g. $\mathbf{H}(x, y, z, t)$.

Summary:

1) $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$ and $\mathbf{B}$ are the four fundamental *vector* field quantities in electromagnetics. All of them can be functions of time.
2) $\epsilon(=\epsilon_0\ \epsilon_r)$ and $\mu(=\mu_0\ \mu_r)$ and σ (conductivity as seen in the next section) form the so-called constitutive parameters, which describe the reaction of a medium to external electromagnetic fields.

3) If both electric and magnetic fields (i.e. electromagnetic fields) exist, a charge will experience a total electromagnetic force given by

$$\mathbf{F}=\mathbf{F}_e + \mathbf{F}_m = q(\mathbf{E} + \mathbf{u}\times\mathbf{B})$$

Ex: A point charge Q with a velocity of $\mathbf{u} = \mathbf{a}_x\, u_o$ enters a region having a uniform magnetic field $\mathbf{B} = \mathbf{a}_x\, B_x + \mathbf{a}_y\, B_y + \mathbf{a}_z\, B_z$. What **E** field should exist in the region so that the charge proceeds without a change of velocity?

Note: Graphically, field lines are used to represent electric and magnetic field spatial distribution. A field line is a continuous line which a field direction is tangential to.

1 – 2 **Maxwell's Equations and boundary conditions**

1. Maxwell's Equations:

Differential form	Integral Form	Significance
$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$	$\oint_C \mathbf{E}\bullet d\mathbf{l} = -\int_S \frac{\partial\mathbf{B}}{\partial t}\bullet d\mathbf{s}$	Faraday's Law
$\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} + \mathbf{J}$	$\oint_C \mathbf{H}\bullet d\mathbf{l} = \int_S (\frac{\partial\mathbf{D}}{\partial t} + \mathbf{J}) \bullet d\mathbf{s}$	Ampere's Circuital Law
$\nabla\bullet\mathbf{D} = \rho$	$\oint_S \mathbf{D}\bullet d\mathbf{s} = Q$	Gauss's Law
$\nabla\bullet\mathbf{B} = 0$	$\oint_S \vec{\mathbf{B}}\bullet d\mathbf{s} = 0$	

OBS: 1) Time-varying magnetic fields will produce or generate electric fields (Faraday's Law),

2) Time-varying electric fields or current flows will produce or generate magnetic fields (Ampere's Law),

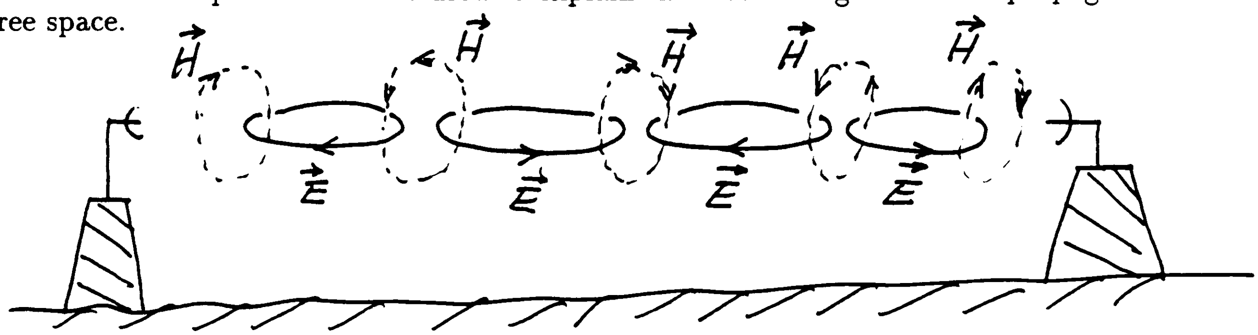
3) There exist isolated electric charges (Gauss's Law),

4) The exist no isolated magnetic charges.

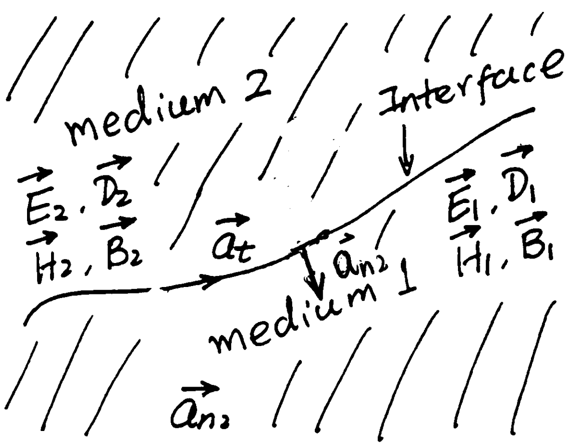
5) $\mathbf{J}(A/m^2)$ is a (surface) current density and therefore, current flowing through surface S is $I=\int_s \mathbf{J}\bullet d\mathbf{s}$. In a conducting medium, $\mathbf{J}=\sigma\mathbf{E}$, σ being the conductivity of the medium.

6) ρ is a volume charge density and therefore, the total charge in volume V is $Q=\int_V \rho\, dv$.

Ex: Maxwell's equations can be used to explain the electromagnetic wave propagation in free space.



2. Electromagnetic Boundary Conditions:



By applying time-varying Maxwell's equations to the interface of two different regions (different = different constitutive parameters), similarly to the processes for the static electric and magnetic field, the following boundary conditions can be obtained. They are exactly the same as those derived in EMT I.

1) The tangential component of an $\mathbf{E}$ field is continuous across an interface

$\mathbf{a}_{n2}\times(\mathbf{E}_1-\mathbf{E}_2)=\mathbf{0}$, or $\mathbf{E}_{1t}=\mathbf{E}_{2t}$ (V/m),

2) The tangential component of an $\mathbf{H}$ field is discontinuous across an interface where a surface current exists. The amount of discontinuity is determined by

$\mathbf{a}_{n2}\times(\mathbf{H}_1-\mathbf{H}_2)=\mathbf{J}_s$ (A/m) ,

3) The normal component of a **D** field is discontinuous across an interface where a surface charge exists. The amount of discontinuity is determined by

$$\mathbf{a}_{n2} \bullet (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \ \ (\mathrm{C/m^2}),$$

4) The normal component of a **B** field is continuous across an interface

$$\mathbf{a}_{n2} \bullet (\mathbf{B}_1 - \mathbf{B}_2) = 0, \quad \text{or } \mathbf{B}_{1n} = \mathbf{B}_{2n} \ (\mathrm{T})$$

Ex: Derive boundary conditions for (a) an interface between two lossless media and (b) an interface between a dielectric and a perfect conductor.

<u>Summary:</u> "Behaviors" of electromagnetic fields are governed by Maxwell's equations and boundary conditions. Therefore, to solve an electromagnetic problem basically means to solve Maxwell's equations + boundary conditions. Mathematically, it is to solve vector differential equations + boundary conditions.

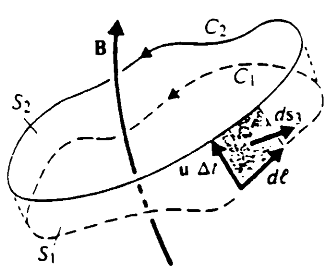
1 – 3 Faraday's Law of Electromagnetic Induction

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

or

$$\oint_C \mathbf{E} \bullet d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \bullet d\mathbf{s}$$

<u>1) A Moving Circuit in a Time-varying Magnetic Field</u>



$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \ \Rightarrow \mathbf{E}' = \mathbf{F}/q = \mathbf{E} + \mathbf{u} \times \mathbf{B}$$

$$\Rightarrow \ \mathbf{E} = \mathbf{E}' - \mathbf{u} \times \mathbf{B}$$

$$\Rightarrow \ \oint_C \mathbf{E}' \bullet d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \bullet d\mathbf{s} + \oint_C (\mathbf{u} \times \mathbf{B}) \bullet d\mathbf{l}$$

It can be shown that (P318, Textbook)

$$\frac{d}{dt}\int_S \mathbf{B} \bullet d\mathbf{s} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \bullet d\mathbf{s} + \oint_C (\mathbf{u} \times \mathbf{B}) \bullet d\mathbf{l}$$

Now we define

$$\mathcal{V} \triangleq \oint_C \mathbf{E} \bullet d\mathbf{l} = \text{emf induced in a circuit with contour C}$$

$$\Phi \triangleq \int_S \mathbf{B} \bullet d\mathbf{s} = \text{magnetic flux crossing surface } S$$

then

$$\mathcal{V} = -\frac{d\Phi}{dt}$$

OBS: The electromotive force induced in a closed circuit is equal to the negative rate of increase of the (external) magnetic flux linking the circuit. In other words, the induced emf tend to cause a current to flow in a direction such that the magnetic flux generated by the current will oppose the change of (external) magnetic flux linking the circuit.

Ex: An electric generator; A $h \times w$ rectangular conducting loop rotating in a changing magnetic field $\mathbf{B}=\mathbf{a_y}\, B_0 \sin \omega t$. The normal of the loop initially makes an angle α with $\mathbf{a_y}$ as shown in the figure below. Find the induced emf in the loop when the loop rotates with an angular velocity ω about the $x-$axis. (P320-321, textbook)

2) A Stationary Circuit in a Time-varying Magnetic Field

OBS: A special case of 1) when $\mathbf{u} = \mathbf{0}$.

Ex: Find the emf in a circular loop of N turns of conducting wire lying in the x-y plane with its center at the origin of a magnetic field, specified by $\mathbf{B}=\mathbf{a_z} \cos(\pi r/2b)$ $\sin(\omega t)$. b is the radius of the loop and ω is the angular frequency. Also mark the polarity of the emf during the first period of change of $\mathbf{B}$. (P310, textbook)

Ex: Explain the working principle of transformer (P311, textbook).

3) A Moving Conductor in a Static Magnetic Field

OBS: A special case of 1) when $\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}$. Note $\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \not\Leftrightarrow \frac{d\Phi}{dt} = 0$.

Ex: A metal bar slides over a pair of conducting rails in a uniform magnetic field $\mathbf{B}=\mathbf{a}_z B_0$ with a constant velocity $\mathbf{u}$, as shown in the figure, (P315, textbook)
a) Determine the open-circuit voltage V_0 that appears across terminals 1 and 2,
b)Assuming that a resistance R is connected between the terminals, find the electric power dissipated in R.

1 – 4 Potential Functions A and *V*

$$\nabla \bullet \mathbf{B} = 0$$

DEF:
$$\Rightarrow \mathbf{B} = \nabla \times \mathbf{A} \quad \text{----- } \mathbf{A} \text{ is called magnetic vector potential}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = -\nabla \times \left(\frac{\partial \mathbf{A}}{\partial t}\right)$$

$$\Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = \mathbf{0}$$

DEF:
$$\Rightarrow \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V \quad \text{-----} V \text{ is called electric scalar potential}$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

For a homogeneous medium, $\mathbf{H}=\mathbf{B}/\mu$ and $\mathbf{D}=\epsilon\mathbf{E}$ with μ and ϵ being constants.

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \Rightarrow \nabla \times \left(\frac{\mathbf{B}}{\mu}\right) = \frac{\partial}{\partial t}(\epsilon \mathbf{E}) + \mathbf{J} \Rightarrow \nabla \times \mathbf{B} = \mu\epsilon \frac{\partial}{\partial t}(\mathbf{E}) + \mu \mathbf{J}$$

$$\Rightarrow \quad \nabla \times \nabla \times \mathbf{A} = \mu\epsilon \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right) + \mu \mathbf{J}$$

$$\Rightarrow \quad \nabla(\nabla \bullet \mathbf{A}) - \nabla^2 \mathbf{A} = -\nabla\left(\mu\epsilon \frac{\partial V}{\partial t}\right) - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} + \mu \mathbf{J}$$

$$\Rightarrow \quad \nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = \nabla\left(\mu\epsilon \frac{\partial V}{\partial t} + \nabla \bullet \mathbf{A}\right) - \mu \mathbf{J}$$

Application of Lorentz condition:

$$\mu\epsilon \frac{\partial V}{\partial t} + \nabla \bullet \mathbf{A} = 0 \quad \text{------ Lorentz condition}$$

leads to:

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} \quad \text{----- Nonhomogeneous wave equation for } \mathbf{A}$$

Note: Nonhomogeneous because the right hand side of the equation $\neq 0$.

OBS: **A** is only generated or excited by current **J**.

$$\nabla \bullet \mathbf{D} = \rho \Rightarrow \epsilon\, \nabla \bullet \mathbf{E} = \rho \;\Rightarrow\; -\nabla \bullet (\nabla V + \frac{\partial \mathbf{A}}{\partial t}) = \frac{\rho}{\epsilon} \;\Rightarrow\; \nabla^2 V + \frac{\partial}{\partial t}(\nabla \bullet \mathbf{A}) = -\frac{\rho}{\epsilon}$$

$$\Rightarrow \qquad \nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon} \quad \text{-------- nonhomogeneous wave equation for } V$$

OBS: V is only generated or excited by charge ρ.

OBS: By solving the above two wave equations separately, we can obtain the vector potential **A** and scalar potential V. Subsequently, we can obtain electromagnetic fields: **B** and **E** ------- this is an alternative solution procedure for EM problems.

Ex: Radiation from a line antenna.

1 – 5 **Solutions of Wave Equations** (in free space)

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} \quad \text{----- Wave equation for vector potential } \mathbf{A}$$

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon} \quad \text{-------- Wave equation for scalar potential } V$$

OBS: The second equation is a normal partial differential equation while the first one is a vector partial differential equations. However, it can be broken down to three scalar partial differential equations as **A** and **J** each consist of three scalar components in the orthogonal coordinates.

In the Cartesian coordinates,

$$\mathbf{A}=\mathbf{a_x}\,A_x + \mathbf{a_y}\,A_y + \mathbf{a_z}\,A_z$$
$$\mathbf{J}=\mathbf{a_x}\,J_x + \mathbf{a_y}\,J_y + \mathbf{a_z}\,J_z$$

$$\nabla^2\mathbf{A} - \mu\epsilon\,\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J}$$

$$\Rightarrow \qquad \nabla^2(\mathbf{a_x}\,A_x + \mathbf{a_y}\,A_y + \mathbf{a_z}\,A_z) - \mu\epsilon\,\frac{\partial^2}{\partial t^2}(\mathbf{a_x}\,A_x + \mathbf{a_y}\,A_y + \mathbf{a_z}\,A_z)$$

$$= -\mu\,(\mathbf{a_x}\,J_x + \mathbf{a_y}\,J_y + \mathbf{a_z}\,J_z)$$

$$\Rightarrow \qquad \mathbf{a_x}\,(\nabla^2 A_x - \mu\epsilon\,\frac{\partial^2 A_x}{\partial t^2}) + \mathbf{a_y}\,(\nabla^2 A_y - \mu\epsilon\,\frac{\partial^2 A_y}{\partial t^2}) + \mathbf{a_z}\,(\nabla^2 A_z - \mu\epsilon\,\frac{\partial^2 A_z}{\partial t^2})$$

$$= \mathbf{a_x}\,(-\mu J_x) + \mathbf{a_y}\,(-\mu J_y) + \mathbf{a_z}\,(-\mu J_z)$$

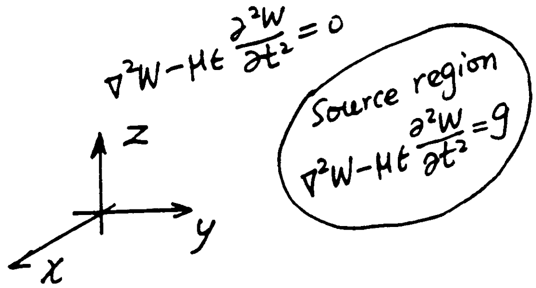
$$\nabla^2 A_x - \mu\epsilon\,\frac{\partial^2 A_x}{\partial t^2} = -\mu J_x$$

$$\Rightarrow \qquad \nabla^2 A_y - \mu\epsilon\,\frac{\partial^2 A_y}{\partial t^2} = -\mu J_y$$

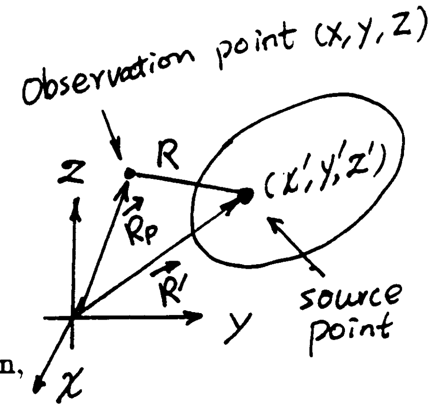
$$\nabla^2 A_z - \mu\epsilon\,\frac{\partial^2 A_z}{\partial t^2} = -\mu J_z$$

OBS: Solution for **A** can be decomposed as solutions of three scalar wave equations of the same type for the three components.

1) General solution of a scalar wave equation (in free space)



$$\nabla^2 W - \mu\epsilon \frac{\partial^2 W}{\partial t^2} = g \leftarrow \text{a force or source term}$$



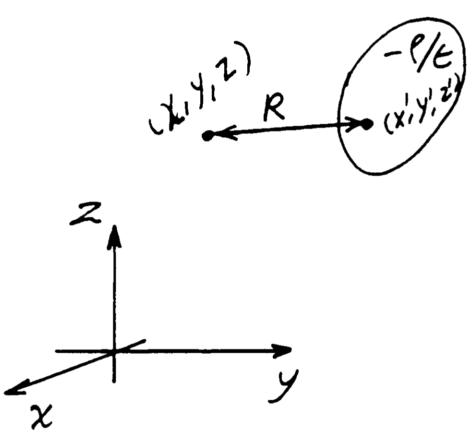
It can be shown (P333 – 334, textbook) that

$$W = \frac{1}{4\pi} \int_{V'} \frac{-g(t - R/c)}{R} \, dv'$$

- Superscript "′" represents integration over the source region,
- $R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$ is the distance between the observation point (x, y, z) and source point (x', y', z') .
- $c = \frac{1}{\sqrt{\mu\epsilon}}$ ---- speed of light in the medium (μ, ϵ). When $\mu = \mu_0$ and $\epsilon = \epsilon_0$, $c = c_0 = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 (\text{m/s})$ ---- speed of light in free space

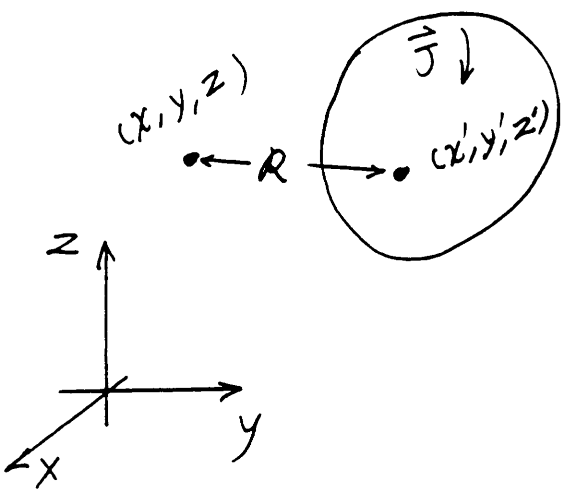
2) Solution for the scalar potential V

$$g = -\rho/\epsilon$$



$$\Rightarrow \quad V = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(t - R/c)}{R} \, dv'$$

3) Solution for the vector potential **A**



For A_x ,

$$A_x = \frac{\mu}{4\pi}\int_{V'} \frac{J_x(t - R/c)}{R}\, dv'$$

For A_y ,

$$A_y = \frac{\mu}{4\pi}\int_{V'} \frac{J_y(t - R/c)}{R}\, dv'$$

For A_z ,

$$A_z = \frac{\mu}{4\pi}\int_{V'} \frac{J_z(t - R/c)}{R}\, dv'$$

$$\mathbf{A} = \mathbf{a_x}\, A_x + \mathbf{a_y}\, A_y + \mathbf{a_z}\, A_z$$

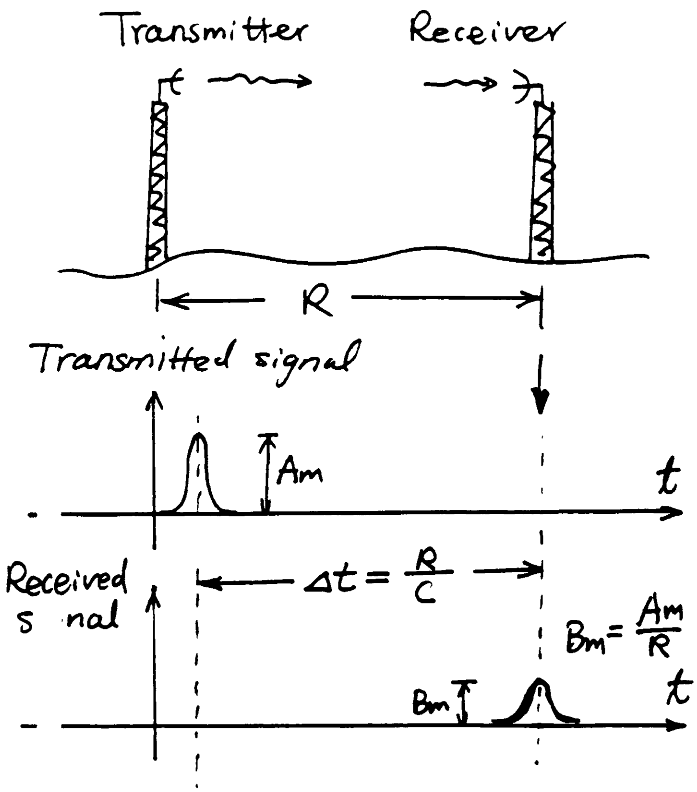
$$\Rightarrow\ \mathbf{A} = \mathbf{a_x}\frac{\mu}{4\pi}\int_{V'} \frac{J_x(t - R/c)}{R}\, dv' + \mathbf{a_y}\frac{\mu}{4\pi}\int_{V'} \frac{J_y(t - R/c)}{R}\, dv' + \mathbf{a_z}\frac{\mu}{4\pi}\int_{V'} \frac{J_z(t - R/c)}{R}\, dv'$$

$$= \frac{\mu}{4\pi}\int_{V'} \frac{\mathbf{a_x} J_x(t - R/c) + \mathbf{a_z}\, J_y(t - R/c) + \mathbf{a_z}\, J_z(t - R/c)}{R}\, dv'$$

$$= \frac{\mu}{4\pi}\int_{V'} \frac{\boldsymbol{J}(t - R/c)}{R}\, dv'$$

OBS: Electromagnetic wave is a retarded wave; it travels at a velocity of c.

Ex: A signal is sent from a source point at t=0. Then it takes 1 second to arrive at a the observation point at 300000 km away from the source.



Chen

For a point source, V' is small. Therefore,

$$A \approx \frac{\mu}{4\pi}\frac{J(t-R/c)}{R}V'$$

$$V \approx \frac{1}{4\pi\epsilon}\frac{\rho(t-R/c)}{R}V'$$

OBS:

– There is a time delay R/c for a EM wave to travel from a source point to an observation point $\Rightarrow$ the velocity=c ,

– A source region can be considered as consisting of many point sources.

Ex: A radar signal sent from earth to the moon is received back on the earth after a delay of 2.562 (s). Determine the distance between the surface of the earth and the moon. (3.843×10^5 km).

1 – 6 Time-harmonic Fields

1. Fourier Transform:

- $F(\omega)=F[f(t)] = \int_{-\infty}^{+\infty} f(t)\, e^{j\omega t}\, dt$ ---- complex number (phasor)
- $f(t) = F^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\, e^{-j\omega t}\, d\omega$
- $f(t) = A\cos(\omega t+\varphi)$, $f(t) = \mathrm{Re}[F(\omega)\, e^{j\omega t}]$ --- time-harmonic instantaneous signal

$\Leftrightarrow$ $A = |F(\omega)|$, $\varphi = \arg[F(\omega)]$

- $F[\frac{d^n}{dt^n} f(t)] = (j\omega)^n F[f(t)] \Rightarrow \frac{d^n}{dt^n} \rightarrow (j\omega)^n$
- $F[f(t-t_o)] = e^{-j\omega t_o} F[f(t)] \Rightarrow t-t_o \rightarrow e^{-j\omega t_o}$

OBS: $F(\omega)$ gives the amplitude and phase of a time-harmonic instantaneous signal $f(t) = |F(\omega)| \cos[\omega t + \arg F(\omega)]$.

Ex: Find the phasor corresponding to 3 $\cos\omega t$ $-4 \sin\omega t$ (Ans.: 5 $e^{j53.13^o}$).

2. Vector Phasors

Suppose $\mathbf{C} = \mathbf{C}(x,y,z,t) = \mathbf{a}_x C_x(x,y,z,t) + \mathbf{a}_y C_y(x,y,z,t) + \mathbf{a}_z C_z(x,y,z,t)$.
Then,

$$\begin{aligned} \mathbf{C}(x,y,z) &= F[\mathbf{C}(x,y,z,t)] \\ &= \mathbf{a}_x F[C_x(x,y,z,t)] + \mathbf{a}_y F[C_y(x,y,z,t)] + \mathbf{a}_z F[C_z(x,y,z,t)] \\ &= \underbrace{\mathbf{a}_x C_x(x,y,z)}_{\text{phsaor}} + \underbrace{\mathbf{a}_y C_y(x,y,z)}_{\text{phasor}} + \underbrace{\mathbf{a}_z C_z(x,y,z)}_{\text{phasor}} \end{aligned}$$

DEF: $\mathbf{C}(x,y,z)$ is called a vector phasor.

3. Time-harmonic Electromagnetics

Denote:

$$\begin{aligned} &\mathbf{E}(x,y,z) = F[\mathbf{E}(x,y,z,t)] \\ &\mathbf{D}(x,y,z) = F[\mathbf{D}(x,y,z,t)] \\ &\mathbf{H}(x,y,z) = F[\mathbf{H}(x,y,z,t)] \\ &\mathbf{B}(x,y,z) = F[\mathbf{B}(x,y,z,t)] \\ &\mathbf{A}(x,y,z) = F[\mathbf{A}(x,y,z,t)] \\ &\mathbf{J}(x,y,z) = F[\mathbf{J}(x,y,z,t)] \\ &V(x,y,z) = F[V(x,y,z,t)] \\ &\rho(x,y,z) = \rho[V(x,y,z,t)] \\ &k = \omega\sqrt{\mu\epsilon} \text{ ----- wave number in medium } (\mu,\epsilon) \end{aligned}$$

Time-harmonic Maxwell's equations:

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \xrightarrow{\frac{\partial}{\partial t}\to j\omega} \nabla\times\mathbf{E} = -j\omega\mathbf{B} \xrightarrow{\text{simple media}} \nabla\times\mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} + \mathbf{J} \xrightarrow{\frac{\partial}{\partial t}\to j\omega} \nabla\times\mathbf{H} = j\omega\mathbf{D} + \mathbf{J} \xrightarrow{\text{simple media}} \nabla\times\mathbf{H} = j\omega\epsilon\mathbf{D} + \mathbf{J}$$

$$\nabla\bullet\mathbf{D} = \rho \rightarrow \nabla\bullet\mathbf{D} = \rho \xrightarrow{\text{simple media}} \nabla\bullet\mathbf{E} = \rho/\epsilon$$

$$\nabla\bullet\mathbf{B} = 0 \rightarrow \nabla\bullet\mathbf{B} = 0 \xrightarrow{\text{simple media}} \nabla\bullet\mathbf{B} = 0$$

Time-harmonic wave equations

$$\nabla^2\mathbf{A} - \mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J} \xrightarrow{\text{simple media}} \nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J}$$

$$\nabla^2 V - \mu\epsilon\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon} \xrightarrow{\text{simple media}} \nabla^2 V + k^2 V = -\frac{\rho}{\epsilon}$$

$$\mathbf{A} = \frac{\mu}{4\pi}\int_{V'}\frac{\boldsymbol{J}(t-R/u)}{R}\,dv' \xrightarrow{\text{simple media}} \mathbf{A} = \frac{\mu}{4\pi}\int_{V'}\frac{\boldsymbol{J}\,e^{-jkR}}{R}\,dv'$$

$$\mathrm{V} = \frac{1}{4\pi\epsilon}\int_{V'}\frac{\rho(t-R/u)}{R}\,dv' \xrightarrow{\text{simple media}} \mathrm{V} = \frac{1}{4\pi\epsilon}\int_{V'}\frac{\rho\,e^{-jkR}}{R}\,dv'$$

Note: All the quantities in time-harmonic equations are phasors ! $u = c = \frac{1}{\sqrt{\mu\epsilon}}$.

1 – 7 Electromagnetic Spectrum

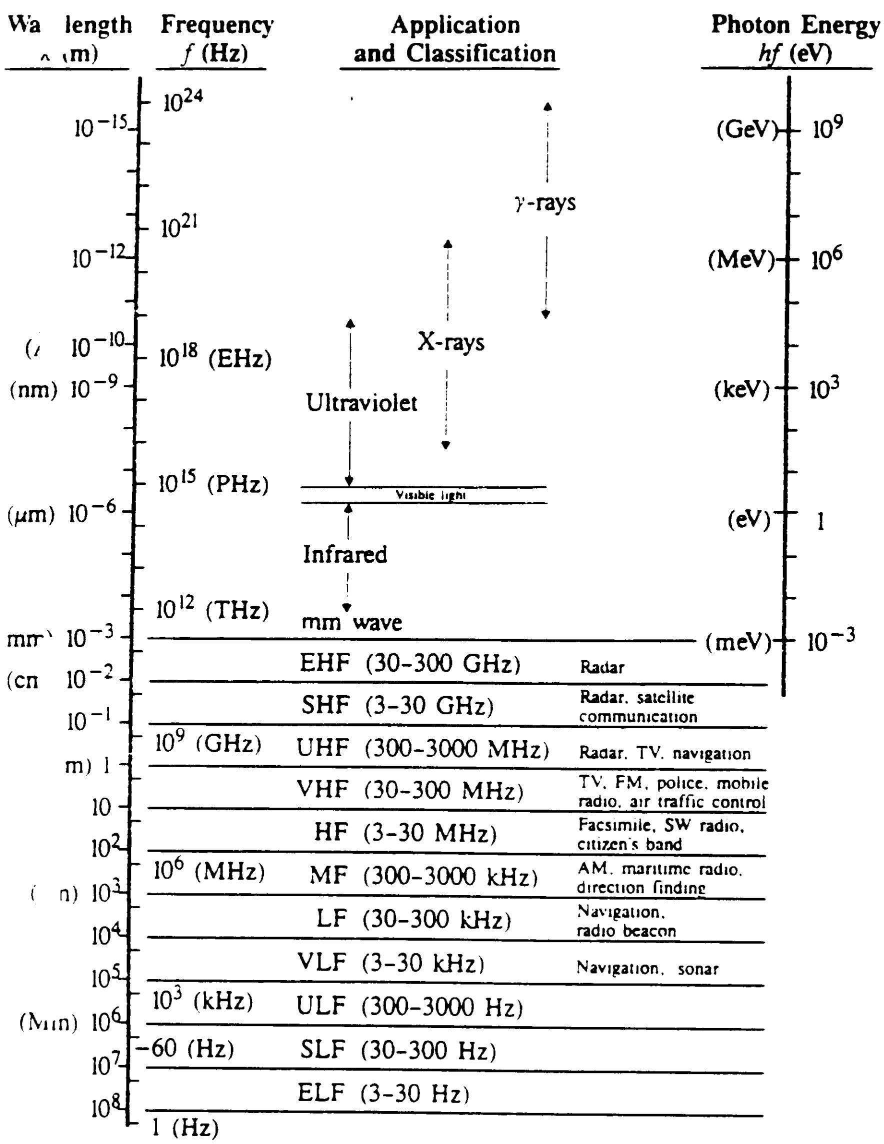


TABLE 7-5
Band Designations for Microwave Frequency Ranges

Old†	New	Frequency Ranges (GHz)
Ka	K	26.5–40
K	K	20–26.5
K	J	18–20
Ku	J	12.4–18
X	J	10–12.4
X	I	8–10
C	H	6–8
C	G	4–6
S	F	3–4
S	E	2–3
L	D	1–2
UHF	C	0.5–1

† Because the old band designations have been in wide use since the early days of radar, they are still in common use because of habit.

SUMMARY

In time-varying situations electric and magnetic fields are coupled and the postulates that we introduced in previous chapters for static fields no longer suffice. In this chapter we

- added a fundamental postulate for electromagnetic induction,
- introduced Faraday's law that relates quantitatively the emf induced in a circuit to the time rate of change of flux linkage,
- explained that the induced emf may be decomposed into two parts: a transformer emf and a motional (flux-cutting) emf,
- discussed the characteristics of ideal transformers,
- obtained a set of four Maxwell's equations (two divergence and two curl equations) that are consistent with the equation of continuity,
- considered the general boundary conditions for field vectors at the interface of contiguous regions of different constitutive parameters,
- expressed electric and magnetic field intensities in terms of a scalar electric potential function V and a vector magnetic potential function **A**,
- derived nonhomogeneous wave equations for V and **A**,
- introduced the concept of retarded potentials,
- converted wave equations into Helmholtz's equations for time-harmonic fields, and
- discussed the electromagnetic spectrum in source-free space.

Chapter 2

Plane Electromagnetic Waves

2 – 1 Introduction

DEF:

– A uniform plane wave: **E** and **H** do not vary in infinite planes perpendicular to the direction of propagation.

– wavefront: surface of constant phase

For a source-free nonconducting simple medium, $\mathbf{J} = 0$ and $\rho = 0$.

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \bullet \mathbf{E} = 0$$

$$\nabla \bullet \mathbf{H} = 0$$

$$\Rightarrow \quad \nabla \times \nabla \times \mathbf{E} = -\mu \, \nabla \times \frac{\partial \mathbf{H}}{\partial t} = -\mu \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) = -\mu \frac{\partial}{\partial t} (\epsilon \frac{\partial \mathbf{E}}{\partial t}) = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

On the other hand,

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \bullet \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$$

$\Rightarrow$ $\qquad \nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$ ----- homogeneous vector wave equation for **E**

or $\qquad \nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad \xRightarrow{\frac{\partial^2}{\partial t^2} \to (j\omega)^2} \quad \nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0$

Similarly,

$\qquad \nabla^2 \mathbf{H} - \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$ ----- homogeneous vector wave equation for **H** or $$\nabla^2 \mathbf{H} - \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \quad \overset{\frac{\partial^2}{\partial t^2} \to (j\omega)^2}{\Rightarrow} \quad \nabla^2 \mathbf{H} + k^2 \mathbf{H} = 0$$

Note: homogeneous because the right hand sides of the equations = 0

2−2 Plane Waves in Lossless Media (σ=0 and ϵ is a real number)

Summary:

- A plane wave consists of a forward (incident) wave and a backward (reflected) wave.
- $u_p = \frac{1}{\sqrt{\mu\epsilon}} = c$ is independent of frequency f
- $\mathbf{H} = \frac{1}{\eta}\, \mathbf{a_n} \times \mathbf{E}$, i.e. $\mathbf{E} \perp \mathbf{H}$ and $\frac{|\mathbf{E}|}{|\mathbf{H}|} = \eta$. $\mathbf{a_n}$ is the unit vector in the propagation direction.
- $\mathbf{E} \times \mathbf{H} = \mathbf{P}$ (Poynting vector) → wave propagation direction ($\mathbf{a}_z$ in this case), i.e. $\mathbf{E} \perp \mathbf{P}$, $\mathbf{H} \perp \mathbf{P}$.

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0 \qquad \left(\nabla^2 = \nabla \bullet \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c}$ ----- wave number in medium (μ,ϵ)

Assume $\mathbf{E} = \mathbf{a}_x\, E_x(z)$ $\qquad (E_y = E_z = 0)$

$$\Rightarrow \quad \frac{d^2 E_x}{\partial z^2} + k^2 E_x = 0$$

$$\Rightarrow \quad E_x = E_x^+(z) + E_x^-(z)$$

$$= E_o^+ e^{-jkz} + E_o^- e^{+jkz}$$

$\uparrow$ forward wave (traveling in $+z$ dir.) $\quad$ $\uparrow$ backward wave (traveling in $-z$ dir.)

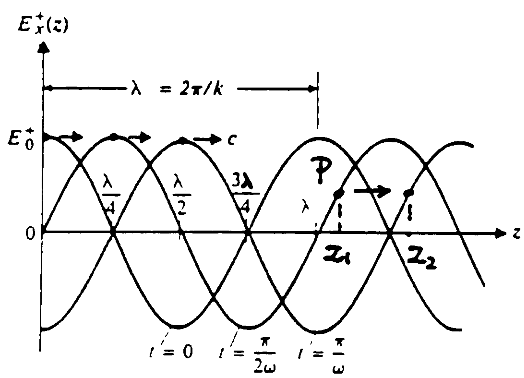
where $E_o^{\pm} = |E_o^{\pm}|\, e^{j\varphi_o^{\pm}}$ ----- a complex constant

Instantaneous forward wave:

$$E_x^+(z,t) = \mathrm{Re}[E_x^+(z)\, e^{j\omega t}]$$

$$= \mathrm{Re}(|E_o^+|\, e^{j\varphi_o^+}\, e^{-jkz}\, e^{j\omega t})$$

$$= |E_o^+| \cos(\omega t - kz + \varphi_o^+)$$



OBS:

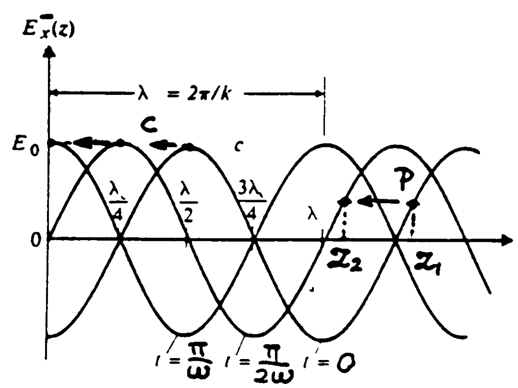
- As t increase, point P moves along the $+z$ direction. The wave travels along $+z$ direction.
- The phase of the wave decreases by kz

Instantaneous backward wave:

$$E_x^-(z,t) = \mathrm{Re}[E_x^-(z)\, e^{j\omega t}]$$

$$= \mathrm{Re}(|E_o^-|\, e^{j\varphi_o^-}\, e^{jkz}\, e^{j\omega t})$$

$$= |E_o^-| \cos(\omega t + kz + \varphi_o^-)$$



OBS:

- As t increase, point P moves along the $-z$ direction. The wave travels along $-z$ direction.
- The phase of the wave increases by kz

Velocity of the Traveling Waves

Forward Waves

at $t=t_1$, point P $\rightarrow \omega t_1 - kz_1 + \varphi_o^+$

at $t=t_2$, point P $\rightarrow \omega t_2 - kz_2 + \varphi_o^+$

$\downarrow$

$\omega t_1 - kz_1 + \varphi_o^+ = \omega t_2 - kz_2 + \varphi_o^+$

$\downarrow$

Backward Waves

at $t=t_1$, point P $\rightarrow \omega t_1 - kz_1 + \varphi_o^-$

at $t=t_2$, point P $\rightarrow \omega t_2 - kz_2 + \varphi_o^-$

$\downarrow$

$\omega t_1 - kz_1 + \varphi_o^- = \omega t_2 - kz_2 + \varphi_o^-$

$\downarrow$

$\rightarrow$ phase velocity $u_p = \frac{z_2 - z_1}{t_2 - t_1} = \pm \frac{\omega}{k}$ $\leftarrow$

DEF: Wavelength λ : the distance between two neighboring points whose phases are only in difference of $\pm 2\pi$ at any *given* time.

$[\omega t - k(z+\lambda) + \varphi_o^+] - (\omega t - kz + \varphi_o^+)$
$= -2\pi$

$[\omega t - k(z+\lambda) + \varphi_o^-] - (\omega t - kz + \varphi_o^-)$
$= -2\pi$

$$\lambda = \frac{u_p}{f} = \frac{2\pi}{k}$$

$$k = \frac{u_p}{f} = \frac{2\pi}{\lambda}$$ ------ wave number

$$f = \frac{\omega}{2\pi}$$ ----- frequency: repetition of changes /second

H of the traveling waves

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\Rightarrow \qquad H_y = \frac{1}{-j\omega\mu}\frac{\partial E_x}{\partial z}, \quad H_x = 0, \quad H_z = 0$$

$$H_y = H_y^+(z) + H_y^-(z)$$

forward wave

$$H_y^+ = \frac{1}{\eta} E_x^+(z)$$

backward wave

$$H_y^- = -\frac{1}{\eta} E_x^-(z)$$

DEF: $\eta = \sqrt{\frac{\mu}{\epsilon}}$ ----- intrinsic impedance of the medium (μ, ϵ). In the free space, $\eta = \sqrt{\mu_0/\epsilon_0} = 120\pi = 377$ (Ω).

Conclusions:

- A plane wave consists of a forward (incident) wave and a backward (reflected) wave.
- $u_p = \frac{1}{\sqrt{\mu\epsilon}} = c$ is independent of frequency f
- $\mathbf{H} = \frac{1}{\eta}\, \mathbf{a_n} \times \mathbf{E}$, i.e. $\mathbf{E} \perp \mathbf{H}$ and $\frac{|\mathbf{E}|}{|\mathbf{H}|} = \eta$. $\mathbf{a_n}$ is the unit vector in the propagation direction.
- $\mathbf{E} \times \mathbf{H} = \mathbf{P}$ (Poynting vector) $\rightarrow$ wave propagation direction ($\mathbf{a}_z$ in this case), i.e. $\mathbf{E} \perp \mathbf{P}$, $\mathbf{H} \perp \mathbf{P}$.
- other components may exist. They can be derived in a entirely similar way or simply by permutation of the subscript indices.

Ex: A uniform plane wave with $\mathbf{E} = \mathbf{a}_x E_x$ propagates in a lossless simple medium ($\epsilon_r=4$, $\mu_r=1$, $\sigma=0$) in the $+z$ direction. Assume that E_x is sinusoidal with a frequency 100MHz and has a maximum value of 10^{-4} (V/m) at $t=0$ and $z=0.125$ m. (P358, textbook)

a) Write the instantaneous expression for $\mathbf{E}$ for any t and z,

b) Write the instantaneous expression for $\mathbf{H}$,

c) Determine the location(s) where E_x is a positive maximum when $t=10^{-8}$ (s)

1. Doppler Effect

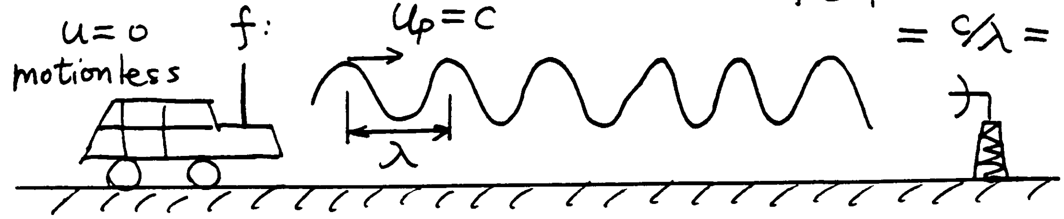
e.g.: changes in the pitch of a fast-moving locomotive whistle as it moves toward you and away from you.

Doppler effect: when there is relative motion between a time-harmonic source and a receiver, the frequency of the wave detected by the receiver tends to be different from that emitted by the source.

Ex: suppose a source is moving with a velocity u toward a receiver along the direct line connecting the source and receiver. If the signal emitted by the source is f, find the frequency detected.

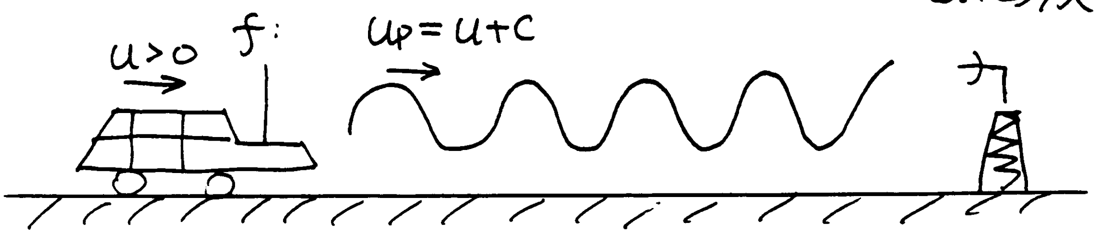
No relative motion:

at receiver: # of ⊕ peaks received/second $= c/\lambda = f$



With relative motion:

of ⊕ peaks received/second $= (u+c)/\lambda = f + \frac{u}{\lambda} = f' > f$



General formulas: (P360, textbook)

If a source of time-harmonic wave of a frequency f moves with a velocity $\boldsymbol{u}$ at an angle θ relative to the direct line to a stationary source receiver, then the frequency f' detected by the receiver is

$$f' = \frac{f}{(1 - \frac{u}{c}\cos\theta)}$$

OBS:

- $\boldsymbol{u}$ should be considered as a relative velocity between the source and the receiver.
- $u\cos\theta > 0$ when the source is moving toward the receiver while $u\cos\theta < 0$ when source is moving away from the receiver.

Ex: Radar speed monitor (Radar gun) (detection of f, f' and $\theta \to u$ of the vehicle) .

2. Transverse Electromagnetic Waves (TEM Waves) – general uniform plane waves (the wave propagation direction is not aligned in $\pm z$ direction)

OBS: for a TEM wave: (1) **E** and **H** are perpendicular to each other (2) both **E** and **H** are perpendicular to the propagation direction.

DEF:

- Wave number vector $\boldsymbol{k} = \mathbf{a}_x k_x + \mathbf{a}_y k_y + \mathbf{a}_z k_z$
- a radius vector $\boldsymbol{R} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z$

Assume:

$$\mathbf{E} = \boldsymbol{E}_o \, e^{-j\boldsymbol{k} \bullet \boldsymbol{R}} \qquad (\, \boldsymbol{E}_o \text{ is a constant vector phasor})$$

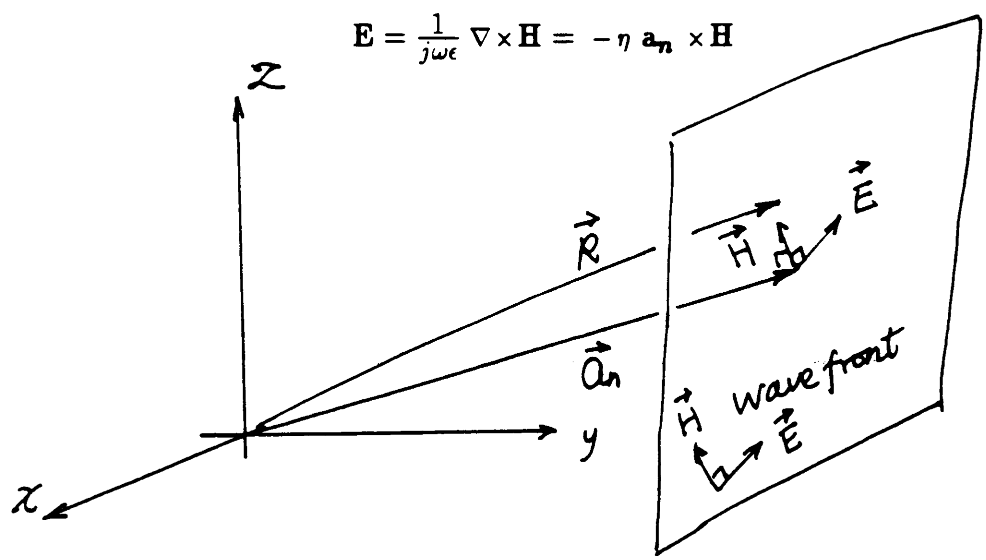
$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0 \Rightarrow |\boldsymbol{k}| = \sqrt{k_x^{\,2} + k_y^{\,2} + k_z^{\,2}} = k = \omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda}$$

$$\Rightarrow \qquad \boldsymbol{k} = \mathbf{a}_n k \qquad \Rightarrow \mathbf{E} = \boldsymbol{E}_o \, e^{-jk\, \mathbf{a}_n \bullet \boldsymbol{R}} \quad (\boldsymbol{E}_o \bullet \mathbf{a}_n = 0)$$

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\Rightarrow \qquad \mathbf{H} = -\frac{1}{j\omega\mu} \nabla \times \mathbf{E} = \frac{1}{\eta} \mathbf{a}_n \times \mathbf{E} = \frac{1}{\eta} (\mathbf{a}_n \times \boldsymbol{E}_o) \, e^{-jk\, \mathbf{a}_n \bullet \boldsymbol{R}}$$

$$\Rightarrow \qquad \mathbf{E} = \frac{1}{j\omega\epsilon} \nabla \times \mathbf{H} = -\eta \, \mathbf{a}_n \times \mathbf{H}$$



OBS:

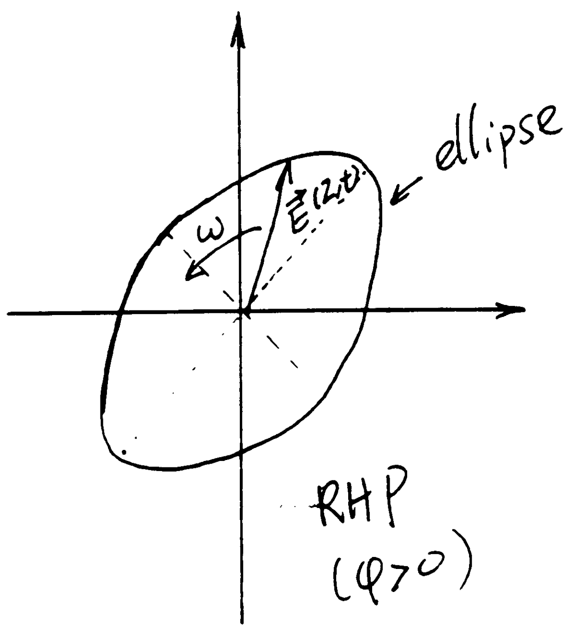
- wavefront (surface of constant phase) is a plane: $\mathbf{a}_n \bullet \boldsymbol{R}$ =constant $\Rightarrow$ $\mathbf{a}_n$ is the direction of propagation.
- *instantaneous* expression $\mathbf{E}(x,y,z,t) = \mathrm{Re}[\mathbf{E}(x,y,z)\ e^{j\omega t}]$
- $\boldsymbol{E}_o$ determine the direction of the vector $\mathbf{E}$

Ex: Write the phasor expression of the z – directed electric field of a uniform plane wave in air having an amplitude E_o , frequency f, and traveling in the $-y$ direction. Find the associated magnetic field.

3. Polarization of Plane Waves

DEF: Polarization: the time-varying behavior of the electric field intensity vector (i.e. locus of the tip of total electric field intensity $\mathbf{E}$)

Ex: Different positioning of radios and wireless telephones results in different degree of reception. It is due to the polarization.

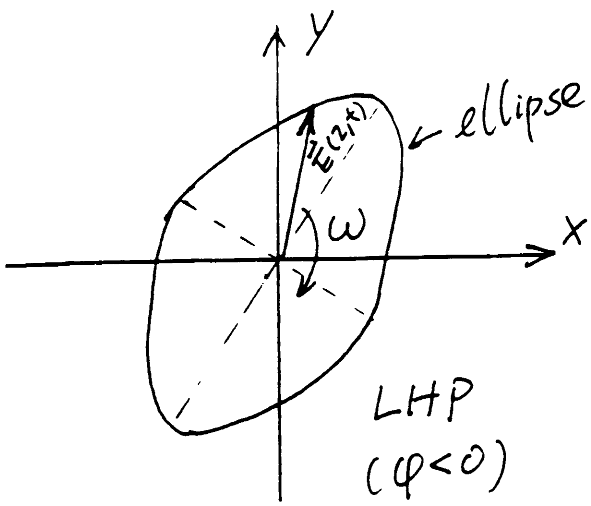


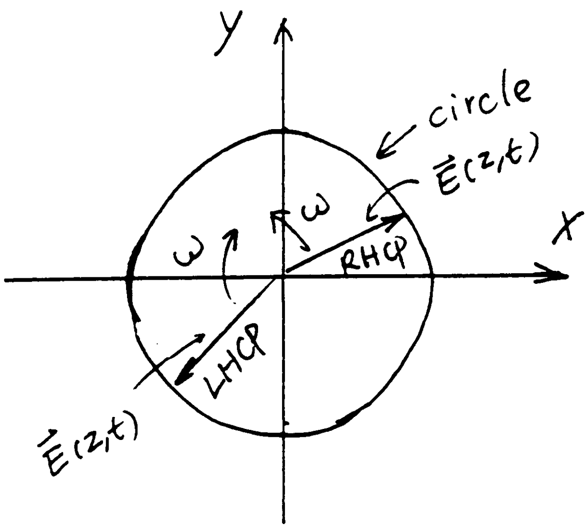
Aligning the propagation direction to $+z$ direction ($\mathbf{a}_n$=$\mathbf{a}_z$) and considering the superposition of two perpendicular plane waves with different amplitude and phases.

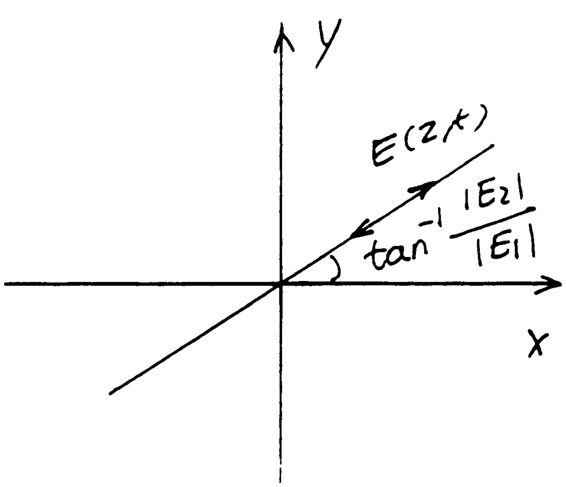
$$\mathbf{E}(z) = \underbrace{\boldsymbol{E}_{10}}\ e^{-jkz} + \underbrace{\boldsymbol{E}_{20}}\ e^{-jkz}$$

$$= \underbrace{\mathbf{a}_x E_1}\ e^{-jkz} + \underbrace{\mathbf{a}_y E_2}\ e^{-jkz}$$

$$= \mathbf{a}_x |E_1| e^{j\varphi_1}\ e^{-jkz} + \mathbf{a}_y |E_2| e^{j\varphi_2}\ e^{-jkz}$$







Instantaneous expression of the electric field then is:

$$\begin{aligned}\mathbf{E}(z,t) &= \mathrm{Re}[\mathbf{E}(z)\ ^{j\omega t}] \\ &= |E_1|\cos(\omega t - kz + \varphi_1) \\ &\qquad + |E_2|\cos(\omega t - kz + \varphi_2)\end{aligned}$$

The locus of the tip of $\mathbf{E}(z,t)$ can be drawn for a given z with varying t.

OBS:

- The difference between φ_1 and φ_2 can be written as $\varphi_1 - \varphi_2 = \mathrm{n}\cdot 360^o + \varphi$, $|\varphi| \leq 180^o$, and n=0, ±1, ±2 ...
- The locus is generally an ellipse ⇒ elliptically polarized wave. When $\varphi > 0^o$, **E** rotates in such a way that if your right hand fingers follow the direction of rotation, your thump points to the propagation direction (+z in this case) ⇒ right-hand elliptically polarized waves. When $\varphi < 0^o$, **E** rotates in such a way that if your left hand fingers follow the direction of rotation, your thump points to the propagation direction (+z in this case) ⇒ left-hand elliptically polarized waves.
- When $|E_1| = |E_2|$ and $|\varphi| = 90^o$, an ellipse becomes a circle ⇒ (right-hand or left-hand) circularly polarized waves.
- When $|\varphi| = 0^o$ or 180^o, the ellipse collapse to a straight line with angle of $\tan^{-1}\frac{E_2}{E_1}$ to the $+x$ axis. The **E** direction is fixed on a line ⇒ linearly polarized waves.

Ex: Describe the polarization effects on the reception of radio and TV signals.

Ex: Describe the polarization of a wave whose $\mathbf{E}(x,t) = (\mathbf{a}_y E_{10} - \mathbf{a}_z E_{20}) \sin(\omega t - kx)$.

2 – 3 Plane Waves in Lossy Media ($\sigma \neq 0$, or permittivity has an nonzero imaginary part)

$$\overbrace{\begin{array}{ccc} \text{lossy dielectric} & & \text{conducting media} \\ (\epsilon = \epsilon' - j\epsilon'',\ \sigma = 0) & (\epsilon = \epsilon' - j\epsilon'',\ \sigma) & (\epsilon'' = 0,\ \sigma) \\ \uparrow & & \uparrow \\ \text{dielectric loss} & & \text{conducting loss} \end{array}}^{\text{lossy media}}$$

$$\begin{aligned} \nabla \times \mathbf{H} &= j\omega\epsilon\mathbf{E} + \mathbf{J} = j\omega(\epsilon' - j\epsilon'')\mathbf{E} + \sigma\mathbf{E} \\ &= j\omega \underbrace{\left[\epsilon' - j(\epsilon'' + \tfrac{\sigma}{\omega})\right]}_{\substack{\epsilon_c \\ \downarrow \\ \text{complex permittivity}}} \mathbf{E} \\ &= j\omega \quad \epsilon_c \quad \mathbf{E} \end{aligned}$$

DEF: loss tangent $\tan\delta_c = \dfrac{\epsilon'' + \frac{\sigma}{\omega}}{\epsilon'} \rightarrow$ loss angle $\delta_c = \tan^{-1} \dfrac{\epsilon'' + \frac{\sigma}{\omega}}{\epsilon'}$.

OBS: Equations obtained for lossless media are still valid with the replacement of ϵ by ϵ_c .

Ex:

Lossless	$\epsilon \to \epsilon_c$	Lossy
$k = \omega\sqrt{\mu\epsilon}$ ----real number		$k_c = j\omega\sqrt{\mu\epsilon_c}$ ---- complex number
$jk = j\omega\sqrt{\mu\epsilon}$ ----complex number ($\beta = k$)		$\underbrace{\gamma}_{\text{propagation const.}} = jk_c = \underbrace{\alpha(\omega)}_{\text{attenuation const.}} + j\underbrace{\beta(\omega)}_{\text{phase constant}}$---complex number
variation in z: e^{-jkz}		$e^{-\gamma z} = \underbrace{e^{-\alpha z}}_{\text{amplitude} \downarrow} \; \underbrace{e^{-j\beta z}}_{\text{phase} \downarrow}$
amplitude=const., phase $\downarrow$ as $z \uparrow$		amplitude $\downarrow$ phase $\downarrow$ (as $z \uparrow$)
intrinsic impedance η ---- real number		η_c ---- complex number
phase velocity $u_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}$ ---- constant		$u_p = \frac{\omega}{\beta(\omega)}$ ---- function of ω (or f)

Note: $\alpha = \alpha(\text{NP/m})$, $\beta = \beta(\text{rad/m})$.

DEF: Skin depth: the distance δ through which the amplitude of a traveling wave decreases by a factor e^{-1} or 0.368.

$$e^{-\alpha(z+\delta)} = e^{-\alpha z}\, e^{-1} \rightarrow \delta = \frac{1}{\alpha}$$

Ex: Assume that in a lossy medium, at a given point z, $|\mathbf{E}| = 1.0$ (mv/m) and at point z+50(m), $|\mathbf{E}| = 0.8$ (mv/m). Find (1) the total attenuation between the two points both in Nepers and Decibel, (2) α in NP/m and dB/m and (3) the skin depth.
Ans.: (1)0.8(NP), 1.94dB, (2) 0.00446(NP/m), 0.0388(dB), (3) 224.1(m)

Ex: (1) formulas for low-loss dielectric (σ=0 and $\epsilon'' << \epsilon'$ or $\sigma/\omega\epsilon' << 1$), (2) formulas for good conductors ($\sigma/\omega\epsilon' >> 1$). (P368 – 372)
OBS: In a general case, α and β for a lossy medium are frequency f dependent.

Application of the concept of propagation constant:

Ex: when a spaceshuttle reenters the earth's atmosphere, its speed and temperature ionize the surrounding atoms and molecules and create a plasma (ionized gases with equal electron and ion identities). It has been found that the permittivity of the plasma is

$$\epsilon_p = \epsilon_o\left(1 - \frac{f_p^2}{f^2}\right) \text{ (F/m)}$$

$f_p = 127\text{(MHz)}$ in this case. Determine at what frequency range that the radio communication can be established between the shuttle and the mission controllers on earth.

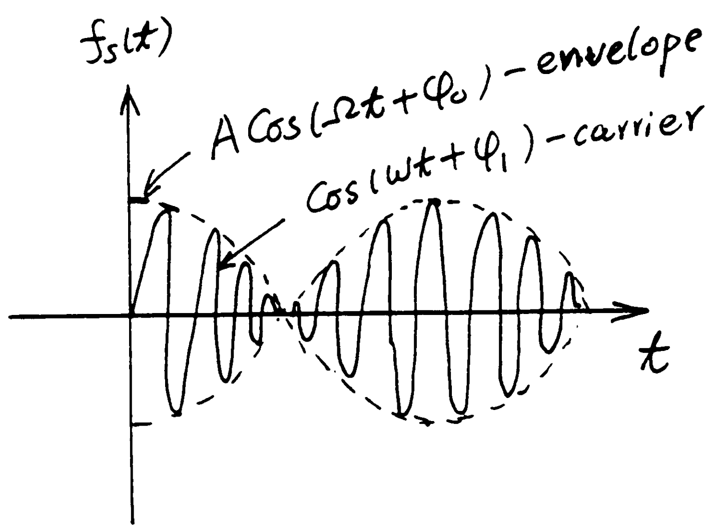
2 – 4 Group Velocity

1. Dispersion:

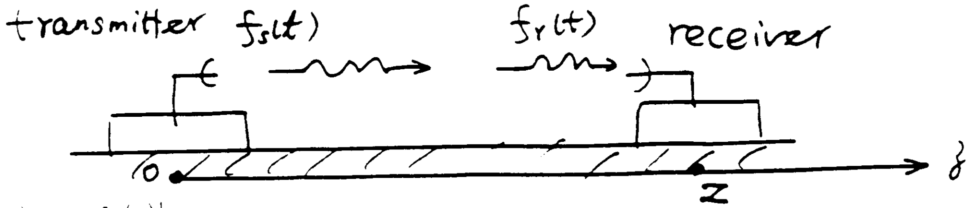
Suppose an AM signal signal on transmitting

$$\begin{aligned} f_s(t) &= \overbrace{A\cos(\Omega t+\varphi_o)}\ \cos(\omega t+\varphi_1) \\ &= \frac{A}{2}\cos(\omega_1 t + \varphi_1 - \varphi_o) \quad \rightarrow \quad f_1(t) \\ &+ \frac{A}{2}\cos(\omega_2 t + \varphi_1 + \varphi_o) \quad \rightarrow \quad f_2(t) \\ &= f_1(t) + f_2(t) \end{aligned}$$

with $\omega_1 = \omega - \Omega$, $\omega_2 = \omega + \Omega$



⇒ A simple AM signal consists of two waves with different frequencies, ω_1 and ω_2.
At the receiver end,



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$$f_r(t) = f_1(t)\big|_{\text{after traveling through the distance } z}$$

$$- f_2(t)\big|_{\text{after traveling through the distance } z}$$

$$= \frac{A}{2}\cos(\omega_1 t + \varphi_1 - \varphi_o - \beta_1 z) + \frac{A}{2}\cos(\omega_2 t + \varphi_1 + \varphi_o - \beta_2 z)$$

$$= A\cos(\Omega t + \varphi_o - \frac{\beta_2 - \beta_1}{2} z)\ \cos(\omega t + \varphi_1 - \frac{\beta_2 + \beta_1}{2} z)$$

$$= \underbrace{A\cos(\Omega t + \varphi_o - \frac{\Delta\beta}{2} z)}\ \cos(\omega t + \varphi_1 - \beta_0 z)$$

where $\beta_1 = \beta(\omega_1) = \dfrac{\omega_1}{u_p(\omega_1)}$, $\beta_2 = \beta(\omega_2) = \dfrac{\omega_2}{u_p(\omega_2)}$, $\Delta\beta = \beta_2 - \beta_1$, $\beta_0 = \dfrac{\beta_2 + \beta_1}{2}$.

$\Rightarrow$ $f_r(t)$ has the same envelope as $f_s(t)$ only when $u_p = \frac{\omega}{\beta}$ = const. (independent of ω or f) $\Leftrightarrow$ $\beta \propto f$ (or ω). In this case:

$$f_r(t) = A\cos[\Omega(t - t') + \varphi_o]\ \cos[\omega(t - t') + \varphi_1]$$

where $t' = \frac{z}{u_p}$ ----- time for the EM signal to travel from the transmitter to the receiver.

<u>OBS:</u> The shape of the received signal is exactly the same as that of the transmitted signal except for the t' delay.

<u>OBS:</u> If the $u_p \neq$ const., the two wave components, namely, $f_1(t)$ and $f_2(t)$ will travel at different velocities $u_p(\omega_1)$ and $u_p(\omega_2)$ and reach the receiver at different time. Consequently, the combination of the two waves will not have the same shape as the one on transmission. Rather, the signal will disperse in space and time.

DEF:

- Dispersion: the phenomenon of signal distortion caused by a dependence of the phase velocity on frequency.
- A dispersive medium: a medium in which dispersion of a signal occurs.

Ex: A lossy dielectric is a dispersive medium.

2. Group Velocity

DEF: Group velocity u_g : velocity of the envelope.

envelope at the receiver: $A \cos(\Omega t + \varphi_o - \frac{\Delta\beta}{2} z)$

$$u_g = \frac{\Omega}{\frac{\Delta\beta}{2}} = \frac{\omega_2 - \omega_1}{\Delta\beta} = \frac{\Delta\omega}{\Delta\beta} \rightarrow \frac{d\omega}{d\beta}$$

$$\Rightarrow \qquad u_g = \frac{d\omega}{d\beta} = \frac{1}{\frac{d\beta}{d\omega}} = \frac{u_p}{1 - \frac{\omega}{u_p}\frac{du_p}{d\omega}}$$

No dispersion: $\frac{du_p}{d\omega} = 0 \quad \rightarrow u_g = u_p$

Normal dispersion: $\frac{du_p}{d\omega} < 0 \quad \rightarrow u_g < u_p$

Anormalous dispersion: $\frac{du_p}{d\omega} > 0 \quad \rightarrow u_g > u_p$

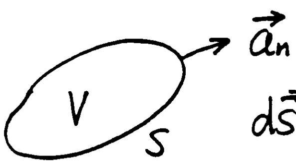
OBS:

- u_g represent the propagation speed of information (or energy) and it has to be less than the speed of light while u_p does not have to be.

• Although the above results and conclusions are derived for a simple AM signal, they apply to any other information bearing signals. The reason is that those signals have small spread of frequencies (sideband) around a high carrier frequency. In other words, those signals comprise of "group" of frequencies and form wave-packet envelopes, very similar to the simple AM signal, except the shape of envelope is not "cos( )" form.

2 – 5 Flow of Electromagnetic Power and the Poynting Vector

1. <u>Power flow in time-domain:</u>



$d\vec{S} = \vec{a}_n \, dS$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

$$\nabla \bullet (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \bullet (\nabla \times \mathbf{E}) - \mathbf{E} \bullet (\nabla \times \mathbf{H}) = -\mathbf{H} \bullet \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \bullet \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \bullet \mathbf{J}$$

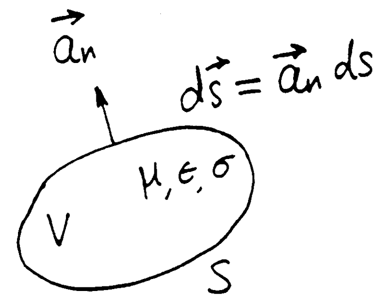
<u>DEF:</u> The Poynting vector $\mathcal{P} = \mathbf{E} \times \mathbf{H}$ ------ $(\mathrm{W/m^2})$

$$\Rightarrow -\oint_S \mathcal{P} \bullet d\mathbf{s} = -\int_V \nabla \bullet \mathcal{P} \, dv = -\int_V \nabla \bullet (\mathbf{E} \times \mathbf{H}) \, dv$$

$$= \int_V \mathbf{H} \bullet \frac{\partial \mathbf{B}}{\partial t} \, dv + \int_V \mathbf{E} \bullet \frac{\partial \mathbf{D}}{\partial t} \, dv + \int_V \mathbf{E} \bullet \mathbf{J} \, dv \qquad \text{(Watt)}$$

$$\underbrace{\left.\begin{matrix}\text{Net total power} \\ \text{flowing into a} \\ \text{closed surface S}\end{matrix}\right\} = \left\{\begin{matrix}\text{increase of stored} \\ \text{magnetic energy in} \\ \text{the volume V}\end{matrix}\right. + \left\{\begin{matrix}\text{increase of stored} \\ \text{electric energy in} \\ \text{the volume V}\end{matrix}\right. + \left\{\begin{matrix}\text{Ohmic power} \\ \text{dissipation} \\ \text{(heat)}\end{matrix}\right.}_{\text{Poynting theorem}}$$

In a simple medium, ϵ, μ and σ are independent of time, t:



$$\mathbf{H} \bullet \frac{\partial \mathbf{B}}{\partial t} = \mathbf{H} \bullet \frac{\partial(\mu \mathbf{B})}{\partial t} = \frac{1}{2}\,\frac{\partial(\mu \mathbf{H} \bullet \mathbf{H})}{\partial t} = \frac{\partial(\frac{1}{2}\,\mu\,|\mathbf{H}|^2)}{\partial t}$$

$$\mathbf{E} \bullet \frac{\partial \mathbf{D}}{\partial t} = \mathbf{E} \bullet \frac{\partial(\epsilon \mathbf{D})}{\partial t} = \frac{1}{2}\,\frac{\partial(\epsilon \mathbf{E} \bullet \mathbf{E})}{\partial t} = \frac{\partial(\frac{1}{2}\,\epsilon\,|\mathbf{E}|^2)}{\partial t}$$

$$\mathbf{E} \bullet \mathbf{J} = \mathbf{E} \bullet (\sigma \mathbf{E}) = \sigma\,|\mathbf{E}|^2$$

$$\Rightarrow -\oint_S \mathcal{P} \bullet d\mathbf{s} = \frac{\partial}{\partial t}\int_V \underbrace{(\tfrac{1}{2}\,\mu\,|\mathbf{H}|^2)}_{\text{magnetic energy density } w_e}\,dv + \frac{\partial}{\partial t}\int_V \underbrace{(\tfrac{1}{2}\,\epsilon\,|\mathbf{E}|^2)}_{\text{electric energy density} w_m}\,dv + \int_V \underbrace{\sigma|\mathbf{E}|^2}_{\text{Ohmic power density } p_\sigma}\,dv$$

$$= \frac{\partial}{\partial t}\int_V w_e\,dv + \frac{\partial}{\partial t}\int_V w_m\,dv + \int_V p_\sigma\,dv$$

Ex: What does

$$\oint_S \mathcal{P}_{\mathbf{av}} \bullet d\mathbf{s} = -\frac{\partial}{\partial t}\int_V (\tfrac{1}{2}\,\mu\,|\mathbf{H}|^2)\,dv - \frac{\partial}{\partial t}\int_V (\tfrac{1}{2}\,\epsilon\,|\mathbf{E}|^2)\,dv - \int_V \sigma|\mathbf{E}|^2\,dv \quad \text{tell us ?}$$

2. Instantaneous and average power densities:

Phasor		Instantaneous
$\mathbf{E}(z)$	$\rightarrow$	$\mathbf{E}(t,z)=\mathrm{Re}[\mathbf{E}(z)\,e^{j\omega t}]$
$\mathbf{H}(z)$	$\rightarrow$	$\mathbf{H}(t,z)=\mathrm{Re}[\mathbf{H}(z)\,e^{j\omega t}]$
		$\downarrow$
$\mathcal{P}_{\mathbf{av}} = \frac{1}{T}\int_0^T \mathcal{P}(z,t)\,dt$	$\leftarrow$	$\underbrace{\mathcal{P}(z,t)=\mathbf{E}(z,t)\times\mathbf{H}(z,t)}$
$= \frac{1}{2}\,\mathrm{Re}[\mathbf{E}(z)\times\mathbf{H}^*(z)]$ (W/m^2)		Instantaneous Poynting vector

$$\underbrace{\mathcal{P}_{\mathbf{av}} = \frac{1}{2}\,\mathrm{Re}[\mathbf{E}(z)\times\mathbf{H}^*(z)]\ \ (\mathrm{W/m^2})}_{\text{average power density vector}}$$

$\Rightarrow$ total average power $P_{\mathrm{av}} = \oint_S \mathcal{P}_{\mathbf{av}} \bullet d\mathbf{s}$ ------ total average power flowing out of S

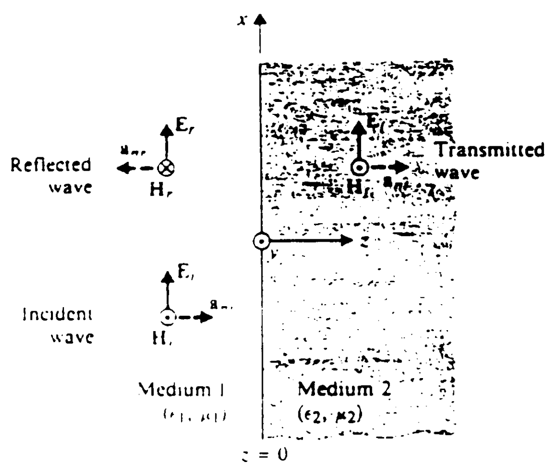
Ex: The far field of a short antenna Idl located at the origin of a spherical coordinate system in free space is

$$\mathbf{E}(R,\theta) = \mathbf{a}_\theta \left(j\frac{60\pi\, I\, dl}{\lambda R}\sin\theta\right) e^{-j\beta R} \qquad (\mathrm{V/m})$$

$$\mathbf{H}(R,\theta) = \mathbf{a}_\phi \left(j\frac{I\, dl}{2\lambda R}\sin\theta\right) e^{-j\beta R} \qquad (\mathrm{A/m})$$

Find the instantaneous Poynting vector and average power density vector (P387, textbook).

2 – 6 Normal Incidence of Plane Waves at Plane Boundaries



Assume that the known incident plane wave is: ($\mathbf{a}_{ni} = \mathbf{a}_z$)

$$\mathbf{E}_i(z) = \mathbf{a}_x E_{io}\, e^{-j\beta_1 z}$$

$$\rightarrow \qquad \mathbf{H}_i(z) = \frac{1}{\eta_1}\mathbf{a}_z \times \mathbf{E}_i(z) = \mathbf{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1 z}$$

$$\beta_1 = k_1 = \omega\sqrt{\mu_1\epsilon_1}, \quad \eta_1 = \sqrt{\mu_1/\epsilon_1}$$

The reflected plane wave can then be expressed as: ($\mathbf{a}_{nr} = -\ \mathbf{a}_z$)

$$\mathbf{E}_r(z) = \mathbf{a}_x\, E_{ro}\, e^{-j\beta_1 z}$$

$$\rightarrow \qquad \mathbf{H}_r(z) = \frac{1}{\eta_1}\,(-\mathbf{a}_z)\times\mathbf{E}_r(z) = -\ \mathbf{a}_y\,\frac{E_{ro}}{\eta_1}\,e^{-j\beta_1 z}$$

E_{ro} is to be determined

The transmitted plane wave can then be expressed as: ($\mathbf{a}_{nt} = \mathbf{a}_z$)

$$\mathbf{E}_t(z) = \mathbf{a}_x\, E_{to}\, e^{-j\beta_2 z}$$

$$\rightarrow \qquad \mathbf{H}_t(z) = \frac{1}{\eta_2}\,\mathbf{a}_z\times\mathbf{E}_t(z) = \mathbf{a}_y\,\frac{E_{to}}{\eta_2}\,e^{-j\beta_2 z}$$

E_{to} is to be determined

$$\beta_2 = k_2 = \omega\,\sqrt{\mu_2\epsilon_2},\quad \eta_1 = \sqrt{\mu_2/\epsilon_2}$$

Enforce the boundary conditions at z=0,

$$\mathbf{a}_z\times(\mathbf{E}_1-\mathbf{E}_2)=0 \Rightarrow \mathbf{E}_i(0) + \mathbf{E}_r(0) = \mathbf{E}_t(0) \Rightarrow E_{io} + E_{ro} = E_{to}$$
$$\mathbf{a}_z\times(\mathbf{H}_1-\mathbf{H}_2)=0 \Rightarrow \mathbf{H}_i(0) + \mathbf{H}_r(0) = \mathbf{H}_t(0) \Rightarrow \frac{1}{\eta_1}\,(E_{io} - E_{ro}) = \frac{E_{to}}{\eta_2}$$

DEF:

– reflection coefficient $\Gamma = \dfrac{\text{reflected wave}}{\text{incident wave}} = \dfrac{E_{ro}}{E_{io}}$ (at z=0)

– transmission coefficient $\tau = \dfrac{\text{transmitted wave}}{\text{incident wave}} = \dfrac{E_{to}}{E_{io}}$ (at z=0)

⇒

$$\Gamma = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \text{(only for normal incidence)}$$

$$\tau = \frac{E_{to}}{E_{io}} = \frac{2\eta_2}{\eta_2 + \eta_1} \qquad \text{(only for normal incidence)}$$

$$1 + \Gamma = \tau \qquad \text{(only for normal incidence)}$$

→ The total field in medium 1:

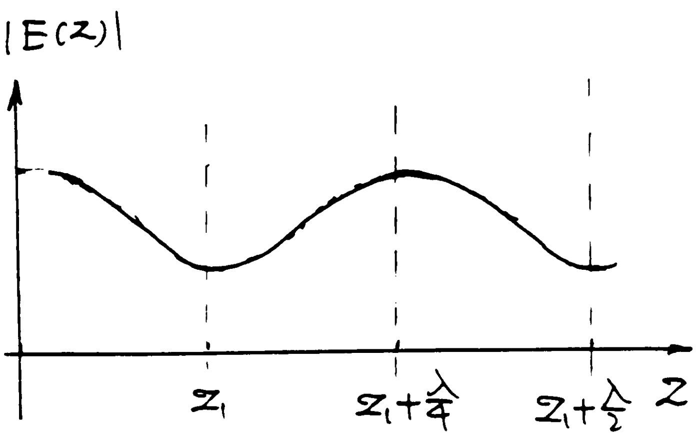
$$\mathbf{E}(z) = \mathbf{E}_i(z) + \mathbf{E}_r(z) = \mathbf{E}_i(z)\ (1 + \Gamma\, e^{+j2\beta_1 z})$$

$$\mathbf{H}(z) = \mathbf{H}_i(z) + \mathbf{H}_r(z) = \mathbf{H}_i(z)\ (1 - \Gamma\, e^{+j2\beta_1 z})$$

→ $$|\mathbf{E}(z)| = |E_{io}|\ |1 + \Gamma\, e^{+j2\beta_1 z}|$$

→ $$|\mathbf{H}(z)| = |\frac{E_{io}}{\eta_1}|\ |1 - \Gamma\, e^{+j2\beta_1 z}|$$

⇒

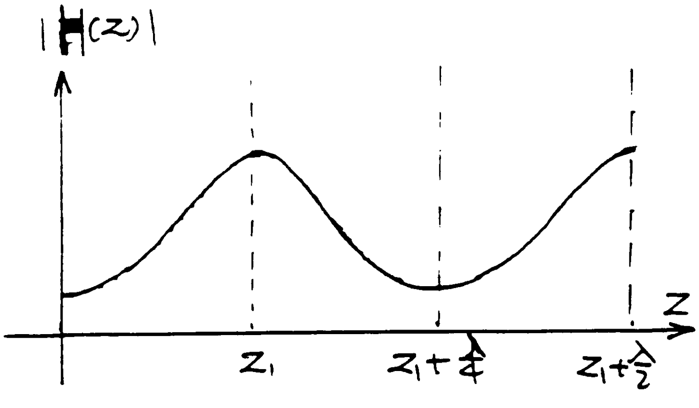


OBS:

- Total fields in medium 1 = incident wave + reflected wave. Its amplitude reaches maximum and minimum at certain locations z
- – $|\mathbf{E}(z)| \rightarrow |\mathbf{E}(z)|_{max} = |E_{io}|(1 + |\Gamma|)$ when z makes $(1 + \Gamma\, e^{+j2\beta_1 z}) = max = 1 + |\Gamma|$

  – $|\mathbf{E}(z)| \rightarrow |\mathbf{E}(z)|_{min} = |E_{io}|(1 - |\Gamma|)$ when z makes $(1 + \Gamma e^{+j2\beta_1 z}) = min = 1 - |\Gamma|$

  – $|\mathbf{H}(z)| \rightarrow |\mathbf{H}(z)|_{min} = |\frac{E_{io}}{\eta_1}|(1 + |\Gamma|)$ when z



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makes $(1+\Gamma\, e^{+j2\beta_1 z})$=max=$1+|\Gamma|$

– $|\mathbf{H}(z)| \to |\mathbf{H}(z)|_{max} = |\frac{E_{io}}{\eta_1}|(1-|\Gamma|)$ when z

makes $(1+\Gamma e^{+j2\beta_1 z})$=min=$1-|\Gamma|$

- $|\mathbf{E}(z)| \to$ max $\Leftrightarrow$ $|\mathbf{H}(z)| \to$ min

  $|\mathbf{E}(z)| \to$ min $\Leftrightarrow$ $|\mathbf{H}(z)| \to$ max

- The successive two maximum points are $\frac{\lambda}{2}$ apart while the successive maximum and minimum points are $\frac{\lambda}{4}$ apart.

<u>DEF:</u> Standing wave ratio (SWR)

$$S = \frac{|\mathbf{E}(z)|_{max}}{|\mathbf{E}(z)|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|} \left(=20\log_{10}\frac{1+|\Gamma|}{1-|\Gamma|}\ \text{dB}\right)$$

$$\Leftrightarrow\ |\Gamma| = \frac{S-1}{S+1}$$

<u>OBS:</u> $S \geq 1.0$ $(|\mathbf{E}(z)|_{max} \geq |\mathbf{E}(z)|_{min}) \to |\Gamma| \leq 1.0$

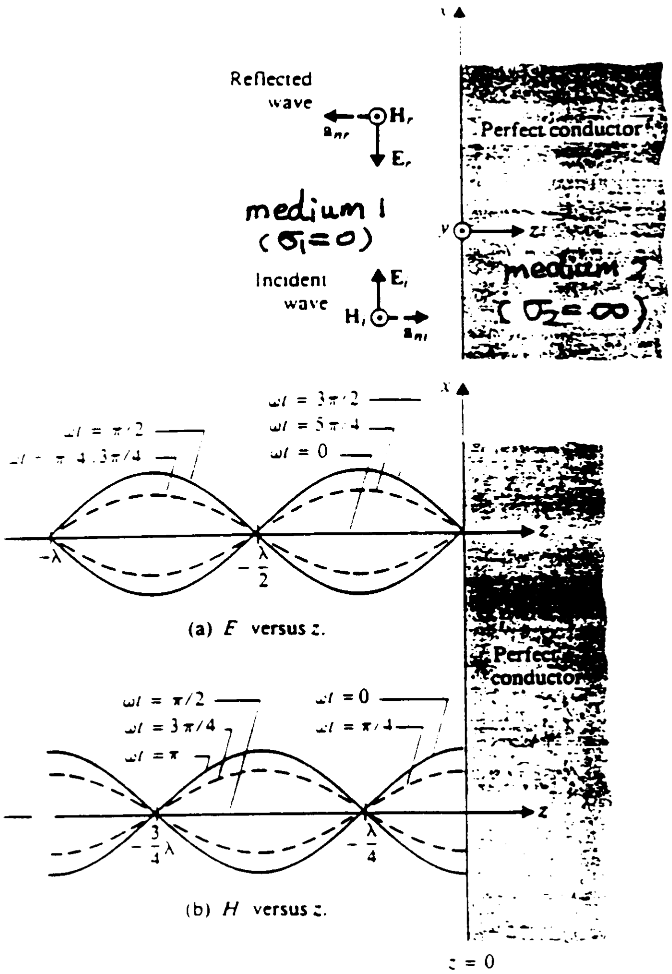
<u>Note:</u> The above results apply to lossy media with $\epsilon \to \epsilon_c$ ($\to j\beta \to \gamma$ and $\eta \to \eta_c$)

Ex: a) convert $\Gamma = 0.20$ into S (dB), b) convert $S = 3$ dB into reflection coefficient. (Ans: 3.52 dB, 0.17)

Ex: Find the time-average power density vectors in medium 1 and medium 2. Show that they are equal at the interface.

Example:

Normal incidence on a perfect conductor ($\sigma = \infty$)



(a) E versus z.

(b) H versus z.

$\sigma = \infty$ while $|\mathbf{J}| = |\sigma \mathbf{E}_t|$ = finite number

$\Rightarrow \mathbf{E}_t = 0$

$\Rightarrow \tau = 0$

$\Rightarrow \Gamma = -1$

Fields in medium 1:

$$\rightarrow \mathbf{E}(z) = \mathbf{E}_i(z)\,(1+\Gamma\, e^{+j2\beta_1 z}) = -\mathbf{a}_x\, 2jE_{io} \sin \beta_1 z$$

$$\rightarrow \mathbf{H}(z) = \mathbf{a}_y \frac{E_{io}}{\eta_1} \cos \beta_1 z$$

Instantaneous fields:

$$\mathbf{E}(z,t) = \mathrm{Re}[\mathbf{E}(z)\, e^{j\omega t}] = \mathbf{a}_x\, 2\,|E_{io}| \sin\beta_1 z \sin(\omega t + \varphi_{io})$$

$$\mathbf{H}(z,t) = \mathrm{Re}[\mathbf{H}(z)\, e^{j\omega t}] = \mathbf{a}_x\, 2\, \left|\frac{E_{io}}{\eta_1}\right| \cos \beta_1 z \cos(\omega t + \varphi_{io})$$

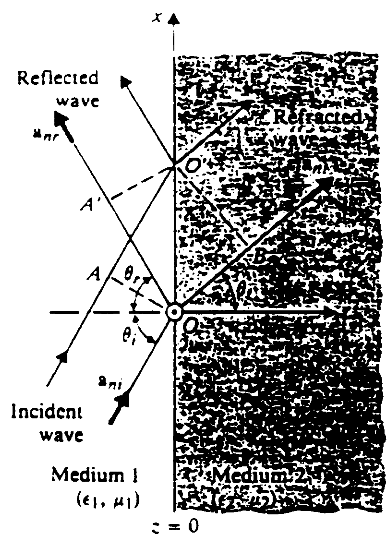
OBS:

- The wave does NOT propagate as the wavefront does not move as time progresses ---- standing wave.
- No energy propagation as $\mathcal{P}_{av} = \frac{1}{2}\,\mathrm{Re}[\mathbf{E}(z) \times \mathbf{H}^*(z)] = 0$. This is due to fact that the power carried by the incident wave is completely reflected by the conductor.

Ex: Determine the locations of $|\mathbf{E}(z)|_{max}$ and $|\mathbf{H}(z)|_{max}$ if a plane wave is normally incident onto a perfect conductor.

(Ans.: $n\frac{\lambda}{2}$, $(2n+1)\frac{\lambda}{4}$)

2 – 7 Oblique Incidence of a Plane Wave at Plane Boundaries



Note: $0^o \le \theta_i \le 90^o$, $0^o \le \theta_r \le 90^o$ and $0^o \le \theta_t \le 90^o$

The incident and reflected wave travel in the same phase velocity (the two are in the same medium 1)

$\Rightarrow$ same time elapse for a wavefront (a phase point) to travel from a) along ray 1 from O→A′ and b) along ray 2 from A→O′. Note $\overline{\text{OA}}$ is a wavefront of the incident wave and $\overline{\text{O'A'}}$ is a

wavefront of the reflected wave

$\Rightarrow \overline{\text{OA'}} = \overline{\text{AO'}}$

$\Rightarrow \overline{\text{OO'}} \sin\theta_r = \overline{\text{OO'}} \sin\theta_i$

$\Rightarrow \theta_r = \theta_i$ ---*Snell's law of reflection*:

Angle of Reflection=Angle of Incidence

time elapse for the incident wave to travel from O→A′ = time elapse for the reflected wave to travel from O→B,

i.e. $\dfrac{\overline{\text{OA'}}}{u_{p1}} = \dfrac{\overline{\text{OB}}}{u_{p2}}$

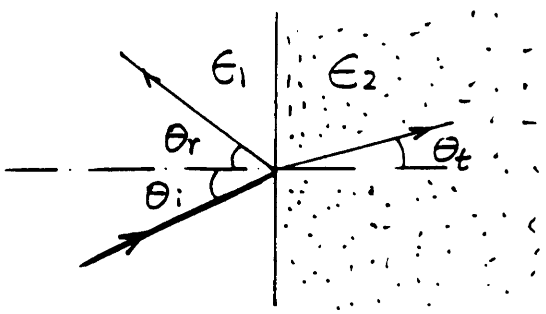
$$\rightarrow \frac{\overline{\text{OB}}}{\overline{\text{OA'}}} = \frac{\overline{\text{OO'}}\sin\theta_t}{\overline{\text{OO'}}\sin\theta_i} = \frac{u_{p2}}{u_{p1}}$$

$$\rightarrow \frac{\sin\theta_t}{\sin\theta_i} = \frac{u_{p2}}{u_{p1}} = \frac{\beta_1}{\beta_2} = \frac{n_1}{n_2}$$

------ ***Snell's law of refraction***

n_1 and n_2 are the indices of refraction for media 1 and 2

DEF: index of refraction for a medium $n = \frac{c_o}{u_p} = \sqrt{\mu_r \epsilon_r}$. $c_o = \frac{1}{\sqrt{\mu_o \epsilon_o}} = 3 \times 10^8$ (m/s) ---- speed of light in free space.

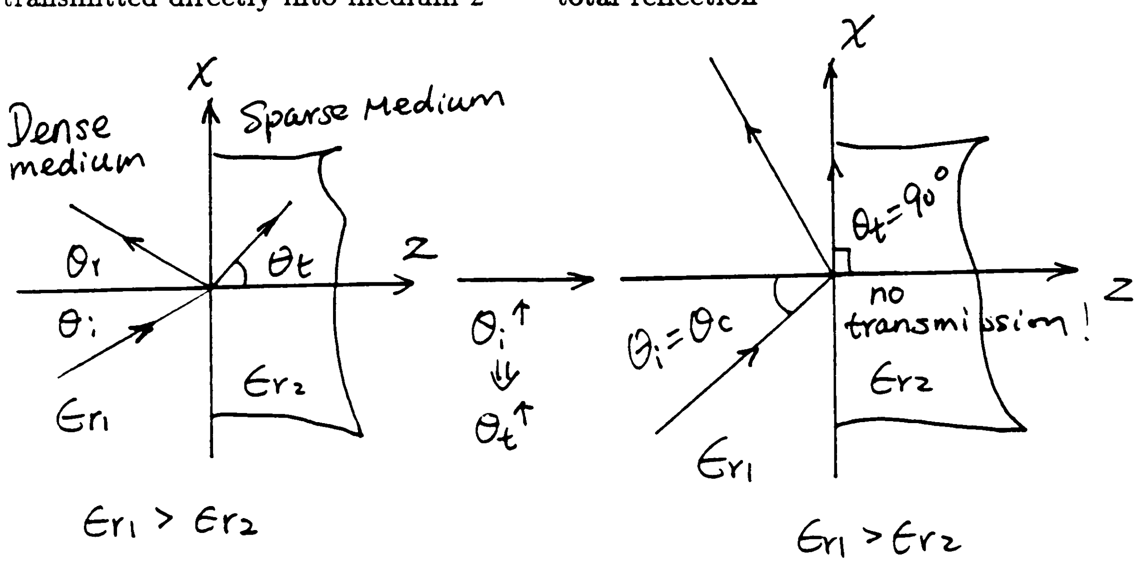


For $\mu_1 = \mu_2$,

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} = \sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2}}} = \frac{\eta_2}{\eta_1}$$

1. Total Reflection

If $n_1 > n_2 \Rightarrow \frac{\sin\theta_t}{\sin\theta_i} > 1 \Rightarrow \theta_t > \theta_i \Rightarrow \theta_i \uparrow \rightarrow \theta_t \uparrow \Rightarrow \theta_i = \theta_c$ when $\theta_t = 90^o$ no waves transmitted directly into medium 2 ----- total reflection



DEF: critical angle θ_c: angle of incidence when $\theta_t = 90^o$ (threshold of the total reflection)

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2} \quad (\theta_i = \theta_c, \theta_t = 90^o) \Rightarrow \theta_c = \sin^{-1}\frac{n_2}{n_1}$$

When $\theta_i > \theta_c$, no waves are propagating in medium 2.

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2} \Rightarrow \sin\theta_t = \frac{n_1}{n_2}\sin\theta_i > 1$$

$$\Rightarrow \mathbf{a}_{nt} = \mathbf{a}_x \sin\theta_t + \mathbf{a}_z \sqrt{1-\sin^2\theta_t} = \mathbf{a}_x \sin\theta_t + \mathbf{a}_z\, j\sqrt{\sin^2\theta_t - 1}$$

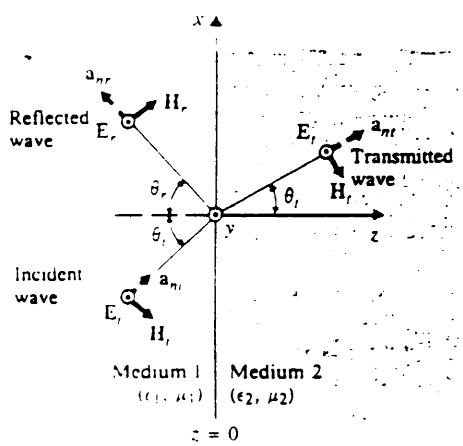
$$\Rightarrow e^{-j\beta_2 \mathbf{a}_{nt}\bullet \boldsymbol{R}} = e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} = e^{-\alpha_2 z}\, e^{-j\beta_{2x}x}$$

$$\alpha_2 = \beta_2\sqrt{\sin^2\theta_t - 1} > 0, \text{ and } \beta_{2x} = \beta_2 \sin\theta_t > 0$$

<u>OBS:</u> when $\theta_i > \theta_c$, the EM wave is attenuated exponentially in medium 2 ($+z$ direction) (forms so called evanescent waves) while propagates along the interface ($+x$ direction) and is therefore bounded to the interface (forms so-called surface waves).

Ex: Determine the minimum dielectric constant of an optical fibre so that a wave incident on one end will be confined within the fibre until it emerges from the other end.
(Ans: $\epsilon_r \geq 2.0$)

<u>2.Perpendicular Polarization (E is perpendicular to the plane in which all the propagation unit vectors lie)</u>



Assume that the known incident plane wave is: ($\mathbf{a}_{ni} = \mathbf{a}_x \sin\theta_i + \mathbf{a}_z \cos\theta_i$)

$$\mathbf{E}_i(z) = \mathbf{a}_y E_{io}\, e^{-j\beta_1 \mathbf{a}_{ni} \bullet \boldsymbol{R}} = \mathbf{a}_y E_{io}\, e^{-j\beta_1 (x \sin\theta_i + z \cos\theta_i)}$$

$$\rightarrow \qquad \mathbf{H}_i(z) = \frac{1}{\eta_1} \mathbf{a}_{ni} \times \mathbf{E}_i(z) = \frac{E_{io}}{\eta_1} (- \mathbf{a}_x \cos\theta_i + \mathbf{a}_z \sin\theta_i)\, e^{-j\beta_1 (x \sin\theta_i + z \cos\theta_i)}$$

$$\beta_1 = k_1 = \omega \sqrt{\mu_1 \epsilon_1}, \quad \eta_1 = \sqrt{\mu_1 / \epsilon_1}$$

The reflected plane wave can then be expressed as: ($\mathbf{a}_{nr} = \mathbf{a}_x \sin\theta_i - \mathbf{a}_z \cos\theta_i$)

$$\mathbf{E}_r(z) = \mathbf{a}_y E_{ro}\, e^{-j\beta_1 \mathbf{a}_{nr} \bullet \boldsymbol{R}} = \mathbf{a}_y E_{ro}\, e^{-j\beta_1 (x \sin\theta_r - z \cos\theta_r)}$$

$$\rightarrow \qquad \mathbf{H}_r(z) = \frac{1}{\eta_1} \mathbf{a}_{nr} \times \mathbf{E}_r(z) = \frac{E_{ro}}{\eta_1} (\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)\, e^{-j\beta_1 (x \sin\theta_r - z \cos\theta_r)}$$

E_{ro} is to be determined

The transmitted plane wave can then be expressed as: ($\mathbf{a}_{nt} = \mathbf{a}_x \sin\theta_t + \mathbf{a}_z \cos\theta_t$)

$$\mathbf{E}_t(z) = \mathbf{a}_y E_{to}\, e^{-j\beta_2 \mathbf{a}_{nt} \bullet \boldsymbol{R}} = \mathbf{a}_y E_{to}\, e^{-j\beta_2 (x \sin\theta_t + z \cos\theta_t)}$$

$$\rightarrow \qquad \mathbf{H}_t(z) = \frac{1}{\eta_2} \mathbf{a}_{nt} \times \mathbf{E}_t(z) = \frac{E_{to}}{\eta_2} (- \mathbf{a}_x \cos\theta_t + \mathbf{a}_z \sin\theta_t)\, e^{-j\beta_2 (x \sin\theta_t + z \cos\theta_t)}$$

$$\beta_2 = k_2 = \omega \sqrt{\mu_2 \epsilon_2}, \quad \eta_2 = \sqrt{\mu_2 / \epsilon_2}$$

E_{to} is to be determined

Enforce the boundary conditions at z=0.

$$(\text{-}\mathbf{a}_z) \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \;\Rightarrow\; E_{io} + E_{ro} = E_{to}$$

$$(\text{-}\mathbf{a}_z) \times (\mathbf{H}_1 - \mathbf{H}_2) = 0 \;\Rightarrow\; \frac{1}{\eta_1} (E_{io} - E_{ro}) \cos\theta_i = \frac{E_{to}}{\eta_2} \cos\theta_t$$

→ reflection coefficient $\Gamma_\perp = \dfrac{\text{reflected wave}}{\text{incident wave}} = \dfrac{(\eta_2/\cos\theta_t) - (\eta_1/\cos\theta_i)}{(\eta_2/\cos\theta_t) + (\eta_1/\cos\theta_i)}$ (at z=0)

→ transmission coefficient $\tau_\perp = \dfrac{\text{transmitted wave}}{\text{incident wave}} = \dfrac{2(\eta_2/\cos\theta_t)}{(\eta_2/\cos\theta_t) + (\eta_1/\cos\theta_i)}$ (at z=0)

$$1+\Gamma_\perp = \tau_\perp \qquad \text{(at } z\text{=0)}$$

<u>OBS:</u> Normal incidence mentioned before is the special case of $\theta_i = \theta_r = \theta_t = 0^o$.

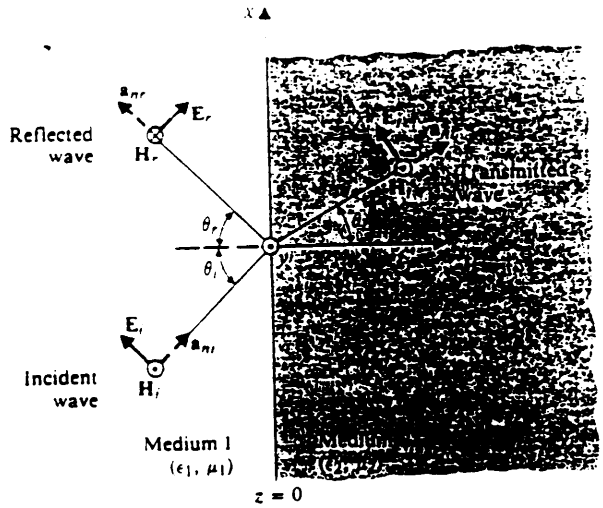
$$\left.\begin{array}{l} \Gamma_\perp = 0 \Leftrightarrow \quad (\eta_2/\cos\theta_t) - (\eta_1/\cos\theta_i) = 0 \\ \qquad\qquad \dfrac{\sin\theta_t}{\sin\theta_i} = \dfrac{\beta_1}{\beta_2} \end{array}\right\} \rightarrow \sin^2\theta_i = \sin^2\theta_{B\perp} = \frac{1 - \mu_1\epsilon_2/\mu_2\epsilon_1}{1-(\mu_1/\mu_2)^2}$$

If $0.0 \le \dfrac{1 - \mu_1\epsilon_2/\mu_2\epsilon_1}{1-(\mu_1/\mu_2)^2} \le 1.0$, $\qquad \theta_{B\perp} = \sin^{-1}\sqrt{\dfrac{1 - \mu_1\epsilon_2/\mu_2\epsilon_1}{1-(\mu_1/\mu_2)^2}}$

<u>DEF:</u> Brewster angle θ_B: the angle of incidence which makes reflection coefficient zero for no reflection.

<u>Note:</u> the concept of θ_B is different from that of θ_c.

3.Parallel Polarization (**E** is in the plane where all the propagation unit vectors lie)



Assume that the known incident plane wave is: ($\mathbf{a}_{ni} = \mathbf{a}_x \sin\theta_i + \mathbf{a}_z \cos\theta_i$)

$$\mathbf{H}_i(z) = \mathbf{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1 \mathbf{a}_{ni} \bullet \boldsymbol{R}} = \mathbf{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1(x \sin\theta_i + z \cos\theta_i)}$$

$$\rightarrow \quad \mathbf{E}_i(z) = -\eta_1\, \mathbf{a}_{ni} \times \mathbf{H}_i(z) = E_{io}\,(\mathbf{a}_x \cos\theta_i - \mathbf{a}_z \sin\theta_i)\, e^{-j\beta_1(x \sin\theta_i + z \cos\theta_i)}$$

$$\beta_1 = k_1 = \omega\sqrt{\mu_1 \epsilon_1}, \quad \eta_1 = \sqrt{\mu_1/\epsilon_1}$$

The reflected plane wave can then be expressed as: ($\mathbf{a}_{nr} = \mathbf{a}_x \sin\theta_r - \mathbf{a}_z \cos\theta_r$)

$$\mathbf{H}_r(z) = -\mathbf{a}_y \frac{E_{ro}}{\eta_1} e^{-j\beta_1 \mathbf{a}_{nr} \bullet \boldsymbol{R}} = -\mathbf{a}_y \frac{E_{ro}}{\eta_1} e^{-j\beta_1(x \sin\theta_r - z \cos\theta_r)}$$

$$\rightarrow \quad \mathbf{E}_r(z) = -\eta_1\, \mathbf{a}_{nr} \times \mathbf{H}_r(z) = E_{ro}\,(\mathbf{a}_x \cos\theta_r + \mathbf{a}_z \sin\theta_r)\, e^{-j\beta_1(x \sin\theta_r - z \cos\theta_r)}$$

E_{ro} is to be determined

The transmitted plane wave can then be expressed as: ($\mathbf{a}_{nt} = \mathbf{a}_x \sin\theta_t + \mathbf{a}_z \cos\theta_t$)

$$\mathbf{H}_t(z) = \mathbf{a}_y \frac{E_{to}}{\eta_2} e^{-j\beta_2 \mathbf{a}_{nt} \bullet \boldsymbol{R}} = \mathbf{a}_y \frac{E_{to}}{\eta_2} e^{-j\beta_2(x \sin\theta_t + z \cos\theta_t)}$$

$$\rightarrow \quad \mathbf{E}_t(z) = -\eta_2\, \mathbf{a}_{nt} \times \mathbf{H}_t(z) = E_{to}\,(\mathbf{a}_x \cos\theta_t - \mathbf{a}_z \sin\theta_t)\, e^{-j\beta_2(x \sin\theta_t + z \cos\theta_t)}$$

$$\beta_2 = k_2 = \omega\sqrt{\mu_2\epsilon_2}, \quad \eta_2 = \sqrt{\mu_2/\epsilon_2}$$

E_{to} is to be determined

Enforce the boundary conditions at $z=0$,

$$\mathbf{a}_z \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \;\Rightarrow\; (E_{io} + E_{ro}) \cos\theta_i = E_{to}\cos\theta_t$$

$$\mathbf{a}_z \times (\mathbf{H}_1 - \mathbf{H}_2) = 0 \;\Rightarrow\; \frac{1}{\eta_1}(E_{io} - E_{ro}) = \frac{E_{to}}{\eta_2}$$

$\rightarrow$ reflection coefficient $\Gamma_{||} = \dfrac{\text{reflected wave}}{\text{incident wave}} = \dfrac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$ (at $z=0$)

$\rightarrow$ transmission coefficient $\tau_{||} = \dfrac{\text{transmitted wave}}{\text{incident wave}} = \dfrac{2\,\eta_2 \cos\theta_t}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$ (at $z=0$)

$$1 + \Gamma_{||} = \tau_{||} \left(\frac{\cos\theta_t}{\cos\theta_i} \right) \qquad (\text{at } z=0)$$

<u>OBS:</u> Normal incidence mentioned before is the special case of $\theta_i = \theta_r = \theta_t = 0^o$.

$$\left.\begin{aligned} \Gamma_{||} = 0 \;\Leftrightarrow\;\; & \eta_2 \cos\theta_t - \eta_1 \cos\theta_i = 0 \\ & \frac{\sin\theta_t}{\sin\theta_i} = \frac{\beta_1}{\beta_2} \end{aligned}\right\} \rightarrow \sin^2\theta_i = \sin^2\theta_{B\,||} = \frac{1 - \mu_2\epsilon_1/\mu_1\epsilon_2}{1 - (\epsilon_1/\epsilon_2)^2}$$

If $0.0 \le \frac{1-\mu_2\epsilon_1/\mu_1\epsilon_2}{1-(\epsilon_1/\epsilon_2)^2} \le 1.0.$ $\qquad \theta_{B\parallel} = \sin^{-1}\sqrt{\frac{1-\mu_2\epsilon_1/\mu_1\epsilon_2}{1-(\epsilon_1/\epsilon_2)^2}}$

Ex: Ab EM wave impinges from air on the surface of the water (ϵ_r=80) (a) Find $\theta_{B\parallel}$ and the corresponding angle of transmission; (b) if the wave is perpendicularly polarized and incident on the water surface at $\theta_i = \theta_{B\parallel}$, find $\Gamma_\perp$ and $\tau_\perp$. (P416, textbook)
Ans: (a) 81.0°, 6.38° (b) −0.967, 0.033.

SUMMARY

At very large distances from a finite source radiating electromagnetic waves, a small portion of the wavefront is very nearly a plane. Thus the study of uniform plane waves is of particular importance. In this chapter we

- examined the behavior of uniform plane waves in both lossless and lossy media,
- explained Doppler effect when there is relative motion between a time-harmonic source and a receiver,
- discussed the polarization of plane waves and showed the relation between linearly polarized and circularly polarized waves,
- explained the significant of a complex wavenumber and a complex propagation constant in a lossy medium,
- introduced the concept of signal dispersion and explained the difference between phase and group velocities,
- discussed the flow of electromagnetic power and Poynting's theorem,
- studied the reflection and refraction of electromagnetic waves at plane boundaries for both normal incidence and oblique incidence,
- derived Snell's laws of reflection and refraction,
- explained the effect of ionosphere on wave propagation, and
- examined the conditions for total reflection and for no reflection.

Chapter 3

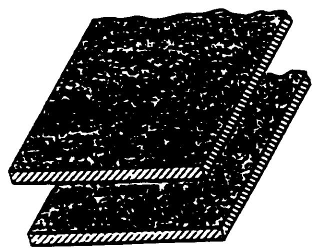
Theory and Applications of Transmission Lines

3 – 1 Introduction

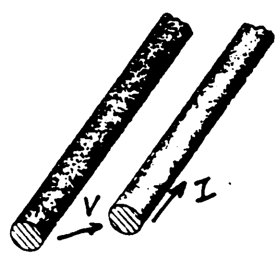
Transmission Lines (TLs): to guide EM propagation for efficient point to point transmission of EM power and information

Two-conductor transmission lines (TLs)

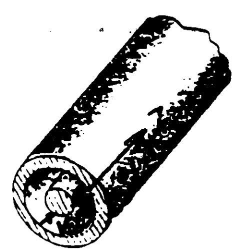
a) Parallel – plate TLs (also called striplines).



b) Two-wire TLs:
e.g. power lines, telephone lines

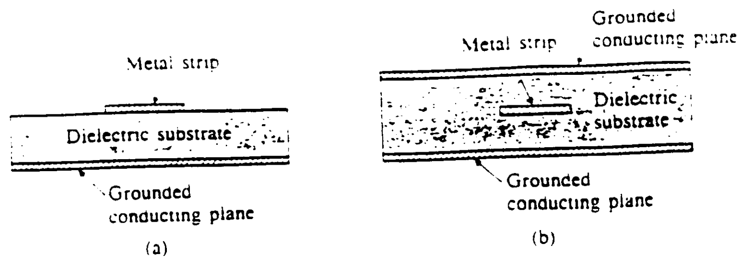


c) Coaxial TLs:
e.g. TV cables,
cables to electronic equip.

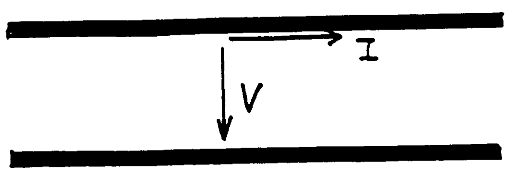


d) Microstrip line:

e.g. PCB

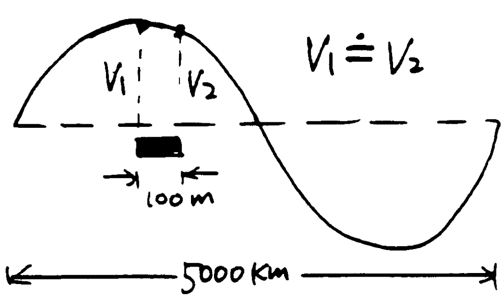


OBS: It has been shown, either by measurement or by theory, that the above structures can be modeled by a two-wire transmission line in terms of voltage between the two conductors and current flowing through the conductors in the structures.



At low frequencies, say f=60 Hz,

$\lambda = u/f = 3.0\times10^8/60 = 5000$ (km)



For a section of wire of l=100m,

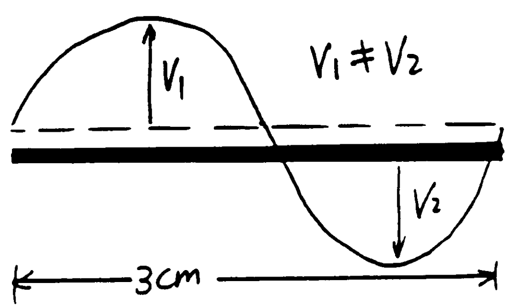
$$\frac{l}{\lambda} = \frac{100\text{m}}{5000\text{ km}} = 2.\times10^{-5} << 1$$

$\Rightarrow V_1 \doteq V_2$, $I_1 \doteq I_2$

voltages and currents are the same everywhere on the line

At high frequencies, f=10 GHz

$\lambda = u/f = 3.0\times10^8/10\times10^9 = 3$ (cm)



For a section of wire of l=3 cm,

$$\frac{l}{\lambda} = \frac{3\text{ cm}}{3\text{ cm}} = 1$$

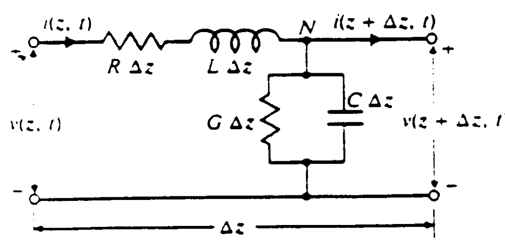
$\Rightarrow V_1 \neq V_2$, $I_1 \neq I_2$

voltages and currents are different on different locations of the line

3 – 2 General Transmission – Line Equations

DEF:

- R --- resistance per unit length (Ω/m) ← resistance of the conductor ← imperfect conductors used
- L --- inductance per unit length (H/m) ← magnetic flux surrounding the conductors ← currents flowing in the conductors
- G --- conductance per unit length (S/m) ← conductance of the materials ← imperfect materials used to fill in between the two conductors
- C --- capacitance per unit length (F/m) ← charge accumulation in the two conductors ← potential difference between the two conductors



Note: for different types of transmission lines, formulas for R, L, G and C can be obtained from electromagnetic field analysis. e.g. P445-447, textbook

OBS: for the TL structure mentioned above, $G/C=\sigma/\epsilon$ and $LC=\mu\epsilon$ where (μ,ϵ,σ) are the constitutive parameters of materials filled in between the two conductors.

Kichhoff's voltage law $\Rightarrow v(z,t)=i(z,t)\ R\ \Delta z + L\ \Delta z\ \dfrac{\partial i(z,t)}{\partial t} + v(z+\Delta z,t)$

$$\Rightarrow \qquad -\frac{v(z+\Delta z,t)-v(z,t)}{\Delta z} = R\ i(z,t) + L\ \frac{\partial i(z,t)}{\partial t}$$

$$\Delta z \to 0: \qquad -\frac{\partial v(z,t)}{\partial z} = R\ i(z,t) + L\ \frac{\partial i(z,t)}{\partial t}$$

Kichhoff's current law $\Rightarrow i(z,t)=v(z,t)\ G\ \Delta z + C\ \Delta z\ \dfrac{\partial v(z+\Delta z,t)}{\partial t} + i(z+\Delta z,t)$

$$\Rightarrow \qquad -\frac{i(z+\Delta z,t) - i(z,t)}{\Delta z} = G\ v(z,t) + C\ \frac{\partial v(z+\Delta z,t)}{\partial t}$$

$$\Delta z \to 0: \qquad -\frac{\partial i(z,t)}{\partial z} = G\ v(z,t) + C\ \frac{\partial v(z,t)}{\partial t}$$

$$\left.\begin{aligned} -\frac{\partial v(z,t)}{\partial z} &= R\ i(z,t) + L\ \frac{\partial i(z,t)}{\partial t} \\ -\frac{\partial i(z,t)}{\partial z} &= G\ v(z,t) + C\ \frac{\partial v(z,t)}{\partial t} \end{aligned}\right\}\ \textit{general TL equations}$$

$\Rightarrow$ time-harmonic TL equations:

$$-\frac{dV(z)}{dz} = R\ I(z) + j\omega L\ I(z) = Z\ I(z)$$

$$-\frac{dI(z)}{dz} = G\ V(z) + j\omega C\ V(z) = Y\ V(z)$$

<u>DEF:</u>

- $Z = Z(\omega) = R+j\omega L$ ---- impedance per unit length in Ω/m
- $Y = Y(\omega) = G+j\omega C$ ---- admittance per unit length in S/m

<u>General solutions of the TL equations – wave characteristics on a TL</u>

$$-\frac{dV(z)}{dz} = Z\ I(z) \Rightarrow -\frac{d^2V(z)}{dz^2} = Z\ \frac{dI(z)}{dz}$$

$$-\frac{dI(z)}{dz} = Y\ V(z): \qquad \frac{d^2V(z)}{dz^2} = \gamma^2\ V(z)\ , \qquad \gamma=\sqrt{ZY} = \alpha(\omega) + j\beta(\omega)$$

$$\Rightarrow \qquad V(z) = \underbrace{V_o^+ e^{-\gamma z}} + \underbrace{V_o^- e^{+\gamma z}}$$

$$= \quad V^+(z) \quad + \quad V^-(z)$$

$$\downarrow \qquad\qquad \downarrow$$

forward (incident) wave backward (reflected) wave

DEF: γ ---- propagation constant, α ---- attenuation constant, and β ----- phase constant

$$-\frac{dV(z)}{dz} = Z\, I(z)$$

$$\Rightarrow \qquad I(z) = -\frac{dV(z)}{Zdz}$$

$$= \underbrace{\frac{V_o^+}{Z_o} e^{-\gamma z}} + \underbrace{\frac{-V_o^-}{Z_o} e^{+\gamma z}}$$

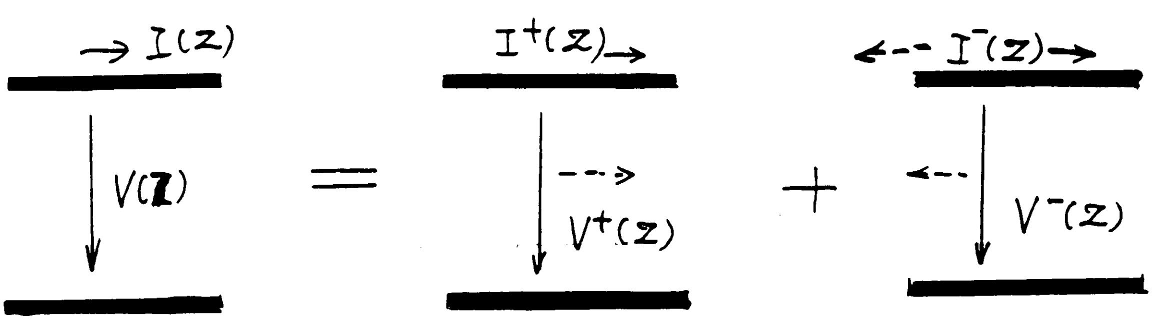
$$= \underbrace{I_o^+ e^{-\gamma z}} + \underbrace{I_o^- e^{+\gamma z}}$$

$$\left(\frac{d^2V(z)}{dz^2} = \gamma^2 V(z) \rightarrow \right) \qquad = \quad I^+(z) \quad + \quad I^-(z)$$

$$\downarrow \qquad\qquad \downarrow$$

forward (incident) wave backward (reflected) wave

$$Z_o = \sqrt{\frac{Z}{Y}}$$



DEF:

- $Z_o = \frac{V^+}{I^+} = -\frac{V^-}{I^-}$ ----- characteristic impedance of the transmission line

- $Y_o = \frac{1}{Z_o}$ ----- characteristic admittance of the transmission line

Note: wave here is referred to voltage or current waves usually, not a plane wave.

OBS:

1) Lossless Line (R=0, G=0) $\Rightarrow$ $\alpha=0$, $\beta=\omega\sqrt{LC}$, $u_p=\frac{\omega}{\beta}=\frac{1}{\sqrt{LC}}$ ----no dispersion, $Z_o=\sqrt{\frac{L}{C}}$

2) Dispersionless Line ($\frac{R}{L}=\frac{G}{C}$) $\Rightarrow$ $\alpha=R\sqrt{\frac{C}{L}}$, $\beta=\omega\sqrt{LC}$, $u_p=\frac{\omega}{\beta}=\frac{1}{\sqrt{LC}}$, $Z_o=\sqrt{\frac{L}{C}}$

3) On an infinite TL,

$$V(z) = V^+(z) = V_o^+ e^{-\gamma z}$$

$$I(z) = I^+(z) = I_o^+ e^{-\gamma z} = \frac{V_o^+}{Z_o} e^{-\gamma z}$$

$$\Rightarrow \text{time-average power } P(z) = \frac{1}{2}\,\mathrm{Re}[V(z)I^*(z)] = \frac{|V_o^+|}{|Z_o|^2}\,\mathrm{Re}(Z_o)\, e^{-2\alpha z}$$

$$\Rightarrow -\frac{\partial P(z)}{\partial z} = P_L(z) = 2\alpha\, P(z) \Rightarrow \alpha = \frac{P_L(z)}{2P(z)}$$

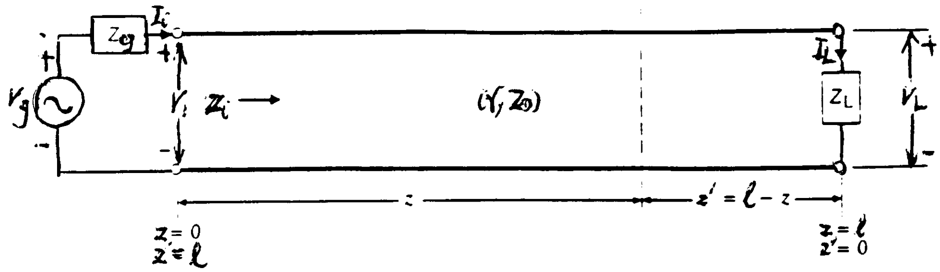
Ex: The α at 10 MHz of a dispersionless coaxial TL is found to be 0.1dB/km. Find α

a) at 50 MHz

b) at 10 MHz if the dielectric constant of the material is doubled.

Ans: 0.224 dB/km, 0.141 dB/km

3 – 3 Wave Characteristics on a Finite TL



1. Impedance

As mentioned before,

$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z}$$

$$I(z) = \frac{V_o^+}{Z_o} e^{-\gamma z} + \frac{-V_o^-}{Z_o} e^{+\gamma z}$$

Enforce the boundary conditions:

$$V_L = V(z{=}L) = I_L Z_L = I(z{=}L)\, Z_L$$

$$\Rightarrow \quad \begin{cases} V(z') = I_L (Z_L \cosh \gamma z' + Z_o \sinh \gamma z') \\ I(z') = \dfrac{I_L}{Z_o} (Z_L \sinh \gamma z' + Z_o \cosh \gamma z') \end{cases}$$

$$\Rightarrow \quad Z(z') = \frac{V(z')}{I(z')} = Z_o \frac{Z_L \cosh \gamma z' + Z_o \sinh \gamma z'}{Z_L \sinh \gamma z' + Z_o \cosh \gamma z'}$$

or $$Z(z') = Z_o \frac{Z_L + Z_o \tanh \gamma z'}{Z_o + Z_L \tanh \gamma z'} = R(z') + jX(z') = \text{Resistance} + j\text{Reactance}$$

Note:

- z' is the distance measured backward from the load Z_L
- $Z(z')$ is the impedance looking into (or toward) the load Z_L
- Input impedance of a TL: $Z_i = Z(z'=l)$
- Admittance $Y(z') = 1/Z(z') = G(z') + jB(z')$=Conductance $+j$Susceptance

For a lossless TL, $\gamma = j\beta$, $\tanh(j\beta z') = j\tan(\beta z')$

$$Z(z') = Z_o \frac{Z_L + jZ_o \tan\beta z'}{Z_o + Z_L j\tan\beta z'}$$

Note: The above four equations are derived from the impedance formula.

2. Reflection coefficient and standing wave ratio

On a T/L,

$$V(z) = V^+(z) + V^-(z) = V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z}$$

$$I(z) = I^+(z) + I^-(z) = \frac{V_o^+}{Z_o} e^{-\gamma z} + \frac{-V_o^-}{Z_o} e^{+\gamma z}$$

DEF:

- reflection coefficient $\Gamma(z') \triangleq \frac{\text{reflected voltage wave}}{\text{incident voltage wave}} = \frac{V^-(z)}{V^+(z)} = -\frac{I^-(z)}{I^+(z)} = \frac{V_o^-}{V_o^+} e^{-2\gamma z}$

- standing wave ratio $S \triangleq \frac{|V(z)|\text{max}}{|V(z)|\text{min}} = \frac{|I(z)|\text{max}}{|I(z)|_{\text{min}}} = \frac{1+|\Gamma|}{1-|\Gamma|}$

Note:

- the notation for reflection coefficient is different from the one in textbook. In the textbook, Γ is denoted as the reflection coefficient at the load end.

$-\Gamma$ is a complex number. $\Gamma = |\Gamma|\, e^{j\theta_\Gamma}$

$$\Rightarrow \quad \begin{cases} V(z') = V^+(z')[1+\Gamma(z')] \\ I(z') = I^+(z')[1-\Gamma(z')] \end{cases}$$

$$\Rightarrow \quad Z(z') = Z_o \frac{1+\Gamma(z')}{1-\Gamma(z')} \quad \text{or} \quad \frac{Z(z')}{Z_o} = \frac{1+\Gamma(z')}{1-\Gamma(z')}$$

$$\Rightarrow \quad \Gamma(z') = \frac{Z(z') - Z_o}{Z(z') + Z_o} \quad \text{or} \quad \Gamma(z') = \frac{\frac{Z(z')}{Z_o} - 1}{\frac{Z(z')}{Z_o} + 1}$$

DEF:

$$\underbrace{\tilde{z}(z')}_{\substack{\text{normalized}\\ \text{impedance}}} = \underbrace{r(z')}_{\substack{\text{normalized}\\ \text{resistance}}} + j\, \underbrace{x(z')}_{\substack{\text{normalized}\\ \text{reactance}}} \triangleq \frac{Z(z')}{Z_o} = \frac{R(z')}{Z_o} + j\frac{X(z')}{Z_o}$$

$$\Rightarrow \quad \begin{cases} \tilde{z}(z') = \dfrac{1+\Gamma(z')}{1-\Gamma(z')} \\ \Gamma(z') = \dfrac{\tilde{z}(z') - 1}{\tilde{z}(z') + 1} \end{cases}$$

It can be found that

$$V^+(z) = \frac{I_L}{2}(Z_L + Z_o)\, e^{\gamma z'}$$

$$V^-(z) = \frac{I_L}{2}(Z_L - Z_o)\, e^{\gamma z'}$$

$$\Rightarrow \quad \Gamma(z') = \frac{Z_L - Z_o}{Z_L + Z_o}\, e^{-2\gamma z'}$$

$$\Rightarrow \qquad \Gamma_L = \Gamma(z'=0) = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{\tilde{z}_L - 1}{\tilde{z}_L + 1} = |\Gamma_L| e^{\theta_L}$$

--- load reflection coefficient

$$\Rightarrow \qquad \Gamma = \Gamma_L\, e^{-2\gamma z'}$$

<u>OBS:</u> load reflection coefficient (reflection coefficient at load) is dependent only on load and characteristic impedance.

For a lossless TL, $\alpha=0$, $\gamma=j\beta$

$$\Gamma(z') = \Gamma_L e^{-j2\beta z'}$$

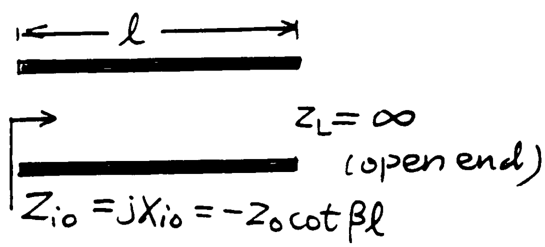
$$\Rightarrow \qquad \begin{cases} |\Gamma(z')| = |\Gamma_L| & \text{----- not dependent on locations } z' \\ \theta_\Gamma = \theta_L - 2\beta z' & \end{cases}$$

<u>OBS:</u> The magnitude of reflection coefficient is the same everywhere on a lossless TL !

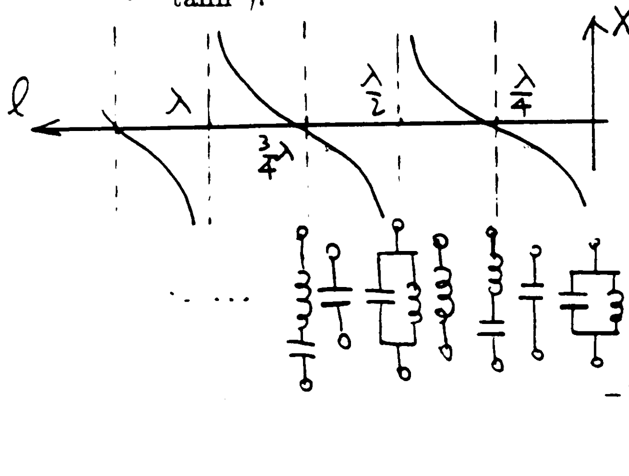
3. Some special aspects of TLs

1) Special sections of TLs

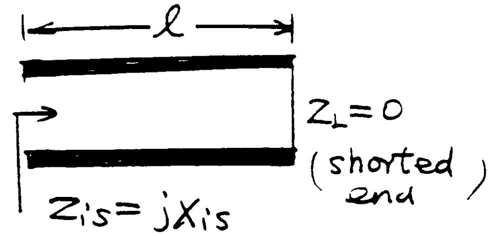
A. Open-circuited line ($Z_L \to \infty$)



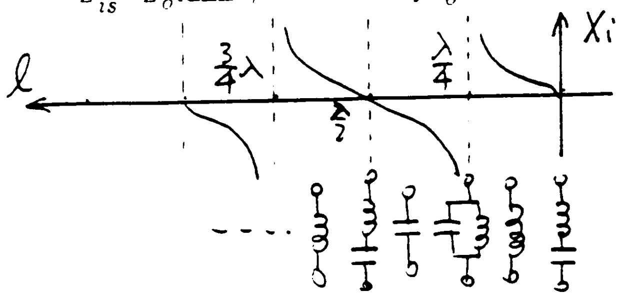
$$Z_{io} = \frac{Z_o}{\tanh \gamma l} \ \underset{=}{\overset{\text{lossless}}{}} \ -jZ_o \cot\beta l$$



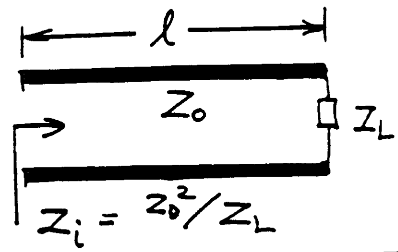
B. Short-circuited line ($Z_L \to 0$)



$Z_{is} = Z_o \tanh \gamma l \overset{\text{lossless}}{=} jZ_o \tan\beta l$

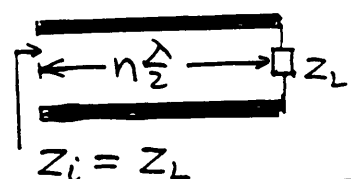


C. Quarter-wave (lossless) line: $l = (2n+1)\lambda/4$



$$Z_i = \frac{Z_o^2}{Z_L}$$

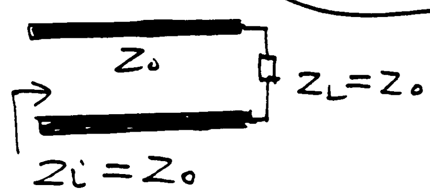
D. Half-wave (lossless) line: $l = n\lambda/2$



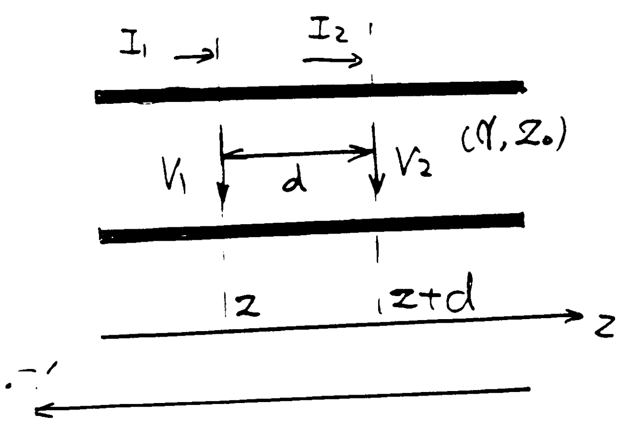
$$Z_i = Z_L$$

E. matched line: $Z_L = Z_o$

$$Z(z') = Z_o$$



2) Relations between voltages (currents) at any two points on a TL



$$V_1 = V_1^+ + V_1^- = V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z}$$

$$V_2 = V_2^+ + V_2^- = V_o^+ e^{-\gamma(z+d)} + V_o^- e^{+\gamma(z+d)}$$

$$\Rightarrow \quad \begin{cases} V_2^+ = V_o^+ e^{-\gamma(z+d)} = (V_o^+ e^{-\gamma z})\, e^{-\gamma d} = V_1^+ \; e^{-\gamma d} \\ V_2^- = V_o^- e^{\gamma(z+d)} = (V_o^- e^{\gamma z})\, e^{\gamma d} = V_1^- \; e^{\gamma d} \end{cases}$$

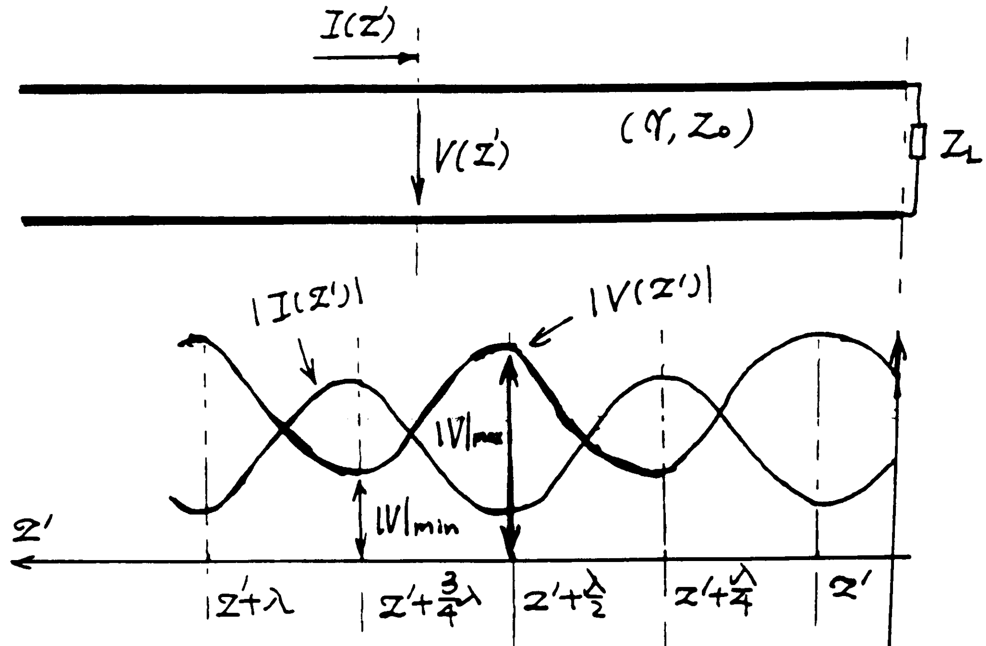
4. Characteristic impedance and propagation constant from input measurement

$$Z_o = \sqrt{Z_{io}\, Z_{is}} \qquad (\Omega)$$

$$\gamma = \frac{1}{l} \tanh^{-1} \sqrt{\frac{Z_{is}}{Z_{io}}}$$

Ex: the open-circuited and short-circuited impedance measured at the input terminals of a lossless TL of length 1.5 cm, which is less than a quarter of wavelength, are $-54.6(\Omega)$ and $j103(\Omega)$, respectively. a) find Z_o and γ of the TL line, b) without changing the operating frequency, find the input impedance of a short-circuited line that is twice the given length. (P457, textbook)

5. Lossless lines with arbitrary terminations



OBS:

1) $\theta_\Gamma = \theta_L - 2\beta z' = -2n\pi \Leftrightarrow |V(z')| \to \max \Leftrightarrow |I(z')| \to \min \Leftrightarrow |Z(z')| \to \max$: $Z(z') = SZ_o \geq Z_o$

2) $\theta_\Gamma = \theta_L - 2\beta z' = -(2n+1)\pi \Leftrightarrow |V(z')| \to \min \Leftrightarrow |I(z')| \to \max \Leftrightarrow |Z(z')| \to \min$: $Z(z') = \frac{Z_o}{S} \leq Z_o$

3) two successive $|V(z')|_{max}$ (or $|I(z')|_{min}$ or $|Z(z')|_{max}$) points are $\frac{\lambda}{2}$ apart

4) two successive $|V(z')|_{min}$ (or $|I(z')|_{max}$ or $|Z(z')|_{min}$) points are $\frac{\lambda}{2}$ apart

5) a $|V(z')|_{max}$ (or $|I(z')|_{min}$ or $|Z(z')|_{max}$) point and its successive $|V(z')|_{min}$ (or $|I(z')|_{max}$ or $|Z(z')|_{min}$) point are $\frac{\lambda}{4}$

Ex: The VSWR on a lossless TL terminated in an unknown impedance is found to be 3.0. The distance between two successive voltage minima is 20 cm and the first minima is located at 5 cm from the load. Determine (a) the load reflection coefficient (b) the load impedance. $Z_o = 50\ \Omega$.
Ans: 0.5, $30 - j40(\Omega)$

3 – 4 The Smith Chart

Smith Chart – a graphical plot of normalized resistance and reactance functions in the reflection – coefficient plan

1. Construction of Impedance (Z) Smith Chart

Let $\Gamma(z') = \Gamma_r + j\,\Gamma_i$

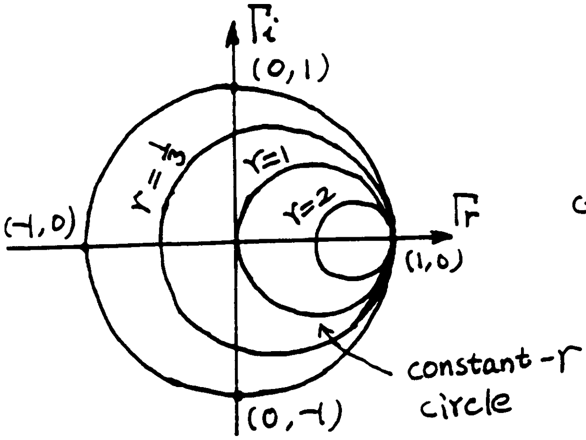
$$\tilde{z}(z') = r(z') + j\,x(z') = \frac{1+\Gamma(z')}{1-\Gamma(z')} = \frac{1+\Gamma_r + j\,\Gamma_i}{1-\Gamma_r - j\,\Gamma_i}$$

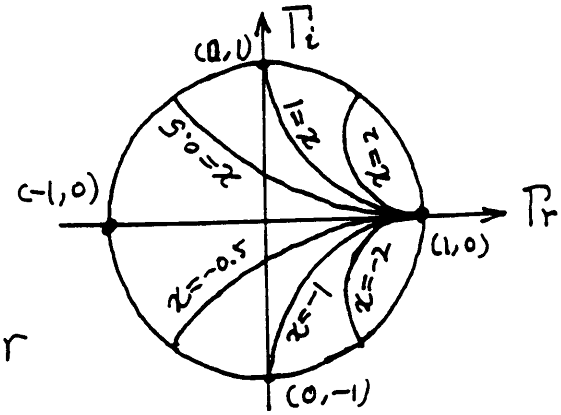
$\Rightarrow r = \mathrm{Re}\left(\dfrac{1+\Gamma_r+j\Gamma_i}{1-\Gamma_r-j\Gamma_i}\right) \Rightarrow \left(\Gamma_r-\dfrac{r}{1+r}\right)^2+\Gamma_i^{\,2}=\left(\dfrac{1}{1+r}\right)^2 \rightarrow$ a circle on the $\Gamma_r-\Gamma_i$ plane

------ constant r circle: locus of Γ for a given r

$\Rightarrow x = \mathrm{Im}\left(\dfrac{1+\Gamma_r+j\Gamma_i}{1-\Gamma_r-j\Gamma_i}\right) \Rightarrow (\Gamma_r-1)^2+\left(\Gamma_i-\dfrac{1}{x}\right)^2=\left(\dfrac{1}{x}\right)^2 \rightarrow$ a circle on the $\Gamma_r-\Gamma_i$ plane

------ constant x circle: locus of Γ for a given x

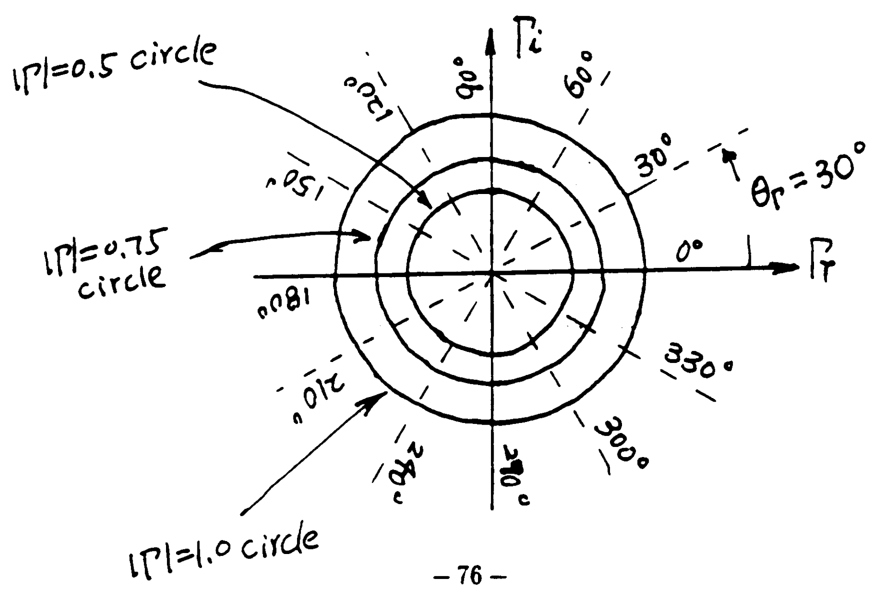




Families of constant-r circles + families pf constant-x circles $\Rightarrow$ Z Smith Chart

OBS:

- On a Smith Chart, $\bar{z}(z') \Leftrightarrow \Gamma(z')$
- For a lossless TL, $|\Gamma(z')| = |\Gamma_L| =$ constant $\rightarrow$ a constant $|\Gamma|$ circle: locus of Γ for a given $|\Gamma|$



DEF: electrical length $\triangleq \frac{z'}{\lambda}$

Ex: $\tilde{z}(z') = 1+j \Rightarrow \Gamma(z') = 0.45\, e^{j63.4^o} = 0.45\ \angle 63.4^o$

$\Gamma(z') = 0.45\, e^{j63.4^o} = 0.45\ \angle 63.4^o \Rightarrow \tilde{z}(z') = 1+j \Rightarrow Z(z') = Z_o\, \tilde{z}(z')$

Ex: Assume $Z_L = 3Z_o$. Find the impedance at $\lambda/24$ and $\lambda/12$ from the load.
Ans: $(1.94\text{-}j1.3)Z_o$, $(0.98 - j1.15)Z_o$

1) Special points on a Smith Chart

$Z_L = \infty \Leftrightarrow \tilde{z}_L = \infty$, $\Gamma_L = 1.0$ (open end)

$Z_L = 0 \Leftrightarrow \tilde{z}_L = 0$, $\Gamma_L = -1.0$ (shorted end)

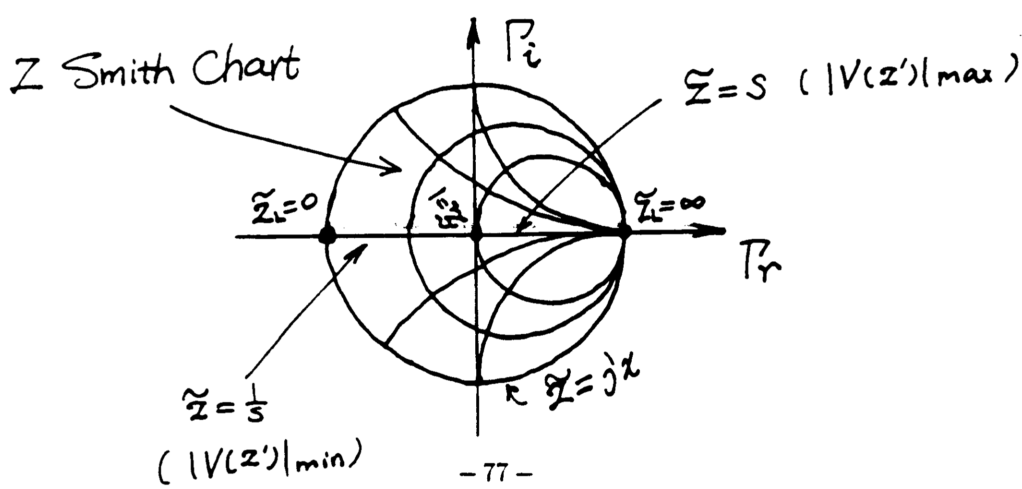
$Z_L = Z_o \Leftrightarrow \tilde{z}_L = 1$, $\Gamma_L = 0.0$ (matched load)

$Z(z') = jX \Leftrightarrow \tilde{z}(z') = jx$, $|\Gamma| = 1.0 \Rightarrow$ outer circle

$Z(z') = SZ_o \Leftrightarrow \tilde{z}(z') = S$, $\Gamma = \frac{S-1}{S+1} \Leftrightarrow$ real axis with $\Gamma_r \geq 0 \Leftrightarrow |V(z')|_{max}$ points

$Z(z') = \frac{Z_o}{S} \Rightarrow \tilde{z}(z') = \frac{1}{S}$, $\Gamma = \frac{S-1}{S+1} \Rightarrow$ real axis with $\Gamma_r \leq 0 \Leftrightarrow |V(z')|_{min}$ points

$\Delta\theta_\Gamma$: $0^o \rightarrow 360^o$ (one turn) $\Leftrightarrow \frac{\Delta z'}{\lambda}$: $0 \rightarrow 0.5$ (since $\theta_\Gamma = \theta_L - 2\beta z' = \theta_L - 2\pi\, \frac{2z'}{\lambda}$)



2) some examples:

P491-495, textbook

2. Construction of Admittance (Y) Smith Chart

DEF:

$$\tilde{y}(z') = g(z') + j\,b(z') \triangleq \frac{Y(z')}{Y_o} = \frac{G(z')}{Y_o} + j\frac{B(z')}{Y_o} = \frac{1}{\tilde{z}(z')}$$

$\tilde{y}(z')$ ↓ normalized admittance; $g(z')$ ↓ normalized conductance; $b(z')$ ↓ normalized susceptance

$$\tilde{z}(z') = \frac{1+\Gamma(z')}{1-\Gamma(z')} \Rightarrow \tilde{y}(z') = \frac{1-\Gamma(z')}{1+\Gamma(z')}$$

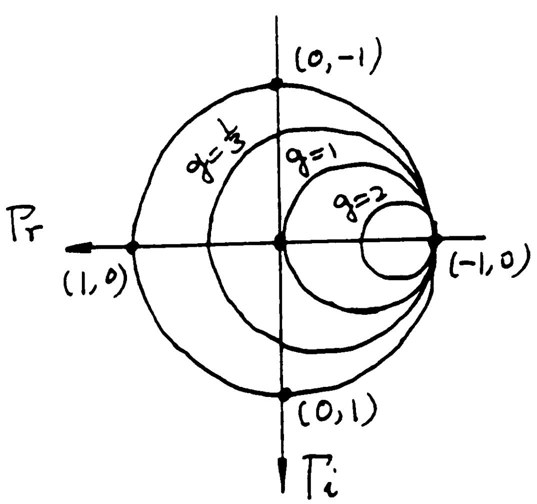
$$\Rightarrow \quad g + jb = \frac{1-\Gamma_r - j\,\Gamma_i}{1+\Gamma_r + j\,\Gamma_i}$$

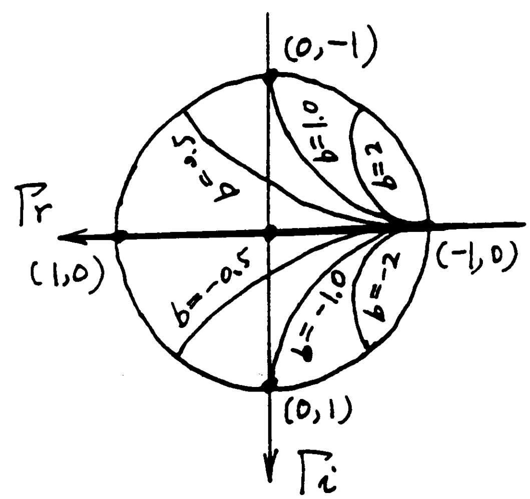
$\Rightarrow g = \mathrm{Re}\left(\frac{1-\Gamma_r-j\,\Gamma_i}{1+\Gamma_r+j\,\Gamma_i}\right) \Rightarrow (\Gamma_r+\frac{g}{1+g})^2+\Gamma_i^2 = (\frac{1}{1+g})^2 \rightarrow$ a circle on the $\Gamma_r-\Gamma_i$ plane

------ constant g circle: locus of Γ for a given g

$\Rightarrow b = \mathrm{Im}\left(\frac{1-\Gamma_r-j\,\Gamma_i}{1+\Gamma_r+j\,\Gamma_i}\right) \Rightarrow (\Gamma_r+1)^2+(\Gamma_i+\frac{1}{x})^2 = (\frac{1}{x})^2 \rightarrow$ a circle on the $\Gamma_r-\Gamma_i$ plane

------ constant b circle: locus of Γ for a given b

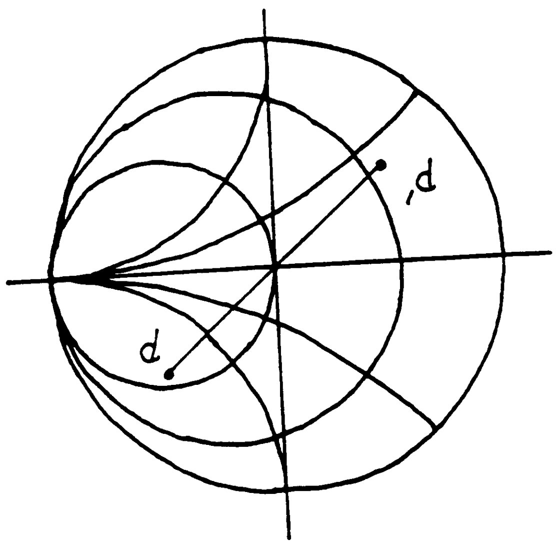




Families of constant-g circles + families of constant-b circles ⇒ Y Smith Chart

OBS: The Y Smith Chart is the same as the Z Smith Chart except a) $r \Rightarrow g$, $x \Rightarrow b$ b) directions of both coordinates, Γ_r and Γ_i, are reversed. In other words, we can use a Smith Chart either as Z Smith Chart or as Y Smith Chart (bearing in mind the above two differences).

3. Conversion between Z Smith Chart and (Y) Smith Chart



OBS: a point on a Z Smith Chart $\overset{\text{rotation of } 180^o}{\Leftrightarrow}$ a point on a Y Smith Chart. More specifically, by rotating a point on a Z Smith Chart by 180^o, you will obtain the corresponding point on the Y Smith Chart (henceforth the Smith Chart becomes Y Smith Chart). The other way around is also true.

Proof:

$$\tilde{z}(z') = \frac{1+\Gamma(z')}{1-\Gamma(z')} = \frac{1-\Gamma(z')\, e^{j180^o}}{1+\Gamma(z')\, e^{j180^o}}$$

compared with the formulas for admittance, $\tilde{y}(z') = \dfrac{1-\Gamma(z')}{1+\Gamma(z')}$

$\Rightarrow$ by rotating a Z point on Z S. C. $\rightarrow$ the corresponding point on Y S. C.

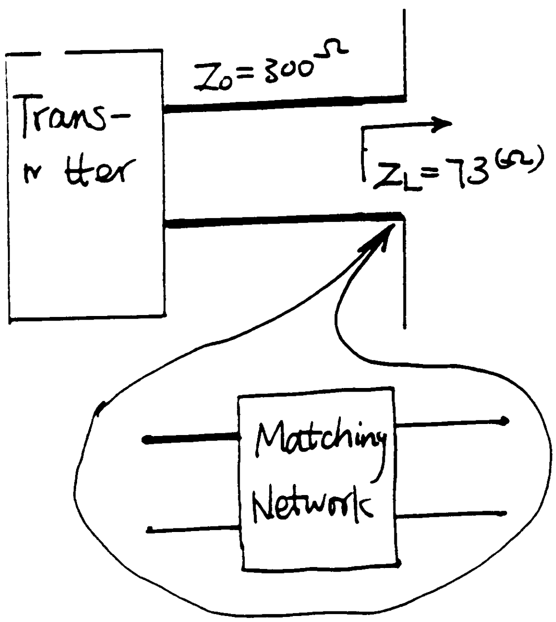
Note: The other way around can be proved in an entirely similar way.
Examples: (P499, textbook)

Ex1: Given $Z_L = 95 + j20$ (Ω), find Y_L. Ans: $Y_L = 10 - j2$ (s)

Ex2: Given $Y = 6 + j11$ (ms), find Z

Ex3: Given $Z_L = 50 + j50$ (Ω), $Z_o = 50\ \Omega$, $l = 0.089\lambda$, find Y_i

4. Transmission-Line impedance matching:



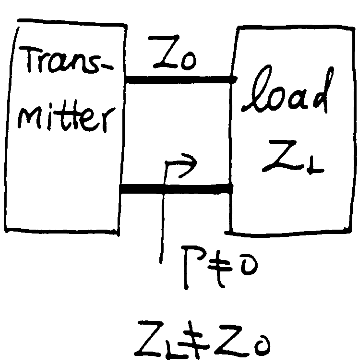
$$
\begin{aligned}
P_{av}(z) &= \tfrac{1}{2}\,\mathrm{Re}[V(z)I^*(z)] \\
&= \tfrac{1}{2}\,\mathrm{Re}[V^+(z)I^{+*}(z)]\,(1-|\Gamma|^2) \\
&= P_{in}\,(1-|\Gamma|^2) = P_{in} - P_{re}
\end{aligned}
$$

$$\Rightarrow P_{re} = P_{in}\,|\Gamma|^2$$

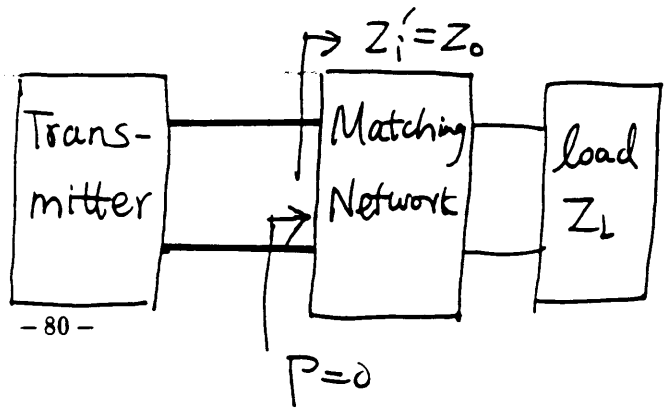
If $Z_L \neq Z_o \Rightarrow |\Gamma| \neq 0 \Rightarrow P_{re} \neq 0$

Solution: Insert a matching (or impedance transformer) network between the load (antenna in this case) and the TL (cable in this case) such that $|\Gamma| \to 0$ or $S \to 1.0$

Generally, we have

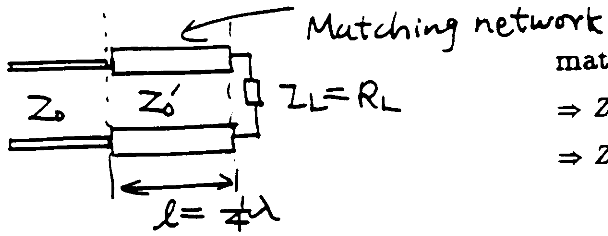


$\Longrightarrow$



1.) Impedance matching by quarter-wave TL

a) For $Z_L = R_L + j0 \neq Z_o$ $(X=0)$

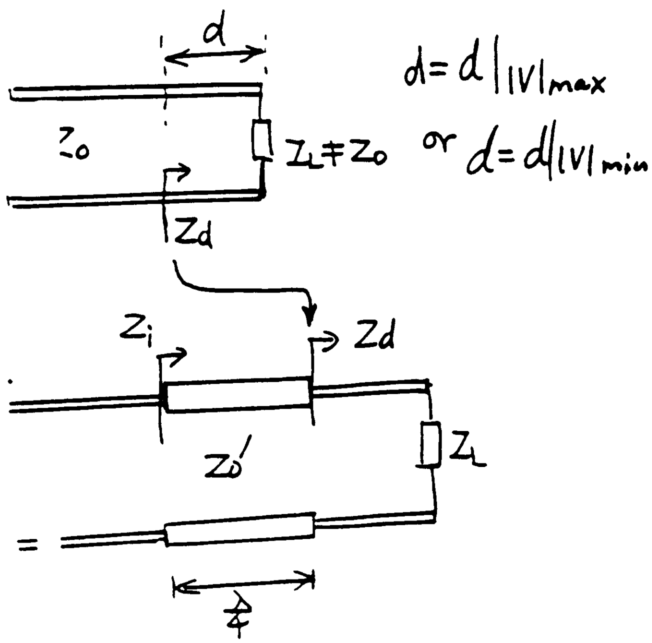


matching requires: $\Gamma_i = 0$

$\Rightarrow Z_i = Z_o$ or $Z_i = Z_o'^2/Z_L = Z_o$

$\Rightarrow Z_o' = \sqrt{Z_L Z_o} = \sqrt{R_L R_o}$

b) For an arbitrary load Z_L or Y_L



• If $d = d|_{|V|_{max}}$ is chosen, $\theta_L - 2\beta d = -2n\pi$.

$\Rightarrow d = \dfrac{\theta_L + 2n\pi}{2\beta}$ (≥ 0), n=0,1,2,3,

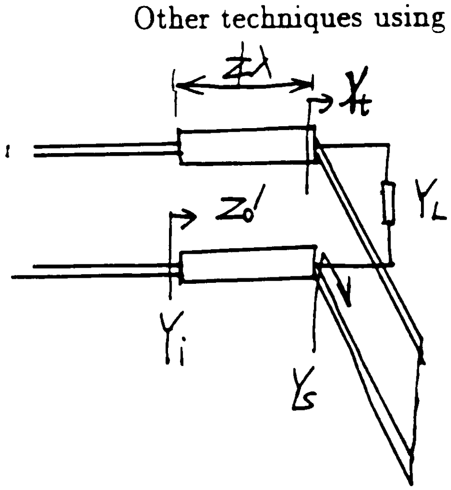
$\Rightarrow Z_d = SZ_o \Rightarrow Z_o' = \sqrt{S}\, Z_o$

• If $d = d|_{|V|_{min}}$ is chosen, $\theta_L - 2\beta d = -(2n+1)\pi$.

$\Rightarrow d = \dfrac{\theta_L + (2n+1)\pi}{2\beta}$ (≥ 0), n=0,1,2.3.

$\Rightarrow Z_d = \dfrac{Z_o}{S} \Rightarrow Z_o' = \dfrac{Z_o}{\sqrt{S}}$

Other techniques using $\lambda/4$ transformer

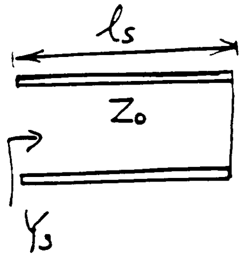


Let $jB_s + jB_L = 0 \Rightarrow B_s = -B_L$

$\Rightarrow Y_t = Y_L + Y_s = G_L + j0 \Rightarrow Z_t = \dfrac{1}{G_L} = R_L$

$\Rightarrow Z_o' = \sqrt{Z_o R_L}$

$Y_s = jB_s = -jB_L$



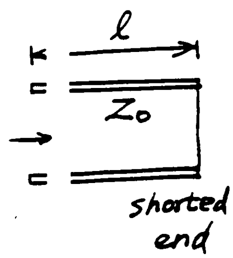
$$\Rightarrow -jB_L = -j\,Z_o \cot\beta l_s$$

$$\Rightarrow l_s = \frac{1}{\beta}\cot^{-1}(B_L Z_o)$$

Ex: use a $\lambda/4$ transformer to match Z_L=50+j50 (Ω) to Z_o=50 Ω.

2.) Single stub matching

Review



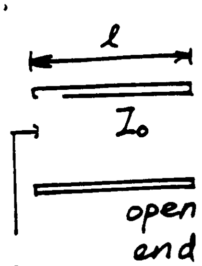
$Z_s = j Z_o \tan\beta\ell$ ($R_s=0$)

$\tilde{z}_s = j\tan\beta\ell$ ($r_s=0$)

$Y_s = -jY_o\cot\beta\ell$ ($G_s=0$)

$\tilde{y}_s = -j\cot\beta\ell$ ($g_s=0$)

- Any short-circuited (lossless) TL (stub) provides only reactance or susceptance, i.e. imaginary part of an impedance



$Z_s = -jZ_o\cot\beta\ell$ ($R_s=0$)

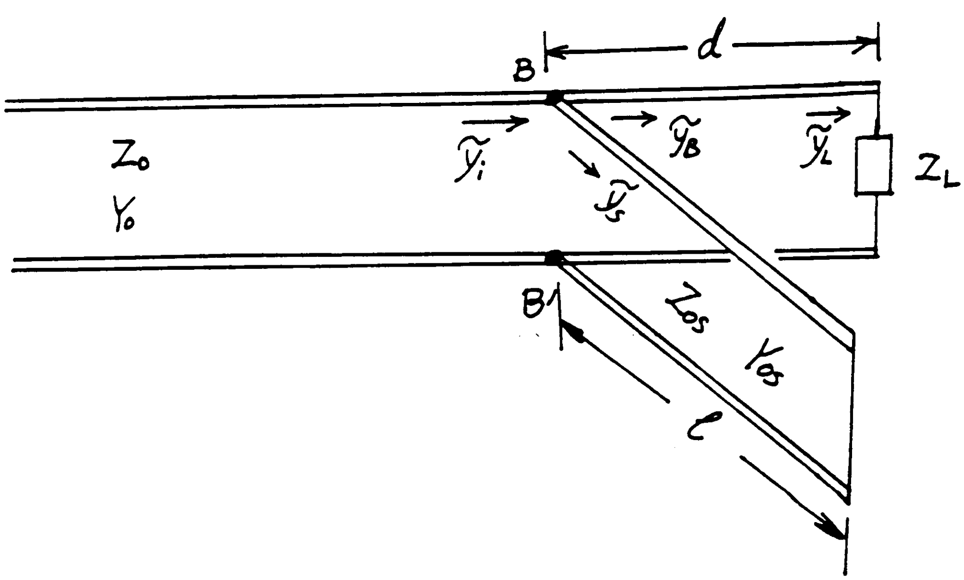
$\tilde{z}_s = -j\cot\beta\ell$ ($r_s=0$)

$Y_s = jZ_o\tan\beta\ell$ ($G_s=0$)

$\tilde{y}_s = j\tan\beta\ell$ ($g_s=0$)

- Any open-circuited (lossless) TL (stub) provides only reactance or susceptance, i.e. imaginary part of an impedance

Shunt-stub matching



Analysis

Analysis		Analysis On Y Smith Chart
a) $Y_s = jB_s$ and $Y_i = Y_s + Y_B$ (or $\tilde{y}_s = jb_s$ and $\tilde{y}_i = \tilde{y}_s + \tilde{y}_B$)	$\Rightarrow$	a) $\tilde{y}_B$ and $\tilde{y}_i$ must be on the same conductance circle: $g = \mathrm{Re}(\tilde{y}_i)$
b) Y_B is the input impedance of a section of TL terminated with Y_L	$\Rightarrow$	b) $\tilde{y}_B$ and $\tilde{y}_L$ must be on the same constant $\lvert\Gamma\rvert = \lvert\Gamma_L\rvert$ circle

a) + b)
$\Rightarrow$ $\tilde{y}_B$ must be at the intersecting point of the circle: $g = \mathrm{Re}(\tilde{y}_i)$ and the circle $|\Gamma| = |\Gamma_L|$
$\Rightarrow \tilde{y}_B$ obtained $\Rightarrow d$
$\Rightarrow \tilde{y}_s = \tilde{y}_i - \tilde{y}_B \;\Rightarrow Y_s = \tilde{y}_s\, Y_{os} \Rightarrow l$

Matching Procedure:

a) Normalized the relevant admittances

b) Enter the point representing the normalized load $\tilde{y}_L$ and draw $|\Gamma| = |\Gamma_L|$ circle

c) Identify $g = \mathrm{Re}(\tilde{y}_i)$ circle. In most case, $\tilde{y}_i = 1$ for matching.

d) Identify the intersecting points of the two circles: $g = \mathrm{Re}(\tilde{y}_i)$ and $|\Gamma| = |\Gamma_L|$. Usually, there are two intersecting points. $\tilde{y}_{B1}$ and $\tilde{y}_{B2}$, and both are possible solutions.

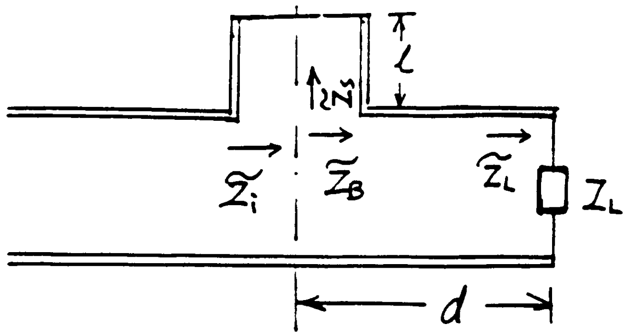
e) Move clockwise backward away from the load $\tilde{y}_L$ (toward the generator) to the two intersecting points ($\tilde{y}_{B1}$ and $\tilde{y}_{B2}$) and determine the distances (you go over), d_1 and d_2.

f) Compute $\tilde{y}_{s1} = \tilde{y}_i - \tilde{y}_{B1}$ and $\tilde{y}_{s2} = \tilde{y}_i - \tilde{y}_{B2}$, then $Y_{s1} = \tilde{y}_{s1}\, Y_{os}$ and $Y_{s2} = \tilde{y}_{s2}\, Y_{os}$.

g) Determine the stub lengths l_1 and l_2 either analytically or graphically using Smith Chart. Be aware that characteristic admittance Y_{os} of the stubs may be different from that of main TL.

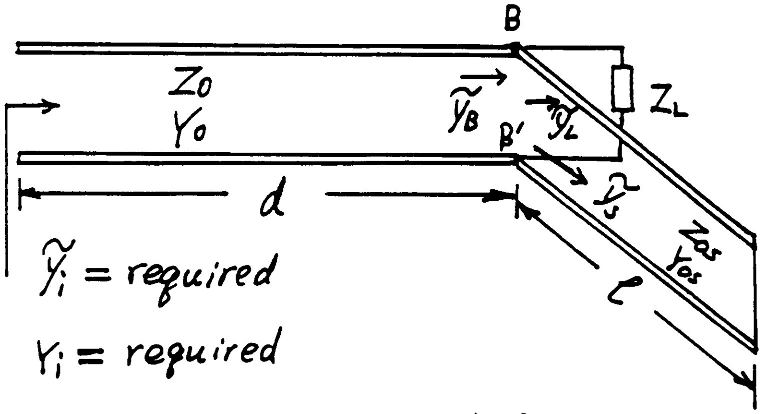
Ex: $Z_L = 35 - j47.5$ (Ω), $Z_o = 50\ \Omega$. $Z_i = 50\ \Omega$. Design a single stub network. (P503, textbook)

Series – stub matching



Analysis and matching are exactly the same as those for the stub placed in parallel with the main TL except that $Y \to Z$ and you need to work on Z Smith Chart.

Alternative shunt – stub matching



Analysis

Analysis		On Y Smith Chart
a) $Y_s = jB_s$ and $Y_B = Y_s + Y_L$ (or $\tilde{y}_s = jb_s$ and $\tilde{y}_B = \tilde{y}_s + \tilde{y}_L$)	$\Rightarrow$	a) $\tilde{y}_B$ and $\tilde{y}_L$ must be on the same conductance circle: $g = \text{Re}(\tilde{y}_L)$
b) Y_i is the input impedance of a section of TL terminated with Y_B	$\Rightarrow$	b) $\tilde{y}_B$ and $\tilde{y}_i$ must be on the same constant $\lvert\Gamma\rvert = \lvert\Gamma_i\rvert$ circle

a) + b)

$\Rightarrow$ $\tilde{y}_B$ must be at the intersecting point of the circle: $g = \text{Re}(\tilde{y}_i)$ and the circle $\lvert\Gamma\rvert = \lvert\Gamma_i\rvert$

$\Rightarrow \tilde{y}_B$ obtained $\Rightarrow d$

$\Rightarrow \tilde{y}_s = \tilde{y}_B - \tilde{y}_L \Rightarrow Y_s = \tilde{y}_s Y_o \Rightarrow l$

Matching Procedure:

a) Normalize the relevant admittances

b) Enter the point representing $\tilde{y}_i$ and draw $|\Gamma|=|\Gamma_i|$ circle

c) Identify $g=\mathrm{Re}(\tilde{y}_L)$ circle.

d) Identify the intersecting points of the two circles: $g=\mathrm{Re}(\tilde{y}_L)$ and $|\Gamma|=|\Gamma_i|$. Usually, there are two intersecting points, $\tilde{y}_{B1}$ and $\tilde{y}_{B2}$, and both are possible solutions.

e) Move clockwise backward away from $\tilde{y}_{B1}$and $\tilde{y}_{B2}$ (toward the generator) to the input point $\tilde{y}_i$ and determine the distances (you went through), d_1 and d_2.

f) Compute $\tilde{y}_{s1}=\tilde{y}_L - \tilde{y}_{B1}$ and $\tilde{y}_{s2}=\tilde{y}_L - \tilde{y}_{B2}$, then $Y_{s1} = \tilde{y}_{s1} Y_{os}$ and $Y_{s2} = \tilde{y}_{s2} Y_{os}$.

g) Determine the stub lengths l_1 and l_2 either analytically or graphically using Smith Chart. Be aware that characteristic admittance Y_{os} of the stub may be different from that of main TL.

Ex: $Z_L=35-j47.5$ (Ω), $Z_o=50\ \Omega$. $Z_i=50\ \Omega$. Design a single stub network. (P503, textbook)

Ex: $\tilde{y}_i = 0.4 - j1.05$, $Y_i = 0.002$ (S), $Y_L=0.002$ (S). Design the single-stub matching network.

SUMMARY

Transmission lines are used for efficient point-to-point transmission of energy and information. We have devoted this chapter to studying the method of analysis and the behavior of the transverse electromagnetic (TEM) waves guided by transmission lines. In particular, we

- discussed the characteristics of the three most common types of transmission lines (the parallel-plate line, the two-wire line, and the coaxial line),
- derived the general transmission-line equations, which combined to yield one-dimensional, second-order, ordinary differential equations under time-harmonic conditions,
- examined the wave characteristics on infinite transmission lines,
- determined the propagation constant, phase velocity, and characteristic impedances of lossless lines and distortionless lines,
- expressed the attenuation constant of a traveling wave on a lossy line in terms of power relations,
- analyzed the wave characteristics on finite transmission lines in terms of propagation constant, input impedance, reflection coefficient, and SWR,
- examined the properties of open-circuited and short-circuited lines,
- solved transmission-line circuit problems,
- studied the principle of construction and the applications of the Smith chart, and
- explained the single-stub method for impedance matching.

Chapter 4

Waveguides and Cavity Resonators

(To be added)