Prob 2.7

(a)

$$\begin{bmatrix} A_x \\ A_y \\ A_y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho\sin\phi \\ \rho\cos\phi \\ -2z \end{bmatrix}$$

$$A_x = \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0$$

$$A_y = \rho\sin^2\phi + \rho\cos^2\phi = \rho = \sqrt{x^2+y^2}$$

$$A_z = -2z$$

Hence,

$$\boldsymbol{A} = \underline{\sqrt{x^2+y^2}\,\boldsymbol{a}_y - 2z\boldsymbol{a}_z}$$

(b)

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} 4r\cos\phi \\ r \\ 0 \end{bmatrix}$$

$$B_x = 4r\sin\theta\cos^2\phi + r\cos\theta\cos\phi$$

$$B_y = 4r\sin\theta\sin\phi\cos\phi + r\cos\theta\sin\phi$$

$$B_z = 4r\cos\theta\cos\phi - r\sin\theta$$

But $r = \sqrt{x^2+y^2+z^2}$, $\quad \sin\theta = \dfrac{\sqrt{x^2+y^2}}{r}$, $\quad \cos\theta = \dfrac{z}{r}$

$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

$$B_x = 4\sqrt{x^2+y^2}\,\frac{x^2}{x^2+y^2} + \frac{zx}{\sqrt{x^2+y^2}}$$

$$B_y = 4\sqrt{x^2+y^2}\,\frac{xy}{x^2+y^2} + \frac{zy}{\sqrt{x^2+y^2}}$$

$$B_z = 4z\frac{x}{\sqrt{x^2+y^2}} - \sqrt{x^2+y^2}$$

$$\mathbf{B} = \frac{1}{\sqrt{x^2+y^2}}\underline{\underline{\left[x(4x+z)\mathbf{a}_x + y(4x+z)\mathbf{a}_y + (4xz-x^2-y^2)\mathbf{a}_z\right]}}$$

Prob 3.16

$$grad\ U = \frac{\partial U}{\partial x}\bar{a}_x + \frac{\partial U}{\partial y}\bar{a}_y + \frac{\partial U}{\partial z}\bar{a}_z$$

$$= (z - 2xy)\bar{a}_x + (2yz^2 - x^2)\bar{a}_y + (x + 2y^2z)\bar{a}_z$$

$$Div\ gradU = \nabla \bullet \nabla U = \frac{\partial}{\partial x}(z - 2xy) + \frac{\partial}{\partial y}(2yz^2 - x^2) + \frac{\partial}{\partial z}(x + 2y^2z)$$

$$= -2y + 2z^2 + 2y^2$$

$$= \underline{\underline{2(z^2 + y^2 - y)}}$$

Prob. 3.17

(a)

$$(\bar{\nabla}\bullet\bar{r})\bar{T} = 3\bar{T} = \underline{\underline{6yz\,\bar{a}_y + 3xy^2\,\bar{a}_y + 3x^2yz\,\bar{a}_z}}$$

(b)

$$x\frac{\partial \bar{T}}{\partial x} + y\frac{\partial \bar{T}}{\partial y} + z\frac{\partial \bar{T}}{\partial z} = x\ (y^2\bar{a}_y + 2xyz\bar{a}_z) + y(2z\bar{a}_x + 2xy\bar{a}_y + x^2z\bar{a}_z)$$

$$+ z(2y\bar{a}_x + x^2y\bar{a}_z)$$

$$= \underline{\underline{4yz\bar{a}_x + 3xy^2\bar{a}_y + 4x^2yz\bar{a}_z}}$$

(c)

$$\nabla \bullet \bar{r}(\bar{r}\bullet\bar{T}) = 3\,(2xyz + xy^3 + x^2yz^2)$$

$$= \underline{\underline{6xyz + 3xy^3 + 3x^2yz^2}}$$

(d)

$$(\bar{r}\bullet\nabla)\bar{r}^2 = (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})(x^2 + y^2 + z^2)$$

$$= x(2x) + y(2y) + z(2z)$$

$$= \underline{\underline{2(x^2 + y^2 + z^2) = 2r^2}}$$

Prob. 3.18

We convert A to cylindrical coordinates; only the ρ-component is needed.

$$A_\rho = A_x\cos\phi + A_y\sin\phi = 2x\cos\phi - z^2\sin\phi$$

But $x = \rho\cos\phi$,

$$A_\rho = 2\rho\cos^2\phi - z^2\sin\phi$$

$$\Psi = \int_S A\cdot dS = \iint A_\rho \rho\, d\phi\, dz = \iint\left[2\rho^2\cos^2\phi - \rho z^2\sin\phi\right]d\phi\, dz$$

$$\int_v \nabla \bullet \bar{F}\, dv = \iiint (3\rho \sin\phi - \frac{z}{\rho}\sin\phi + \rho)\, \rho\, d\phi\, d\rho\, dz$$

$$= 0 + 0 + \int_0^5 dz \int_0^{2\pi} d\phi \int_2^3 \rho^2 d\rho$$

$$= \underline{\underline{\frac{190\,\pi}{3}}}$$

Prob. 3.26

(a) $$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2 & -xz \end{vmatrix} = z\mathbf{a}_y - x\mathbf{a}_z$$

(b)
$$\nabla \times \mathbf{B} = \left(\frac{1}{\rho} 2\rho z 2 \sin\phi \cos\phi - 0\right)\mathbf{a}_\rho + (2\rho z - 2z\sin^2\phi)\mathbf{a}_\phi + \frac{1}{\rho}\left(2\rho \sin^2\phi - 0\right)\mathbf{a}$$

$$= 4z\sin\phi\cos\phi\mathbf{a}_\rho + 2(\rho z - z\sin^2\phi)\mathbf{a}_\phi + 2\sin^2\phi\mathbf{a}_z$$

$$= \underline{\underline{2z\sin 2\phi\cos\phi\mathbf{a}_\rho + 2z(\rho - \sin^2\phi)\mathbf{a}_\phi + 2\sin^2\phi\mathbf{a}_z}}$$

(c)
$$\nabla \times \mathbf{C} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(r\cos^2\theta\sin\theta\right]\mathbf{a}_r - \frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\cos^2\theta\right]\mathbf{a}_\theta$$

$$= \frac{r}{r\sin\theta}\left[(2\cos\theta)(-\sin\theta)\sin\theta + \cos\theta(\cos^2\theta)\right]\mathbf{a}_r - \frac{\cos^2\theta}{r}(2r)\mathbf{a}_\theta$$

$$= \underline{\underline{\frac{(\cos^3\theta - 2\sin^2\theta\cos\theta)}{\sin\theta}\mathbf{a}_r - 2\cos^2\theta\mathbf{a}_\theta}}$$

Prob. 3.27

(a)

$$\nabla \times \bar{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & -2xz \end{vmatrix} = \underline{\underline{-y^2\bar{a}_x + 2z\bar{a}_y - x^2\bar{a}_z}}.$$

$$\nabla \bullet \nabla \times \bar{A} = \underline{\underline{0}}$$

(b)

$$\nabla \times \bar{A} = (\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z})\bar{a}_\rho + (\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho})\bar{a}_\phi + \frac{1}{\rho}(\frac{\partial(\rho A_\rho)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi})\bar{a}_z$$

$$= (0-0)\bar{a}_\rho + (\rho^2 - 3z^2)\bar{a}_\phi + \frac{1}{\rho}(4\rho^3 - 0)\bar{a}_z$$

$$= \underline{\underline{(\rho^2 - 3z^2)\bar{a}_\phi + 4\rho^2\bar{a}_z}}$$

$$\nabla \bullet \nabla \times \bar{A} = \underline{\underline{0}}$$

(c)

$$\nabla \times A = \frac{1}{r\sin\theta}\left[0 - \frac{\sin\phi}{r^2}\right]a_r + \frac{1}{r}\left[\frac{-1}{\sin\theta}\frac{\cos\phi}{r^2} - 0\right]a_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(\frac{\cos\phi)}{r} - 0\right]a_\phi$$

$$= -\frac{\sin\phi}{r^3\sin\theta}a_r + \frac{\cos\phi}{r^3\sin\theta}a_\theta + \frac{\cos\phi}{r^3}a_\phi$$

$$\nabla \bullet \nabla \times A = \frac{-\sin\phi}{r^4\sin\theta} + 0 + \frac{\sin\phi}{r^4\sin\theta} = 0$$

$$\nabla \bullet \nabla \times \bar{A} = \underline{\underline{0}}$$

Prob. 3.28

$$\nabla \ln\rho = (\frac{\partial}{\partial x}\ln\rho)\bar{a}_x + (\frac{\partial}{\partial y}\ln\rho)\bar{a}_y + (\frac{\partial}{\partial z}\ln\rho)\bar{a}_z$$

$$= \frac{x}{\rho^2}\bar{a}_x + \frac{y}{\rho^2}\bar{a}_y$$

$$\nabla \times \phi\bar{a}_z = \nabla \times \tan^{-1}\frac{y}{x}\bar{a}_z$$

$$= \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial \mathbf{x}} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \tan^{-1}\frac{y}{x} \end{vmatrix}$$

$$= \frac{x}{x^2+y^2}\bar{a}_x + \frac{y}{x^2+y^2}\bar{a}_y$$

$$= \frac{x}{\rho^2}\bar{a}_x + \frac{y}{\rho^2}\bar{a}_y$$

$$= \underline{\underline{\nabla \ln\rho, \quad \text{as expected !}}}$$

$$\nabla x\nabla(\nabla\cdot\mathbf{A})=\begin{vmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 16e^{-y} & -16xe^{-y} & 0\end{vmatrix}=(-16e^{-y}+16e^{-y})\mathbf{a}_z=\underline{\underline{0}}$$

Should be expected since $\nabla x\nabla V=0$.

Prob. 3.37

(a) $\underline{\underline{\nabla V=-\dfrac{\sin\theta\cos\phi}{r^2}\mathbf{a}_r+\dfrac{\cos\theta\cos\phi}{r^2}\mathbf{a}_\theta-\dfrac{\sin\phi}{r^2}\mathbf{a}_\phi}}$

(b) $\nabla x\nabla V=\underline{\underline{0}}$

(c)

$$\nabla\cdot\nabla V=\nabla^2V=\frac{1}{r^2}\frac{\partial}{\partial r}(-\sin\theta\cos\phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\cos\theta\cos\phi}{r})+\frac{1}{r^2\sin^2\theta}(-\frac{\sin\theta\cos\phi}{r})$$

$$=0+\frac{\cos\phi}{r^3\sin\theta}(1-2\sin^2\theta)-\frac{\cos\phi}{r^3\sin\theta}$$

$$=\underline{\underline{-\frac{2\sin\theta\cos\phi}{r^3}}}$$

Prob.3.38

$$\bar{Q}=\frac{r}{r\sin\theta}r\sin\theta[(\cos\phi-\sin\phi)\bar{a}_x+(\cos\phi+\sin\phi)\bar{a}_y]$$

$$=r(\cos\phi-\sin\phi)\bar{a}_x+r(\cos\phi+\sin\phi)\bar{a}_y$$

$$\begin{bmatrix}Q_r\\Q_\theta\\Q_\phi\end{bmatrix}=\begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}Q_x\\Q_y\\Q_z\end{bmatrix}$$

(a) $\bar{Q}=r\sin\theta\bar{a}_r+r\cos\theta\bar{a}_\theta+r\bar{a}_\phi$

$$d\bar{l} = \rho d\phi \bar{a}_\phi, \quad \rho = r\sin 30° = 2(\frac{1}{2}) = 1$$

$$z = r\cos 30° = \sqrt{3}$$

$$Q_\phi = r = \sqrt{\rho^2 + z^2}$$

$$\oint \bar{Q} \bullet d\bar{l} = \int_0^{2\pi} \sqrt{\rho^2 + z^2}\, \rho d\phi = 2(1)(2\pi) = \underline{\underline{4\pi}}$$

(b)

$$\nabla \times \bar{Q} = \cot\theta\, \bar{a}_r - 2\bar{a}_\theta + \cos\theta\, \bar{a}_\phi$$

For S_1, $d\bar{S} = r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r$

$$\int_{S_1} (\nabla \times \bar{Q}) \bullet d\bar{S} = \int r^2 \sin\theta \cot\theta\, d\theta\, d\phi \Big|_{r=2}$$

$$= 4\int_0^{2\pi} d\phi \int_0^{30°} \cos\theta\, d\theta = \underline{\underline{4\pi}}$$

(c)

For S_2, $d\bar{S} = r\sin\theta\, d\theta\, dr\, \bar{a}_\theta$

$$\int_{S_2} (\nabla \times \bar{Q}) \bullet d\bar{S} = -2\int r\sin\theta\, d\phi\, dr \Big|_{\theta=30°}$$

$$= -2\sin 30 \int_0^2 r dr \int_0^{2\pi} d\phi$$

$$= \underline{\underline{-4\pi}}$$

(d)

For S_1, $d\bar{S} = r^2 \sin\theta\, d\phi\, d\theta\, \bar{a}_r$

$$\int_{S_1} \bar{Q} \bullet d\bar{S} = r^3 \int \sin^2\theta\, d\theta\, d\phi \Big|_{r=2}$$

$$= 8\int_0^{2\pi} d\phi \int_0^{30°} \sin^2\theta\, d\theta$$

$$= 4\pi[\frac{\pi}{3} - \frac{\sqrt{3}}{2}] = \underline{\underline{2.2767}}$$

(e)

For S_2, $d\bar{S} = r\sin\theta\, d\phi\, dr\, \bar{a}_\theta$

$$\int_{S_2} \bar{Q} \bullet d\bar{S} = \int r^2 \sin\theta \cos\theta\, d\phi\, dr \Big|_{\theta=30°}$$

$$= \frac{4\pi\sqrt{3}}{3} = \underline{\underline{7.2552}}$$

(f)

$$\nabla \bullet \bar{Q} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^3 \sin\theta) + \frac{r}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta) + 0$$

$$= 2\sin\theta + \cos\theta\cot\theta$$

$$\int \nabla \bullet \bar{Q}dV = \int (2\sin\theta + \cos\theta\cot\theta)r^2 \sin\theta d\theta d\phi dr$$

$$= \frac{r^3}{3}\Big|_0^2 (2\pi)\int_0^{30}(1+\sin^2\theta)d\theta$$

$$= \frac{4\pi}{3}(\pi - \frac{\sqrt{3}}{2}) = \underline{\underline{9.532}}$$

Check: $\int \nabla \bullet \bar{Q}dV = (\int_{S_1} + \int_{S_2} \bar{Q} \bullet d\bar{S}$

$$= 4\pi[\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3}]$$

$$= \frac{4\pi}{3}[\pi - \frac{\sqrt{3}}{2}] \quad \text{(It checks!)}$$

Prob. 3.39

Since $\bar{u} = \bar{\omega} \times \bar{r}$, $\nabla \times \bar{u} = \nabla \times (\bar{\omega} \times \bar{r})$. From Appendix A.10,

$$\nabla \times (\bar{A} \times \bar{B}) = \bar{A}(\nabla \bullet \bar{B}) - \bar{B}(\nabla \bullet \bar{A}) + (B \bullet \nabla)\bar{A} - (\bar{A} \bullet \nabla)\bar{B}$$

1.24) $A = 2x\,\hat{a}_x + y\,\hat{a}_y - z^2\,\hat{a}_z$

$B = 3x^2\,\hat{a}_x + 6\,\hat{a}_y + \hat{a}_z$ at point $(1, 2, -4)$

a) $A \cdot B = (2, 2, -16) \cdot (3, 6, 1) = (6 + 12 - 16) = 2$

b) $\varphi_{AB} = \cos^{-1} \dfrac{A \cdot B}{|A||B|} = \cos^{-1} \dfrac{2}{\sqrt{264.46}} = \cos^{-1} \dfrac{2}{110.19} = 88.95^\circ$

c) vector component of A on B $= \dfrac{(A \cdot B)\, B}{|B|^2} = \dfrac{2\,(3\hat{a}_x + 6\hat{a}_y + \hat{a}_z)}{46}$

$= 0.13\,\hat{a}_x + 0.26\,\hat{a}_y + 0.043\,\hat{a}_z$

2.25) $A = (2z - \sin\varphi)\,\hat{a}_\rho + (4\rho + 2\cos\varphi)\,\hat{a}_\varphi - 3\rho z\,\hat{a}_z$

$B = \rho\cos\varphi\,\hat{a}_\rho + \sin\varphi\,\hat{a}_\varphi + \hat{a}_z$

a) $\cos\varphi_{AB} = \left.\dfrac{A \cdot B}{|A|\,|B|}\right|_{point = (1, 60^\circ, 1)} =$

$A\Big|_{point} = \left(-2\sqrt{\tfrac{3}{2}}\right)\hat{a}_\rho + \left(4 + 2\left(\tfrac{1}{2}\right)\right)\hat{a}_\varphi + 3\hat{a}_z$

$A = -2.066\,\hat{a}_\rho + 5\,\hat{a}_\varphi + 3\hat{a}_z \Rightarrow |A| = 6.5$

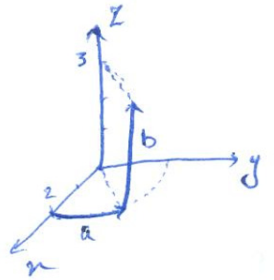
$B\Big|_{point} = \tfrac{1}{2}\,\hat{a}_\rho + \sqrt{\tfrac{3}{2}}\,\hat{a}_\varphi + \hat{a}_z \Rightarrow |B| = \sqrt{2} = 1.41$

$\Rightarrow \cos\varphi_{AB} = \dfrac{5.9}{6.5 \times 1.41} = 49.92$

b) $A\Big|_{point\,(1, 90^\circ, 0)} = -\hat{a}_\rho + 4\,\hat{a}_\varphi$, $B\Big|_{point\,(1, 90^\circ, 0)} = \hat{a}_\rho + \hat{a}_z$

$\vec{A} \times \vec{B} = \begin{vmatrix} a_\rho & a_\varphi & a_z \\ -1 & 4 & 0 \\ \ldots \end{vmatrix} = 4\,\hat{a}_\rho + \hat{a}_\varphi - \hat{a}_z \Rightarrow a_n = \dfrac{1}{3\sqrt{2}}\left(4\hat{a}_\rho + \hat{a}_\varphi - a_z\right)$

3.9)



$$\int_L F\cdot dl = \int_a F\cdot dl + \int_b F\cdot dl =$$

$$= \int_a F\cdot d\varphi + \int_b F\cdot dz = \int_0^{\pi/4} 0\, d\varphi + \int_0^3 \cos\tfrac{\pi}{4}\, dz$$

$$= 0 + \int_0^3 \tfrac{\sqrt{2}}{2}\, dz = \tfrac{\sqrt{2}}{2}\,(z)\Big|_0^3 = \frac{3\sqrt{2}}{2}$$

3.12)

$$\nabla U = \frac{dU}{dx}\hat{a_x} + \frac{dU}{dy}\hat{a_y} + \frac{dU}{dz}\hat{a_z}$$

a) $V_1 = 6xy - 2xz + z \Rightarrow \nabla V_1 = (6y - 2z)\hat{a_x} + 6x\hat{a_y} + (-2z+1)\hat{a_z}$

b) $V_2 = 10\rho\cos\varphi - \rho z$ $\qquad \nabla V_2 = \frac{dV_2}{d\rho}\hat{a_\rho} + \frac{1}{\rho}\frac{dV_2}{d\varphi}\hat{a_\varphi} + \frac{dV_2}{dz}\hat{a_z}$

$$\nabla V_2 = (10\cos\varphi - z)\hat{a_\rho} - 10\sin\varphi\,\hat{a_\varphi} - \rho\hat{a_z}$$

c) $V_3 = \frac{2}{r}\cos\varphi$ $\qquad \nabla V_3 = \frac{dV_3}{dr}\hat{a_r} + \frac{1}{r}\frac{dV_3}{d\theta}\hat{a_\theta} + \frac{1}{r\sin}\frac{dV_3}{d\varphi}\hat{a_\varphi}$

$$\nabla V_3 = \frac{-2}{r^2}\cos\varphi\,\hat{a_r} - \frac{2\sin\varphi}{r^2\sin\theta}\hat{a_\varphi}$$