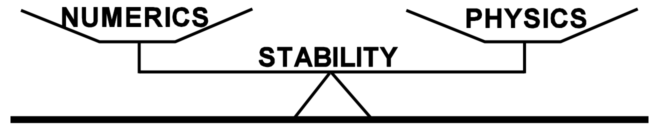
Chapter 4

Numerical Stability



4.1 Introduction

This chapter completes the mathematics required for analyzing numerical procedures. A method to be of practical use must be both accurate and stable. Accuracy was dealt with in the previous chapter. Numerical stability implies that small errors do not grow fast enough to render the solution to be meaningless. Unstable numerical procedures can amplify small solution errors exponentially fast. Several examples using the von Neumann stability analysis approach will be given.

4.2 Fundamental Numerical Requirements

A finite difference equation or set of finite difference equations should satisfy the following two fundamental requirements

(1) Consistency

Difference Equations $\rightarrow$ Differential Equations as $\Delta x, \Delta y, \cdots, \Delta t \rightarrow 0$ *in any manner*

(2) Stability

No component of the solution grows exponentially fast.

Lax Equivalence Theorem (proved for *Linear Equations only*)

Consistency + Stability $\Leftrightarrow$ Convergence

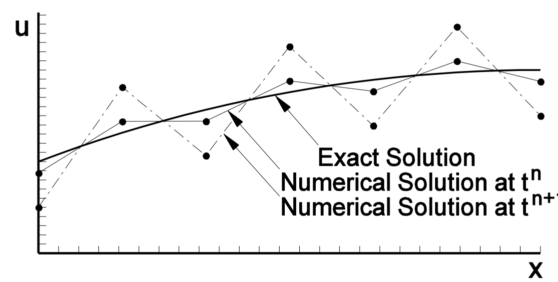


Figure 4.1 Numerical error growth

4.3 Numerical Stability

We will develop algorithms in later chapters to solve hyperbolic equations of the general form

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = Q$$

In Chapter 2 we analyzed this equation by linearizing it to form (See Section 2.6)

$$\frac{\partial U}{\partial t} + A_0 \frac{\partial U}{\partial x} + B_0 \frac{\partial U}{\partial y} + C_0 \frac{\partial U}{\partial z} = Q_0$$

This equation has an exact solution.

$$U(x,y,z,t\ ) = \sum_j \underbrace{e^{-i(t-t_0)(k_{x_j}A_0+k_{y_j}B_0+k_{z_j}C_0)}}_{exact\ amplification\ factor\ G_{exact}} c_j e^{i(k_{x_j}x+k_{y_j}y+k_{z_j}z)} + (t-t_0)Q_0$$

No component of the solution will have exponential growth if the matrix $M = k_{x_j}A_0 + k_{y_j}B_0 + k_{z_j}C_0$ has only real eigenvalues for any set of real k_{x_j}, k_{y_j} and k_{z_j} (see hyperbolic requirement Section 2.8). We now must check that the difference equation used to approximate the differential equation, representing the algorithm, also has no exponential growth components. Remember, the difference equation is equal to the sum of the original differential equation plus the truncation error, a combined differential equation of infinite degree (See Sections 3.5.1 and 3.5.3). As for the exact solution, linear growth of the numerical solution is allowed. The solution of the original differential equation is sought over the time interval $t_0 \le t \le T$, given an initial value at t_0. The norm of the numerical amplification must therefore satisfy for one time step advance

$$\|G\| \le 1 + \alpha\Delta t, \quad where \quad \Delta t = \frac{T-t_0}{N}$$

and α is a constant that the norm of G must satisfy during any time step and N is the number of time steps needed to cover the time interval. The bound, B, on G after n time steps is given by

$$\|G\|^n \le (1+\alpha\Delta t)^n \le e^{\alpha n\Delta t} \le e^{\alpha N\Delta t} = e^{\alpha(T-t_0)} = B$$

Thus $\|G\|^n$ is bounded, as $N \to \infty$ and $\Delta t \to 0$, and the numerical algorithm is stable if it can be shown to satisfy $\|G\| \le 1+\alpha\Delta t$. We examine a few algorithms for stability using the analysis of John von Neumann below.

4.4 von Neumann Stability Analysis

In addition to approximating the solution to the set of governing partial differential equations, the numerical solution carries along with it parasitic components. These unwanted components, though not necessarily present initially, can be introduced through truncation and roundoff error. A stable numerical approximation to the governing equations will not allow the parasitic components to grow without unbounded exponentially fast. The following von Neumann analysis can be used to determine the numerical stability of a solution algorithm.

1. Apply the algorithm to the linearized partial differential system of equations.

2. Determine the growth of an arbitrary Fourier component, $c_j e^{i(k_{x_j}x+k_{y_j}y+k_{z_j}z)}$, of the numerical solution.

Examples of the von Neumann stability analysis follow.

4.4.1 Explicit in Time - Central in Space Difference Method

Stability analysis for the explicit difference method given by Equation (3.2) applied to solve Equation (3.1), - *central in space explicit in time* -.

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$$
$$\text{at } t = t_0\,, \quad u_i^0 = u(x_i, t_0) = g(x_i) = \sum_j c_j e^{ik_j x_i}\,,$$

where we have expanded the initial condition function into a Fourier Series. Assume all components are present (i.e., $c_j \neq 0$ *for any* j). If not present initially, roundoff error will surely introduce them into the numerical solution eventually. By linearity, each component grows or decays independently of the others. Thus, we need consider only a single arbitrary component of the solution to test for the numerical stability for all. Let our test component be given as follows.

$$v_i^0 = c_j^0 e^{ik_j x_i}$$

On substitution of this component into the difference equation above, we obtain

$$v_i^{n+1} = v_i^n - \frac{c\Delta t}{2\Delta x}\left(c_j^n e^{ik_j(x_i+\Delta x)} - c_j^n e^{ik_j(x_i-\Delta x)}\right)$$
$$\text{or } v_i^{n+1} = v_i^n - \frac{c\Delta t}{2\Delta x}\left(e^{ik_j\Delta x} - e^{-ik_j\Delta x}\right)v_i^n = \underbrace{\left\{1 - \frac{c\Delta t}{2\Delta x} 2i\sin(k_j\Delta x)\right\}}_{G_j} v_i^n$$

where G_j is the numerical amplification factor for one time step.

$$v_i^{n+1} = G_j v_i^n = \left(G_j\right)^n v_i^0\,, \text{ for } n \text{ time steps}$$

The exact amplification factor for this component for one time step is

$$G_{j_{exact}} = e^{-i\Delta t k_j c}\,, \text{ hence, the magnitude } \left\|G_{j_{exact}}\right\| = 1$$

In comparison, the numerical amplification factor is, with $c\Delta t \approx \Delta x$,

$$G_j = 1 - \frac{c\Delta t}{\Delta x} i\sin(k_j\Delta x) \text{ and } \left\|G_j\right\|^2 = 1 + \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2(k_j\Delta x) \approx 1 + \alpha$$

Thus, for all non-zero wave numbers k_j the corresponding Fourier components will amplify exponentially fast.

$$\left\|v_i^{n+1}\right\| = \left\|\left(G_j\right)^n\right\| \left\|v_i^0\right\| = \left\|G_j\right\|^n \left\|v_i^0\right\| \to \infty \quad (\text{Note: } (1+\alpha)^{n/2} \simeq e^{\alpha n/2})$$

$\Rightarrow$ The finite difference method of Equation (3.2) is unstable.

4.4.2 Implicit in Time - Central in Space Difference Method

Stability analysis for the implicit difference method given by Equation (3.3) applied to Equation (3.1)), - *central in space implicit in time* -.

$$u_i^{n+1} + \frac{c\Delta t}{4\Delta x}(u_{i+1}^{n+1} - u_{i-1}^{n+1}) = u_i^n - \frac{c\Delta t}{4\Delta x}(u_{i+1}^n - u_{i-1}^n)$$

By substitution of an arbitrary test component we obtain

$$v_i^{n+1} + \frac{c\Delta t}{4\Delta x} 2i\sin(k_j\Delta x)v_i^{n+1} = v_i^n - \frac{c\Delta t}{4\Delta x} 2i\sin(k_j\Delta x)v_i^n$$

$$\text{or } v_i^{n+1} = \frac{1 - \frac{c\Delta t}{2\Delta x} i\sin(k_j\Delta x)}{1 + \frac{c\Delta t}{2\Delta x} i\sin(k_j\Delta x)} v_i^n = G_j v_i^n$$

$$\left\|G_j\right\|^2 = \frac{1 + \left(\frac{c\Delta t}{2\Delta x}\right)^2 \sin^2(k_j\Delta x)}{1 + \left(\frac{c\Delta t}{2\Delta x}\right)^2 \sin^2(k_j\Delta x)} = 1$$

$$\Rightarrow \left\|v_i^{n+1}\right\| \text{ remains bounded as } n \to \infty$$

$\Rightarrow$ The finite difference method of Equation (1.3) is stable.

This method is said to be neutrally stable because $\left\|G_j\right\| = 1$ exactly.

4.4.3 Explicit in Time - Backward in Space Difference Method

Stability analysis for a *backward in space and explicit in time* method for solving Equation (3.3)

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{\Delta x}(u_i^n - u_{i-1}^n)$$

By substitution of test component $v_i = c_j e^{ik_j x_i}$

$$v_i^{n+1} = v_i^n - \frac{c\Delta t}{\Delta x}\left(1-e^{-ik_j\Delta x}\right)v_i^n = \underbrace{\left\{1-\frac{c\Delta t}{\Delta x}\left[1-\cos(k_j\Delta x)+i\sin(k_j\Delta x)\right]\right\}}_{G_j} v_i^n$$

$$G_j = \underbrace{1-\frac{c\Delta t}{\Delta x}\left[1-\cos(k_j\Delta x)\right]}_{\text{Real Part}} - \underbrace{\frac{c\Delta t}{\Delta x} i\sin(k_j\Delta x)}_{\text{Imaginary Part}}$$

$$\left\|G_j\right\|^2 = \left(1-\frac{c\Delta t}{\Delta x}\left[1-\cos(k_j\Delta x)\right]\right)^2 + \left(\frac{c\Delta t}{\Delta x}\right)^2 \sin^2(k_j\Delta x)$$

$$\left\|G_j\right\|^2 = 1-2\frac{c\Delta t}{\Delta x}\left[1-\cos(k_j\Delta x)\right] + \left(\frac{c\Delta t}{\Delta x}\right)^2\left[1-\cos(k_j\Delta x)\right]^2 + \left(\frac{c\Delta t}{2\Delta x}\right)^2 \sin^2(k_j\Delta x)$$

$$\left\|G_j\right\|^2 = 1-2\frac{c\Delta t}{\Delta x}\left[1-\cos(k_j\Delta x)\right] + 2\left(\frac{c\Delta t}{\Delta x}\right)^2\left[1-\cos(k_j\Delta x)\right]$$

$$\left\|G_j\right\|^2 = 1-\underbrace{2\left[1-\cos(k_j\Delta x)\right]}_{\geq 0}\ \underbrace{\frac{c\Delta t}{\Delta x}}_{\substack{\geq 0 \\ if\ c\geq 0}}\ \underbrace{\left(1-\frac{c\Delta t}{\Delta x}\right)}_{\substack{\geq 0 \\ if\ 0\leq\frac{c\Delta t}{\Delta x}\leq 1}}$$

Hence, $\left\|G_j\right\| \leq 1$ if $c \geq 0$ and $\left|\frac{c\Delta t}{\Delta x}\right| \leq 1$. The latter inequality is called the CFL condition in honor of Courant, Friedrichs and Lewy.

4.4.3.1 The CFL Number

The CFL condition for the explicit in time - backward in space difference method has the following meaning.

$$CFL\ Number = \left|\frac{c\Delta t}{\Delta x}\right| \leq 1 \Rightarrow \begin{cases} Numerical\ Domain\ of\ \ Dependence \\ contains \\ Physical\ or\ Mathematical\ Domain\ of\ \ Dependence \end{cases}$$

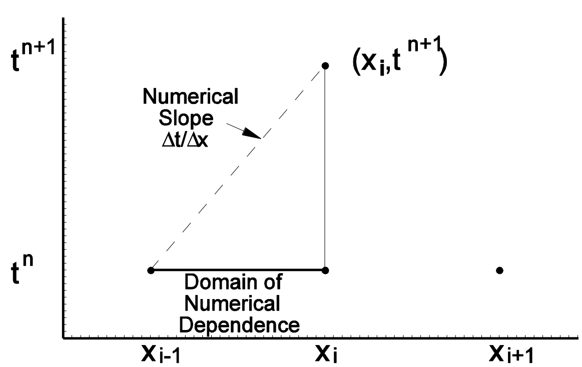


Figure 4.2 Numerical dependence

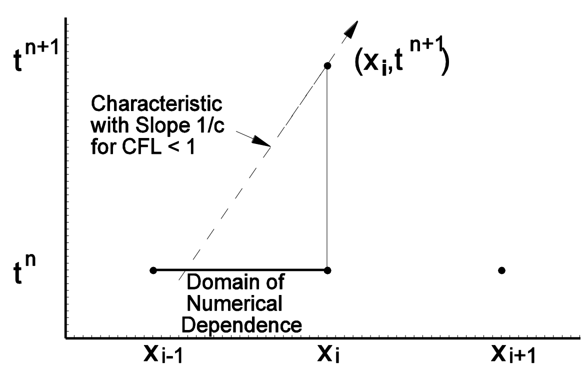


Figure 4.3 Mathematical dependence

The numerical domain of dependence of the algorithm at time t^n is given by the spatial interval $x_{i-1} \le x \le x_i$, but the mathematical domain of dependence, expressed by the characteristic path through the point (x_i, t^{n+1}), at time t^n, is just the intersection point of the characteristic path with the spatial interval, as shown in Figure 4.3. See also Figure 2.12.

4.4.3.2 Stability and The Unit Circle

It is often useful to graph the amplification factor in the complex plane. If the curve for $G_j(\xi)$ lies within the unit circle, then $\|G_j\| \le 1$. The figure below shows $G_j(\xi)$ for the explicit in time - backward in space difference method for $0 \le \xi \le 2\pi$ and CFL numbers 1.0, 0.9, 0.5 and 0.1.

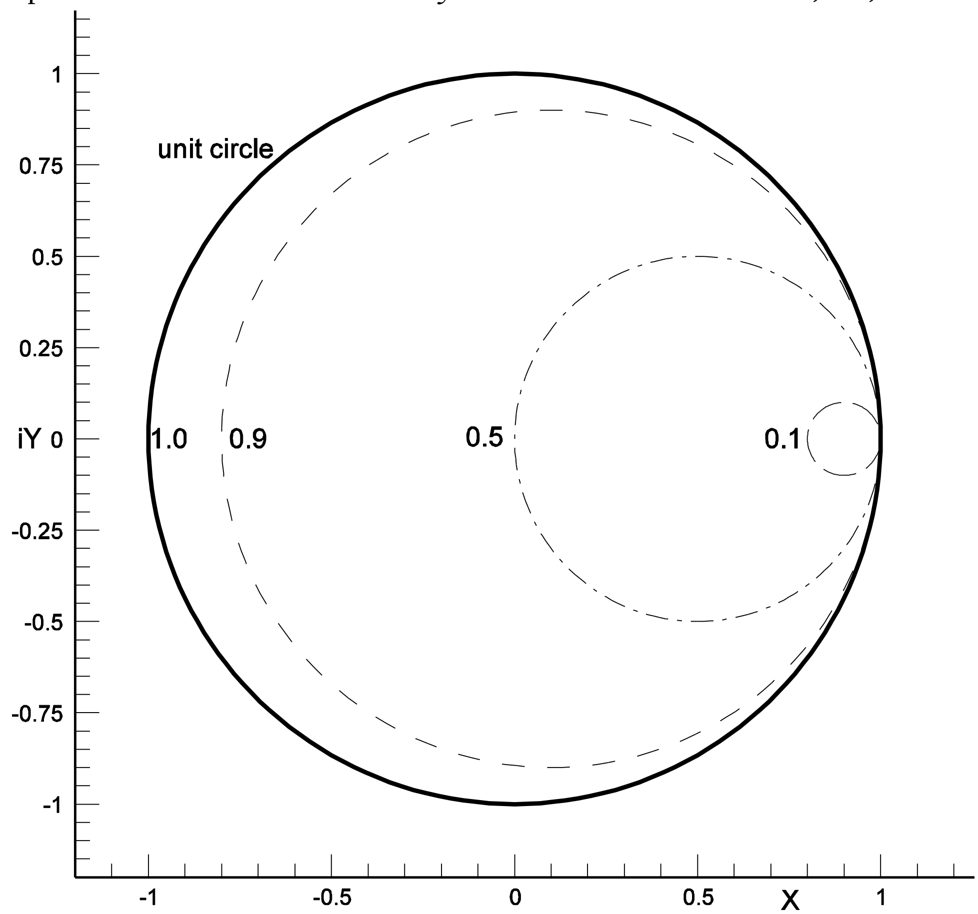


Figure 4.4 Graph of $G_j(\xi)$ for $0 \le \xi \le 2\pi$ compared with the unit circle

The figure below re-plots the data from above, unwrapped as $\|G_j\|^2$ versus ξ.

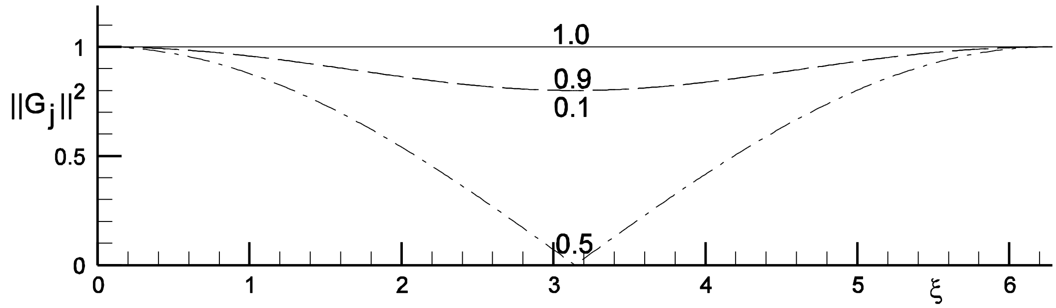


Figure 4.5 Graph of $\left\|G_j\right\|^2$ versus ξ for $0 \le \xi \le 2\pi$

Note that the curves for CFL numbers 0.9 and 0.1 are the same, as can be seen from the equation

$$\left\|G_j\right\|^2 = 1 - 2[1-\cos\xi]\frac{c\Delta t}{\Delta x}\left(1-\frac{c\Delta t}{\Delta x}\right)$$

4.4.4 Stability Analysis for a System of Equations

Stability analysis applied to a system of m equations in one dimension

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0, \quad F = F(U)$$

Consider the following explicit backward spatial difference method

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}(F_i^n - F_{i-1}^n)$$

where $F_i^n = F(U_i^n)$), i.e. F is a function of U, and F and U are vectors of length m.

Linearizing the above differential equation, we obtain

$$\frac{\partial U}{\partial t} + A_0\frac{\partial U}{\partial x} = 0, \quad where \quad A_0 = \left.\frac{\partial F}{\partial U}\right|_{x_0,t_0}$$

and A_0 is *frozen* locally at (x_0, t_0). The method applied to the linear equation is

$$U_i^{n+1} = U_i^n - \frac{A_0\Delta t}{\Delta x}(U_i^n - U_{i-1}^n)$$

Using $V_i = C_j e^{ik_j x_i}$ as a test component of the solution, where $C_j = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$, we obtain the following

$$V_i^{n+1} = V_i^n - \frac{A_0 \Delta t}{\Delta x}\left(1 - e^{-ik_j \Delta x}\right)V_i^n = \underbrace{\left\{ I - \frac{A_0 \Delta t}{\Delta x}\left[1 - \cos(k_j \Delta x) + i \sin(k_j \Delta x)\right]\right\}}_{G_j} V_i^n$$

Assuming A_0 is diagonalizable by matrix S and its inverse , $A_0 = S^{-1} \Lambda \, S$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}, \qquad \text{Hyperbolic Requirement: } \textit{eigenvalue } \lambda_l \textit{ is real for each } l$$

$$G_j = S^{-1}\left\{ I - \frac{\Lambda \Delta t}{\Delta x}\left[1 - \cos(k_j \Delta x) + i \sin(k_j \Delta x)\right]\right\} S$$

Note that repeated iterations of the algorithm to time $n\Delta t$ will have amplification

$$\left(G_j\right)^n = \underbrace{G_j G_j G_j G_j \cdots G_j G_j G_j}_{n\ factors} = S^{-1}\left\{ I - \frac{\Lambda \Delta t}{\Delta x}\left[1 - \cos(k_j \Delta x) + i \sin(k_j \Delta x)\right]\right\}^n S$$

where cancellation of jacobian matrices S and its inverse has occurred within the telescopic product. Therefore

$$\left\| \left(G_j\right)^n \right\| = \left\| G_j \right\|^n \le \left\| S^{-1} \right\| \left\| I - \frac{\Lambda \Delta t}{\Delta x}\left[1 - \cos(k_j \Delta x) + i \sin(k_j \Delta x)\right]\right\|^n \left\| S \right\|$$

Using the spectral norm (largest eigenvalue)

$$\left\| G_j \right\|^n \le \left\| S^{-1} \right\| \left\| S \right\| \underbrace{\max_l \left| 1 - \frac{\lambda_l \Delta t}{\Delta x}\left[1 - \cos(k_j \Delta x) + i \sin(k_j \Delta x)\right]\right|^n}_{\le 1 \ if \ 0 \le \frac{\lambda_l \Delta t}{\Delta x} \le 1}$$

Thus $\left\| G_j \right\|^n$ is bounded by $B = \left\| S^{-1} \right\| \left\| S \right\|$ as $n \to \infty$ if $0 \le \frac{\lambda_l \Delta t}{\Delta x} \le 1$ for each l. Therefore no exponential growth is possible, proving that the algorithm is stable.

4.4.5 Stability Analysis for Model Parabolic Equation

The model parabolic equation, or diffusion equation, is (see Section 1.8) given by

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}, \quad \text{with} \quad \nu > 0 \text{ and constant.}$$

Using a central difference approximation for the spatial derivative,

$$u_i^{n+1} = u_i^n + \Delta t \nu \frac{D_{xx} \cdot}{\Delta x^2} u_i^n = u_i^n + \frac{\nu \Delta t}{\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

In terms of the test component $v_i = c_j e^{ik_j x_i}$

$$v_i^{n+1} = \left\{1 + \frac{\nu \Delta t}{\Delta x^2}(2\cos\xi - 2)\right\} v_i^n = G v_i^n$$

$$G = 1 - \frac{2\nu \Delta t}{\Delta x^2}(1 - \cos\xi) \qquad \text{and therefore} \qquad \|G\| \le 1 \quad if \quad \frac{2\nu\Delta t}{\Delta x^2} \le 1$$

4.4.6 Stability Analysis for Burgers Equation

Burgers equation is given by

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2},$$

with constant viscosity ν. After linearization, we obtain

$$\frac{\partial u}{\partial t} + u_0 \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

Assuming $u_0 \ge 0$, we use a backward difference approximation for the advection term and a central difference approximation for the viscous term

$$u_i^{n+1} = u_i^n - \Delta t u_0 \frac{D_{-} \cdot}{\Delta x} u_i^n + \Delta t \nu \frac{D_{xx} \cdot}{\Delta x^2} u_i^n = u_i^n - \frac{u_0 \Delta t}{\Delta x}(u_{i+1}^n - u_i^n) + \frac{\nu \Delta t}{\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

In terms of the test component $v_i = c_j e^{ik_j x_i}$

$$v_i^{n+1} = \left\{1 - \frac{u_0 \Delta t}{\Delta x}(1 - \cos\xi + i\sin\xi) + \frac{\nu \Delta t}{\Delta x^2}(2\cos\xi - 2)\right\} v_i^n = G v_i^n$$

The real and imaginary parts of the amplification factor G are

$$\mathrm{Re}\{G\} = 1 - \frac{u_0 \Delta t}{\Delta x}(1 - \cos\xi) - \frac{2\nu\Delta t}{\Delta x^2}(1 - \cos\xi) \quad \text{and} \quad \mathrm{Im}\{G\} = -\frac{u_0 \Delta t}{\Delta x}\sin\xi$$

$$\left|G\right|^2 = \mathrm{Re}^2 + \mathrm{Im}^2 = 1 - 2\frac{u_0\Delta t}{\Delta x}(1-\cos\xi) + \left(\frac{u_0\Delta t}{\Delta x}\right)^2\left((1-\cos\xi)^2 + \sin^2\xi\right)$$
$$- 2\frac{2\nu\Delta t}{\Delta x^2}(1-\cos\xi) + \left(\frac{2\nu\Delta t}{\Delta x^2}\right)^2(1-\cos\xi)^2$$
$$+ 2\frac{u_0\Delta t}{\Delta x}\frac{2\nu\Delta t}{\Delta x^2}(1-\cos\xi)^2$$

or

$$\left\|G\right\|^2 = 1 - 2\frac{u_0\Delta t}{\Delta x}\left(1-\frac{u_0\Delta t}{\Delta x}\right)(1-\cos\xi) - 2\frac{2\nu\Delta t}{\Delta x^2}(1-\cos\xi)\left(1-\frac{1}{2}\frac{2\nu\Delta t}{\Delta x^2}(1-\cos\xi)\right) + 2\frac{u_0\Delta t}{\Delta x}\frac{2\nu\Delta t}{\Delta x^2}(1-\cos\xi)^2$$

The second and third terms on the right hand side above are negative if $\frac{u_0\Delta t}{\Delta x} \le 1$ and $\frac{2\nu\Delta t}{\Delta x^2} \le 1$. The fourth term is positive. If the magnitude of the sum of the second and third terms is greater than the fourth, then $\left\|G\right\|^2$ will be less than or equal to one. This will be true if $\frac{u_0\Delta t}{\Delta x} + \frac{2\nu\Delta t}{\Delta x^2} \le 1$, as shown below.

$$\cancel{2}\frac{u_0\Delta t}{\Delta x}\left(1-\frac{u_0\Delta t}{\Delta x}\right)\cancel{(1-\cos\xi)} + \cancel{2}\frac{2\nu\Delta t}{\Delta x^2}\cancel{(1-\cos\xi)}\left(1-\frac{1}{2}\frac{2\nu\Delta t}{\Delta x^2}(1-\cos\xi)\right) \quad \ge ? \quad \cancel{2}\frac{u_0\Delta t}{\Delta x}\frac{2\nu\Delta t}{\Delta x^2}(1-\cos\xi)^{\cancel{2}}$$

$$\frac{u_0\Delta t}{\Delta x} - \left(\frac{u_0\Delta t}{\Delta x}\right)^2 + \frac{2\nu\Delta t}{\Delta x^2} - \frac{1}{2}\left(\frac{2\nu\Delta t}{\Delta x^2}\right)^2(1-\cos\xi) \quad \ge ? \quad \frac{u_0\Delta t}{\Delta x}\frac{2\nu\Delta t}{\Delta x^2}(1-\cos\xi)$$

Note that $\cos\xi = -1$ will minimize the left hand side and maximize the right hand side. Using $\cos\xi = -1$ and moving two terms to the right hand side, we obtain

$$\frac{u_0\Delta t}{\Delta x} + \frac{2\nu\Delta t}{\Delta x^2} \quad \ge ? \quad \left(\frac{u_0\Delta t}{\Delta x}\right)^2 + 2\frac{u_0\Delta t}{\Delta x}\frac{2\nu\Delta t}{\Delta x^2} + \left(\frac{2\nu\Delta t}{\Delta x^2}\right)^2 = \left(\frac{u_0\Delta t}{\Delta x} + \frac{2\nu\Delta t}{\Delta x^2}\right)^2$$

Thus the difference equation will be stable if $1 \;\ge\; \frac{u_0\Delta t}{\Delta x} + \frac{2\nu\Delta t}{\Delta x^2}$ or $\Delta t \;\le\; \dfrac{1}{\frac{u_0}{\Delta x} + \frac{2\nu}{\Delta x^2}}$

4.4.7 Explicit in Time – Second Order Backward in Space Difference Method

The stability analysis for a second order accurate *backward in space and explicit first order in time* method for solving the model hyperbolic equation follows

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad \textit{with} \quad c > 0 \quad \text{is approximated by}$$

$$
\begin{aligned}
u_i^{n+1} &= u_i^n - c\Delta t \frac{3u_i^n - 4u_{i-1}^n + u_{i-2}^n}{2\Delta x} \\
&= u_i^n - c\Delta t \left\{ \underbrace{\frac{u_i^n - u_{i-1}^n}{\Delta x}}_{1^{st}\ order\ approx.} + \underbrace{\frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{2\Delta x}}_{2^{nd}\ order\ correction} \right\}
\end{aligned}
$$

By substitution of test component $v_i = c_j e^{ik_j x_i}$

$$
v_i^{n+1} = \left\{ 1 - \frac{c\Delta t}{\Delta x}\left[\left(1 - e^{-i\xi}\right) + e^{-i\xi}\left(\cos\xi - 1\right) \right] \right\} v_i^n = G_j(\xi) v_i^n
$$

$$
G_j(\xi) = \underbrace{1 - \frac{c\Delta t}{\Delta x}\left(1 - \cos\xi\right)^2}_{\text{Real Part}} - \underbrace{\frac{c\Delta t}{\Delta x} i \sin\xi \left(2 - \cos\xi\right)}_{\text{Imaginary Part}}
$$

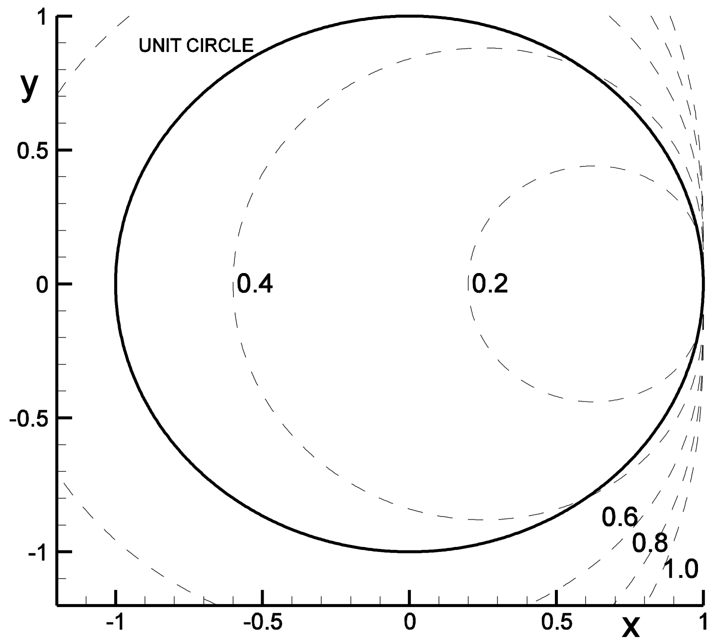


Figure 4.6 Graph of $G_j(\xi)$ for $0 \le \xi \le 2\pi$ at CFL numbers 0.2, 0.4, 0.6. 0.8 and 1.0

The amplification factor graphed in the complex plane is shown in Figure 4.6 for $0 \le \xi \le 2\pi$ and CFL numbers 1.0, 0.9, 0.5 and 0.1. Curves for $G_j(\xi)$ lie outside the unit circle, which implies that the method is unstable at CFL numbers 0.5 and above. However, it is not clear if $\left\|G_j\right\| \le 1$ for CFL numbers below 0.5. The graph of $\left\|G_j\right\|^2$ versus ξ for $0 \le \xi \le 2\pi$ CFL numbers 0.4, 0.3, 0.2 and 0.1

is also plotted below and it is evident that $\|G_j\| > 1$ for CFL numbers 0.4 and 0.3. Finally, the detailed view near $\xi = 2\pi$ reveals that $\|G_j\| > 1$, even for impractically small CFL numbers.

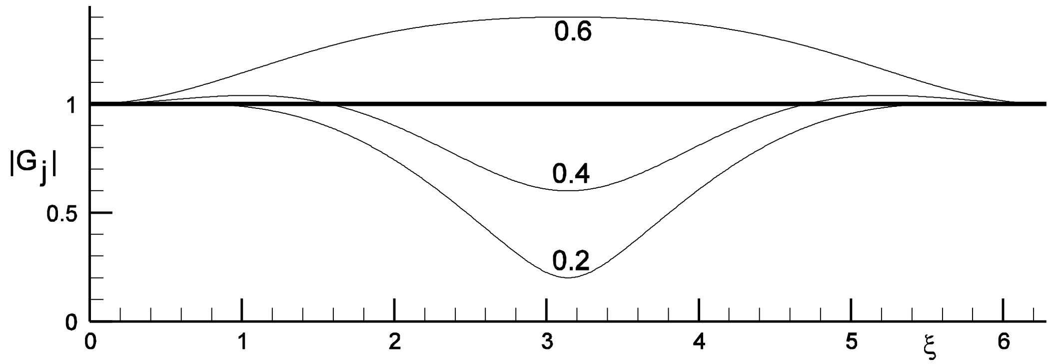


Figure 4.7 Graph of $\|G_j\|^2$ versus ξ for $0 \le \xi \le 2\pi$ CFL numbers 0.2, 0.4 and 0.6

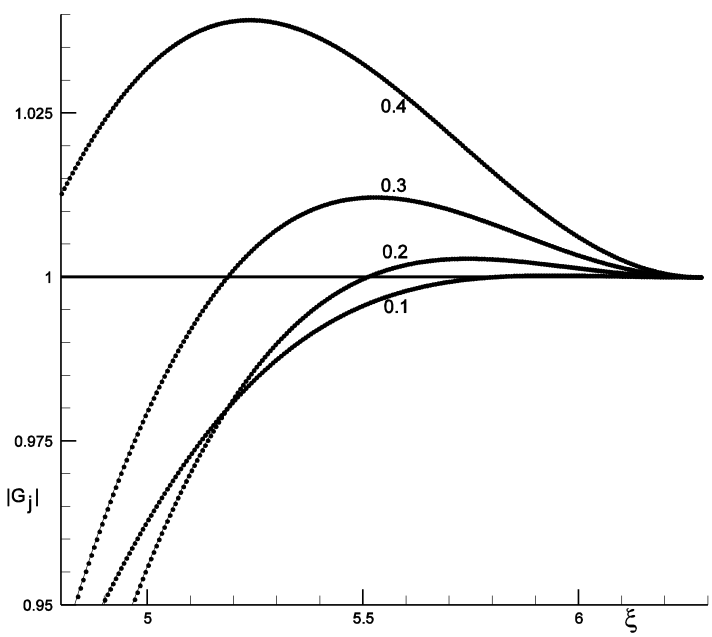


Figure 4.8 Detailed view of $|G_j|$ versus ξ for $0 \le \xi \le 2\pi$ at CFL numbers 0.4, 0.3, 0.2 and 0.1

A few words of caution should be expressed on using approximations of different orders for the time and spatial derivatives. The overall order of accuracy of the solution is equal to the lowest order approximation used for any term in the governing equations. The time and spatial derivatives are linked at least by the time dependent equation itself. There is no advantage to using higher order approximations for the spatial derivatives than for the time derivatives for time dependent solutions. For solutions going to a steady state, however, the order of accuracy will be that of the lowest order spatial derivative approximation.

4.4.8 The Warming-Beam Method – Second Order in Space and Time

The Warming-Beam method is second order accurate in space and time and is stable for a CFL number up to 2. It is shown below as a two step method applied to solve the model hyperbolic equation $\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0, \quad with \quad c>0$.

$$u_i^{n+1/2}=u_i^n-\frac{c\Delta t}{2}\frac{D_-\cdot}{\Delta x}u_i^n$$

$$u_i^{n+1}=u_i^n-c\Delta t\frac{D_-\cdot}{\Delta x}\left\{u_i^{n+1/2}+\frac{\Delta x}{2}\frac{D_-\cdot}{\Delta x}u_i^n\right\}$$

It can for this linear equation case also be shown as a single step method

$$u_i^{n+1}=u_i^n-c\Delta t\frac{D_-\cdot}{\Delta x}\left\{u_i^n+\frac{\Delta x}{2}(1-\frac{c\Delta t}{\Delta x})\frac{D_-\cdot}{\Delta x}u_i^n\right\}$$

The amplification factor, using the test component $v_i=c_je^{ik_jx_i}$, is

$$G_j(\xi)=1-\lambda\left(1-e^{-i\xi}\right)\left(1+\frac{1}{2}(1-\lambda)\left(1-e^{-i\xi}\right)\right)$$

where $\lambda=\frac{c\Delta t}{\Delta x}$. The amplification factor is graphed in the complex plane below for $0\le\xi\le 2\pi$ and CFL numbers 1.0, 0.9, 0.5 and 0.1. Curves for $G_j(\xi)$ lie inside the unit circle, which implies that the method is stable at CFL numbers up to 2.

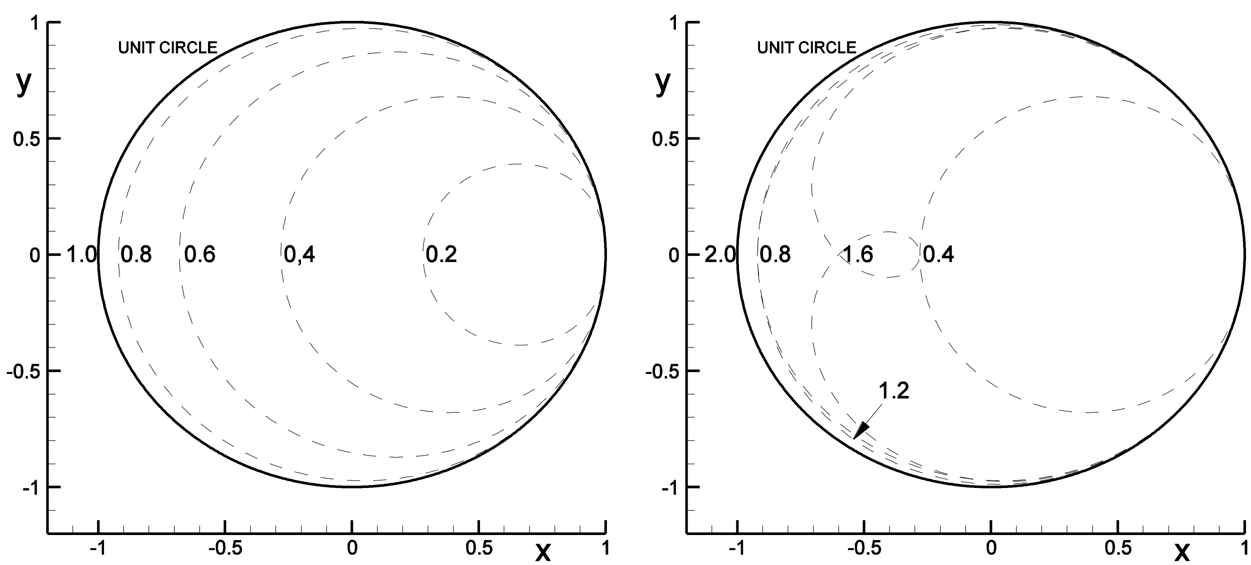


Figure 4.9 Unit circle graph of $G_j(\xi)$ for $0\le\xi\le 2\pi$ at CFL numbers $\lambda=$ 0.2 to 2.0

Note that if $\lambda = 1$, then $G_j(\xi) = e^{-i\xi} = e^{-ik_j\Delta x} = e^{-ik_j c\Delta t}$, the exact amplification for the solution to model wave equation after one time step, and that if $\lambda = 2$, then $G_j(\xi) = e^{-i2\xi} = e^{-i2k_j\Delta x} = e^{-ik_j c 2\Delta t}$, remarkably the exact amplification for the solution after one time step of size equal to $2\Delta t$. These observations are also apparent in Figure 4.9 for $\lambda =$ 1.0 and 2.0.