# *Chapter 2*

# *Integral and Conservation Law Form, Discontinuities, Jacobians, Linearization and Characteristic Relations*

## *2.1 Introduction*

The basic mathematics for solving the equations of compressible flow are presented in this chapter Integral and conservation law form of the governing equations is defined and then used to predict relations occurring across possible discontinuities within the flow. Linearization, used for analysis of the in general nonlinear flow equations, is obtained using Jacobian matrices. Characteristic equations are derived from the governing flow equations that will be useful later for devising correct boundary condition procedures. Finally, the definition for a set of hyperbolic equations will be presented. These mathematical procedures are fundamental for the analysis and understanding of the numerical methods to be presented in later chapters.

## *2.2 Conservation Law Form*

Equations are said to be in conservation law form, or more precisely in divergence law form, if they are written as follows:

$$\frac{\partial U}{\partial t} + \vec{\nabla} \cdot \vec{F} = Q \tag{2.1}$$

where

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i}_x + \frac{\partial}{\partial y}\vec{i}_y + \frac{\partial}{\partial z}\vec{i}_z$$

$$\vec{F} = F\vec{i}_x + G\vec{i}_y + H\vec{i}_z$$

where $\vec{i}_x$, $\vec{i}_y$ and $\vec{i}_z$ are unit vectors in the $x$, $y$ and $z$ directions. If the vector $Q$ is equal to zero the equation is said to be in *strong* conservation law form.

Almost all of the equations presented earlier in Chapter 1 were or can be put into conservation law form. For example, the *Transonic Small Disturbance Equation* can be put into strong conservation law form if we define $\vec{F}$ as follows:

$$\vec{F} = \left[ (1 - M_\infty^2)\hat{\phi}_x - (\gamma + 1)M_\infty^2 \frac{\hat{\phi}_x^2}{2q_\infty} \right]\vec{i}_x + \hat{\phi}_y \vec{i}_y$$

The Navier Stokes Equations in strong conservation form in two dimensions are as follows:

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=0$$

Where $U=\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ e\end{pmatrix}$, $F=\begin{pmatrix}\rho u\\ \rho u^2+\sigma_x\\ \rho uv+\tau_{yx}\\ (e+\sigma_x)u+\tau_{yx}v-k\dfrac{\partial T}{\partial x}\end{pmatrix}$ and $G=\begin{pmatrix}\rho v\\ \rho vu+\tau_{xy}\\ \rho v^2+\sigma_y\\ (e+\sigma_y)v+\tau_{xy}u-k\dfrac{\partial T}{\partial y}\end{pmatrix}$

Although the equations are mathematically equivalent in their conservation and non-conservation forms, the numerical solution can depend on which form is chosen, which we will discuss later on.

## ***2.3 Integral Form***

Integrating Equation (2.1) over an arbitrary volume $V$ enclosed by surface $S$, we obtain

$$\int_V \frac{\partial U}{\partial t}dv+\int_V \vec{\nabla}\cdot\vec{F}dv=\int_V Qdv$$

By use of the Divergence (Gauss) theorem we have

$$\frac{\partial \overline{U}}{\partial t}+\frac{1}{V}\int_S \vec{F}\cdot\vec{ds}=\frac{1}{V}\int_V Qdv \qquad (2.2)$$

where $\overline{U}=\frac{1}{V}\int_V Udv$ and $\vec{ds}$ is a surface element of $S$ with direction given by the outer normal to $S$.

Equation (2.2) represents the rate of change of the mean value of $U$ over the volume $V$ caused by the net flux of $\vec{F}$ crossing surface $S$ plus volume source $Q$. The flux vector represents the convection of mass, momentum and energy across the bounding surface as well as the stresses acting on the surface.

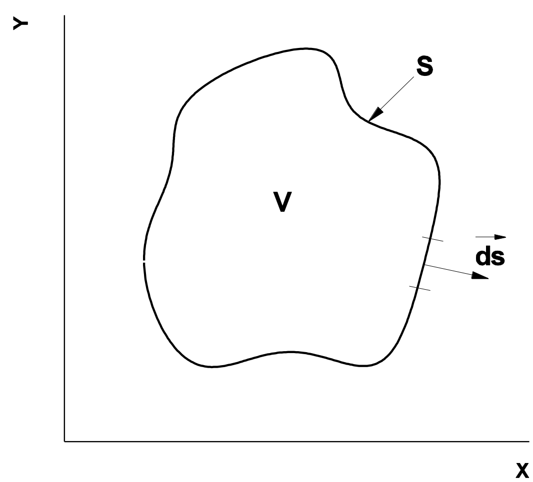


**Figure 2.1** Arbitrary fluid volume

## *2.4 Relations at Discontinuities*

### *2.4.1 Stationary Discontinuities*

Using the steady version of Equation (2.2), we can determine the "jump" in flow values across a surface located at a possible discontinuity within the fluid. Let's assume that $f(x,y,z)=0$ represents such a surface and that the flow is continuous within each of the two sub-domains shown in the sketch. The volume $V$ is placed symmetrically about an arbitrary point on the surface and is allowed to shrink to zero. Equation (2.2) becomes for steady flow. $\int_S \vec{F}\cdot\vec{ds} = \int_V Q dv$

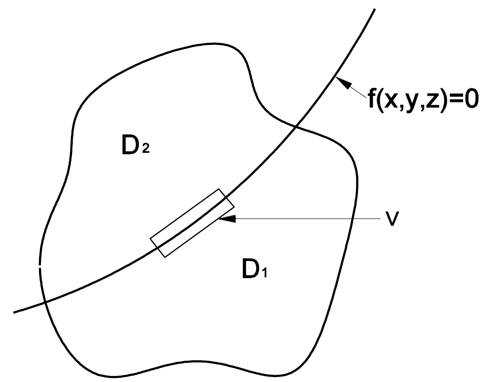


**Figure 2.2** Volume divided by surface $f$

As $V \rightarrow 0$ the term on the right goes to zero at a faster rate than the surface integration term ($h^3$vs. $h^2$, where $h \simeq V^{1/3} \simeq S^{1/2}$). For sufficiently small $V$ we obtain the following.

$$\int_S \vec{F}\cdot\vec{ds} = 0 \qquad \text{(for infinitesimal } V\text{)}$$

Assume the volume is a small box aligned with the surface $f$ (see Fig 2.3). The above equation for two-dimensional flow consists of the integration of four surface flux terms. By continuity within either side of surface $f$, the contributions from the top and bottom surfaces cancel, leaving only the contributions from the sides parallel to the surface.. The resulting equation is

$$\int_S \vec{F}\cdot\vec{ds} = \vec{F_1}\cdot\vec{n_1}ds_1 + \vec{F_2}\cdot\vec{n_2}ds_2 = 0$$

or $\left(\vec{F_1}-\vec{F_2}\right)\cdot\vec{n_1} = 0$, with $\vec{n_1} = \dfrac{\vec{\nabla} f}{\left|\vec{\nabla} f\right|}$

Hence, $\left(\vec{F_1}-\vec{F_2}\right)\cdot\vec{\nabla} f = 0$

Defining the jump across surface $f$ as $\left(\vec{F_1}-\vec{F_2}\right) = \left[\vec{F}\right]_1^2$, we have $\left[\vec{F}\right]_1^2\cdot\vec{\nabla} f = 0$, or finally, where $\vec{F} = F\vec{i_x} + G\vec{i_y} + H\vec{i_z}$,

$$\left[F\right]_1^2 f_x + \left[G\right]_1^2 f_y + \left[H\right]_1^2 f_z = 0 \quad (2.3)$$

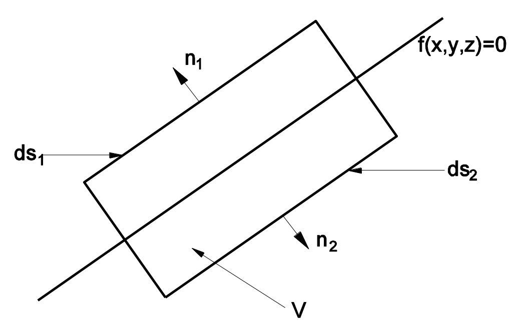


**Figure 2.3** Infinitesimal volume about $f$

Equation (2.3) represents the steady jump relations at surface $f(x,y,z)=0$. For the Euler equations they represent the *Rankine-Hugoniot* relations across a shock wave.

## 2.4.2 Moving Discontinuities

The above discussion can be generalized to obtain the unsteady jump relations for moving discontinuities by including time as a dimension, as defined by the following.

$$\vec{\nabla}=\frac{\partial}{\partial t}\vec{i}_t+\frac{\partial}{\partial x}\vec{i}_x+\frac{\partial}{\partial y}\vec{i}_y+\frac{\partial}{\partial z}\vec{i}_z, \qquad \vec{F}=U\vec{i}_t+F\vec{i}_x+G\vec{i}_y+H\vec{i}_z \qquad \text{and} \qquad f(x,y,z,t)=0$$

The jump relation becomes $\left[\vec{F}\right]_1^2\cdot\vec{\nabla}f=\left[U\right]_1^2 f_t+\left[F\right]_1^2 f_x+\left[G\right]_1^2 f_y+\left[H\right]_1^2 f_z=0$

## 2.4.3 Shock Wave Examples

### 2.4.3.1 The Simple Wave Equation

Consider the following model hyperbolic equation with constant wave speed $c$

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0,$$

(a) Stationary discontinuity surface

$$f(x)=x-x_0=0.$$

$$\vec{F}=u\vec{i}_t+cu\vec{i}_x \text{ and } \vec{n}_1=\frac{\vec{\nabla}f}{\left|\vec{\nabla}f\right|}=\vec{i}_x$$

$$\left(\vec{F}_1-\vec{F}_2\right)\cdot\vec{n}_1=c(u_1-u_2)=0, \text{ or } u_1=u_2,$$

which implies that no jump or discontinuity is possible. This is not surprising for a linear equation.

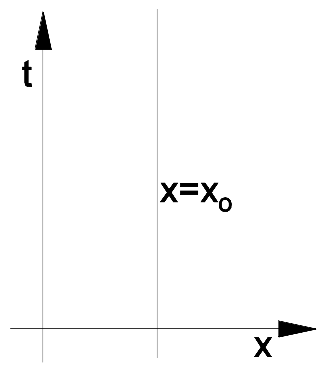


**Figure 2.4** $f(x)=x-x_0=0$

(b) Discontinuity surface moving at constant speed $w$, $f(x,t)=x-x_0-w(t-t_0)=0$.

$$\vec{n}_1=\frac{\vec{\nabla}f}{\left|\vec{\nabla}f\right|}=\frac{\vec{i}_x-w\vec{i}_t}{\sqrt{1+w^2}} \qquad \text{and}$$

$$\left(\vec{F}_1-\vec{F}_2\right)\cdot\vec{n}_1=\frac{1}{\sqrt{1+w^2}}\left(c(u_1-u_2)-w(u_1-u_2)\right)=0$$

which implies that any jump or discontinuity is possible as long as it moves with speed $c$.

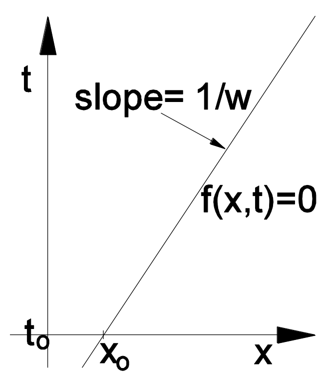


**Figure 2.5** $f(x,t)=x-x_0-w(t-t_0)=0$

### *2.4.3.2 Burgers Equation*

The Burgers equation is given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2},$$

with constant viscosity $\nu \geq 0$. We need to put this nonlinear equation in conservation law form

$$\frac{\partial u}{\partial t} + \frac{\partial \frac{1}{2}u^2}{\partial x} = \frac{\partial \nu \frac{\partial u}{\partial x}}{\partial x}$$

Because the viscous term will try to smooth any possible discontinuity in the flow, we will take for this example $\nu = 0$.

(a) Steady possible discontinuity surface $f(x) = x - x_0 = 0$

$$\vec{F} = u\vec{i}_t + \frac{1}{2}u^2\vec{i}_x \text{ and } \vec{n}_1 = \frac{\vec{\nabla} f}{\left|\vec{\nabla} f\right|} = \vec{i}_x$$

$$\left(\vec{F}_1 - \vec{F}_2\right)\cdot\vec{n}_1 = \frac{1}{2}(u_1^2 - u_2^2) = 0$$

The above quadratic equation has two solutions , the no jump solution $u_1 = u_2$ and the jump solution $u_1 = -u_2$

(a) Unsteady possible discontinuity surface moving at constant speed $f(x) = x - x_0 - w(t - t_0) = 0$.

$$\vec{n}_1 = \frac{\vec{\nabla} f}{\left|\vec{\nabla} f\right|} = \frac{\vec{i}_x - w\vec{i}_t}{\sqrt{1+w^2}} \quad \text{and} \quad \begin{aligned}\left(\vec{F}_1 - \vec{F}_2\right)\cdot\vec{n}_1 &= \frac{1}{\sqrt{1+w^2}}\left(\frac{1}{2}(u_1^2 - u_2^2) - w(u_1 - u_2)\right) \\ &= \frac{u_1 - u_2}{\sqrt{1+w^2}}\left(\frac{1}{2}(u_1 + u_2) - w\right) = 0\end{aligned},$$

which implies that any jump or discontinuity is possible as long as it moves with speed $w = \frac{u_1 + u_2}{2}$.

### *2.4.3.3 Laplace Like Equations*

$$(1 - M_\infty^2)\phi_{xx} + \phi_{yy} = 0$$

where for purely subsonic flows, $M_\infty < 0.8$, the equation is elliptic and $\phi$ is the velocity potential, $u = \phi_x$ and $v = \phi_y$ (see Section 1.7). However, if $M_\infty > 1.2$, the equation is hyperbolic and can be used to describe purely supersonic flows with small perturbations about a supersonic free stream with velocity $\vec{V} = V_\infty\vec{i}_x$. For this latter case, the potential can be used to calculated the small disturbance velocity components $u = \phi_x$ and $v = \phi_y$ past thin bodies, such that $\vec{V} = (V_\infty + u)\vec{i}_x + v\vec{i}_y$

Let's consider as a possible discontinuity surface $f(x,y) = a(x - x_0) - b(y - y_0) = 0$, where $a$ and $b$ are constants.

$$\vec{F} = (1 - M_\infty^2)\phi_x \vec{i}_x + \phi_y \vec{i}_y \text{ and } \vec{n}_1 = \frac{\vec{\nabla} f}{\left|\vec{\nabla} f\right|} = \frac{a\vec{i}_x - b\vec{i}_y}{\sqrt{a^2 + b^2}}$$

$$\left(\vec{F}_1 - \vec{F}_2\right) \cdot \vec{n}_1 = \frac{1}{\sqrt{a^2 + b^2}}\left((1 - M_\infty^2) a(\phi_{x_1} - \phi_{x_2}) - b(\phi_{y_1} - \phi_{y_2})\right) = 0$$

$$\text{or } (1 - M_\infty^2) a(u_1 - u_2) = b(v_1 - v_2)$$

Clearly, if $a$ or $b$ equal zero there are no permissible jump solutions. Also, purely subsonic flow will contain no discontinuities, i.e., shock waves.

Small perturbation jumps can occur for the supersonic case across Mach lines with slope $tan\,\theta = \frac{a}{b} = \frac{1}{\sqrt{M_\infty^2 - 1}}$. The flow can turn through an angle $\delta$ from the free stream direction, see figure at right, such that $v_2 - v_1 = -\sqrt{M_\infty^2 - 1}(u_2 - u_1)$. For the flow illustrated in the figure to the right $u_1 = v_1 = 0$, $v_2 = V_\infty \tan\delta$ and $u_2 = -\frac{V_\infty \tan\delta}{\sqrt{M_\infty^2 - 1}}$

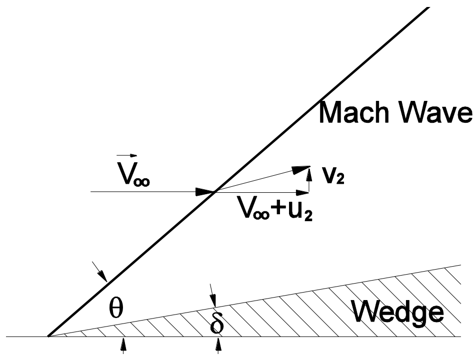


**Figure 2.6** Linearized supersonic flow past a thin wedge

***2.4.3.4 The Transonic Small Disturbance Equations***

$$\left[1 - M_\infty^2 - (\gamma + 1) M_\infty^2 \frac{\hat{\phi}_x}{q_\infty}\right]\phi_{xx} + \phi_{yy} = 0$$

for $0.8 \le M_\infty^2 \le 1.2$, where $\phi$ represents the small disturbance velocity potential, $u' = \hat{\phi}_x$, $v = v' = \hat{\phi}_y$ and $u = q_\infty + u'$ (see Section 1.6). Supersonic flow is possible for this equation only in the $x$ direction and we take $f(x) = x - x_0 = 0$ for a possible surface of a discontinuity. We first put the equation in conservation law form (see Section 2.2).

$$\frac{\partial}{\partial x}\left[(1 - M_\infty^2)\hat{\phi}_x - (\gamma + 1) M_\infty^2 \frac{\hat{\phi}_x^2}{2q_\infty}\right] + \frac{\partial}{\partial x}\hat{\phi}_y = 0$$

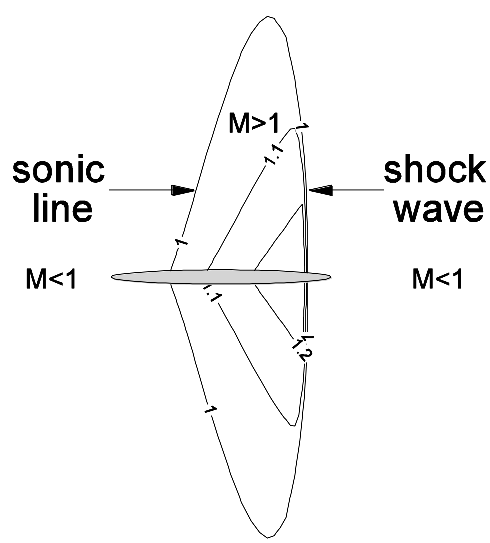


**Figure 2.7** Transonic flow

Then $$\vec{F}=\left[(1-M_\infty^2)\hat{\phi}_x-(\gamma+1)M_\infty^2\frac{\hat{\phi}_x^2}{2q_\infty}\right]\vec{i}_x+\hat{\phi}_y\vec{i}_y \text{ and } \vec{n}_1=\frac{\vec{\nabla}f}{\left|\vec{\nabla}f\right|}=\vec{i}_x$$

$$\left(\vec{F}_1-\vec{F}_2\right)\cdot\vec{n}_1=(1-M_\infty^2)(\hat{\phi}_{x_1}-\hat{\phi}_{x_2})-\frac{(\gamma+1)M_\infty^2}{2q_\infty}(\hat{\phi}_{x_1}^2-\hat{\phi}_{x_2}^2)=0$$

or $$(\hat{\phi}_{x_1}-\hat{\phi}_{x_2})\left(1-M_\infty^2-\frac{(\gamma+1)M_\infty^2}{2q_\infty}(\hat{\phi}_{x_1}+\hat{\phi}_{x_2})\right)=0$$

Thus, there are two solutions, the trivial no jump solution $\hat{\phi}_{x_1}=\hat{\phi}_{x_2}$ and the jump solution

$$\frac{\hat{\phi}_{x_1}+\hat{\phi}_{x_2}}{2}=\frac{1-M_\infty^2}{(\gamma+1)M_\infty^2}q_\infty=\hat{\phi}_x^*$$

Note that $\hat{\phi}_x^*$ is the value of $\hat{\phi}_x$ at which the bracketed coefficient term before the $\hat{\phi}_{xx}$ derivative vanishes. Starred terms in fluid dynamics often imply conditions where the flow speed equals the speed of sound. In the case of the Transonic Small Disturbance equation, the speed of the flow at Mach one is given by $u^*=q_\infty+\hat{\phi}_x^*=a^*$ and the above jump relation expresses the condition that the average of the flow speeds across the discontinuity equals $a^*$, $\frac{u_1+u_2}{2}=a^*$.

***2.4.3.5 The Full Potential Equation***

The Full Potential equation in conservation law form is ( see Chapter 1, Section 1.5)

$$\frac{\partial\rho\phi_x}{\partial x}+\frac{\partial\rho\phi_y}{\partial y}=\frac{\partial\rho u}{\partial x}+\frac{\partial\rho v}{\partial y}=0$$

with $\rho=\rho_\infty\left[1-\frac{\gamma-1}{2}M_\infty^2\left(\frac{\phi_x^2+\phi_y^2}{q_\infty^2}-1\right)\right]^{\frac{1}{\gamma-1}}$

Consider the shock wave terminating the supersonic region at the upper surface of an airfoil, as shown in the adjacent figure. The shock wave surface is normal to the airfoil surface and therefore to the flow following the surface, otherwise it would be an oblique shock wave causing the flow to change direction and turn away from the airfoil surface. Therefore,

$$\vec{n}_1=\frac{u_1\vec{i}_x+v_1\vec{i}_y}{\sqrt{u_1^2+v_1^2}}=\frac{u_1\vec{i}_x+v_1\vec{i}_y}{V_1}=\frac{u_2\vec{i}_x+v_2\vec{i}_y}{V_2},$$

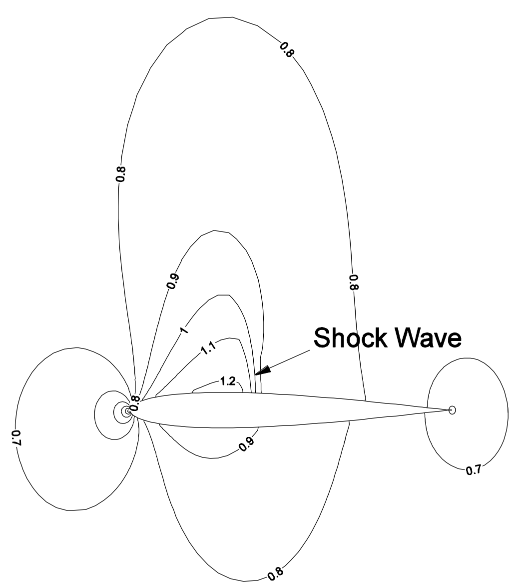


**Figure 2.8** Mach contours about a NACA 0012 airfoil foil in $M_\infty=0.75$ flow at $1^o$ degree angle of attack

$$\vec{F} = \rho u \vec{i}_x + \rho v \vec{i}_y \quad \text{and} \quad \left(\vec{F_1} - \vec{F_2}\right) \cdot \vec{n_1} = \rho_1 V_1 - \rho_2 V_2 = 0$$

We can express the isentropic density relation also as $\dfrac{\rho_2}{\rho_1} = \left(1 + \dfrac{\gamma - 1}{2} M_1^2 (1 - \dfrac{V_2^2}{V_1^2})\right)^{\frac{1}{\gamma-1}}$.

If we assume that $1 - \dfrac{V_2^2}{V_1^2}$ is small, we can approximate the density ratio with $\dfrac{\rho_2}{\rho_1} \simeq 1 + \dfrac{M_1^2}{2}(1 - \dfrac{V_2^2}{V_1^2})$, (i.e., $(1+\varepsilon)^{\frac{1}{\gamma-1}} \simeq 1 + \dfrac{\varepsilon}{\gamma - 1}$ for small $\varepsilon$). Using this last result within the above jump relation $\dfrac{V_1}{V_2} = 1 + \dfrac{M_1^2}{2}(1 - \dfrac{V_2^2}{V_1^2}) \approx 1 + M_1^2 (1 - \dfrac{V_2}{V_1})$ or $\dfrac{V_2}{V_1} - 1 + M_1^2 \dfrac{V_2}{V_1}(1 - \dfrac{V_2}{V_1}) = 0$ which has two roots, (a) the no jump solution $\dfrac{V_2}{V_1} = 1$ and (b) the jump solution $\dfrac{V_2}{V_1} = \dfrac{1}{M_1^2}$.

### ***2.4.3.6 The Steady Euler Equations in Two Dimensions***

The steady Euler equations in two dimension are (see also Chapter1, Section 1.4).

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \quad \text{where} \quad F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (e+p)u \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} \rho v \\ \rho v u \\ \rho v^2 + p \\ (e+p)v \end{pmatrix}$$

These equations can describe shock waves if the flow is supersonic, such as flow past a wedge in which an oblique shock wave forms, separating two regions of uniform flow. We take as the surface of the discontinuity $f(x,y) = (x - x_0)\sin\theta - (y - y_0)\cos\theta = 0$, where $\theta$ is the angle the oblique shock wave makes with the $x$ coordinate.

$$\vec{F} = F\vec{i}_x + G\vec{i}_y, \quad \vec{n}_1 = \frac{\vec{\nabla} f}{\left|\vec{\nabla} f\right|} = \sin\theta \vec{i}_x - \cos\theta \vec{i}_y \quad \text{and} \quad \left(\vec{F_1} - \vec{F_2}\right) \cdot \vec{n_1} = \sin\theta (F_1 - F_2) - \cos\theta (G_1 - G_2)$$

The oblique shock relations for the steady Euler equations are as follows.

$$\begin{aligned} \rho_2 q_{n2} &= \rho_1 q_{n1} \\ \rho_2 q_{n2}^2 + p_2 &= \rho_1 q_{n1}^2 + p_1 \\ \rho_2 q_{n2} q_{t2} &= \rho_1 q_{n1} q_{t1} \\ (e_2 + p_2) q_{n2} &= (e_1 + p_1) q_{n1} \end{aligned}$$

where
$$\begin{aligned} q_n &= u\sin\theta - v\cos\theta \\ q_t &= u\cos\theta + v\sin\theta \end{aligned}$$

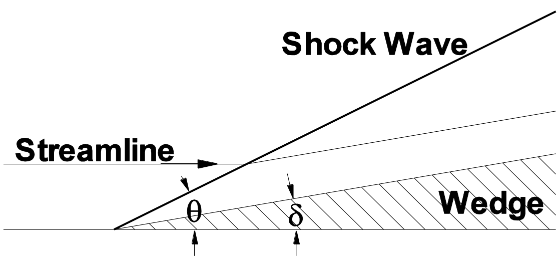


**Figure 2.9** Supersonic flow past a wedge

The velocity components $q_n$ and $q_t$ are the normal and tangential to the shock wave surface.

The above relations specify the jump conditions for a contact or tangential discontinuity if $q_{n2} = q_{n1}$, a shock wave if $q_{n2} < q_{n1}$, and a nonphysical (though mathematically possible, because the Euler equations lack an entropy condition) expansion jump if $q_{n2} > q_{n1}$. For the contact discontinuity $p_2 = p_1$, $q_{n2} = q_{n1} = 0$, and the allowed jumps in $\rho$ and $q_t$ are arbitrary. For each shock jump solution denoted by $(\rho_1, u_1, v_1, p_1)$, $(\rho_2, u_2, v_2, p_2)$ and $\theta$, there is also a nonphysical expansion jump given by $(\rho_1, -u_1, -v_1, p_1)$, $(\rho_2, -u_2, -v_2, p_2)$ and $\theta$.

### ***2.4.3.7 The Unsteady Euler Equations***

The Euler equations in two dimensions are (see also Chapter1, Section 1.4).

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \quad \text{where} \quad U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ e \end{pmatrix}, \quad F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (e+p)u \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} \rho v \\ \rho v u \\ \rho v^2 + p \\ (e+p)v \end{pmatrix}$$

Consider the surface $f(x,t) = x - x_0 - w(t - t_0) = 0$, where $w$ represents the shock velocity.

$$\vec{F} = U\vec{i}_t + F\vec{i}_x + G\vec{i}_y, \qquad \vec{n}_1 = \frac{\vec{\nabla} f}{\left|\vec{\nabla} f\right|} = \frac{\vec{i}_x - w\vec{i}_t}{\sqrt{1+w^2}} \quad \text{and} \quad \left(\vec{F}_1 - \vec{F}_2\right)\cdot\vec{n}_1 = \frac{1}{\sqrt{1+w^2}}\left(F_1 - F_2 - w(U_1 - U_2)\right)$$

$$\begin{aligned}
(\rho_2 - \rho_1)w &= \rho_2 u_2 - \rho_1 u_1 \\
(\rho_2 u_2 - \rho_1 u_1)w &= \rho_2 u_2^2 + p_2 - \rho_1 u_1^2 - p_1 \\
(\rho_2 v_2 - \rho_1 v_1)w &= \rho_2 u_2 v_2 - \rho_1 u_1 v_1 \\
(e_2 - e_1)w &= (e_2 + p_2)u_2 - (e_1 + p_1)u_1
\end{aligned}$$

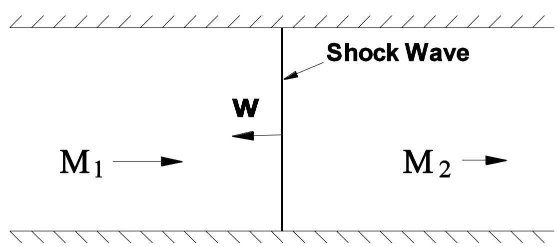


**Figure 2.10a** Moving shock wave

The above Euler relations specify the jump conditions for a contact discontinuity if $u_2 = u_1$, a shock if $u_2 < u_1$, and a nonphysical expansion jump if $u_2 > u_1$. For the contact discontinuity, $p_2 = p_1$, $u_2 = u_1 = w$, and the allowed jumps in $\rho$ and $v$ are arbitrary. The nonphysical expansion jumps are again possible solutions, because the Euler equations impose no restrictions against entropy decreases. For each shock solution, represented by $(\rho_1, u_1, v_1, p_1)$, $(\rho_2, u_2, v_2, p_2)$ and $w$, there is a corresponding expansion jump solution represented by $(\rho_1, -u_1, -v_1, p_1)$, $(\rho_2, -u_2, -v_2, p_2)$ and $w$.

If $w=0$ , then the shock wave is stationary.

$$\rho_2 u_2 = \rho_1 u_1 = m$$
$$\rho_2 u_2^2 + p_2 = \rho_1 u_1^2 + p_1 = n$$
$$\frac{e_2 + p_2}{\rho_2} = \frac{e_1 + p_1}{\rho_1} = h$$

where $m,\ n\ and\ h$ are constants evaluated from the flow variables ahead of the shock wave.

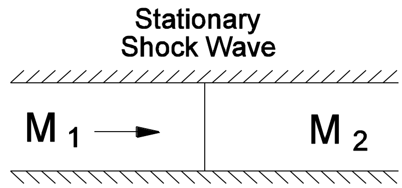


**Figure 2.10b** Stationary shock wave.

From the three relations above and using the perfect gas equation of state, $p=(\gamma-1)\left(e-\frac{1}{2}\rho u^2\right)$, we can obtain a quadratic equation for $u_2$, the velocity behind the shock wave, from the known values ahead of the shock wave.

$$\underbrace{\left(\frac{1}{2}-\frac{\gamma}{\gamma-1}\right)}_{a}u_2^2 \;+\; \underbrace{\frac{\gamma}{\gamma-1}\frac{n}{m}}_{b}u_2 \;\underbrace{-h}_{c} \;=0$$

with solutions $u_{2\pm} = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

One solution is the trivial no shock continuous solution and the other is the shock wave solution.

$$u_2 = \min\left\{u_{2-}, u_{2+}\right\}$$

The values for density, pressure and energy behind the shock wave are then found from the three shock jump relations shown above.

## ***2.5 Jacobians***

We can write Equation (2.1) in non-conservation form as follows

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial U}\frac{\partial U}{\partial x} + \frac{\partial G}{\partial U}\frac{\partial U}{\partial y} + \frac{\partial H}{\partial U}\frac{\partial U}{\partial z} = Q$$

or

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} + C\frac{\partial U}{\partial z} = Q \qquad (2.4)$$

where $$A = \frac{\partial F}{\partial U}, \quad B = \frac{\partial G}{\partial U} \text{ and } \quad C = \frac{\partial H}{\partial U}$$

The matrices $A$, $B$, and $C$ are called the *Jacobians* of $F$, $G$ and $H$ with respect to $U$. For the Euler equations in two dimensions they are each $4 \times 4$ matrices. In general for $m$-dimensional vectors $U$ and $F$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial u_1} & \cdots & \frac{\partial f_m}{\partial u_m} \end{pmatrix}, \text{ an } m \times m \text{ matrix, where } U = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \text{ and } F = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$$

The ***fundamental rule*** to obtain the *Jacobian* of $F$ with respect to $U$ is to first write $F$ in terms of the elements of $U$ and then take the partial derivatives indicated above.

If $U$ can be defined in terms of the elements of another vector $V$, $U = U(V)$, then Equation (2.4) can be transformed as follows:

$$\frac{\partial V}{\partial t} + A'\frac{\partial V}{\partial x} + B'\frac{\partial V}{\partial y} + C'\frac{\partial V}{\partial z} = Q' \tag{2.5}$$

where $A' = T^{-1}AT$, $B' = T^{-1}BT$, $C' = T^{-1}CT$, $Q' = T^{-1}Q$ and $T = \frac{\partial U}{\partial V}$. The matrix $T$ represents the *Jacobian* of $U$ with respect to $V$.

To find the *Jacobians* with respect to $U$ it is often convenient to choose a simple $V$ then

$$\frac{\partial}{\partial U} = \frac{\partial}{\partial V}\frac{\partial V}{\partial U} = \frac{\partial}{\partial V}T^{-1}$$

***Definition*: Homogeneous of Degree** $p$

Consider $F = F(U)$, $F$ is a function of the elements of $U$, and $U' = sU$ for any nonzero scalar $s$. If $F(U') = s^p F(U)$, then $F$ is said to be homogeneous of degree $p$ with respect to $U$.

If $F$ is homogeneous of degree $p$ with respect to $U$, then $AU = pF$, where $A = \frac{\partial F}{\partial U}$.

***Proof***: Let $f_i$ be an element of $F$, then in general

$$f_i = \sum_j t_{i,j}\,, \quad \text{where } t_{i,j} = c_{i,j} u_1^{\alpha_{i,j_1}} u_2^{\alpha_{i,j_2}} \cdots u_m^{\alpha_{i,j_m}}$$

If $F$ is homogeneous of degree $p$, then $\sum_k \alpha_{i,j_k} = p$ for all $i, j$.

$$\frac{\partial t_{i,j}}{\partial u_k} = \alpha_{i,j_k} c_{i,j} u_1^{\alpha_{i,j_1}} u_2^{\alpha_{i,j_2}} \cdots u_m^{(\alpha_{i,j_k}-1)} \cdots u_m^{\alpha_{i,j_m}}$$

$$\sum_k \frac{\partial t_{i,j}}{\partial u_k} = \sum_k \alpha_{i,j_k} t_{i,j} = p t_{i,j}\,, \quad \text{for all } i, j$$

Finally, $$\sum_k \frac{\partial f_i}{\partial u_k} u_k = p f_i \quad \Diamond$$

*Exercises:*

(1) For $F = \begin{pmatrix} \rho \\ \rho u^2 \end{pmatrix}$ and $U = \begin{pmatrix} \rho \\ \rho u \end{pmatrix}$ show that $A = \dfrac{\partial F}{\partial U} = \begin{pmatrix} 1 & 0 \\ -u^2 & 2u \end{pmatrix}$

(2) Write the Euler equations in conservation law form in two dimensions to find the flux vectors $F$ and $G$ and the state vector $U$. Assume a perfect gas equation of state for the following.

a) Calculate the *Jacobian* of $F$ with respect to $U$ (i) directly and (ii) indirectly using the transformation matrix $T$, as shown above, and the vector $V = (\rho, u, v, p)^T$.
b) Show that the flux vectors $F$ and $G$ have the remarkable property of being homogeneous of degree 1 with respect to $U$ and that, by actual multiplication, the flux vector $F = AU$.

## ***2.6 Linearization***

It is often convenient to linearize the governing equations for analysis. Equation (2.4) is easily linearized by ``freezing" the coefficient matrices $A$, $B$ and $C$ at some localized flow condition. The equation becomes

$$\frac{\partial U}{\partial t} + A_0 \frac{\partial U}{\partial x} + B_0 \frac{\partial U}{\partial y} + C_0 \frac{\partial U}{\partial z} = Q_0 \tag{2.6}$$

where $A_0$, $B_0$ and $C_0$ are constant matrices and $Q_0$ is a constant vector. Linearized equations have exact solutions. Given an initial value for $U$ at time $t_o$ on a compact domain

$$U(x,y,z,t_0) = \sum_j c_j e^{i(k_{x_j}x + k_{y_j}y + k_{z_j}z)}$$

where we have expanded the initial solution by Fourier decomposition with wave numbers $k_{x_j}$, $k_{y_j}$ and $k_{z_j}$.

The exact solution of Equation (2.6) for $t > t_0$ is

$$U(x,y,z,t\ ) = \sum_j e^{-i(t-t_0)(k_{x_j}A_0 + k_{y_j}B_0 + k_{z_j}C_0)} c_j e^{i(k_{x_j}x + k_{y_j}y + k_{z_j}z)} + (t-t_0)Q_0$$

Note that the solution has both a linear growth term and, depending on the eigenvalues of the matrices

$$M_j = k_{x_j}A_0 + k_{y_j}B_0 + k_{z_j}C_0 ,$$

possible exponential growth in time components.

## *2.7 Characteristic Relations*

The method of characteristics is a precise and powerful tool for solving hyperbolic sets of equations in two independent coordinates. Its extension to solve multi-dimensional problems encountered significant difficulties in the latter 1960's when computers became powerful enough for their consideration. General finite difference methods largely took over, although characteristic relations were often employed to update points along boundaries. Characteristic theory again received strong interest since the mid-1970's with the development of Moretti's *``Lambda'' scheme*, Steger and Warming's *Flux Split schemes*, Roe's *Flux Difference Vector splitting scheme* and the use of the eigenvalue-eigenvector structure of the governing equations to develop efficient implicit methods.

We start with the unsteady non-conservation form of the governing equations in coordinates $x$ and $t$. (Note: For the steady equations in two dimensions we can replace $(t,x)$ with $(x,y)$ in the following.)

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = Q$$

where $A = \dfrac{\partial F}{\partial U}$. Let similarity transformation $S$ diagonalize matrix $A$. The columns of $S^{-1}$ are the eigenvectors of $A$.

$$SAS^{-1} = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_m \end{bmatrix}$$

Multiplying the above time dependent equation by $S$ we obtain

$$S\frac{\partial U}{\partial t}+\Lambda S\frac{\partial U}{\partial x}=SQ$$

This equation can now be expressed as a set of $m$ ordinary differential equations, which define the set of characteristic equations

$$\sum_{n=1}^{m} s_{k,n}\frac{du_n}{d\xi_k}=h_k\,,\quad \text{for } k=1,2,3,\cdots,m\,,\quad \text{where } \frac{d}{d\xi_k}=\frac{\partial}{\partial t}+\lambda_k\frac{\partial}{\partial x}\,,$$

$$S=\begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{m1} & \cdots & s_{mn} \end{pmatrix},\quad U=\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}\quad \text{and } SQ=\begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix}$$

***<u>2.7.1 Application to the One-Dimensional Euler Equations</u>***

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}=0\text{, where } U=\begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix},\ F=\begin{bmatrix} \rho u \\ \rho u^2+p \\ (e+p)u \end{bmatrix}$$

$$\text{with } p=(\gamma-1)\left(e-\rho\frac{u^2}{2}\right) \text{ and } \frac{\gamma p}{\rho}=c^2 \text{ for sound speed } c\,.$$

We choose the non-conservation form of the equation (see Equations (2.4) and (2.5)) and for convenience use the vector $V$ and the Jacobian $T=\frac{\partial U}{\partial V}$ to derive the equation.

$$V=\begin{bmatrix} \rho \\ u \\ p \end{bmatrix},\quad \frac{\partial V}{\partial t}+A'\frac{\partial V}{\partial x}=0 \quad \text{and} \quad A'=T^{-1}AT=\begin{pmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \gamma p & u \end{pmatrix}$$

Similarity transformation $S$ diagonalizes $A'$.

$$SA'S^{-1}=\Lambda=\begin{pmatrix} u & 0 & 0 \\ 0 & u+c & 0 \\ 0 & 0 & u-c \end{pmatrix},\quad S=\begin{pmatrix} 1 & 0 & \frac{-1}{c^2} \\ 0 & \rho c & 1 \\ 0 & -\rho c & 1 \end{pmatrix}\quad \text{and } S^{-1}=\begin{pmatrix} 1 & \frac{1}{2c^2} & \frac{1}{2c^2} \\ 0 & \frac{1}{2\rho c} & \frac{-1}{2\rho c} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

The three characteristic relations are

$$\frac{d\rho}{d\xi_0}-\frac{1}{c^2}\frac{dp}{d\xi_0}=0 \quad \& \quad \frac{d}{d\xi_0}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}$$
$$\frac{dp}{d\xi_+}+\rho c\frac{du}{d\xi_+}=0 \quad \& \quad \frac{d}{d\xi_+}=\frac{\partial}{\partial t}+(u+c)\frac{\partial}{\partial x}$$
$$\frac{dp}{d\xi_-}-\rho c\frac{du}{d\xi_-}=0 \quad \& \quad \frac{d}{d\xi_-}=\frac{\partial}{\partial t}+(u-c)\frac{\partial}{\partial x}$$

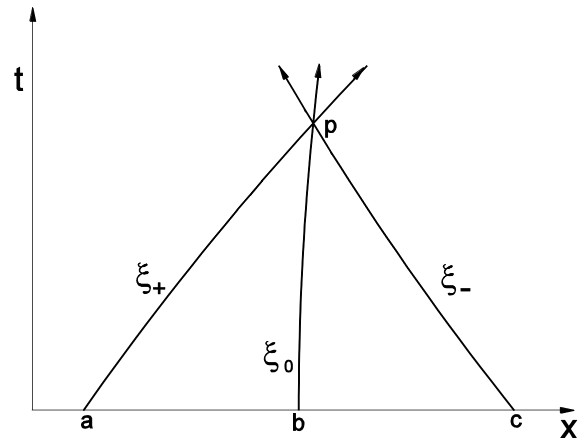


**Figure 2.11** Characteristic paths

***Example (1):*** Use the characteristic relations to find $\rho$, $u$ and $p$ at point p from known values at points a, b and c shown in Fig.2.11.

***Solution:*** The relations evaluated along the three characteristic paths are

$$\rho_p-\rho_b-(p_p-p_b)/c^2=0$$
$$p_p-p_a+\rho c(u_p-u_a)=0$$
$$p_p-p_c-\rho c(u_p-u_c)=0$$

Above are three relations in three unknowns $\rho_p$, $u_p$ and $p_p$. The unscripted variables in the equations could be first evaluated at points a, b and c to obtain a predicted solution at point p and then a corrected solution could be obtained using averaged values, including the point p, along each characteristic path segment.

***Example (2):*** Solve the model hyperbolic equation using characteristic theory. The model hyperbolic equation is its own characteristic relation, with characteristic speed $c>0$.

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0, \quad or \quad \frac{du}{d\xi}=0 \quad with \quad \frac{d}{d\xi}=\frac{\partial}{\partial t}+c\frac{\partial}{\partial x},$$

Consider Fig.2.12, with the solution $u$ known at points a and b. The solution at point p, using the characteristic relation $\frac{du}{d\xi}=0$ is $u_p = u_{x_0}$, where $u_{x_0}$ can be found by interpolation from the values of $u$ at a and b, $u_p = u_{x_0} = \big((c\Delta t)u_a + (\Delta x - c\Delta t)u_b\big)/\Delta x$.

If we, instead, approximated the model hyperbolic equation using a finite difference equation, as shown below.

$$\frac{u_p - u_b}{\Delta t} + c\frac{u_b - u_a}{\Delta x} = 0$$

Solving this equation for $u_p$, we obtain

$$\begin{aligned} u_p &= u_b + c\Delta t\frac{u_b - u_a}{\Delta x} \\ &= \big((c\Delta t)u_a + (\Delta x - c\Delta t)u_b\big)/\Delta x \end{aligned}$$

which is the same as the solution obtained using the characteristic relation.

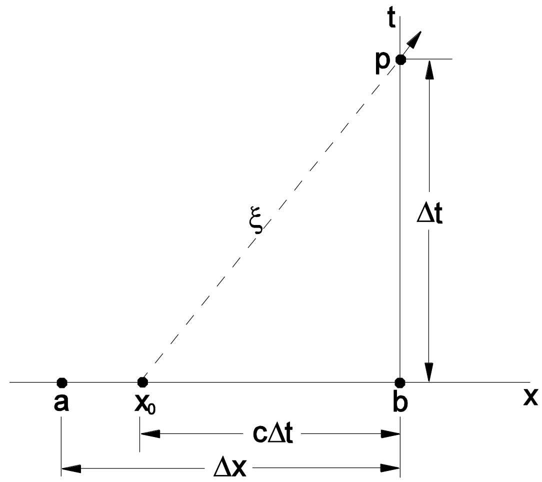


**Figure 2.12** Characteristic path for model hyperbolic equation

***2.7.2 Application to the steady Euler equations in two dimensions***

Consider the steady Euler equations in two dimensions

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0,$$

where

$$F = \begin{pmatrix} \rho \\ \rho u^2 + p \\ \rho uv \\ (e+p)u \end{pmatrix} \text{ and } G = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (e+p)v \end{pmatrix},$$

$$p = (\gamma - 1)\rho\varepsilon \text{ and } \varepsilon = c_v T = \frac{e}{\rho} - \frac{1}{2}u^2$$

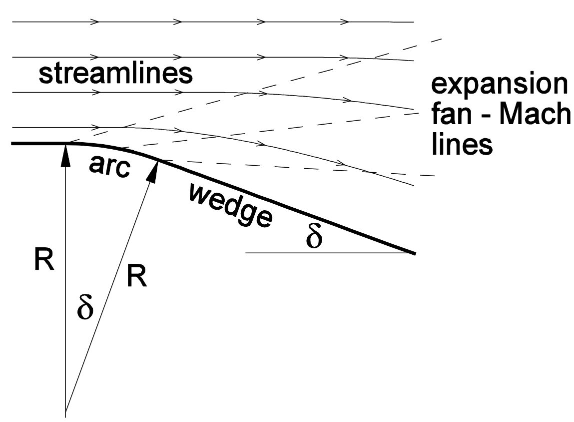


**Figure 2.13** 2-D steady flow over an expansion surface

In non-conservation form these equations become

$$A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0 \quad \text{or} \quad A''\frac{\partial V}{\partial x} + B''\frac{\partial V}{\partial y} = 0 \quad \text{with} \quad A'' = T^{-1}AT, \quad B'' = T^{-1}BT \quad \text{and} \quad T = \frac{\partial U}{\partial V}$$

where

$$V=\begin{bmatrix}\rho\\u\\v\\p\end{bmatrix},\quad A''=\begin{pmatrix}u&\rho&0&0\\0&u&0&\frac{1}{\rho}\\0&0&u&0\\0&\gamma p&0&u\end{pmatrix}\ \text{ and }\ B''=\begin{pmatrix}v&0&\rho&0\\0&v&0&0\\0&0&v&\frac{1}{\rho}\\0&0&\gamma p&v\end{pmatrix}\ \text{ with }\ p=(\gamma-1)\left(e-\rho\frac{u^2+v^2}{2}\right)$$

Multiplying by the inverse of $A''$, we obtain

$$\frac{\partial V}{\partial x}+A'''\frac{\partial V}{\partial y}=0,\quad \text{where } A'''=A''^{-1}B''=\begin{pmatrix}\frac{v}{u}&\frac{-\rho v}{u^2-c^2}&\frac{\rho u}{u^2-c^2}&\frac{v}{u(u^2-c^2)}\\0&\frac{uv}{u^2-c^2}&\frac{-c^2}{u^2-c^2}&\frac{-v/\rho}{u^2-c^2}\\0&0&\frac{v}{u}&\frac{1}{\rho u}\\0&\frac{-\rho c^2 v}{u^2-c^2}&\frac{\rho c^2 u}{u^2-c^2}&\frac{uv}{u^2-c^2}\end{pmatrix}\ \text{ and again, } c^2=\frac{\gamma p}{\rho}.$$

The matrix $A'''$ can be diagonalized by transformation $S'$.

$$S'A'''S'^{-1}=\Lambda'=\begin{pmatrix}\frac{v}{u}&0&0&0\\0&\frac{uv+c^2\beta}{u^2-c^2}&0&0\\0&0&\frac{v}{u}&0\\0&0&0&\frac{uv-c^2\beta}{u^2-c^2}\end{pmatrix},$$

$$S'=\begin{pmatrix}1&0&0&\frac{-1}{c^2}\\0&\frac{-\rho v}{\beta}&\frac{\rho u}{\beta}&1\\0&\rho u&\rho v&1\\0&\frac{\rho v}{\beta}&\frac{-\rho u}{\beta}&1\end{pmatrix}\quad\text{and}\quad S'^{-1}=\begin{pmatrix}1&\frac{1}{2c^2}&0&\frac{1}{2c^2}\\0&-\frac{uv+c^2\beta}{2\rho c^2(v+u\beta)}&\frac{u}{\rho q^2}&-\frac{uv-c^2\beta}{2\rho c^2(v-u\beta)}\\0&\frac{u^2-c^2}{2\rho c^2(v+u\beta)}&\frac{v}{\rho q^2}&\frac{u^2-c^2}{2\rho c^2(v-u\beta)}\\0&\frac{1}{2}&0&\frac{1}{2}\end{pmatrix}$$

$$q^2 = u^2 + v^2, \quad \beta = \sqrt{M^2 - 1} \text{ and } M = \frac{q}{c}$$

The resulting four characteristic relations are given as follows. Note that the following relations make sense only if the flow is supersonic.

$$\frac{d\rho}{d\xi_1} - \frac{1}{c^2}\frac{dp}{d\xi_1} = 0, \quad \frac{d}{d\xi_1} = \frac{\partial}{\partial x} + \frac{v}{u}\frac{\partial}{\partial y}$$

$$\frac{dp}{d\xi_2} + \frac{\rho u^2}{\beta}\frac{dv/u}{d\xi_2} = 0, \quad \frac{d}{d\xi_2} = \frac{\partial}{\partial x} + \frac{uv + c^2\beta}{u^2 - c^2}\frac{\partial}{\partial y}$$

$$\frac{dp}{d\xi_3} + \rho\frac{d(q^2/2)}{d\xi_3} = 0, \quad \frac{d}{d\xi_3} = \frac{\partial}{\partial x} + \frac{v}{u}\frac{\partial}{\partial y}$$

$$\frac{dp}{d\xi_4} - \frac{\rho u^2}{\beta}\frac{dv/u}{d\xi_4} = 0, \quad \frac{d}{d\xi_4} = \frac{\partial}{\partial x} + \frac{uv - c^2\beta}{u^2 - c^2}\frac{\partial}{\partial y}$$

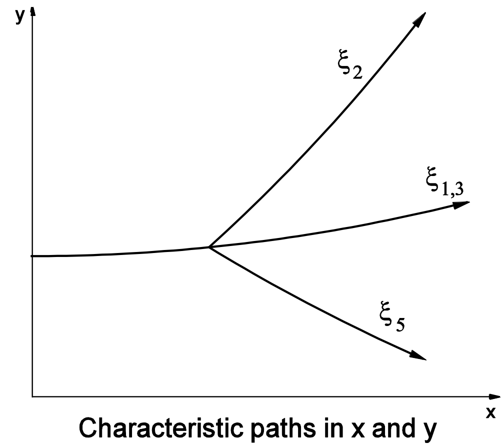


**Figure 2.14** Characteristic paths in $x$ and $y$

## ***2.8 Hyperbolic Requirement***

Consider the following equation

$$\frac{\partial E}{\partial x_\alpha} + \frac{\partial F}{\partial x_1} = 0$$

where $x_\alpha$ is the direction in which the solution is marched and $E$ is the state or solution vector. The coordinate $x_\alpha$ is time-like and is actually $t$ for unsteady flow, and could be, for example, $x$ for the steady Euler equations describing supersonic flow. The vector $E$ is initialized at $x_{\alpha_0}$ and then determined for $x_\alpha > x_{\alpha_0}$ by solving the above equation.

***Examples (3):***

**(1) Unsteady Euler Equations in one spatial dimension**

$$x_\alpha = t, \quad x_1 = x, \quad E = U = \begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix} \text{ and } F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (e+p)u \end{pmatrix}$$

**(2) Steady Euler Equations in two dimensions**

$$x_\alpha = x, \quad x_1 = y, \quad E = \begin{pmatrix} \rho \\ \rho u^2 + p \\ \rho uv \\ (e+p)u \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (e+p)v \end{pmatrix}$$

Let similarity transformations $S$ and $S^{-1}$ diagonalize matrix $A$ below

$$A = \frac{\partial F}{\partial E} = S^{-1}\Lambda S \quad \text{and} \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_m \end{bmatrix}$$

The above equation is hyperbolic if

(1) $\lambda_k$ is real for each $k$, and
(2) the matrix $A$ has a complete set of eigenvectors.

In the general multidimensional case, Equation (2.4), the system is hyperbolic if the matrix

$$M = k_x A + k_y B + k_z C$$

has only real eigenvalues and a complete set of eigenvectors, for all sets of real numbers $(k_x, k_y, k_z)$ (see Section 2.7).

## ***<u>2.9 Boundary Conditions for Hyperbolic Equation</u>***

The characteristic paths coming to or from the boundaries determine the number and nature of the required boundary conditions for solving a given hyperbolic equation.

In general, a flow field solution depends only upon the

1. initial conditions, (at $t = t_0$ in Fig 2.14)
2. governing equations, and
3. boundary conditions (at $x = x_0$ and $x = x_1$).

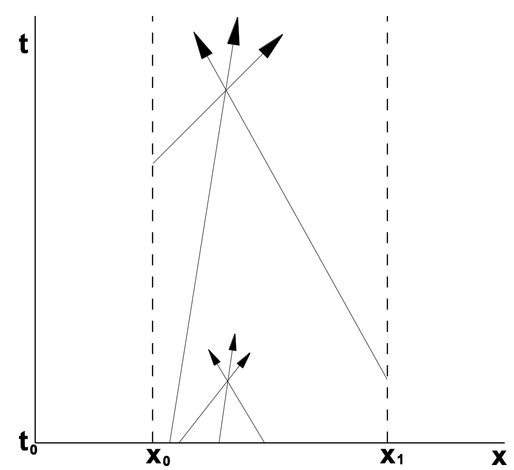

**Figure 2.15** Characteristic paths in $x$ and $t$

For each characteristic starting at a boundary point, a boundary value must be specified for the quantity being carried along the characteristic path. On the other hand, for each characteristic ending at a boundary the boundary point a quantity must be determined by an equation, generally a characteristic relation, relating the boundary point value to the interior of the domain. As the solution evolves in time its dependence on the choice of initial conditions diminishes, depending upon only the governing equations and the boundary conditions. This is very fortunate because all that is needed initially is a plausible starting solution and not a precise one.