Chapter 12

Boundary Conditions for Fluid Flow

12.1 Introduction

As previously pointed out in Section 2.9 for solving hyperbolic equations, characteristics relations and their paths determine where and what kind of boundary conditions are required and that the solution to an initial value problem should depend only upon three items (1) the initial conditions, (2) the governing equations, and (3) the boundary conditions. As the solution evolves in time its dependence on the choice of initial conditions diminishes, leaving only the governing equations and the boundary conditions.. We now discuss the boundary conditions, first for inviscid and then for viscous flow. The equations of compressible flow, the Euler equations in two spatial dimensions are

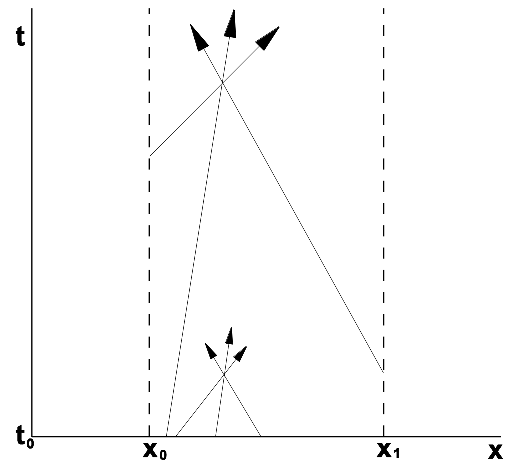


Figure 12.1 Characteristic paths in x and t

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0$$

where

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ e \end{pmatrix}, \quad F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho v u \\ (e+p)u \end{pmatrix} \quad and \quad G = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (e+p)v \end{pmatrix}$$

12.2 Impermeable Boundaries---Solid Walls or Streamlines

At an impermeable boundary the flow must be tangent to the boundary surface. The flow tangency condition is expressed as follows

$$\vec{q} \cdot \vec{n} = 0$$

where in two dimensions, $\vec{q} = u\vec{i_x} + v\vec{i_y}$ is the fluid velocity and $\vec{n}$ represents the unit normal vector to the boundary surface. As a result of the tangency condition, the only dependent flow variable required at an impermeable boundary for the Euler equations written in conservation law form is pressure ("***All you need is** p **!***"). This is in agreement with the number of characteristics, namely one, that brings information to an impermeable boundary point from the interior of the flow. See, for example, Figure 12.5. For inviscid flow, a solid wall boundary can only do two things (1) prevent the flow from crossing through it, handled by the flow tangency condition and (2) push on the flow through pressure at the wall. You may want to know other quantities at the wall, such as density, velocity and temperature, but these are not required to advance the flow solution. For viscous flow, boundary conditions for velocity and temperature will be needed to advance the flow in time

The following example, using the Euler equations in two dimensions for flow past a solid surface aligned with the x-axis, illustrates the ``**all you need is** p **!**" requirement. To approximate the governing equations on a finite difference mesh at interior points with $j = 2$ requires that the vector G be given at boundary points $(i,1)$.

At a solid wall G for the Euler equations is

$$G = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (e+p)v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p_{i,1} \\ 0 \end{pmatrix}_{at\ wall} ,$$

but because of the flow tangency condition $v = 0$.

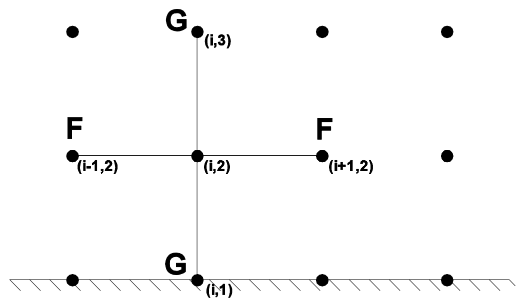


Figure 12.2 Mesh point stencil about (i,2)

This result is also true in curvilinear coordinate systems with impermeable boundaries placed arbitrarily in the flow. The rotated flux vector G' is (see Section 7.6)

$$G'^{n}_{i,j+1/2} = \left[\left(-F^n \frac{\partial y}{\partial \xi} + G^n \frac{\partial x}{\partial \xi}\right)\Delta\xi\right]_{i,j+1/2} = \begin{bmatrix} \rho qs \\ \rho u\, qs - \dfrac{\partial y}{\partial \xi} p \\ \rho v\, qs + \dfrac{\partial x}{\partial \xi} p \\ (e+p)qs \end{bmatrix}_{i,j+1/2} , \text{ where } \quad qs = -\frac{\partial y}{\partial \xi} u + \frac{\partial x}{\partial \xi} v$$

If the flux is at an impermeable boundary, flow tangency requires $\vec{q} \cdot \vec{n} = 0$, which implies $qs = 0$, leaving again only pressure to be determined at the boundary.

Two approaches are given to find pressure at solid wall boundaries - (1) the normal momentum equation for the unsteady the Euler equations and (2) a characteristic relation for solving the steady Euler equations for supersonic flow.

12.2.1 The Normal Momentum Equation

For unit normal vector $\vec{n} = n_x \vec{i}_x + n_y \vec{i}_y$ and velocity vector $\vec{q} = u\vec{i}_x + v\vec{i}_y$, the normal momentum equation can be defined by the following vector inner product of the normal vector with the Euler equations.

$$(0, n_x, n_y, 0) \cdot \left(\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \right),$$

which yields, using the tangency condition to simplify the equation,

$$n_x \frac{\partial p}{\partial x} + n_y \frac{\partial p}{\partial y} - \rho \vec{q} \cdot \left(u \frac{\partial \vec{n}}{\partial x} + v \frac{\partial \vec{n}}{\partial y} \right) = 0 \qquad \text{or} \qquad \frac{\partial p}{\partial n} = \rho \vec{q} \cdot \frac{\partial \vec{n}}{\partial \xi}$$

With the partial derivatives taken in directions normal and tangential to the wall as follows

$$\frac{\partial}{\partial n} = n_x \frac{\partial}{\partial x} + n_y \frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial \xi} = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}$$

Two cases can be considered, one for flat walls and one for flow along curved surfaces.

Case (1) Flat Surface (no curvature)

$$\frac{\partial \vec{n}}{\partial \xi} = 0$$

$$\frac{\partial p}{\partial n} = 0 \;\Rightarrow\; p_B \simeq p_A$$

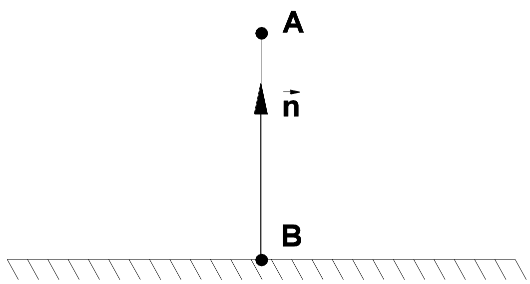


Figure 12.3 Boundary point on a flat wall

Case (2) Boundary of Curvature R

$$\vec{n} = \frac{x_0 - x}{R} \vec{i}_x + \frac{y_0 - y}{R} \vec{i}_y$$

$$\frac{\partial \vec{n}}{\partial \xi} = u \frac{\partial \vec{n}}{\partial x} + v \frac{\partial \vec{n}}{\partial y}$$

$$= u \frac{-1}{R} \vec{i}_x + v \frac{-1}{R} \vec{i}_y = -\frac{\vec{q}}{R}$$

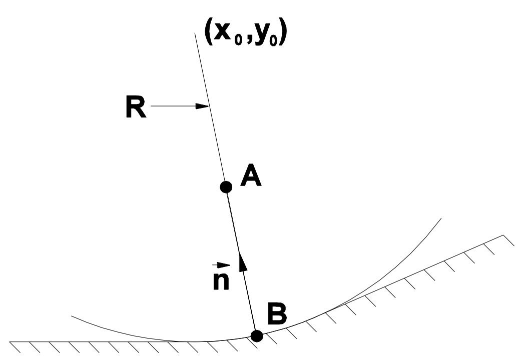


Figure 12.4 Boundary point on a curved wall

Thus $\frac{\partial p}{\partial n} = -\frac{\rho q^2}{R} \;\Rightarrow\; p_B \simeq p_A + \Delta n \frac{\rho_A q_A^2}{R}$, where Δn is the distance between points A and B.

The above expressions for pressure at wall point B, for both cases, require that the pressure at point A is known, either because point A is located at a mesh point or that a value for pressure at this point can be interpolated from values at the surrounding mesh points. The radius of curvature R can be found for a curved boundary given as a function of x, $y = f(x)$, by the following formula

$$R = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\left(\frac{d^2 y}{dx^2}\right)}$$

The sign of R is positive for concave surfaces, as shown in the figure above, and negative for convex surfaces. Mesh point values along the surface can be used to evaluate the derivatives if the equation for the surface is not given.

12.2.2 Characteristic Relations for the Steady Euler Equations

If the flow is steady and supersonic, the steady Euler equations can be marched in the supersonic flow direction. These equations are in two dimensions

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0$$

Assuming that the flow is supersonic in the x direction, i.e., $u > c$, a relation along characteristic path ξ_4, discussed in Section 2.7.2 and shown again in Fig.12.5, can be used relate wall pressure to the interior of the flow. This characteristic relation is

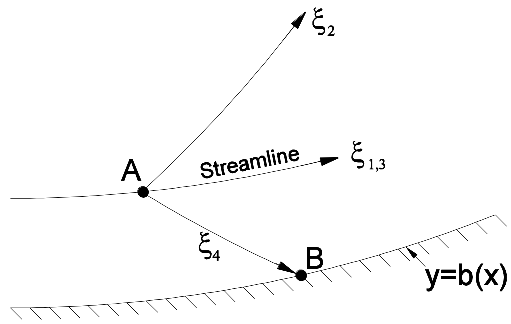


Figure 12.5 Characteristic paths near a solid wall

$$\frac{dp}{d\xi_4} - \frac{\rho u^2}{\beta}\frac{dv/u}{d\xi_4} = 0, \qquad \text{where} \qquad \frac{d}{d\xi_4} = \frac{\partial}{\partial x} + \frac{uv - c^2\beta}{u^2 - c^2}\frac{\partial}{\partial y},$$

$$\text{which is approximated by} \quad p_B \simeq p_A + \frac{\rho_A u_A^2}{\beta_A}\left(\frac{v_B}{u_B} - \frac{v_A}{u_A}\right)$$

But, from the flow tangency condition, $\frac{v_B}{u_B} = \frac{db(x)}{dx} = b'(x) =$ the slope of the boundary at point B.

Thus $$p_B \simeq p_A + \frac{\rho_A u_A^2}{\beta_A}\left(b'(x) - \frac{v_A}{u_A}\right) \quad \text{with} \quad \beta_A = \sqrt{\frac{u_A^2 + v_A^2}{c_A^2} - 1}$$

The values of the flow variables at point A can be found by interpolation using nearby mesh point values.

12.3 Entrance and Exit Boundaries

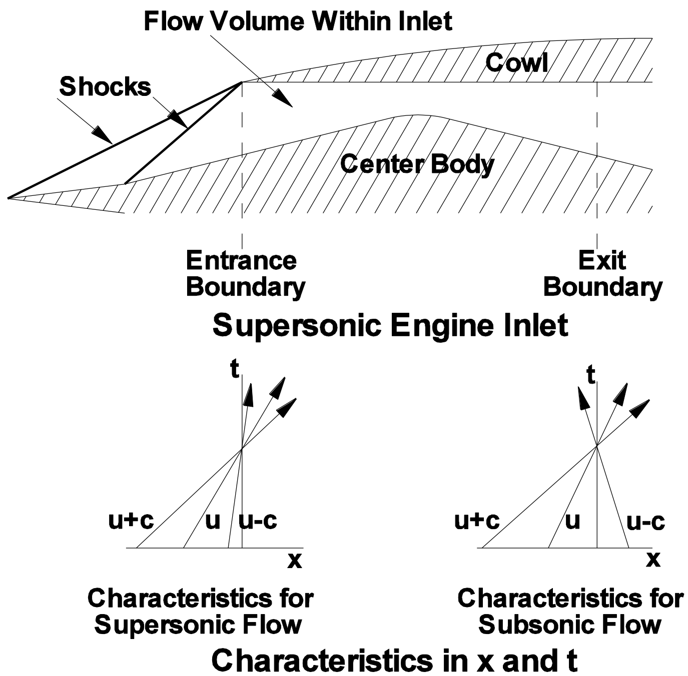


Figure 12.6 Entrance and exit boundary conditions

Consider the calculation within flow volume of the supersonic inlet shown in Figure 12.6. Four independent pieces of information must be given at each entrance and exit boundary point to solve the Euler equations in two dimensions. The number and type of conditions that need to be specified from information exterior to the flow, as well as that which must be determined from information from the interior of the flow volume, can be determined from an examination of the characteristic paths bringing information to each boundary. It is also always helpful, when numerically simulating an experiment, to examine the manner in which the experimenter controlled the flow at the boundaries. The following table summarizes the quantities to be specified from information external to the flow and the number of characteristic relations from the interior of the flow that need to be solved at both entrance and exit boundaries.

	Type	Quantities to be specified	Characteristic Relations to be solved - "characteristic speeds"
Entrance	Subsonic	$p_t,\ T_t,\ \theta$ or $h_t,\ m,\ \theta$	1 relation - "$u-c$"
	Supersonic	all	0 relations
Exit	Subsonic	p_{exit}	3 relations - "$u,\ u,\ u+c$"
	Supersonic	none	4 relations - "$u,\ u,\ u-c,\ u+c$"

Where p_t and T_t are the total pressure and temperature, θ is the flow angle of the entering fluid, which may vary across the entrance, h_t is the total enthalpy, m is mass flow rate and p_{exit} is the pressure at the exit. Supersonic entrance boundaries are the simplest, specifying all flow variables, usually at freestream conditions. Subsonic entrance boundaries are discussed next for both cases shown in the above table. Exit boundary conditions are given in Section 12.5.

12.4 Subsonic Entrance Boundary

Two cases are considered herein, (1) specified total pressure, temperature and entering flow angle and (2) specified total enthalpy, mass flow rate and entering flow angle. Both explicit and implicit procedures for their implementation are given. It is suggested that semi-implicit boundary conditions should always be used instead of purely explicit ones. We begin with the first case.

12.4.1 Total Pressure, Total Temperature and Flow Angle Given

The figure at the right shows the grid and the fluid velocity at the entrance. The boundary points have index i=1 and are aligned vertically with respect the x axis. Assume p_t, T_t and θ are known at the entrance. We can express pressure and temperature as functions of only the flow velocity u, as follows (see, for example, Anderson, *Fundamentals of Aerodynamics* or the isentropic relations given in Section 1.5).

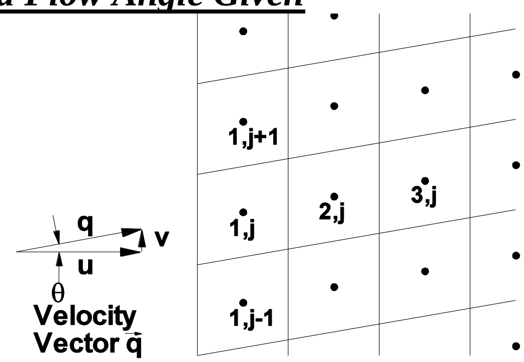


Figure 12.7 Grid at entrance

$$p = p_t\left[1-\frac{\gamma-1}{\gamma+1}\left(1+\tan^2\theta\right)\frac{u^2}{a_*^2}\right]^{\frac{\gamma}{\gamma-1}} = p(u) \tag{12.1}$$

$$T = T_t\left[1-\frac{\gamma-1}{\gamma+1}\left(1+\tan^2\theta\right)\frac{u^2}{a_*^2}\right] = T(u) \tag{12.2}$$

where we have also used the relations $v = \tan\theta\, u$ and $a_*^2 = 2\gamma\frac{\gamma-1}{\gamma+1}c_v T_t$, the speed of sound at sonic points.

To close the system of equations, we select the characteristic relation that couples the boundary with the interior flow volume (see for example Chapter 2, Section 2.7.1). We approximate the characteristic relation connecting points (1, j) and (2, j) below, where u is the velocity component normal to the entrance boundary. Variation in the y direction is neglected.

$$\frac{\partial p}{\partial t} - \rho c\frac{\partial u}{\partial t} = -(u-c)\left(\frac{\partial p}{\partial x} - \rho c\frac{\partial u}{\partial x}\right)$$

The total pressure relation, Equation (12.1), as a function of u, can be differentiated with respect to u to yield a relation between the incremental change in pressure to the change in velocity at the boundary.

$$\delta p_1 - \left.\frac{\partial p}{\partial u}\right|_1 \delta u_1 = 0, \tag{12.3}$$

where the operator δ is defined by $\delta z_1 = \Delta t \left.\frac{\partial z}{\partial t}\right|_1 = z_1^{n+1} - z_1^n$, $\left.\frac{\partial p}{\partial u}\right|_1$ is the derivative of Equation (12.1) and the subscript 1 indicates evaluation at a boundary point. The subscript j has been suppressed.

12.4.1.1 Semi-Implicit Entrance Boundary Condition

A semi-implicit finite difference relation that approximates the characteristic relation given above is shown below.

$$\frac{p_1^{n+1} - p_1^n}{\Delta t} - \rho c \frac{u_1^{n+1} - u_1^n}{\Delta t} = -(u-c)\left(\frac{p_2^n - p_1^{n+1}}{\Delta x} - \rho c \frac{u_2^n - u_1^{n+1}}{\Delta x} \right)$$

Note that the values at the boundary point are evaluated at time t^{n+1} within the spatial difference approximations. This equation can be written in "delta" law form by expressing $p_1^{n+1} = p_1^n + \delta p_1$ and $u_1^{n+1} = u_1^n + \delta u_1$.

$$\delta p_1 - \rho c\, \delta u_1 = \frac{-\lambda_4}{1-\lambda_4}\left(p_2^n - p_1^n - \rho c (u_2^n - u_1^n) \right) = R \tag{12.4}$$

where $\lambda_4 = (u-c)\frac{\Delta t}{\Delta x}$. The subscript j has also been suppressed and the missing subscripts on ρ, c and u indicate that they can evaluated by using either the data at $i=1$, $i=2$ or an average of the two at time t^n.

Solving for δu_1 from Equations (12.3) and (12.4) yields

$$\delta u_1 = \frac{R}{\left.\frac{\partial p}{\partial u}\right|_1 - \rho c}$$

Hence, the updated boundary point values at the mesh entrance, all determined from δu_1, are

$$u_1^{n+1} = u_1^n + \delta u_1, \qquad v_1^{n+1} = \tan\theta\, u_1^{n+1}, \qquad p_1^{n+1} = p(u_1^{n+1}) \quad \text{and} \quad T_1^{n+1} == T(u_1^{n+1})$$

Thus

$$\rho_1^{n+1} = \frac{p_1^{n+1}}{(\gamma-1)c_v T_1^{n+1}} \qquad \text{and} \qquad e_1^{n+1} = \rho_1^{n+1}\left(c_v T_1^{n+1} + \frac{1}{2}\left((u_1^{n+1})^2 + (v_1^{n+1})^2 \right) \right)$$

The changes in the conservative variables are given by

$$\left.\begin{aligned}\delta\rho_1 &= \rho_1^{n+1} - \rho_1^n \\ \delta(\rho u)_1 &= \rho_1^{n+1}u_1^{n+1} - \rho_1^n u_1^n \\ \delta(\rho v)_1 &= \rho_1^{n+1}v_1^{n+1} - \rho_1^n v_1^n \\ \delta e_1 &= e_1^{n+1} - e_1^n\end{aligned}\right\} \quad or \quad \delta U_{1,j}^{n+1} = U_{1,j}^{n+1} - U_{1,j}^n, \quad for\ all\ j$$

12.4.1.2 Note on explicit and semi-implicit boundary conditions

Let's use the model hyperbolic equation as an example for a characteristic relation to be used as a boundary condition at the entrance.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad \text{with} \quad c < 0$$

This equation brings information to the left boundary from the interior of the flow volume. An explicit approximation to this equation is

$$u_i^{n+1} = u_i^n - c\Delta t\frac{D_+ \cdot}{\Delta x}u_i^n = u_i^n - \lambda(u_{i+1}^n - u_i^n) \quad \text{with} \quad \lambda = \frac{c\Delta t}{\Delta x} \text{ and } i = 1 \text{ for the boundary point.}$$

In "delta law" form, with $\delta u_1 = u_1^{n+1} - u_1^n$, this explicit equation becomes

$$\delta u_1 = -\lambda(u_2^n - u_1^n)\,.$$

A semi-implicit approximation to this equation evaluated at the left boundary is

$$u_1^{n+1} = u_1^n - \lambda(u_2^n - u_1^{n+1}) = u_1^n - \lambda(u_2^n - u_1^n) + \lambda(u_1^{n+1} - u_1^n)$$

In "delta law" form, this semi-implicit equation becomes

$$\delta u_1 = \frac{-\lambda}{1-\lambda}(u_2^n - u_1^n)$$

Note the similarity of the two expressions for δu_1. The semi-implicit formula has built in protection, because the factor $\frac{-\lambda}{1-\lambda}$ is bounded within the interval $(0,1)$ and therefore errors in the spatial difference approximation can not be amplified (i.e., multiplied by a factor greater than one). Unlike the explicit formula above, the semi-implicit expression is stable for any size Δt and is therefore preferred. This expression can be used as a boundary condition procedure for solving both explicit and implicit difference equations. However, the next sections shows how it is possible to implement fully implicit procedures.

12.4.1.3 Fully Implicit Entrance Boundary Condition

For implicit algorithms it is recommended that fully implicit boundary conditions be used. Implicit boundary condition procedures are more complex, but always possible. Again assuming that the total pressure and total temperature are given, Equations (12.1) and (12.2) can be used to obtain an equation for density as a function of u, using the perfect gas equation of state.

$$\rho(u) = \frac{p(u)}{(\gamma - 1)c_v T(u)}$$

We can differentiate this equation for $\rho(u)$ and the equation for $p(u)$ with respect to u at x_1 to obtain the following two equations.

$$\delta\rho_1 = \left.\frac{\partial\rho}{\partial u}\right|_1 \delta u_1 \quad \text{and} \quad \delta p_1 = \left.\frac{\partial p}{\partial u}\right|_1 \delta u_1$$

The equation for the angle θ of the entering flow, $v_1 = \tan\theta\, u_1$, yields

$$\delta v_1 = \tan\theta\, \delta u_1$$

The characteristic equation can be implicitly approximated as follows.

$$\frac{p_1^{n+1} - p_1^n}{\Delta t} - \rho c\,\frac{u_1^{n+1} - u_1^n}{\Delta t} = -(u-c)\left(\frac{p_2^{n+1} - p_1^{n+1}}{\Delta x} - \rho c\,\frac{u_2^{n+1} - u_1^{n+1}}{\Delta x}\right)$$

This equation can be written in "delta" law form by expressing $p_i^{n+1} = p_i^n + \delta p_i$ and $u_i^{n+1} = u_i^n + \delta u_i$. Retaining the lowest order terms

$$\frac{\delta p_1}{\Delta t} - \rho c\,\frac{\delta u_1}{\Delta t} + (u-c)\left(\frac{\delta p_2 - \delta p_1}{\Delta x} - \rho c\,\frac{\delta u_2 - \delta u_1}{\Delta x}\right) = -(u-c)\left(\frac{p_2^n - p_1^n}{\Delta x} - \rho c\,\frac{u_2^n - u_1^n}{\Delta x}\right) = R$$

Defining $\lambda_4 = (u-c)\dfrac{\Delta t}{\Delta x}$ and then combining the four equations for $\delta\rho$, δu, δv and , δp we obtain the following matrix equation, in terms of $\delta V = \begin{bmatrix} \delta\rho \\ \delta u \\ \delta v \\ \delta p \end{bmatrix}$.

$$\underbrace{\begin{bmatrix} 1 & -\left.\dfrac{\partial \rho}{\partial u}\right|_1 & 0 & 0 \\ 0 & -\left.\dfrac{\partial p}{\partial u}\right|_1 & 0 & 1 \\ 0 & -\tan\theta & 0 & 1 \\ 0 & -\rho c(1-\lambda_4) & 0 & 1-\lambda_4 \end{bmatrix}}_{A'} \delta V_1^{n+1} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\rho c\lambda_4 & 0 & \lambda_4 \end{bmatrix}}_{B'} \delta V_2^{n+1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta t R \end{bmatrix}}_{\Delta U_1^n}$$

Using the matrix $S = \dfrac{\partial V}{\partial U}$ given in Section 9.6 to transform from δV to δU, we obtain

$$A' S \delta U_1^{n+1} + B' S \delta U_2^{n+1} = \Delta U_1^n$$

$$\text{or} \quad \bar{\mathrm{A}}_1 \delta U_1^{n+1} + \bar{\mathrm{B}}_1 \delta U_2^{n+1} = \Delta U_1^n, \quad \text{where} \quad \bar{\mathrm{A}}_1 = A'S \quad \text{and} \quad \bar{\mathrm{B}}_1 = B'S$$

This last equation represents the boundary condition given by the bottom row in a typical implicit matrix equation of form,

$$\begin{bmatrix} \bar{\mathrm{A}}_I & \bar{\mathrm{C}}_I & 0 & 0 & 0 & 0 & 0 \\ \bar{\mathrm{B}}_{I-1} & \bar{\mathrm{A}}_{I-1} & \bar{\mathrm{C}}_{I-1} & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 & 0 \\ 0 & 0 & \bar{\mathrm{B}}_i & \bar{\mathrm{A}}_i & \bar{\mathrm{C}}_i & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \bar{\mathrm{B}}_2 & \bar{\mathrm{A}}_2 & \bar{\mathrm{C}}_2 \\ 0 & 0 & 0 & 0 & 0 & \bar{\mathrm{B}}_1 & \bar{\mathrm{A}}_1 \end{bmatrix} \begin{bmatrix} \delta U_I^{n+1} \\ \delta U_{I-1}^{n+1} \\ \vdots \\ \delta U_i^{n+1} \\ \vdots \\ \delta U_2^{n+1} \\ \delta U_1^{n+1} \end{bmatrix} = \begin{bmatrix} \Delta U_I^n \\ \Delta U_{I-1}^n \\ \vdots \\ \Delta U_i^n \\ \vdots \\ \Delta U_2^n \\ \Delta U_1^n \end{bmatrix}$$

This matrix equation can be solved for all δU using the implicit solution procedures presented in Chapter 11.

12.4.2 Total Enthalpy, Mass Flow Rate and Flow Angle Given

If the total enthalpy, $h_t = (e+p)/\rho$, and the mass flow rate, $m = \rho u$, are used to control the flow at the entrance, then we can replace the "delta" relations for total pressure and density used above as follows.

$$h_t = (e+p)/\rho = (\rho\varepsilon + \rho(u^2+v^2)/2 + p)/\rho,$$

where ε is the internal energy per unit mass and $\varepsilon = c_v T$. Using the perfect gas equation of state, $p = (\gamma-1)\rho\varepsilon$, we obtain

$$h_t = \frac{\gamma}{\gamma-1}\frac{p}{\rho} + \frac{u^2+v^2}{2} \quad \text{and} \quad m = \rho u$$

The new "delta" relations are therefore

$$\delta m = u_1 \delta\rho_1 + \rho_1 \delta u_1$$

and

$$\delta h_t = \frac{\gamma}{\gamma-1}\frac{\delta p_1}{\rho_1} - \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1^2}\delta\rho_1 + u_1\delta u_1 + v_1 \delta v_1$$

The equation for the angle θ of the entering flow, $v_1 = \tan\theta\, u_1$, yields

$$\delta v_1 = \tan\theta\, \delta u_1$$

If total enthalpy and mass flow rate are fixed in time $\delta h_t = 0$ and $\delta m = 0$.

$$\frac{\partial p}{\partial t} - \rho c \frac{\partial u}{\partial t} = -(u-c)\left(\frac{\partial p}{\partial x} - \rho c \frac{\partial u}{\partial x}\right)$$

12.4.2.1 Semi-Implicit Entrance Boundary Condition

A semi-implicit approximation to this characteristic equation, derived in Sec.12.4.1.1, written in "delta" law form is

$$\delta p_1 - \rho c\, \delta u_1 = \frac{-\lambda_4}{1-\lambda_4}\left(p_2^n - p_1^n - \rho c(u_2^n - u_1^n)\right) = R, \quad \text{where} \quad \lambda_4 = (u-c)\frac{\Delta t}{\Delta x}.$$

These four "delta" form equations can be combined into matrix form as follows.

$$\begin{bmatrix} u_1 & \rho_1 & 0 & 0 \\ -\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1^2} & u_1 & v_1 & \frac{\gamma}{\gamma-1}\frac{1}{\rho_1} \\ 0 & -\tan\theta & 1 & 0 \\ 0 & -\rho c & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta\rho_1 \\ \delta u_1 \\ \delta v_1 \\ \delta p_1 \end{bmatrix} = \begin{bmatrix} \delta m \\ \delta h_t \\ 0 \\ R \end{bmatrix}$$

Once solved for δV_1, the inverse of the matrix $S = \frac{\partial V}{\partial U}$, given in Sec.9.5, can be used to find the elements of $\delta U_1 = S_1^{-1} \delta V_1$.

12.4.2.2 Fully Implicit Entrance Boundary Condition

The above characteristic equation approximated fully implicitly, as derived in Sec.12.4.1.3, written in "delta" law form is

$$\frac{\delta p_1}{\Delta t}-\rho c\frac{\delta u_1}{\Delta t}+(u-c)\left(\frac{\delta p_2-\delta p_1}{\Delta x}-\rho c\frac{\delta u_2-\delta u_1}{\Delta x}\right)=-(u-c)\left(\frac{p_2^n-p_1^n}{\Delta x}-\rho c\frac{u_2^n-u_1^n}{\Delta x}\right)=R$$

This equation can be combined with the "delta" form of the equations for total enthalpy, mass flow rate and flow angle to form the following matrix equation

$$\underbrace{\begin{bmatrix} u_1 & \rho_1 & 0 & 0 \\ -\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1^2} & u_1 & v_1 & \frac{\gamma}{\gamma-1}\frac{1}{\rho_1} \\ 0 & -\tan\theta & 0 & 1 \\ 0 & -\rho c(1-\lambda_4) & 0 & 1-\lambda_4 \end{bmatrix}}_{A'}\delta V_1^{n+1}+\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\rho c\lambda_4 & 0 & \lambda_4 \end{bmatrix}}_{B'}\delta V_2^{n+1}=\underbrace{\begin{bmatrix} \delta m \\ \delta h_t \\ 0 \\ \Delta tR \end{bmatrix}}_{\Delta U_1^n}$$

Again, using the matrix $S=\frac{\partial V}{\partial U}$ given in Section 9.5 to transform from δV to δU, we obtain

$$A'S\delta U_1^{n+1}+B'S\delta U_2^{n+1}=\Delta U_1^n$$

$$\text{Or}\qquad \bar{A}_1\delta U_1^{n+1}+\bar{B}_1\delta U_2^{n+1}=\Delta U_1^n,\quad \text{where } \bar{A}_1=A'S \quad\text{and}\quad \bar{B}_1=B'S$$

12.5 Subsonic or Supersonic Exit Boundary

Assume p_{exit} is specified at the exit if the flow is subsonic. The mesh and boundary point I,j at the exit are shown in Figure 12.8. Four characteristic relations connecting points I,j and $I-1,j$ are given by

$$\frac{\partial\rho}{\partial t}-\frac{1}{c^2}\frac{\partial p}{\partial t}=-u\left(\frac{\partial\rho}{\partial x}-\frac{1}{c^2}\frac{\partial p}{\partial x}\right)$$
$$\frac{\partial p}{\partial t}+\rho c\frac{\partial u}{\partial t}=-(u+c)\left(\frac{\partial p}{\partial x}+\rho c\frac{\partial u}{\partial x}\right)$$
$$\frac{\partial v}{\partial t}=-u\frac{\partial v}{\partial x}$$
$$\frac{\partial p}{\partial t}-\rho c\frac{\partial u}{\partial t}=-(u-c)\left(\frac{\partial p}{\partial x}-\rho c\frac{\partial u}{\partial x}\right)$$

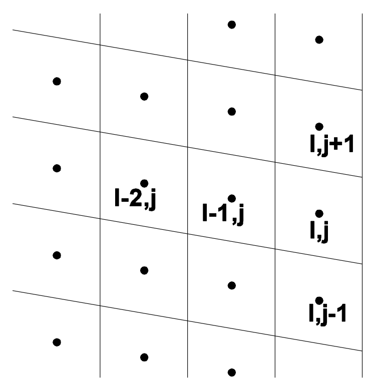


Figure 12.8 Grid at exit

Note that only the time and the x spatial derivatives are retained. It is also assumed that the exit boundary is located in a region where the y spatial derivatives are negligible.

12.5.1 Semi-Implicit Exit Boundary Condition

Semi-implicit finite difference approximations for the above relations are given as follows (corresponding explicit approximations result if the quotients of form $\dfrac{\lambda}{1+\lambda}$ are replaced simply by λ below).

$$\delta\rho - \frac{1}{c^2}\,\delta p = -\frac{\lambda_1}{1+\lambda_1}\left(\rho_I^n - \rho_{I-1}^n - \frac{1}{c^2}(p_I^n - p_{I-1}^n)\right) = R_1, \quad \lambda_1 = u\,\frac{\Delta t}{\Delta x}$$

$$\delta p + \rho c\,\delta u = -\frac{\lambda_2}{1+\lambda_2}\left(p_I^n - p_{I-1}^n + \rho c(u_I^n - u_{I-1}^n)\right) = R_2, \quad \lambda_2 = (u+c)\,\frac{\Delta t}{\Delta x}$$

$$\delta v = -\frac{\lambda_1}{1+\lambda_1}(v_I^n - v_{I-1}^n) = R_3$$

$$\delta p - \rho c\,\delta u = -\frac{\lambda_4}{1+\lambda_4}\left(p_I^n - p_{I-1}^n - \rho c(u_I^n - u_{I-1}^n)\right) = R_4, \quad \lambda_4 = (u-c)\,\frac{\Delta t}{\Delta x}$$

The fourth characteristic relation cannot be used if the flow at the exit is subsonic. Therefore solving for δp

$$\delta p = \begin{cases} \dfrac{R_2 + R_4}{2}, & if \;\; M = \dfrac{u_{I-1}}{c_{I-1}} > 1 \\[2ex] 0 \quad , & if \;\; M < 1 \;\; \text{(assuming } \dfrac{\partial p_{\text{exit}}}{\partial \text{t}} = 0) \end{cases}$$

Then

$$\delta\rho = R_1 + \frac{\delta p}{c^2}, \quad \delta u = \frac{R_2 - \delta p}{\rho c} \quad \text{and} \quad \delta v = R_3$$

Thus

$$\begin{aligned} \rho_I^{n+1} &= \rho_I^n + \delta\rho \\ u_I^{n+1} &= u_I^n + \delta u \\ v_I^{n+1} &= v_I^n + \delta v \\ p_I^{n+1} &= p_I^n + \delta p \end{aligned}$$

$$T_I^{n+1} = \frac{p_I^{n+1}}{(\gamma-1)c_v\rho_I^{n+1}} \quad \text{and} \quad e_I^{n+1} = \rho_I^{n+1}\left(c_v T_I^{n+1} + \frac{1}{2}\left((u_I^{n+1})^2 + (v_I^{n+1})^2\right)\right)$$

The conserved variable changes for momentum and energy at exit grid point I, j are

$$\left.\begin{aligned} \delta\rho_I &= \rho_I^{n+1} - \rho_I^n \\ \delta(\rho u)_I &= \rho_I^{n+1}u_I^{n+1} - \rho_I^n u_I^n \\ \delta(\rho v)_I &= \rho_I^{n+1}v_I^{n+1} - \rho_I^n v_I^n \\ \delta e_I &= e_I^{n+1} - e_I^n \end{aligned}\right\} \quad or \quad \delta U_{I,j}^{n+1} = U_{I,j}^{n+1} - U_{I,j}^n, \quad for\ all\ j$$

12.5.2 Fully Implicit Exit Boundary Condition

The four characteristic equations given earlier at the exit can be written in matrix form as

$$C'\frac{\partial V}{\partial t} = -\Lambda C'\frac{\partial V}{\partial x}, \quad \text{where} \quad V = \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix},$$

$$C' = \begin{bmatrix} 1 & 0 & 0 & -1/c^2 \\ 0 & \rho c & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -\rho c g & 0 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & u+c & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & (u-c)g \end{bmatrix} \text{ and } g = \begin{cases} 1, if\ u \ge c \\ 0, if\ u < c \end{cases},$$

assuming that the exit pressure is fixed in time if the flow at the exit is subsonic. This equation can be approximated implicitly by

$$C'\frac{V_I^{n+1} - V_I^n}{\Delta t} = -\Lambda C'\frac{V_I^{n+1} - V_{I-1}^{n+1}}{\Delta x},$$

which in “delta “ law form becomes

$$\left(C' + \frac{\Delta t}{\Delta x}\Lambda C'\right)\delta V_I^{n+1} - \frac{\Delta t}{\Delta x}\Lambda C'\delta V_{I-1}^{n+1} = -\Delta t \Lambda C'\frac{V_I^n - V_{I-1}^n}{\Delta x}$$

Using the matrix S given earlier in Section 9.5 to transform from δV to δU, we obtain

$$\left(I + \frac{\Delta t}{\Delta x}\Lambda\right)C'S\delta U_I^{n+1} - \frac{\Delta t}{\Delta x}\Lambda C'S\delta U_{I-1}^{n+1} = -\Delta t \Lambda C'\frac{V_I^n - V_{I-1}^n}{\Delta x}$$

Therefore, we can define the top row of the matrix equation of Section 12.4.1.3 for this fully implicit boundary condition as

$$\bar{\mathrm{A}}_I = \left(C' + \frac{\Delta t}{\Delta x}\Lambda C'\right)S, \quad \bar{\mathrm{C}}_I = -\frac{\Delta t}{\Delta x}\Lambda C'S \quad \text{and} \quad \Delta U_I^n = -\Delta t \Lambda C'\frac{V_I^n - V_{I-1}^n}{\Delta x}$$

12.6 Boundary Conditions for the Navier-Stokes Equations

In addition to pressure at impermeable boundaries, additional conditions are needed for the Navier-Stokes equations. These include temperature for heat transfer and each velocity component to insure zero velocity at solid walls, i.e. the no slip condition. We will consider viscous flow conditions at a solid surface and at an axis of symmetry. We assume $\vec{n}$ is the unit normal to the surface and v is the velocity component in this direction.

12.6.1 Solid Wall Boundary Conditions

pressure	$\frac{\partial p}{\partial n} = 0$	Boundary layer equation
velocity	$\vec{q} = 0$	no slip condition
temperature	$k\frac{\partial T}{\partial n} = 0$	Adiabatic wall
	$T = T_{wall}$	Isothermal wall

The normal momentum equation for viscous flow at solid walls is the same equation that is used in the boundary layer theory equations across the entire layer. It is used for Navier-Stokes calculations just at wall boundary points and is consistent with that given earlier, for either flat or curved walls, because of the no slip condition on velocity. At standard temperatures and pressures, with molecular mean free paths insignificantly small in comparison to relative body lengths, there is no velocity slip at solid walls. Under the rarefied conditions of high altitudes slip is present, however. Boundary conditions for temperature are usually adiabatic (no heat transfer) or the wall temperature is specified, either as constant (isothermal) or as a given function along the wall.

12.6.2 Axis of Symmetry Boundary Conditions

pressure	$\frac{\partial p}{\partial n} = 0$	Pressure is symmetric
velocity	$\frac{\partial u}{\partial n} = 0$ and $v = 0$	Tangential velocity u is symmetric Normal velocity v is anti-symmetric
temperature	$\frac{\partial T}{\partial n} = 0$	Temperature is symmetric

Flow variables across symmetry boundaries are either symmetric or anti-symmetric. The flow on one side is the mirror image of that on the other side.

12.6.3 Finite Difference and Finite Volume Implementations at a Solid Wall

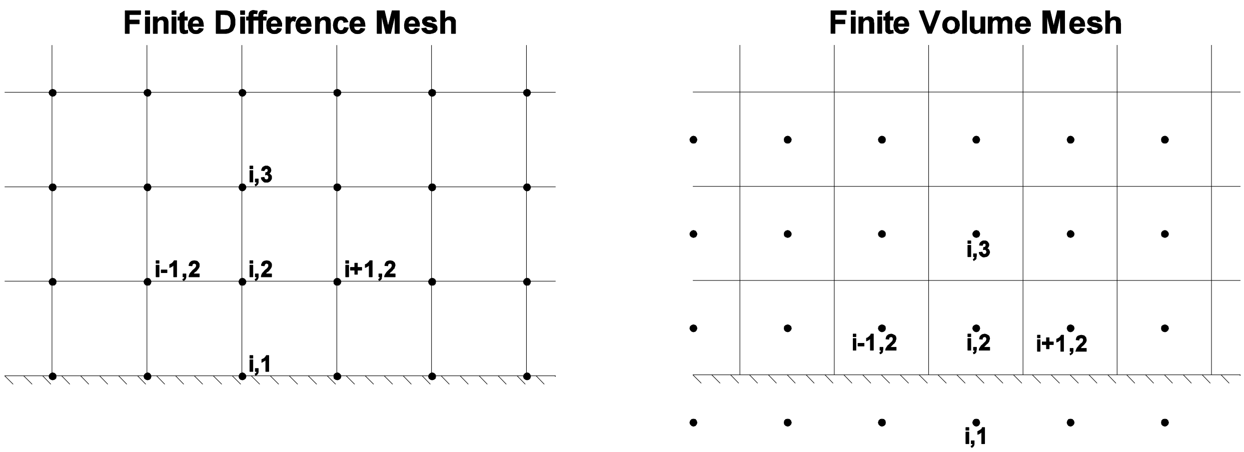


Figure 12.9 Finite difference and finite volume meshes

Finite difference and finite volume calculations can use the same grids. The chief distinction between the two approaches is that the finite difference approach places its mesh points at the intersection of the grid lines and the finite volume approach places its points at the volume centers of the cells formed by the grid lines.

Finite Difference boundary conditions

$p_{i,1} = p_{i,2}$

$u_{i,1} = v_{i,1} = 0$, no slip

$$T_{i,1} = \begin{cases} T_{i,2} & , adiabatic \\ T_{wall} & , isothermal \end{cases}$$

Finite Volume boundary conditions

$p_{i,1} = p_{i,2}$

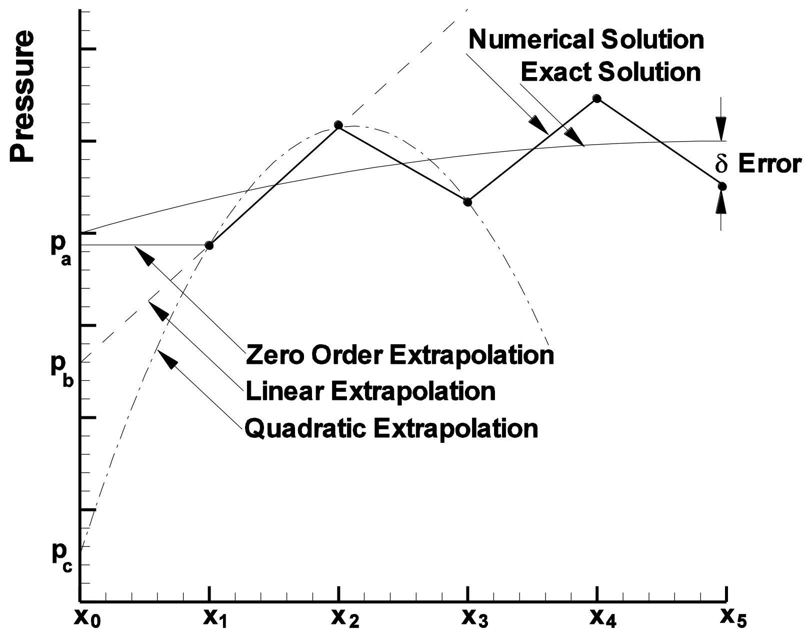
$u_{i,1} = -u_{i,2}\ , v_{i,1} = -v_{i,2}$, no slip

$$T_{i,1} = \begin{cases} T_{i,2} & , adiabatic \\ 2T_{wall} - T_{i,2} & , isothermal \end{cases}$$

The finite volume mesh has no point on the wall. The "no slip" and isothermal wall boundary conditions given above assure that the approximation of derivatives using the boundary points are correct. Note that at some conditions, for example at cold walls, $T_{i,1}$ could become negative and caution should to taken not to use these values at so the called "ghost points" other than to approximate derivatives, such as $\left.\frac{\partial T}{\partial n}\right|_{wall}$, required by the flow. It is good to keep in mind that the interior points are the important points and that the boundary points only have meaning in that they feed the interior points the required information about conditions at the boundary.

12.7 Boundary Conditions to Avoid

The following sketch illustrates the danger in using extrapolation to determine boundary conditions. The numerical solution shown in the sketch has a typical *"saw tooth"* error of amplitude δ. The error in determining pressure at wall point x_0 increases with the order of extrapolation used. Only the zero order extrapolation formula has the possibility of containing some flow physics (see the inviscid flat wall and boundary layer relations for pressure given earlier in Section 12.2.1.1). The other extrapolations are purely mathematical and should be only used to determine flow values that will not be fed back into the calculation via boundary conditions.



$$(a)\quad \frac{\partial p}{\partial x}=0,\quad p_a=p_1 \quad - \text{ zero order extrapolation}$$

$$no\ physics\begin{cases}(b)\quad p_b=2p_1-\mathrm{p}_2, & - \text{ first order extrapolation}\\ (c)\quad p_b=\cdots\quad , & - \text{ second order extrapolation}\end{cases}$$

Figure 12.10 Extrapolated boundary conditions

12.8 The Order of Accuracy Required for Boundary Conditions

If the algorithm used to update the interior points of the mesh, for solving the Euler equations, is of the order of accuracy p then the order of accuracy required for boundary point procedures is $p-1$. The following discussion supporting this statement came from a conversation with H.O. Kreiss on a bus in Belgium in 1973. Let the truncation error for updating interior points and boundary points be given by $\tau_i=\alpha\,\Delta h^p$ and $\tau_b=\beta\,\Delta h^{p-1}$, respectively, where Δh is a measure of the mesh point spacing. The order of accuracy of τ_i is given by $\tau_i\approx O\left(\Delta x^r,\Delta y^s,\Delta z^t,\Delta t^q\right)=O\left(\Delta h^p\right)$, where $p=\min(r,s,t,q)$ and $\Delta t=O(\Delta h)$ through the CFL condition, i.e. $\Delta t\approx CFL\dfrac{\Delta h}{|u|+c}$. The parameters α and β are order of one. Consider information traveling through the mesh along a characteristic path, shown in the Figure 12.11, encountering a boundary point perhaps once each time traveling across the mesh of length L.

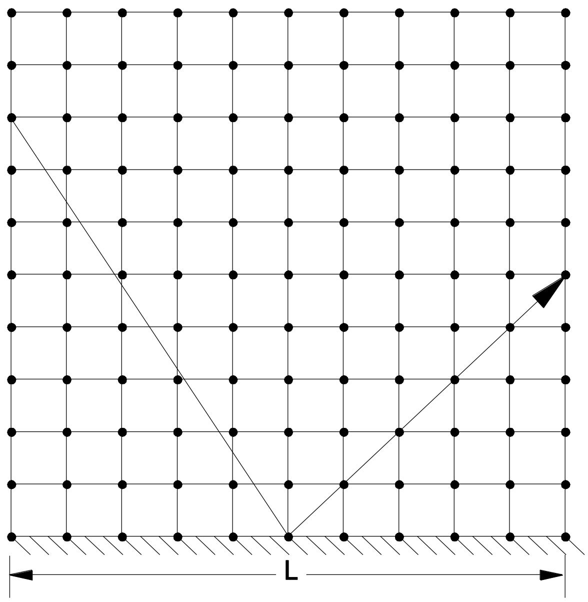


Figure 12.11 Characteristic path through mesh

The information passing through the mesh along the characteristic path is solved for or updated N times, $N \approx \dfrac{L}{\Delta t\left(|u|+c\right)} \approx \dfrac{L}{CFL\,\Delta h}$, at interior points and once at a boundary point. The accumulated error on completion of travel across the mesh is

$$Error \approx N\tau_i + \tau_b \approx \frac{L}{\Delta h}\alpha\,\Delta h^p + \beta\,\Delta h^{p-1} = O\left(\Delta h^{p-1}\right)$$

Therefore, a boundary condition procedure of order of accuracy $p-1$ is compatible with an interior point solution algorithm of order of accuracy p. The set of boundary points constitute a *thin* set of points relative to the set of interior points. The same is true for the set of points describing interior sharp features of the flow, such as shock waves, where the order of accuracy of the algorithm may drop by one without compromising the overall accuracy of the solution. This is utilized by the high resolution TVD procedures discussed earlier in Chapter 10, which become lower order procedures at discontinuities to preserve solution monotonicity.