Chapter 7

Transformations and the Finite Volume Approach

7.1 Introduction

The applications of the previous chapters only required Cartesian meshes. The flow fields were either one dimensional or allowed the use of thin airfoil boundary conditions. The applications to be considered include multidimensional flows past complex geometries. In this chapter we present transformations for both the Cartesian space and equations presented in Chapter 1 into arbitrary curvilinear coordinate systems.

7.2 Computational Meshes and Transformations

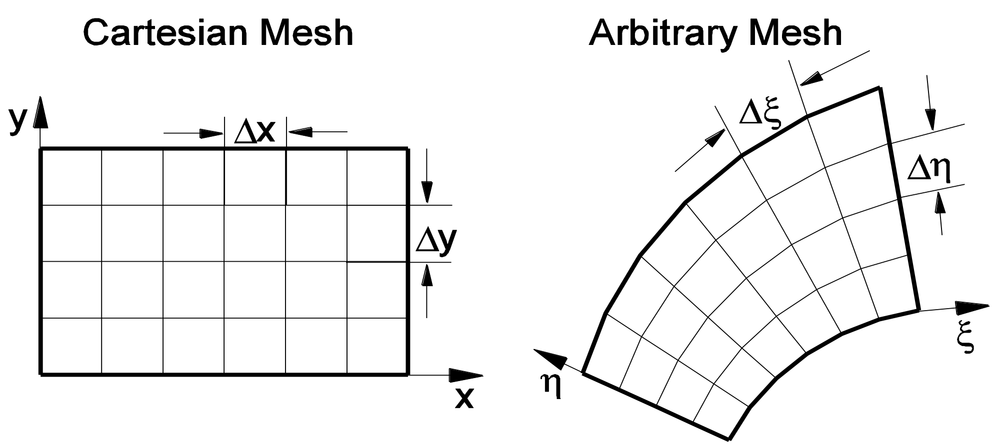


Figure 7.1 Cartesian and arbitrary meshes

The above sketches represent two types of descretizations of the $x-y$ plane in physical space, one Cartesian and one fitted to curved surface.

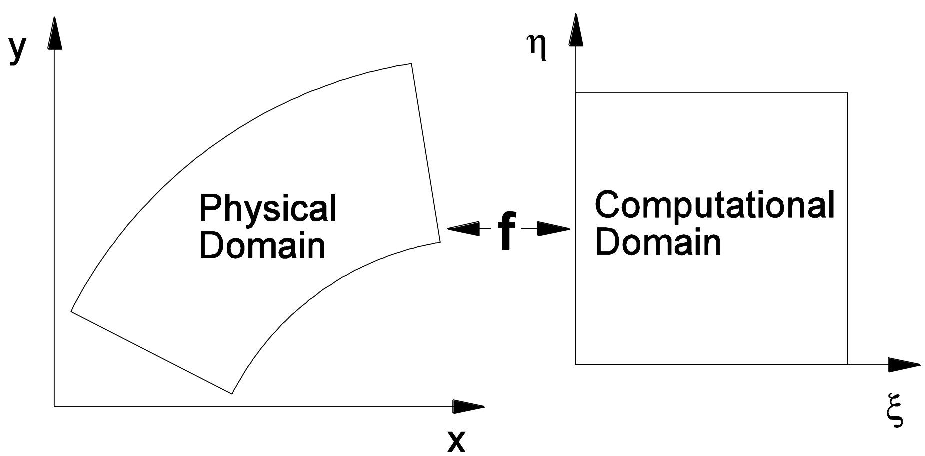


Figure 7.2 Mapping physical domain to computational domain

Consider the transformation f shown above for mapping an arbitrary domain in physical space into a rectangular domain in computational space. The function f, either analytic or numerical, is also used to transform the governing equations for use in the new computational coordinate system. The following sections discuss transformations in both stationary and moving coordinate transformations.

7.3 Transformations - Stationary Case

Consider the transformation and its inverse shown below

$$\xi = \xi(x, y), \quad \eta = \eta(x, y)$$

and

$$x = x(\xi, \eta), \quad y = y(\xi, \eta)$$

The two coordinate systems, (x, y) and (ξ, η), are not moving with respect to each other.

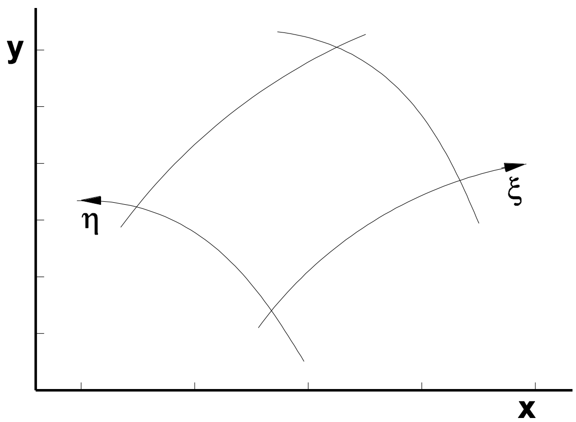


Figure 7.3 Transformation $(x, y) \Leftrightarrow (\xi, \eta)$

Using the ``chain" rule to transform partial derivatives,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} \qquad \text{or} \qquad \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix}}_{T} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y}$$

Similarly, the inverse transformation is given by

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi} \qquad \text{or} \qquad \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{T^{-1}} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta}$$

We can obtain directly on the computational mesh approximations for derivatives with respect to ξ and η, but not those with respect to x and y. However, using the fact that $T = \left(T^{-1}\right)^{-1}$ and equating like elements, we obtain the following identities that relate the two sets of derivatives.

$$\frac{\partial x}{\partial \xi}=\frac{1}{d_{\xi\eta}}\frac{\partial \eta}{\partial y},\quad \frac{\partial y}{\partial \xi}=\frac{-1}{d_{\xi\eta}}\frac{\partial \eta}{\partial x},\quad \frac{\partial x}{\partial \eta}=\frac{-1}{d_{\xi\eta}}\frac{\partial \xi}{\partial y},\quad \frac{\partial y}{\partial \eta}=\frac{1}{d_{\xi\eta}}\frac{\partial \xi}{\partial x}$$

$$\frac{\partial \xi}{\partial x}=\frac{1}{d_{xy}}\frac{\partial y}{\partial \eta},\quad \frac{\partial \eta}{\partial x}=\frac{-1}{d_{xy}}\frac{\partial y}{\partial \xi},\quad \frac{\partial \xi}{\partial y}=\frac{-1}{d_{xy}}\frac{\partial x}{\partial \eta},\quad \frac{\partial \eta}{\partial y}=\frac{1}{d_{xy}}\frac{\partial x}{\partial \xi}$$

$$d_{\xi\eta}=\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}-\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x},\quad d_{xy}=\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta}-\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi},\quad and\quad d_{xy}d_{\xi\eta}=1$$

7.3.1 Transformation of Equations

We start with the conservation law form of the governing equations in the $x-y$ coordinate system.

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=0 \tag{7.1}$$

In $\xi-\eta$ space, using the *chain* rule for differentiation, the equations transform into the following non-conservation law form

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial x}+\frac{\partial G}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial G}{\partial \eta}\frac{\partial \eta}{\partial y}=0$$

We define the following *primed* vectors as follows

$$U'=\frac{1}{d_{\xi\eta}}U,\quad F'=\frac{1}{d_{\xi\eta}}\left(F\frac{\partial \xi}{\partial x}+G\frac{\partial \xi}{\partial y}\right),\quad G'=\frac{1}{d_{\xi\eta}}\left(F\frac{\partial \eta}{\partial x}+G\frac{\partial \eta}{\partial y}\right)$$

with $d_{\xi\eta}$ given in the previous section. These vectors can also be expressed in terms of derivatives with respect to the computational coordinates using the above identities, as follows.

$$U'=d_{xy}U,\quad F'=F\frac{\partial y}{\partial \eta}-G\frac{\partial x}{\partial \eta},\quad G'=-F\frac{\partial y}{\partial \xi}+G\frac{\partial x}{\partial \xi}$$

The vectors F' and G' are flux vectors rotated in the new coordinate system. The vectors F' and G' are not necessarily in the ξ and η directions, respectively, unless the transformed coordinate system is orthogonal. The vector F' is orthogonal to a line of constant η and the vector G' is similarly orthogonal to a line of constant ξ. With these definitions and the above identities, we can transform Equation (7.1) back again into conservation law form in the new coordinate space as follows

$$\frac{\partial U'}{\partial t}+\frac{\partial F'}{\partial \xi}+\frac{\partial G'}{\partial \eta}=0 \tag{7.2}$$

"Conservation laws map into conservation laws"

Exercise: Show that Equation (7.2) is equivalent to Equation (7.1).

7.3.2 Transformation of Derivatives

We can obtain the partial derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in terms of the derivatives that can be approximated directly on the $\xi - \eta$ mesh as follows.

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \frac{1}{d_{xy}} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix}$$

7.4 Transformations - Non-Stationary Case

Consider the unsteady transformation and its inverse given below.

$$\xi = \xi(x, y, t), \quad \eta = \eta(x, y, t) \quad and \quad \tau = t$$

and

$$x = x(\xi, \eta, \tau), \quad y = y(\xi, \eta, \tau) \quad and \quad t = \tau$$

The two coordinate systems are moving relative to one another with velocity $\vec{w}$.

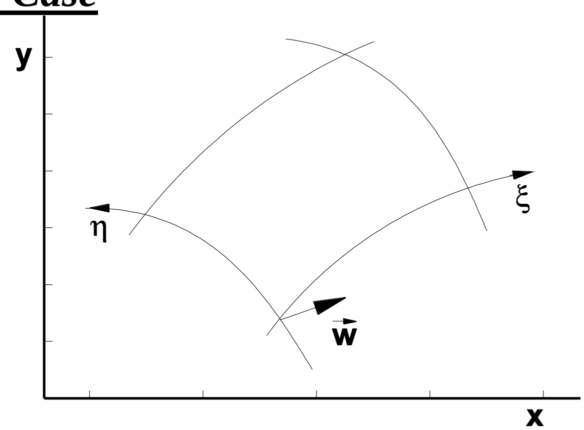


Figure 7.4 Transformation $(x, y, t) \Leftrightarrow (\xi, \eta, \tau)$

Again using the *chain* rule we obtain the following transformations for partial derivatives.

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & 0 \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & 0 \\ \frac{\partial \xi}{\partial t} & \frac{\partial \eta}{\partial t} & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \tau} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \tau} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & 0 \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & 0 \\ \frac{\partial x}{\partial \tau} & \frac{\partial y}{\partial \tau} & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial t} \end{bmatrix}$$

Four additional identities

$$\frac{\partial x}{\partial \tau} = \frac{1}{d_{\xi\mu}}\left(\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial t} - \frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial t}\right) \qquad \frac{\partial y}{\partial \tau} = \frac{1}{d_{\xi\mu}}\left(\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial t} - \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial t}\right)$$

$$\frac{\partial \xi}{\partial t} = \frac{1}{d_{xy}}\left(\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \tau} - \frac{\partial y}{\partial \eta}\frac{\partial x}{\partial \tau}\right) \qquad \frac{\partial \eta}{\partial t} = \frac{1}{d_{xy}}\left(\frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \tau} - \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \tau}\right)$$

Again, the conservation law form of the governing equations in (x, y, t) space maps into conservation law form in (ξ, η, τ) space.

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial U'}{\partial \tau} + \frac{\partial F'}{\partial \xi} + \frac{\partial G'}{\partial \eta} = 0$$

where

$$U' = \frac{1}{d_{\xi\eta}}U, \quad F' = \frac{1}{d_{\xi\eta}}\left(F\frac{\partial \xi}{\partial x} + G\frac{\partial \xi}{\partial y} + U\frac{\partial \xi}{\partial t}\right), \quad G' = \frac{1}{d_{\xi\eta}}\left(F\frac{\partial \eta}{\partial x} + G\frac{\partial \eta}{\partial y} + U\frac{\partial \eta}{\partial t}\right)$$

Again, these vectors can be expressed in terms of derivatives with respect to the computational coordinates.

$$U' = d_{xy}U, \quad F' = F\frac{\partial y}{\partial \eta} - G\frac{\partial x}{\partial \eta} + U\left(\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \tau} - \frac{\partial y}{\partial \eta}\frac{\partial x}{\partial \tau}\right), \quad G' = -F\frac{\partial y}{\partial \xi} + G\frac{\partial x}{\partial \xi} + U\left(\frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \tau} - \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \tau}\right)$$

The terms $Q_\xi = \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \tau} - \frac{\partial y}{\partial \eta}\frac{\partial x}{\partial \tau}$ and $Q_\eta = \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \tau} - \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \tau}$ represent the movement of the (ξ, η, τ) coordinate system through the fluid in the (x, y, t) coordinate system. The total flux across a surface in (ξ, η, τ) space consists of that of the Cartesian velocity components u and v moving the fluid within the flow volume and that caused by the moving coordinates.

7.5 Finite Volume Formulation

There are two approaches to discretizing a flow field, finite difference that uses a set of mesh points and finite volume that divides the volume into a set of small cells. The finite difference approach approximates the differential form of the governing equations. The finite volume approach uses the integral form of the equations for approximation. Both approaches are illustrated in below.

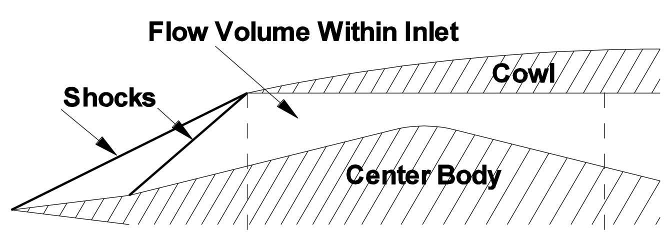


Figure 7.5 Supersonic Inlet

Consider the Euler equations in two dimensions

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=0, \quad \text{where} \quad U=\begin{bmatrix}\rho \\ \rho u \\ \rho v \\ e\end{bmatrix}, \quad F=\begin{bmatrix}\rho u \\ \rho u^2+p \\ \rho vu \\ (e+p)u\end{bmatrix} \quad and \quad G=\begin{bmatrix}\rho v \\ \rho uv \\ \rho v^2+p \\ (e+p)v\end{bmatrix} \qquad (7.3)$$

$$p=(\gamma-1)\rho\varepsilon, \quad \varepsilon=\frac{e}{\rho}-\frac{1}{2}(u^2+v^2)$$

Consider the control volume V in Figure 7.6. The right and left sides, S_R and S_L, are vertical and the top and bottom sides, S_T and S_B need not be parallel. Flow can enter and leave through the left and right sides, but the top and bottom sides are impermeable. The flux difference entering and leaving the volume is $F_L S_L - F_R S_R$. The pressure p acting on the top and bottom surfaces can increase the momentum of the flow in the x-direction by $p(S_R - S_L)$.

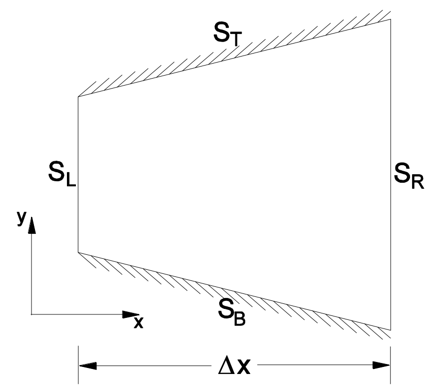


Figure 7.6 Control volume V

The integral form of Equation 7.3, see Section 2.3, for a volume V enclosed by surface S is $\frac{\partial \overline{U}}{\partial t}+\frac{1}{V}\int_S \vec{F}\cdot d\vec{s}=0$, where $\overline{U}$ is the volume averaged value of U. A finite volume approximation to this equation, applied to the volume in Figure 7.6, becomes

$$\overline{U}^{n+1}=\overline{U}^{n}-\frac{\Delta t}{V}\left(\vec{F}_R\cdot\vec{S}_R-\vec{F}_L\cdot\vec{S}_L+\vec{F}_T\cdot\vec{S}_T-\vec{F}_B\cdot\vec{S}_B\right)$$

We have now abandoned the outer normal convention (i.e., surface normals that always point to the outside of volume). For example, the left and right side normals now both point to the right and the plus or minus signs indicate either flow entering or leaving the volume laterally. The four surfaces are each given by form $\vec{S}=sx\,\vec{i}_x+sy\,\vec{i}_y$

$$\vec{S}_R=S_R\vec{i}_x\,, \quad \vec{S}_L=S_L\vec{i}_x\,, \quad \vec{S}_T=-\left.\frac{\partial y}{\partial x}\right|_T \Delta x\,\vec{i}_x+\Delta x\,\vec{i}_y \quad \text{and} \quad \vec{S}_T=-\left.\frac{\partial y}{\partial x}\right|_B \Delta x\,\vec{i}_x+\Delta x\,\vec{i}_y$$

The vector $\vec{F}=F\,\vec{i}_x+G\,\vec{i}_y$ and $\vec{F}\cdot\vec{S}=F\,sx+G\,sy=\begin{bmatrix}\rho\vec{q}\cdot\vec{n} \\ \rho u\vec{q}\cdot\vec{n}+p\,sx/S \\ \rho v\vec{q}\cdot\vec{n}+p\,sy/S \\ (e+p)\vec{q}\cdot\vec{n}\end{bmatrix}S$, where $\vec{q}=u\,\vec{i}_x+v\,\vec{i}_y$

$\vec{S}=\sqrt{sx^2+sy^2}$ and $\vec{n}=(sx\,\vec{i}_x+sy\,\vec{i})/S$. Note that at the top and bottom impermeable walls

$\vec{F}\cdot\vec{S}\Big|_{wall} = \begin{bmatrix} 0 \\ p\, sx \\ p\, sy \\ 0 \end{bmatrix}_{wall}$ because of the flow tangency condition $\vec{q}\cdot\vec{n}=0$. The finite volume approximation therefore becomes

$$\bar{U}^{n+1} = \bar{U}^{n} - \frac{\Delta t}{V}\left(F_R S_R - F_L S_L\right) + Q,$$

$$\text{where} \quad Q = \frac{\Delta t}{V}\begin{bmatrix} 0 \\ \bar{p}(sx_T - sx_B) \\ \bar{p}(sy_T - sy_B) \\ 0 \end{bmatrix} = \frac{\Delta t}{V}\begin{bmatrix} 0 \\ \bar{p}(sx_T - sx_B) \\ 0 \\ 0 \end{bmatrix} = \frac{\Delta t}{V}\begin{bmatrix} 0 \\ \bar{p}(S_R - S_L) \\ 0 \\ 0 \end{bmatrix}$$

and $\bar{p}$ represents the average pressure within the volume $V = \frac{1}{2}(S_R + S_L)\Delta x = \bar{S}\Delta x$. We present the Quasi 1-D Euler equations in the next section that correspond to the control volume equations just derived above. These equations assume that the y-momentum equation is not needed if the flow is contained within a channel with small changes in cross sectional area.

7.5.1 The Quasi-1-D Euler Equations

Consider the quasi one dimensional Euler equations for calculating the flow within a channel in which the cross-sectional area change is small (see for example the flow volume within the dashed lines of the supersonic inlet shown in Figure 7.5). Only one momentum equation is required.

$$\frac{\partial U}{\partial t} + \frac{1}{S}\frac{\partial FS}{\partial x} = Q \tag{7.4}$$

where

$$U = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}, \quad F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (e+p)u \end{bmatrix}, \quad \text{and} \quad Q = \begin{bmatrix} 0 \\ \dfrac{p}{S}\dfrac{\partial S}{\partial x} \\ 0 \end{bmatrix}$$

$$p = (\gamma - 1)\rho\varepsilon, \quad \varepsilon = \frac{e}{\rho} - \frac{1}{2}u^2 \quad \text{and} \quad S = S(x)$$

S represents the cross-sectional area of the channel as a function of x.

A discrete approximation to this equation in generic form is given by

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{S_i}\frac{F_{i+1/2}^n S_{i+1/2} - F_{i-1/2}^n S_{i-1/2}}{\Delta x} + \frac{\Delta t}{S_i}\begin{bmatrix} 0 \\ p_i^n \dfrac{S_{i+1/2} - S_{i-1/2}}{\Delta x} \\ 0 \end{bmatrix}$$

Choices need to be made on how to evaluate the flux terms $F_{i+1/2}^{n}$ and the metric terms $S_{i+1/2}$. Two approaches are the finite volume approach and the finite difference approach.

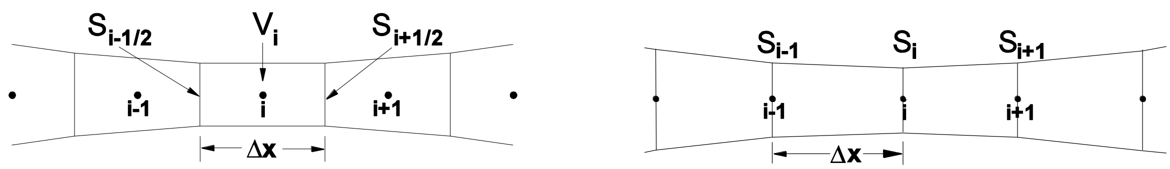


Figure 7.7 Finite volume approach Finite difference approach

A finite volume approximation, using a backward difference approximation for the flux vector, is given by

$$\bar{U}_i^{n+1} = \bar{U}_i^{n} - \frac{\Delta t}{V_i}\left(F_i^{n} S_{i+1/2} - F_{i-1}^{n} S_{i-1/2}\right) + \frac{\Delta t}{V_i}\begin{bmatrix} 0 \\ p_i^n\left(S_{i+1/2} - S_{i-1/2}\right) \\ 0 \end{bmatrix}$$

with the volume is defined by $V_i = \Delta x \dfrac{S_{i+1/2} + S_{i-1/2}}{2}$

A corresponding finite difference approximation for Equation (7.4) is perhaps

$$U_i^{n+1} = U_i^{n} - \frac{\Delta t}{S_i \Delta x}\left(F_i^{n} S_i - F_{i-1}^{n} S_{i-1}\right) + \frac{\Delta t}{S_i \Delta x}\begin{bmatrix} 0 \\ p_i^n\left(S_i - S_{i-1}\right) \\ 0 \end{bmatrix}$$

The differences between the two approximations for the 1-D Quasi Euler equations are small, $\bar{U}$ versus U and the subscripts on the metric surface terms, but more significant differences occur in two or three dimensional approximations. The finite volume approach places mesh points at the center of the finite volumes and the finite difference approach places them along mesh lines. Finite difference mesh points will therefore lie on body surfaces, while finite volume mesh points will be offset by about a half mesh point spacing. Consequently, finite volume meshes sometimes require “ghost” points located across body surfaces for implementing boundary condition

A finite volume approximation uses the exact geometrical elements that surround each mesh point. These elements include a volume and the surfaces that enclose the volume. The result is a geometric conservation. The finite difference approach has more liberty in choosing how to evaluate metric terms. Geometric conservation will ensure that a uniform flow remains uniform until acted upon by an outside disturbance, because the flow across all surfaces for this case sum to zero net change. A finite difference approximation may not have this property if the surface metric terms do not exactly enclose an equivalent volume metric term, because of either surface overlap or incomplete closure.

7.5.2 The 2-D Euler Equations

Consider the Euler equations in two dimensions in conservation law form, Equations (7.3).

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=0$$

Approximating the integral equation on the arbitrary mesh shown below, using backward spatial difference operators, we obtain the following.

$$\bar{U}_{i,j}^{n+1}=\bar{U}_{i,j}^{n}-\frac{\Delta t}{V_{i,j}}\left(\vec{F}_{i,j}^{n}\cdot\vec{S}_{i+1/2}-\vec{F}_{i-1,j}^{n}\cdot\vec{S}_{i-1/2}+\vec{F}_{i,j}^{n}\cdot\vec{S}_{j+1/2}-\vec{F}_{i,j-1}^{n}\cdot\vec{S}_{j-1/2}\right)$$

where,

$$\vec{S}_{i+1/2}=\left(y_{i+1,j+1}-y_{i+1,j}\right)\vec{i}_x-\left(x_{i+1,j+1}-x_{i+1,j}\right)\vec{i}_y\,,\qquad \vec{S}_{i-1/2}=\left(y_{i,j+1}-y_{i,j}\right)\vec{i}_x-\left(x_{i,j+1}-x_{i,j}\right)\vec{i}_y$$

$$\vec{S}_{j+1/2}=-\left(y_{i+1,j+1}-y_{i,j+1}\right)\vec{i}_x+\left(x_{i+1,j+1}-x_{i,j+1}\right)\vec{i}_y\,,\qquad \vec{S}_{j-1/2}=-\left(y_{i+1,j}-y_{i,j}\right)\vec{i}_x+\left(x_{i+1,j}-x_{i,j}\right)\vec{i}_y$$

and

$$\begin{aligned}V_{i,j}=\frac{1}{2}\{&\left|(x_{i,j}-x_{i+1,j})y_{i+1,j+1}+(x_{i+1,j}-x_{i+1,j+1})y_{i,j}+(x_{i+1,j+1}-x_{i,j})y_{i+1,j}\right|\\&+\left|(x_{i,j}-x_{i+1,j+1})y_{i,j+1}+(x_{i+1,j+1}-x_{i,j+1})y_{i,j}+(x_{i,j+1}-x_{i,j})y_{i+1,j+1}\right|\}\end{aligned}$$

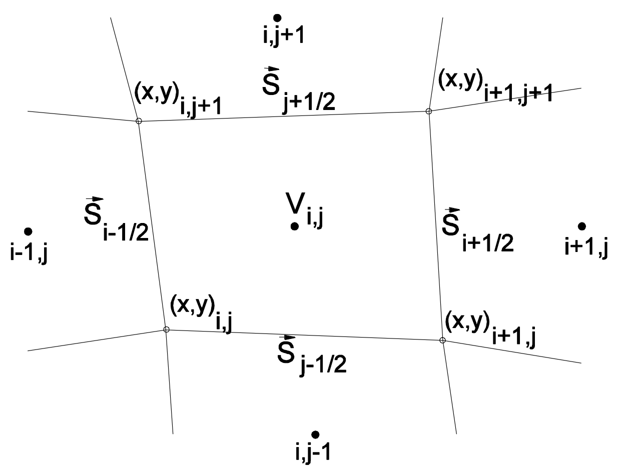


Figure 7.8 Mesh points and surfaces enclosing volume $V_{i,j}$

The corresponding finite difference approximation requires a mapping function f to transform the above arbitrary mesh into a rectangular one in the computational space. The mapping function need not be analytic. A numerical function, which relates the two coordinate systems, can be used as well. In fact, the finite volume procedure can be considered to be a numerical mapping procedure, though one that treats the geometric terms with unique respect, "geometric conservation". The next section shows the equivalence between the two approaches.

7.6 Equivalence of Finite Difference and Finite Volume Forms

Let's approximate the two dimensional Euler equations on an arbitrary stationary mesh using both formulations.

Finite Difference Approximation

$$U'^{n+1}_{i,j} = U'^{n}_{i,j} - \Delta t\left(\frac{D\cdot F'^{n}_{i,j}}{\Delta\xi} + \frac{D\cdot G'^{n}_{i,j}}{\Delta\eta}\right)$$

Finite Volume Approximation

$$\bar{U}^{n+1}_{i,j} = \bar{U}^{n}_{i,j} - \frac{\Delta t}{V_{i,j}}\left((\vec{F}^{n}\cdot\vec{S})_{i+1/2} - (\vec{F}^{n}\cdot\vec{S})_{i-1/2} + (\vec{F}^{n}\cdot\vec{S})_{j+1/2} - (\vec{F}^{n}\cdot\vec{S})_{j-1/2}\right)$$

We will now show that the above two formulations are equivalent, though not necessarily exactly the same, at all interior points. They would be exactly the same if the metric terms used in the finite difference formulation were evaluated exactly as in the finite volume formulation. The finite volume formulation has only one choice for determining the metric terms. The surfaces $\vec{S}_{i+1/2,j}$, $\vec{S}_{i-1/2,j}$, $\vec{S}_{i,j+1/2}$ and $\vec{S}_{i,j-1/2}$ must exactly enclose volume $V_{i,j}$. There is no such requirement for the finite difference approach, which often leads to freestream anomalies. We begin by starting with the finite volume formulation and evaluating terms such as $(\vec{F}^{n}\cdot\vec{S})_{i+1/2,j}$ (see Figures 7.8 and 7.9).

$$\vec{S}_{i+1/2,j} = \left(\frac{\partial y}{\partial\eta}\Delta\eta\,\vec{i}_x - \frac{\partial x}{\partial\eta}\Delta\eta\,\vec{i}_y\right)_{i+1/2,j}$$

$$\vec{S}_{i-1/2,j} = \left(\frac{\partial y}{\partial\eta}\Delta\eta\,\vec{i}_x - \frac{\partial x}{\partial\eta}\Delta\eta\,\vec{i}_y\right)_{i-1/2,j}$$

$$\vec{S}_{i,j+1/2} = \left(-\frac{\partial y}{\partial\xi}\Delta\xi\,\vec{i}_x + \frac{\partial x}{\partial\xi}\Delta\xi\,\vec{i}_y\right)_{i,j+1/2}$$

$$\vec{S}_{i,j-1/2} = \left(-\frac{\partial y}{\partial\xi}\Delta\xi\,\vec{i}_x + \frac{\partial x}{\partial\xi}\Delta\xi\,\vec{i}_y\right)_{i,j-1/2}$$

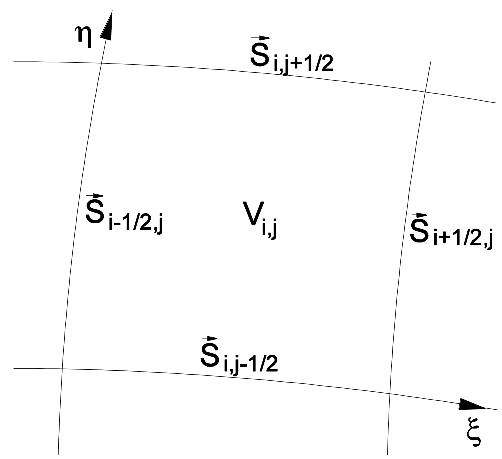


Figure 7.9 Surfaces enclosing volume $V_{i,j}$.

$$(\vec{F}^{n}\cdot\vec{S})_{i+1/2,j} = \left[\left(F^{n}\frac{\partial y}{\partial\eta} - G^{n}\frac{\partial x}{\partial\eta}\right)\Delta\eta\right]_{i+1/2,j} = \left[\left(F^{n}\frac{\partial\xi}{\partial x} + G^{n}\frac{\partial\xi}{\partial y}\right)\frac{\Delta\eta}{d_{\xi\eta}}\right]_{i+1/2,j} = F'^{n}_{i+1/2,j}\Delta\eta$$

and similarly

$$(\vec{F}^{n}\cdot\vec{S})_{i-1/2,j} = F'^{n}_{i-1/2,j}\Delta\eta,\ \ (\vec{F}^{n}\cdot\vec{S})_{i,j+1/2} = G'^{n}_{i,j+1/2}\Delta\xi \ \ and\ \ (\vec{F}^{n}\cdot\vec{S})_{i,j-1/2} = G'^{n}_{i,j-1/2}\Delta\xi$$

By direct substitution of the above terms into the finite volume approximation, we obtain, using $V_{i,j} = \frac{\Delta\xi\Delta\eta}{d_{\xi\eta}}$

$$U_{i,j}^{n+1} = U_{i,j}^{n} - \frac{\Delta t}{1/d_{\xi\eta}}\left(\frac{F'^n_{i+1/2,j} - F'^n_{i-1/2,j}}{\Delta\xi} + \frac{G'^n_{i,j+1/2} - G'^n_{i,j-1/2}}{\Delta\eta}\right)$$

$$U'^{n+1}_{i,j} = U'^n_{i,j} - \Delta t\left(\frac{D\cdot F'^n_{i,j}}{\Delta\xi} + \frac{D_{-}\cdot G'^n_{i,j}}{\Delta\eta}\right)$$

Note the subscripts $i+1/2$ appearing on the dependent flow variables equal $i+1$ or i for a forward or backward difference operator, respectively. For a central difference operator it represents the average of the flow variable at points $i+1$ and i, etc. We also should note again, however, that the equivalence shown above is not valid at boundary points. Finite difference methods place mesh points along boundaries and finite volume methods place surface elements along boundaries.

Finally, to show that the vectors $(\vec{F}^n \cdot \vec{S})_{i+1/2,j} = F'^n_{i+1/2,j}\Delta\eta$ and $(\vec{F}^n \cdot \vec{S})_{i,j+1/2} = G'^n_{i,j+1/2}\Delta\xi$ are not too much more complex than the Cartesian F and G, we can express them as follows

$$(\vec{F}^n \cdot \vec{S})_{i+1/2,j} = F'^n_{i+1/2,j}\Delta\eta = \left[\left(F^n\frac{\partial y}{\partial\eta} - G^n\frac{\partial x}{\partial\eta}\right)\Delta\eta\right]_{i+1/2,j} = \begin{bmatrix} \rho qs \\ \rho u\, qs + \frac{\partial y}{\partial\eta}p \\ \rho v\, qs - \frac{\partial x}{\partial\eta}p \\ (e+p)qs \end{bmatrix}_{i+1/2,j} \text{, where } qs = \frac{\partial y}{\partial n}u - \frac{\partial x}{\partial n}v$$

$$(\vec{F}^n \cdot \vec{S})_{i,j+1/2} = G'^n_{i,j+1/2}\Delta\xi = \left[\left(-F^n\frac{\partial y}{\partial\xi} + G^n\frac{\partial x}{\partial\xi}\right)\Delta\xi\right]_{i,j+1/2} = \begin{bmatrix} \rho qs \\ \rho u\, qs - \frac{\partial y}{\partial\xi}p \\ \rho v\, qs + \frac{\partial x}{\partial\xi}p \\ (e+p)qs \end{bmatrix}_{i,j+1/2} \text{,where } qs = -\frac{\partial y}{\partial\xi}u + \frac{\partial x}{\partial\xi}v$$

7.7 A Common Formulation for Both Finite Difference and Finite Volume

It will be very convenient to write the algorithms in a common notation for use by either the finite difference of finite volume approaches. The finite difference equation shown at the beginning of Section 7.6 appears to be more simple and even perhaps more aesthetic than the finite volume equation compared to it just below. Integral equations appear less friendly than the more familiar differential equations. The next few sections will lay the ground work for a common notation for both approaches.

7.7.1 The Surface Normal Convention

In Section 2.3 we used the outer normal convention to orient the direction of surface vectors. We have now abandoned this convention in this chapter and hereafter. Instead we choose the direction of the surface normal vectors from the direction of the increasing mesh indices themselves. For example, the surface $\vec{S}_{i+1/2,j}$, between mesh points i,j and $i+1,j$, will have its surface normal vector pointing in the increasing i direction, from the surface into the $i+1,j$ volume, regardless if it is considered to be a surface of volume i,j or of volume $i+1,j$.

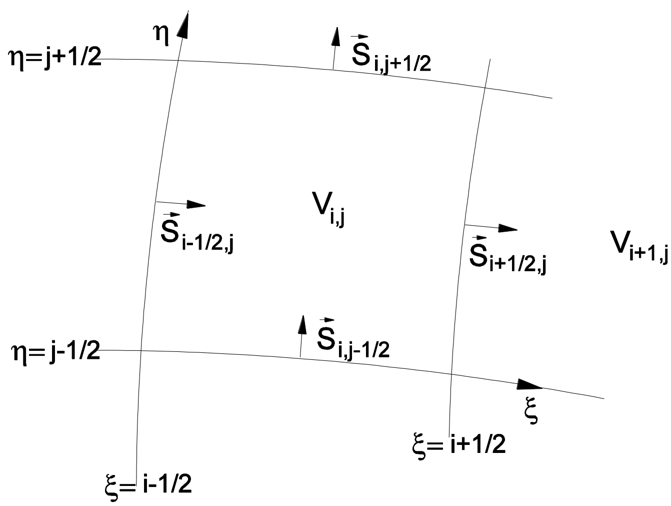


Figure 7.10 New surface normal convention and mesh line orientation

Similarly, the surface $\vec{S}_{i,j+1/2}$, between mesh points i,j and $i,j+1$, will have its surface normal vector pointing in the increasing j direction, from the surface into the $i,j+1$ volume, regardless if it is considered to be a surface of volume i,j or of volume $i,j+1$, as shown in Figure 7.10.

7.7.2 A Mesh Induced Metric

In addition to the change in convention for the surface normal vectors, we define for convenience the metric for the computational space to be that induced by the mesh indices i and j. Therefore the coordinates ξ and η are dimensionless real numbers that achieve integer values at mesh points and half integer values at surfaces between mesh points, also as shown in the above figure. This metric convention will be used for both the finite difference and volume formulations. Note that with this metric $\Delta\xi = 1$ and $\Delta\eta = 1$ everywhere in the mesh and that the dimensions of length carried by Δx and Δy in Cartesian meshes must be carried by the other metric terms than $\Delta\xi$ and $\Delta\eta$ in the difference equations. This is a small, but troublesome, complication necessary to achieve a greater benefit.

7.7.3 Generic Equation for Both Finite Difference and Volume Approaches

The *generic difference equation* for both the finite difference and volume approaches can be expressed in arbitrary curvilinear coordinates as

$$U_{i,j}^{n+1} = U_{i,j}^{n} - \frac{\Delta t}{V_{i,j}} \left(\frac{F_{i+1/2,j}'^{n} - F_{i-1/2,j}'^{n}}{\Delta \xi} + \frac{G_{i,j+1/2}'^{n} - G_{i,j-1/2}'^{n}}{\Delta \eta} \right)$$

where we have taken the bar off $\bar{U}_{i,j}^{n+1}$ and $\bar{U}_{i,j}^{n}$ and divided Δt by the volume $V_{i,j}$. The bar previously indicated a value that was volume averaged over the cell volume and the unbarred variable indicated the value at a mesh point. The new bar less variable now indicates the value at the cell center.

Often in finite difference formulations the Jacobian of the transformation, usually named $J_{i,j}$, is used in place of $V_{i,j}$, but they are equivalent. Note that now $V_{i,j}$ has the dimensions of volume (area for our two space dimensional equation) because the mesh induced metric for ξ and η is dimensionless.

$$V_{i,j} = J_{i,j} = \frac{1}{d_{\xi\eta}} = d_{xy} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}.$$

In Cartesian coordinates the difference equation is

$$U_{i,j}^{n+1} = U_{i,j}^{n} - \Delta t \left(\frac{F_{i+1/2,j}^{n} - F_{i-1/2,j}^{n}}{\Delta x} + \frac{G_{i,j+1/2}^{n} - G_{i,j-1/2}^{n}}{\Delta y} \right)$$

Note that the dimensions of F' and G' are not the same as those for F and G, again because ξ and η are dimensionless.

$$F' = F\frac{\partial y}{\partial \eta} - G\frac{\partial x}{\partial \eta}, \quad G' = -F\frac{\partial y}{\partial \xi} + G\frac{\partial x}{\partial \xi}$$

However, the equations do not abandon their Cartesian roots. The vectors U, F' and G' retain the Cartesian velocities u and v and the momentum equations solve for the x and y momentum per unit volume.

Exercise: Express F', G' and $V_{i,j}$ on a Cartesian mesh.

7.7.4 Order of Accuracy for Finite Difference and Volume Approaches

It is mistakenly believed by some that finite volume approximations are limited to second order accuracy, while finite difference approximations are not so limited. It is true that if the exact fluxes for $F_{i+1/2,j}'^{n}$ and $G_{i,j+1/2}'^{n}$ are used in a finite volume formulation the resulting accuracy would only be of second order. However, as in a finite difference calculation, the order of accuracy of a derivative approximation depends upon the difference between the two fluxes divided by the metric difference in location between them, i.e. $\dfrac{F'^{n}_{i+1/2,j} - F'^{n}_{i-1/2,j}}{\Delta\xi}$. The whole term must be considered, not just an individual flux $F'^{n}_{i+1/2,j}$. Finite volume approximations can use the same formulas for determining $F'^{n}_{i+1/2,j}$ as finite difference approximations. In addition, the bar has been removed from the state vector $\bar{U}^{n}_{i,j}$ and is now defined as the value at a mesh point, similar to that of finite difference approximations. The only remaining difference between the two approaches is the location of mesh points, cell centers or at mesh line intersections, along surfaces or just off surfaces a half of a mesh interval away.

7.8 A Common Computational Mesh for Both the Finite Difference and Finite Volume Approaches

A computational mesh for a finite difference calculation can be used as is for a finite volume calculation and vice versa. However, the finite difference approach defines mesh point locations at the intersections of the mesh lines and the finite volume approach defines the mesh points at the centers of volumes created by the mesh lines, or in three dimensions by the six mesh surfaces. Even for the finite difference approach it is often convenient to consider a virtual volume about each mesh point with flux surfaces midway between adjacent mesh points. Algorithms are usually presented with the flux vectors defined at such surfaces. Because of the difference in mesh point locations relative to the mesh for the two approaches, mesh points are located along body surfaces for body fitted meshes in the finite difference approach and volume surfaces are aligned with body surfaces in the finite volume approach. Examples of each are shown below, including the virtual volume surrounding finite difference mesh point i, j.

7.8.1 Finite Difference Evaluation of Flow Variables and Metric Terms

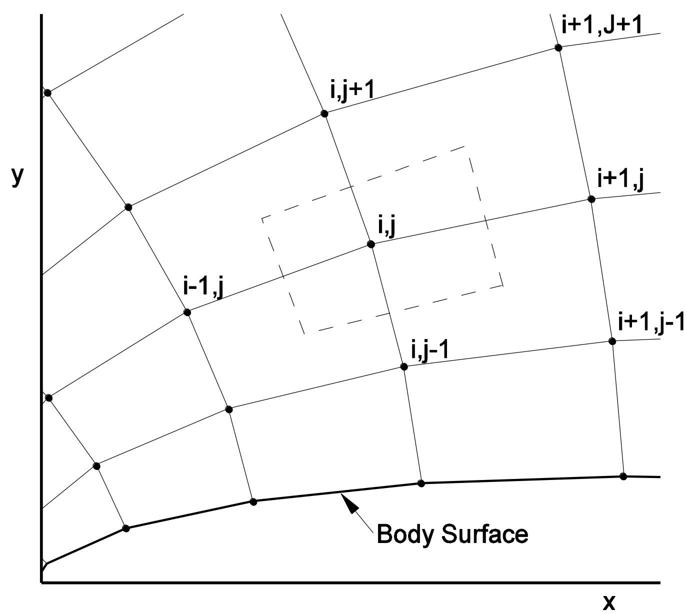


Figure 7.11 Finite difference mesh

The flow variables, the elements of U , are defined at the whole integer mesh points shown in the figure above and the metric terms , $\frac{\partial x}{\partial \xi}$, $\frac{\partial y}{\partial \xi}$, $\frac{\partial x}{\partial \eta}$ and $\frac{\partial y}{\partial \eta}$, are defined to be centered relative to the specific flux surface. For example, in the *generic difference equation* of the last section we need a rule to evaluate $F'^{n}_{i+1/2,j}$ at the dashed surface between mesh points i,j and $i+1,j$ and also $G'^{n}_{i,j+1/2}$ at the dashed surface between mesh points i,j and $i,j+1$.

The flow variables within $F'^{n}_{i+1/2,j}$ and $G'^{n}_{i,j+1/2}$ will depend upon the chosen algorithm, but the metric variables will depend upon the location of the flux surface, as shown below.

$$F'^{n}_{i+1/2,j} = F^{n}_{i+1/2,j} \left.\frac{\partial y}{\partial \eta}\right|_{i+1/2} - G^{n}_{i+1/2,j} \left.\frac{\partial x}{\partial \eta}\right|_{i+1/2}$$

with $$\left.\frac{\partial x}{\partial \eta}\right|_{i+1/2} \simeq \frac{x_{i+1/2,j+1/2} - x_{i+1/2,j-1/2}}{\Delta\eta} \quad \text{and} \quad \left.\frac{\partial y}{\partial \eta}\right|_{i+1/2} \simeq \frac{y_{i+1/2,j+1/2} - y_{i+1/2,j-1/2}}{\Delta\eta}$$

and

$$G'^{n}_{i,j+1/2} = -F^{n}_{i,j+1/2} \left.\frac{\partial y}{\partial \xi}\right|_{j+1/2} + G^{n}_{i,j+1/2} \left.\frac{\partial x}{\partial \xi}\right|_{j+1/2}$$

with $$\left.\frac{\partial x}{\partial \xi}\right|_{j+1/2} \simeq \frac{x_{i+1/2,j+1/2} - x_{i-1/2,j+1/2}}{\Delta\xi} \quad \text{and} \quad \left.\frac{\partial y}{\partial \xi}\right|_{j+1/2} \simeq \frac{y_{i+1/2,j+1/2} - y_{i-1/2,j+1/2}}{\Delta\xi}$$

where $\Delta\xi = 1$ and $\Delta\eta = 1$.

The x and y coordinate variables with half integer subscripts are located at the corners of the dashed virtual volume above, and are evaluated as follows.

$$x_{i+1/2,k+1/2} = \frac{x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1}}{4} \text{ and } y_{i+1/2,k+1/2} = \frac{y_{i,j} + y_{i+1,j} + y_{i,j+1} + y_{i+1,j+1}}{4}, \text{ etc.}$$

7.8.2 Finite Volume Evaluation of Flow Variables and Metric Terms

The flow variables, the elements of U , are also defined at the whole integer mesh points shown in the figure below, but now they are located at volume centers. The metric terms, $\frac{\partial x}{\partial \xi}$, $\frac{\partial y}{\partial \xi}$, $\frac{\partial x}{\partial \eta}$ and $\frac{\partial y}{\partial \eta}$, are defined by the corner points of the relevant flux surface.

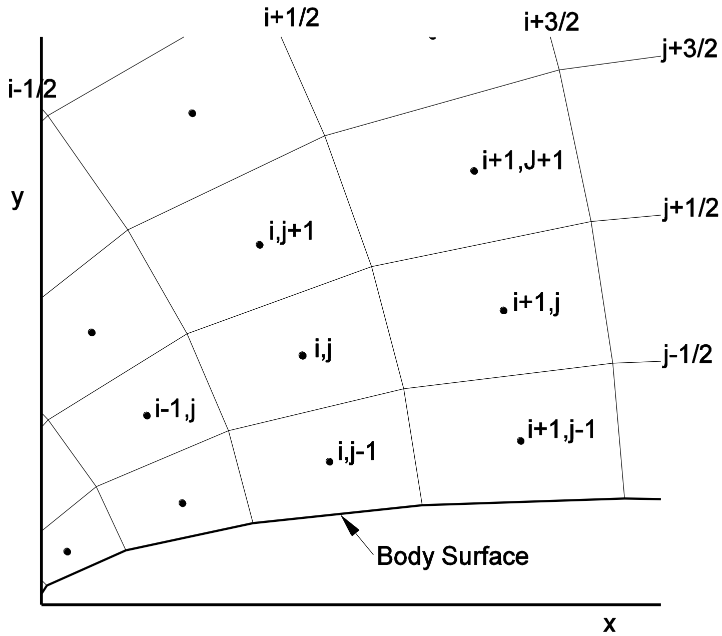


Figure 7.12 Finite volume mesh

The evaluation of $F'^n_{i+1/2,j}$ and $G'^n_{i,j+1/2}$ are the same as before for the finite difference mesh, although the locations of the mesh points are different.

$$F'^n_{i+1/2,j} = F^n_{i+1/2,j} \left.\frac{\partial y}{\partial \eta}\right|_{i+1/2} - G^n_{i+1/2,j} \left.\frac{\partial x}{\partial \eta}\right|_{i+1/2}$$

with $\left.\frac{\partial x}{\partial \eta}\right|_{i+1/2} \simeq \frac{x_{i+1/2,j+1/2} - x_{i+1/2,j-1/2}}{\Delta \eta}$ and $\left.\frac{\partial y}{\partial \eta}\right|_{i+1/2} \simeq \frac{y_{i+1/2,j+1/2} - y_{i+1/2,j-1/2}}{\Delta \eta}$

and

$$G'^n_{i,j+1/2} = -F^n_{i,j+1/2} \left.\frac{\partial y}{\partial \xi}\right|_{j+1/2} + G^n_{i,j+1/2} \left.\frac{\partial x}{\partial \xi}\right|_{j+1/2}$$

with $\left.\frac{\partial x}{\partial \xi}\right|_{j+1/2} \simeq \frac{x_{i+1/2,j+1/2} - x_{i-1/2,j+1/2}}{\Delta \xi}$ and $\left.\frac{\partial y}{\partial \xi}\right|_{j+1/2} \simeq \frac{y_{i+1/2,j+1/2} - y_{i-1/2,j+1/2}}{\Delta \xi}$

again with $\Delta\xi = 1$ and $\Delta\eta = 1$.

7.8.3 The Dual Mesh

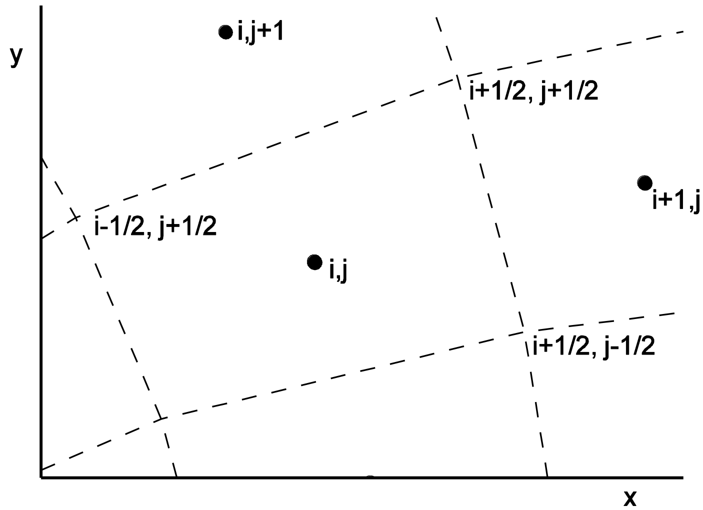


Figure 7.13 Common mesh setup for either finite difference or finite volume approach

Both the finite difference and finite volume methods can use the same mesh. Let's call this mesh the **original mesh** and define **grid points** to be the set of locations at the intersections of the **original mesh** lines or surfaces. We have been using mesh points indiscriminately to be both **grid points** and the location where the flow variables are known. Again, the same grid point locations serve in both the finite difference and finite volume approaches. Also, both approaches will form a **dual mesh** consisting of the centers of cells formed by the original mesh surfaces. The cell center coordinates are the average of the surrounding 2^d grid point coordinates, where d is the dimension of the space. The finite difference approach uses the **dual mesh** to form virtual volumes and surfaces for metric term and flux evaluations. The finite volume approach uses the **dual mesh** as the points where the flow variables are located.

7.8.3.1 Finite Difference Approach

(1) The flow variable $U_{i,j}$ is co-located at **grid point** $(x_{i,j}, y_{i,j})$.

(2) The metric variables are defined by the set of coordinates $x_{i+1/2,j+1/2}$ and $y_{i+1/2,j+1/2}$, located at the corners of surfaces defining virtual volumes and are calculated from the average value of the surrounding **grid points**. This set of corner points is the **dual mesh** to the **original mesh**.

7.8.3.1 Finite Volume Approach

(1) The flow variable $U_{i,j}$ is located at the center of the volume and its coordinate location, $(x_{i,j}, y_{i,j})$, is calculated from the average value of the surrounding **grid points**. This set of volume center points is the **dual mesh** to the **original mesh**.

(2) The metric variables are defined by the set of coordinates $(x_{i+1/2,j+1/2}, y_{i+1/2,j+1/2})$, located at **grid points**.

7.8.4 Evaluation of the Metric Terms for Both the Finite Difference and Finite Volume Approaches

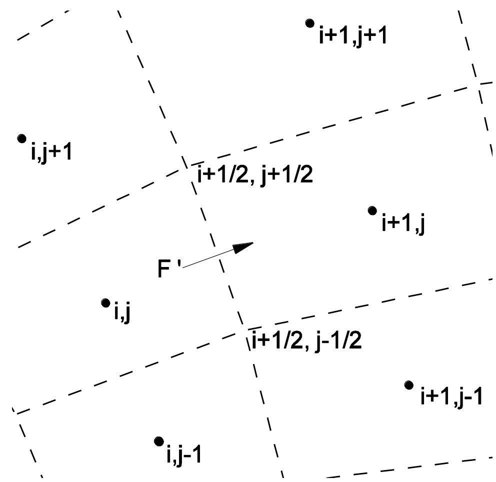
The evaluation of the metric derivative terms need to be centered with respect to the surface at which the flux crosses. Therefore, for the derivatives needed at surface $\xi = i + 1/2$ are defined as follows

$$\left.\frac{\partial x}{\partial \eta}\right|_{i+1/2,j} \simeq \frac{D_- \cdot x_{i+1/2,j+1/2}}{\Delta \eta} = \frac{x_{i+1/2,j+1/2} - x_{i+1/2,j-1/2}}{\Delta \eta}$$

$$\left.\frac{\partial y}{\partial \eta}\right|_{i+1/2,j} \simeq \frac{D_- \cdot y_{i+1/2,j+1/2}}{\Delta \eta} = \frac{y_{i+1/2,j+1/2} - y_{i+1/2,j-1/2}}{\Delta \eta}$$

$$\left.\frac{\partial x}{\partial \xi}\right|_{i+1/2,j} \simeq \frac{D_- \cdot x_{i+1,j}}{\Delta \xi} = \frac{x_{i+1,j} - x_{i,j}}{\Delta \xi} \quad \text{and}$$

$$\left.\frac{\partial y}{\partial \xi}\right|_{i+1/2,j} \simeq \frac{D_- \cdot y_{i+1,j}}{\Delta \xi} = \frac{y_{i+1,j} - y_{i,j}}{\Delta \xi}$$



Similarly, for the derivatives needed at surface $\eta = j + 1/2$ are defined as follows

$$\left.\frac{\partial x}{\partial \xi}\right|_{i,j+1/2} \simeq \frac{D_- \cdot x_{i+1/2,j+1/2}}{\Delta \xi} = \frac{x_{i+1/2,j+1/2} - x_{i-1/2,j+1/2}}{\Delta \xi}$$

$$\left.\frac{\partial y}{\partial \xi}\right|_{i,j+1/2} \simeq \frac{D_- \cdot y_{i+1/2,j+1/2}}{\Delta \xi} = \frac{y_{i+1/2,j+1/2} - y_{i-1/2,j+1/2}}{\Delta \xi}$$

$$\left.\frac{\partial x}{\partial \eta}\right|_{i+1/2,j} \simeq \frac{D_- \cdot x_{i,j+1}}{\Delta \eta} = \frac{x_{i,j+1} - x_{i,j}}{\Delta \eta} \quad \text{and}$$

$$\left.\frac{\partial y}{\partial \eta}\right|_{i+1/2,j} \simeq \frac{D_- \cdot y_{i,j+1}}{\Delta \eta} = \frac{y_{i,j+1} - y_{i,j}}{\Delta \eta}$$