Chapter 18

Algorithms for the Navier-Stokes Equations - In 3-D Arbitrary Curvilinear Coordinates

18.1 Introduction

The Navier-Stokes equations in three dimensional Cartesian coordinate can be transformed into arbitrary curvilinear coordinates in the same manner as the two dimensional equations were in Chapter 7.

18.2 Transformations in 3-D

Consider the general unsteady transformation from (x, y, z, t) coordinates to $(\xi, \eta, \varsigma, \tau)$ coordinates and the inverse transformation given below.

$$\xi = \xi(x, y, z, t), \quad \eta = \eta(x, y, z, t), \quad \varsigma = \varsigma(x, y, z, t) \quad and \quad \tau = t$$

and

$$x = x(\xi, \eta, \varsigma, \tau), \quad y = y(\xi, \eta, \varsigma, \tau), \quad z = z(\xi, \eta, \varsigma, \tau) \quad and \quad t = \tau$$

Using the *chain* rule, as in Sections 7.3 and 7.4 we obtain the following transformations for partial derivatives.

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial t} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial \varsigma}{\partial x} & 0 \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial \varsigma}{\partial y} & 0 \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \eta}{\partial z} & 0 \\ \frac{\partial \xi}{\partial t} & \frac{\partial \eta}{\partial t} & \frac{\partial \varsigma}{\partial t} & 1 \end{bmatrix}}_{T} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \varsigma} \\ \frac{\partial}{\partial \tau} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \varsigma} \\ \frac{\partial}{\partial \tau} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} & 0 \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} & 0 \\ \frac{\partial x}{\partial \varsigma} & \frac{\partial y}{\partial \varsigma} & \frac{\partial z}{\partial \varsigma} & 0 \\ \frac{\partial x}{\partial \tau} & \frac{\partial y}{\partial \tau} & \frac{\partial z}{\partial \tau} & 1 \end{bmatrix}}_{T^{-1}} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ \frac{\partial}{\partial t} \end{bmatrix}$$

Again, the conservation law form of the governing equations in (x, y, z, t) space maps into conservation law form in $(\xi, \eta, \varsigma, \tau)$ space.

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial U'}{\partial \tau} + \frac{\partial F'}{\partial \xi} + \frac{\partial G'}{\partial \eta} + \frac{\partial H'}{\partial \varsigma} = 0$$

where

$$U' = \frac{1}{d_{\xi\eta\varsigma}} U, \qquad d_{\xi\eta\varsigma} = \det T = \frac{1}{\det T^{-1}} = \frac{1}{d_{xyz}} = \frac{1}{\mathrm{V}},$$

$$F' = \frac{1}{d_{\xi\eta\varsigma}} \left(F \frac{\partial \xi}{\partial x} + G \frac{\partial \xi}{\partial y} + H \frac{\partial \xi}{\partial z} + U \frac{\partial \xi}{\partial t} \right),$$

$$G' = \frac{1}{d_{\xi\eta\varsigma}} \left(F \frac{\partial \eta}{\partial x} + G \frac{\partial \eta}{\partial y} + H \frac{\partial \eta}{\partial z} + U \frac{\partial \eta}{\partial t} \right),$$

$$\text{and } H' = \frac{1}{d_{\xi\eta\varsigma}} \left(F \frac{\partial \varsigma}{\partial x} + G \frac{\partial \varsigma}{\partial y} + H \frac{\partial \varsigma}{\partial z} + U \frac{\partial \varsigma}{\partial t} \right)$$

The derivatives with respect to x, y and z need to be expressed in terms of derivatives with respect to the computational coordinates ξ, η and ς, which can be approximated directly on the $\xi-\eta-\varsigma$ mesh. This can be done by determining the matrix inverse of T^{-1} and matching its elements directly with those of the matrix T, $\left(T^{-1}\right)^{-1} = T$, as done before in Section 7.3.

$$T = \left(T^{-1}\right)^{-1} = \frac{1}{\det T^{-1}} \begin{bmatrix} S_{\xi_x} & S_{\eta_x} & S_{\varsigma_x} & 0 \\ S_{\xi_y} & S_{\eta_y} & S_{\varsigma_y} & 0 \\ S_{\xi_z} & S_{\eta_z} & S_{\varsigma_z} & 0 \\ q_{\xi} & q_{\eta} & q_{\varsigma} & \det T^{-1} \end{bmatrix}, \quad \det T^{-1} = S_{\xi_x} \frac{\partial x}{\partial \xi} + S_{\eta_x} \frac{\partial x}{\partial \eta} + S_{\varsigma_x} \frac{\partial x}{\partial \varsigma},$$

$$S_{\xi_x} = \left(\frac{\partial y}{\partial \eta} \frac{\partial z}{\partial \varsigma} - \frac{\partial z}{\partial \eta} \frac{\partial y}{\partial \varsigma} \right), \quad S_{\eta_x} = -\left(\frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \varsigma} - \frac{\partial z}{\partial \xi} \frac{\partial y}{\partial \varsigma} \right), \quad S_{\varsigma_x} = \left(\frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \eta} - \frac{\partial z}{\partial \xi} \frac{\partial y}{\partial \eta} \right),$$

$$S_{\xi_y} = -\left(\frac{\partial x}{\partial \eta} \frac{\partial z}{\partial \varsigma} - \frac{\partial z}{\partial \eta} \frac{\partial x}{\partial \varsigma} \right), \quad S_{\eta_y} = \left(\frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \varsigma} - \frac{\partial z}{\partial \xi} \frac{\partial x}{\partial \varsigma} \right), \quad S_{\varsigma_y} = -\left(\frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \eta} - \frac{\partial z}{\partial \xi} \frac{\partial x}{\partial \eta} \right),$$

$$S_{\xi_z} = \left(\frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \varsigma} - \frac{\partial y}{\partial \eta} \frac{\partial x}{\partial \varsigma} \right), \quad S_{\eta_z} = -\left(\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \varsigma} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \varsigma} \right), \quad S_{\varsigma_z} = \left(\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta} \right),$$

$$Q_{\xi} = -\left(S_{\xi_x} \frac{\partial x}{\partial \tau} + S_{\xi_y} \frac{\partial y}{\partial \tau} + S_{\xi_z} \frac{\partial z}{\partial \tau} \right),$$

$$Q_{\eta} = -\left(S_{\eta_x} \frac{\partial x}{\partial \tau} + S_{\eta_y} \frac{\partial y}{\partial \tau} + S_{\eta_z} \frac{\partial z}{\partial \tau} \right),$$

and

$$Q_{\varsigma} = -\left(S_{\varsigma_x} \frac{\partial x}{\partial \tau} + S_{\varsigma_y} \frac{\partial y}{\partial \tau} + S_{\varsigma_z} \frac{\partial z}{\partial \tau} \right)$$

Often the determinant of the transformation inverse, $\det T^{-1}$, is called J. Herein it will be named V. We will continue to let the mesh indices form the metric for the computational coordinates ξ, η and ς. Thus, V has the dimension of a volume. The terms Q_ξ, Q_η and Q_ς represent the movement of the $(\xi,\eta,\varsigma,\tau)$ coordinate system through the fluid in the (x,y,z,t) coordinate system.

Using the above identities, $\quad U' = U\mathrm{V} \quad$ with $\quad \mathrm{V} = \det T^{-1}$

and the rotated flux vectors are

$$F' = S_{\xi_x} F + S_{\xi_y} G + S_{\xi_z} H + Q_\xi U \,,$$

$$G' = S_{\eta_x} F + S_{\eta_y} G + S_{\eta_z} H + Q_\eta U \,,$$

and

$$H' = S_{\varsigma_x} F + S_{\varsigma_y} G + S_{\varsigma_z} H + Q_\varsigma U$$

The first three terms of each rotated flux above represents the vector "dot" product of $\vec{F} = F\vec{i}_x + G\vec{i}_y + H\vec{i}_z$ with the surface vectors $\vec{S}_\xi$, $\vec{S}_\eta$ and $\vec{S}_\varsigma$. This is the fluid flux moving through the surface elements. The surface element vectors are given by

$$\vec{S}_\xi = S_{\xi_x}\vec{i}_x + S_{\xi_y}\vec{i}_y + S_{\xi_z}\vec{i}_z$$

$$\vec{S}_\eta = S_{\eta_x}\vec{i}_x + S_{\eta_y}\vec{i}_y + S_{\eta_z}\vec{i}_z$$

$$\vec{S}_\varsigma = S_{\varsigma_x}\vec{i}_x + S_{\varsigma_y}\vec{i}_y + S_{\varsigma_z}\vec{i}_z$$

The fourth term in each rotated flux above represents the last term of "dot" product of the four dimensional vector $\vec{F} = F\vec{i}_x + G\vec{i}_y + H\vec{i}_z + U\vec{i}_t$ with the surface vectors $\vec{S}_\xi + Q_\xi\vec{i}_t$, $\vec{S}_\eta + Q_\eta\vec{i}_t$ and $\vec{S}_\varsigma + Q_\varsigma\vec{i}_t$. The last term of each rotated flux represents the flux of mass, momentum and energy from the movement of the surface elements through the fluid, which can increase or decrease volume V in time.

18.3 Stationary Coordinate Transformation

Consider the coordinate systems (x,y,z,t) and $(\xi,\eta,\varsigma,\tau)$ not moving with respect to one another, i.e., $Q_\xi = 0$, $Q_\eta = 0$ and $Q_\varsigma = 0$. The volume V does not change with time and the Navier-Stokes equations can be written as

$$\frac{\partial U}{\partial t}+\frac{1}{\mathrm{V}}\frac{\partial F'}{\partial \xi}+\frac{1}{\mathrm{V}}\frac{\partial G'}{\partial \eta}+\frac{1}{\mathrm{V}}\frac{\partial H'}{\partial \varsigma}=0, \quad \text{with} \quad \mathrm{U}=\begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ e \end{bmatrix}$$

The equation above still has its Cartesian roots in that the state vector U is still written in terms of the Cartesian velocities u , v and w. The Cartesian fluxes are

$$F=\underbrace{\begin{bmatrix} \rho u \\ \rho u^2+p \\ \rho vu \\ \rho wu \\ (e+p)u \end{bmatrix}}_{Euler}+\underbrace{\begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{yx} \\ \tau_{zx} \\ \tau_{xx}u+\tau_{yx}v+\tau_{zx}w-k\frac{\partial T}{\partial x} \end{bmatrix}}_{viscous}=F_e+\begin{bmatrix} 0 \\ -\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)-2\mu\frac{\partial u}{\partial x} \\ -\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \\ -\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) \\ \tau_{xx}u+\tau_{xy}v+\tau_{xz}w-k\frac{\partial T}{\partial x} \end{bmatrix}$$

$$G=\underbrace{\begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2+p \\ \rho wv \\ (e+p)v \end{bmatrix}}_{Euler}+\underbrace{\begin{bmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ \tau_{zy} \\ \tau_{xy}u+\tau_{yy}v+\tau_{zy}w-k\frac{\partial T}{\partial y} \end{bmatrix}}_{viscous}=G_e+\begin{bmatrix} 0 \\ -\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ -\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)-2\mu\frac{\partial u}{\partial y} \\ -\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) \\ \tau_{xy}u+\tau_{yy}v+\tau_{zy}w-k\frac{\partial T}{\partial y} \end{bmatrix},$$

$$H=\underbrace{\begin{bmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2+p \\ (e+p)w \end{bmatrix}}_{Euler}+\underbrace{\begin{bmatrix} 0 \\ \tau_{xz} \\ \tau_{yz} \\ \tau_{zz} \\ \tau_{xz}u+\tau_{yz}v+\tau_{zz}w-k\frac{\partial T}{\partial z} \end{bmatrix}}_{viscous}=G_e+\begin{bmatrix} 0 \\ -\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ -\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ -\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)-2\mu\frac{\partial u}{\partial z} \\ \tau_{xz}u+\tau_{yz}v+\tau_{zz}w-k\frac{\partial T}{\partial z} \end{bmatrix}$$

The three Cartesian flux vectors can also be expressed as

$$F = F_e - M_{xx}\frac{\partial V}{\partial x} - M_{xy}\frac{\partial V}{\partial y} - M_{xz}\frac{\partial V}{\partial z}$$
$$G = G_e - M_{yx}\frac{\partial V}{\partial x} - M_{yy}\frac{\partial V}{\partial y} - M_{yz}\frac{\partial V}{\partial z}\ , \qquad V = \begin{bmatrix} \rho \\ u \\ v \\ w \\ T \end{bmatrix}$$
$$H = H_e - M_{zx}\frac{\partial V}{\partial x} - M_{zy}\frac{\partial V}{\partial y} - M_{zz}\frac{\partial V}{\partial z}$$

$$M_{xx} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & u(\lambda+2\mu) & v\mu & w\mu & k \end{bmatrix},\ M_{xy} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & v\mu & u\lambda & 0 & 0 \end{bmatrix},\ M_{xz} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & w\mu & 0 & u\lambda & 0 \end{bmatrix}$$

$$M_{yx} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & v\lambda & u\mu & 0 & 0 \end{bmatrix},\ M_{yy} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & \lambda+2\mu & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & u\mu & v(\lambda+2\mu) & w\mu & k \end{bmatrix},\ M_{yz} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & w\mu & v\lambda & 0 \end{bmatrix}$$

$$M_{zx} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & w\lambda & 0 & u\mu & 0 \end{bmatrix},\ M_{zy} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & w\lambda & v\mu & 0 \end{bmatrix},\ M_{zz} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \lambda+2\mu & 0 \\ 0 & u\mu & v\mu & w(\lambda+2\mu) & k \end{bmatrix}$$

The rotated flux vectors can now be written completely in the three dimensional computational arbitrary coordinate system as

$$F' = F'_e - M_{\xi\xi}\frac{\partial V}{\partial \xi} - M_{\xi\eta}\frac{\partial V}{\partial \eta} - M_{\xi\varsigma}\frac{\partial V}{\partial \varsigma}\ ,$$
$$G' = G'_e - M_{\eta\xi}\frac{\partial V}{\partial \xi} - M_{\eta\eta}\frac{\partial V}{\partial \eta} - M_{\eta\varsigma}\frac{\partial V}{\partial \varsigma}\ ,$$
$$H' = H'_e - M_{\varsigma\xi}\frac{\partial V}{\partial \xi} - M_{\varsigma\eta}\frac{\partial V}{\partial \eta} - M_{\varsigma\varsigma}\frac{\partial V}{\partial \varsigma}\ ,$$

where

$$\begin{bmatrix} F_e' \\ G_e' \\ H_e' \end{bmatrix} = \begin{bmatrix} S_{\xi_x} & S_{\xi_y} & S_{\xi_z} \\ S_{\eta_x} & S_{\eta_y} & S_{\eta_z} \\ S_{\varsigma_x} & S_{\varsigma_y} & S_{\varsigma_z} \end{bmatrix} \begin{bmatrix} F_e \\ G_e \\ H_e \end{bmatrix}$$

$$\begin{bmatrix} M_{\xi\xi} & M_{\xi\eta} & M_{\xi\varsigma} \\ M_{\eta\xi} & M_{\eta\eta} & M_{\eta\varsigma} \\ M_{\varsigma\xi} & M_{\varsigma\eta} & M_{\varsigma\varsigma} \end{bmatrix} = \frac{1}{\mathrm{V}} \begin{bmatrix} S_{\xi_x} & S_{\xi_y} & S_{\xi_z} \\ S_{\eta_x} & S_{\eta_y} & S_{\eta_z} \\ S_{\varsigma_x} & S_{\varsigma_y} & S_{\varsigma_z} \end{bmatrix} \begin{bmatrix} M_{xx} & M_{xy} & M_{xz} \\ M_{yx} & M_{yy} & M_{yz} \\ M_{zx} & M_{zy} & M_{zz} \end{bmatrix} \begin{bmatrix} S_{\xi_x} & S_{\eta_x} & S_{\varsigma_x} \\ S_{\xi_y} & S_{\eta_y} & S_{\varsigma_y} \\ S_{\xi_z} & S_{\eta_z} & S_{\varsigma_z} \end{bmatrix}$$

Similarly, the rotated jacobian matrices become

$$\begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = \begin{bmatrix} S_{\xi_x} & S_{\xi_y} & S_{\xi_z} \\ S_{\eta_x} & S_{\eta_y} & S_{\eta_z} \\ S_{\varsigma_x} & S_{\varsigma_y} & S_{\varsigma_z} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

The jacobian matrices A, B and C are defined in three dimensions by

$$A = \frac{\partial F_e}{\partial U} = S^{-1} Ch_A^{-1} \begin{bmatrix} u & 0 & 0 & 0 & 0 \\ 0 & u+c & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & u-c \end{bmatrix} Ch_A S \ ,$$

$$B = \frac{\partial G_e}{\partial U} = S^{-1} Ch_B^{-1} \begin{bmatrix} v & 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 & 0 \\ 0 & 0 & v+c & 0 & 0 \\ 0 & 0 & 0 & v & 0 \\ 0 & 0 & 0 & 0 & v-c \end{bmatrix} Ch_B S \ ,$$

$$C = \frac{\partial H_e}{\partial U} = S^{-1} Ch_C^{-1} \begin{bmatrix} w & 0 & 0 & 0 & 0 \\ 0 & w & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & w+c & 0 \\ 0 & 0 & 0 & 0 & w-c \end{bmatrix} Ch_C S$$

The matrices S and S^{-1} are the three dimensional analogs of those given in Section 9.6 for two dimensional flow. They are defined by

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -u/\rho & 1/\rho & 0 & 0 & 0 \\ -v/\rho & 0 & 1/\rho & 0 & 0 \\ -w/\rho & 0 & 0 & 1/\rho & 0 \\ \alpha\beta & -u\beta & -v\beta & -w\beta & \beta \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ u & \rho & 0 & 0 & 0 \\ v & 0 & \rho & 0 & 0 \\ w & 0 & 0 & \rho & 0 \\ \alpha & u\rho & v\rho & w\rho & 1/\beta \end{bmatrix},$$

$$\text{with} \qquad \alpha = \frac{u^2+v^2+w^2}{2} \qquad \text{and} \qquad \beta = \gamma - 1$$

The matrices Ch and Ch^{-1} with subscripts, A, B and C, are the three dimensional analogs of those given in Section 9.6, named C and C^{-1}, for two dimensional flow. Their named has been changed to avoid confusion with the flux jacobian matrix C. They are defined by

$$Ch_A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1/c^2 \\ 0 & \rho c & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -\rho c & 0 & 0 & 1 \end{bmatrix}, \qquad Ch_A^{-1} = \begin{bmatrix} 1 & \frac{1}{2c^2} & 0 & 0 & \frac{1}{2c^2} \\ 0 & \frac{1}{2\rho c} & 0 & 0 & \frac{-1}{2\rho c} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \end{bmatrix}$$

$$Ch_B = \begin{bmatrix} 1 & 0 & 0 & 0 & -1/c^2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \rho c & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\rho c & 0 & 1 \end{bmatrix}, \qquad Ch_B^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{2c^2} & 0 & \frac{1}{2c^2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2\rho c} & 0 & \frac{-1}{2\rho c} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

$$Ch_C = \begin{bmatrix} 1 & 0 & 0 & 0 & -1/c^2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \rho c & 1 \\ 0 & 0 & 0 & -\rho c & 1 \end{bmatrix}, \qquad Ch_C^{-1} = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2c^2} & \frac{1}{2c^2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\rho c} & \frac{-1}{2\rho c} \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

These matrices S and S^{-1} convert changes in the conservative variables to changes in the primitive variables, $\delta V = S\,\delta U$, and vice versa. The matrices Ch_A and Ch_A^{-1}, etc., convert changes in the primitive variables to changes in the characteristic variables (see Section 2.7). For example,

$$\delta V = \begin{bmatrix} \delta\rho \\ \delta u \\ \delta v \\ \delta w \\ \delta p \end{bmatrix} = \frac{\partial V}{\partial U}\delta U = S \begin{bmatrix} \delta\rho \\ \delta\rho u \\ \delta\rho v \\ \delta\rho w \\ \delta e \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \delta\rho - 1/c^2\delta p \\ \rho c\delta u + \delta p \\ \delta v \\ \delta w \\ \rho c\delta u - \delta p \end{bmatrix} = Ch_A\,\delta V = Ch_A \begin{bmatrix} \delta\rho \\ \delta u \\ \delta v \\ \delta w \\ \delta p \end{bmatrix}$$

These similarity transformations are used to diagonalize the jacobian matrices A, B and C to reveal their eigenvalue structure, which is needed for flux vector splitting or Roe's flux difference vector splitting. In three dimensional arbitrary curvilinear coordinates the eigenvalues of the rotated jacobian matrices A', B' and C' need to be obtained. As in two dimensional arbitrary curvilinear coordinates (see Sections 7.2 – 7.4), this can be done by the use of rotation matrices.

$$A' = S^{-1}R_{A'}^{-1}Ch_A^{-1}\Lambda_{A'}Ch_A R_{A'}S\ ,$$

$$B' = S^{-1}R_{B'}^{-1}Ch_B^{-1}\Lambda_{B'}Ch_B R_{B'}S\ ,$$

and

$$C' = S^{-1}R_{C'}^{-1}Ch_C^{-1}\Lambda_{C'}Ch_C R_{C'}S$$

The diagonal matrices $\Lambda_{A'}$, $\Lambda_{B'}$ and $\Lambda_{C'}$ are given by

$$\Lambda_{A'} = \left|\vec{S}_\xi\right| \begin{bmatrix} u' & 0 & 0 & 0 & 0 \\ 0 & u'+c & 0 & 0 & 0 \\ 0 & 0 & u' & 0 & 0 \\ 0 & 0 & 0 & u' & 0 \\ 0 & 0 & 0 & 0 & u'-c \end{bmatrix}, \quad \Lambda_{B'} = \left|\vec{S}_\eta\right| \begin{bmatrix} v' & 0 & 0 & 0 & 0 \\ 0 & v' & 0 & 0 & 0 \\ 0 & 0 & v'-c & 0 & 0 \\ 0 & 0 & 0 & v' & 0 \\ 0 & 0 & 0 & 0 & v'-c \end{bmatrix}$$

$$\text{and} \quad \Lambda_{C'} = \left|\vec{S}_\varsigma\right| \begin{bmatrix} w' & 0 & 0 & 0 & 0 \\ 0 & w' & 0 & 0 & 0 \\ 0 & 0 & w' & 0 & 0 \\ 0 & 0 & 0 & w'+c & 0 \\ 0 & 0 & 0 & 0 & w'-c \end{bmatrix},$$

where

$$\left|\vec{S}_\xi\right| = \sqrt{(S_{\xi_x})^2 + (S_{\xi_y})^2 + (S_{\xi_z})^2}\,, \quad u' = (S_{\xi_x} u + S_{\xi_y} v + S_{\xi_z} w)/\left|\vec{S}_\xi\right|\,,$$

$$\left|\vec{S}_\eta\right| = \sqrt{(S_{\eta_x})^2 + (S_{\eta_y})^2 + (S_{\eta_z})^2}\,, \quad v' = (S_{\eta_x} u + S_{\eta_y} v + S_{\eta_z} w)/\left|\vec{S}_\eta\right|\,,$$

and

$$\left|\vec{S}_\varsigma\right| = \sqrt{(S_{\varsigma_x})^2 + (S_{\varsigma_y})^2 + (S_{\varsigma_z})^2}\,, \quad w' = (S_{\varsigma_x} u + S_{\varsigma_y} v + S_{\varsigma_z} w)/\left|\vec{S}_\varsigma\right|$$

Note the importance of the surface vectors $\vec{S}_\xi$, $\vec{S}_\eta$ and $\vec{S}_\varsigma$ in the above definitions. These vectors are not in general orthogonal to each other in an arbitrary curvilinear coordinate system.

Consider the rotation matrices $R_{A'}$ and $R_{A'}^{-1}$

$$R_{A'} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & s_{\xi_x} & s_{\xi_y} & s_{\xi_z} & 0 \\ 0 & s''_{\eta_x} & s''_{\eta_y} & s''_{\eta_z} & 0 \\ 0 & s''_{\varsigma_x} & s''_{\varsigma_y} & s''_{\varsigma_z} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad R_{A'}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & s_{\xi_x} & s''_{\eta_x} & s''_{\varsigma_x} & 0 \\ 0 & s_{\xi_y} & s''_{\eta_y} & s''_{\varsigma_y} & 0 \\ 0 & s_{\xi_z} & s''_{\eta_z} & s''_{\varsigma_z} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\text{where} \quad s_{\xi_x} = \frac{S_{\xi_x}}{\left|\vec{S}_\xi\right|}, \quad s_{\xi_y} = \frac{S_{\xi_y}}{\left|\vec{S}_\xi\right|} \quad \text{and} \quad s_{\xi_z} = \frac{S_{\xi_z}}{\left|\vec{S}_\xi\right|}$$

The lower case quantities s_{ξ_x}, s_{ξ_y} and s_{ξ_z} are components of the unit vector $\vec{s}_\xi$ that is normal to the surface $\vec{S}_\xi$. This is the surface that the flux F' passes through. The quantities with double primes are components of unit vectors tangential to the surface $\vec{S}_\xi$ and orthogonal to the unit vector $\vec{s}_\xi$. The directions of the unit tangential vectors are arbitrary as long as they are orthogonal to each other and lie in the surface $\vec{S}_\xi$. They are not necessarily related to the vectors $\vec{S}_\eta$ and $\vec{S}_\varsigma$ and we will see that it is possible that they do not need to be defined at all. They will, however, be kept for convenience and can be defined as follows. At least one of the components of the unit vector $\vec{s}_\xi$ must be greater than ½ (actually greater or equal to $1/\sqrt{3} = 0.577$) in magnitude. For example, if $\left|s_{\xi_x}\right| > 1/2$, then the vectors

$$\vec{s}''_\eta = s''_{\eta_x}\vec{i}_x + s''_{\eta_y}\vec{i}_y + s''_{\eta_z}\vec{i}_z = \left(-(s_{\xi_y} + s_{\xi_z})\vec{i}_x + s_{\xi_x}\vec{i}_y + s_{\xi_x}\vec{i}_z\right)/\sqrt{(s_{\xi_y} + s_{\xi_z})^2 + 2s_{\xi_x}^2}$$

and

$$\vec{s}''_\varsigma = \vec{s}_\xi \times \vec{s}''_\eta = (s_{\xi_y} - s''_{\eta_z})\vec{i}_x + (s_{\xi_z} - s''_{\eta_x})\vec{i}_y + (s_{\xi_x} - s''_{\eta_y})\vec{i}_z$$

complete the set of three mutually orthogonal unit vectors.

Similarly

$$R_{B'}=\begin{bmatrix}1&0&0&0&0\\0&s''_{\xi_x}&s''_{\xi_y}&s''_{\xi_z}&0\\0&s_{\eta_x}&s_{\eta_y}&s_{\eta_z}&0\\0&s''_{\varsigma_x}&s''_{\varsigma_y}&s''_{\varsigma_z}&0\\0&0&0&0&1\end{bmatrix}\quad\text{and}\quad R_{B'}^{-1}=\begin{bmatrix}1&0&0&0&0\\0&s''_{\xi_x}&s_{\eta_x}&s''_{\varsigma_x}&0\\0&s''_{\xi_y}&s_{\eta_y}&s''_{\varsigma_y}&0\\0&s''_{\xi_z}&s_{\eta_z}&s''_{\varsigma_z}&0\\0&0&0&0&1\end{bmatrix},$$

$$\text{where}\quad s_{\eta_x}=\frac{S_{\eta_x}}{\left|\vec{S}_\eta\right|},\quad s_{\eta_y}=\frac{S_{\eta_y}}{\left|\vec{S}_\eta\right|}\quad\text{and}\quad s_{\eta_z}=\frac{S_{\eta_z}}{\left|\vec{S}_\eta\right|}$$

Again, the quantities s_{η_x}, s_{η_y} and s_{η_z} are components of the unit vector $\vec{s}_\eta$ normal to surface $\vec{S}_\eta$. This is the surface that the flux G' passes through and the quantities with double primes are components of unit vectors tangential to the surface $\vec{S}_\eta$ and orthogonal to the unit vector $\vec{s}_\eta$.

Finally,

$$R_{C'}=\begin{bmatrix}1&0&0&0&0\\0&s''_{\xi_x}&s''_{\xi_y}&s''_{\xi_z}&0\\0&s''_{\eta_x}&s''_{\eta_y}&s''_{\eta_z}&0\\0&s_{\varsigma_x}&s_{\varsigma_y}&s_{\varsigma_z}&0\\0&0&0&0&1\end{bmatrix}\quad\text{and}\quad R_{C'}^{-1}=\begin{bmatrix}1&0&0&0&0\\0&s''_{\xi_x}&s''_{\eta_x}&s_{\varsigma_x}&0\\0&s''_{\xi_y}&s''_{\eta_y}&s_{\varsigma_y}&0\\0&s''_{\xi_z}&s''_{\eta_z}&s_{\varsigma_z}&0\\0&0&0&0&1\end{bmatrix},$$

$$\text{where}\quad s_{\varsigma_x}=\frac{S_{\varsigma_x}}{\left|\vec{S}_\varsigma\right|},\quad s_{\varsigma_y}=\frac{S_{\varsigma_y}}{\left|\vec{S}_\varsigma\right|}\quad\text{and}\quad s_{\varsigma_z}=\frac{S_{\varsigma_z}}{\left|\vec{S}_\varsigma\right|}$$

where the quantities s_{ς_x}, s_{ς_y} and s_{ς_z} are components of the unit vector $\vec{s}_\varsigma$ normal to surface $\vec{S}_\varsigma$. This is the surface that the flux H' passes through and the quantities with double primes are components of unit vectors tangential to the surface $\vec{S}_\varsigma$ and orthogonal to the unit vector $\vec{s}_\varsigma$.

18.3.1 The Jacobian Matrix A'

The factors matrix $A'=S^{-1}R_{A'}^{-1}Ch_A^{-1}\Lambda_{A'}Ch_AR_{A'}S$ can be written as $A'=T_{A'}^{-1}\Lambda_{A'}T_{A'}$ with and

$$T_{A'}^{-1} = S^{-1}R_{A'}^{-1}Ch_A^{-1} = \begin{bmatrix} 1 & 1/(2c^2) & 0 & 0 & 1/(2c^2) \\ u & (u+s_{\xi_x}c)/(2c^2) & s''_{\eta_x} & s''_{\varsigma_x} & (u-s_{\xi_x}c)/(2c^2) \\ v & (v+s_{\xi_y}c)/(2c^2) & s''_{\eta_y} & s''_{\varsigma_y} & (v-s_{\xi_y}c)/(2c^2) \\ w & (w+s_{\xi_z}c)/(2c^2) & s''_{\eta_z} & s''_{\varsigma_z} & (w-s_{\xi_z}c)/(2c^2) \\ \alpha & (\alpha+u'c)/(2c^2)+1/(2\beta) & v'' & w'' & (\alpha-u'c)/(2c^2)+1/(2\beta) \end{bmatrix}$$

and

$$T_{A'} = Ch_A R_{A'} S = \begin{bmatrix} 1-\alpha\beta/c^2 & u\beta/c^2 & v\beta/c^2 & w\beta/c^2 & -\beta/c^2 \\ -u'c+\alpha\beta & s_{\xi_x}c-u\beta & s_{\xi_y}c-v\beta & s_{\xi_z}c-w\beta & \beta \\ -v'' & s''_{\eta_x} & s''_{\eta_y} & s''_{\eta_z} & 0 \\ -w'' & s''_{\varsigma_x} & s''_{\varsigma_y} & s''_{\varsigma_z} & 0 \\ u'c+\alpha\beta & -s_{\xi_x}c-u\beta & -s_{\xi_y}c-v\beta & -s_{\xi_z}c-w\beta & \beta \end{bmatrix}$$

$$u' = s_{\xi_x}u + s_{\xi_y}v + s_{\xi_z}w, \quad v'' = s''_{\eta_x}u + s''_{\eta_y}v + s''_{\eta_z}w \quad \text{and} \quad w'' = s''_{\varsigma_x}u + s''_{\varsigma_y}v + s''_{\varsigma_z}w$$

Note (1) Density, ρ , does not appear in either $T_{A'}$ or $T_{A'}^{-1}$.
Note (2) The double primed quantities only appear in the third and fourth rows matrix $T_{A'}$, only in the third and fourth columns of $T_{A'}^{-1}$, and not at all in the matrix $\Lambda_{A'}$.

Possible removal of the double primed quantities

The diagonal matrix $\Lambda_{A'}$ can be partitioned according to its eigenvalues as follows

$$\Lambda_{A'} = \underbrace{\left|\vec{S}_\xi\right| \begin{bmatrix} u' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u' & 0 & 0 \\ 0 & 0 & 0 & u' & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\Lambda_{u'}} + \underbrace{\left|\vec{S}_\xi\right| \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & u'+c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\Lambda_{u'+c}} + \underbrace{\left|\vec{S}_\xi\right| \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u'-c \end{bmatrix}}_{\Lambda_{u'-c}}$$

Using the orthogonal properties of the unit vectors $\vec{s}_\xi$, $\vec{s}''$ and $\vec{s}''$ and the identities found within $R_{A'}R_{A'}^{-1} = I$ and $R_{A'}^{-1}R_{A'} = I$, we find that for the first eigenvalue u' of A'

$$A'_{u'} = T^{-1}\Lambda_{u'}T =$$

$$u'\left|\vec{S}_\xi\right|\begin{bmatrix} 1-\alpha\beta/c^2 & u\beta/c^2 & v\beta/c^2 & w\beta/c^2 & -\beta/c^2 \\ s_{\xi_x}u'-u\alpha\beta/c^2 & 1-s_{\xi_x}s_{\xi_x}+uu\beta/c^2 & -s_{\xi_x}s_{\xi_y}+uv\beta/c^2 & -s_{\xi_x}s_{\xi_z}+uw\beta/c^2 & -u\beta/c^2 \\ s_{\xi_y}u'-v\alpha\beta/c^2 & -s_{\xi_y}s_{\xi_x}+vu\beta/c^2 & 1-s_{\xi_y}s_{\xi_y}+vv\beta/c^2 & -s_{\xi_y}s_{\xi_z}+vw\beta/c^2 & -v\beta/c^2 \\ s_{\xi_z}u'-w\alpha\beta/c^2 & -s_{\xi_z}s_{\xi_x}+wu\beta/c^2 & -s_{\xi_z}s_{\xi_y}+wv\beta/c^2 & 1-s_{\xi_z}s_{\xi_z}+ww\beta/c^2 & -w\beta/c^2 \\ u'u'-\alpha-\alpha^2\beta/c^2 & u-s_{\xi_x}u'+u\alpha\beta/c^2 & v-s_{\xi_y}u'+v\alpha\beta/c^2 & w-s_{\xi_z}u'+w\alpha\beta/c & -\alpha\beta/c^2 \end{bmatrix}$$

Notice that the double primed quantities have disappeared in the above matrix for $A'_{u'}$. Similarly for the second and third eigenvalues of A'

$$A'_{u'+c} = T^{-1}\Lambda_{u'+c}T = (u'+c)\left|\vec{S}_\xi\right|T^{-1}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -u'c+\alpha\beta & s_{\xi_x}c-u\beta & s_{\xi_y}c-v\beta & s_{\xi_z}c-w\beta & \beta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$A'_{u'-c} = T^{-1}\Lambda_{u'-c}T = (u'-c)\left|\vec{S}_\xi\right|T^{-1}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ u'c+\alpha\beta & -s_{\xi_x}c-u\beta & -s_{\xi_y}c-v\beta & -s_{\xi_z}c-w\beta & \beta \end{bmatrix}$$

Because the matrix T^{-1} has double primed quantities only in the third and fourth columns, the matrices $A'_{u'+c}$ and $A'_{u'-c}$ will have no double primed quantities upon multiplication by T^{-1}. However, it may be convenient to retain the double primed quantities used earlier to define the rotation matrices, such as in $R_{A'}$, etc..

18.4 Non-Stationary Coordinate Transformation

Consider the general case where the coordinate systems (x,y,z,t) and $(\xi,\eta,\varsigma,\tau)$ may be moving with respect to one another, i.e., $Q_\xi \neq 0$, $Q_\eta \neq 0$ or $Q_\varsigma \neq 0$. The volume V may change with time and the Navier-Stokes equations should be written as (see Section 18.2)

$$\frac{\partial U \mathrm{V}}{\partial t}+\frac{\partial F'}{\partial \xi}+\frac{\partial G'}{\partial \eta}+\frac{\partial H'}{\partial \varsigma}=0, \quad \text{with} \quad \mathrm{U}=\begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ e \end{bmatrix}$$

and the rotated flux vectors are given by

$$\begin{aligned} F' &= S_{\xi_x} F + S_{\xi_y} G + S_{\xi_z} H + Q_\xi U \\ G' &= S_{\eta_x} F + S_{\eta_y} G + S_{\eta_z} H + Q_\eta U \\ H' &= S_{\varsigma_x} F + S_{\varsigma_y} G + S_{\varsigma_z} H + Q_\varsigma U \end{aligned} \quad \text{or} \quad \begin{bmatrix} F' \\ G' \\ H' \end{bmatrix} = \begin{bmatrix} S_{\xi x} & S_{\xi_y} & S_{\xi_z} & Q_\xi \\ S_{\eta x} & S_{\eta_y} & S_{\eta z} & Q_\eta \\ S_{\varsigma x} & S_{\varsigma_y} & S_{\varsigma z} & Q_\varsigma \end{bmatrix} \begin{bmatrix} F \\ G \\ H \\ U \end{bmatrix}$$

Similarly the rotated jacobian matrices are

$$\begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = \begin{bmatrix} S_{\xi x} & S_{\xi_y} & S_{\xi_z} & Q_\xi \\ S_{\eta x} & S_{\eta_y} & S_{\eta z} & Q_\eta \\ S_{\varsigma x} & S_{\varsigma_y} & S_{\varsigma z} & Q_\varsigma \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ I \end{bmatrix}$$

The eigenvalues of the diagonal matrices $\Lambda_{A'}$, $\Lambda_{B'}$ and $\Lambda_{C'}$ need to be changed to include the moving coordinate terms, the eigenvalue speeds u', v' and w' now need to be

$$u' = (S_{\xi_x} u + S_{\xi_y} v + S_{\xi_z} w + Q_\xi) / \left|\vec{S}_\xi\right|,$$

$$v' = (S_{\eta_x} u + S_{\eta_y} v + S_{\eta_z} w + Q_\eta) / \left|\vec{S}_\eta\right|,$$

and

$$w' = (S_{\varsigma_x} u + S_{\varsigma_y} v + S_{\varsigma_z} w + Q_\varsigma) / \left|\vec{S}_\varsigma\right|$$

with $\left|\vec{S}_\xi\right|$, $\left|\vec{S}_\eta\right|$ and $\left|\vec{S}_\varsigma\right|$ as previously defined.

18.5 All You Need is F' and A'

The forgoing discussion presented three rotated flux vectors and three rotated jacobian matrices. Because the rotations are completely arbitrary any one set rotated flux vector and jacobian matrix can represent the others. In practical terms, this means that in the coding of an algorithm only one subroutine need be written for the evaluation of three fluxes and only one subroutine need be written for the three rotated jacobians. This is may be significant because changing from one algorithm to another, such as Modified-Steger-Warming to Roe, will only require changes to to two subroutines as opposed to making correct parallel changes to six. However, the metric terms will be different for the three fluxes and jacobians.

Metric terms in three dimensions

In the same manner as in Chapter 7, the flux and jacobian matrices are surface centric For example, if the flux surface is at $\vec{S}_{\xi_{i+1/2,j,k}}$ located between mesh points i,j,k and $i+1,j,k$ all derivatives x, y and z with respect to ξ, η and ς appearing in the metric terms for the flux and jacobian matrices should be approximated using mesh point centered about the flux surface. This is also is true for derivatives of the flow variables appearing in the viscous stress terms.

Subroutine construction

Each flux F', G' and H' or jacobian matrix A', B' and C' is a linear is a linear combination of F, G, H and U or the matrices A, B, C and I. If S_x, S_y, S_z and q represent the quantities at surface $\vec{S}_{\xi_{i+1/2,j,k}}$, $\vec{S}_{\eta_{i,j+1/2,k}}$ or $\vec{S}_{\varsigma_{i,j,k+1/2}}$, then $F' = S_x F + S_y G + S_z H + QU$ and $A' = S_x A + S_y B + S_z C + QI$ can represent the rotated flux and jacobian at any surface. Therefore, in the construction of a subroutine to determine either the rotated flux or jacobian, the input to the subroutine should include the mesh points, centered about the flux surface, to be used to evaluate the elements of the flux or jacobian only. All else is common in structure for evaluation of all rotated fluxes and jacobians.

18.6 Implicit Algorithm for the Navier-Stokes Equations in 3-D

We return to using the fluxes F', G' and H' and jacobian matrices A', B' and C' to describe the generic algorithm for approximating the Navier-Stokes equations in three dimensions, although they are only F' and A' at different surfaces. We again present the difference equation as if a split flux or split flux difference vector algorithm is used.

18.6.1 Stationary Coordinate Transformation Case

For an arbitrary curvilinear coordinate system not moving with respect to the Cartesian coordinate system, the governing equation to be solved is (i.e., $Q_\xi = 0$, $Q_\eta = 0$ and $Q_\varsigma = 0$, see Section 18.3)

$$\frac{\partial U}{\partial t} + \frac{1}{\mathrm{V}}\frac{\partial F'}{\partial \xi} + \frac{1}{\mathrm{V}}\frac{\partial G'}{\partial \eta} + \frac{1}{\mathrm{V}}\frac{\partial H'}{\partial \varsigma} = 0$$

The generic implicit algorithm for the solution is given by

$$\left\{ I + \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k}} \left(\frac{D_- \cdot}{\Delta\xi} \overline{A'}^n_{+_{i+1/2,j,k}} + \frac{D_+ \cdot}{\Delta\xi} \overline{A'}^n_{-_{i-1/2,j,k}} + \frac{D_- \cdot}{\Delta\eta} \overline{B'}^n_{+_{i,j+1/2,k}} + \frac{D_+ \cdot}{\Delta\eta} \overline{B'}^n_{-_{i,j-1/2,k}} \right.\right.$$

$$+ \frac{D_- \cdot}{\Delta\varsigma} \overline{C'}^n_{+_{i,j,k+1/2}} + \frac{D_+ \cdot}{\Delta\varsigma} \overline{C'}^n_{-_{i,j,k-1/2}}$$

$$- \frac{D_- \cdot}{\Delta\xi} M_{\xi\xi_{i+1/2,j,k}} \frac{D_+ \cdot N_{i,j,k}}{\Delta\xi} - \frac{D_- \cdot}{\Delta\eta} M_{\eta\eta_{i,j+1/2,k}} \frac{D_+ \cdot N_{i,j,k}}{\Delta\eta}$$

$$\left.\left. - \frac{D_- \cdot}{\Delta\varsigma} M_{\varsigma\varsigma_{i,j,k+1/2}} \frac{D_+ \cdot N_{i,j,k}}{\Delta\varsigma} \right) \right\} \delta U^{n+1}_{i,j,k} = \Delta U^n_{i,j,k}$$

$$= -\frac{\Delta t}{\mathrm{V}_{i,j,k}} \left(\frac{F'^n_{i+1/2,j,k} - F'^n_{i-1/2,j,k}}{\Delta\xi} + \frac{G'^n_{i,j+1/2,k} - G'^n_{i,j-1/2,k}}{\Delta\eta} + \frac{H'^n_{i,j+1/2,k} - H'^n_{i,j-1/2,k}}{\Delta\varsigma} \right)$$

The definitions of the rotated fluxes and jacobian matrices are given in Section 18.3.

Block matrix equation

The algorithm can be expressed as a block matrix equation, a line of which is given by

$$\overline{\mathrm{B}}_{i,j,k} \delta U^{n+1}_{i,j+1,k} + \overline{\mathrm{A}}_{i,j,k} \delta U^{n+1}_{i,j,k} + \overline{\mathrm{C}}_{i,j,k} \delta U^{n+1}_{i,j-1,k} + \overline{\mathrm{D}}_{i,j,k} \delta U^{n+1}_{i+1,j,k} + \overline{\mathrm{E}}_{i,j,k} \delta U^{n+1}_{i-1,j,k}$$

$$+ \overline{\mathrm{F}}_{i,j,k} \delta U^{n+1}_{i,j,k+1} + \overline{\mathrm{G}}_{i,j,k} \delta U^{n+1}_{i,j,k-1} = \Delta U^n_{i,j,k}$$

with the block elements defined by

$$\overline{\mathrm{A}}_{i,j,k} = I + \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k} \Delta\xi} \left(\overline{A'}^n_{+_{i+1/2,j,k}} - \overline{A'}^n_{-_{i-1/2,j,k}} + \frac{1}{\Delta\xi} \left(M_{\xi\xi_{i+1/2,j,k}} + M_{\xi\xi_{i-1/2,j,k}} \right) N_{i,j,k} \right)$$

$$+ \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k} \Delta\eta} \left(\overline{B'}^n_{+_{i,j+1/2,k}} - \overline{B'}^n_{-_{i,j-1/2,k}} + \frac{1}{\Delta\eta} \left(M_{\eta\eta_{i,j+1/2,k}} + M_{\eta\eta_{i,j-1/2,k}} \right) N_{i,j,k} \right)$$

$$+ \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k} \Delta\varsigma} \left(\overline{C'}^n_{+_{i,j,k+1/2}} - \overline{C'}^n_{-_{i,j,k-1/2}} + \frac{1}{\Delta\varsigma} \left(M_{\varsigma\varsigma_{i,j,k+1/2}} + M_{\varsigma\varsigma_{i,j,k-1/2}} \right) N_{i,j,k} \right)$$

$$\overline{\mathrm{B}}_{i,j,k} = \frac{\Delta t}{\mathrm{V}_{i,j,k} \Delta\eta} \left(+\overline{B'}^n_{-_{i,j+1/2,k}} - \frac{1}{\Delta\eta} M_{\eta\eta_{i,j+1/2,k}} N_{i,j+1,k} \right)$$

$$\overline{\mathrm{C}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k} \Delta\eta} \left(-\overline{B'}^n_{-_{i,j-1/2,k}} + \frac{1}{\Delta\eta} M_{\eta\eta_{i,j-1/2,k}} N_{i,j-1,k} \right)$$

$$\overline{\mathrm{D}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k} \Delta\xi} \left(+\overline{A'}^n_{-_{i+1/2,j,k}} + \frac{1}{\Delta\xi} M_{\xi\xi_{i+1/2,j,k}} N_{i+1,j,k} \right)$$

$$\overline{\mathrm{E}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k} \Delta\xi} \left(-\overline{A'}^n_{+_{i-1/2,j,k}} + \frac{1}{\Delta\xi} M_{\xi\xi_{i-1/2,j,k}} N_{i-1,j,k} \right)$$

$$\overline{\mathrm{F}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k} \Delta\varsigma} \left(+\overline{C'}^n_{-_{i,j,k+1/2}} + \frac{1}{\Delta\varsigma} M_{\varsigma\varsigma_{i,j,k+1/2}} N_{i,j,k+1} \right)$$

$$\overline{\mathrm{G}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k}\Delta\varsigma}\left(-\overline{C'}^{n}_{+_{i,j,k-1/2}} + \frac{1}{\Delta\varsigma} M_{\varsigma\varsigma_{i,j,k-1/2}} N_{i,j,k-1} \right)$$

18.6.2 Non-Stationary Coordinate Transformation Case

For an arbitrary curvilinear coordinate system moving with respect to the Cartesian coordinate system, the governing equation to be solved is

$$\frac{\partial U'}{\partial t} + \frac{\partial F'}{\partial \xi} + \frac{\partial G'}{\partial \eta} + \frac{\partial H'}{\partial \varsigma} = 0 \quad \text{with } U' = U\mathrm{V}$$

Derivation of the implicit equation

We start with the following general algorithm for the non-stationary case, omitting the viscous terms for convenience.

$$\delta U'^{n+1}_{i,j,k} + \Delta t \left(\frac{D_- \cdot}{\Delta\xi} \overline{A'}^{n}_{i+1/2,j,k} \delta U^{n+1}_{i+1/2,j,k} + \frac{D_- \cdot}{\Delta\eta} \overline{B'}^{n}_{i,j+1/2,k} \delta U^{n+1}_{i,j+1/2,k} + \frac{D_- \cdot}{\Delta\varsigma} \overline{C'}^{n}_{i,j,k+1/2} \delta U^{n+1}_{i,j,k+1/2} \right)$$
$$= -\Delta t \left(\frac{D_- \cdot}{\Delta\xi} F'^{n}_{i+1/2,j,k} + \frac{D_- \cdot}{\Delta\eta} G'^{n}_{i,j+1/2,k} + \frac{D_- \cdot}{\Delta\varsigma} H'^{n}_{i,j+1/2,k} \right)$$
$$= -\Delta t \left(\frac{D_- \cdot}{\Delta\xi} \overline{A'}^{n}_{i+1/2,j,k} U^{n+1}_{i+1/2,j,k} + \frac{D_- \cdot}{\Delta\eta} \overline{B'}^{n}_{i,j+1/2,k} U^{n+1}_{i,j+1/2,k} + \frac{D_- \cdot}{\Delta\varsigma} \overline{C'}^{n}_{i,j,k+1/2} U^{n+1}_{i,j,k+1/2} \right)$$

Notice that the only primed $\delta U'$ is the first term of the above equation, where in general

$$\delta U' = U^{n+1}\mathrm{V}^{n+1} - U^{n}\mathrm{V}^{n} = \mathrm{V}^{n+1}(U^{n+1} - U^{n}) + U^{n}(\mathrm{V}^{n+1}\text{-}\mathrm{V}^{n}) = \mathrm{V}^{n+1}\delta U + U^{n}\delta\mathrm{V}$$
$$\text{or} \quad \delta U = \frac{1}{\mathrm{V}^{n+1}}(\delta U' - U^{n}\delta\mathrm{V})$$

Case (1) algorithm

Upon substitution of this last result in the above algorithm

$$\delta U'^{n+1}_{i,j,k} + \Delta t \left(\frac{D_- \cdot}{\Delta\xi} \frac{\overline{A'}^{n}_{i+1/2,j,k}}{\mathrm{V}^{n+1}_{i+1/2,j,k}} \delta U'^{n+1}_{i+1/2,j,k} + \frac{D_- \cdot}{\Delta\eta} \frac{\overline{B'}^{n}_{i,j+1/2,k}}{\mathrm{V}^{n+1}_{i,j+1/2,k}} \delta U'^{n+1}_{i,j+1/2,k} + \frac{D_- \cdot}{\Delta\varsigma} \frac{\overline{C'}^{n}_{i,j,k+1/2}}{\mathrm{V}^{n+1}_{i,j,k+1/2}} \delta U'^{n+1}_{i,j,k+1/2} \right)$$
$$= -\Delta t \left(\frac{D_- \cdot}{\Delta\xi} \frac{\overline{A'}^{n}_{i+1/2,j,k}}{\mathrm{V}^{n+1}_{i+1/2,j,k}} U'^{n}_{i+1/2,j,k} + \frac{D_- \cdot}{\Delta\eta} \frac{\overline{B'}^{n}_{i,j+1/2,k}}{\mathrm{V}^{n+1}_{i,j+1/2,k}} U'^{n}_{i,j+1/2,k} + \frac{D_- \cdot}{\Delta\varsigma} \frac{\overline{C'}^{n}_{i,j,k+1/2}}{\mathrm{V}^{n+1}_{i,j,k+1/2}} U'^{n}_{i,j,k+1/2} \right)$$

Notice that now primes appear on all variables $\delta U'$ and U'.

Case (2) algorithm

On the other hand, we could have replaced the lone $\delta U'$ appearing the first equation with $\mathrm{V}^{n+1}\delta U + U^{n}\delta\mathrm{V}$ and obtained the following algorithm case (2) without the primed variables.

$$\delta U_{i,j,k}^{n+1} + \frac{\Delta t}{\mathrm{V}_{i,j,k}^{n+1}}\left(\frac{D_-\cdot}{\Delta\xi}\overline{A'}^n_{i+1/2,j,k}\delta U_{i+1/2,j,k}^{n+1} + \frac{D_-\cdot}{\Delta\eta}\overline{B'}^n_{i,j+1/2,k}\delta U_{i,j+1/2,k}^{n+1} + \frac{D_-\cdot}{\Delta\varsigma}\overline{C'}^n_{i,j,k+1/2}\delta U_{i,j,k+1/2}^{n+1}\right)$$

$$= -U_{i,j,k}^{n}\frac{\delta\mathrm{V}_{i,j,k}^{n+1}}{\mathrm{V}_{i,j,k}^{n+1}} - \frac{\Delta t}{\mathrm{V}_{i,j,k}^{n+1}}\left(\frac{D_-\cdot}{\Delta\xi}\overline{A'}^n_{i+1/2,j,k}U_{i+1/2,j,k}^{n} + \frac{D_-\cdot}{\Delta\eta}\overline{B'}^n_{i,j+1/2,k}U_{i,j+1/2,k}^{n} + \frac{D_-\cdot}{\Delta\varsigma}\overline{C'}^n_{i,j,k+1/2}U_{i,j,k+1/2}^{n}\right)$$

Geometric conservation

Consider the coordinate system moving through a uniform flow. This implies that the part of the flux caused by the movement of the fluid will have no net effect upon a closed surface, one with no gaps or overlaps. We assume that the mesh volumes are each completely enclosed by mesh surfaces and that this part of the flux can be neglected. The remaining part of the flux is caused by movement of the mesh. It should also have no net effect if the mesh movement has the property of geometric conservation. Consider only the part of the flux caused by movement of the mesh. For the case (1) algorithm, with $A' = Q_\xi I$, $B' = Q_\eta I$, $C' = Q_\varsigma I$ and U constant

$$\delta\mathrm{V}_{i,j,k}^{n+1} + \Delta t\left(\frac{D_-\cdot}{\Delta\xi}\frac{Q_{\xi}{}^{n}_{i+1/2,j,k}}{\mathrm{V}_{i+1/2,j,k}^{n+1}}\delta\mathrm{V}_{i+1/2,j,k}^{n+1} + \frac{D_-\cdot}{\Delta\eta}\frac{Q_{\eta}{}^{n}_{i,j+1/2,k}}{\mathrm{V}_{i,j+1/2,k}^{n+1}}\delta\mathrm{V}_{i,j+1/2,k}^{n+1} + \frac{D_-\cdot}{\Delta\varsigma}\frac{Q_{\varsigma}{}^{n}_{i,j,k+1/2}}{\mathrm{V}_{i,j,k+1/2}^{n+1}}\delta\mathrm{V}_{i,j,k+1/2}^{n+1}\right)$$

$$= -\Delta t\left(\frac{D_-\cdot}{\Delta\xi}\frac{Q_{\xi}{}^{n}_{i+1/2,j,k}}{\mathrm{V}_{i+1/2,j,k}^{n+1}}\mathrm{V}_{i+1/2,j,k}^{n} + \frac{D_-\cdot}{\Delta\eta}\frac{Q_{\eta}{}^{n}_{i,j+1/2,k}}{\mathrm{V}_{i,j+1/2,k}^{n+1}}\mathrm{V}_{i,j+1/2,k}^{n} + \frac{D_-\cdot}{\Delta\varsigma}\frac{Q_{\varsigma}{}^{n}_{i,j,k+1/2}}{\mathrm{V}_{i,j,k+1/2}^{n+1}}\mathrm{V}_{i,j,k+1/2}^{n}\right)$$

$$\text{or}\qquad \delta\mathrm{V}_{i,j,k}^{n+1} + \Delta t\left(\frac{D_-\cdot}{\Delta\xi}Q_{\xi}{}^{n}_{i+1/2,j,k} + \frac{D_-\cdot}{\Delta\eta}Q_{\eta}{}^{n}_{i,j+1/2,k} + \frac{D_-\cdot}{\Delta\varsigma}Q_{\varsigma}{}^{n}_{i,j,k+1/2}\right) = 0$$

Thus, geometric conservation requires that

$$\delta\mathrm{V}_{i,j,k}^{n+1} = -\Delta t\left(\frac{D_-\cdot}{\Delta\xi}Q_{\xi}{}^{n}_{i+1/2,j,k} + \frac{D_-\cdot}{\Delta\eta}Q_{\eta}{}^{n}_{i,j+1/2,k} + \frac{D_-\cdot}{\Delta\varsigma}Q_{\varsigma}{}^{n}_{i,j,k+1/2}\right)$$

Similarly, for the case (2) algorithm, with constant $U \Rightarrow \delta U = 0$, the above equation is also satisfied,. However, the evaluation of the metric terms appearing in the algorithm, either case (1) or (2), must be chosen so that the above constraint on $\delta\mathrm{V}_{i,j,k}^{n+1}$ holds everywhere in the mesh during movement in time by Δt .

The generic implicit algorithm for case (1)

The generic implicit algorithm for case (1) is given by

$$
\left\{ I + \alpha\Delta t \left(\frac{D_- \cdot}{\Delta\xi} \overline{A'}^n_{+i+1/2,j,k} + \frac{D_+ \cdot}{\Delta\xi} \overline{A'}^n_{-i-1/2,j,k} + \frac{D_- \cdot}{\Delta\eta} \overline{B'}^n_{+i,j+1/2,k} + \frac{D_+ \cdot}{\Delta\eta} \overline{B'}^n_{-i,j-1/2,k} \right.\right.
$$

$$
+ \frac{D_- \cdot}{\Delta\varsigma} \overline{C'}^n_{+i,j,k+1/2} + \frac{D_+ \cdot}{\Delta\varsigma} \overline{C'}^n_{-i,j,k-1/2}
$$

$$
- \frac{D_- \cdot}{\Delta\xi} M_{\xi\xi_{i+1/2,j,k}} \frac{D_+ \cdot N_{i,j,k}}{\Delta\xi} - \frac{D_- \cdot}{\Delta\eta} M_{\eta\eta_{i,j+1/2,k}} \frac{D_+ \cdot N_{i,j,k}}{\Delta\eta}
$$

$$
\left.\left. - \frac{D_- \cdot}{\Delta\varsigma} M_{\varsigma\varsigma_{i,j,k+1/2}} \frac{D_+ \cdot N_{i,j,k}}{\Delta\varsigma} \right) \frac{1}{\mathrm{V}_{i,j,k}} \right\} \delta U'^{n+1}_{i,j,k} = \Delta U''^{n}_{i,j,k}
$$

$$
= -\Delta t \left(\frac{F''^n_{i+1/2,j,k} - F''^n_{i-1/2,j,k}}{\Delta\xi} + \frac{G''^n_{i,j+1/2,k} - G''^n_{i,j-1/2,k}}{\Delta\eta} + \frac{H''^n_{i,j+1/2,k} - H''^n_{i,j-1/2,k}}{\Delta\varsigma} \right)
$$

The definitions of the jacobian matrices are as given in Section 18.4. However, notice the double prime appearing on the explicit flux vectors, which are defined below.

$$
\begin{aligned}
F''^n_{i+1/2,j.k} &= F'^n_{i+1/2,j,k} \mathrm{V}^n_{i+1/2,j,k} / \mathrm{V}^{n+1}_{i+1/2,j,k}\,, \\
G''^n_{i,j+1/2,k} &= G'^n_{i,j+1/2,k} \mathrm{V}^n_{i,j+1/2,k} / \mathrm{V}^{n+1}_{i,j+1/2,k}\,, \\
\text{and} \quad H''^n_{i,j,k+1/2} &= H'^n_{i,j,k+1/2} \mathrm{V}^n_{i,j,k+1/2} / \mathrm{V}^{n+1}_{i,j,k+1/2}
\end{aligned}
$$

The single primed fluxes are defined as before. The essential difference between the equations of Section 18.6.1 and the present section is the placement of the volume terms V and the subscripts on them.

Block matrix equation for case (1)

The algorithm can be expressed as a block matrix equation, a line of which is given by

$$
\overline{\mathrm{B}}_{i,j,k} \delta U''^{n+1}_{i,j+1,k} + \overline{\mathrm{A}}_{i,j,k} \delta U'^{n+1}_{i,j,k} + \overline{\mathrm{C}}_{i,j,k} \delta U'^{n+1}_{i,j-1,k} + \overline{\mathrm{D}}_{i,j,k} \delta U'^{n+1}_{i+1,j,k} + \overline{\mathrm{E}}_{i,j,k} \delta U'^{n+1}_{i-1,j,k}
$$

$$
+ \overline{\mathrm{F}}_{i,j,k} \delta U'^{n+1}_{i,j,k+1} + \overline{\mathrm{G}}_{i,j,k} \delta U'^{n+1}_{i,j,k-1} = \Delta U''^{n}_{i,j,k}
$$

with the block elements defined by

$$
\begin{aligned}
\overline{\mathrm{A}}_{i,j,k} = I &+ \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k} \Delta\xi} \left(\overline{A'}^n_{+i+1/2,j,k} - \overline{A'}^n_{-i-1/2,j,k} + \frac{1}{\Delta\xi} \left(M_{\xi\xi_{i+1/2,j,k}} + M_{\xi\xi_{i-1/2,j,k}} \right) N_{i,j,k} \right) \\
&+ \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k} \Delta\eta} \left(\overline{B'}^n_{+i,j+1/2,k} - \overline{B'}^n_{-i,j-1/2,k} + \frac{1}{\Delta\eta} \left(M_{\eta\eta_{i,j+1/2,k}} + M_{\eta\eta_{i,j-1/2,k}} \right) N_{i,j,k} \right) \\
&+ \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k} \Delta\varsigma} \left(\overline{C'}^n_{+i,j,k+1/2} - \overline{C'}^n_{-i,j,k-1/2} + \frac{1}{\Delta\varsigma} \left(M_{\varsigma\varsigma_{i,j,k+1/2}} + M_{\varsigma\varsigma_{i,j,k-1/2}} \right) N_{i,j,k} \right)
\end{aligned}
$$

$$
\overline{\mathrm{B}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i,j+1,k} \Delta\eta} \left(+ \overline{B'}^n_{-i,j+1/2,k} - \frac{1}{\Delta\eta} M_{\eta\eta_{i,j+1/2,k}} N_{i,j+1,k} \right)
$$

$$\overline{\mathrm{C}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i,j-1,k}\Delta\eta}\left(-\overline{B'}^{n}_{-_{i,j-1/2,k}} + \frac{1}{\Delta\eta} M_{\eta\eta_{i,j-1/2,k}} N_{i,j-1,k} \right)$$

$$\overline{\mathrm{D}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i+1,j,k}\Delta\xi}\left(+\overline{A'}^{n}_{-_{i+1/2,j,k}} + \frac{1}{\Delta\xi} M_{\xi\xi_{i+1/2,j,k}} N_{i+1,j,k} \right)$$

$$\overline{\mathrm{E}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i-1,j,k}\Delta\xi}\left(-\overline{A'}^{n}_{+_{i-1/2,j,k}} + \frac{1}{\Delta\xi} M_{\xi\xi_{i-1/2,j,k}} N_{i-1,j,k} \right)$$

$$\overline{\mathrm{F}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k+1}\Delta\varsigma}\left(+\overline{C'}^{n}_{-_{i,j,k+1/2}} + \frac{1}{\Delta\varsigma} M_{\varsigma\varsigma_{i,j,k+1/2}} N_{i,j,k+1} \right)$$

$$\overline{\mathrm{G}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k-1}\Delta\varsigma}\left(-\overline{C'}^{n}_{+_{i,j,k-1/2}} + \frac{1}{\Delta\varsigma} M_{\varsigma\varsigma_{i,j,k-1/2}} N_{i,j,k-1} \right)$$

<u>The generic implicit algorithm for case (2)</u>

The generic implicit algorithm for case (2) is given by

$$\begin{aligned}
&\Bigg\{ I + \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k}} \Bigg(\frac{D_-\cdot}{\Delta\xi}\overline{A'}^{n}_{+_{i+1/2,j,k}} + \frac{D_+\cdot}{\Delta\xi}\overline{A'}^{n}_{-_{i-1/2,j,k}} + \frac{D_-\cdot}{\Delta\eta}\overline{B'}^{n}_{+_{i,j+1/2,k}} + \frac{D_+\cdot}{\Delta\eta}\overline{B'}^{n}_{-_{i,j-1/2,k}} \\
&\qquad + \frac{D_-\cdot}{\Delta\varsigma}\overline{C'}^{n}_{+_{i,j,k+1/2}} + \frac{D_+\cdot}{\Delta\varsigma}\overline{C'}^{n}_{-_{i,j,k-1/2}} \\
&\qquad - \frac{D_-\cdot}{\Delta\xi} M_{\xi\xi_{i+1/2,j,k}} \frac{D_+\cdot N_{i,j,k}}{\Delta\xi} - \frac{D_-\cdot}{\Delta\eta} M_{\eta\eta_{i,j+1/2,k}} \frac{D_+\cdot N_{i,j,k}}{\Delta\eta} \\
&\qquad - \frac{D_-\cdot}{\Delta\varsigma} M_{\varsigma\varsigma_{i,j,k+1/2}} \frac{D_+\cdot N_{i,j,k}}{\Delta\varsigma} \Bigg) \Bigg\} \delta U^{n+1}_{i,j,k} = \Delta U'^{n}_{i,j,k} \\
&= -U^{n}_{i,j,k}\frac{\delta \mathrm{V}^{n+1}_{i,j,k}}{\mathrm{V}_{i,j,k}} - \frac{\Delta t}{\mathrm{V}_{i,j,k}}\left(\frac{F'^{n}_{i+1/2,j,k} - F'^{n}_{i-1/2,j,k}}{\Delta\xi} + \frac{G'^{n}_{i,j+1/2,k} - G'^{n}_{i,j-1/2,k}}{\Delta\eta} + \frac{H'^{n}_{i,j+1/2,k} - H'^{n}_{i,j-1/2,k}}{\Delta\varsigma} \right)
\end{aligned}$$

The definitions of the jacobian matrices and single primed fluxes are as given in Section 18.4.

<u>Block Matrix Equation for case (2)</u>

The algorithm can be expressed as a block matrix equation, a line of which is given by

$$\begin{aligned}
&\overline{\mathrm{B}}_{i,j,k}\delta U''^{n+1}_{i,j+1,k} + \overline{\mathrm{A}}_{i,j,k}\delta U'^{n+1}_{i,j,k} + \overline{\mathrm{C}}_{i,j,k}\delta U'^{n+1}_{i,j-1,k} + \overline{\mathrm{D}}_{i,j,k}\delta U'^{n+1}_{i+1,j,k} + \overline{\mathrm{E}}_{i,j,k}\delta U'^{n+1}_{i-1,j,k} \\
&\qquad + \overline{\mathrm{F}}_{i,j,k}\delta U'^{n+1}_{i,j,k+1} + \overline{\mathrm{G}}_{i,j,k}\delta U'^{n+1}_{i,j,k-1} = \Delta U'^{n}_{i,j,k}
\end{aligned}$$

with the block elements defined by

$$\overline{\mathrm{A}}_{i,j,k} = I + \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k}\Delta\xi}\left(\overline{A'}^{n}_{+_{i+1/2,j,k}} - \overline{A'}^{n}_{-_{i-1/2,j,k}} + \frac{1}{\Delta\xi}\left(M_{\xi\xi_{i+1/2,j,k}} + M_{\xi\xi_{i-1/2,j,k}} \right) N_{i,j,k} \right)$$

$$+ \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k}\Delta\eta}\left(\overline{B'}^{n}_{+_{i,j+1/2,k}} - \overline{B'}^{n}_{-_{i,j-1/2,k}} + \frac{1}{\Delta\eta}\left(M_{\eta\eta_{i,j+1/2,k}} + M_{\eta\eta_{i,j-1/2,k}} \right) N_{i,j,k} \right)$$

$$+ \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k}\Delta\varsigma}\left(\overline{C'}^{n}_{+_{i,j,k+1/2}} - \overline{C'}^{n}_{-_{i,j,k-1/2}} + \frac{1}{\Delta\varsigma}\left(M_{\varsigma\varsigma_{i,j,k+1/2}} + M_{\varsigma\varsigma_{i,j,k-1/2}} \right) N_{i,j,k} \right)$$

$$\overline{\mathrm{B}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k}\Delta\eta}\left(+\overline{B'}^{n}_{-_{i,j+1/2,k}} - \frac{1}{\Delta\eta} M_{\eta\eta_{i,j+1/2,k}} N_{i,j+1,k} \right)$$

$$\overline{\mathrm{C}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k}\Delta\eta}\left(-\overline{B'}^{n}_{-_{i,j-1/2,k}} + \frac{1}{\Delta\eta} M_{\eta\eta_{i,j-1/2,k}} N_{i,j-1,k} \right)$$

$$\overline{\mathrm{D}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k}\Delta\xi}\left(+\overline{A'}^{n}_{-_{i+1/2,j,k}} + \frac{1}{\Delta\xi} M_{\xi\xi_{i+1/2,j,k}} N_{i+1,j,k} \right)$$

$$\overline{\mathrm{E}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k}\Delta\xi}\left(-\overline{A'}^{n}_{+_{i-1/2,j,k}} + \frac{1}{\Delta\xi} M_{\xi\xi_{i-1/2,j,k}} N_{i-1,j,k} \right)$$

$$\overline{\mathrm{F}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k}\Delta\varsigma}\left(+\overline{C'}^{n}_{-_{i,j,k+1/2}} + \frac{1}{\Delta\varsigma} M_{\varsigma\varsigma_{i,j,k+1/2}} N_{i,j,k+1} \right)$$

$$\overline{\mathrm{G}}_{i,j,k} = \alpha \frac{\Delta t}{\mathrm{V}_{i,j,k}\Delta\varsigma}\left(-\overline{C'}^{n}_{+_{i,j,k-1/2}} + \frac{1}{\Delta\varsigma} M_{\varsigma\varsigma_{i,j,k-1/2}} N_{i,j,k-1} \right)$$