Chapter 10

Higher Resolution in Space and Time - The TVD Method

10.1 Introduction

The computed results, shown in Chapters 5 and 9, using second order accurate methods contained oscillations, some needing artificial dissipation to control them, while those of the first order methods were smooth, but tended to smear discontinuities in the flow. For example, the results of using the second order Warming and Beam method and the first order upwind method, Methods (10) and (11) of Section 5.2, applied to the model hyperbolic equation, $\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0$ with $c=1$, are shown below. Both methods are upwind methods.

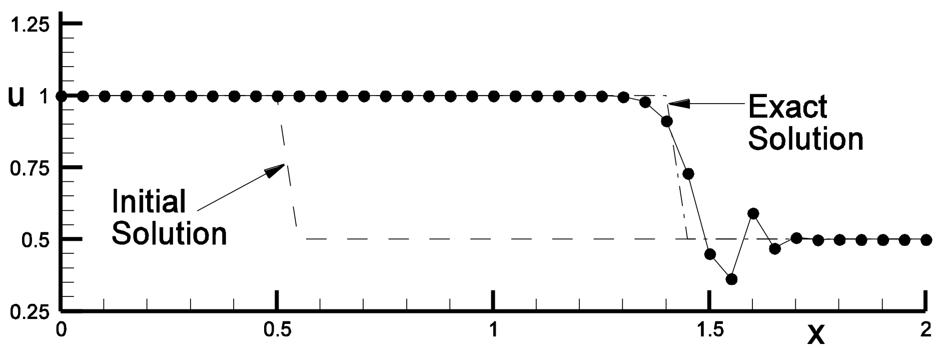

Figure 10.1 Second Order Warming-Beam method, CFL=0.9, 20 time steps

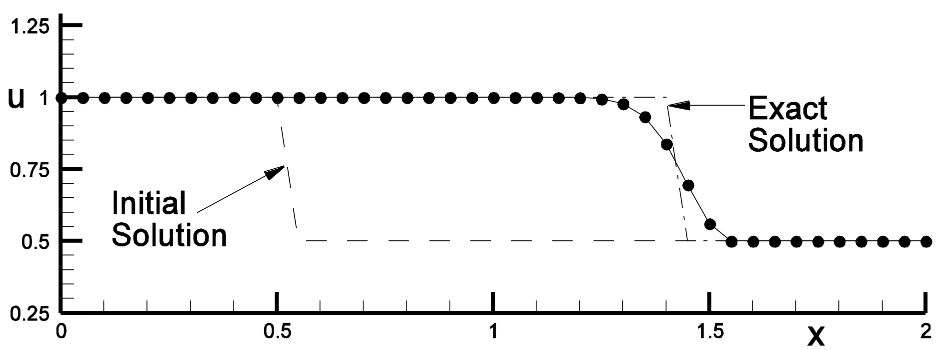

Figure 10.2 First order upwind method, CFL=0.9, 20 time steps

The second order Warming-Beam method, written as a single step method, is shown below.

$$u_i^{n+1}=u_i^n-c\Delta t\frac{D_-\cdot}{\Delta x}\left\{u_i^n+\frac{\Delta x}{2}(1-\frac{c\Delta t}{\Delta x})\frac{D_-\cdot}{\Delta x}u_i^n\right\}$$

The first order accurate upwind method is

$$u_i^{n+1} = u_i^n - \Delta t \frac{D_- \cdot}{\Delta x} f_{i+1/2}^n \quad \text{with} \quad f_{i+1/2}^n = \frac{1}{2} c(u_i^n + u_{i+1}^n) - \frac{1}{2}|c|(u_{i+1}^n - u_i^n)$$

or

$$u_{i+1}^n = u_i^n - \frac{c\Delta t}{\Delta x}\frac{1+\operatorname{sgn}(c)}{2}(u_i^n - u_{i-1}^n) - \frac{c\Delta t}{\Delta x}\frac{1-\operatorname{sgn}(c)}{2}(u_{i+1}^n - u_i^n), \quad \text{where} \quad \operatorname{sgn}(c) = \begin{cases} +1, & if \quad c \ge 0 \\ -1, & if \quad c < 0 \end{cases}$$

The results using the first order accurate method, shown in the above figure, are smooth, monotonically decreasing with x, but show a fairly smeared discontinuity, while the second order results are oscillatory, but with a sharper representation of the discontinuity. This chapter will show how to obtain monotonical and sharper results using the TVD, Total Variation Diminishing method, first proposed by Ami Harten. The TVD method will be applied to the single step version of second order Warming-Beam method. The two step version could serve as well. The two volume set “*Numerical Computation of Internal and External Flows*” by C. Hirsch is recommended for a complete discussion on TVD and higher resolution methods.

10.2 Harten’s TVD Method

The total variation of u_i^n over the interval $x_0 \le x \le x_1$ is defined to be

$$TV^n = \sum_i \left|u_i^n - u_{i-1}^n\right|$$

Total Variation Diminishing is defined by $TV^{n+1} \le TV^n$. If the initial solution is monotone then $TV^0 = \sum_i \left|u_i^0 - u_{i-1}^0\right| = \left|u_I^0 - u_1^0\right|$, where u_1^0 and u_I^0 are the values of the solution at the end points. If the solution algorithm is TVD no new extrema, either over or under-shoots, will be formed. For example, the first order upwind method above, applied to the linear hyperbolic model equation for $c > 1$, becomes $u_i^{n+1} = u_i^n - \frac{|c|\Delta t}{\Delta x}\left\{u_i^n - u_{i-1}^n\right\}$ and the total variation is then

$$\begin{aligned} TV^{n+1} &= \sum_i \left|u_i^{n+1} - u_{i-1}^{n+1}\right| = \sum_i \left|u_i^n - u_{i-1}^n - \frac{|c|\Delta t}{\Delta x}(u_i^n - u_{i-1}^n) + \frac{|c|\Delta t}{\Delta x}(u_{i-1}^n - u_{i-2}^n)\right| \\ &\le \sum_i \left\{\left(1 - \frac{|c|\Delta t}{\Delta x}\right)\left|u_i^n - u_{i-1}^n\right| + \frac{|c|\Delta t}{\Delta x}\left|u_{i-1}^n - u_{i-2}^n\right|\right\} \end{aligned}$$

We assume that the CFL condition is satisfied, $\frac{|c|\Delta t}{\Delta x} \le 1$, to keep the factor $\left(1 - \frac{|c|\Delta t}{\Delta x}\right)$ positive. The above summation is a telescopic sum because the term $\frac{|c|\Delta t}{\Delta x}\left|u_k^n - u_{k-1}^n\right|$ will appear twice, once with a negative sign with summation index $i = k$, and then with a positive sign with summation index $i = k+1$. Therefore

$$TV^{n+1} = \sum_i \left| u_i^{n+1} - u_{i-1}^{n+1} \right| \le \sum_i \left\{ \left(1 - \cancel{\frac{|c|\Delta t}{\Delta x}} \right) \left| u_i^n - u_{i-1}^n \right| + \cancel{\frac{|c|\Delta t}{\Delta x} \left| u_{i-1}^n - u_{i-2}^n \right|} \right\} = \sum_i \left| u_i^n - u_{i-1}^n \right| = TV^n$$

It can be shown that all methods of second order accuracy are not TVD.

10.2.1 First Order Accurate TVD for a General Scalar Non-linear Equation

The proof above can be extended to show that the upwind method applied to a general nonlinear equation

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \text{, where } f = f(u)$$

is also TVD. This equation becomes, via the chain rule for derivatives,

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \text{, where } c = \frac{\partial f}{\partial u} = c(u) \text{, which can be either positive or negative.}$$

A generic explicit algorithm for solving this equation is

$$u_i^{n+1} = u_i^n - \Delta t \frac{f_{i+1/2}^n - f_{i-1/2}^n}{\Delta x}$$

A characteristic speed can be defined, using a capital C, as follows

$$C_i^n = \frac{f_{i+1/2}^n - f_{i-1/2}^n}{u_{i+1/2}^n - u_{i-1/2}^n}, \quad if \quad u_{i+1/2}^n - u_{i-1/2}^n \ne 0 \quad \text{or} \quad C_i^n = c_i^n, \quad if \quad u_{i+1/2}^n - u_{i-1/2}^n = 0$$

Then the explicit equation can be re written as

$$u_i^{n+1} = u_i^n - C_i^n \frac{\Delta t}{\Delta x}(u_{i+1/2}^n - u_{i-1/2}^n)$$

The upwind algorithm applied (see Section 5.2) to this equation can be represented by

$$u_i^{n+1} = u_i^n - \frac{C_{+,i}\Delta t}{\Delta x}(u_i^n - u_{i-1}^n) - \frac{C_{-,i}\Delta t}{\Delta x}(u_{i+1}^n - u_i^n)$$

where

$$C_{+,i} = \frac{1 + \operatorname{sgn}(C_i^n)}{2} C_i^n \ge 0 \text{ and } C_{-,i} = \frac{1 - \operatorname{sgn}(C_i^n)}{2} C_i^n \le 0$$

Then

$$TV^{n+1} = \sum_i \left| u_i^{n+1} - u_{i-1}^{n+1} \right| =$$

$$\sum_i \left| u_i^n - u_{i-1}^n - \frac{C_{+,i}\Delta t}{\Delta x}(u_i^n - u_{i-1}^n) + \frac{C_{+,i-1}\Delta t}{\Delta x}(u_{i-1}^n - u_{i-2}^n) - \frac{C_{-,i}\Delta t}{\Delta x}(u_{i+1}^n - u_i^n) + \frac{C_{-,i-1}\Delta t}{\Delta x}(u_i^n - u_{i-1}^n) \right|$$

$$\leq \sum_i \left\{ \left(1 - \frac{C_{+,i}\Delta t}{\Delta x} + \frac{C_{-,i-1}\Delta t}{\Delta x} \right) \left| u_i^n - u_{i-1}^n \right| + \frac{C_{+,i-1}\Delta t}{\Delta x} \left| u_{i-1}^n - u_{i-2}^n \right| - \frac{C_{-,i}\Delta t}{\Delta x} \left| u_{i+1}^n - u_i^n \right| \right\}$$

Where we now use the CFL condition, $\frac{C_{+,i}\Delta t}{\Delta x} - \frac{C_{-,i-1}\Delta t}{\Delta x} \leq 1$. The signs of the characteristic speeds $C_{+,i-1}$ and $C_{-,i}$ are such that the factors appearing before the absolute signs are all positive. Again, the above summation is telescopic and $TV^{n+1} = \sum_i \left| u_i^{n+1} - u_{i-1}^{n+1} \right| \leq \sum_i \left| u_i^n - u_{i-1}^n \right| = TV^n$.

Therefore, any algorithm that can be written as $u_i^{n+1} = u_i^n - \frac{C_{+,i}\Delta t}{\Delta x}(u_i^n - u_{i-1}^n) - \frac{C_{-,i}\Delta t}{\Delta x}(u_{i+1}^n - u_i^n)$ ***with*** $C_{+,i} \geq 0$ ***and*** $C_{-,i} \leq 0$***, that satisfies the CFL condition*** $\frac{C_{+,i}\Delta t}{\Delta x} - \frac{C_{-,i-1}\Delta t}{\Delta x} \leq 1$ ***is TVD.***

10.2.2 Flux Limitation and the minmod Function

Again, second order methods are in general non-monotone, but may be made so by limiting the fluxes passing through the interfaces between mesh points. The limiters are non-linear. They depend upon the gradients of the solution and when invoked can reduce the local order of accuracy to first order. Fortunately, they need be invoked infrequently and only when a new extrema occurs, thereby retaining the higher order of accuracy almost everywhere. The TVD property is more than cosmetic. It may prevent the numerical solution from straying into non-physical states, such as negative densities, pressures and temperatures, and thereby terminating the calculation. Let's take a look again at the second order upwind method of Warming and Beam applied to solve the non-linear equation

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0, \text{ where } f = f(u), \ c = \frac{\partial f}{\partial u} \text{ and } c(u) \geq 0$$

The characteristic speed for the equation is c and the characteristic information carried along at this speed is u. Assuming that c may vary with x and t, but will remain positive, the following equation approximates the above equation to second order accuracy, again using C_i^n as defined previously.

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left\{ f_{i+1/2}^n - f_{i-1/2}^n \right\} = u_i^n - C_i^n \frac{\Delta t}{\Delta x}\left\{ u_{i+1/2}^n - u_{i-1/2}^n \right\}$$

$$\text{with } u_{i+1/2}^n = u_i^n + \frac{1}{2}\left(1 - \frac{c_i^n \Delta t}{\Delta x}\right)(u_i^n - u_{i-1}^n)$$

then

$$u_i^{n+1} = u_i^n - C_i^n \Delta t \frac{D_- \cdot}{\Delta x} \left\{ u_i^n + \frac{\Delta x}{2}(1 - \frac{c_i^n \Delta t}{\Delta x}) \frac{D_- \cdot}{\Delta x} u_i^n \right\}$$

or

$$u_i^{n+1} = u_i^n - \frac{C_i^n \Delta t}{\Delta x} \left\{ u_i^n + \frac{1}{2}(1 - \frac{c_i^n \Delta t}{\Delta x})(u_i^n - u_{i-1}^n) - u_{i-1}^n - \frac{1}{2}(1 - \frac{c_{i-1}^n \Delta t}{\Delta x})\left(u_{i-1}^n - u_{i-2}^n\right) \right\}$$

The characteristic speed at the flux surface, midway between mesh points i and $i+1$, is $c_{i+1/2}^n$ and the characteristic information carried across this surface is given by $u_i^n + \frac{1}{2}(1 - \frac{c_i^n \Delta t}{\Delta x})(u_i^n - u_{i-1}^n)$. The second term makes the algorithm of second order accuracy and non-TVD. This part of the flux will be limited to make the algorithm TVD. We define the function ψ_+ as follows

$$\psi_+(u_i - u_{i-1}) = \min\bmod\left(u_i - u_{i-1}, u_{i+1} - u_i\right)$$

The minmod function is defined by $$\min\bmod\left(r,s\right) = \begin{cases} 0, \ \mathit{if}\ \mathrm{sgn}(r) \neq \mathrm{sgn}(s) \\ \mathit{otherwise} \\ r, \ \mathit{if}\ |r| \leq |s| \\ s, \ \mathit{if}\ |r| > |s| \end{cases}$$

or $$\min\bmod\left(r,s\right) = \frac{1}{2}\left\{\mathrm{sgn}(r) + \mathrm{sgn}(s)\right\}\min(|r|,|s|)$$

The flux is therefore limited by

$$u_{i+1/2}^n = u_i^n + \frac{1}{2}(1 - \frac{c_i^n \Delta t}{\Delta x})\psi_+(u_i^n - u_{i-1}^n)$$

The flux limiter ψ_+ will return a zero value, limiting the flux to first order accuracy, only when the solution interval segments, $u_i - u_{i-1}$, change sign, which occurs at solution extrema. At other points, where the gradients of the solution have the same sign, the limiter will limit the flux by taking the milder gradient. Where the solution is smoothly changing its gradient the limiter will not decrease the formal order of accuracy. Its purpose is to damp only the high frequency error oscillations, such as the "saw tooth error" often observed in numerical calculations, but not damp well resolved features of the flow. The second order accuracy of the method should make it far less dissipative than first order procedures.

10.2.3 Second Order Accurate TVD for a Scalar Non-linear Equation

The explicit equation of the last section, now using the flux limiter, becomes

$$u_i^{n+1} = u_i^n - \frac{C_i^n \Delta t}{\Delta x}\left\{ u_i^n - u_{i-1}^n + \frac{1}{2}(1 - \frac{c_i^n \Delta t}{\Delta x})\psi_+(u_i^n - u_{i-1}^n) - \frac{1}{2}(1 - \frac{c_{i-1}^n \Delta t}{\Delta x})\psi_+(u_{i-1}^n - u_{i-2}^n) \right\}$$

Assuming that $u_i^n - u_{i-1}^n \neq 0$, otherwise $\psi_+(u_i^n - u_{i-1}^n)$ and $\psi_+(u_{i-1}^n - u_{i-2}^n)$ would both vanish, we can write

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\underbrace{C_i^n \left\{ 1 + \frac{1}{2}(1 - \frac{c_i^n \Delta t}{\Delta x})\frac{\psi_+(u_i^n - u_{i-1}^n)}{u_i^n - u_{i-1}^n} - \frac{1}{2}(1 - \frac{c_{i-1}^n \Delta t}{\Delta x})\frac{\psi_+(u_{i-1}^n - u_{i-2}^n)}{u_i^n - u_{i-1}^n} \right\}}_{C_i'}(u_i^n - u_{i-1}^n)$$

All we need to show now is that C_i', defined by the equation above, satisfies the CFL condition, $0 \leq \frac{C_i' \Delta t}{\Delta x} \leq 1$, and therefore, as in the last section, the algorithm $u_i^{n+1} = u_i^n - \frac{C_i' \Delta t}{\Delta x}(u_i^n - u_{i-1}^n)$ is TVD. We first set Δt such that the CFL condition $0 \leq \lambda^n = \frac{c_i^n \Delta t}{\Delta x} \leq 1$ is satisfied for all i. The minmod definition implies that $0 \leq \frac{\psi_+(u_i^n - u_{i-1}^n)}{u_i^n - u_{i-1}^n} \leq 1 \leq \frac{2}{\lambda_i^n}$ and $0 \leq \frac{\psi_+(u_{i-1}^n - u_{i-2}^n)}{u_i^n - u_{i-1}^n} \leq 1 \leq \frac{2}{1 - \lambda_{i-1}^n}$. Therefore,

$$\lambda_i' = \frac{C_i' \Delta t}{\Delta x} = \frac{C_i^n \Delta t}{\Delta x}\left\{ 1 + \frac{1}{2}(1 - \frac{c_i^n \Delta t}{\Delta x})\frac{\psi_+(u_i^n - u_{i-1}^n)}{u_i^n - u_{i-1}^n} - \frac{1}{2}(1 - \frac{c_{i-1}^n \Delta t}{\Delta x})\frac{\psi_+(u_{i-1}^n - u_{i-2}^n)}{u_i^n - u_{i-1}^n} \right\}$$

or

$$\begin{aligned} \lambda_i' &= \lambda_i^n \left\{ 1 + \frac{1}{2}(1 - \lambda_i^n)\frac{\psi_+(u_i^n - u_{i-1}^n)}{u_i^n - u_{i-1}^n} - \frac{1}{2}(1 - \lambda_{i-1}^n)\frac{\psi_+(u_{i-1}^n - u_{i-2}^n)}{u_i^n - u_{i-1}^n} \right\} \\ &\leq \lambda_i^n \left\{ 1 + \frac{1}{2}(1 - \lambda_i^n)\frac{\psi_+(u_i^n - u_{i-1}^n)}{u_i^n - u_{i-1}^n} \right\} \leq \lambda_i^n \left\{ 1 + \frac{1}{2}(1 - \lambda_i^n)\frac{2}{\lambda_i^n} \right\} \leq 1 \end{aligned}$$

Where we have assumed $\lambda_i^n = \frac{c_i \Delta t}{\Delta x} \approx \frac{C_i^n \Delta t}{\Delta x}$, otherwise we could have replaced the lower case c_i^n with the capital C_i^n in the above expressions, ending with the same result.

Similarly,

$$\begin{aligned} \lambda_i' &= \lambda_i^n \left\{ 1 + \frac{1}{2}(1 - \lambda_i^n)\frac{\psi_+(u_i^n - u_{i-1}^n)}{u_i^n - u_{i-1}^n} - \frac{1}{2}(1 - \lambda_{i-1}^n)\frac{\psi_+(u_{i-1}^n - u_{i-2}^n)}{u_i^n - u_{i-1}^n} \right\} \\ &\geq \lambda_i^n \left\{ 1 - \frac{1}{2}(1 - \lambda_{i-1}^n)\frac{\psi_+(u_{i-1}^n - u_{i-2}^n)}{u_i^n - u_{i-1}^n} \right\} \geq \lambda_i^n \left\{ 1 - \frac{1}{2}(1 - \lambda_{i-1}^n)\frac{2}{1 - \lambda_{i-1}^n} \right\} = 0 \end{aligned}$$

Therefore, we have shown that $0 \leq \lambda_i' \leq 1$ and the algorithm is therefore TVD.

The Warming and Beam algorithm, modified for TVD was applied to solve the model hyperbolic equation and the results are shown below. This solution is somewhat sharper than the first order upwind method and is without the oscillations of the second order method. However, for this simple problem the only feature is a discontinuity where the method becomes first order accurate. There are no other features here to exploit the higher resolution of the method.

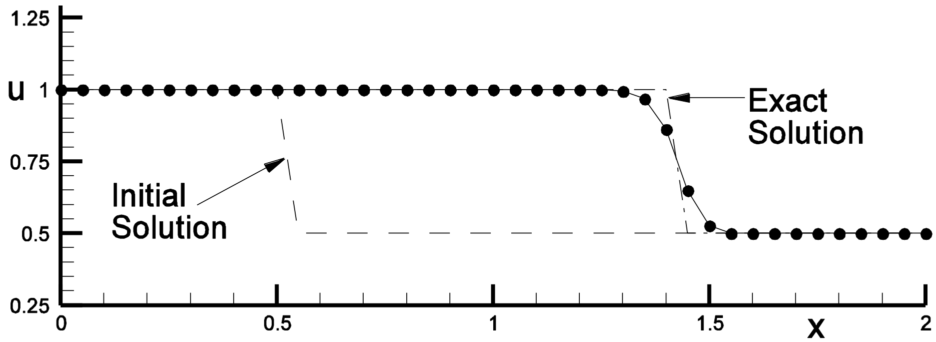

Figure 10.3 Second Order Warming-Beam method with TVD, CFL=0.9, 20 time steps

Note: The Warming-Beam explicit algorithm was used above to illustrate the use of flux limitation with the minmod function to remove spurious oscillations from the computed solution. The limited flux was given by

$$f_{i+1/2}^n = c_{i+1/2}^n \left\{ u_i^n + \frac{1}{2}\left(1 - \frac{c_{i+1/2}^n \Delta t}{\Delta x}\right)\psi_+ (u_i^n - u_{i-1}^n) \right\}, \text{ with } c_{i+1/2}^n = c$$

This flux will solve the governing equation with $c > 0$ to second order accuracy in both space and time (see Section 5.2).

A second order accurate spatial flux was given in Section 9.11.1.

$$f_{i+1/2}^n = c_{i+1/2}^n \left\{ u_i^n + \frac{1}{2}(u_i^n - u_{i-1}^n) \right\}$$

As pointed out in Chapter 9 and demonstrated in Section 4.4.7, this would result in an unstable algorithm if used in the following explicit method.

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left\{ f_{i+1/2}^n - f_{i-1/2}^n \right\} = u_i^n - C_i^n \frac{\Delta t}{\Delta x}\left\{ u_{i+1/2}^n - u_{i-1/2}^n \right\}$$

However, if the above flux was limited, $f_{i+1/2}^n = c_{i+1/2}^n \left\{ u_i^n + \frac{1}{2}\psi_+ (u_i^n - u_{i-1}^n) \right\}$, then the algorithm can be shown to be stable by the same reasoning given above for the Warming-Beam algorithm, but with a more restrictive CFL stability condition, $\lambda_i^n = \frac{c_i \Delta t}{\Delta x} \approx \frac{C_i^n \Delta t}{\Delta x} \le \frac{2}{3}$. This somewhat simpler flux should be used in an implicit algorithm, to be discussed later, instead of the one given by the

Warming-Beam algorithm, because of the possible change in sign of the term $(1-\frac{c_{i+1/2}^n \Delta t}{\Delta x})$ with the large time steps allowed with implicit methods.

10.2.4 Second Order Accurate TVD for a General Scalar Non-linear Equation

The Warming and Beam upwind method can be extended to apply to a general nonlinear equation of form

$$\frac{\partial u}{\partial t}+\frac{\partial f}{\partial x}=0 \text{, where } f=f(u)$$

We will apply the upwind second order Warming-Beam algorithm with TVD to this equation, where the characteristic speed c can be either positive or negative. The generic form of the difference equation is

$$u_i^{n+1}=u_i^n-\frac{\Delta t}{\Delta x}\left\{f_{i+1/2}^n-f_{i-1/2}^n\right\}=u_i^n-C_i^n\frac{\Delta t}{\Delta x}\left\{u_{i+1/2}^n-u_{i-1/2}^n\right\}$$

First, the unlimited second order accurate flux is, assuming as for the Euler equations $f=\frac{\partial f}{\partial u}u$,

$$f_{i+1/2}^n=c_{+,i+1/2}\left\{u_i^n+\frac{1}{2}\left(1-\frac{c_{+,i+1/2}\Delta t}{\Delta x}\right)(u_i^n-u_{i-1}^n)\right\}+c_{-,i+1/2}\left\{u_{i+1}^n-\frac{1}{2}\left(1+\frac{c_{-,i+1/2}\Delta t}{\Delta x}\right)(u_{i+2}^n-u_{i+1}^n)\right\},$$

$$\text{with} \quad c_{\pm,i+1/2}=\frac{1\pm\operatorname{sgn}(c_{i+1/2}^n)}{2}c_{i+1/2}^n \text{ and } c_{i+1/2}^n=\frac{c_i^n+c_{i+1}^n}{2}$$

The limited flux is then

$$f_{i+1/2}^n=c_{+,i+1/2}\left\{u_i^n+\frac{1}{2}\left(1-\frac{c_{+,i+1/2}\Delta t}{\Delta x}\right)\psi_+(u_i^n-u_{i-1}^n)\right\}+c_{-,i+1/2}\left\{u_{i+1}^n-\frac{1}{2}\left(1+\frac{c_{-,i+1/2}\Delta t}{\Delta x}\right)\psi_-(u_{i+2}^n-u_{i+1}^n)\right\}$$

$$\text{with} \quad \psi_+(u_i-u_{i-1})=\min\operatorname{mod}\left(u_i-u_{i-1},u_{i+1}-u_i\right)$$

$$\text{and} \quad \psi_-(u_{i+1}-u_i)=\min\operatorname{mod}\left(u_i-u_{i-1},u_{i+1}-u_i\right).$$

Note that $\psi_+(u_i-u_{i-1})=\psi_-(u_{i+1}-u_i)$. We can make use of this relation to simplify the flux calculation.

$$f_{i+1/2}^n=c_{+,i+1/2}\left\{u_i^n+\frac{1}{2}(1-\frac{c_{+,i+1/2}\Delta t}{\Delta x})\psi_+(u_i^n-u_{i-1}^n)\right\}+c_{-,i+1/2}\left\{u_{i+1}^n-\frac{1}{2}(1+\frac{c_{-,i+1/2}\Delta t}{\Delta x})\psi_+(u_{i+1}^n-u_i^n)\right\}$$

Using a similar analysis to that of the last section, it can be shown that this flux is also TVD if $c_{+,i+1/2} \ge 0$, $c_{-,i+1/2} \le 0$ and Δt is chosen so that $\frac{\left|c_{i+1/2}^n\right| \Delta t}{\Delta x} \le 1$ for all i. This flux can also be written as

$$f_{i+1/2}^n = c_{+,i+1/2} \underbrace{\left(u_i^n + \frac{\text{sgn}(c_{i+1/2}^n)}{2} \left(1 - \frac{\left|c_{i+1/2}^n\right| \Delta t}{\Delta x} \right) \psi_+ (u_i^n - u_{i-1}^n) \right)}_{g_{+,i+1/2}}$$

$$+ c_{-,i+1/2} \underbrace{\left(u_{i+1}^n + \frac{\text{sgn}(c_{i+1/2}^n)}{2} \left(1 - \frac{\left|c_{i+1/2}^n\right| \Delta t}{\Delta x} \right) \psi_+ (u_{i+1}^n - u_i^n) \right)}_{g_{-,i+1/2}}$$

or

$$f_{i+1/2}^n = c_{+,i+1/2} g_{+,i+1/2} + c_{-,i+1/2} g_{-,i+1/2}\text{, with } g_{+,i+1/2} \text{ and } g_{-,i+1/2} \text{ defined as above.}$$

Finally,

$$f_{i+1/2}^n = \frac{c_{i+1/2}}{2} \left\{ (1 + \text{sgn}(c_{i+1/2}^n)) g_{+,i+1/2} + (1 - \text{sgn}(c_{i+1/2}^n)) g_{-,i+1/2} \right\}$$

An example of a program for calculating the flux $f_{i+1/2}$ at all interior surfaces, from $i = 1 + 1/2$ to $i = I - 1/2$, follows.

1) Begin for $i = 1, \cdots, I-1$ do the following

2) Calculate $g_1 = u_i^n + \frac{\text{sgn}(c_{i+1/2}^n)}{2} (1 - \frac{\left|c_{i+1/2}^n\right| \Delta t}{\Delta x}) \min\text{mod}(u_{i+1}^n - u_i^n, u_i^n - u_{i-1}^n)$

3) Calculate $g_2 = u_{i+1}^n + \frac{\text{sgn}(c_{i+1/2}^n)}{2} (1 - \frac{\left|c_{i+1/2}^n\right| \Delta t}{\Delta x}) \min\text{mod}(u_{i+2}^n - u_{i+1}^n, u_{i+1}^n - u_i^n)$

4) Set $\bar{c} = \frac{c_i^n + c_{i+1}^n}{2}$

5) Set $f_{i+1/2}^n = \frac{\bar{c}}{2} \left\{ \left(1 + \text{sgn}(\bar{c})\right) g_1 + \left(1 - \text{sgn}(\bar{c})\right) g_2 \right\}$

6) End if $i = I - 1$

There are many flux limiters in addition to the minmod function to achieve the TVD property. (see Hirsch cited above for an extensive description of several flux limiters).

10.3 TVD for the Euler Equations

Copies of Figures 9.27 and 9.30 from Chapter 9 for both the Roe and Modified Steger Warming Version (2) methods for the stationary shock wave case are shown in Figures 10.4 and 10.5. Neither method needed correction for either entropy violation or numerical difficulty. They are both first order with good representation of the shock wave, but not so good for the contact discontinuity or rarefaction.

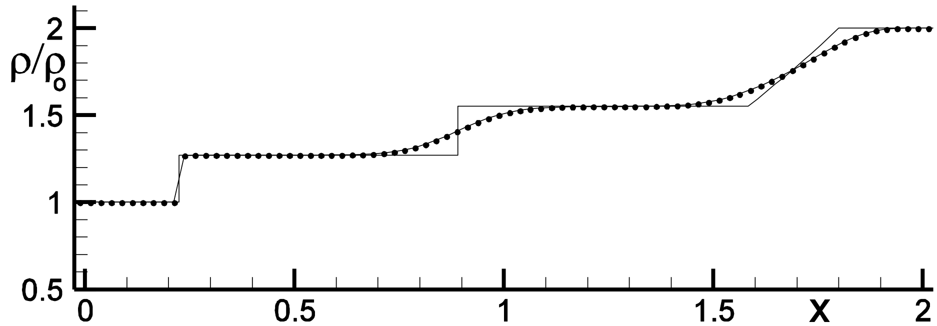

Figure 10.4 Density comparison for the Roe method, with no entropy correction, stationary shock wave, (CFL=0.9, 35 time steps)

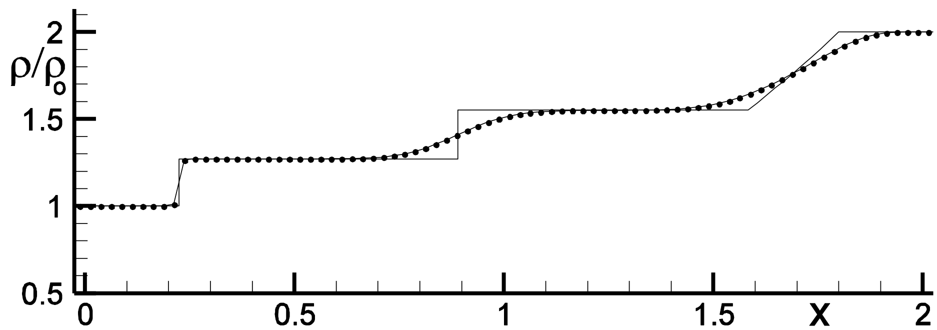

Figure 10.5 Density comparison for the Modified Steger-Warming method, Version (2), stationary shock wave, (CFL=0.9, 35 time steps)

Second order accuracy should help resolve the contact discontinuity and rarefaction, but TVD will be needed to prevent the formation of extraneous extrema. Note that here again the exact solution is monotonic, increasing with x. However, the important quantities for the Euler equations are the characteristics variables, which can also be shown to be monotone. These will be the variables whose fluxes will be limited to achieve TVD solutions. Again, we start with the conservative generic algorithm form in one spatial dimension

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left\{F_{i+1/2}^n - F_{i-1/2}^n\right\},$$

$$\text{where } U = \begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix} \text{ and } F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (e+p)u \end{pmatrix}$$

The Roe and Modified Steger Warming Version (2) methods for evaluating the flux vector are shown below. They are both first order methods at present.

$$F^n_{i+1/2} = \begin{cases} F^{(Roe)}_{i+1/2} = \dfrac{F_i + F_{i+1}}{2} - \dfrac{1}{2}\left|\hat{A}_{i+1/2}\right|\left(U_{i+1} - U_i\right) \\ F^{(M-S-W-II)}_{i+1/2} = \overline{\mathcal{A}}_{+_{i+1/2}} F_i + \overline{\mathcal{A}}_{-_{i+1/2}} F_{i+1} \end{cases}$$

Where we define $\left|\hat{A}\right| = \hat{A}_+ - \hat{A}_-$, $\hat{A}_\pm = \hat{S}^{-1}\hat{C}_A^{-1}\hat{\Lambda}_{A_\pm}\hat{C}_A\hat{S}$ and $\overline{\mathcal{A}}_\pm = \overline{S}^{-1}\overline{C}_A^{-1}\overline{\Lambda}_{A_\pm}\overline{\Lambda}_A^{-1}\overline{C}_A\overline{S} = \overline{S}^{-1}\overline{C}_A^{-1}\overline{D}_{A_\pm}\overline{C}_A\overline{S}$ (see Chapter 9). The "hats" and "bars" indicate that matrices use, respectively, either "Roe" averaged data or arithmetic average data. Note after some algebraic manipulation the Modified Steger Warming method above can be written in a similar form as the Roe method

$$F^{(M-S-W-II)}_{i+1/2} = \overline{\mathcal{A}}_{+_{i+1/2}} F_i + \overline{\mathcal{A}}_{-_{i+1/2}} F_{i+1}$$

$$\begin{aligned} F^{(M-S-W-II)}_{i+1/2} &= -\frac{1}{2}\overline{\mathcal{A}}_{+_{i+1/2}}(F_{i+1} - F_i) + \frac{1}{2}\overline{\mathcal{A}}_{-_{i+1/2}}(F_{i+1} - F_i) \\ &\quad + \frac{1}{2}\overline{\mathcal{A}}_{+_{i+1/2}}(F_{i+1} + F_i) + \frac{1}{2}\overline{\mathcal{A}}_{-_{i+1/2}}(F_{i+1} + F_i) \end{aligned}$$

$$F^{(M-S-W-II)}_{i+1/2} = \underbrace{\left(\overline{\mathcal{A}}_{+_{i+1/2}} + \overline{\mathcal{A}}_{-_{i+1/2}}\right)}_{I}\frac{F_i + F_{i+1}}{2} - \frac{1}{2}\underbrace{\left(\overline{\mathcal{A}}_{+_{i+1/2}} - \overline{\mathcal{A}}_{-_{i+1/2}}\right)}_{\left|\overline{\mathcal{A}}_{i+1/2}\right|}(F_{i+1} - F_i)$$

or

$$F^{(M-S-W-II)}_{i+1/2} = \frac{F_i + F_{i+1}}{2} - \frac{1}{2}\left|\overline{\mathcal{A}}_{i+1/2}\right|(F_{i+1} - F_i)$$

10.3.1 TVD for the Roe Method

Analogous to the expression for the second order accurate flux approximation for the general non-linear scalar equation given at the beginning of Section 10.2.4, we extend to the vector case as follows.

$$F^n_{i+1/2} = A_{+,i+1/2}\left\{U^n_i + \frac{1}{2}\left(I - \frac{A_{+,i+1/2}\Delta t}{\Delta x}\right)(U^n_i - U^n_{i-1})\right\} + A_{-,i+1/2}\left\{U^n_{i+1} - \frac{1}{2}\left(I + \frac{A_{-,i+1/2}\Delta t}{\Delta x}\right)(U^n_{i+2} - U^n_{i+1})\right\}$$

To obtain the analogue for the Roe method, we first note that

$$\begin{aligned} A_+U_i + A_-U_{i+1} &= +\frac{1}{2}A_+(U_{i+1} + U_i) + \frac{1}{2}A_-(U_{i+1} + U_i) \\ &\quad - \frac{1}{2}A_+(U_{i+1} - U_i) + \frac{1}{2}A_-(U_{i+1} - U_i) = \frac{F_i + F_{i+1}}{2} - \frac{1}{2}|A|(U_{i+1} - U_i) \end{aligned}$$

Guided by this last expression, we extend the Roe method to second order, as follows

$$F^n_{i+1/2} = \frac{F_i + F_{i+1}}{2} - \frac{1}{2}\left|\hat{A}_{i+1/2}\right|(U_{i+1} - U_i) + \frac{\hat{A}_{+,i+1/2}}{2}(I - \frac{\hat{A}_{+,i+1/2}\Delta t}{\Delta x})(U^n_i - U^n_{i-1})$$
$$- \frac{\hat{A}_{-,i+1/2}}{2}(I + \frac{\hat{A}_{-,i+1/2}\Delta t}{\Delta x})(U^n_{i+2} - U^n_{i+1})$$

The first two terms on the right hand side are the first order Roe method, which gave the monotone results shown earlier. The next two terms increase the spatial and temporal resolution, but may contribute to oscillations unless limited. The matrix $\hat{A}_{\pm}$ can be diagonalized (see Section 9.6) as follows $\hat{A}_{\pm} = \hat{S}^{-1}\hat{C}_A^{-1}\hat{\Lambda}_{A_{\pm}}\hat{C}_A\hat{S}$. Therefore, we can write the flux as

$$F^n_{i+1/2} = \frac{F_i + F_{i+1}}{2} - \frac{1}{2}\left|\hat{A}_{i+1/2}\right|(U_{i+1} - U_i) + \frac{1}{2}\hat{S}^{-1}_{i+1/2}\hat{C}^{-1}_{A_{i+1/2}}\hat{\Lambda}_{A_{+,i+1/2}}(I - \frac{\Delta t}{\Delta x}\hat{\Lambda}_{A_{+,i+1/2}})\hat{C}_{A_{i+1/2}}\hat{S}_{i+1/2}(U^n_i - U^n_{i-1})$$
$$- \frac{1}{2}\hat{S}^{-1}_{i+1/2}\hat{C}^{-1}_{A_{i+1/2}}\hat{\Lambda}_{A_{-,i+1/2}}(I + \frac{\Delta t}{\Delta x}\hat{\Lambda}_{A_{-,i+1/2}})\hat{C}_{A_{i+1/2}}\hat{S}_{i+1/2}(U^n_{i+2} - U^n_{i+1})$$

Let the eigenvalues of $\hat{A}$ be ordered as $\hat{\lambda}_1 = \hat{u}$, $\hat{\lambda}_2 = \hat{u} + \hat{c}$ and $\hat{\lambda}_3 = \hat{u} - \hat{c}$, then

$$\hat{\Lambda}_{\pm} = \begin{bmatrix} d_{\pm,1}\hat{\lambda}_1 & 0 & 0 \\ 0 & d_{\pm,2}\hat{\lambda}_3 & 0 \\ 0 & 0 & d_{\pm,3}\hat{\lambda}_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \left(1 \pm \text{sgn}(\hat{\lambda}_1)\right)\hat{\lambda}_1 & 0 & 0 \\ 0 & \left(1 \pm \text{sgn}(\hat{\lambda}_2)\right)\hat{\lambda}_2 & 0 \\ 0 & 0 & \left(1 \pm \text{sgn}(\hat{\lambda}_3)\right)\hat{\lambda}_3 \end{bmatrix}$$

We define the changes in characteristic variables from k to $k+1$ for $k = i-1,\ i\ \mathit{and}\ i+1$, by

$$\delta K_k = \begin{bmatrix} \delta k_1 \\ \delta k_2 \\ \delta k_3 \end{bmatrix}_k = \hat{C}_{A_{i+1/2}}\hat{S}_{i+1/2}(U^n_{k+1} - U^n_k),$$

$$F^n_{i+1/2} = \frac{F_i + F_{i+1}}{2} - \frac{1}{2}\hat{S}^{-1}_{i+1/2}\hat{C}^{-1}_{A_{i+1/2}}\left\{\left|\hat{\Lambda}_{A_{i+1/2}}\right|\delta K_i - \left|\hat{\Lambda}_{A_{+,i+1/2}}\right|(I - \left|\hat{\Lambda}_{A_{+,i+1/2}}\right|\frac{\Delta t}{\Delta x})\delta K_{i-1}\right.$$
$$\left. - \left|\hat{\Lambda}_{A_{-,i+1/2}}\right|(I - \left|\hat{\Lambda}_{A_{-,i+1/2}}\right|\frac{\Delta t}{\Delta x})\delta K_{i+1}\right\}$$

We now apply the flux limiter to define the TVD flux for the Roe method.

$$F_{i+1/2}^{n} = \frac{F_i + F_{i+1}}{2} - \frac{1}{2}\hat{S}_{i+1/2}^{-1}\hat{C}_{A_{i+1/2}}^{-1}\begin{bmatrix} \left|\hat{\lambda}_{1,i+1/2}\right|\left\{\delta k_{1,i} - \left(1-\left|\hat{\lambda}_{1,i+1/2}\right|\frac{\Delta t}{\Delta x}\right)\left(d_{+,1}\psi_+(\delta k_{1,i-1}) + d_{-,1}\psi_+(\delta k_{1,i})\right)\right\} \\ \left|\hat{\lambda}_{2,i+1/2}\right|\left\{\delta k_{2,i} - \left(1-\left|\hat{\lambda}_{2,i+1/2}\right|\frac{\Delta t}{\Delta x}\right)\left(d_{+,2}\psi_+(\delta k_{2,i-1}) + d_{-,2}\psi_+(\delta k_{2,i})\right)\right\} \\ \left|\hat{\lambda}_{3,i+1/2}\right|\left\{\delta k_{3,i} - \left(1-\left|\hat{\lambda}_{3,i+1/2}\right|\frac{\Delta t}{\Delta x}\right)\left(d_{+,3}\psi_+(\delta k_{3,i-1}) + d_{-,3}\psi_+(\delta k_{3,i})\right)\right\} \end{bmatrix}$$

An example of a program for calculating the flux $F_{i+1/2}^{n}$ at all interior surfaces, from $i = 1+1/2$ to $i = I-1/2$, using temporary variables g_l, for $l = 1, 2\ and\ 3$, and vectors δK_k, with elements $\delta k_{k,l}$, for $k = 1, 2\ and\ 3$ and $l = 1, 2\ and\ 3$, etc, follows.

1) Begin for $i = 1, \cdots, I-1$ do the following
2) Calculate $\delta K_1 = \hat{C}_{A_{i+1/2}}\hat{S}_{i+1/2}(U_i^n - U_{i'}^n)$, where $i' = \max(1, i-1)$
3) Calculate $\delta K_2 = \hat{C}_{A_{i+1/2}}\hat{S}_{i+1/2}(U_{i+1}^n - U_i^n)$
4) Calculate $\delta K_3 = \hat{C}_{A_{i+1/2}}\hat{S}_{i+1/2}(U_{i''+2}^n - U_{i+1}^n)$, where $i'' = \min(I-2, i)$

5) Calculate, for $l = 1, 2\ and\ 3$,

$$if\ \operatorname{sgn}(\hat{\lambda}_{l,i+1/2}) = 1\ then\ \ g_l = \left|\hat{\lambda}_{l,i+1/2}\right|\left\{\delta k_{2,l} - \frac{1}{2}\left(1-\frac{\left|\hat{\lambda}_{l,i+1/2}\right|\Delta t}{\Delta x}\right)\min\bmod(\delta k_{1,l}, \delta k_{2,l})\right\}$$

$$else\ \ g_l = \left|\hat{\lambda}_{l,i+1/2}\right|\left\{\delta k_{2,l} - \frac{1}{2}\left(1-\frac{\left|\hat{\lambda}_{l,i+1/2}\right|\Delta t}{\Delta x}\right)\min\bmod(\delta k_{2,l}, \delta k_{3,l})\right\},$$

6) Calculate $F_{i+1/2}^{n} = \dfrac{F_i + F_{i+1}}{2} - \dfrac{1}{2}\hat{S}_{i+1/2}^{-1}\hat{C}_{A_{i+1/2}}^{-1}\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$

7) End if $i = I-1$

The Roe method should always be used with its entropy correction. The eigenvalues used above are assumed to have been corrected as follows.

For $l = 1, 2\ and\ 3$ calculate $\hat{\lambda}_{l,i+1/2}$ and $\varepsilon_{l,i+1/2} = \sigma_0 \max\left\{0,\ \lambda_{l,i+1/2} - \lambda_{l,i},\ \lambda_{l,i+1} - \lambda_{l,i+1/2}\right\}$, where σ_0 is a constant, usually equal to 1, but may need to be increased for some flow problems. Then

$$\text{If } \left|\lambda_{l,i+1/2}\right| < \varepsilon_{l,i+1/2} \text{ then reset } \lambda_{l,i+1/2} \leftarrow \frac{\operatorname{sgn}(\lambda_{l,i+1/2})}{2}\left(\frac{\lambda_{l,i+1/2}^{2}}{\varepsilon_{l,i+1/2}} + \varepsilon_{l,i+1/2}\right) \text{ etc.}$$

The figures below shows the results for the Roe method with TVD and entropy correction for both the moving and stationary shock problems (see Chapter 9). The results are seen to be shaper than the corresponding first order results shown in Figures 9.26 and 9.27.

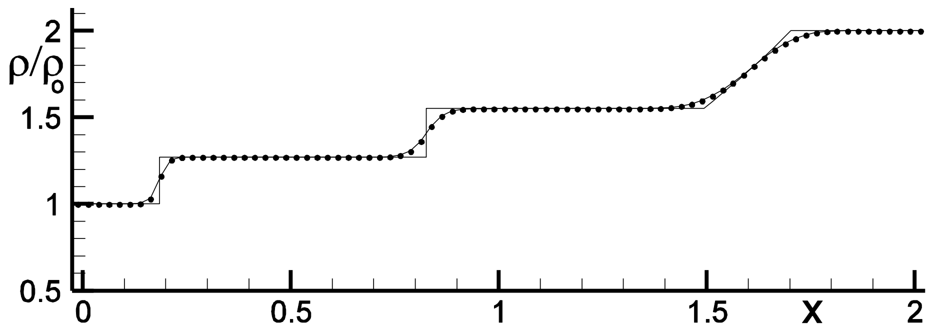

Figure 10.6 Density comparison for the Roe method, with TVD, moving shock wave, (CFL=0.9, 40 time steps)

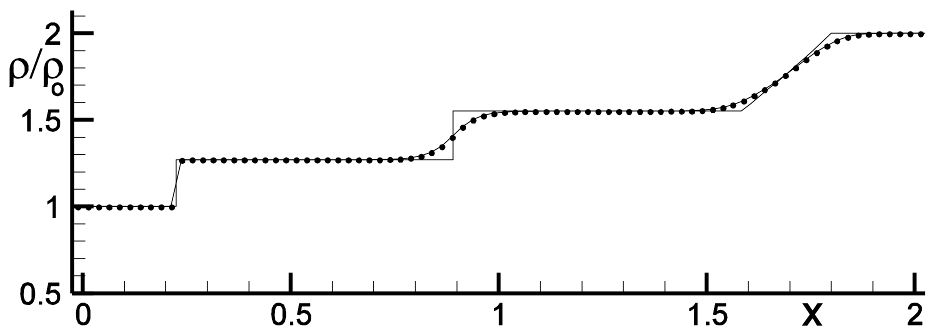

Figure 10.7 Density comparison for the Roe method, with TVD, stationary shock wave, (CFL=0.9, 35 time steps)

Some Variations

There are many variations in applying TVD to the Roe method. One is to include the characteristic speeds, eigenvalues within the limiter as shown below

$$F_{i+1/2}^{n} = \frac{F_i + F_{i+1}}{2} - \frac{1}{2}\hat{S}_{i+1/2}^{-1}\hat{C}_{A_{i+1/2}}^{-1}\begin{bmatrix} \left|\hat{\lambda}_{1,i+1/2}\right|\delta k_{1,i} - \left(1-\left|\hat{\lambda}_{1,i+1/2}\right|\frac{\Delta t}{\Delta x}\right)\left(\psi_{+}(d_{+,1}\hat{\lambda}_{1,i-1/2}\delta k_{1,i-1}) - \psi_{+}(d_{-,1}\hat{\lambda}_{1,i+1/2}\delta k_{1,i})\right) \\ \left|\hat{\lambda}_{2,i+1/2}\right|\delta k_{2,i} - \left(1-\left|\hat{\lambda}_{2,i+1/2}\right|\frac{\Delta t}{\Delta x}\right)\left(\psi_{+}(d_{+,2}\hat{\lambda}_{2,i-1/2}\delta k_{2,i-1}) - \psi_{+}(d_{-,2}\hat{\lambda}_{2,i+1/2}\delta k_{2,i})\right) \\ \left|\hat{\lambda}_{3,i+1/2}\right|\delta k_{3,i} - \left(1-\left|\hat{\lambda}_{3,i+1/2}\right|\frac{\Delta t}{\Delta x}\right)\left(\psi_{+}(d_{+,3}\hat{\lambda}_{3,i-1/2}\delta k_{3,i-1}) - \psi_{+}(d_{-,3}\hat{\lambda}_{3,i+1/2}\delta k_{3,i})\right) \end{bmatrix}$$

10.3.2 TVD for the Modified Steger-Warming Method Version (2)

The analogous expression for the second order accurate flux approximation using the Modified Steger-Warming method to that for the general non-linear scalar equation given earlier is

$$\bar{F}^n_{i+1/2} = \bar{\mathcal{A}}_{+_{i+1/2}} \left\{ F_i^n + \frac{1}{2}\left(I - \bar{A}_{+_{i+1/2}} \frac{\Delta t}{\Delta x} \right)(F_i^n - F_{i-1}^n) \right\} + \bar{\mathcal{A}}_{-_{i+1/2}} \left\{ F_{i+1}^n - \frac{1}{2}\left(I + \bar{A}_{-_{i+1/2}} \frac{\Delta t}{\Delta x} \right)(F_{i+2}^n - F_{i+1}^n) \right\}$$

The matrix $\bar{\mathcal{A}}_\pm$ can be diagonalized (see Section 9.6) as follows $\bar{\mathcal{A}}_\pm = \bar{S}^{-1}\bar{C}_A^{-1}\bar{D}_\pm\bar{C}_A\bar{S}$. Therefore we can write flux vector as

$$\begin{aligned}\bar{F}^n_{i+1/2} = \bar{\mathcal{A}}_{+_{i+1/2}} F_i^n + \bar{\mathcal{A}}_{-_{i+1/2}} F_{i+1}^n &+ \frac{1}{2}\bar{S}^{-1}_{i+1/2}\bar{C}^{-1}_{A,i+1/2}\bar{D}_{+,i+1/2}\bar{C}_{A,i+1/2}\bar{S}_{i+1/2}\left(I - \bar{A}_{+_{i+1/2}} \frac{\Delta t}{\Delta x} \right)(F_i^n - F_{i-1}^n) \\ &- \frac{1}{2}\bar{S}^{-1}_{i+1/2}\bar{C}^{-1}_{A,i+1/2}\bar{D}_{-,i+1/2}\bar{C}_{A,i+1/2}\bar{S}_{i+1/2}\left(I + \bar{A}_{-_{i+1/2}} \frac{\Delta t}{\Delta x} \right)(F_{i+2}^n - F_{i+1}^n)\end{aligned}$$

or

$$\begin{aligned}\bar{F}^n_{i+1/2} = \bar{\mathcal{A}}_{+_{i+1/2}} F_i^n + \bar{\mathcal{A}}_{-_{i+1/2}} F_{i+1}^n &+ \frac{1}{2}\bar{S}^{-1}_{i+1/2}\bar{C}^{-1}_{A,i+1/2}\bar{D}_{+,i+1/2}\left(I - \left|\bar{\Lambda}_{+,i+1/2}\right| \frac{\Delta t}{\Delta x} \right)\bar{C}_{A,i+1/2}\bar{S}_{i+1/2}(F_i^n - F_{i-1}^n) \\ &- \frac{1}{2}\bar{S}^{-1}_{i+1/2}\bar{C}^{-1}_{A,i+1/2}\bar{D}_{-,i+1/2}\left(I - \left|\bar{\Lambda}_{-,i+1/2}\right| \frac{\Delta t}{\Delta x} \right)\bar{C}_{A,i+1/2}\bar{S}_{i+1/2}(F_{i+2}^n - F_{i+1}^n)\end{aligned}$$

Let the eigenvalues of $\bar{A}$ be ordered as $\bar{\lambda}_1 = \bar{u}$, $\bar{\lambda}_2 = \bar{u} + \bar{c}$ and $\bar{\lambda}_3 = \bar{u} - \bar{c}$, then

$$\bar{D}_\pm = \begin{bmatrix} d_{\pm,1} & 0 & 0 \\ 0 & d_{\pm,2} & 0 \\ 0 & 0 & d_{\pm,3} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \pm \mathrm{sgn}(\bar{\lambda}_1) & 0 & 0 \\ 0 & 1 \pm \mathrm{sgn}(\bar{\lambda}_2) & 0 \\ 0 & 0 & 1 \pm \mathrm{sgn}(\bar{\lambda}_3) \end{bmatrix}$$

We define the vector G_k for $k = i-1,\ i,\ i+1\ \mathit{and}\ i+2$ to $i = k+1$, by

$$G_k = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}_k = \bar{C}_{A_{i+1/2}}\bar{S}_{i+1/2}F_k^n \quad ,$$

Then

$$\begin{aligned}\bar{F}^n_{i+1/2} = \bar{S}^{-1}_{i+1/2}\bar{C}^{-1}_{A,i+1/2}\Big\{\bar{D}_{+,i+1/2}\,G_i + \bar{D}_{-,i+1/2}G_i &+ \frac{1}{2}\bar{D}_{+,i+1/2}\left(I - \left|\bar{\Lambda}_{i+1/2}\right| \frac{\Delta t}{\Delta x} \right)(G_i - G_{i-1}) \\ &- \frac{1}{2}\bar{D}_{-,i+1/2}\left(I - \left|\bar{\Lambda}_{i+1/2}\right| \frac{\Delta t}{\Delta x} \right)(G_{i+2} - G_{i+1})\ \Big\}\end{aligned}$$

We now apply the flux limiter to define the TVD flux for the Modified Steger-Warming method.

$$F_{i+1/2}^{n}=\hat{S}_{i+1/2}^{-1}\hat{C}_{A_{i+1/2}}^{-1}\begin{bmatrix} d_{+,1}g_{1,i}+d_{-,1}g_{1,i+1}\ +\frac{1}{2}\left(1-\left|\bar{\lambda}_{1,i+1/2}\right|\frac{\Delta t}{\Delta x}\right)\left(d_{+,1}\psi_{+}(g_{1,i}-g_{1,i-1})\ -d_{-,1}\psi_{+}(g_{1,i+1}-g_{1,i})\right) \\ d_{+,1}g_{2,i}+d_{-,1}g_{2,i+1}+\frac{1}{2}\left(1-\left|\bar{\lambda}_{2,i+1/2}\right|\frac{\Delta t}{\Delta x}\right)\left(d_{+,2}\psi_{+}(g_{2,i}-g_{2,i-1})-d_{-,2}\psi_{+}(g_{2,i+1}-g_{2,i})\right) \\ d_{+,1}g_{3,i}+d_{-,1}g_{3,i+1}+\frac{1}{2}\left(1-\left|\bar{\lambda}_{3,i+1/2}\right|\frac{\Delta t}{\Delta x}\right)\left(d_{+,3}\psi_{+}(g_{3,i}-g_{3,i-1})-d_{-,3}\psi_{+}(g_{3,i+1}-g_{3,i})\right) \end{bmatrix}$$

where

$$\psi_{+}(g_{l,i}-g_{l,i-1})=\min\operatorname{mod}\left(g_{l,i}-g_{l,i-1},g_{l,i+1}-g_{l,i}\right)$$

An example of a program for calculating the flux $F_{i+1/2}^{n}$ at all interior surfaces, from $i=1+1/2$ to $i=I-1/2$, using temporary variables h_l, for $l=1,2\ and\ 3$, and vectors G_k, with elements $g_{k,l}$, for $k=1,2,3\ and\ 4$ and $l=1,2\ and\ 3$, etc, follows.

1) Begin for $i=1,\cdots,I-1$ do the following
2) Calculate $G_1=\bar{C}_{A_{i+1/2}}\bar{S}_{i+1/2}F_{i'-1}^{n}$, where $i'=\max(2,i)$
3) Calculate $G_2=\bar{C}_{A_{i+1/2}}\bar{S}_{i+1/2}F_{i}^{n}$
4) Calculate $G_3=\bar{C}_{A_{i+1/2}}\bar{S}_{i+1/2}F_{i+1}^{n}$
5) Calculate $G_4=\bar{C}_{A_{i+1/2}}\bar{S}_{i+1/2}F_{i''+2}^{n}$, where $i''=\min(I-2,i)$

6) Calculate, for $l=1,2\ and\ 3$,

$$if\ \operatorname{sgn}(\hat{\lambda}_{l,i+1/2})=1\ \ then\ \ h_l=g_{2,l}+\frac{1}{2}\left(1-\left|\bar{\lambda}_{l,i+1/2}\right|\frac{\Delta t}{\Delta x}\right)\min\operatorname{mod}(g_{2,l}-g_{1,l},g_{3,l}-g_{2,l})$$

$$else\ \ h_l=g_{3,l}-\frac{1}{2}\left(1-\left|\bar{\lambda}_{l,i+1/2}\right|\frac{\Delta t}{\Delta x}\right)\min\operatorname{mod}(g_{3,l}-g_{2,l},g_{4,l}-g_{3,l})\ ,$$

7) Calculate $F_{i+1/2}^{n}=\bar{S}_{i+1/2}^{-1}\bar{C}_{A_{i+1/2}}^{-1}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$
8) End if $i=I-1$

The figures below shows the results for the Modified Steger-Warming Version (2) method with TVD for both the moving and stationary shock problems (see Chapter 9). The results are seen to be shaper than the corresponding first order results shown in Figs.9.26 and 9.27.

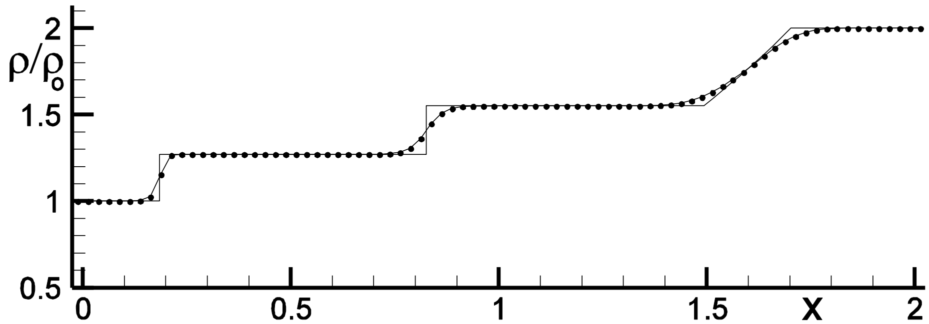

Figure 10.10 Density comparison for the Modified Steger-Warming Version (2) method, with TVD, moving shock wave, (CFL=0.9, 40 time steps)

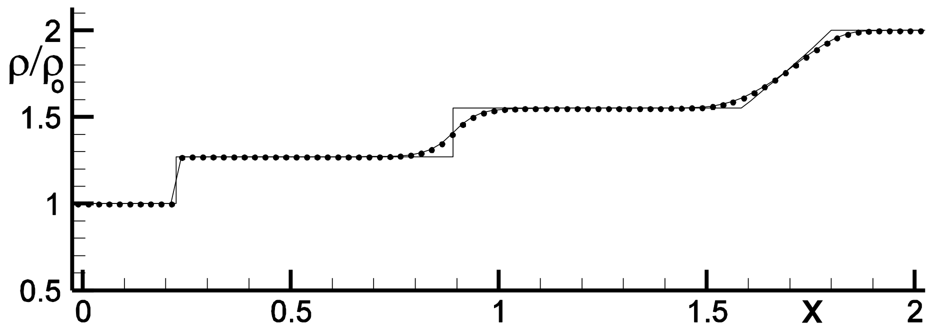

Figure 10.11 Density comparison for the Modified Steger-Warming method, with TVD, stationary shock wave, (CFL=0.9, 35 time steps)

10.4 Implicit TVD Methods

The previous TVD algorithms were all explicit. In one dimension the implicit Modified-Steger-Warming or Roe methods can be written in the following form (see Section 9.12.1).

$$\left\{ I + \alpha\Delta t \left(\frac{D_-}{\Delta x} \cdot \overline{A}^n_{+_{i+1/2}} + \frac{D_+}{\Delta x} \cdot \overline{A}^n_{-_{i-1/2}} \right) \right\} \delta U_i^{n+1} = -\Delta t \left(\frac{F^n_{i+1/2} - F^n_{i-1/2}}{\Delta x} \right)$$

As seen in the previous Section 10.3, flux limitation is applied to each of the characteristic equations that form the Euler equations. Instead of considering the Euler equations as a whole, we now consider for the implicit algorithm just one such characteristic equation, with characteristic variable u . Let's start as before in Section 10.2.1 with the equation.

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0, \text{ where } f = f(u) \text{ and } c = \frac{\partial f}{\partial u}$$

.

A fully implicit first order accurate algorithm for solving this equation is

$$u_i^{n+1} = u_i^n - c_{+,i} \frac{\Delta t}{\Delta x}(u_i^{n+1} - u_{i-1}^{n+1}) - c_{-,i} \frac{\Delta t}{\Delta x}(u_{i+1}^{n+1} - u_i^{n+1}) \quad \text{with} \quad c_{\pm,i} = \frac{1 \pm \text{sgn}(c_i^n)}{2} c_i^n$$

To show that the above algorithm is TVD, we subtract from the equation for u_i^{n+1} a similar one for u_{i-1}^{n+1}

$$u_i^{n+1} - u_{i-1}^{n+1} = u_i^n - u_{i-1}^n - \frac{c_{+,i}\Delta t}{\Delta x}(u_i^{n+1} - u_{i-1}^{n+1}) + \frac{c_{+,i-1}\Delta t}{\Delta x}(u_{i-1}^{n+1} - u_{i-2}^{n+1}) - \frac{c_{-,i}\Delta t}{\Delta x}(u_{i+1}^{n+1} - u_i^{n+1}) + \frac{c_{-,i-1}\Delta t}{\Delta x}(u_i^{n+1} - u_{i-1}^{n+1})$$

Rearranging terms so that all factors before terms of form $(u_k - u_{k-1})$ are positive

$$\left(1 + \frac{c_{+,i}\Delta t}{\Delta x} - \frac{c_{-,i-1}\Delta t}{\Delta x}\right)(u_i^{n+1} - u_{i-1}^{n+1}) = u_i^n - u_{i-1}^n + \frac{c_{+,i-1}\Delta t}{\Delta x}(u_{i-1}^{n+1} - u_{i-2}^{n+1}) - \frac{c_{-,i}\Delta t}{\Delta x}(u_{i+1}^{n+1} - u_i^{n+1})$$

Summing over *i* and cancelling like terms on either side of the inequality below

$$\sum_i \left\{\left(1 + \cancel{\frac{c_{+,i}\Delta t}{\Delta x}} - \cancel{\frac{c_{-,i-1}\Delta t}{\Delta x}}\right)\left|u_i^{n+1} - u_{i-1}^{n+1}\right|\right\} \le \sum_i \left\{\left|u_i^n - u_{i-1}^n\right| + \cancel{\frac{c_{+,i-1}\Delta t}{\Delta x}\left|u_{i-1}^{n+1} - u_{i-2}^{n+1}\right|} - \cancel{\frac{c_{-,i}\Delta t}{\Delta x}\left|u_{i+1}^{n+1} - u_i^{n+1}\right|}\right\}$$

Again, the above summation yields $TV^{n+1} = \sum_i \left|u_i^{n+1} - u_{i-1}^{n+1}\right| \le \sum_i \left|u_i^n - u_{i-1}^n\right| = TV^n$.

The above first order algorithm can be written in delta law form as

$$\left\{1 + \Delta t\left(c_{+,i}\frac{D_-^{(1)}\cdot}{\Delta x} + c_{-,i}\frac{D_+^{(1)}\cdot}{\Delta x}\right)\right\}\delta u_i^{n+1} = -\Delta t\left(c_{+,i}\frac{D_-^{(1)}\cdot}{\Delta x}u_i^n + c_{-,i}\frac{D_+^{(1)}\cdot}{\Delta x}u_i^n\right)$$

where as in Section 9.12.2, the superscripts on the operators D indicate order of spatial accuracy and the "dots" indicate that the difference operator applies to all factors to the right. In Section 9.12.2 we also defined an implicit algorithm using first order difference approximations on the left hand, or implicit side, of the equation and second order on the right hand, explicit side of the equation. Extending this to the present section, using Van Leer's MUSCL approach (See Section 9.11) on the right hand side of the equal sign, we obtain

$$\left\{1 + \alpha\Delta t\left(c_{+,i}\frac{D_-^{(1)}\cdot}{\Delta x} + c_{-,i}\frac{D_+^{(1)}\cdot}{\Delta x}\right)\right\}\delta u_i^{n+1} = -\Delta t\left(c_{+,i}\frac{D_-^{(2)}\cdot}{\Delta x}u_{-,i}^n + c_{-,i}\frac{D_+^{(2)}\cdot}{\Delta x}u_{+,i}^n\right)$$

Where $1 < \alpha \le 2$, $u_{-,i}^n = u_i^n + \frac{1}{2}\psi_+(u_i^n - u_{i-1}^n)$ and $u_{+,i}^n = u_{i+1}^n - \frac{1}{2}\psi_+(u_{i+1}^n - u_i^n)$

Alternatively, we define the algorithm, in conservation form, as

$$\left\{1 + \alpha\Delta t\left(\frac{D_-^{(1)}\cdot c_{+,i}}{\Delta x} + \frac{D_+^{(1)}\cdot c_{-,i}}{\Delta x}\right)\right\}\delta u_i^{n+1} = -\Delta t\,\frac{f_{i+1/2}^n - f_{i-1/2}^n}{\Delta x}$$

Where $$f_{i+1/2}^n = c_{+,i+1/2}\left\{u_i^n + \frac{1}{2}\psi_+(u_i^n - u_{i-1}^n)\right\} + c_{-,i+1/2}\left\{u_{i+1}^n - \frac{1}{2}\psi_+(u_{i+1}^n - u_i^n)\right\}$$

$$\text{with}\quad c_{\pm,i+1/2} = \frac{1 \pm \operatorname{sgn}(c_{i+1/2}^n)}{2} c_{i+1/2}^n \ \text{ and } \ c_{i+1/2}^n = \frac{c_i^n + c_{i+1}^n}{2}$$

or finally

$$f_{i+1/2}^n = c_{+,i+1/2}\left(u_i^n + \frac{\operatorname{sgn}(c_{i+1/2}^n)}{2}\psi_+(u_i^n - u_{i-1}^n)\right) + c_{-,i+1/2}\left(u_{i+1}^n + \frac{\operatorname{sgn}(c_{i+1/2}^n)}{2}\psi_+(u_{i+1}^n - u_i^n)\right)$$

The above algorithm is second order accurate in space and TVD, but only first order accurate in time. Second order in time can be achieved via the procedure shown in Section 9.13.

10.4.1 Implicit TVD for the Euler Equations

In one dimension the implicit Modified-Steger-Warming or Roe methods can be written in the following form (see Section 9.12.1).

$$\left\{I + \alpha\Delta t\left(\frac{D_-^{(1)}\cdot}{\Delta x}\overline{A}_{+\,i+1/2}^{\,n} + \frac{D_+^{(1)}\cdot}{\Delta x}\overline{A}_{-\,i-1/2}^{\,n}\right)\right\}\delta U_i^{n+1} = -\Delta t\left(\frac{F_{i+1/2}^n - F_{i-1/2}^n}{\Delta x}\right)$$

We define the fluxes $F_{i+1/2}^n$ below to be TVD and second order accurate, but the overall accuracy is of first order because of the left hand implicit side. Again higher order time accuracy can be obtained using the procedure of Section 9.13

10.4.1 Implicit Roe TVD Algorithm

For the Roe implicit method, using variables defined in Section 10.3.1

$$F_{i+1/2}^n = \frac{F_i + F_{i+1}}{2} - \frac{1}{2}\hat{S}_{i+1/2}^{-1}\hat{C}_{A_{i+1/2}}^{-1}\begin{bmatrix} \left|\hat{\lambda}_{1,i+1/2}\right|\left\{\delta k_{1,i} - \left(d_{+,1}\psi_+(\delta k_{1,i-1}) + d_{-,1}\psi_+(\delta k_{1,i})\right)\right\} \\ \left|\hat{\lambda}_{2,i+1/2}\right|\left\{\delta k_{2,i} - \left(d_{+,2}\psi_+(\delta k_{2,i-1}) + d_{-,2}\psi_+(\delta k_{2,i})\right)\right\} \\ \left|\hat{\lambda}_{3,i+1/2}\right|\left\{\delta k_{3,i} - \left(d_{+,3}\psi_+(\delta k_{3,i-1}) + d_{-,3}\psi_+(\delta k_{3,i})\right)\right\} \end{bmatrix}$$

Note that the factor used in the Warming-Beam algorithm has been deleted from the flux given in Section 10.3.1 (modification of step 5 of Roe TVD procedure required).

10.4.2 Implicit Modified-Steger-Warming Version 2 TVD Algorithm

For the Modified-Steger-Warming Version 2 implicit method, using variables defined in Section 10.3.2

$$F^n_{i+1/2} = \hat{S}^{-1}_{i+1/2}\hat{C}^{-1}_{A_{i+1/2}} \begin{bmatrix} d_{+,1}g_{1,i} + d_{-,1}g_{1,i+1} + \frac{1}{2}\left(d_{+,1}\psi_{+}(g_{1,i} - g_{1,i-1}) - d_{-,1}\psi_{+}(g_{1,i+1} - g_{1,i})\right) \\ d_{+,1}g_{2,i} + d_{-,1}g_{2,i+1} + \frac{1}{2}\left(d_{+,2}\psi_{+}(g_{2,i} - g_{2,i-1}) - d_{-,2}\psi_{+}(g_{2,i+1} - g_{2,i})\right) \\ d_{+,1}g_{3,i} + d_{-,1}g_{3,i+1} + \frac{1}{2}\left(d_{+,3}\psi_{+}(g_{3,i} - g_{3,i-1}) - d_{-,3}\psi_{+}(g_{3,i+1} - g_{3,i})\right) \end{bmatrix}$$

Note again that the factor used in the Warming-Beam algorithm has been deleted the flux given in Section 10.3.2 (modification of step 6 of M-S-W-2 TVD procedure required).