Chapter 1

The Equations Governing Fluid Flow

1.1 Introduction

The equations describing compressible viscous flow have been known for more than a century. Until the late 1960's they were taught to graduate students in engineering and science with little hope of analytic solution, except for the simplest of flows. Experimentation, particularly within wind tunnels, was the only method to simulate solutions to these equations. The availability of advanced scientific computers and the development of computational fluid dynamics have allowed these equations to be solved now routinely. This capability is now also available on personal computers. The equations governing fluid flow are listed and briefly discussed in this chapter. Methods for solving them are presented in later chapters.

1.2 Equation Hierarchy

The equations governing fluid dynamics can be placed on a hierarchical ladder, starting with the most complete description and ending with the simplest on the bottom rungs.

1) The Boltzmann equations
2) The Burnett equations
3) The Navier-Stokes equations
4) The Euler equations
5) The Full Potential equation
6) The Transonic Small Disturbance equation
7) The Laplace equation
8) The model equations

The Boundary Layer equations should also be included. They would be placed below the Navier-Stokes equations. However, although they contain viscous terms, they are not as complete as the Euler equations and therefore their position in the above hierarchy is not clear. The Reynolds' Averaged Navier-Stokes Equations, used to include the effects of turbulence without actually resolving the turbulent scale motions, also deserve mention. This set of equations has essentially the same form as the Navier-Stokes equations. They are solved together with a turbulence model, which enhances the values of the molecular viscous coefficients to account for the macroscopic fluid mixing caused by turbulent eddies.

The top two sets of equations on the hierarchical ladder are, for practical reasons, beyond the scope of the solution procedures to be covered. The Boltzmann equations give the most complete description of fluid flow, even within the free molecular range. The Burnett equations are similar to the Navier-Stokes equations in that they are based upon the continuum hypothesis and describe compressible viscous flow. But they, unlike the Navier Stokes equations, include non-linear viscous stress strain relations. The bottom three equations are included for both historical interest and because their numerical treatment has produced useful procedures for solving the Euler and

Navier-Stokes equations, the primary interest of these work. We begin by presenting these equations below.

1.3 The Navier-Stokes Equations

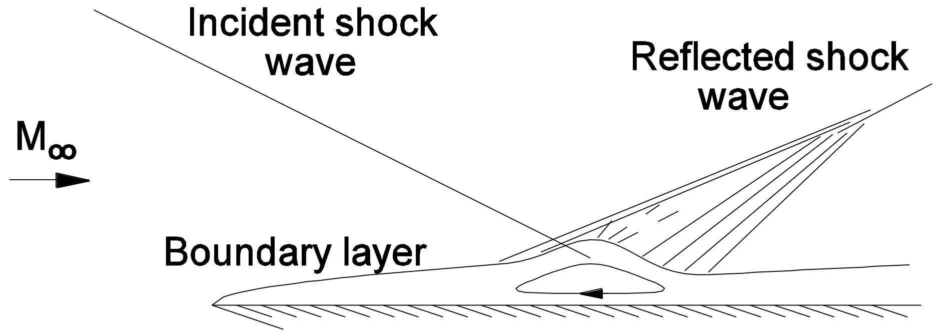

Figure 1.1 Shock wave-boundary layer interaction

The Navier-Stokes equations are generally accepted as an adequate description for aerodynamic flows at standard temperatures and pressures, such as the flow illustrated above. The figure shows an oblique shock wave incident upon a boundary layer. The adverse pressure gradient produced by the shock wave can propagate upstream through the subsonic part of the boundary layer and, if sufficiently strong, separate the flow, forming a circulation within a separation bubble. The boundary layer thickens near the incident shock wave and then necks down where the flow reattaches to the wall, generating two sets of compression waves bounding a rarefaction fan, which eventually form the reflected shockwave. The equations that describe this inviscid-viscous interaction can be written as follows.

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}=0, \qquad \text{(continuity equation)}$$

$$\frac{\partial \rho u}{\partial t}+\frac{\partial(\rho u^2+\sigma_x)}{\partial x}+\frac{\partial(\rho uv+\tau_{xy})}{\partial y}+\frac{\partial(\rho uw+\tau_{xz})}{\partial z}=0 \qquad \text{(x-momentum equation)}$$

$$\frac{\partial \rho v}{\partial t}+\frac{\partial(\rho vu+\tau_{yx})}{\partial x}+\frac{\partial(\rho v^2+\sigma_y)}{\partial y}+\frac{\partial(\rho vw+\tau_{yz})}{\partial z}=0 \qquad \text{(y-momentum equation)}$$

$$\frac{\partial \rho w}{\partial t}+\frac{\partial(\rho wu+\tau_{zx})}{\partial x}+\frac{\partial(\rho wv+\tau_{zy})}{\partial y}+\frac{\partial(\rho w^2+\sigma_z)}{\partial z}=0 \qquad \text{(z-momentum equation)}$$

$$\frac{\partial e}{\partial t}+\frac{\partial((e+\sigma_x)u+v\tau_{yx}+w\tau_{zx}-k\frac{\partial T}{\partial x})}{\partial x}+\frac{\partial((e+\sigma_y)v+u\tau_{xy}+w\tau_{zy}-k\frac{\partial T}{\partial y})}{\partial y} \qquad \text{(energy equation)}$$
$$+\frac{\partial((e+\sigma_z)w+u\tau_{xz}+v\tau_{yz}-k\frac{\partial T}{\partial z})}{\partial z}=0$$

where $p = p(\rho,\varepsilon)$, $\varepsilon = \frac{e}{\rho} - \frac{u^2+v^2+w^2}{2}$, $\sigma_x = p - \lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) - 2\mu\frac{\partial u}{\partial x}$

$$\sigma_y = p - \lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) - 2\mu\frac{\partial v}{\partial y}, \qquad \sigma_z = p - \lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) - 2\mu\frac{\partial w}{\partial z},$$

$$\tau_{xy} = \tau_{yx} = -\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}), \quad \tau_{xz} = \tau_{zx} = -\mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \quad \text{and} \quad \tau_{yz} = \tau_{zy} = -\mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})$$

ρ is the density, u, v and w are the x, y and z velocity components, e is the total energy per unit volume, ε is the internal energy per unit mass, p is the pressure, μ and λ are the first and second coefficients of viscosity ($\lambda = -\frac{2}{3}\mu$ for most fluids), k is the coefficient of heat conductivity, and T is the temperature. For a perfect gas $p = (\gamma - 1)\rho\varepsilon$ where γ is the ratio of the specific heats of the gas $\frac{c_p}{c_v}$, and $\gamma = 1.4$ for air. If the gas is calorically perfect then $\varepsilon = c_v T$ and $k = \frac{\gamma\mu}{P_r} c_v$, where P_r is the Prandtl number $\simeq 0.72$ for air.

1.4 The Euler Equations

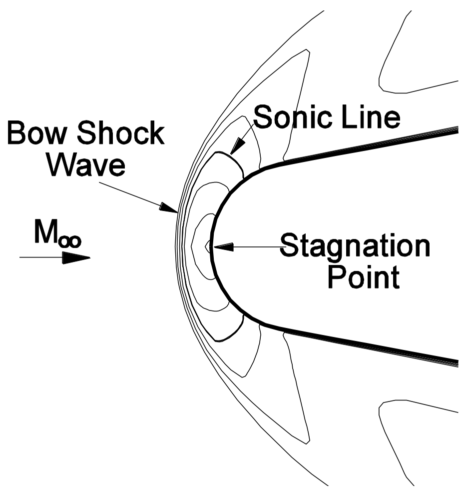

Figure 1.2 Mach contours for inviscid supersonic flow past a blunt body

Flows with negligible viscous and thermal conduction effects ($\mu = \lambda = k = 0$) can be described by the Euler equations. The figure above illustrates supersonic flow past a sphere-cone body. The free stream flow is disturbed ahead of the body by a detached shock wave. A region of subsonic flow is bounded by the sonic line about the nose of the body. The governing Euler equations can be written as follows.

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \qquad \text{(the continuity equation)}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial(\rho u^2 + p)}{\partial x} + \frac{\partial \rho uv}{\partial y} + \frac{\partial \rho uw}{\partial z} = 0 \qquad \text{(the x-momentum equation)}$$

$$\frac{\partial \rho v}{\partial t} + \frac{\partial \rho vu}{\partial x} + \frac{\partial(\rho v^2 + p)}{\partial y} + \frac{\partial \rho vw}{\partial z} = 0 \qquad \text{(the y-momentum equation)}$$

$$\frac{\partial \rho w}{\partial t} + \frac{\partial \rho wu}{\partial x} + \frac{\partial \rho wv}{\partial y} + \frac{\partial(\rho w^2 + p)}{\partial z} = 0 \qquad \text{(the z-momentum equation)}$$

$$\frac{\partial e}{\partial t} + \frac{\partial(e + p)u}{\partial x} + \frac{\partial(e + p)v}{\partial y} + \frac{\partial(e + p)w}{\partial z} = 0 \qquad \text{(the energy equation)}$$

where $p = p(\rho, \varepsilon)$; $\varepsilon = \dfrac{e}{\rho} - \dfrac{u^2 + v^2 + w^2}{2}$ and ρ, u, v, w and e are as defined above for the Navier-Stokes equations.

1.5 The Full Potential Equation

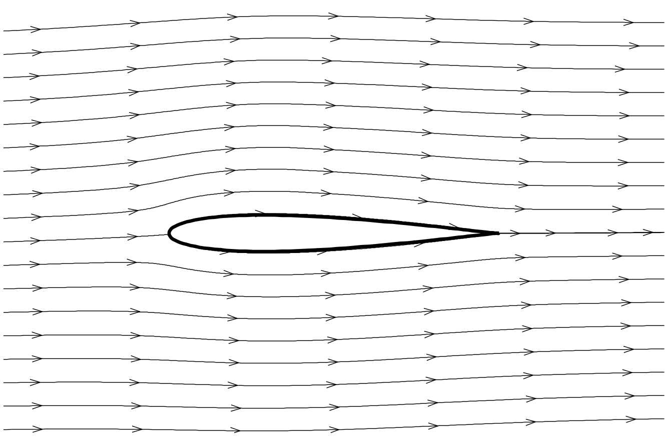

Figure 1.3 Streamlines for potential flow past a NACA 0012 airfoil

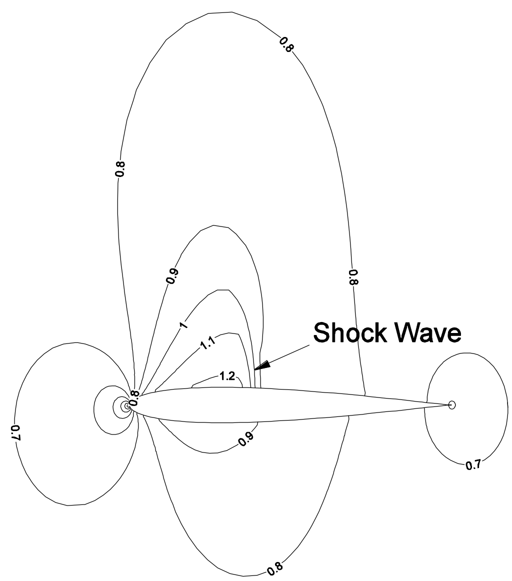

Figure 1.4 Mach contours, $M_\infty = 0.75$ at 1^o angle of attack

If the flow is isentropic and perfect, then the following relations apply (see *NACA Report 1135, Equations, Tables, and Charts for Compressible Flow*, or *Anderson, J.D. jr. Fundamentals of Aerodynamics, Fourth Edition, McGraw-Hill*).

$$T = T_\infty \left[1 - \frac{\gamma - 1}{2} M_\infty^2 \left(\frac{u^2 + v^2 + w^2}{q_\infty^2} - 1 \right) \right] \qquad \text{(adiabatic, perfect)}$$

$$p = p_\infty \left[1 - \frac{\gamma - 1}{2} M_\infty^2 \left(\frac{u^2 + v^2 + w^2}{q_\infty^2} - 1 \right) \right]^{\frac{\gamma}{\gamma - 1}} \qquad \text{(isentropic, perfect)}$$

$$\rho = \rho_\infty \left[1 - \frac{\gamma - 1}{2} M_\infty^2 \left(\frac{u^2 + v^2 + w^2}{q_\infty^2} - 1 \right) \right]^{\frac{1}{\gamma - 1}} \qquad \text{(isentropic, perfect)}$$

where $T_\infty, p_\infty, \rho_\infty, M_\infty$ and q_∞ represent temperature, pressure, density, Mach number, and velocity at free stream conditions.

If the flow is also irrotational, we can represent the velocity components as derivatives of a scalar potential function ϕ,

$$u = \frac{\partial \phi}{\partial x} = \phi_x, \quad v = \frac{\partial \phi}{\partial y} = \phi_y \quad \text{and} \quad w = \frac{\partial \phi}{\partial z} = \phi_z .$$

Using the steady Euler continuity equation and the relations above, we can write the *Full Potential Equation* in conservative form as follows.

$$(\rho\phi_x)_x + (\rho\phi_y)_y + (\rho\phi_z)_z = 0, \quad \text{with} \quad \rho = \rho_\infty \left[1 - \frac{\gamma - 1}{2} M_\infty^2 \left(\frac{\phi_x^2 + \phi_y^2 + \phi_z^2}{q_\infty^2} - 1 \right) \right]^{\frac{1}{\gamma - 1}}$$

This equation can also be written in non-conservative form as follows.

$$(a^2 - \phi_x^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (a^2 - \phi_y^2)\phi_{yy} - 2\phi_y\phi_z\phi_{yz} + (a^2 - \phi_z^2)\phi_{zz} - 2\phi_z\phi_x\phi_{zx} = 0,$$

where the speed of sound $a = \sqrt{\dfrac{\gamma p}{\rho}}$.

The Full Potential equation adequately describes irrotational isentropic flows, from incompressible to compressible transonic conditions, as long as any shock wave contained in the flow is not strong enough to increase entropy significantly.

1.6 Transonic Small Disturbance Equation

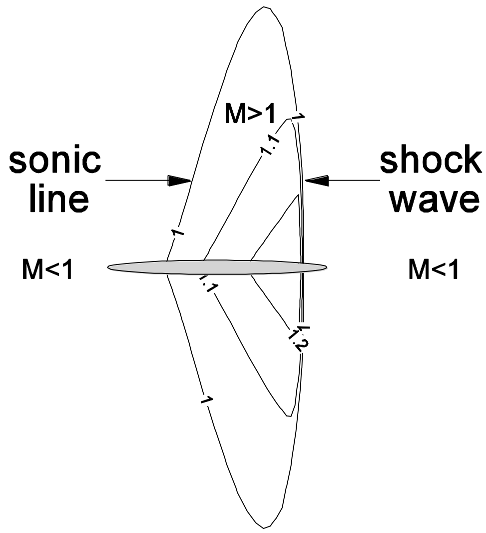

Figure 1.5 Transonic flow past a thin airfoil

If the flow, in addition to being irrotational and isentropic, is only slightly disturbed from a uniform free stream flow near Mach one, $0.8 \le M_\infty \le 1.2$, by the presence of a thin body, as shown in the above figure, the *Transonic Small Disturbance Equation*, as follows, can be used to describe the flow.

$$\left[1 - M_\infty^2 - (\gamma+1) M_\infty^2 \frac{\phi_x}{q_\infty}\right] \phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$

where M_∞ is the free stream Mach number and ϕ represents the small disturbance velocity potential, $u' = \phi_x$, $v = v' = \phi_y$, $w = w' = \phi_z$ and $u = q_\infty + u'$.

The equation is of mixed type. It can be either elliptic, if $u < a$, or hyperbolic, if $u > a$.

$$0 < \left[1 - M_\infty^2 - (\gamma+1) M_\infty^2 \frac{\phi_x}{q_\infty}\right] < 0$$

Elliptic or Hyperbolic

The velocity components u, v and w can be obtained by differentiation of ϕ. Then the above isentropic perfect gas relations of Section 1.5, or the following small disturbance relations can be used to obtain density and pressure.

$$\rho \simeq \rho_\infty \left(1 - M_\infty^2 \left(\frac{u}{q_\infty} - 1\right)\right) \text{ and } p \simeq p_\infty \left(1 - \gamma M_\infty^2 \left(\frac{u}{q_\infty} - 1\right)\right)$$

The pressure coefficient given by small disturbance theory is $c_p = \dfrac{p - p_\infty}{\frac{1}{2}\rho_\infty q_\infty^2} \simeq -2\dfrac{u'}{q_\infty}$

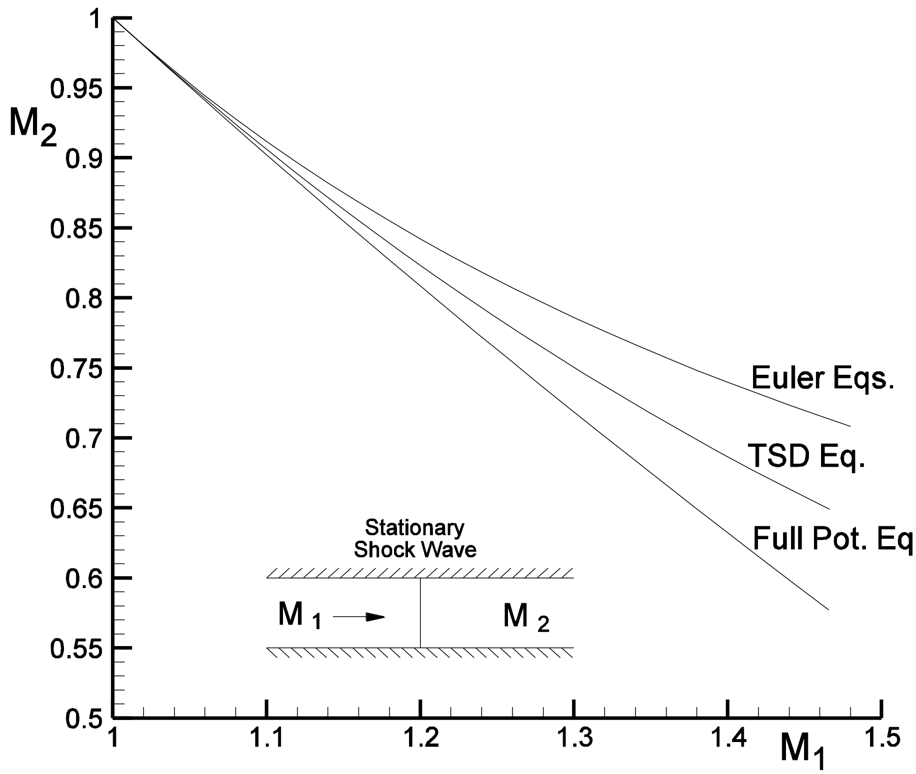

Figure 1.6 Isentropic and Euler Equation shock jump relations

Unlike the *Transonic Small Disturbance Equation*, the *Full Potential Equation* can describe large disturbances to the free stream caused, for example, by blunt nosed airfoils. Both equations assume isentropic flow and are therefore limited to flows containing only weak shocks. Figure 1.6 shows the discontinuous jump in Mach number across a normal shock wave for these two equations compared with that for the Euler equations, which we may accept as exact. Shock waves described by the *Navier-Stokes Equations* would represent the jump as a smooth transition, of length equal to a few mean free paths, between the same values given by the *Euler Equations*. The isentropic equations are in general limited to flows with peak Mach numbers below 1.3. Shock waves, with Mach numbers just ahead of the shock greater than 1.3, are usually strong enough to cause boundary layer separation, for which the *Navier-Stokes Equations* are required. However, this Mach number range below 1.3 is extremely important because all commercial aircraft cruise and all military aircraft must maneuver within it.

1.7 Laplace Like Equations

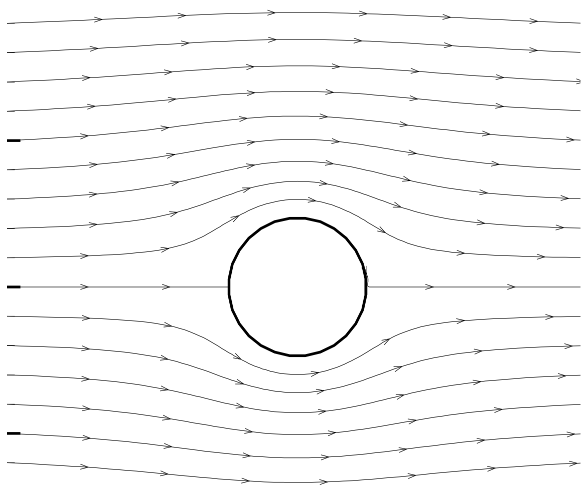

Figure 1.7 Streamlines about non-lifting cylinder at Mach 0.2

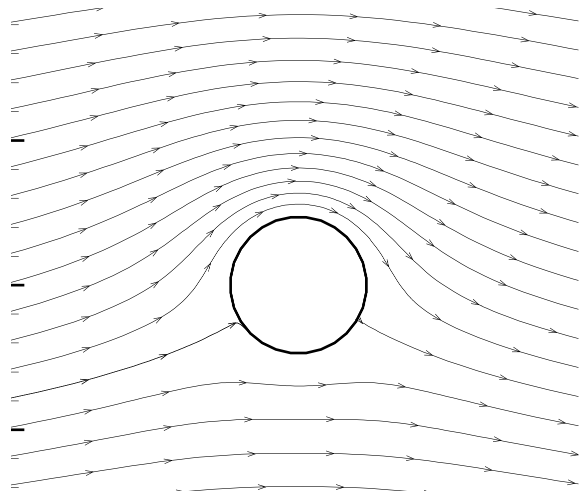

Figure 1.8 Streamlines about a lifting cylinder at Mach 0.2

Purely subsonic or supersonic, steady, isentropic and irrotational flow can be described by Laplace's equation with the Prandtl-Glauert correction factor shown below. This linear potential flow equation describes the subsonic flow about the right circular cylinder with and without circulation shown in the figures above. Circulation can move the fore and aft stagnation points down to create lift.

$$(1-M_\infty^2)\phi_{xx}+\phi_{yy}+\phi_{zz}=0$$

where again ϕ is the velocity potential, $u=\phi_x$, $v=\phi_y$ and $w=\phi_z$, $M_\infty=\left.\sqrt{\dfrac{u^2+v^2+w^2}{a^2}}\right|_\infty$ is the free stream Mach number, and a is the speed of sound. This equation is valid for incompressible flow ($M_\infty=0$, *Laplace's Equation*), subsonic flow ($M_\infty<0.8$), and supersonic flow ($M_\infty>1.2$) for which non-linear effects are negligible (no transonic flow). For supersonic flow ϕ is the small disturbance velocity potential, which describes small disturbances to the free stream for flow past slender bodies ($u=V_\infty+\phi_x$, $v=\phi_y$ and $w=\phi_z$). The solution of this equation has been important in the design of almost all flight vehicles to date. Because of its basic simplicity, it has been solved for flows past highly complex geometries. Numerical methods for solving it, namely *Panel Methods*, represent a mature technology today. There is adequate literature available that describes these methods and little attention will be given to them here.

1.8 The Model Equations

It is often useful to test numerical procedures on simple model equations rather than the complete complex set of equations governing fluid flow. The following are model equations for hyperbolic, elliptic, and parabolic equations.

{a) Hyperbolic model equation

$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0$, where c is constant.

Initial condition at $t=t_0$

$$u(x,t_0)=g(x)$$

Boundary condition at $x=0$

$$u(0,t)=u_0$$

Exact solution at $t>t_0$

$$u(x,t)=g(x-c(t-t_0))$$

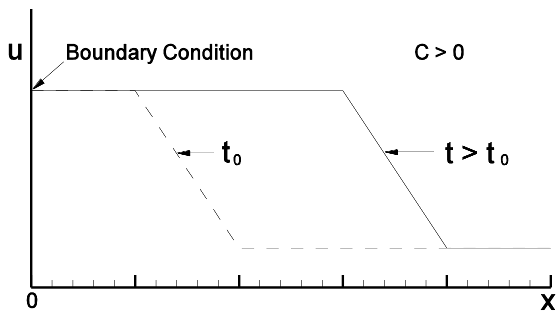

Figure 1.9 Hyperbolic solution

{b) Elliptic model equation

$A\phi_{xx}+\phi_{yy}=0$, with $A>0$ and constant

Boundary conditions

(1) ϕ on far field boundary C_{ff}

(2) $\frac{\partial \phi}{\partial n}$ on body surface boundary C_b

If $A<0$, this equation is hyperbolic.

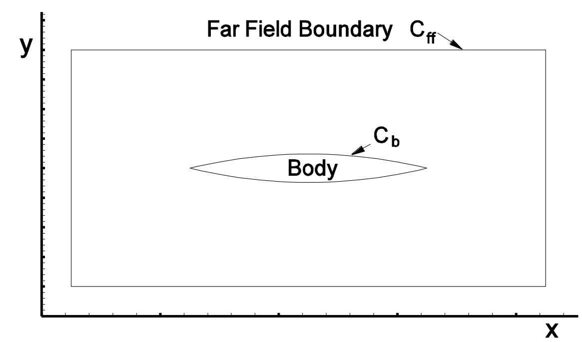

Figure 1.10 Elliptic boundary value problem

c) Parabolic model equation

$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial x^2}$, with constant $\nu>0$.

Initial condition at $t=t_0$

$$u(x,t_0)=f(x)$$

Boundary conditions at $x=x_0$ and $x=x_1$

$$u(x_0,t)=u_0 \text{ and } u(x_1,t)=u_1$$

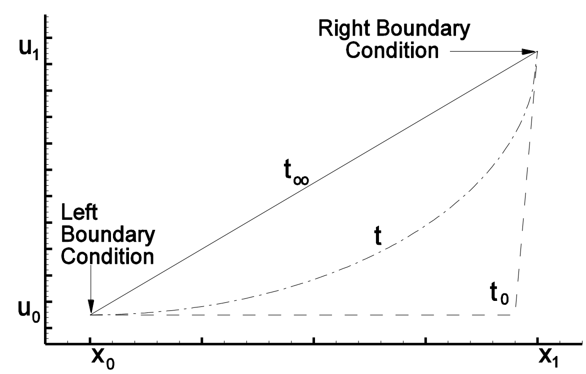

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Figure 1.11 Parabolic solution