Chapter 15

The Navier-Stokes Equations and the Reynolds Averaged Navier-Stokes Equations (RANS)

15.1 Introduction

The Navier-Stokes equations, known for more than a century, are now routinely solved for many flow of aerodynamic interest. However, practical limitations of computer speed and memory prevent their solution for turbulent flows of present engineering interest. Turbulence needs to be modeled so that significant motions of the flow below present grid size limitations are accounted for. The models essentially bump up viscosity and thermal diffusion to account for sub-grid scale turbulent mixing. The resulting equations are called the Reynolds-Averaged Navier-Stokes (RANS) equations.

15.2 The Navier-Stokes Equations

The Navier-Stokes equations written in tensor notation are as follows (repeated indices imply summation).

$$\text{Continuity} \qquad \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0$$

$$\text{Momentum} \qquad \frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j}$$

$$\text{Energy} \qquad \frac{\partial e}{\partial t} + \frac{\partial e u_j}{\partial x_j} = -\frac{\partial p u_j}{\partial x_j} - \frac{\partial \tau_{ij} u_i}{\partial x_j} - \frac{\partial q_j}{\partial x_j}$$

Where

$$e = \rho\left(\varepsilon + \tfrac{1}{2} u_i u_i\right), \quad or \quad \varepsilon = \frac{e}{\rho} - \frac{1}{2} u_i u_i$$

$$p = p(\rho, \varepsilon) = (\gamma - 1)\rho\varepsilon, \quad \varepsilon = c_v T,$$

$$\tau_{ij} = -\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \delta_{ij}\lambda \frac{\partial u_k}{\partial x_k},$$

$$\lambda = -\tfrac{2}{3}\mu, \quad \mu = \mu(T), \quad q_j = -k\frac{\partial T}{\partial x_j} \quad and \quad k = \frac{\mu}{P_r} c_p = \frac{\gamma\mu}{P_r} c_v$$

For air at standard conditions $\gamma = 1.4$, the Prandtl Number, $\mathrm{P}_r = 0.7$, and c_v and μ_l, using Sutherland's formula, given below in both English and metric units

$$c_v = 4290 \frac{ft^2}{\sec^2 \cdot {}^o R}, \qquad \mu_l = 2.270 \times 10^{-8} \frac{T^{\frac{3}{2}}}{T + 198.6^o R} \frac{lb \cdot \sec}{ft^2}, \qquad \text{in English units,}$$

or

$$c_v = 717.5 \frac{J}{kg \cdot {}^oK}, \qquad \mu_l = 1.458 \times 10^{-6} \frac{T^{\frac{3}{2}}}{T + 110.4^o K} \frac{N \cdot \sec}{m^2}, \qquad \text{in metric units .}$$

15.3 Time Averaging for the RANS Equations

The Reynolds Averaged Navier-Stokes equations have been devised to describe turbulent flow. Turbulent flows have a range of length scales, from the characteristic length of the body the flow is passing about, or through, all the way down to the size of the smallest turbulent eddy. This range can cover many orders of magnitude for a high Reynolds number aerodynamic flow. The Reynolds number of a transport aircraft in flight is of the order of

$$R_L = \frac{\rho_\infty u_\infty L}{\mu_\infty} \simeq 30 \times 10^6$$

where L is the characteristic length of the aircraft. A computational mesh about such an aircraft, designed to resolve the complete range of length scales of a turbulent flow, is impractical at present and will continue to be so into the near future. Instead, only the largest features of the flow plus the viscous boundary layer are attempted to be resolved by a mesh that is very fine across the boundary layer and then stretched to resolve the inviscid features of the flow outside the boundary layer. The turbulent scales of motion, not resolvable by the mesh, the sub-grid scales, will need to be modeled.

An unsteady turbulent flow can often be idealized as consisting of two parts: a slowly varying mean flow plus a rapidly fluctuating component related to turbulence. The more disparate these two time scales are the more realistic is this description.

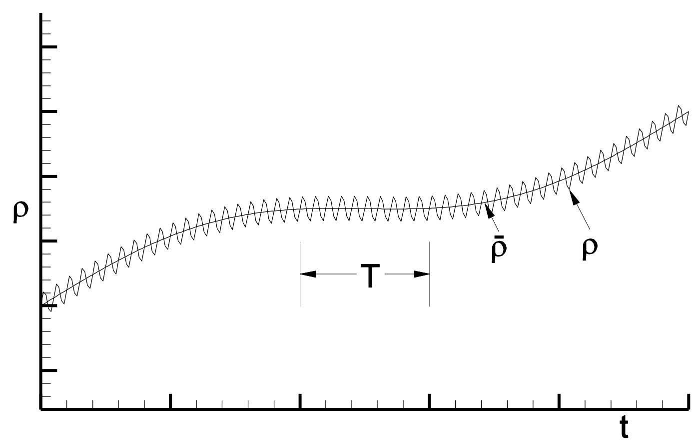

Figure 15.1 Mean and local space-time variations

Let us write for a typical flow variable $z = \overline{z} + z'$ and define the mean flow quantity $\overline{z}$ by a time average over an interval T, an interval assumed long with respect to the time scales of turbulence and short with respect to those of the mean flow.

$$\overline{z}(t_0) = \frac{1}{T}\int_{t_0-T/2}^{t_0+T/2} z\,dt, \qquad z' = z - \overline{z}$$

$$\text{Note: } \frac{1}{T}\int_{t_0-T/2}^{t_0+T/2} z'\,dt = 0$$

An averaging of the primitive variables ρ, u_i and p, as well as viscous stress τ_{ij} and heat transfer q_j, yields

$$\rho = \overline{\rho} + \rho', \quad u_i = \overline{u}_i + u_i', \quad p = \overline{p} + p', \quad \tau_{ij} = \overline{\tau}_{ij} + \tau_{ij}' \quad \textit{and} \quad q_j = \overline{q}_j + q_j'$$

The time or, as it is usually called, Reynolds averaged continuity equation is

$$\frac{1}{T}\int_{t_0-T/2}^{t_0+T/2}\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j}\right)dt = 0 \quad \text{or} \quad \frac{\partial \overline{\rho}}{\partial t} + \frac{\partial\left(\overline{\rho}\overline{u}_j + \overline{\rho' u_j'}\right)}{\partial x_j} = 0$$

Similarly, the Reynolds averaged momentum equation is

$$\frac{\partial\left(\overline{\rho}\overline{u}_i + \overline{\rho' u_i'}\right)}{\partial t} + \frac{\partial\left(\overline{\rho}\overline{u}_i\overline{u}_j + \overline{u}_i\overline{\rho' u_j'}\right)}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} - \frac{\partial\left(\overline{\tau}_{ij} + \overline{u}_j\overline{\rho' u_i'} + \overline{\rho}\overline{u_i' u_j'} + \overline{\rho' u_i' u_j'}\right)}{\partial x_j}$$

The above time averaging introduces more than 40 correlations (closure moments) of the form $\overline{z'y'}$, each of which needs to be determined.

For incompressible flow $\rho' = 0$ and the above averaged equations simplify to

$$\left.\begin{aligned}
&\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{\rho}\overline{u}_j}{\partial x_j} = 0 \\
&\frac{\partial \overline{\rho}\overline{u}_i}{\partial t} + \frac{\partial \overline{\rho}\overline{u}_i\overline{u}_j}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} - \frac{\partial}{\partial x_j}(\overline{\tau}_{i,j} + \underbrace{\overline{\rho}\overline{u_i' u_j'}}_{\text{Reynolds stress tensor}})
\end{aligned}\right\} \textit{incompressible flow}$$

The Reynolds stress term tensor contains six independent components.

15.4 Favre Averaging for Compressible Flow

Favre (see *Rubesin and Rose* 1973) using a time averaging of the conservative variables was able to preserve the same simplified form, as above for the *incompressible* Reynolds averaged equations, for the *compressible* Reynolds averaged equations as well.

$$\rho = \overline{\rho} + \rho', \quad p = \overline{p} + p', \quad \tau_{ij} = \overline{\tau}_{ij} + \tau_{ij}', \quad q_j = \overline{q}_j + q_j'$$

$$\rho u_i = \overline{\rho u_i} + (\rho u_i)', \quad \tilde{u}_i = \frac{\overline{\rho u_i}}{\bar{\rho}}, \quad u_i'' = u_i - \tilde{u}_i,$$

$$\rho H = \overline{\rho H} + (\rho H)', \quad \tilde{H} = \frac{\overline{\rho H}}{\bar{\rho}}, \quad H'' = H - \tilde{H}, \quad H = \frac{e+p}{\rho},$$

$$\rho h = \overline{\rho h} + (\rho h)', \quad \tilde{h} = \frac{\overline{\rho h}}{\bar{\rho}}, \quad h'' = h - \tilde{h}, \quad h = H - \frac{1}{2} u_i u_i,$$

The Reynolds averaged equations for a compressible turbulent flow become

$$\left.\begin{aligned}
&\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \bar{\rho} \tilde{u}_j}{\partial x_j} = 0 \\
&\frac{\partial \bar{\rho} \tilde{u}_i}{\partial t} + \frac{\partial \bar{\rho} \tilde{u}_i \tilde{u}_j}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} - \frac{\partial}{\partial x_j} (\bar{\tau}_{ij} + \underbrace{\overline{\rho u_i'' u_j''}}_{\substack{\text{Reynolds} \\ \text{stress terms}}}) \\
&\frac{\partial \bar{e}}{\partial t} + \frac{\partial \bar{e} \tilde{u}_j}{\partial x_j} = -\frac{\partial \bar{p} \tilde{u}_j}{\partial x_j} - \underbrace{\frac{\partial \overline{\rho u_j'' h''}}{\partial x_j}}_{\substack{\text{Reynolds} \\ \text{heat flux terms}}} - \frac{\partial}{\partial x_j} \left[\tilde{u}_i \Big(\bar{\tau}_{ij} + \underbrace{\overline{\rho u_i'' u_j''}}_{\substack{\text{Reynolds} \\ \text{stress tensor}}} + \underbrace{\overline{u_i'' (\tau_{ij} - \tfrac{1}{2} \rho u_i'' u_j'')}}_{\substack{\text{Reynolds} \\ \text{dissipation terms}}} \Big) \right]
\end{aligned}\right\} \textit{compressible flow}$$

From the equations of continuity and momentum of the Navier-Stokes equations given earlier we can obtain the following equation

$$\frac{\partial \rho u_i u_j}{\partial t} + \frac{\partial \rho u_i u_j u_k}{\partial x_k} = \cdots$$

By time averaging the above equation and subtracting a similar relation obtained from the Reynolds averaged equations given above we can obtain an equation for the six Reynolds stress terms shown below

$$\frac{\partial \overline{\rho u_i'' u_j''}}{\partial t} + \frac{\partial \overline{\rho u_i'' u_j''} \tilde{u}_k}{\partial x_k} = \cdots - \frac{\partial \overline{\rho u_i'' u_j'' u_k''}}{\partial x_k} + \cdots$$

Unfortunately, we have also introduced higher order moments that will also need additional equations to close the system. A different approach credited to Boussinesq is discussed below.

15.5 Boussinesq Eddy Viscosity

An alternative approach is to use the Boussinesq concept of an eddy viscosity to include the effect of additional mixing caused by turbulent flow. First, the Navier-Stokes viscous stress tensor is given by

$$\tau_{ij} = -\mu_l \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \delta_{ij} \lambda_l \frac{\partial u_k}{\partial x_k}, \quad \lambda_l = \frac{2}{3} \mu_l$$

where the subscript on the viscous coefficients implies laminar flow and the coefficients are derived from molecular viscosity relationships. The Reynolds stress using the Boussinesq concept of an ``eddy'' viscosity is then given by a similar formula.

$$\overline{\rho u_i'' u_j''} = -\mu_t \left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right) - \delta_{ij} \lambda_t \frac{\partial \tilde{u}_k}{\partial x_k}, \quad \lambda_t = \frac{2}{3} \mu_t$$

The subscripts on the viscous coefficients here imply turbulent flow. The total effective viscosity is given by the sum of the two viscosities

$$\mu = \mu_l + \mu_t$$

The turbulent viscosity has units of density times length times velocity

$$\mu_t \propto \rho \cdot l \cdot v$$

Morkovin's hypothesis states that compressibility should not affect the descriptions of the scales of turbulent flow and therefore the local fluid density can be used for the density factor. The remaining scales of the local turbulent motion, length and velocity, need to be modeled as follows.

15.6 Prandtl-van Driest Model - Near Walls

In Prandtl's mixing length concept for boundary layers the length scale is proportional to the distance from the wall, $l = k\,y$, because the turbulent eddy sizes will be limited by their proximity to the wall. The coefficient k is called the *von Karman constant* and equals 0.4 (not to be confused with the same symbol used earlier for the coefficient of heat conduction). Van Driest observed a stronger wall damping effect on the length scale and added an additional factor to the length scale relation.

$$l = k\,y\left(1 - e^{-y^+/A^+}\right),$$

$$\text{where } y^+ = \frac{\sqrt{\rho_w |\tau_w|}}{\mu_w} y, \quad \tau_w = \mu_w \left.\frac{\partial u}{\partial y}\right|_w, \quad A^+ = 26.0$$

The subscript w indicates values at the wall and the value of y^+ will change with position along the wall. Note: This model for turbulent flow was developed for boundary layers along flat plates, where the velocity derivative $\frac{\partial u}{\partial y}$ has its maximum value at the wall. The extension to flows with adverse pressure gradients may cause trouble because $\frac{\partial u}{\partial y}$ could be less than or equal to zero for some points along the wall. For more complex flows, y represents the distance to the nearest point along the wall, say point w, and it is suggested that $\left.\frac{\partial u}{\partial y}\right|_w$ be replaced by the $\max\left\{\left|\frac{\partial u}{\partial y}\right|\right\}$ within the boundary layer above point w.

The velocity scale is given by the product of the length scale with the local vorticity of the flow. Hence, the eddy viscosity near a wall is modeled as

$$\mu_t = \rho\left(k\,y\left(1-e^{-y^+/A^+}\right)\right)^2 |\omega|$$

$$\text{with } |\omega| = \sqrt{\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\right)^2}$$

15.7 Cebeci-Smith Model

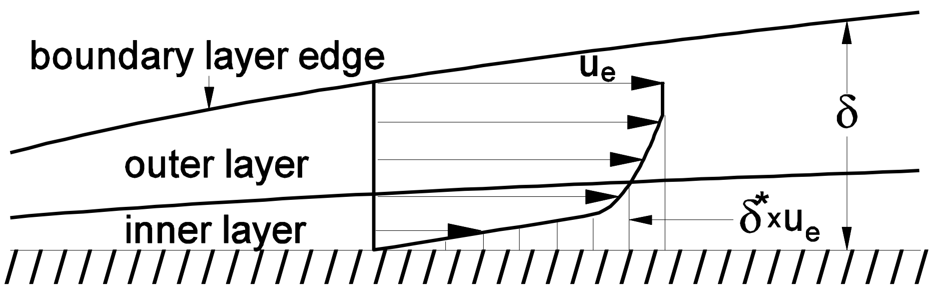

Figure 15.2 Turbulent boundary layer

The Cebeci-Smith model contains two layers, an inner layer near the wall and an outer layer away from the wall. The inner layer uses the *Prandtl-van Driest formulation* and the outer uses a *Clauser wake formulation.*

$$\mu_t = \begin{cases} \rho\left(k\,y\left(1-e^{-y^+/A^+}\right)\right)^2 |\omega|, & y \le y_c,\ \ \textit{inner region} \\ \rho K u_e \delta^*, & y > y_c,\ \ \textit{outer region} \end{cases}$$

where $K = 0.0168$, u_e is the velocity at the boundary layer edge, and the displacement thickness δ^* is given by

$$\delta^* = \int_0^{y_e}\left(1-\frac{u}{u_e}\right)dy$$

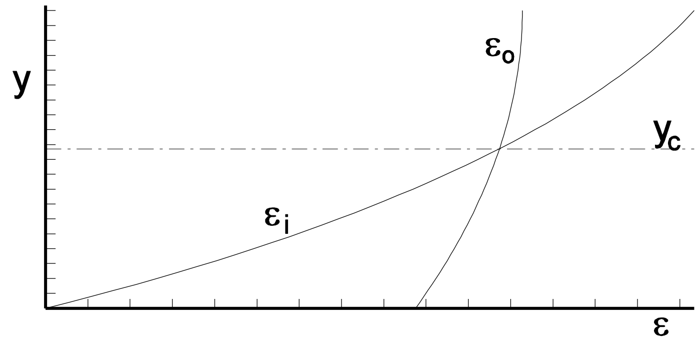

Figure 15.3 Inner and outer eddy viscosities

The distance to the wall y_c represents the point where ε_i first exceeds ε_o

An alternative expression for the Cebeci-Smith model is given by

$$\varepsilon_i = \rho\left(k\,y\left(1-e^{-y^+/A^+}\right)\right)^2|\omega|,$$
$$\varepsilon_o = \rho K u_e \delta^*$$
$$\mu_t = \min\{\varepsilon_i, \varepsilon_o\}$$

This turbulence model was originally formulated using data for flows past flat plates. A major difficulty encountered in using the above model for flow passed curved wall surfaces is in determining the location of the boundary layer edge, y_e. Baldwin and Lomax modified the *Cebeci-Smith model* to remove this difficulty.

15.8 Baldwin-Lomax Model

The inner layer of the *Baldwin-Lomax model* is the same as the *Cebeci-Smith model*, but the outer layer is restructured to remove the model dependence on the boundary layer edge location. Baldwin noticed upon plotting the velocity scale $F(y) = y\left(1-e^{-y^+/A^+}\right)|\omega|$ against the distance from the wall y that there appeared a clear maximum that could be used to replace δ^* in the outer model. He also added a Klebanoff term so that the turbulent viscosity would return to zero far outside the boundary layer. At each point along the wall, a search is made for the first maximum, starting at the wall surface, in the velocity scale function, $F(y) = y\left(1-e^{-y^+/A^+}\right)|\omega|$, see Fig.15.4. Also, during this preliminary search $\max\left\{\left|\frac{\partial u}{\partial y}\right|\right\}$, $\max\{|u(y)|\}$ and $\min\{|u(y)|\}$ are found. The Baldwin-Lomax model is defined as follows.

$$\mu_t = \begin{cases} \rho\left(k\,y\left(1-e^{-y^+/A^+}\right)\right)^2|\omega|, & y \le y_c, \ \textit{inner region} \\ \rho K C_{cp} F_{wake} F_{Kleb}(y), & y > y_c, \ \textit{outer region} \end{cases}$$

where

$$F_{wake} = \min\left\{ y_{\max} F_{\max}, C_{wk} y_{\max} u_{dif}^2 / F_{\max} \right\}$$

$$F(y) = y\left(1 - e^{-y^+/A^+}\right)|\omega|$$

$$F_{Kleb}(y) = \frac{1}{1 + 5.5\left(\dfrac{C_{Kleb}\, y}{y_{\max}}\right)^6}$$

$$u_{dif} = \max\left\{|u(y)|\right\} - \underbrace{\min\left\{|u(y)|\right\}}_{0,\ except\ in\ wakes}$$

$$C_{cp} = 1.6,\ \ C_{Kleb} = 0.3,\ \ C_{wk} = 0.25$$

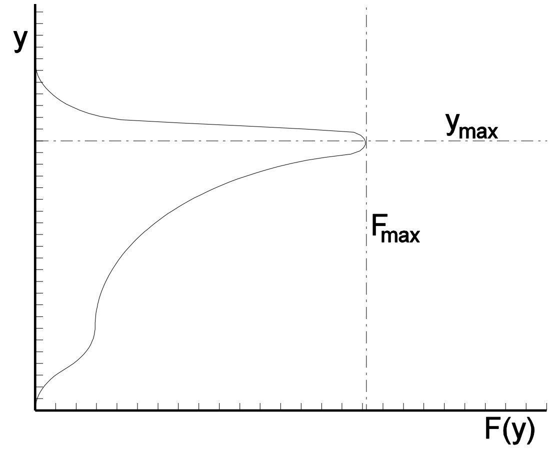

Figure 15.4 The function $F(y)$

Transition to turbulence can be simulated in the *Baldwin-Lomax model* as follows. First, μ_t is calculated as above, then

$$\mu_t = 0 \ \ if \ \ \max_{in\ profile} \left\{\mu_t(y)\right\} < c_{mutm} \times \underbrace{\mu_\infty}_{\substack{freestream \\ viscosity}} \ \ , \ \ c_{mutm} = 14.0$$

15.9 Other Turbulence Models

The models just presented are all algebraic models that calculate turbulent viscosity from mean flow quantities. Currently there are several promising models using one, two, and more differential equations to calculate the convection, production, and dissipation of turbulence quantities associated with the flow, from which more physically realistic determinations of turbulent viscosity can be made. A current search of the literature is suggested to locate these promising models. David C. Wilcox's book "Turbulence Modeling for CFD" (Third Edition, DCW Industries) covers the field extensively. The long range solution to turbulence simulation, when sufficiently large computer resources become available, is the direct solution of the un-averaged Navier-Stokes equations on such a fine spatial and temporal scale that all significant turbulent motions are resolved. Until then, we will need to model all turbulent motion smaller than that resolvable by the mesh.

15.9.1 One Equation Models

Turbulence models that require the solution of an additional partial differential flow equation are of form

$$\frac{\partial \tilde{\nu}}{\partial t} + u_i \frac{\partial \tilde{\nu}}{\partial x_i} = \underbrace{\cdots\cdots\cdots\cdots}_{production} + \underbrace{\cdots\cdots\cdots\cdots}_{dissipation} + \underbrace{\frac{\partial}{\partial x_k}\left[(\nu + \tilde{\nu})\frac{\partial \tilde{\nu}}{\partial x_k}\right]}_{diffusion}$$

where $\nu \approx \nu_t = \dfrac{\mu_t}{\rho}$ is the kinematic turbulent viscosity

These models contain terms for modeling turbulent production, dissipation and diffusion. An early one equation turbulence model was developed by Baldwin and Barth (1990). A very popular model today is the Spalart-Allmaris model (1992).

15.9.2 Two Equation Models

There are two types of two equation turbulence models:

1) $k-\omega$ models that solve for turbulent kinetic energy, $k = \frac{1}{2}\overline{u'_i u'_i} = \frac{1}{2}\overline{(u'^2 + v'^2 + w'^2)}$, and the dissipation rate of turbulent kinetic energy, ω, suggested by Kolmogorov (1942). (See, for example, the turbulence models by Wilcox (1988), Mentor (1992), …)

$$\frac{\partial k}{\partial t} + u_i \frac{\partial k}{\partial x_i} = \cdots\cdots\cdots\cdots + \cdots\cdots\cdots\cdots + \frac{\partial}{\partial x_k}\left[\left(\nu + \sigma^* \frac{k}{\omega}\right)\frac{\partial k}{\partial x_k}\right]$$

$$\frac{\partial \omega}{\partial t} + u_i \frac{\partial \omega}{\partial x_i} = \cdots\cdots\cdots\cdots + \cdots\cdots\cdots\cdots + \frac{\partial}{\partial x_k}\left[\left(\nu + \sigma \frac{k}{\omega}\right)\frac{\partial \omega}{\partial x_k}\right]$$

where $\nu_t = \frac{k}{\omega}$ is the kinematic turbulent viscosity, $\sigma = \frac{1}{2}$ and $\sigma^* = \frac{3}{5}$

and

2) $k-\varepsilon$ models that solve for turbulent kinetic energy, k, and turbulent dissipation, $\varepsilon = \nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}}$. (See, for example, Jones & Launder (1972), Launder & Sharma (1974), …)

$$\frac{\partial k}{\partial t} + u_i \frac{\partial k}{\partial x_i} = \cdots\cdots\cdots\cdots + \cdots\cdots\cdots\cdots + \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_k}\right]$$

$$\frac{\partial \varepsilon}{\partial t} + u_i \frac{\partial \varepsilon}{\partial x_i} = \cdots\cdots\cdots\cdots + \cdots\cdots\cdots\cdots + \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_k}\right]$$

where $\nu_t = C_\mu \frac{k^2}{\varepsilon}$ is the kinematic turbulent viscosity, $C_\mu = 0.09$, $\sigma_k = 1.0$ and $\sigma_\varepsilon = 1.3$

15.9.3 Reynolds Stress-Transport Models

These models contain one equation for each independent component of the Reynolds stress, 6 in all (see Lumley (1978)…). The equations are solved for each Reynolds stress $\tau_{i,j} = -\overline{u'_i u'_j}$, which can then be used directly in the RANS equations without the need for the Boussinesq approximation.

$$\frac{\partial \tau_{i,j}}{\partial t} + u_k \frac{\partial \tau_{i,j}}{\partial x_k} = \cdots\cdots\cdots\cdots + \cdots\cdots\cdots\cdots + \frac{\partial}{\partial x_k}\left[\nu \frac{\partial \tau_{i,j}}{\partial x_k}\right]$$

15.9.4 Direct Numerical Simulation (DNS)

DNS models solve the Navier-Stokes equations on fine enough meshes to resolve all turbulent scales of motion. They are limited to low Reynolds numbers, far from the range of interest for the simulation of flow past aerodynamic shapes of practical engineering interest today. However, this approach can be used to create data bases for the testing and development of better turbulence models for the RANS equations and, with the continuing advance of computer resources, this is the future solution to the grand challenge of fluid dynamics, the simulation of turbulent flow. See review article by Moin and Mahesh (1998) for further discussion of DNS

15.9.5 Large Eddy Simulations (LES)

LES simulations solve the Reynolds Averaged Navier-Stokes (RANS) equations on meshes fine enough to resolve all the significant energy bearing eddies in the energy spectrum for turbulent flows. The sub-grid scale (SGS) turbulent motion effects are modeled using simpler and less critical models than those required for RANS simulations on courser meshes. See the review article by Ferziger (1996) for further discussion of LES

15.9.6 Detached Eddy Simulation (DES)

The DES approach, introduced by Spalart, Jou, Strelets and Allmaras in 1997, solves the RANS equations on a mesh fine enough to resolve only the largest turbulent eddies. The rest are modeled using a modified Spalart-Allmaris model. As time proceeds this approach will naturally evolve into LES and then DNS with future computational resources. Present resources enable DES simulations of flows past aerodynamic elements of practical engineering interest

15.10 Summary

The Reynolds Averaged Navier-Stokes (RANS) equations for a compressible turbulent flow, using the *Boussinesq concept* of an eddy viscosity, are presented below, where we have dropped for convenience the *"bars"*, "*tildes" and "primes"* so that they have same form as the un-averaged Navier-Stokes equations given earlier. The difference is now contained only in the definition of the momentum and heat transfer transport coefficients μ and k .

$$\text{Continuity}\quad \frac{\partial \rho}{\partial t}+\frac{\partial \rho u_j}{\partial x_j}=0$$

$$\text{Momentum}\quad \frac{\partial \rho u_i}{\partial t}+\frac{\partial \rho u_i u_j}{\partial x_j}=-\frac{\partial p}{\partial x_i}-\frac{\partial \tau_{ij}}{\partial x_j}$$

$$\text{Energy}\quad \frac{\partial e}{\partial t}+\frac{\partial e u_j}{\partial x_j}=-\frac{\partial p u_j}{\partial x_j}-\frac{\partial \tau_{ij} u_i}{\partial x_j}-\frac{\partial q_j}{\partial x_j}$$

where

$$\tau_{ij}=-\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-\delta_{ij}\lambda\frac{\partial u_k}{\partial x_k},\quad q_j=-k\frac{\partial T}{\partial x_j},$$

$$\mu=\mu_l+\mu_t,\quad \lambda=-\frac{2}{3}\mu,\quad k=\gamma\left(\frac{\mu_l}{\Pr_l}+\frac{\mu_t}{\Pr_t}\right)c_v,$$

$$\text{and}\quad p=(\gamma-1)\left(e-\tfrac{1}{2}\rho u_i u_i\right)$$

For air at standard conditions $\gamma=1.4,\ \Pr_l=0.7\ \ \textit{and}\ \ \Pr_t\simeq 0.9$