Řešení cvičení 12: Určitý integrál

Snadné integrály

Spočtěte následující integrály

(a) $\int_0^5 x^3 + 2x^2 + \frac{x}{3}\ \mathrm{d}x$,

(b) $\int_0^2 \frac{x}{(1+2x^2)^2}\ \mathrm{d}x$,

(c) $\int_4^1 \sqrt{x}e^{1-\sqrt{x^3}}\ \mathrm{d}x$,

(d) $\int_0^\infty \frac{3}{5+2x}\ \mathrm{d}x$,

(e) $\int_5^5 \frac{\arctan(x^{0.75}+3)}{e^{x^3}+x+2}\ \mathrm{d}x$,

(f) $\int_0^{\frac{\pi}{2}} \frac{\sin(x)\cos^2(x)}{1+\cos^2(x)}\ \mathrm{d}x$,

(g) $\int_0^1 \frac{x^3}{3+x}\ \mathrm{d}x$,

(h) $\int_0^1 x^2(2-3x^2)^2\ \mathrm{d}x$.

(a)

$$\int_0^5 x^3+2x^2+\frac{x}{3}\ \mathrm{d}x = \int_0^5 x^3\ \mathrm{d}x + \int_0^5 2x^2\ \mathrm{d}x + \int_0^5 \frac{x}{3}\ \mathrm{d}x = \left[\frac{x^4}{4}+\frac{2x^3}{3}+\frac{x^2}{6}\right]_0^5 = \frac{5^4}{4}+\frac{2\cdot 5^3}{3}+\frac{5^2}{6}-0.$$

(b)

$$\int_0^2 \frac{x}{(1+2x^2)^2}\ \mathrm{d}x = \left\{\begin{array}{c} 2x^2 = y \\ 4x\mathrm{d}x = \mathrm{d}y \end{array}\right\} = \frac{1}{4}\int_0^4 \frac{1}{(1+y)^2}\ \mathrm{d}y = \frac{1}{4}\left[-\frac{1}{1+y}\right]_0^4 = -\frac{1}{4}\left[\frac{1}{1+4}-1\right] = \frac{1}{5}.$$

(c)

$$\int_4^1 \sqrt{x}e^{1-\sqrt{x^3}}\ \mathrm{d}x = -\int_1^4 \sqrt{x}e^{1-\sqrt{x^3}}\ \mathrm{d}x = \left\{\begin{array}{c} x^{\frac{3}{2}} = y \\ \frac{3}{2}\sqrt{x}\mathrm{d}x = \mathrm{d}y \end{array}\right\} = -\frac{2}{3}\int_1^{4^{\frac{3}{2}}} e^{1-y}\ \mathrm{d}y =$$

$$-\frac{2}{3}\left[-e^{1-y}\right]_1^{4^{\frac{3}{2}}} = -\frac{2}{3}\left[1-e^{1-4^{\frac{3}{2}}}\right].$$

(d)

$$\int_0^\infty \frac{3}{5+2x}\ \mathrm{d}x = \frac{3}{5}\int_0^\infty \frac{1}{1+\frac{2x}{5}}\ \mathrm{d}x = \left\{\begin{array}{c} \frac{2x}{5} = y \\ \frac{2}{5}\mathrm{d}x = \mathrm{d}y \end{array}\right\} = \frac{3}{2}\int_0^\infty \frac{1}{1+y}\ \mathrm{d}y = \frac{3}{2}\left[\ln(1+y)\right]_0^\infty = \infty.$$

(e)

$$\int_5^5 \frac{\arctan(x^{0.75}+3)}{e^{x^3}+x+2}\ \mathrm{d}x = 0,$$

protože integrujeme přes interval nulové délky.

(f)

$$\int_0^{\frac{\pi}{2}} \frac{\sin(x)\cos^2(x)}{1+\cos^2(x)}\ \mathrm{d}x = \left\{\begin{array}{c} \cos(x) = y \\ \sin(x)\ \mathrm{d}x = \mathrm{d}y \end{array}\right\} = \int_1^0 \frac{y^2}{1+y^2}\ \mathrm{d}y = \int_1^0 1-\frac{1}{1+y^2}\ \mathrm{d}y =$$

$$\left[y-\arctan(y)\right]_1^0 = \arctan(1)-1.$$

(g)

$$\int_0^1 \frac{x^3}{3+x}\ \mathrm{d}x = \left\{\frac{x^3}{3+x} = x^2-3x+9-\frac{27}{3+x}\right\} = \int_0^1 x^2-3x+9-\frac{27}{3+x}\ \mathrm{d}x =$$

$$\left[\frac{1}{3}x^3-\frac{3}{2}x^2+9x-27\ln(3+x)\right]_0^1 = \frac{1}{3}-\frac{3}{2}+9-27\ln\left(\frac{4}{3}\right).$$

(h)

$$\int_0^1 x^2(2-3x^2)^2 \ \mathrm{d}x = \int_0^1 x^2(4-12x^2+9x^4) \ \mathrm{d}x = \int_0^1 4x^2-12x^4+9x^6 \ \mathrm{d}x = \left[\frac{4}{3}x^3-\frac{12}{5}x^5+\frac{9}{7}x^7\right]_0^1 = \frac{4}{3}-\frac{12}{5}+\frac{9}{7}.$$

Složitější integrály

Spočtšte následující integrály

(a) $\int_0^{\ln(2)} \sqrt{e^x-1} \ \mathrm{d}x$,

(b) $\int_0^1 x\ln(x) \ \mathrm{d}x$,

(c) $\int_0^\infty e^{-ax}\cos(bx) \ \mathrm{d}x$,

(d) $\int_0^\infty x^3 e^{\frac{x^2}{2}} \ \mathrm{d}x$.

(a)

$$\int_0^{\ln(2)} \sqrt{e^x-1} \ \mathrm{d}x = \int_0^{\ln(2)} \frac{e^x}{e^x}\sqrt{e^x-1} \ \mathrm{d}x = \left\{ \begin{array}{c} e^x=y \\ e^x\mathrm{d}x=\mathrm{d}y \end{array} \right\} = \int_1^2 \frac{\sqrt{y-1}}{y} \ \mathrm{d}y =$$

$$\int_1^2 \frac{\sqrt{y-1}^2}{\sqrt{y-1}^2+1}\frac{1}{\sqrt{y-1}} \ \mathrm{d}y = \left\{ \begin{array}{c} y-1=z^2 \\ \mathrm{d}y=2z \ \mathrm{d}z \end{array} \right\} = 2\int_0^1 \frac{z^2}{z^2+1} \ \mathrm{d}z =$$

$$2\int_0^1 1-\frac{1}{z^2+1} \ \mathrm{d}z = 2\left[z-\arctan(z)\right]_0^1 = 2(1-\arctan(1)).$$

(b)

$$\int_0^1 x\ln(x) \ \mathrm{d}x \overset{pp.}{=} \left[\frac{x^2}{2}\ln(x)\right]_0^1 - \int_0^1 \frac{x^2}{2}\frac{1}{x} \ \mathrm{d}x = \left[\frac{x^2}{2}\ln(x)\right]_0^1 - \int_0^1 \frac{x}{2} \ \mathrm{d}x = \left[\frac{x^2}{2}\ln(x)-\frac{x^2}{4}\right]_0^1 = -\frac{1}{4},$$

kde jsme použili

$$\lim_{x\to0^+}\frac{x^2}{2}\ln(x) = \frac{1}{2}\lim_{x\to0^+}\frac{\ln(x)}{\frac{1}{x^2}} \overset{l'H}{=} \frac{1}{2}\lim_{x\to0^+}\frac{\frac{1}{x}}{-\frac{2}{x^3}} = -\frac{1}{2}\lim_{x\to0^+}\frac{x^2}{2} = 0.$$

(c)

$$I = \int_0^\infty e^{-ax}\cos(bx) \ \mathrm{d}x \overset{pp.}{=} -\left[\frac{e^{-ax}}{a}\cos(bx)\right]_0^\infty + \int_0^\infty \frac{e^{-ax}}{a}b\sin(bx) \ \mathrm{d}x =$$

$$\frac{1}{a}+\frac{b}{a}\int_0^\infty e^{-ax}\sin(bx) \ \mathrm{d}x \overset{pp.}{=} \frac{1}{a}-\frac{b}{a}\left[\frac{e^{-ax}}{a}\sin(bx)\right]_0^\infty - \frac{b^2}{a^2}\int_0^\infty \frac{e^{-ax}}{a}\cos(bx) \ \mathrm{d}x = \frac{1}{a}-\frac{b}{a^2}-\frac{b^2}{a^2}I,$$

tedy

$$I = \frac{\frac{1}{a}-\frac{b}{a^2}}{1+\frac{b^2}{a^2}} = \frac{\frac{a-b}{a^2}}{\frac{a^2-b^2}{a^2}} = \frac{1}{a+b}.$$

(d)

$$\int_0^\infty x^3 e^{-\frac{x^2}{2}} \ \mathrm{d}x = \left\{ \begin{array}{c} \frac{x^2}{2}=y \\ x\mathrm{d}x=\mathrm{d}y \end{array} \right\} = 2\int_0^\infty ye^{-y} \ \mathrm{d}y \overset{pp.}{=} 2\left[-ye^{-y}\right]_0^\infty + 2\int_0^\infty e^{-y} \ \mathrm{d}y =$$

$$2\left[-ye^{-y}-e^{-y}\right]_0^\infty = 2.$$

Integrální kritérium konvergence

Vyšetřete konvergenci následujících řad

(a) $\sum_{i=1}^{\infty} \frac{1}{n}$,

(b) $\sum_{i=2}^{\infty} \frac{1}{n\ln(n)}$,

(c) $\sum_{i=1}^{\infty} \frac{n}{(n^2+1)\ln(n^2+1)}$,

(d) $\sum_{i=1}^{\infty} \frac{e^n}{1+e^{2n}}$.

(a)

$$\sum_{i=1}^{\infty} \frac{1}{n} \geq \int_1^{\infty} \frac{1}{x} \, \mathrm{d}x = [\ln(x)]_1^{\infty} = \infty.$$

(b)

$$\sum_{i=2}^{\infty} \frac{1}{n\ln(n)} \geq \int_2^{\infty} \frac{1}{x\ln(x)} \, \mathrm{d}x = \left\{ \begin{array}{c} \ln(x) = y \\ \frac{1}{x} \, \mathrm{d}x = \mathrm{d}y \end{array} \right\} = \int_{\ln(2)}^{\infty} \frac{1}{y} \, \mathrm{d}y = [\ln(y)]_{\ln(2)}^{\infty} = \infty.$$

(c)

$$\sum_{i=1}^{\infty} \frac{n}{(n^2+1)\ln(n^2+1)} \geq \int_1^{\infty} \frac{x}{(x^2+1)\ln(x^2+1)} \, \mathrm{d}x = \left\{ \begin{array}{c} x^2+1 = y \\ 2x \, \mathrm{d}x = \mathrm{d}y \end{array} \right\} = \int_2^{\infty} \frac{1}{2y\ln(y)} \, \mathrm{d}y =$$

$$\left\{ \begin{array}{c} \ln(y) = z \\ \frac{1}{y} \, \mathrm{d}y = \mathrm{d}z \end{array} \right\} = \int_{\ln(2)}^{\infty} \frac{1}{2z} \, \mathrm{d}z = \infty.$$

(d)

$$0 \leq \sum_{i=1}^{\infty} \frac{e^n}{1+e^{2n}} \leq \int_1^{\infty} \frac{e^{x+1}}{1+e^{2x+2}} \, \mathrm{d}x \overset{x+1=y}{=} \int_2^{\infty} \frac{e^y}{1+e^{2y}} \, \mathrm{d}y = \left\{ \begin{array}{c} e^y = z \\ e^y \, \mathrm{d}y = \mathrm{d}z \end{array} \right\} =$$

$$\int_{e^2}^{\infty} \frac{1}{1+z^2} \, \mathrm{d}z = [\arctan(z)]_{e^2}^{\infty} = \frac{\pi}{2} - \arctan(e^2).$$