Řešení domácího úkolu 6

Spočtěte následující limity pro $m, n \in \mathbb{N}$ a $a > 0$

1.
$$\lim_{x\to -8} \frac{\sqrt{1-x}-3}{2+\sqrt[3]{x}},$$

(1 bod)

$$\lim_{x\to -8} \frac{\sqrt{1-x}-3}{2+\sqrt[3]{x}} = \lim_{x\to -8} \frac{\sqrt{1-x}-3}{2+\sqrt[3]{x}} \frac{\sqrt{1-x}+3}{\sqrt{1-x}+3} \frac{4-2\sqrt[3]{x}+x^{\frac{2}{3}}}{4-2\sqrt[3]{x}+x^{\frac{2}{3}}} =$$
$$\lim_{x\to -8} \frac{1-x-9}{8-x} \frac{4-2\sqrt[3]{x}+x^{\frac{2}{3}}}{\sqrt{1-x}+3} = -\frac{12}{6} = -2.$$

2.
$$\lim_{x\to 1} \left(\frac{m}{1-x^m} - \frac{n}{1-x^n}\right),$$

(2 body)

$$\lim_{x\to 1} \left(\frac{m}{1-x^m} - \frac{n}{1-x^n}\right) = \lim_{x\to 1} \frac{m(1-x^n)-n(1-x^m)}{(1-x^m)(1-x^n)} \overset{x=1+h}{=}$$
$$\lim_{h\to 0} \frac{m(1-(1+h)^n)-n(1-(1+h)^m)}{(1-(1+h)^m)(1-(1+h)^n)} =$$
$$\lim_{h\to 0} \frac{m(1-(1+nh+\frac{n(n-1)}{2}h^2+o(h^2)))-n(1-(1+mh+\frac{m(m-1)}{2}h^2+o(h^2)))}{(1-(1+mh+o(h)))(1-(1+nh+o(h)))} =$$
$$\lim_{h\to 0} \frac{-m\frac{n(n-1)}{2}h^2+n\frac{m(m-1)}{2}h^2+o(h^2)}{(mh+o(h))(nh+o(h))} = \lim_{h\to 0} \frac{-mn(n-1)+nm(m-1)+o(1)}{2(m+o(1))(n+o(1))} = \frac{m-n}{2}.$$

3.
$$\lim_{x\to a} \left(\frac{\sin(x)}{\sin(a)}\right)^{\frac{1}{x-a}}.$$

(2 body)

$$\lim_{x\to a} \left(\frac{\sin(x)}{\sin(a)}\right)^{\frac{1}{x-a}} = \lim_{x\to a} e^{\frac{1}{x-a}\ln\left(\frac{\sin(x)}{\sin(a)}\right)} \overset{x=a+h}{=} \lim_{h\to 0} e^{\frac{1}{h}\ln\left(\frac{\sin(a+h)}{\sin(a)}\right)} = \lim_{h\to 0} e^{\frac{1}{h}\ln\left(1-1+\frac{\sin(a+h)}{\sin(a)}\right)} =$$
$$\lim_{h\to 0} e^{\frac{\frac{\sin(a+h)}{\sin(a)}-1}{h}\underbrace{\frac{\ln\left(1+\frac{\sin(a+h)}{\sin(a)}-1\right)}{\frac{\sin(a+h)}{\sin(a)}-1}}_{\to 1}} = \lim_{h\to 0} e^{\frac{\sin(a+h)-\sin(a)}{h\sin(a)}} = \lim_{h\to 0} e^{\frac{\sin(a)\cos(h)+\sin(h)\cos(a)-\sin(a)}{h\sin(a)}} =$$
$$\lim_{h\to 0} e^{\frac{\cos(h)-1}{h}+\frac{\sin(h)}{h}\cot(a)} = \lim_{h\to 0} e^{\underbrace{\frac{\cos(h)-1}{h^2}}_{\to 1}h+\underbrace{\frac{\sin(h)}{h}}_{\to 1}\cot(a)} = e^{\cot(a)}.$$