

Solutions to Erdős Algebra

1. Alice has 10 apples that Bob wants to turn into apple cider. It takes 2 apples to make 1 liter of cider. How many liters of cider can Bob make using Alice's apples?

Proposed by Julian Kovalovsky.

Answer: 5

Solution: If 2 apples make 1 liter of cider, 10 apples can make $\frac{10}{2} = 5$ liters of cider.

2. If $\frac{27}{45} = \sqrt{\frac{9}{x}}$, what is the value of x ?

Proposed by William Roe.

Answer: 25

Solution: Squaring both sides gives us $\frac{9}{x} = \left(\frac{27}{45}\right)^2 = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$. Multiplying both sides of the equation by $25x$ gives $9x = 9 \cdot 25$, so $x = 25$.

3. Mr. Bramble teaches a World History class with 30 students. On a quiz, every student initially scored 10 points. Half of the class then retook the quiz, and each student who retook the quiz improved their score by 3 points. By how many points did the class average increase after the retakes?

Proposed by Rakshay Narayanan.

Answer: $\frac{3}{2}$

Solution: Before the retakes the class average was 10. After the retakes, 15 students have a score of 10 and 15 students have a score of 13, so the total score among all students is $10 \cdot 15 + 13 \cdot 15 = 345$ and the class average is $\frac{345}{30} = \frac{23}{2}$. So, the class average increased by $\frac{23}{2} - 10 = \frac{3}{2}$ points.

4. William has a bag consisting of marbles that come in 4 different colors: red, blue, green, and yellow. If $\frac{1}{2}$ of the marbles are red, $\frac{1}{3}$ are blue, $\frac{1}{12}$ are green, and the remaining 6 marbles are yellow, how many total marbles are there in the bag?

Proposed by Evan Zhang.

Answer: 72

Solution: $\frac{1}{2} + \frac{1}{3} + \frac{1}{12} = \frac{11}{12}$ of the marbles are either red, blue, or green, so $1 - \frac{11}{12} = \frac{1}{12}$ of the marbles are yellow. Since there are 6 yellow marbles, $\frac{1}{12}$ of the total amount of marbles is equal to 6, so there are $6 \cdot 12 = 72$ total marbles.

5. Farmer Yunyi has a farm with chickens and chimeras. Chickens have 1 head and 2 legs, and chimeras have 3 heads and 4 legs. If the farm has a total of 200 heads and 350 legs (not counting Farmer Yunyi), how many animals are on the farm?

Proposed by Evan Zhang.

Answer: 150

Solution: Let c represent the number of chickens on Farmer Yunyi's farm, and let m represent the number of chimeras on Farmer Yunyi's farm. We have

$$c + 3m = 200$$

$$2c + 4m = 350$$

Subtracting the two equations gives $c + m = 150$, so there are 150 animals on the farm.

6. Pedro is trying to find rectangles with positive integer side lengths whose areas are numerically equal to their perimeters. To his surprise, he discovers that there are only two unique rectangles that have this property. What is the sum of the areas of these two rectangles?

Proposed by Evan Zhang.

Answer: 34

Solution: Let the side lengths of the rectangle be a and b . Our rectangles satisfy the equation $ab = 2a + 2b$, which we can factor into $(a - 2)(b - 2) = 4$.

Since a and b are both positive, then either $(a - 2)$ and $(b - 2)$ are 1 and 4 in some order, or are 2 and 2 in some order. The two rectangles are a 3×6 rectangle and a 4×4 rectangle. The sum of the areas of these two rectangles is $3 \cdot 6 + 4 \cdot 4 = 18 + 16 = 34$.

7. A falling ping-pong ball always bounces up to $\frac{3}{4}$ of the height it falls from. If Stanley drops a ball from 40 feet above the ground, what is the total distance the ball will travel?

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Proposed by Evan Zhang.

Answer: [280]

Solution: The ball travels 40 feet during the first fall, then a total of $2\left(\frac{3}{4} \cdot 40\right)$ feet during the first bounce and resulting fall, then a total of $2\left(\frac{3}{4}\right)^2 \cdot 40$ feet during the second bounce and resulting fall, and so on.

The total distance the ball travels is then

$$40 + 2\left(\frac{3}{4} \cdot 40 + \left(\frac{3}{4}\right)^2 \cdot 40 + \dots\right) = 40 + 2\left(\frac{30}{1 - \frac{3}{4}}\right) = 40 + 2 \cdot 120 = 280.$$

8. If $\frac{1}{x^3} + \frac{1}{y^3} = 4$ and $\frac{1}{x} + \frac{1}{y} = 2$, what is $\frac{1}{x^2} + \frac{1}{y^2}$?

Proposed by Valerie Song.

Answer: $\frac{8}{3}$

Solution: Cubing $\frac{1}{x} + \frac{1}{y} = 2$ gives $\frac{1}{x^3} + \frac{3}{x^2y} + \frac{3}{xy^2} + \frac{1}{y^3} = 8$. Since $\frac{1}{x^3} + \frac{1}{y^3} = 4$, we know $4 + \frac{3}{xy} \left(\frac{1}{x} + \frac{1}{y}\right) = 8$, so $\frac{6}{xy} = 4$ and $\frac{1}{xy} = \frac{2}{3}$.

Squaring $\left(\frac{1}{x} + \frac{1}{y}\right) = 2$ gives $\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = 4$, so $\frac{1}{x^2} + \frac{1}{y^2} = 4 - \frac{2}{xy} = 4 - \frac{4}{3} = \frac{8}{3}$.