MAE309

Lecture 3 & 4

Conservation Law

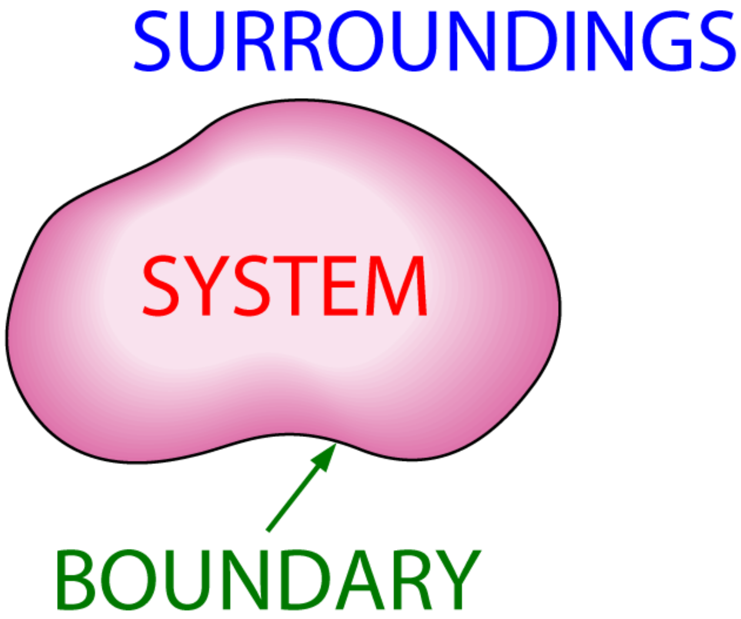
General Principle of Transport Phenomena

Learning Objectives

- To understand
  - System, control volume, boundary
  - Eulerian and Lagragian descriptions of fluid flow
  - Classification of fluid flows: viscous and inviscid flows; laminar and turbulent flows; compressible and incompressible flows; steady and unsteady flows; internal and external flows; one-, two- and three-dimensional flows
  - Material derivative, acceleration of fluid particle
  - Flow visualization: streamline, pathline, streakline, timeline
  - Mass conservation: mass flow rate, continuity equation (integral and differential forms)
  - Momentum conservation: Navier-Stokes equations, Euler equations

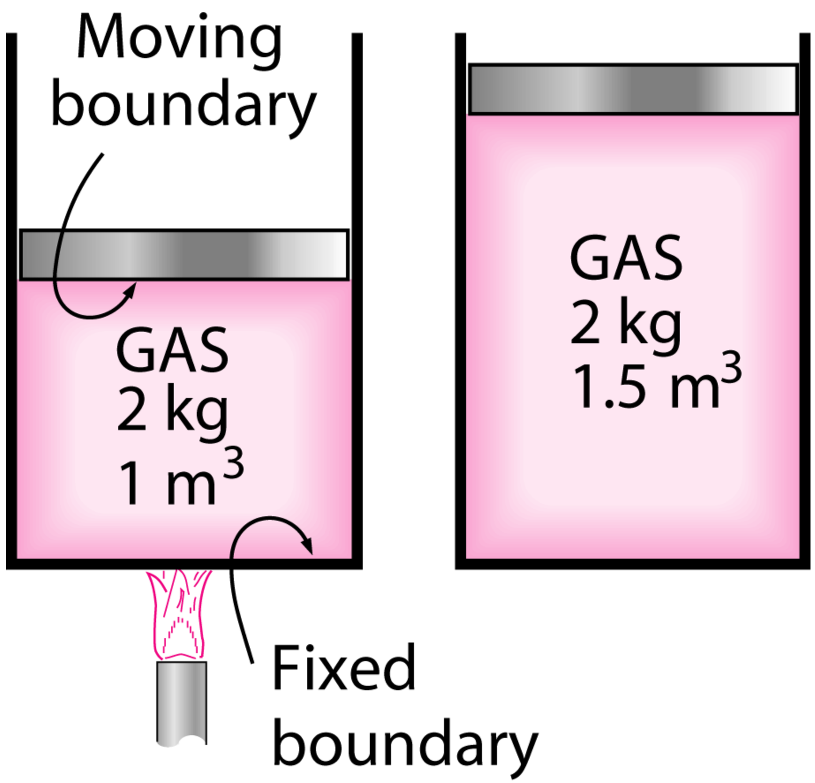
System and Control Volume

- System
  - Quantity of matter of a region in space chosen for study
- Surroundings
  - Mass or region outside the system
- Boundary
  - Real or imaginary surface that separates system from its surroundings
  - Can be either movable or fixed



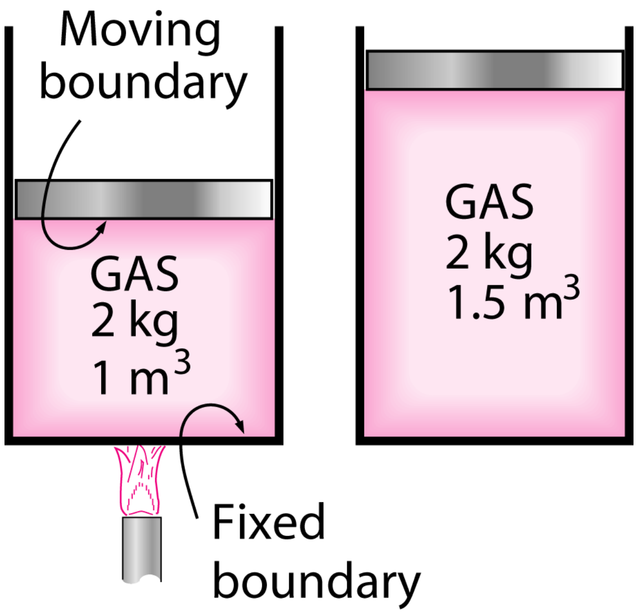
System and Control Volume

- Closed System
  - Consists of fixed amount of mass
  - No mass can cross its boundary
  - Energy, in the form of heat or work, can cross boundary
  - Volume does not have to be fixed



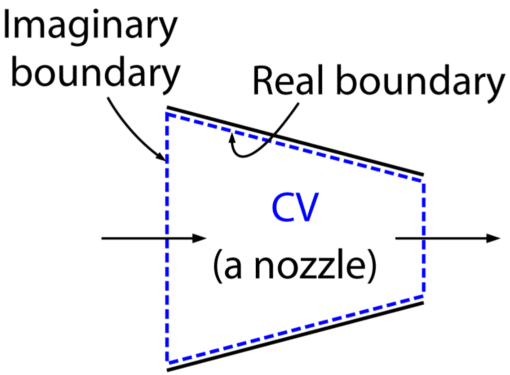
System and Control Volume

- Closed System
  - Example: Piston-cylinder device
    - ✓ System: gas trapped in cylinder by piston
    - ✓ Energy: inner surfaces of piston and cylinder
    - ✓ Surroundings: everything outside the gas, including piston and cylinder
    - ✓ closed system: No mass crossing boundary. Energy may cross boundary
    - ✓ Part of boundary (inner surface of the piston) may move

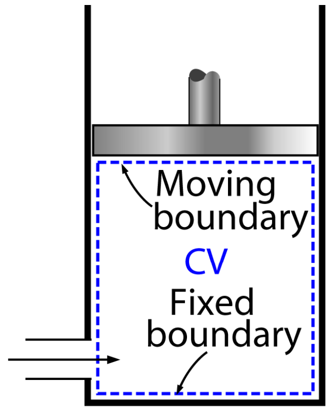


System and Control Volume

- Control volume/Open system
  - Both mass and energy can cross boundary
  - Usually encloses a device that involves mass flow such as a compressor, turbine, or nozzle
  - Any arbitrary region in space can be selected as a control volume, but proper and good choice often makes analysis much easier
  - Can be fixed in size and shape or involve a moving boundary



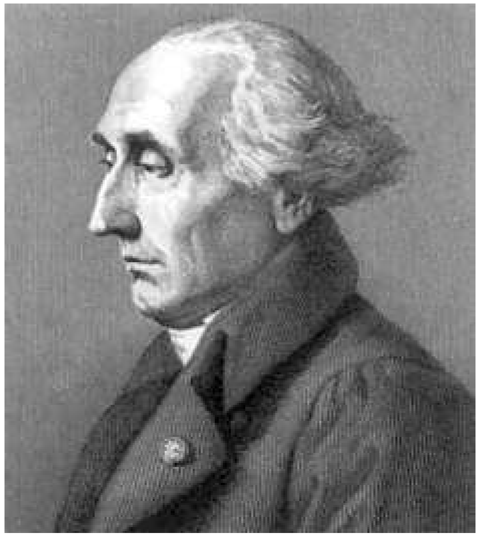
(a) A control volume (CV) with real and imaginary boundaries



(b) A control volume (CV) with fixed and moving boundaries

Methods to Describe Fluid Flows

- Two Methods to Describe Fluid Flows
  - Lagrangian description
  - Eulerian description

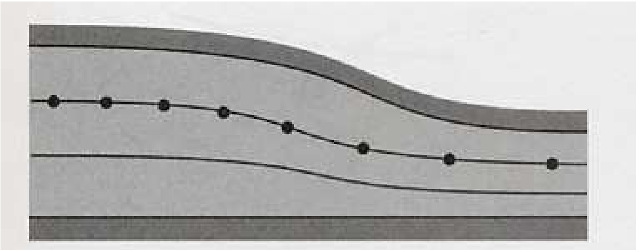


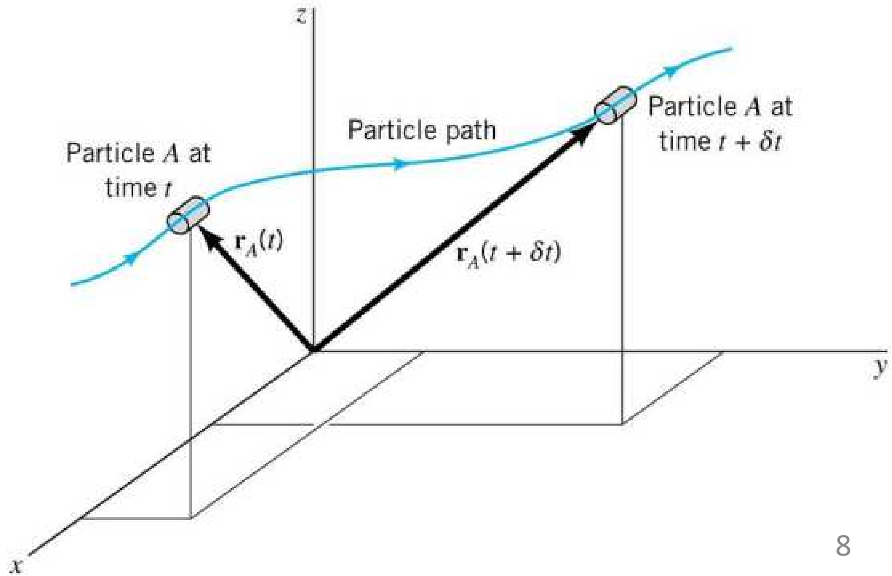
Lagrange
1736-1813

Euler
1707-1783

Methods to Describe Fluid Flows

- Lagrangian Description
  - Tracks the motion (position and velocity vectors) of a generic individual fluid particle
  - Observer moves with the fluid
  - The main difficulty is that the observer moves with the fluid

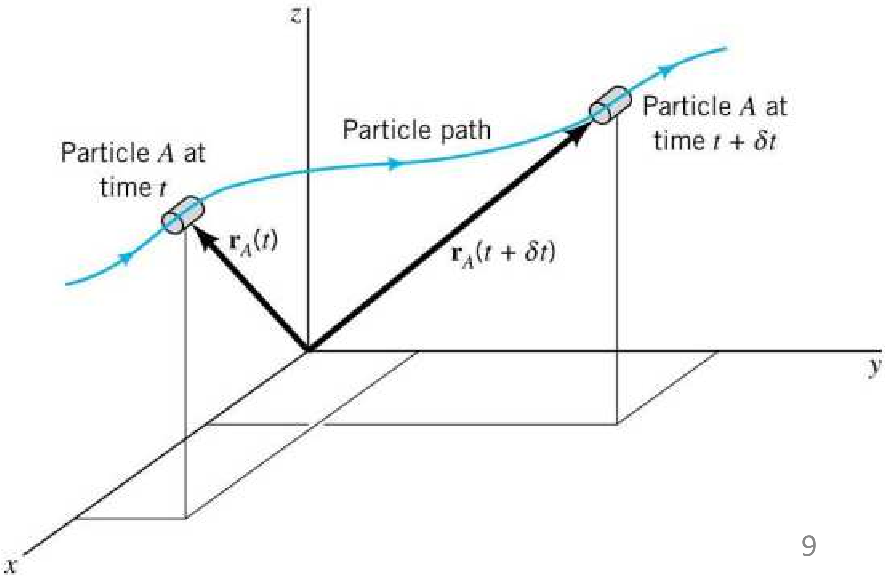




Methods to Describe Fluid Flows

- Lagrangian Description
  - The initial position of fluid particles is $r_A(t_0)$
  - The variation of position of fluid particles with time is described by $r_A(t)$
  - The velocity $\boldsymbol{u}$ and acceleration $\boldsymbol{a}$ of each fluid particle are obtained from the first and second temporal derivatives of particle position

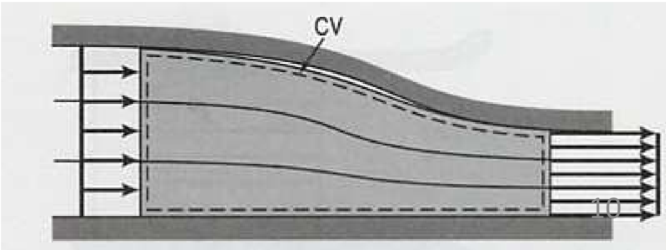
$$\boldsymbol{u} = dr_A(t)/dt$$
$$\boldsymbol{a} = d^2r_A(t)/dt^2$$



  - All the variables are valid for the fluid particle when it moves along its trajectory through flow field

Methods to Describe Fluid Flows

- Eulerian Description
  - A finite control volume (CV) is defined, through which particles flow in and out
  - Individual fluid particles are not identified and tracked
  - Define field variables (functions of space and time) within CV: pressure field $P(x, y, z, t)$, velocity field $V(x, y, z, t)$, temperature field $T(x, y, z, t)$, etc.
  - Field variables define the flow field
  - All field variables are defined at any location (x, y, z) within CV and at any instant of time t

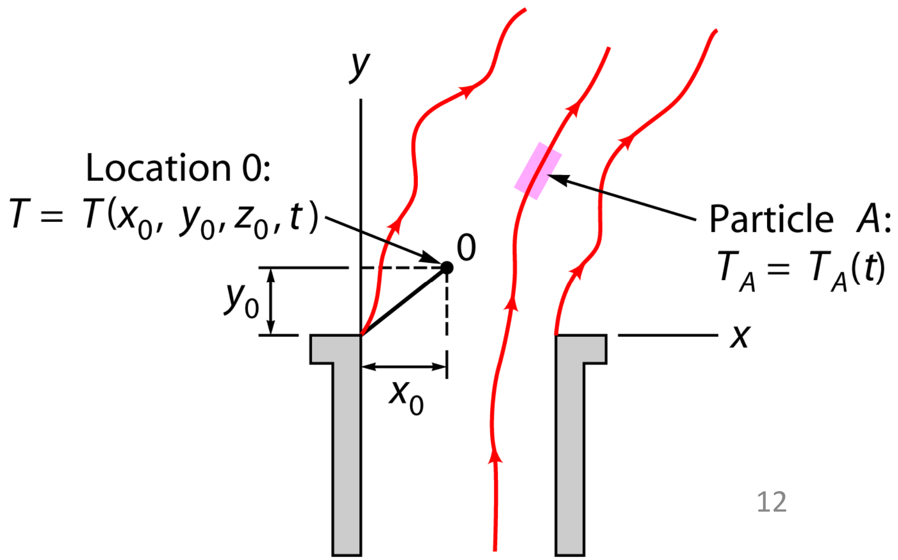


Methods to Describe Fluid Flows

- Eulerian V.S. Lagragian Description
  - It is generally more common to use Eulerian approach to fluid flows. Measuring water temperature, or pressure at a point in a pipe.
  - Lagrangian methods sometimes used in experiments. Throwing tracers into moving water bodies to determine currents.
  - Eulerian description can be converted to Lagrangian description and vice versa

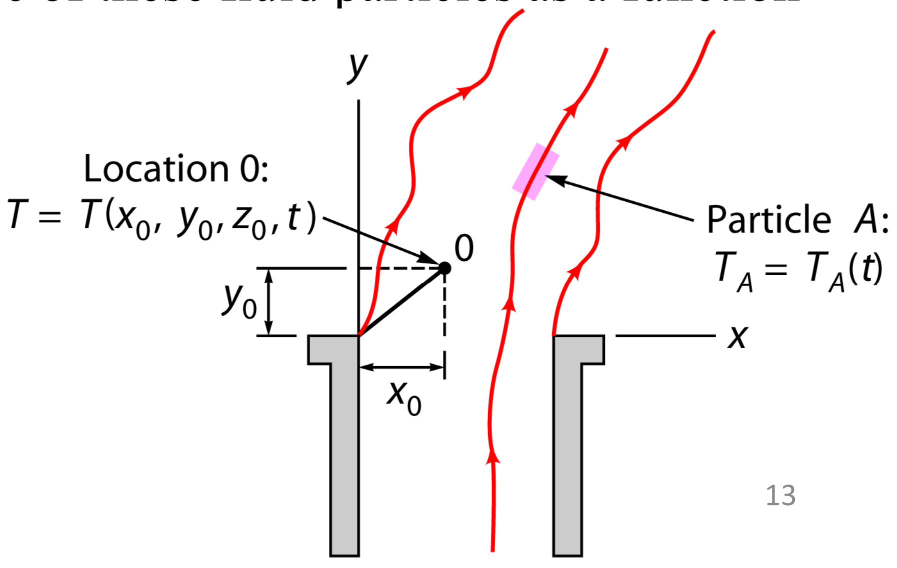
Methods to Describe Fluid Flows

- Example: Smoke Discharging From a Chimney
  - Eulerian description:
    - ✓ Attach a temperature-measuring device to top of chimney (point 0)
    - ✓ Record temperature at point 0 as a function of time
    - ✓ At different times, different fluid particles pass by stationary device
    - ✓ Obtain temperature, T, for point 0 as a function of time, $T(x_0, y_0, z_0, t)$
    - ✓ Use of numerous temperature-measuring devices fixed at various locations yields the temperature field, $T(x, y, z, t)$



Methods to Describe Fluid Flows

- Example: Smoke Discharging From a Chimney
  - Lagrangian description:
    - ✓ Attach temperature-measuring device to a particular fluid particle (particle A)
    - ✓ Record particle A's temperature as it moves about
    - ✓ Obtain particle A's temperature as a function of time, $T_A = T_A(t)$
    - ✓ Use of many such measuring devices moving with various fluid particles yields the temperature of these fluid particles as a function of time



Methods to Describe Fluid Flows

- Example 1
  - Question
    - ✓ Assume that the fluid motion is described by Eulerian method.

$$u = ax + t^2$$
$$v = by - t^2 \qquad (a + b = 0)$$
$$w = 0$$

    - ✓ The initial condition is $x_0 = \alpha$, $y_0 = \beta$, and $z_0 = \gamma$
    - ✓ Obtain the fluid motion in Lagrangian description and find the acceleration

Methods to Describe Fluid Flows

- Example 1
  - Solution:

$$u = \frac{dx}{dt} = ax + t^2$$

$$v = \frac{dy}{dt} = by - t^2$$

$$w = \frac{dz}{dt} = 0$$

$$x = c_1 e^{at} - \frac{1}{a}t^2 - \frac{2}{a^2}t - \frac{2}{a^3}$$

$$y = c_2 e^{bt} + \frac{1}{b}t^2 + \frac{2}{b^2}t + \frac{2}{b^3}$$

$$z = c_3$$

$$x_0 = \alpha$$
$$y_0 = \beta$$
$$z_0 = \gamma$$

$$x = \left(\alpha + \frac{1}{a^3}\right)e^{at} - \frac{1}{a}t^2 - \frac{2}{a^2}t - \frac{2}{a^3}$$

$$y = \left(\beta - \frac{1}{b^3}\right)e^{bt} + \frac{1}{b}t^2 + \frac{2}{b^2}t + \frac{2}{b^3}$$

$$z = \gamma$$

Methods to Describe Fluid Flows

- Example 1
  - Solution:

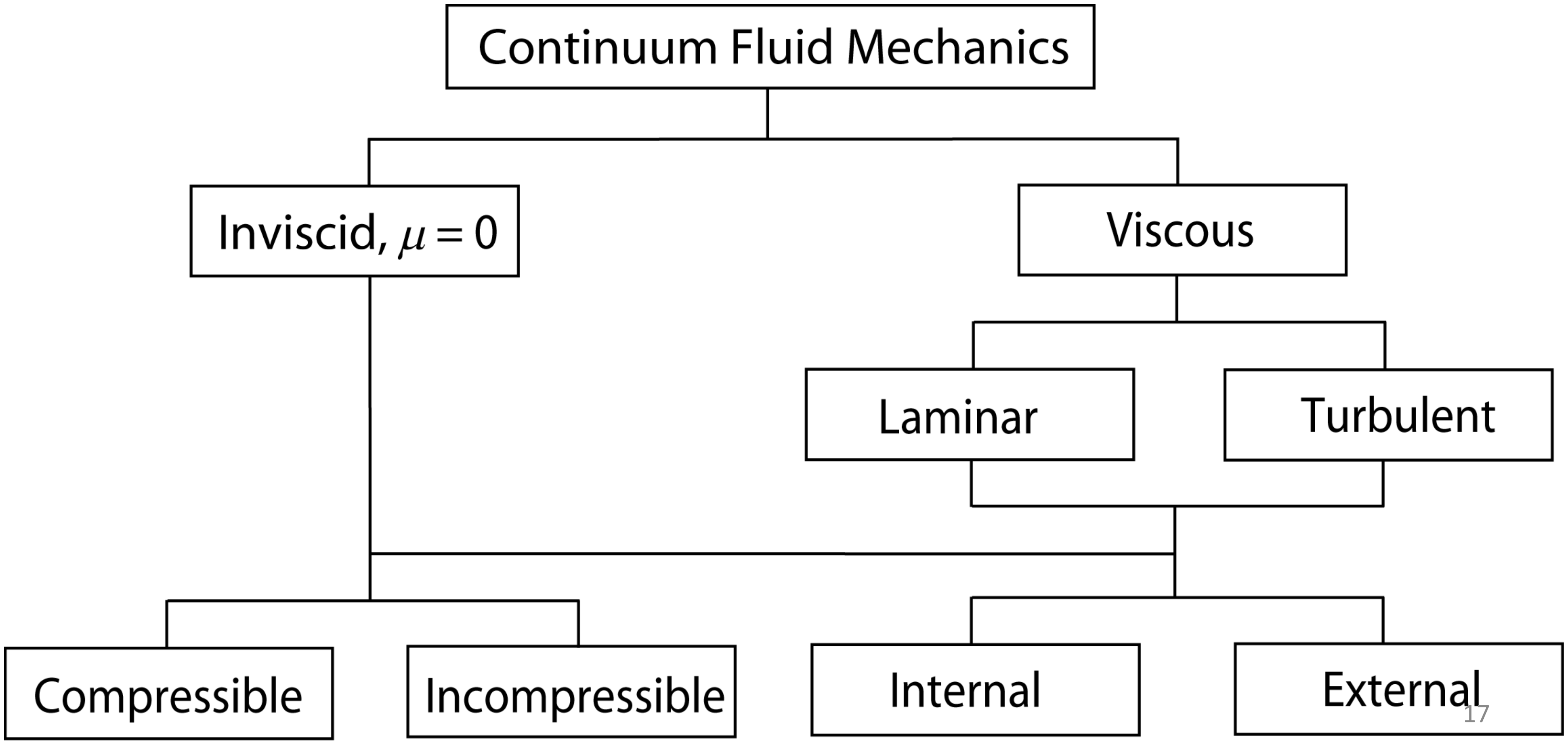
$$u_L = a\left(\alpha + \frac{1}{a^3}\right)e^{at} - \frac{2}{a}t - \frac{2}{a^2} \longrightarrow a_{xL} = a^2\left(\alpha + \frac{1}{a^3}\right)e^{at} - \frac{2}{a}$$

$$v_L = b\left(\beta - \frac{1}{b^3}\right)e^{bt} + \frac{2}{b}t + \frac{2}{b^2} \qquad b_{yL} = b^2\left(\beta - \frac{1}{b^3}\right)e^{bt} + \frac{2}{b}$$

$$w_L = 0 \qquad a_{zL} = 0$$

Classification of Fluid Flows

- There are many ways to classify fluid flows
  - One possible classification:

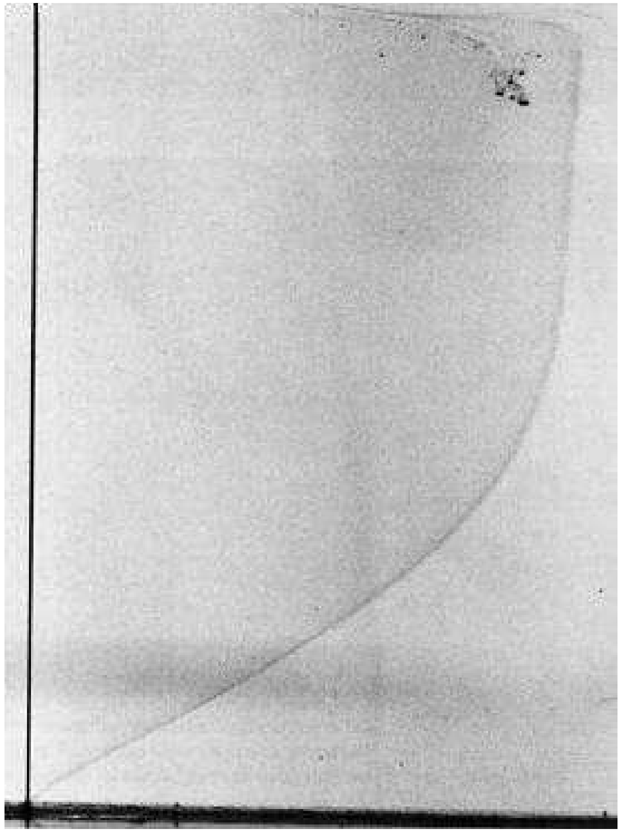


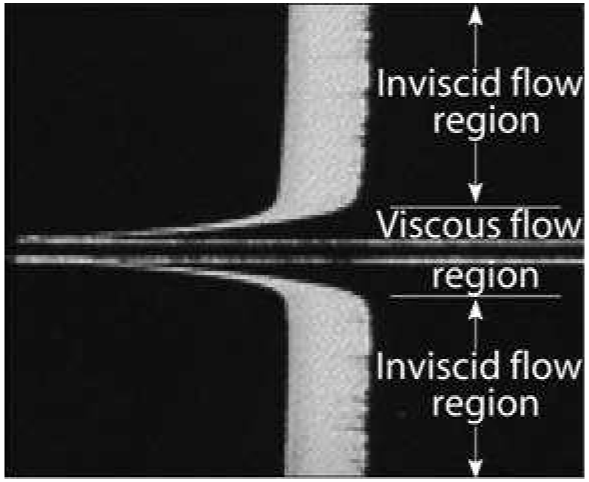
Classification of Fluid Flows

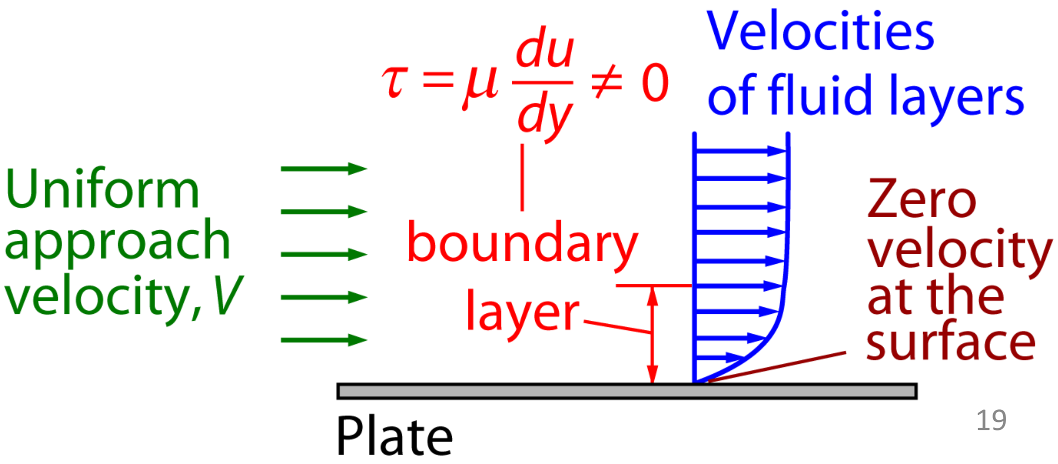
- Viscous Flow versus Inviscid Flow
  - Viscous flows: flows in which frictional or viscous effects are significant
  - There is no fluid with zero viscosity $\Rightarrow$ all fluid flows involve viscous effects to some degree
  - In many practical flows, there are regions (typically away from solid surfaces) where viscous forces are negligible compared to other forces (inertia, pressure, gravity) $\Rightarrow$ neglect viscous effects in these regions $\Rightarrow$ inviscid (ideal) flow
  - Inviscid flow $\Rightarrow$ assume $\mu = 0$
  - Generally easier to analyze an inviscid flow than a viscous flow

Classification of Fluid Flows

- Viscous Flow versus Inviscid Flow

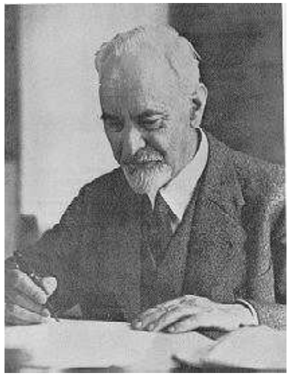






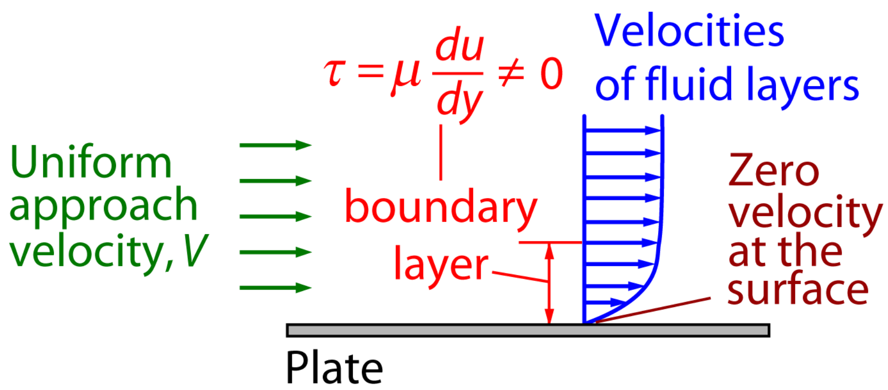
Classification of Fluid Flows

- Viscous Flow versus Inviscid Flow
  - No-slip condition ⇒ fluid has zero velocity at wall
  - Fluid velocity approaches V far away from wall
  - Fluid velocity increases from zero at wall to V far away from wall ⇒ non-zero velocity gradient in a thin layer adjacent to wall ⇒ boundary layer



Prandtl

(1875-1953)

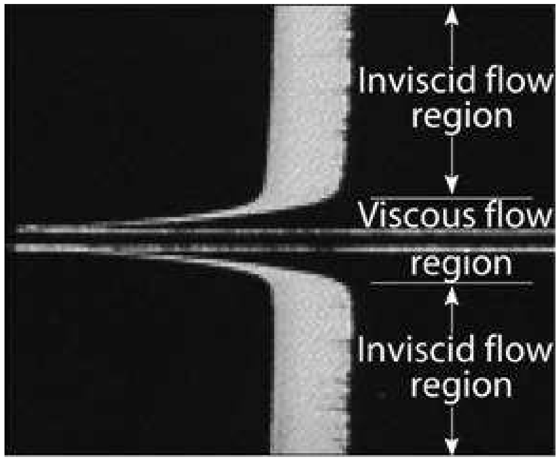


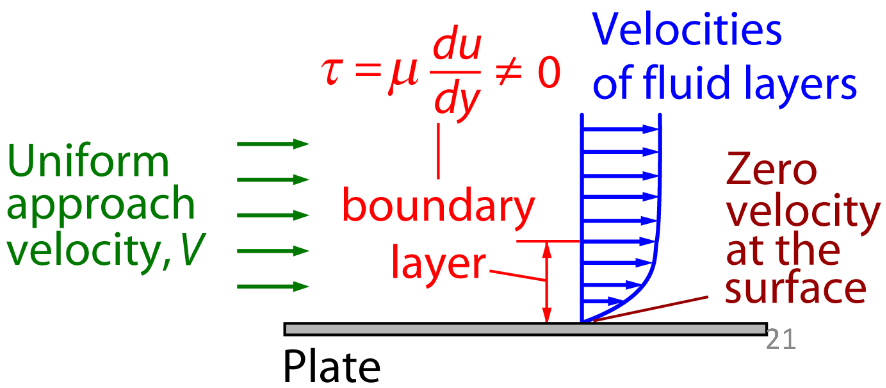
Classification of Fluid Flows

- Viscous Flow versus Inviscid Flow
  - Boundary layer
    - ✓ flow region adjacent to solid surface in which viscous effects and velocity gradients are significant
    - ✓ velocity gradient $\neq 0 \Rightarrow$ shear stress $\neq 0$

$$\tau = \mu \frac{du}{dy} \neq 0$$

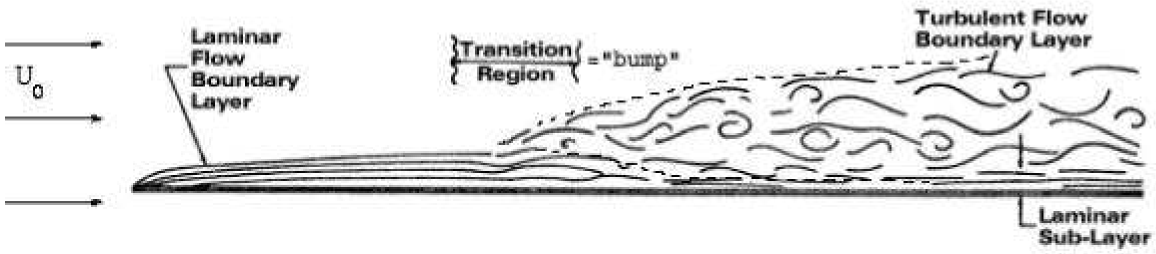
  - Outside boundary layer $\Rightarrow$ flow unaffected by presence of plate $\Rightarrow$ viscous effects unimportant $\Rightarrow$ assume flow to be inviscid





Classification of Fluid Flows

- Laminar Flow versus Turbulent Flow
  - Laminar flow: highly ordered fluid motion characterized by smoothly flowing layers of fluid
  - Turbulent flow: highly disordered fluid motion typically occurring at high velocities and is characterized by random three-dimensional velocity fluctuations
  - Transitional flow: flow that alternates between being laminar and turbulent



Classification of Fluid Flows

- Laminar Flow versus Turbulent Flow
  - Reynolds number
    - ✓ Defined as the ratio of inertial forces to viscous forces

$$\mathrm{Re} = \frac{\text{inertial forces}}{\text{viscous forces}} = \frac{(\text{mass})(\text{acceleration})}{(\text{dynamic viscosity})(\text{velocity}/\text{distance})(\text{area})} = \frac{(\rho L^3)(v^2/L)}{\mu(v/L)L^2} = \frac{\rho v L}{\mu}$$

μ and ρ are fluid viscosity and density respectively; v is the reference velocity (the velocity of the object or free stream velocity of fluid); L is the reference length.

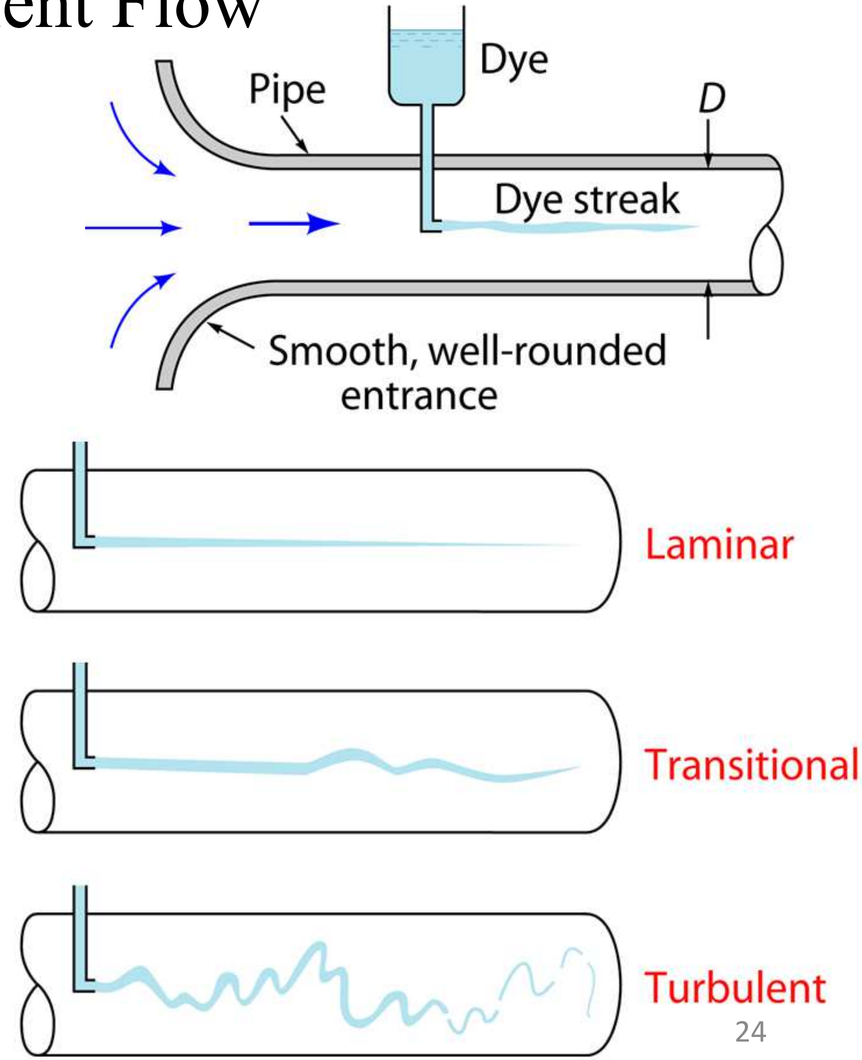
- ✓ Nature of flow can be characterized by Reynolds number
- ✓ Laminar flow occurs at low Reynolds numbers, where viscous forces are dominant
- ✓ Turbulent flow occurs at high Reynolds numbers and is dominated by inertial force

Classification of Fluid Flows

- Laminar Flow versus Turbulent Flow
  - Flow in the circular pipe
    - ✓ Reynolds number is defined as

$$Re = \frac{\rho V D}{\mu}$$

    - ✓ $Re < 2300$: Laminar flow
    - ✓ $2300 < Re < 10^5$: Transitional flow
    - ✓ $Re > 10^5$: Turbulent flow



Classification of Fluid Flows

- Laminar Flow versus Turbulent Flow
  - Flow in the circular pipe

Laminar

Turbulent

Classification of Fluid Flows

- Compressible Flow versus Incompressible Flow
  - Incompressible flow
    - flows in which variations in density are negligible
    - Liquids $\Rightarrow$ incompressible
    - Gases $\Rightarrow$ need to consider Mach number, Ma
  - Mach number

$$Ma = \frac{V}{c} = \frac{\text{Speed of flow}}{\text{Speed of sound}}$$

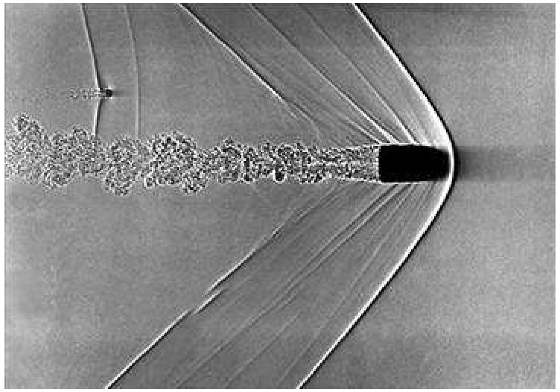
- Speed of sound $c = 346$ m/s in air at room temperature at sea level
- Gases $\Rightarrow$ incompressible if $Ma < 0.3 \Rightarrow$ changes in density less than 5%
- Compressibility effects of air can be neglected at speeds < 100 m/s

Classification of Fluid Flows

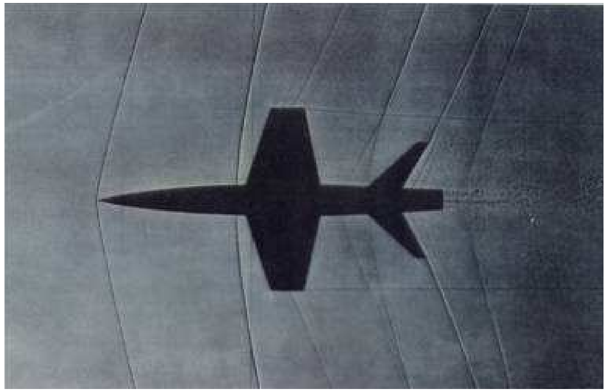
- Compressible Flow versus Incompressible Flow
  - Compressible flow
    - Density variations within flow are not negligible $\Rightarrow$ $Ma > 0.3$
    - $Ma < 1 \Rightarrow$ subsonic flow
    - $Ma = 1 \Rightarrow$ sonic flow
    - $Ma > 1 \Rightarrow$ supersonic flow $\Rightarrow$ shock waves can form $\Rightarrow$ abrupt change in fluid properties across a shock wave

Classification of Fluid Flows

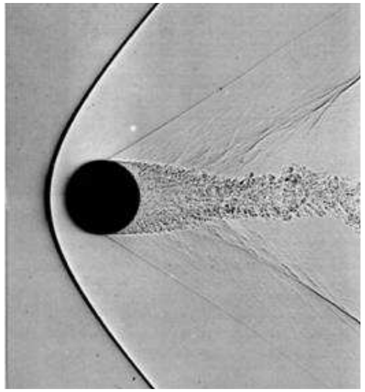
- Compressible Flow versus Incompressible Flow
  - Shock waves



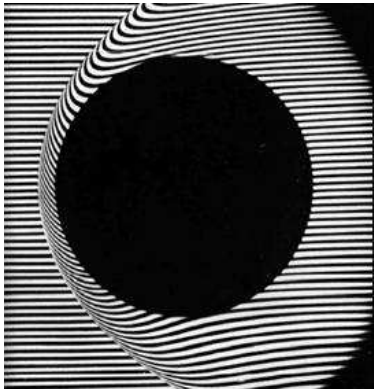
Bullet at Mach 1.5



Airplane model at Mach 1.1



Sphere (Mach 1.53)



Sphere (Mach 5.7)

F/A-18 Hornet

Classification of Fluid Flows

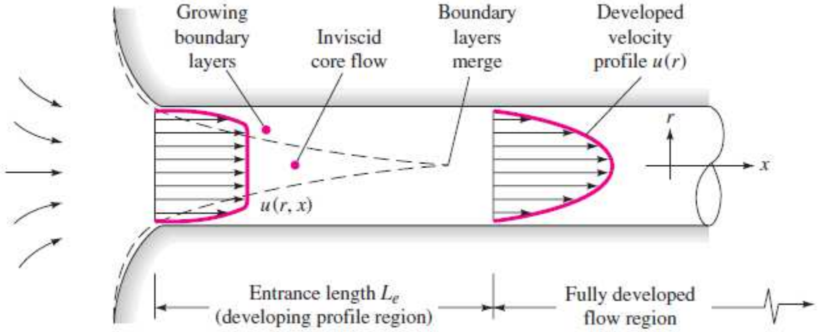
- Compressible Flow versus Incompressible Flow
  - Shock waves

F18

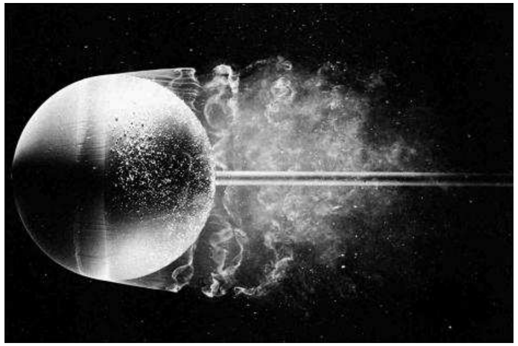
F14

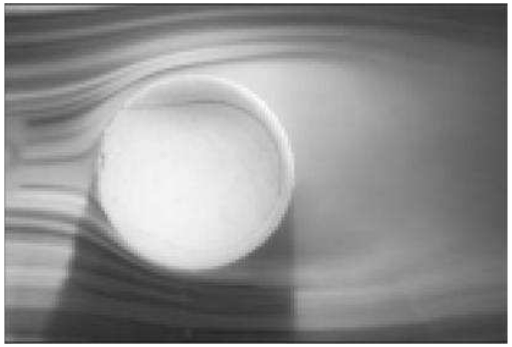
Classification of Fluid Flows

- Internal versus External Flow
  - Internal flow: fluid flow is completely bounded by solid surfaces; e.g. flow in a pipe or duct



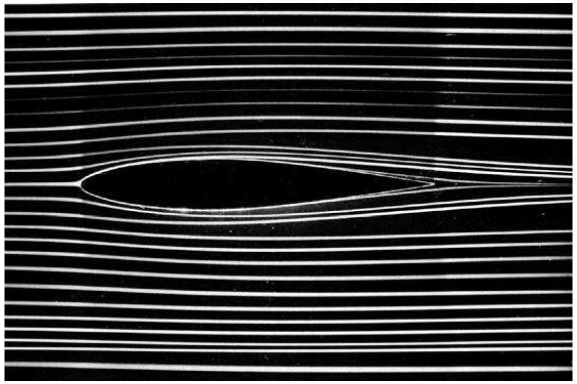
  - External flow: flow of an unbounded fluid over a surface; e.g. flow past sphere



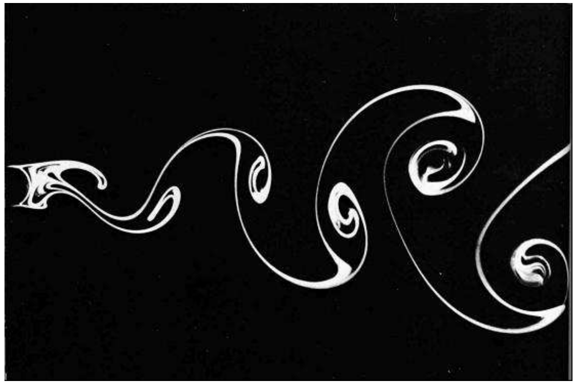


Classification of Fluid Flows

- Steady Flow versus Unsteady Flow
  - Steady flow: flow at any point does not change with time

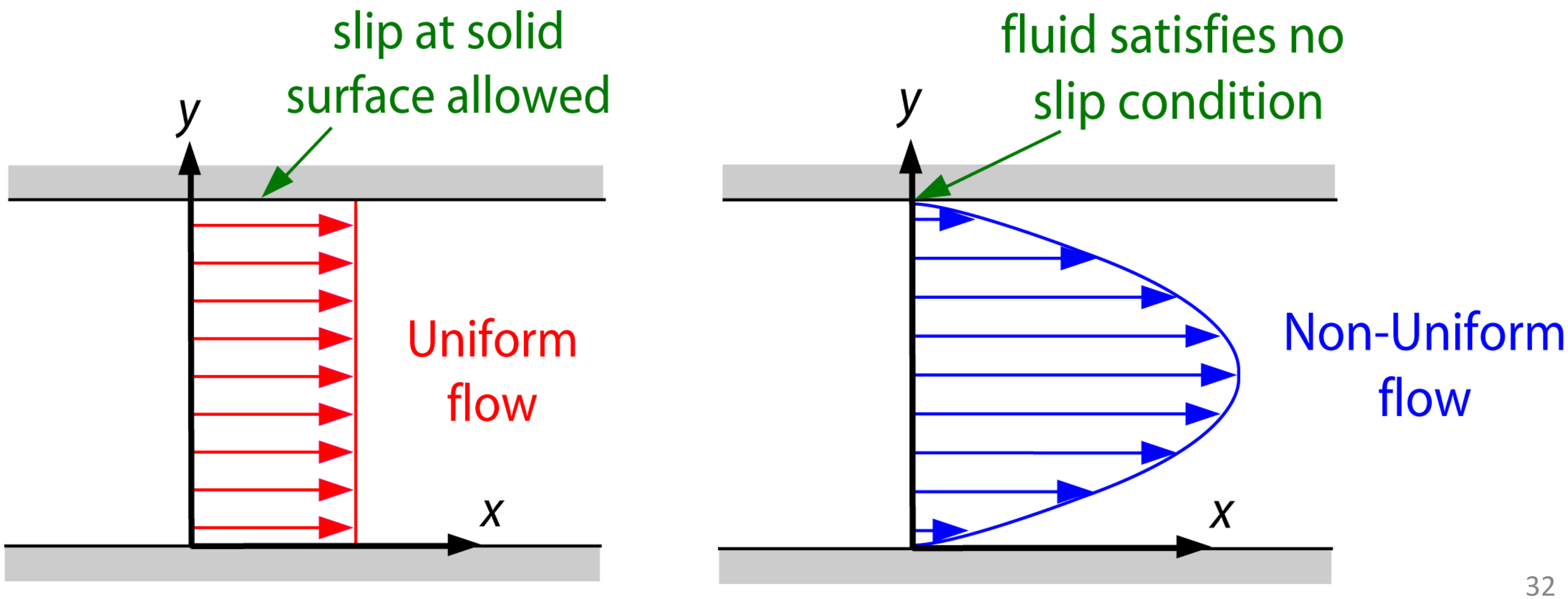


  - Unsteady flow: flow pattern varies with time



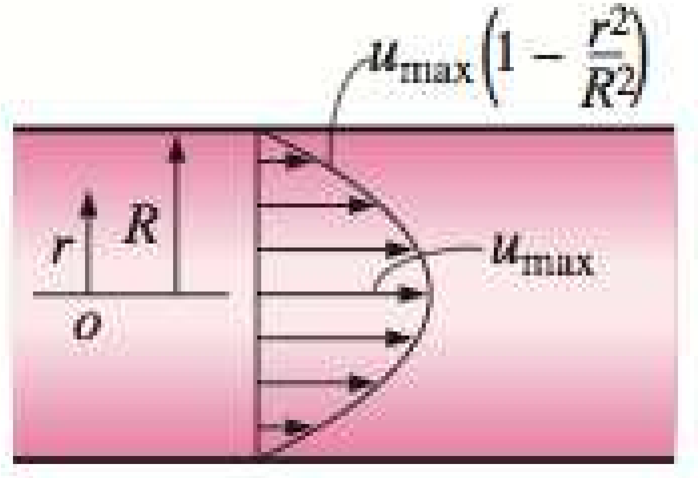
Classification of Fluid Flows

- Uniform versus Non-uniform Flow
  - Uniform flow $\Rightarrow$ flow in which velocity does not vary with location $\Rightarrow$ eg. inviscid flow through a channel:
  - Non-uniform flow $\Rightarrow$ flow in which velocity varies with location $\Rightarrow$ eg. viscous flow through a channel:



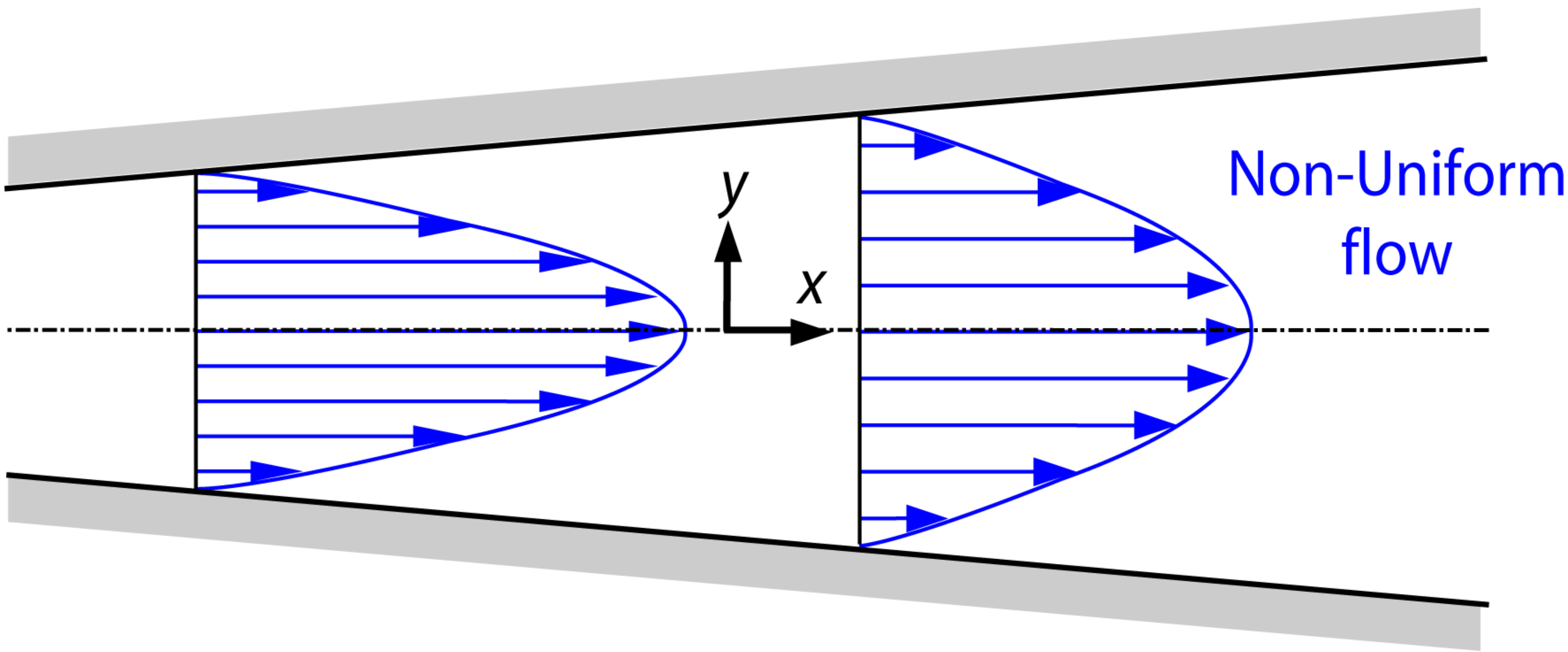
Classification of Fluid Flows

- One-, Two- and Three-Dimensional Flow
  - A flow is said to be one-, two- or three dimensional if flow velocity varies in one, two or three primary dimensions, respectively
  - One-dimensional flow: Fully developed flow in a circular pipe $\Rightarrow$ flow velocity only varies in the r direction



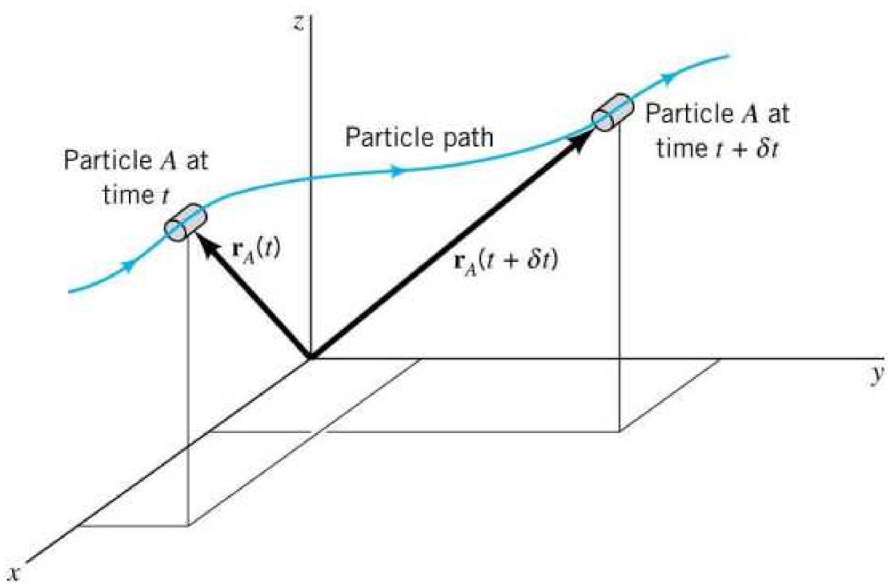
Classification of Fluid Flows

- One-, Two- and Three-Dimensional Flow
  - Two-dimensional $\Rightarrow$ flow Velocity varies in both x- and y-directions



Basic Concept for Fluid Motion

- Material Derivative
  - In the Eulerian method, the fundamental property is the velocity field. The velocity field does not track the behaviour of individual particle, it describes the velocity of whatever happens to be at a given location. To do dynamics, need to apply $F = ma$. Getting the acceleration is not trivial



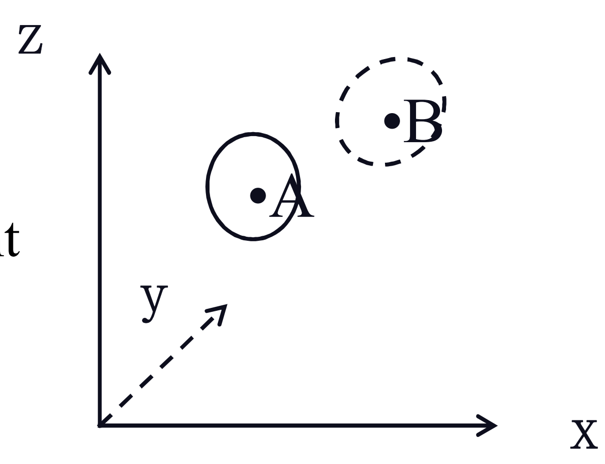
Basic Concept for Fluid Motion

- Material Derivative
  - Also called substantial derivative
  - Assume a fluid particle in a system with physical property $N(x,y,z,t)$ moves from position A to position B after a time interval Δt
  - The material derivative can be calculated as,

$$\frac{DN}{Dt} = \lim_{\Delta t \to 0} \frac{N(B, t+\Delta t) - N(A, t)}{\Delta t}$$

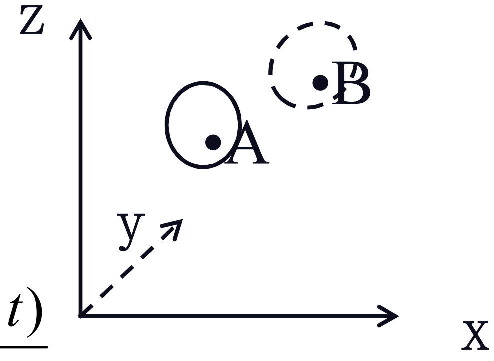
$\frac{D(\ )}{Dt}$ is used here because it is different

from the simple derivative $\frac{d(\ )}{dt}$



Basic Concept for Fluid Motion

- Material Derivative
  - Two types of contributions for the variation in *N*:
    - ✓ Unsteady effect: Variation in *N* due to the time rate-of-change during the time interval Δt
    - ✓ Spatial effect: Variation in *N* due to the non-uniform feature of the field when the fluid particle moves from positions A to B



$$\frac{DN}{Dt} = \lim_{\Delta t \to 0} \frac{N(B, t+\Delta t) - N(A,t)}{\Delta t}$$

$$= \underbrace{\lim_{\Delta t \to 0} \frac{N(B, t+\Delta t) - N(B,t)}{\Delta t}}_{\text{Unsteady Effect}} + \underbrace{\lim_{\Delta t \to 0} \frac{N(B,t) - N(A,t)}{\Delta t}}_{\text{Non-uniform Effect}}$$

$$= \lim_{\Delta t \to 0} \frac{N(B, t+\Delta t) - N(B,t)}{\Delta t} + \lim_{\Delta t \to 0} \frac{AB}{\Delta t} \bullet \lim_{AB \to 0} \frac{N(B,t) - N(A,t)}{AB}$$

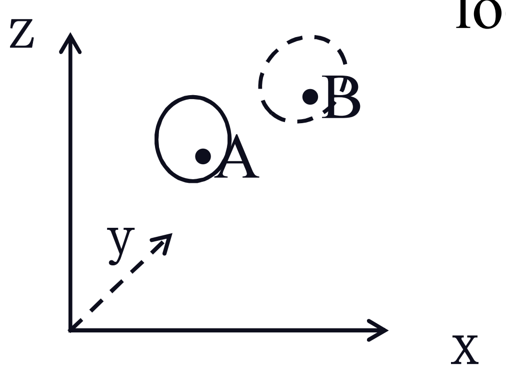
Basic Concept for Fluid Motion

- Material Derivative
  - Two types of contributions for the variation in *N*:

$$\frac{DN}{Dt}=\lim_{\Delta t\to 0}\frac{N(B,t+\Delta t)-N(B,t)}{\Delta t}+\lim_{\Delta t\to 0}\frac{AB}{\Delta t}\bullet\lim_{AB\to 0}\frac{N(B,t)-N(A,t)}{AB}$$

$\Delta t\to 0$ $\Big\downarrow$ $A\to B$ $\qquad$ $\Big\downarrow$ *velocity* $\qquad$ $\Big\downarrow$ $A\to B$

$$\underbrace{\frac{\partial N(B,t)}{\partial t}}_{\text{local derivative}}\qquad\qquad \underbrace{\vec{V}\bullet\frac{\partial N(B,t)}{\partial s}}_{\text{convective derivative}}$$

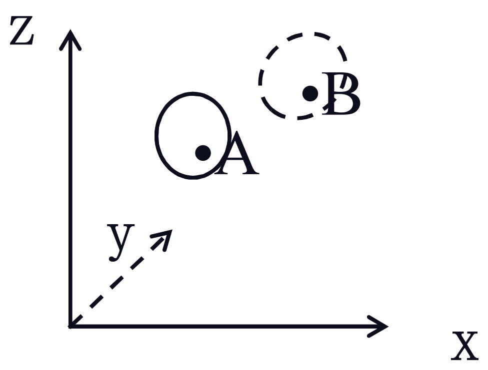


Basic Concept for Fluid Motion

- Material Derivative
  - Mathematical deduction by using the chain rule:

$$\frac{DN}{Dt}=\frac{\partial N}{\partial t}+\frac{\partial N}{\partial x}\frac{dx}{dt}+\frac{\partial N}{\partial y}\frac{dy}{dt}+\frac{\partial N}{\partial z}\frac{dz}{dt}$$

$$=\frac{\partial N}{\partial t}+u\frac{\partial N}{\partial x}+v\frac{\partial N}{\partial y}+w\frac{\partial N}{\partial z}=\frac{\partial N}{\partial t}+(\vec{V}\bullet\nabla)N$$



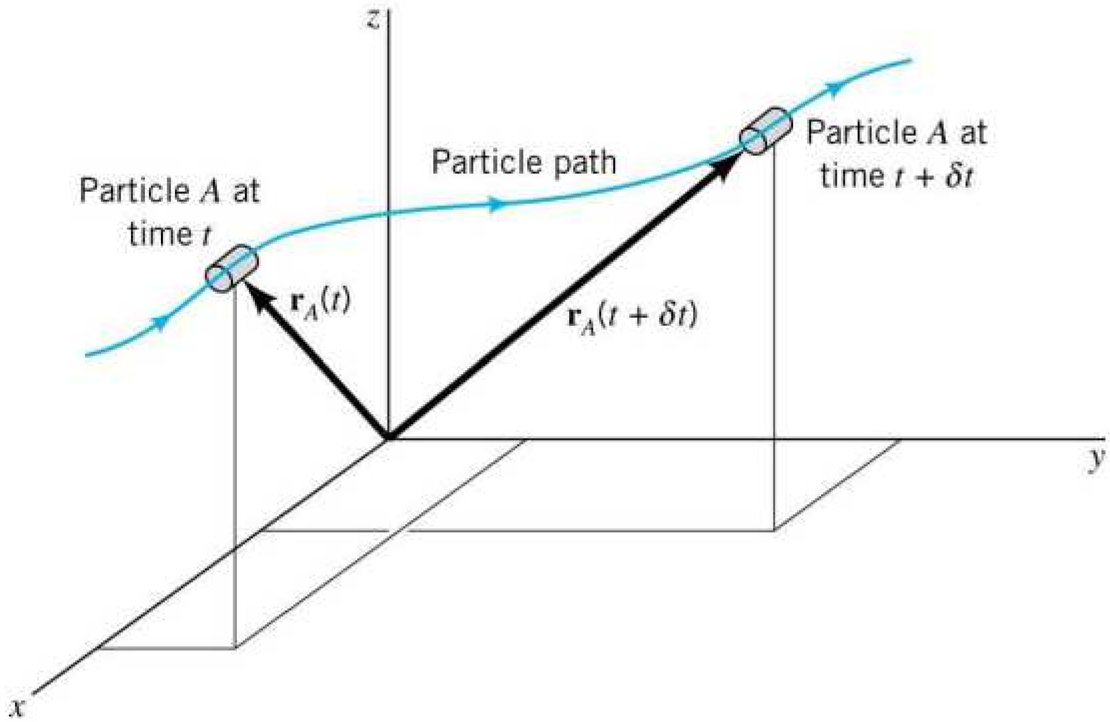
$$\nabla=\vec{i}\,\frac{\partial}{\partial x}+\vec{j}\,\frac{\partial}{\partial y}+\vec{k}\,\frac{\partial}{\partial z}$$

为哈密顿算子符（Hamiltonian)

Basic Concept for Fluid Motion

- Material Derivative
  - Acceleration of Fluid Particle:
    - ✓ For particle A , $x_A(t)$, $y_A(\mathrm{t})$, $z_A(t)$ describe the motion of the particle.

$$\vec{V}_A = \vec{V}_A\left(x_A\left(t\right), y_A\left(t\right), z_A\left(t\right), t\right)$$



Basic Concept for Fluid Motion

- Material Derivative
  - Acceleration of Fluid Particle:

$$
\begin{aligned}
a_A &= \frac{D\vec{V}_A\left(x_A(t), y_A(t), z_A(t), t\right)}{Dt} \\
&= \frac{\partial \vec{V}_A}{\partial t} + \frac{\partial \vec{V}_A}{\partial x}\frac{dx_A(t)}{dt} + \frac{\partial \vec{V}_A}{\partial y}\frac{dy_A(t)}{dt} + \frac{\partial \vec{V}_A}{\partial z}\frac{dz_A(t)}{dt} \\
&= \frac{\partial \vec{V}_A}{\partial t} + u_A(t)\frac{\partial \vec{V}_A}{\partial x} + v_A(t)\frac{\partial \vec{V}_A}{\partial y} + w_A(t)\frac{\partial \vec{V}_A}{\partial z}
\end{aligned}
$$

  - Since A is any particle, the acceleration field is

$$
\vec{a} = \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}} + \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}}
$$

Basic Concept for Fluid Motion

- Material Derivative
  - Acceleration of Fluid Particle:
    - ✓ The component of the vector ***a***:

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

- ➢ Unsteady effect: the time rate-of-change of the velocity component at any specified spatial location
- ➢ Spatial effect: spatial changes in velocity (or any other fluid property) due to motion of a fluid parcel being carried (convected) by the flow field (u, v, w) $\Rightarrow$ fluid parcels may be accelerating even in a steady (time-independent) flow field

Basic Concept for Fluid Motion

- Material Derivative
  - General Form:

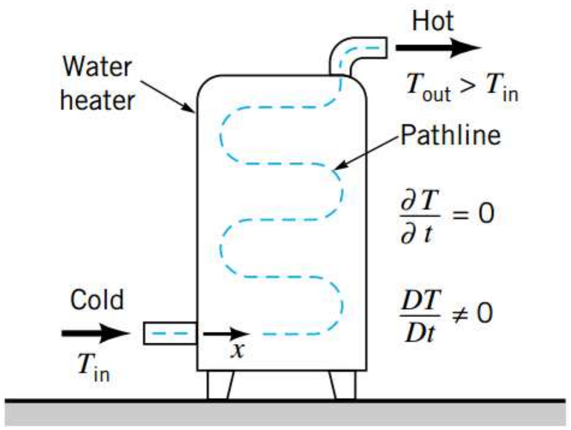
$$\frac{D(\;)}{Dt} = \frac{\partial(\;)}{\partial t} + u\frac{\partial(\;)}{\partial x} + v\frac{\partial(\;)}{\partial y} + w\frac{\partial(\;)}{\partial z} = \frac{\partial(\;)}{\partial t} + (\vec{V} \bullet \nabla)(\;)$$

  - One can define the material derivative for other properties for a fluid, e.g. temperature or pressure

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{\partial T}{\partial t} + (\vec{V} \bullet \nabla)(T)$$

Basic Concept for Fluid Motion

- Material Derivative
  - Example for convective derivative
    - ✓Consider water going through a water heater under steady state flow conditions
    - ✓The water temperature at any fixed location is fixed



$$\frac{\partial T}{\partial t} = 0$$

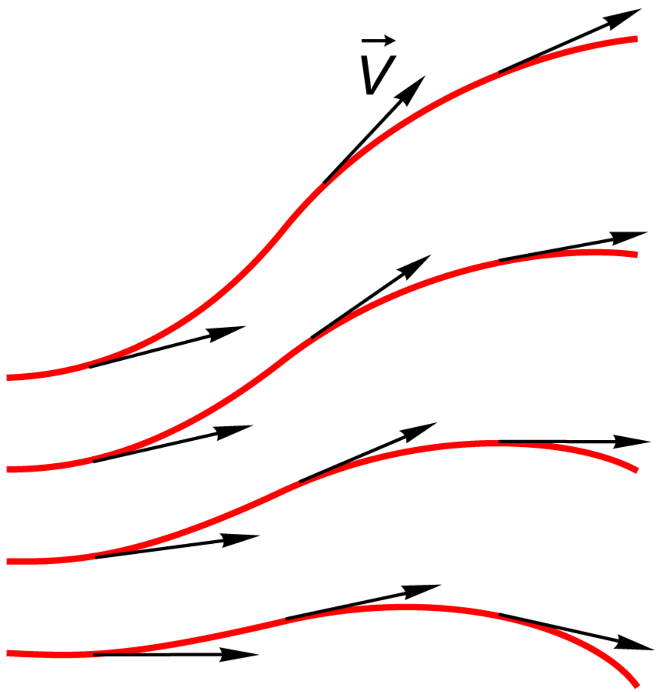
- ✓However, the water temperature for a given piece of water will increase as it progresses through the heater. The rate of change is

$$\frac{dT}{dt} = \begin{pmatrix} \text{Rate at which} \\ T \text{ changes} \\ \text{with postion} \end{pmatrix} \times \begin{pmatrix} \text{How quickly} \\ \text{water changes} \\ \text{postion} \end{pmatrix}$$

$$= \frac{\partial T}{\partial s} u_s$$

Flow Visualization

- Streamlines
  - A streamline is a curve that is everywhere tangent to the instantaneous local velocity vector – a mathematical concept
  - Typical set of streamlines

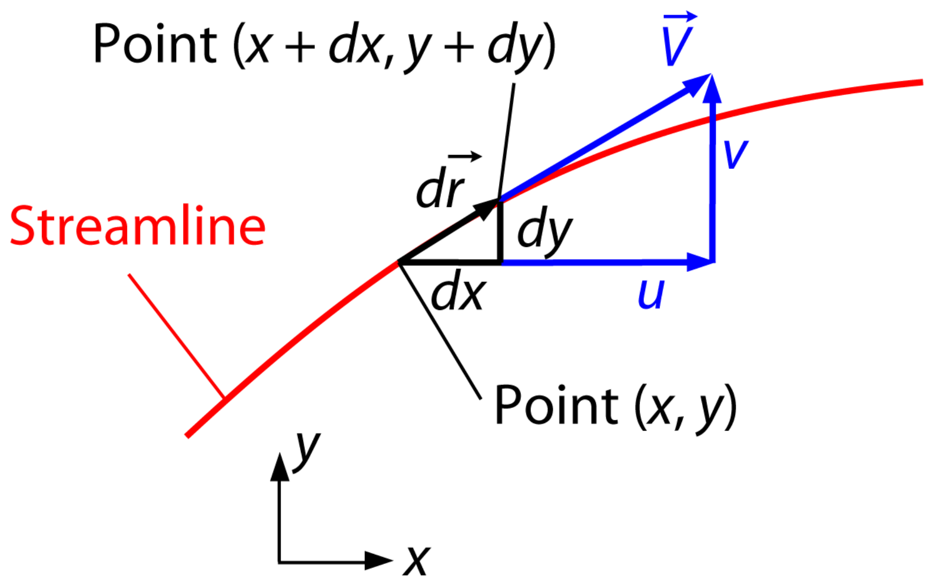


Flow Visualization

- Streamlines
  - Streamlines are everywhere parallel to the local velocity
  - Fluid cannot cross a streamline by definition
  - Any particle starting on one streamline will stay on that same streamline
  - Streamlines cannot cross each other
  - Fluid flowing past a solid boundary does not flow into or out of the solid surface
  - Close to a solid boundary, streamlines are parallel to that boundary
  - Streamlines are difficult to generate experimentally

Flow Visualization

- Streamlines
  - Calculation of streamlines
    - ✓ Consider an infinitesimal arc length $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ along a streamline



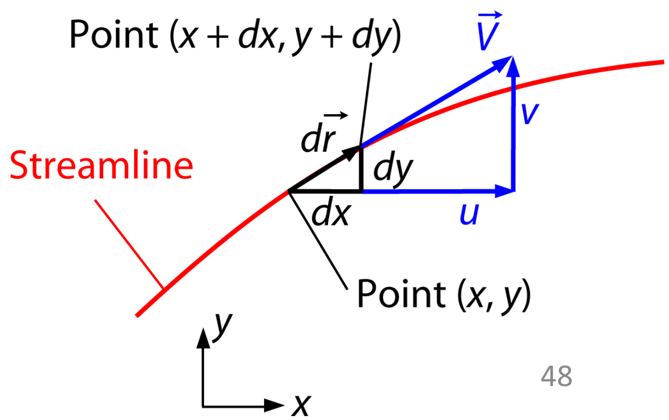
Flow Visualization

- Streamlines
  - Calculation of streamlines
    - ✓ According to definition of a streamline, $d\vec{r}$ is parallel to local velocity vector $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$
    - ✓ Similar triangles ⇒ components of $d\vec{r}$ proportional to components of $\vec{V}$

$$\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

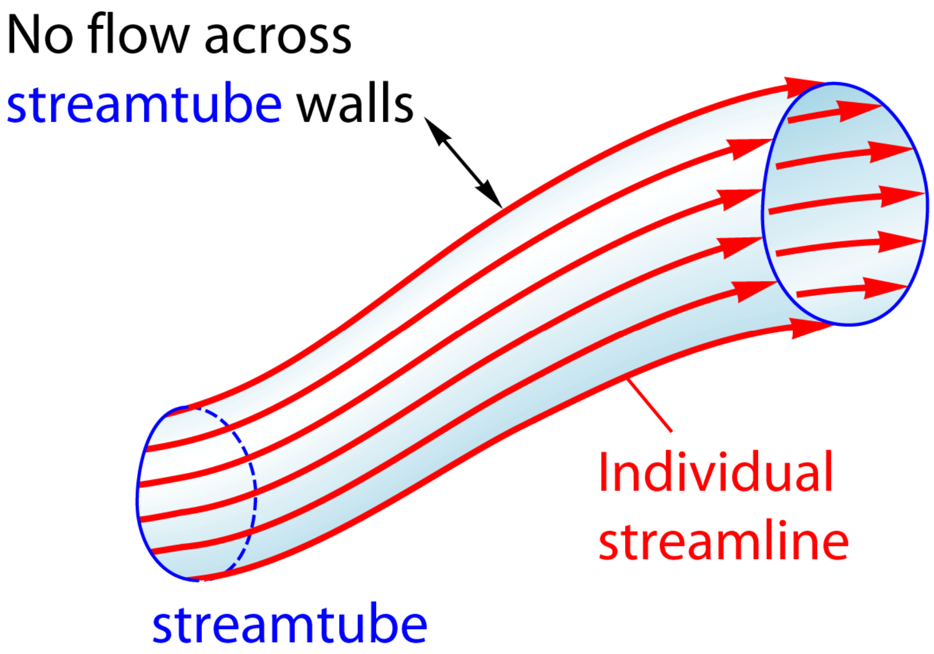
where dr and V are magnitudes of $d\vec{r}$ and $\vec{V}$, respectively

    - ✓ If the velocities (u, v, w) are known functions of position and time, the above equation can be integrated to find the streamline passing through the initial point (x_0, y_0, z_0, t_0)



Flow Visualization

- Streamtube
  - Streamtube consists of a bundle of streamlines
    - ✓ The walls of a streamtube are streamlines
    - ✓ Fluid cannot flow across a streamline, so fluid cannot cross a streamtube wall
    - ✓ Streamtube walls need not be solid but may be fluid surfaces



Flow Visualization

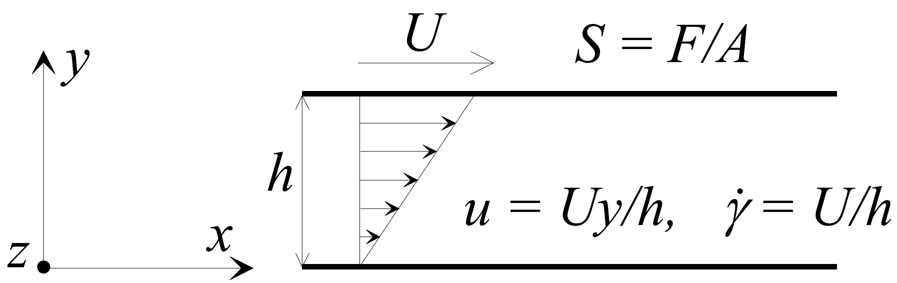
- Streamline and Streamtube
  - Both streamlines and streamtubes are instantaneous quantities, defined at a particular instant in time according to the velocity field at that instant
  - In an unsteady flow, streamline and streamtube pattern may change significantly with time
  - In a steady flow, the positions of streamlines and streamtubes do not change
  - Streamlines are visualized by taking a short-time exposure of fluid particles – each will trace out a velocity vector

Flow Visualization

- Example 2
  - Question
    - ✓ Consider the 1-dimensional steady shear flow

$$u = U\frac{y}{h}, \quad v = w = 0$$

    - ✓ To find the streamlines

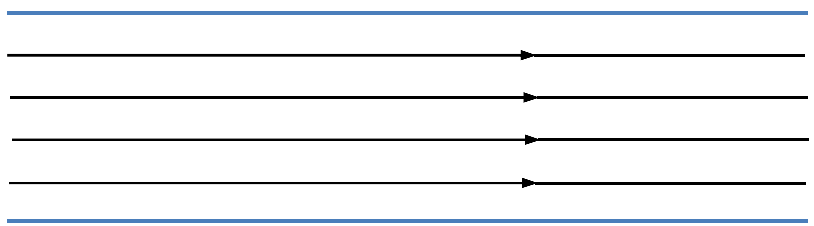


Flow Visualization

- Example 2
  - Solution
    - ✓ The streamlines are found by solving

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad \frac{dy}{dx} = 0 \qquad\qquad u = U\frac{y}{h}, \quad v = w = 0$$

    - ✓ This yields y = constant for the streamlines (straight lines parallel to the flow direction)

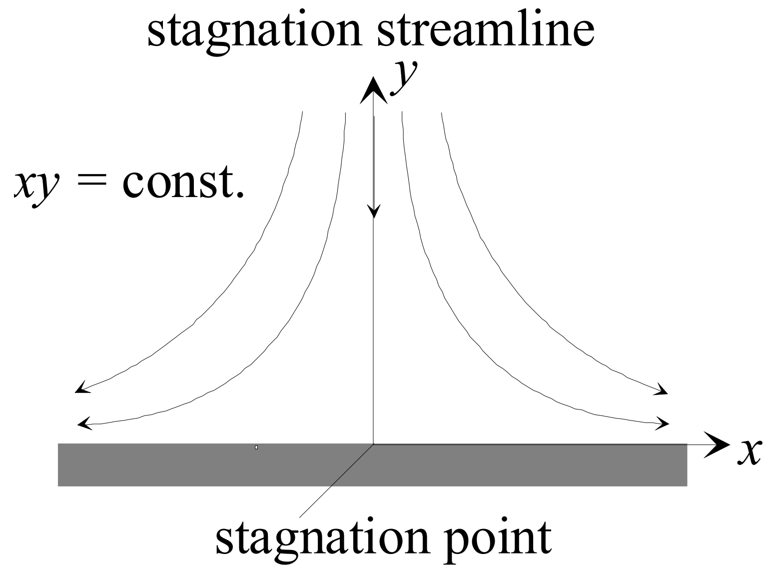


Flow Visualization

- Example 3
  - Question
    - ✓ Consider the following steady 2-dimensional flow (bi-axial flow)

$$\mathbf{v} = \{u, v, w\}, \quad u = \dot{\gamma} x, \quad v = -\dot{\gamma} y, \quad w = 0$$

    - ✓ To find the streamlines



Flow Visualization

- Example 3
  - Solution

✓ The streamlines are found by solving

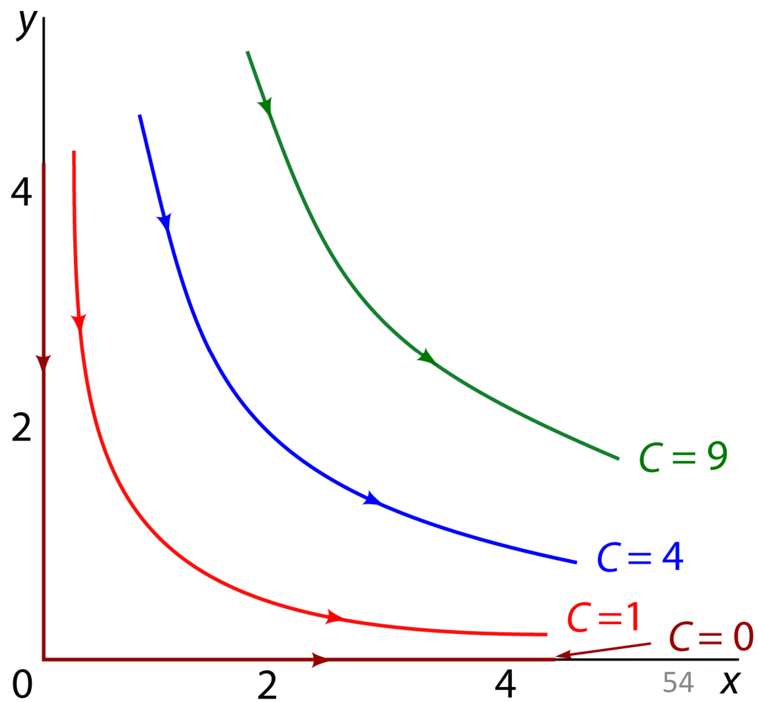
$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad \frac{dx}{\dot{\gamma} x} = -\frac{dy}{\dot{\gamma} y} \quad \text{or} \quad \ln y = -\ln x + \text{constant}$$

$$\therefore \quad xy = C$$

✓ These streamlines are hypebolas

✓ Use different values of C $\Rightarrow$ plot various lines in x-y plane $\Rightarrow$ streamlines

✓ Arrows indicate flow direction



Flow Visualization

- Example 4
  - Question
    - ✓ Consider the 2-dimensional unsteady flow

$$\mathbf{v} = \{u, v, w\}, \quad u = x, \quad v = yt, \quad w = 0$$

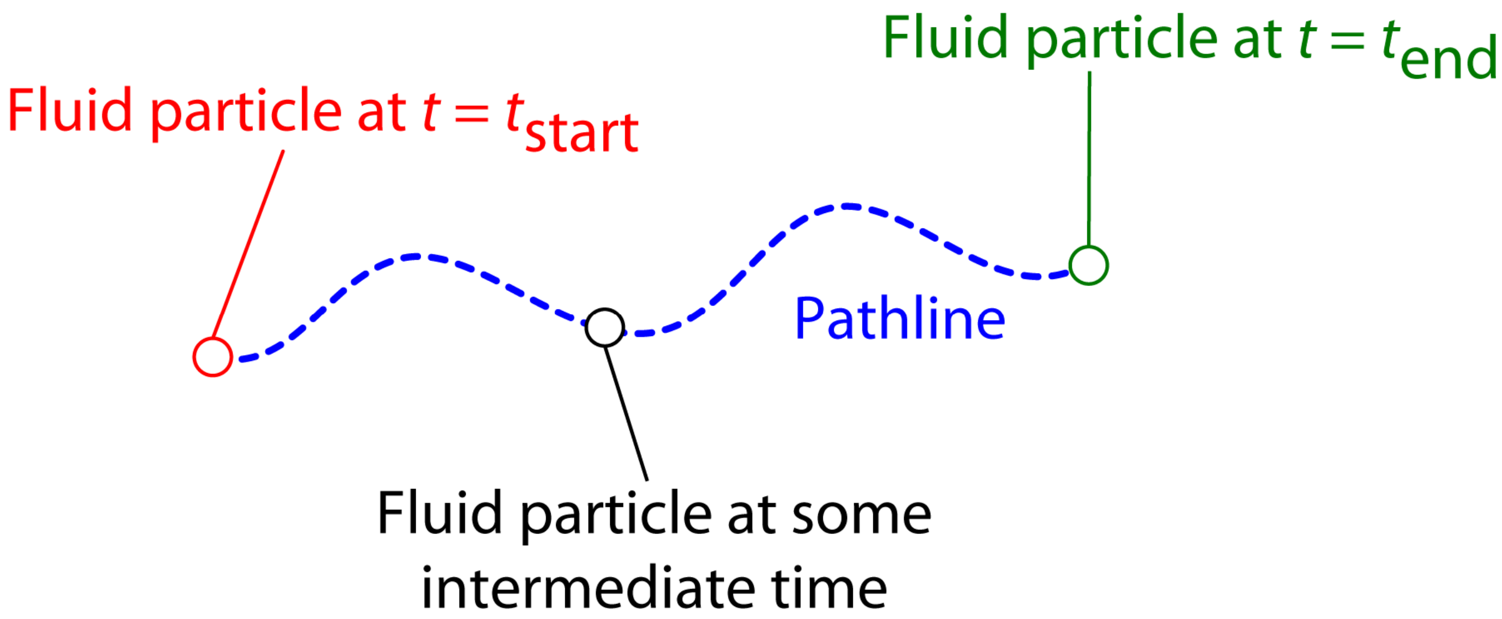
    - ✓ To find the streamlines (assume streamline pass (1,1))
  - Solution
    - ✓ The streamlines are found by solving

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad \frac{dx}{x} = \frac{dy}{ty} \quad \therefore \ \ln y = t \ln x + \text{constant} \quad \therefore \ y = Cx^t$$

    - ✓ The streamline passing through (1,1) has C = 1

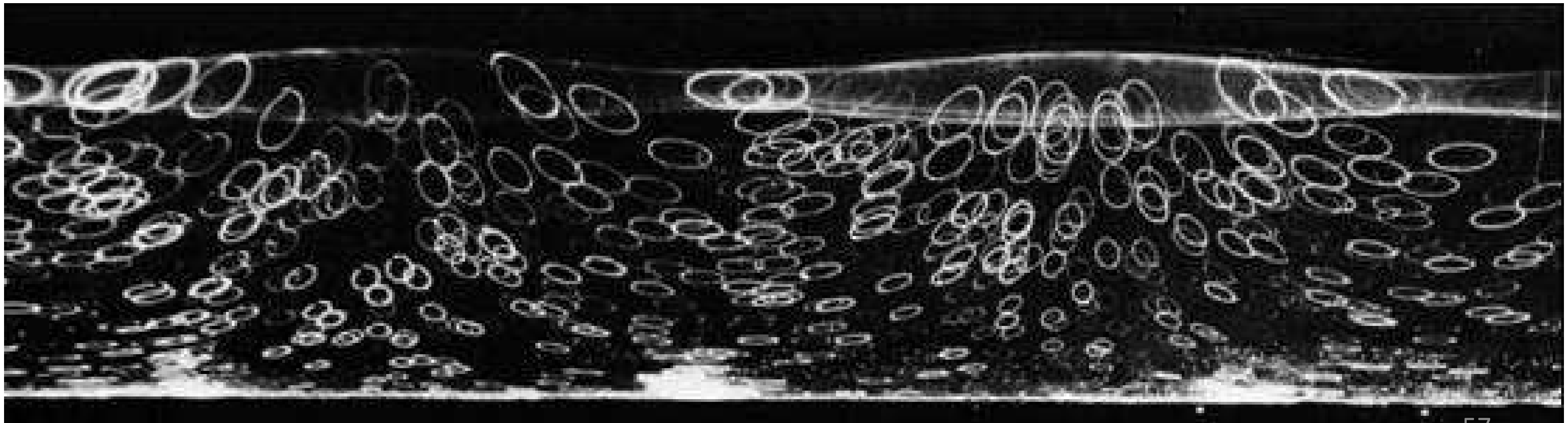
Flow Visualization

- Pathline
  - A pathline is the actual path travelled by an individual fluid over some time interval
  - Pathline is a Lagrangian concept $\Rightarrow$ the path of an individual fluid particle is tracked as it moves around in the flow field



Flow Visualization

- Pathline
  - Pathlines can be generated experimentally by marking a fluid particle (dying a small fluid element) and taking a long-time time exposure photograph of its motion through the flow
  - Example: waves moving along surface of water in a tank $\Rightarrow$ pathlines are elliptical in shape



Flow Visualization

- Pathline
  - Calculation of pathlines
    - ✓ Pathline is defined by integration of the velocity components:

$$\frac{d}{dt}\vec{x} = \vec{V}, \quad \vec{x}(t_0) = \vec{x}_0 \quad \begin{cases} x(t) = x_0 + \int_{t_0}^{t} u\,dt \\ y(t) = y_0 + \int_{t_0}^{t} v\,dt \\ z(t) = z_0 + \int_{t_0}^{t} w\,dt \end{cases}$$

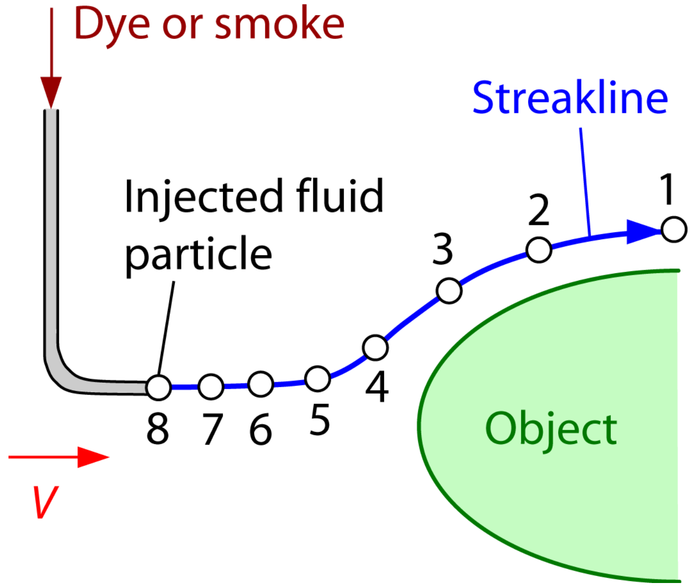
or in vector notation:

$$\vec{x}(t) = \vec{x}_0 + \int_{t_0}^{t} \vec{V}\,dt$$

    - ✓ Given velocity field (u, v, w, t), the integration is started at a specified initial position and time (x_0, y_0, z_0, t_0)

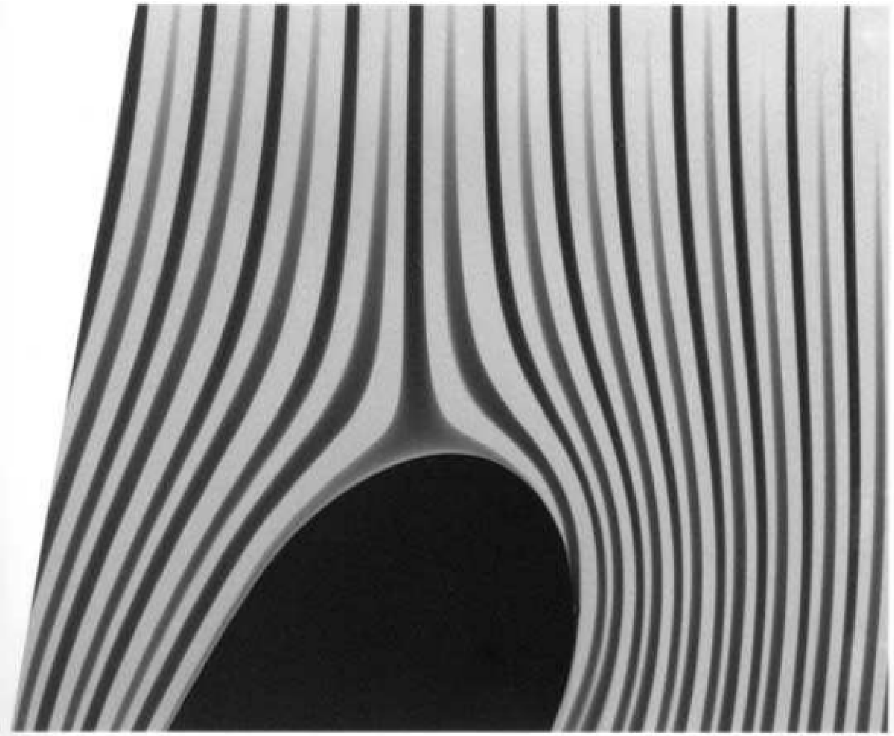
Flow Visualization

- Streaklines
  - A streakline is the locus of fluid particles that have passed sequentially through a prescribed point in the flow
  - Main point: locus of particles passing through one common point in space
  - Streaklines are the most flow pattern generated in physical experiments

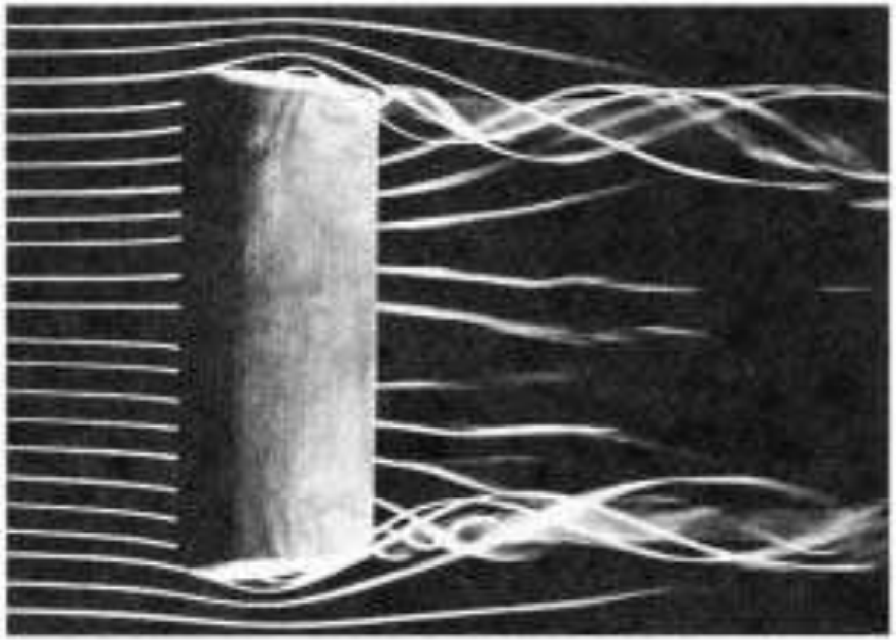


Flow Visualization

- Streaklines



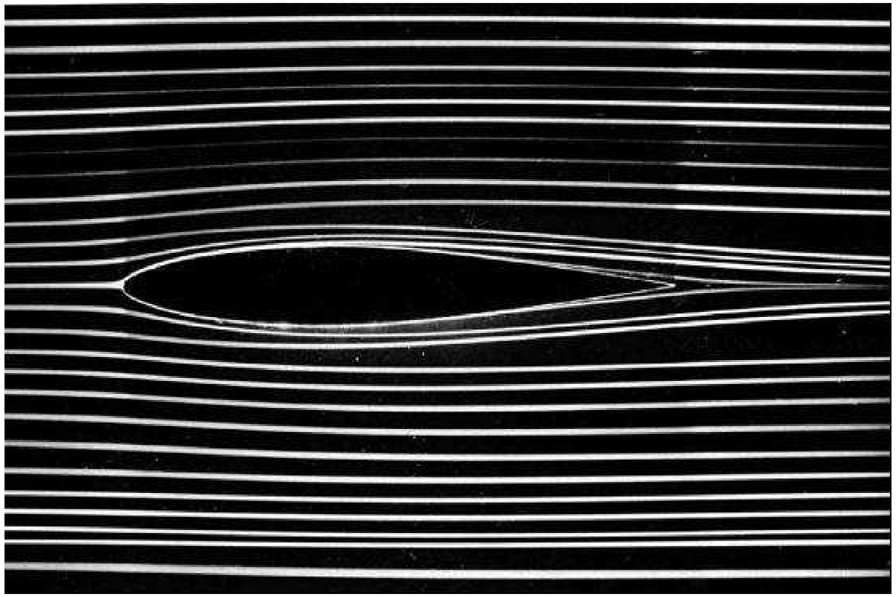
Streaklines of dye moving past obstruction.
They are also the streamlines for the flow.



Streaklines of smoke moving past obstruction

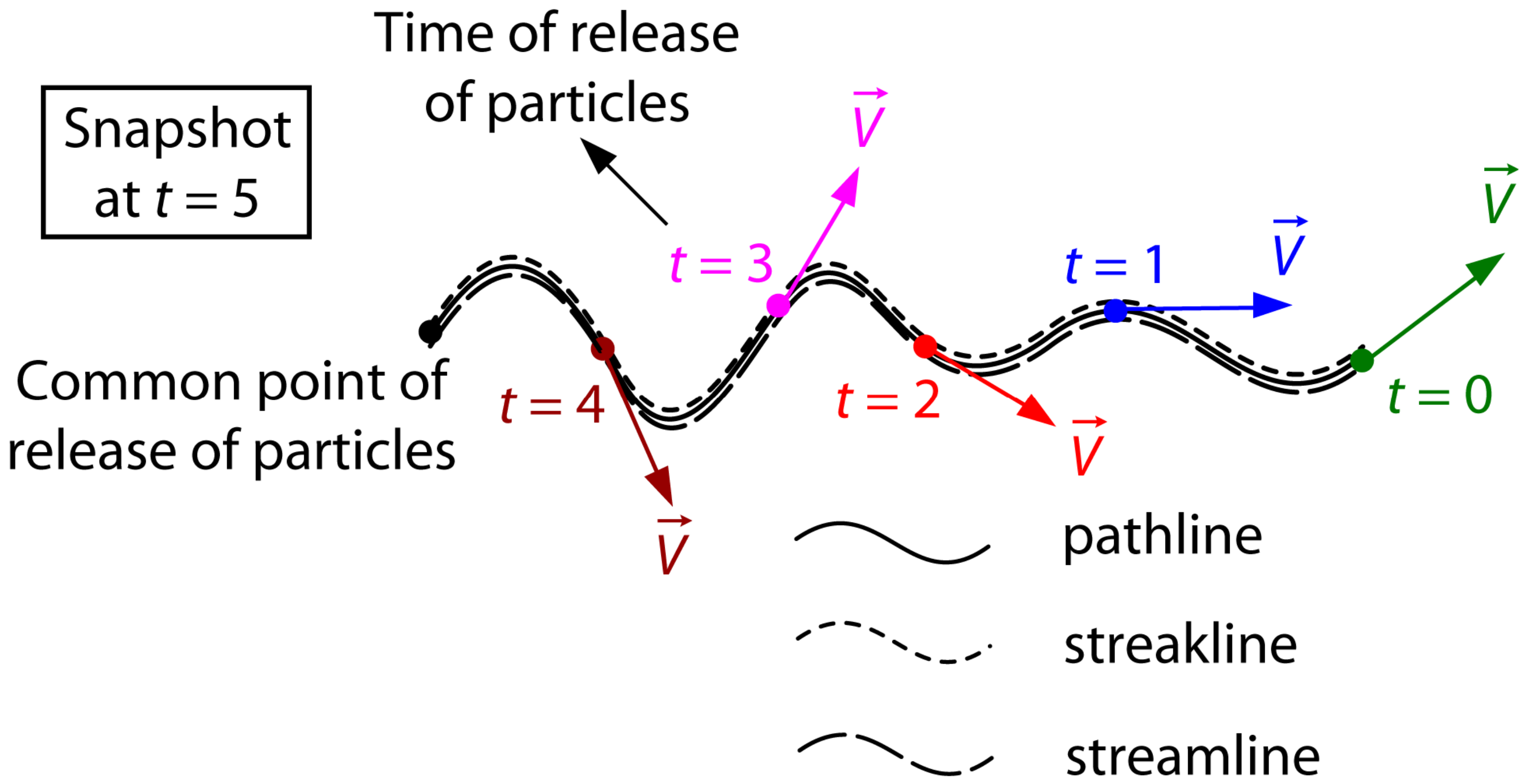
Flow Visualization

- Steady Flow
  - Streamlines, pathlines and streaklines are identical
  - Path taken by a marked particle (pathline) is the same as line formed by all other particles that previously passed through point of injection (streakline) $\Rightarrow$ these lines are in turn tangent to the velocity field (streamlines)



Flow Visualization

- Steady Flow
  - Example

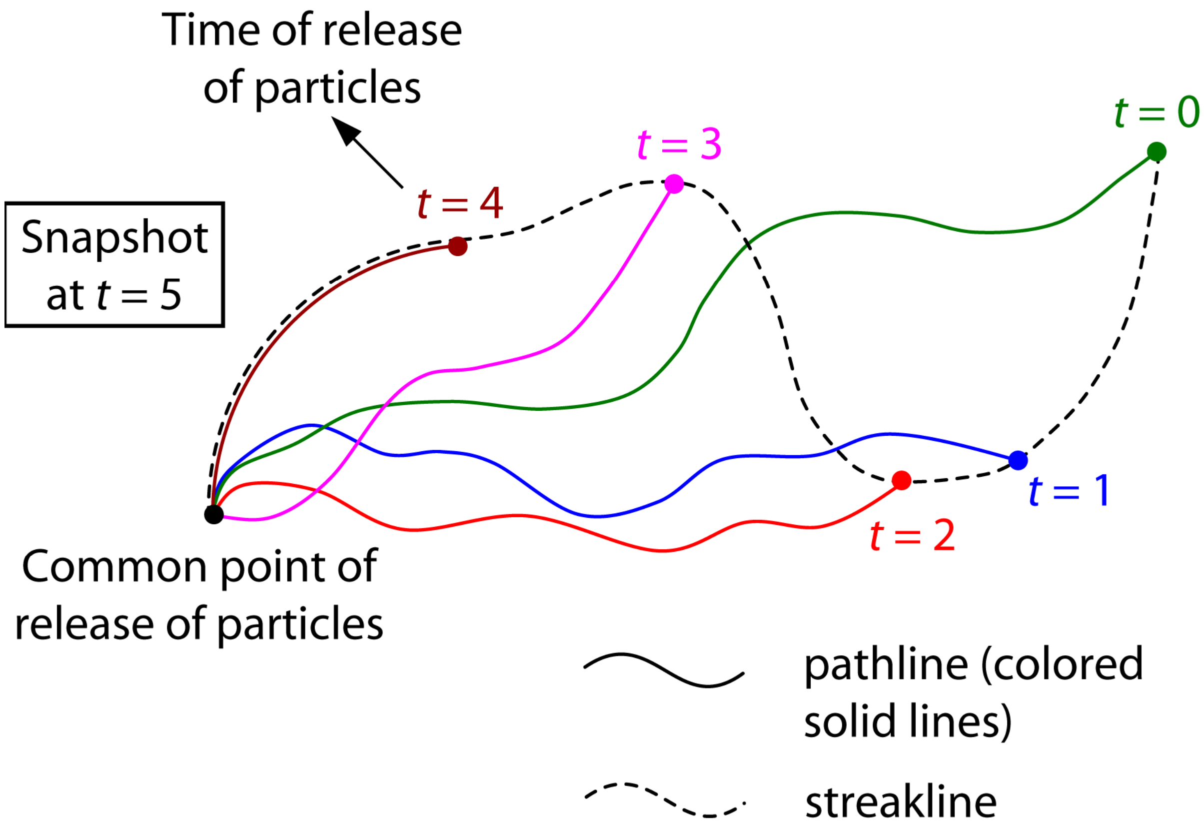


Flow Visualization

- Unsteady Steady Flow
  - Streamline represents instantaneous flow pattern at given instant of time
  - Streakline and pathline are flow patterns generated by passage of time, which means age and time history associated with flow pattern
  - Streakline is instantaneous snapshot of time-integrated flow pattern
  - Pathline is time-exposed flow path of individual particle over some time interval

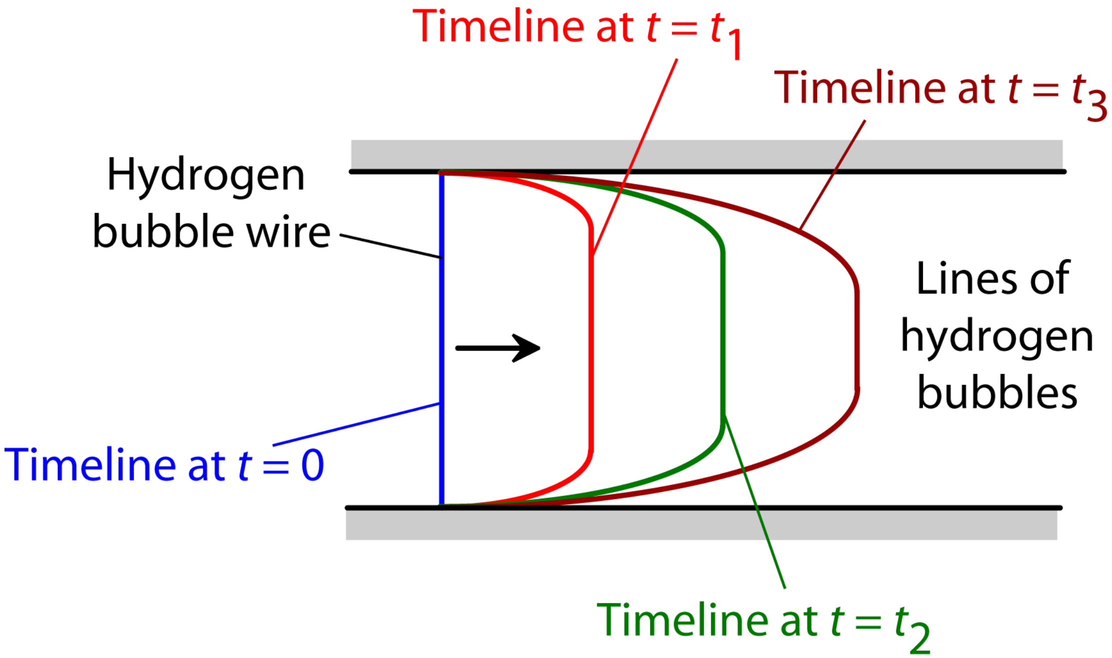
Flow Visualization

- Unsteady Steady Flow
  - Example



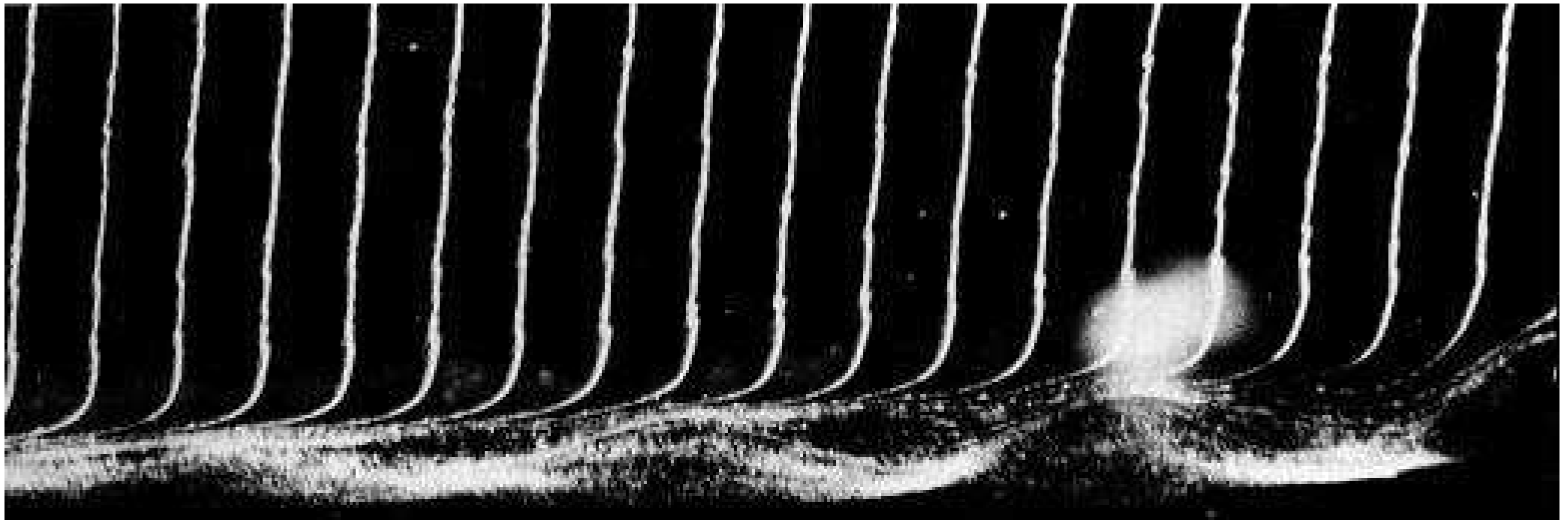
Flow Visualization

- Timelines
  - A timeline is a set of adjacent fluid particles that were marked at the same (earlier) instant in time
    - ✓ Timeline is useful for investigating uniformity of flow
    - ✓ Can be generated experimentally in water using hydrogen bubble wire



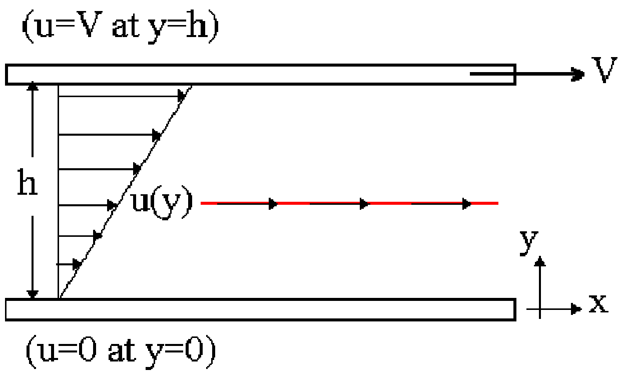
Flow Visualization

- Timelines
  - Example: Timelines produced by a hydrogen bubble wire in a boundary layer

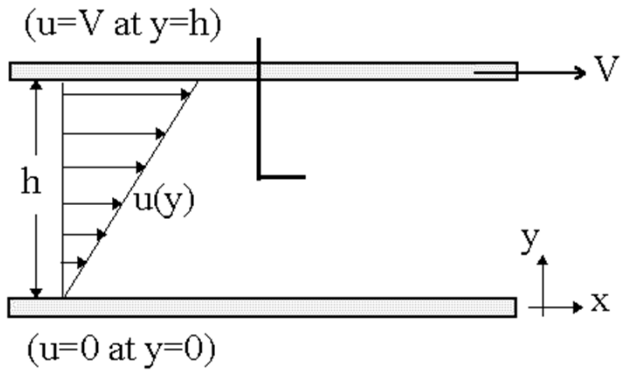


Flow Visualization

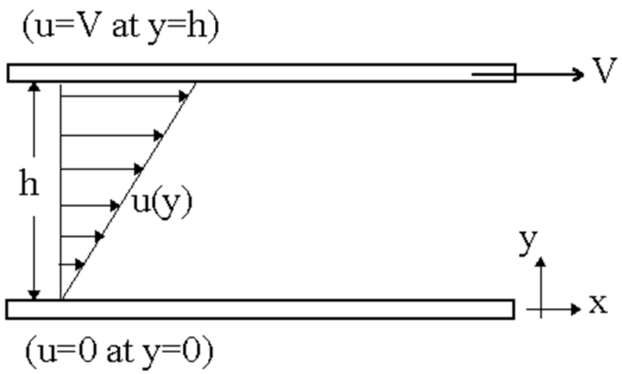
- Simple Shear Flow between Parallel Plates $u = V\dfrac{y}{h}$



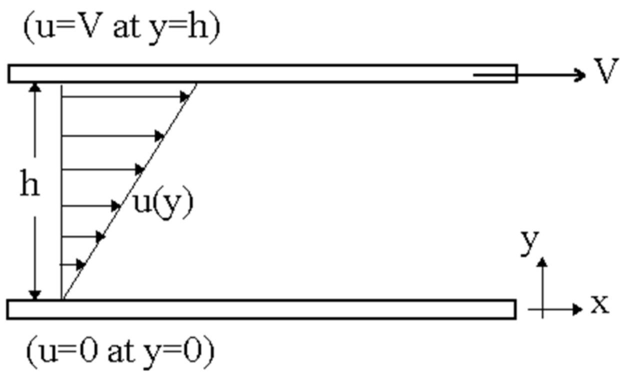
streamline



streakline



pathline



timeline

Flow Visualization

- Example 5
  - Question
    - ✓ Consider the two-dimensional velocity field

$$\mathbf{v} = \{u, v, w\}, \quad u = x, \quad v = yt, \quad w = 0$$

    - ✓ Find out the pathline passing through point (1,1)
  - Solution
    - ✓ The equation for pathlines

$$\frac{dx}{dt} = u = x \therefore \frac{dx}{x} = dt, \quad \frac{dy}{dt} = v = ty \therefore \frac{dy}{y} = tdt$$

$$x(0) = X, \; y(0) = Y$$

    - ✓ Solving these

$$\ln x = t + C_1, \quad \ln y = \frac{1}{2}t^2 + C_2$$

$$x = e^{t+C_1} = Xe^t, \quad y = e^{t^2/2+C_2} = Ye^{t^2/2}$$

Flow Visualization

- Example 5
  - Solution
    - ✓ The pathline passing through point (1,1) at time $t = 0$ is

$$1 = Xe^0 = X,\ 1 = Ye^0 = Y \therefore x = e^t,\ \ y = e^{t^2/2}$$

    - ✓ To find the streakline passing through (1,1) we note that any pathline passing through (1,1) at some point in time t_0 satisfies

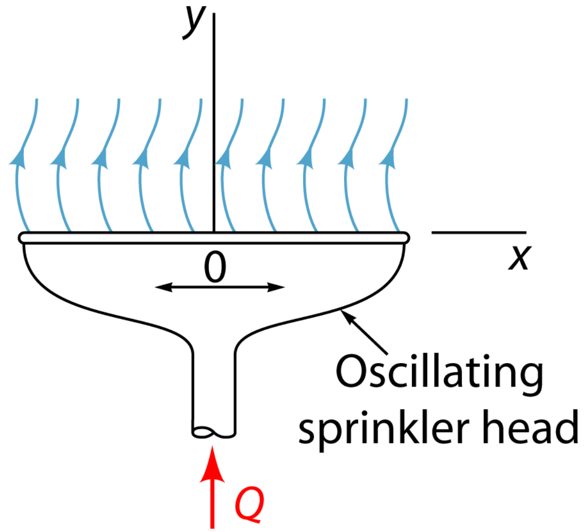
$$1 = Xe^{t_0},\ 1 = Ye^{t_0^2/2} \quad \therefore \quad X = e^{-t_0},\ Y = e^{-t_0^2/2}$$

    - ✓ This yields

$$x = e^{t-t_0},\ \ y = e^{\left(t^2-t_0^2\right)/2}$$

    - ✓ A plot of this for different t_0 will yield a family of streaklines, all pass through (1,1) at some point in time

Flow Visualization



- Example 6
  - Question
    - ✓ Water flows from oscillating slit produces velocity field

$$\vec{V} = u_0 \sin\left[\omega\left(t - y/v_0\right)\right]\hat{i} + v_0\hat{j}$$

where u_0, v_0, and ω are positive constants

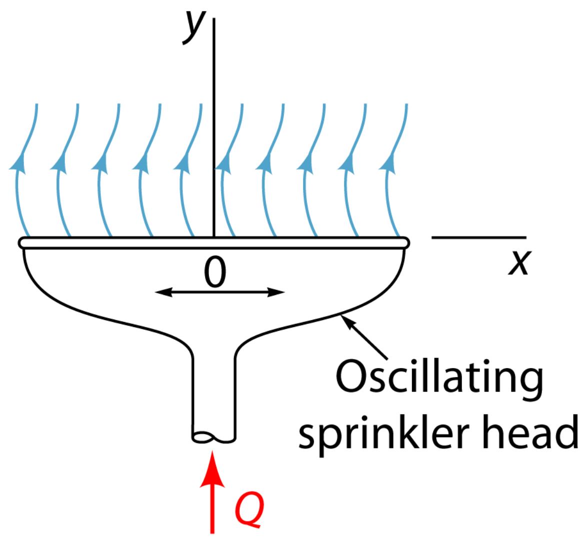
    - ✓ y-component of velocity remains constant
    - ✓ x-component of velocity at $y = 0$ coincides with velocity of oscillating sprinkler head

$$u = u_0 \sin\left(\omega t\right) \ @\ y = 0$$

  - Determine
    - a) streamline passing through origin at $t = 0$ and $t = \pi/2\omega$
    - b) pathline of particle at origin at $t = 0$ and $t = \pi/2\omega$
    - c) discuss shape of streakline passing through origin

Flow Visualization

- Example 6
  - Solution: Part (a)



✓Streamlines:

$$\frac{dy}{dx}=\frac{v}{u}=\frac{v_0}{u_0\sin\left[\omega\left(t-y/v_0\right)\right]}$$

✓Separating variables and integrating:

$$u_0\int\sin\left[\omega\left(t-y/v_0\right)\right]dy=v_0\int dx$$

$$u_0\left(v_0/\omega\right)\cos\left[\omega\left(t-y/v_0\right)\right]=v_0x+C$$

where C is a constant

✓Streamline passing through origin ($x = y = 0$) at $t = 0$:

$$C=u_0v_0/\omega$$

✓Equation of streamline passing through $x = y = 0$, at $t = 0$

$$x=\frac{u_0}{\omega}\left[\cos\left(\frac{\omega y}{v_0}\right)-1\right]$$

Flow Visualization

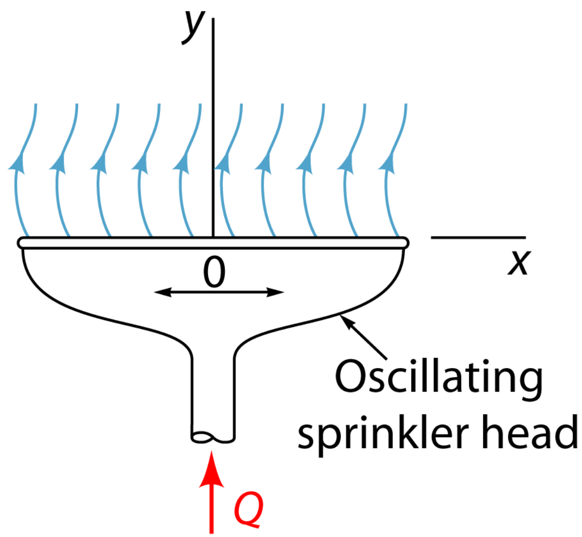
- Example 6
  - Solution: Part (a)
    - ✓ Streamline passing through origin ($x = y = 0$) at $t = \pi/2\omega$:

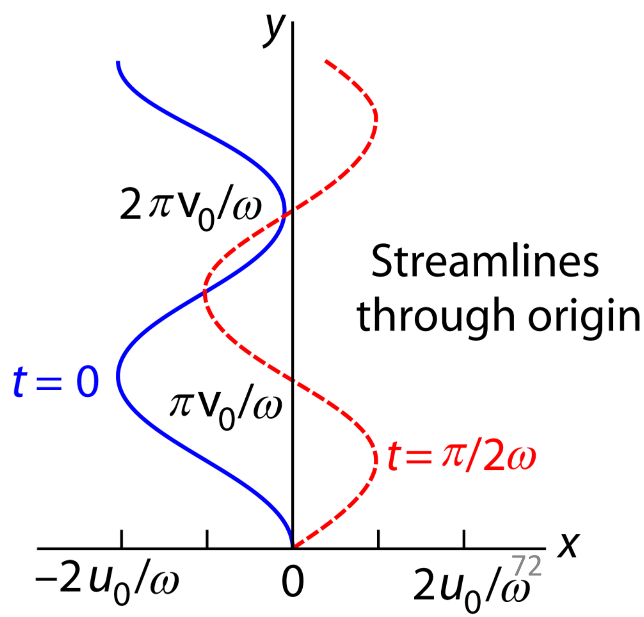
      $$C = 0$$

    - ✓ Equation of streamline:

      $$x = \frac{u_0}{\omega}\sin\left(\frac{\omega y}{v_0}\right)$$

    - ✓ Flow is unsteady: streamlines vary with time





Flow Visualization

- Example 6
  - Solution: Part (b)

✓ Pathlines:

$$u = \frac{dx}{dt} = u_0 \sin\left[\omega\left(t - \frac{y}{v_0}\right)\right]$$

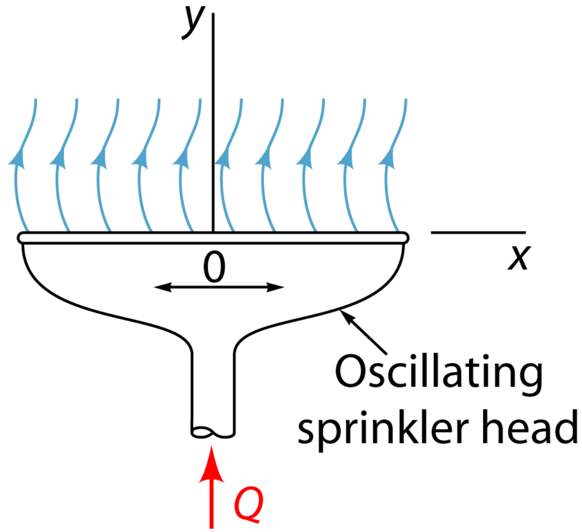
$$v = \frac{dy}{dt} = v_0 \Rightarrow y = v_0 t + C_1$$

where C_1 is a constant

$$\frac{dx}{dt} = u_0 \sin\left[\omega\left(t - \frac{v_0 t + C_1}{v_0}\right)\right] = -u_0 \sin\left(\frac{C_1 \omega}{v_0}\right)$$

$$\Rightarrow x = -\left[u_0 \sin\left(\frac{C_1 \omega}{v_0}\right)\right] t + C_2$$

where C_2 is a constant



Flow Visualization

- Example 6
  - Solution: Part (b)
    - ✓ Particle at origin (x = y = 0) at t = 0:

$$C_1 = C_2 = 0$$

    - ✓ Pathline:

$$x = 0 \qquad\qquad y = v_0 t$$

    - ✓ Particle at origin (x = y = 0) at t = π/2ω:

$$C_1 = -\frac{\pi v_0}{2\omega} \qquad\qquad C_2 = -\frac{\pi u_0}{2\omega}$$

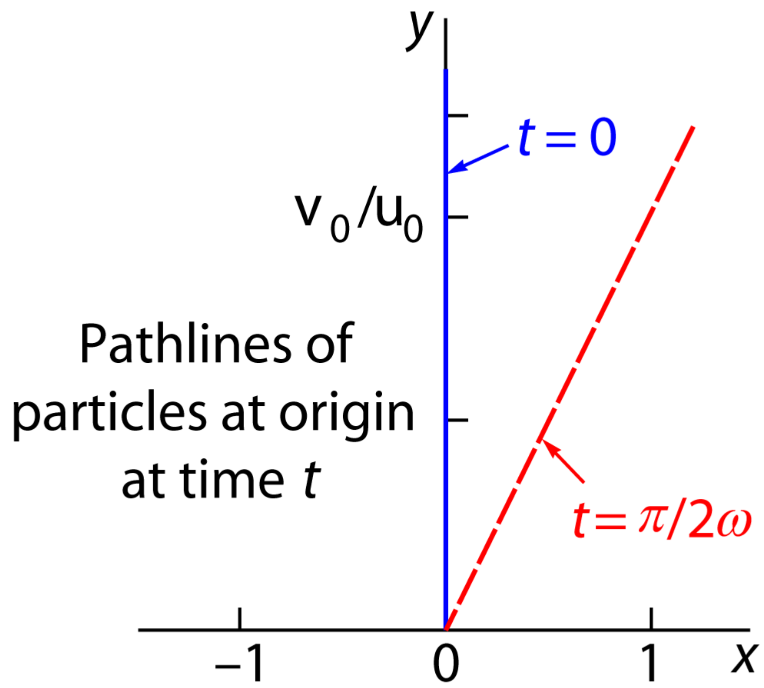
    - ✓ Pathline:

$$x = u_0\left(t - \frac{\pi}{2\omega}\right) \qquad\qquad y = v_0\left(t - \frac{\pi}{2\omega}\right)$$

$$\Rightarrow y = \frac{v_0}{u_0}x$$

Flow Visualization

- Example 6
  - Solution: Part (b)



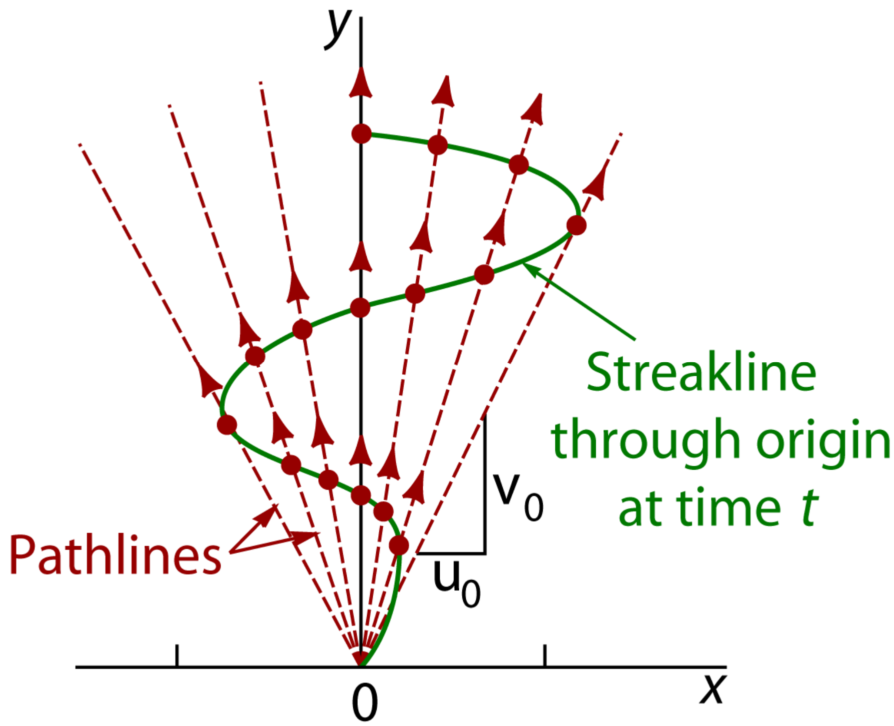
✓ Flow is unsteady: pathlines do not coincide

Flow Visualization

- Example 6
  - Solution: Part (c)
    - ✓ Streakline through origin at time $t = 0$ is locus of particles at $t = 0$ that previously ($t < 0$) passed through origin
    - ✓ Each particle flowing through origin travels in a straight line (pathlines are rays from the origin), the slope of which lies between $\pm v_0/u_0$
    - ✓ Particles passing through origin at different times located on different rays from origin and at different distances from origin
    - ✓ Flow is unsteady: streakline varies with time, although it always has the oscillating, sinuous character

Flow Visualization

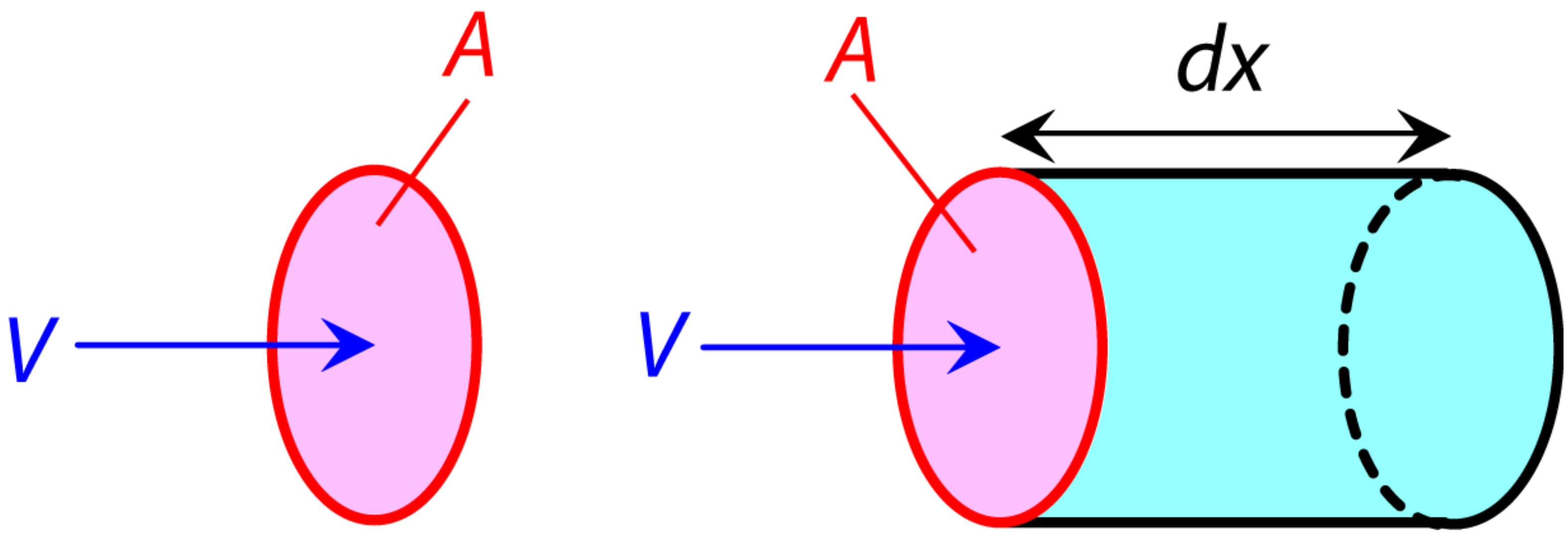
- Example 6
  - Solution: Part (c)



✓ Unsteady flow: streamlines, pathlines, and streaklines do not coincide

Conservation of Mass

- Mass and Volume Flow Rates
  - Mass flow rate
    - ✓ Amount of mass flowing through a cross section per unit time
    - ✓ Usually denoted by $\dot{m}$
    - ✓ Consider a plane surface of arbitrary shape with area A
    - ✓ ρ is fluid density, V is uniform velocity normal to A.



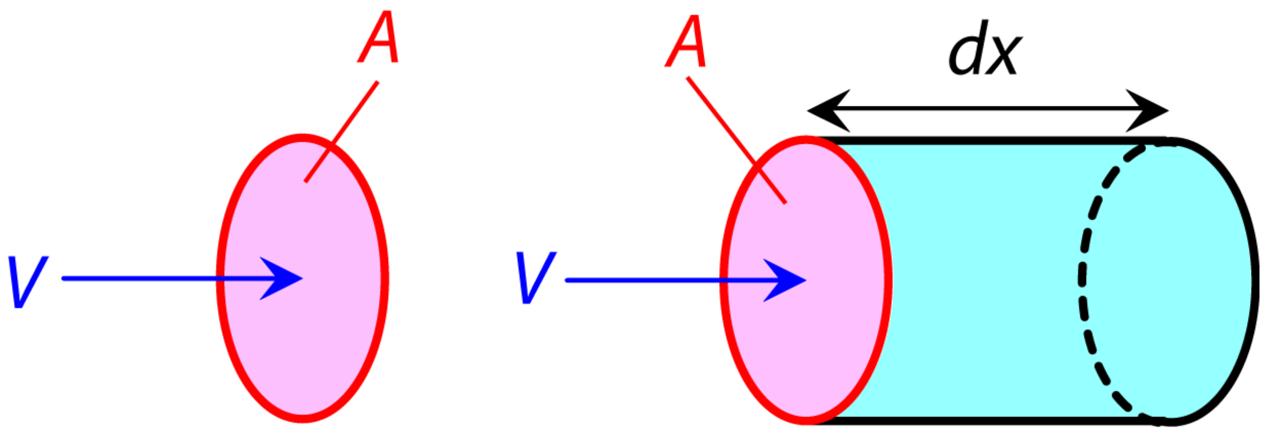
Conservation of Mass

- Mass and Volume Flow Rates
  - Mass flow rate
    - ✓ In time dt, fluid passing through A sweeps through distance dx
    - ✓ Mass of fluid contained in fluid column of length dx:

$$dm = \rho A dx$$

    - ✓ and

$$dx = V dt$$



Conservation of Mass

- Mass and Volume Flow Rates
  - Mass flow rate
    - ✓ Hence:

$$dm = \rho A V dt$$

    - ✓ *dm* is the mass of fluid which has passed through area *A* in time *dt*
    - ✓ Mass flow rate ($\dot{m}$) = mass of fluid passing through area *A* per unit time

$$\dot{m} = \frac{dm}{dt} = \rho A V$$

  - Volume flow rate
    - ✓ Volume flow rate (Q) = volume of fluid passing through area *A* per unit time

$$Q = \frac{\dot{m}}{\rho} = A V$$

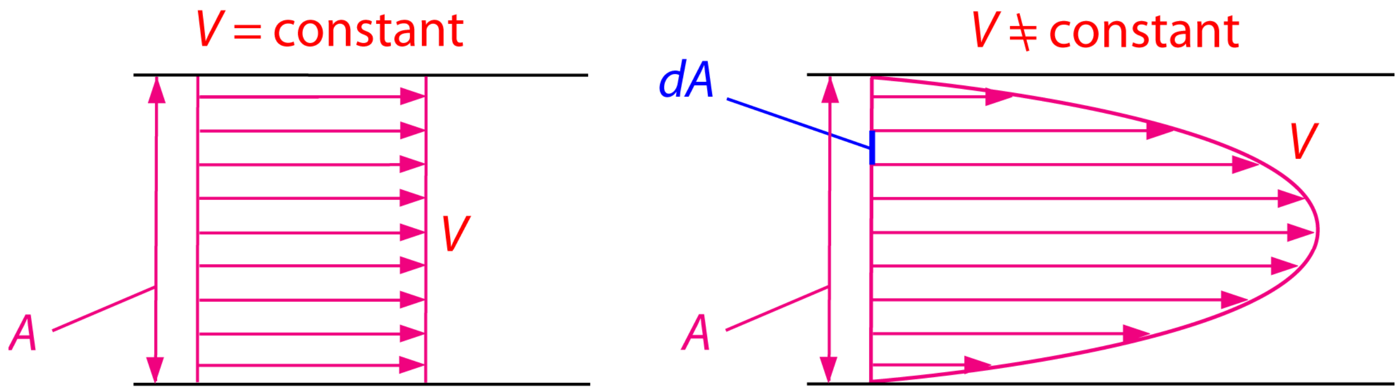
Conservation of Mass

- Mass and Volume Flow Rates
  - Mass flow rate

$$d\dot{m} = \rho V dA \Rightarrow \dot{m} = \int_A d\dot{m} = \int_A \rho V dA$$

  - Volume flow rate

$$dQ = VdA \Rightarrow Q = \int_A dQ = \int_A VdA$$



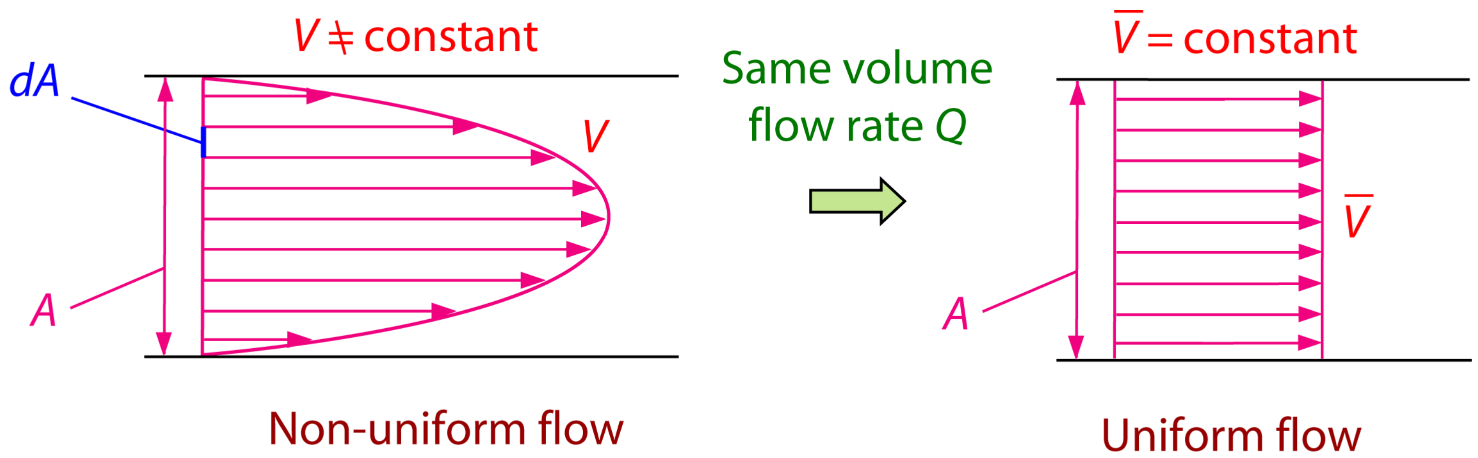
Uniform flow

Non-uniform flow

Conservation of Mass

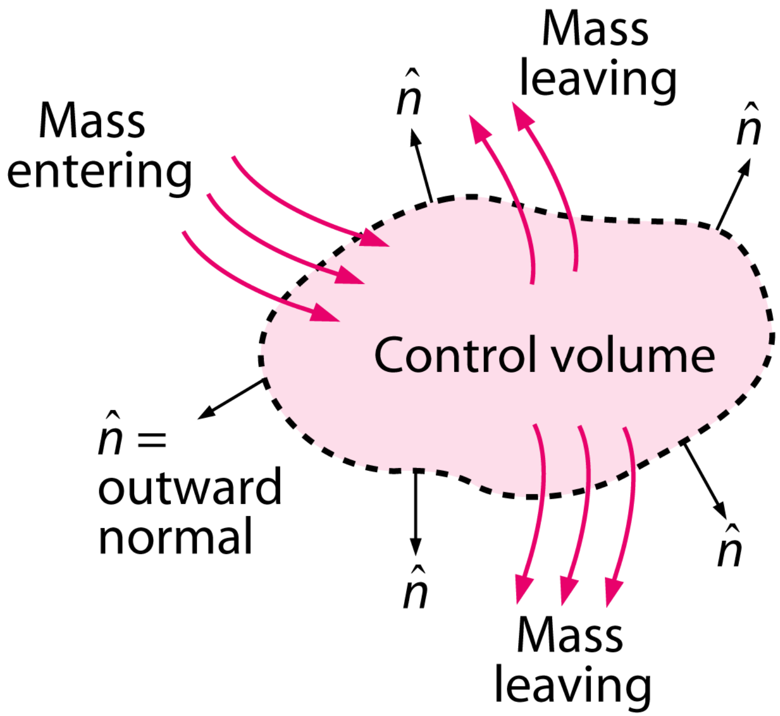
- Mass and Volume Flow Rates
  - Average velocity

$$\bar{V} = \frac{Q}{A} = \frac{\int_A V dA}{A}$$



Conservation of Mass

- Integral Form of Continuity Equation
  - Consider a general control volume (CV):
    - ✓ There are several inlet and outlet ports or streams
    - ✓ Velocities may not be normal to surfaces



Conservation of Mass

- Integral Form of Continuity Equation
  - Conservation of mass:
    - ✓ The net mass transfer to or from a CV during a time interval Δt is equal to the net change (increase or decrease) in the total mass within the CV during Δt

$$\begin{pmatrix} \text{Total mass entering} \\ \text{CV during } \Delta \text{t} \end{pmatrix} - \begin{pmatrix} \text{Total mass leaving} \\ \text{CV during } \Delta \text{t} \end{pmatrix} = \begin{pmatrix} \text{Net change in mass} \\ \text{within CV during } \Delta \text{t} \end{pmatrix}$$

or

$$m_{in} - m_{out} = \Delta m_{CV}$$

where

$$\Delta m_{CV} = m_{final} - m_{initial}$$

is the change in mass of CV

Conservation of Mass

- Integral Form of Continuity Equation

(4.5.1)

  - Conservation of mass:

    ✓ In rate form

$$\underbrace{\dot{m}_{in} - \dot{m}_{out}}_{A} = \underbrace{\frac{dm_{CV}}{dt}}_{B}$$

where $\dot{m}_{\text{in}}$ and $\dot{m}_{\text{out}}$ are the total mass flow rates into and out of the CV, respectively, and dm_{CV}/dt is the rate of change of mass within the CV

✓ Note: $\dot{m}_{\text{in}}$ and $\dot{m}_{\text{out}}$ are positive

Conservation of Mass

- Integral Form of Continuity Equation
  - Evaluation of B :
    - ✓ Consider CV of arbitrary shape
    - ✓ Mass of differential volume $d\Omega$ within CV:

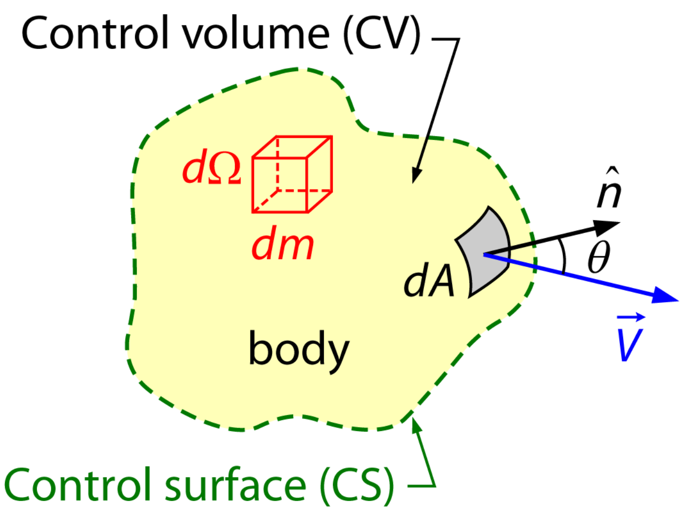
$$dm = \rho d\Omega$$

    - ✓ Total mass within CV at time t:

$$m_{CV} = \int_{CV} \rho d\Omega$$

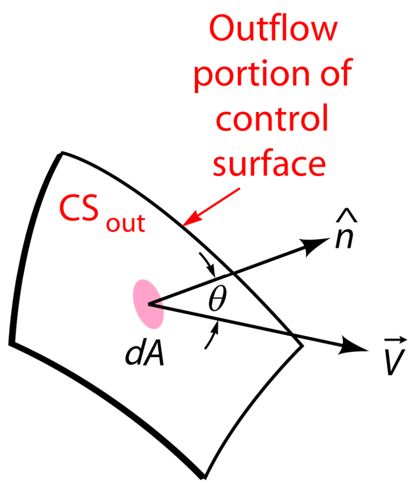
    - ✓ Rate of change of mass within CV:

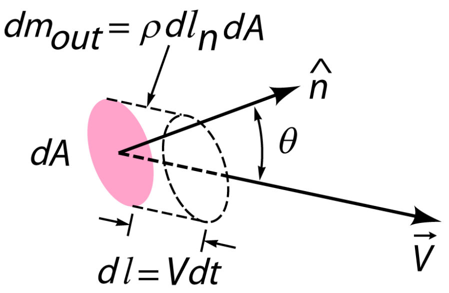
$$\frac{dm_{CV}}{dt} = \frac{d}{dt}\int_{CV} \rho d\Omega$$

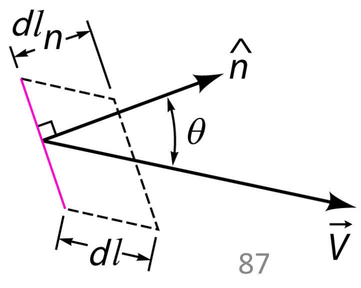


Conservation of Mass

- Integral Form of Continuity Equation
  - Evaluation of A :
    - ✓ Consider mass flow out of CV through differential area dA on control surface of CV
    - ✓ Let $\hat{n}$ be outward unit vector of dA normal to dA :
    - ✓ Let $\vec{V}$ be the flow velocity at dA :
    - ✓ In general, velocity may cross at angle θ off normal:





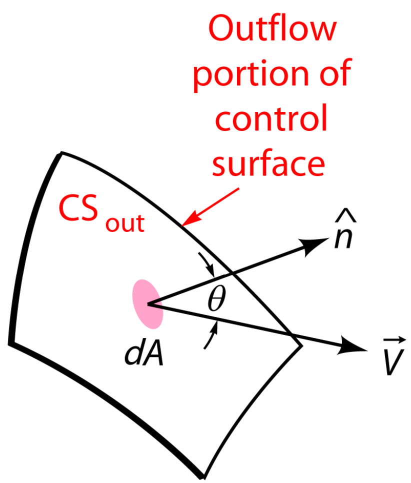


Conservation of Mass

- Integral Form of Continuity Equation
  - Evaluation of A :
    - ✓ In time dt mass of fluid coming out of CV and passing across each area element

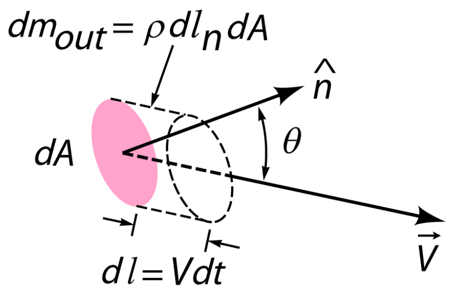
$$dm_{out} = \rho dl_n dA$$

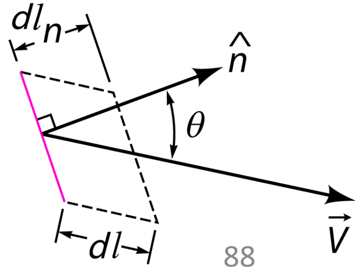
where $dl_n = dl\cos\theta$ is height (normal to the base) of small volume element



- ✓ However, $dl = Vdt$
- ✓ Hence:

$$dm_{out} = \rho\left(dl\cos\theta\right)dA$$
$$= \rho\left(V\cos\theta dt\right)dA$$





Conservation of Mass

- Integral Form of Continuity Equation
  - Evaluation of A :
    - ✓ Outflow mass flow rate across small area element

$$d\dot{m}_{out} = \frac{dm_{out}}{dt} = \rho V \cos\theta dA$$

    - ✓ Integrating over entire outflow portion of control surface CS_{out}:

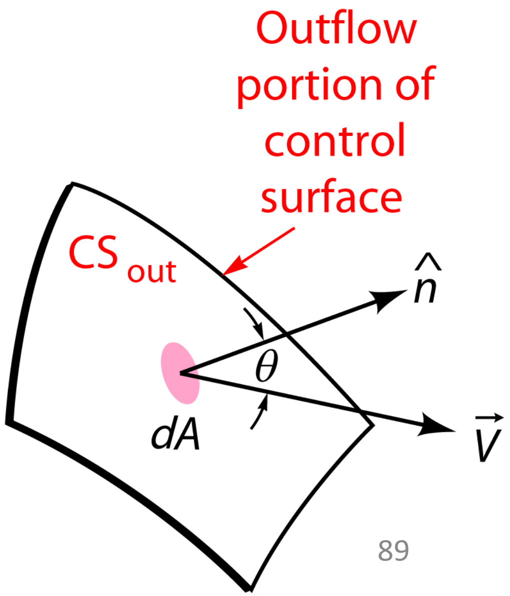
$$\dot{m}_{out} = \int_{CS_{out}} d\dot{m}_{out} = \int_{CS_{out}} \rho V \cos\theta dA$$

    - ✓ Definition of dot product

$$V \cos\theta = \vec{V} \cdot \hat{n}$$

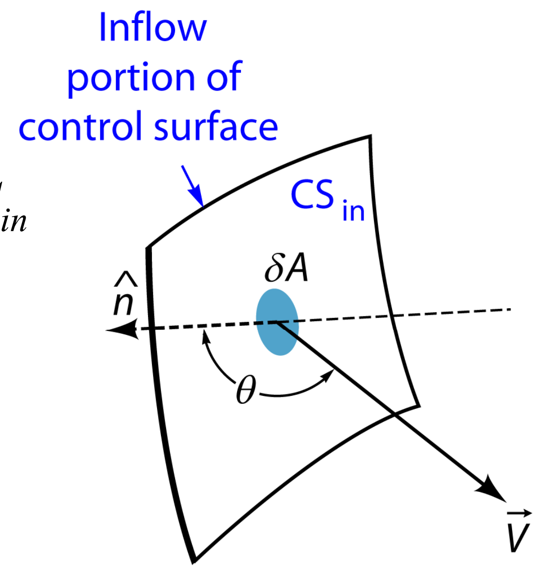
    - ✓ Hence

$$\dot{m}_{out} = \int_{CS_{out}} \rho\left(\vec{V} \cdot \hat{n}\right) dA$$



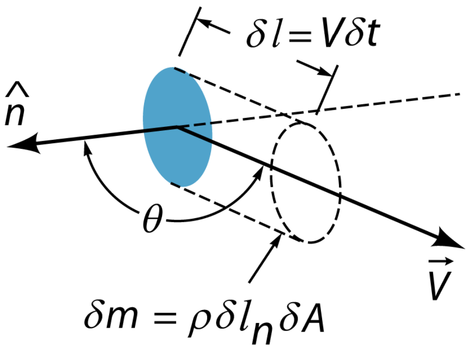
Conservation of Mass

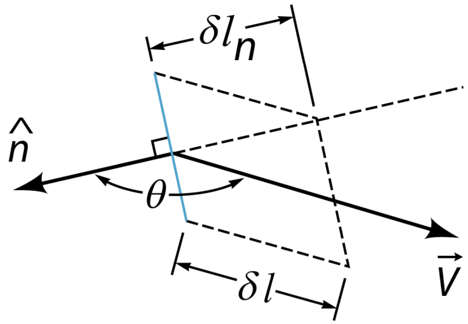
- Integral Form of Continuity Equation
  - Evaluation of (A) :
    - ✓ Similarly, consider the inflow portion of the control surface CS_{in}
    - ✓ Inflow mass flow rate into CV



$$\dot{m}_{in} = -\int_{CS_{in}} \rho V \cos\theta dA$$

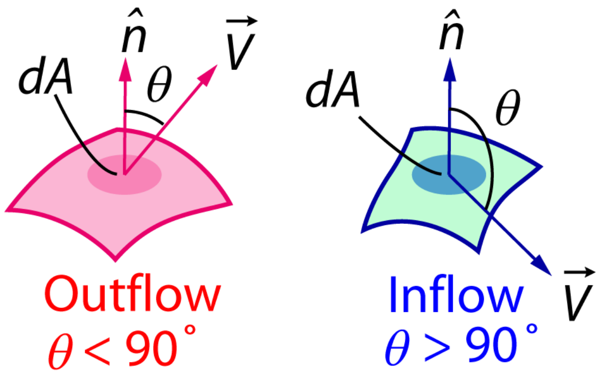
$$\dot{m}_{in} = -\int_{CS_{in}} \rho \left(\vec{V} \cdot \hat{n}\right) dA$$





Conservation of Mass

- Integral Form of Continuity Equation
  - Evaluation of A :



$\vec{V}$: Velocity vector

$\hat{n}$: Outer normal vector

$$\vec{V} \cdot \hat{n} = |\vec{V}||\hat{n}|\cos\theta = V\cos\theta$$

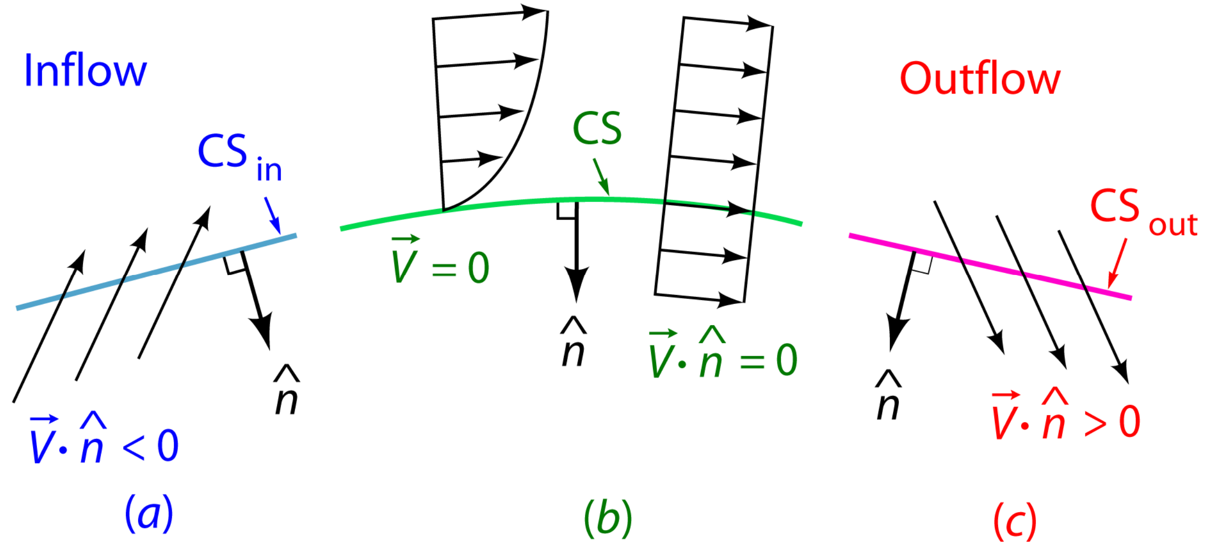
If $\theta < 90^\circ$, then $\cos\theta > 0$ (outflow).

If $\theta > 90^\circ$, then $\cos\theta < 0$ (inflow).

If $\theta = 90^\circ$, then $\cos\theta = 0$ (no flow).

Conservation of Mass

- Integral Form of Continuity Equation
  - Evaluation of A :
    - ✓ $\theta > 90^\circ \Rightarrow \cos\theta < 0 \Rightarrow \vec{V}\cdot\hat{n} = V\cos\theta < 0 \Rightarrow$ inflow of mass into CV
    - ✓ $\vec{V}\cdot\hat{n} = V\cos\theta = 0 \Rightarrow$ no inflow or outflow $\Rightarrow$ either $V = 0$ (fluid "sticks" to surface) or $\cos\theta = 0$ (fluid "slides" along surface without crossing it)
    - ✓ $\theta < 90^\circ \Rightarrow \cos\theta > 0 \Rightarrow \vec{V}\cdot\hat{n} = V\cos\theta > 0 \Rightarrow$ outflow of mass from CV



Conservation of Mass

- Integral Form of Continuity Equation
  - Evaluation of A :
    - ✓ Net mass flow rate across entire control surface

$$\dot{m}_{net} = \dot{m}_{out} - \dot{m}_{in}$$

$$\dot{m}_{net} = \int_{CS_{out}} \rho\left(\vec{V}\cdot\hat{n}\right)dA - \left(-\int_{CS_{in}} \rho\left(\vec{V}\cdot\hat{n}\right)dA\right)$$

$$\dot{m}_{net} = \int_{CS} \rho\left(\vec{V}\cdot\hat{n}\right)dA$$

✓ $\dot{m}_{\text{net}} > 0 \Rightarrow$ net outflow of mass from CV

✓ $\dot{m}_{\text{net}} < 0 \Rightarrow$ net inflow of mass from CV

$$\underbrace{\dot{m}_{in} - \dot{m}_{out}}_{A} = -\dot{m}_{net} = -\int_{CS} \rho\left(\vec{V}\cdot\hat{n}\right)dA$$

Conservation of Mass

- Integral Form of Continuity Equation
  - Put everything together:

$$\dot{m}_{in} - \dot{m}_{out} = \frac{dm_{CV}}{dt}$$

$$\underbrace{-\int_{CS} \rho \left(\vec{V} \cdot \hat{n} \right) dA}_{A} = \underbrace{\frac{d}{dt} \int_{CV} \rho d\Omega}_{B}$$

$$\frac{d}{dt} \int_{CV} \rho d\Omega + \int_{CS} \rho \left(\vec{V} \cdot \hat{n} \right) dA = 0$$

  - The above equation is known as the integral form of the mass conservation equation or integral form of continuity equation

Conservation of Mass

- Integral Form of Continuity Equation
  - Time rate of change of mass within CV plus net mass flow rate through control surface is equal to zero
  - Integral form of continuity equation can be also written as

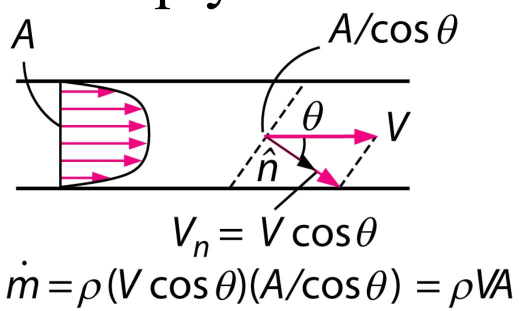
$$\frac{d}{dt}\int_{CV}\rho d\Omega+\underbrace{\sum_{out}\int_{A}\rho\left|V_{n}\right|dA}_{\text{Outflow streams}}-\underbrace{\sum_{in}\int_{A}\rho\left|V_{n}\right|dA}_{\text{Inflow streams}}=0$$

Or

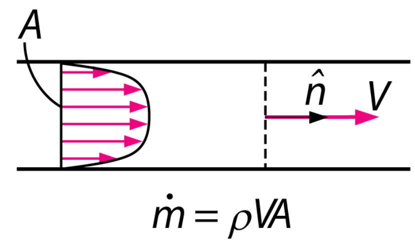
$$\frac{d}{dt}\int_{CV}\rho d\Omega=\sum_{out}\dot{m}-\sum_{in}\dot{m}$$

Conservation of Mass

- Integral Form of Continuity Equation
  - Proper choice of CV helps to simplify analysis
  - Choose control surface normal to flow $\Rightarrow$ dot product $\vec{V}\cdot\hat{n}$ becomes magnitude of the velocity $\Rightarrow$ integral $\int_A \rho\left(\vec{V}\cdot\hat{n}\right)dA$ becomes simply



(a) Control surface at an angle to the flow



(b) Control surface normal to the flow

Conservation of Mass

- Integral Form of Continuity Equation
  - Conservation of mass for steady flow processes
    - ✓ Steady flow: "(.)" means "anything"

$$\frac{d}{dt}(.) = 0 \Longrightarrow \frac{dm_{CV}}{dt} = 0$$

$$\frac{d}{dt}\int_{CV} \rho d\Omega = 0$$

$$\int_{CS} \rho\left(\vec{V} \cdot \hat{n}\right) dA = 0$$

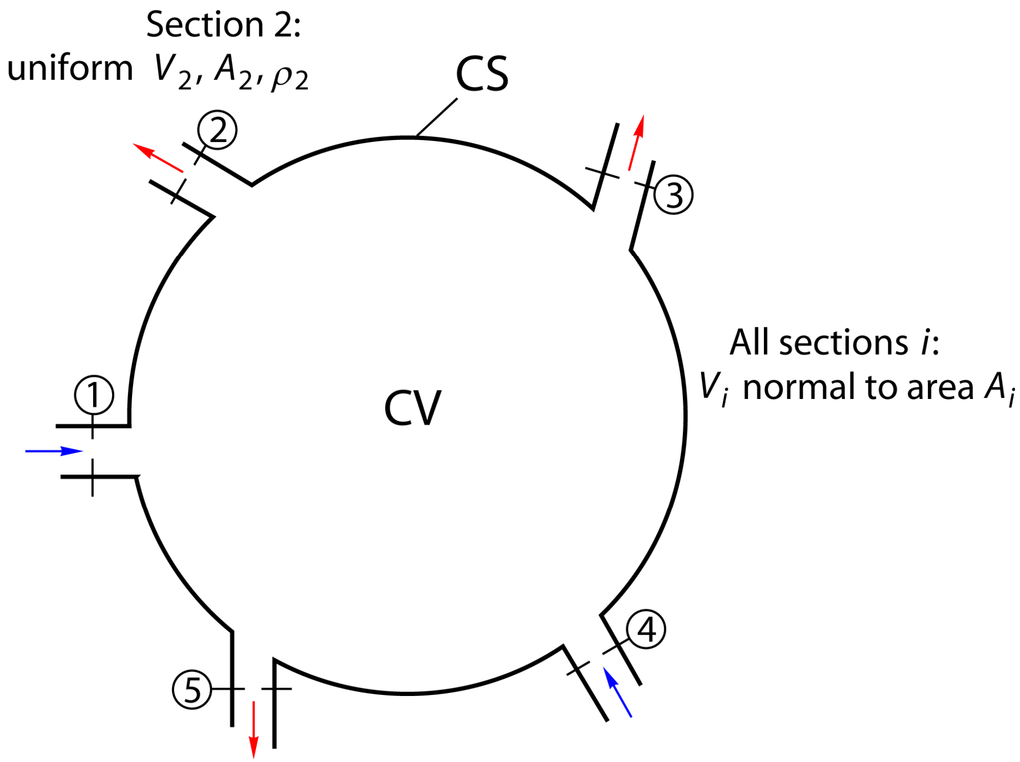
$$\sum_{in} \dot{m} - \sum_{out} \dot{m} = 0 \Rightarrow \sum_{in} \dot{m} = \sum_{out} \dot{m}$$

$$\sum_{in} \rho A V = \sum_{out} \rho A V$$

    - ✓ Total mass flow rate entering CV = Total mass flow rate leaving CV

Conservation of Mass

- Integral Form of Continuity Equation
  - Steady flow with multiple inlets and outlets



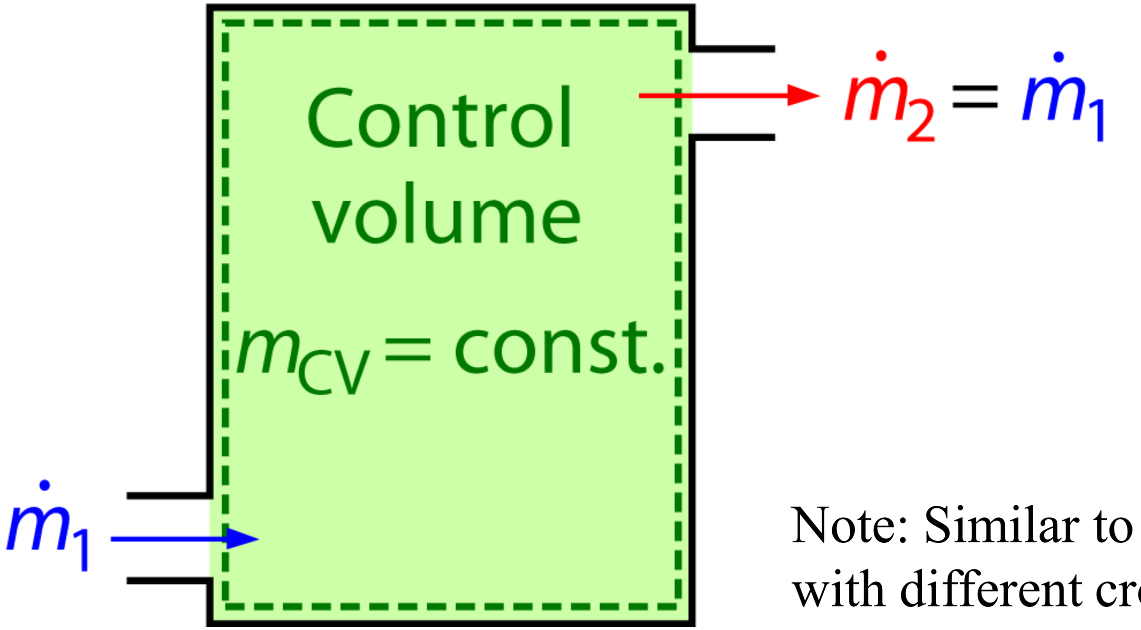
$$\rho_2 A_2 V_2 + \rho_3 A_3 V_3 + \rho_5 A_5 V_5 = \rho_1 A_1 V_1 + \rho_4 A_4 V_4$$

Conservation of Mass

- Integral Form of Continuity Equation
  - Single stream ⇒ one inlet (station 1) and one outlet (station 2), steady flow

$$\dot{m}_1 = \dot{m}_2$$

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$



Note: Similar to pipe flow with different cross-section

Conservation of Mass

- Integral Form of Continuity Equation
  - Steady flow, incompressible flow

$$\frac{d}{dt}(.) = 0 \quad \text{and} \qquad \rho = \text{constant}$$

$$\sum_{in} \dot{m} = \sum_{out} \dot{m}$$

  - Mass flow rate $\dot{m}$ and volume flow rate Q are related via

$$\dot{m} = \rho Q$$

$$\sum_{in} \rho Q = \sum_{out} \rho Q$$

$$\sum_{in} Q = \sum_{out} Q$$

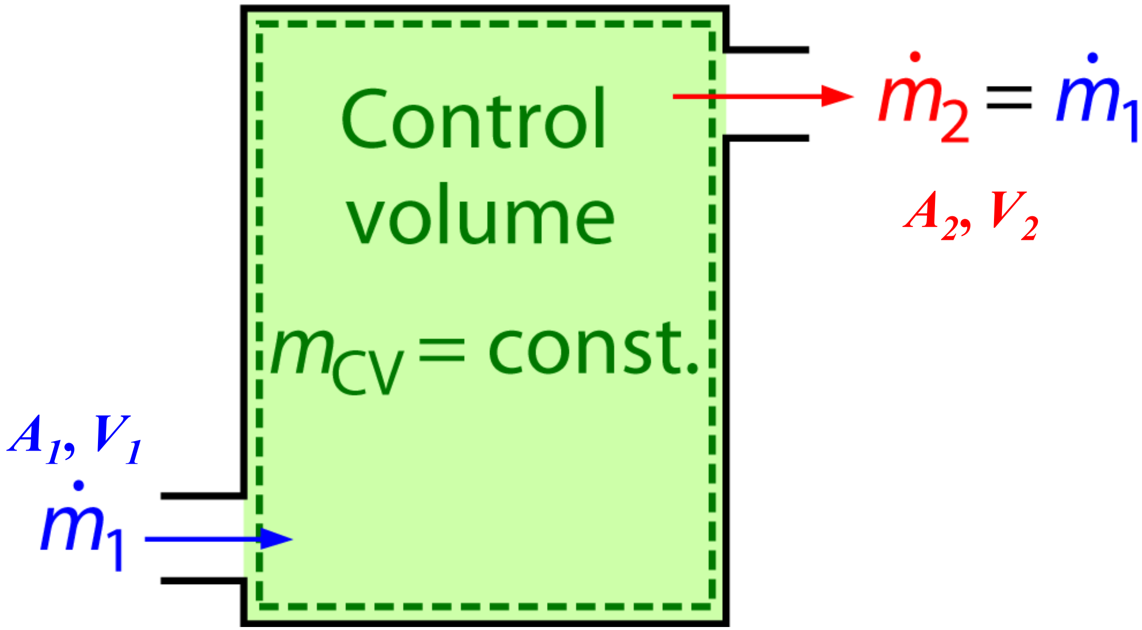
$$\sum_{in} AV = \sum_{out} AV$$

Conservation of Mass

- Integral Form of Continuity Equation
  - Single stream ⇒ one inlet (station 1) and one outlet (station 2), steady, ρ = constant

$$Q_1 = Q_2$$

$$A_1 V_1 = A_2 V_2$$



Conservation of Mass

- Integral Form of Continuity Equation
  - "Conservation of volume" is meaningless
  - Example: flow through air compressor
  - Mass flow rate of air through compressor is constant:

$$\dot{m}_1 = \dot{m}_2$$

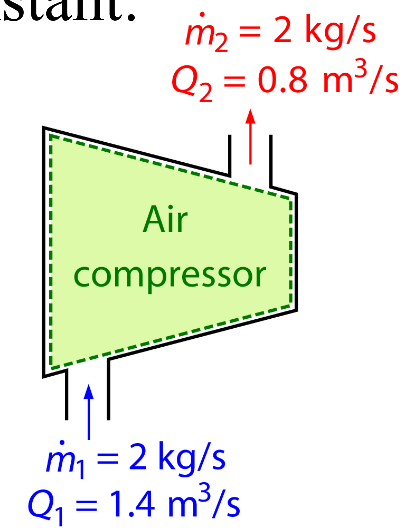
$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

  - Since $\rho_2 > \rho_1$,

$$A_1 V_1 > A_2 V_2$$

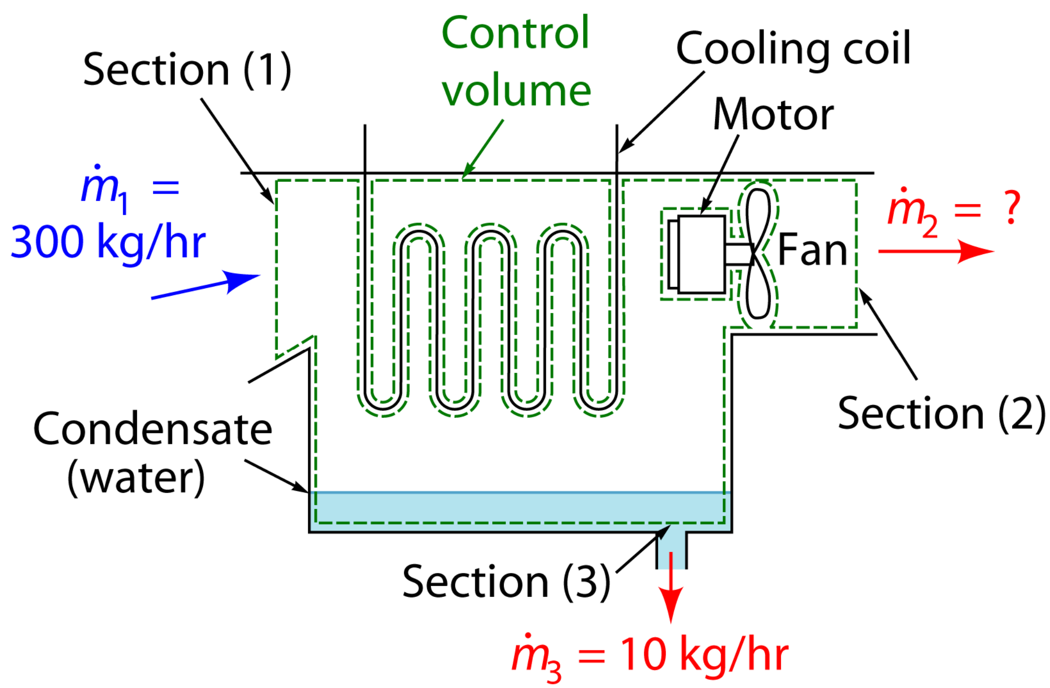
$$Q_1 > Q_2$$

  - Volume flow rate at outlet much less than that at inlet (volume flow rate is not conserved)



Conservation of Mass

- Example 7
  - Question
    - ✓ Moist air enters dehumidifier at rate of 300 kg/hr
    - ✓ Liquid water drains out of dehumidifier at rate of 10 kg/hr
    - ✓ Determine mass flow rate of dry air and water vapor leaving dehumidifier



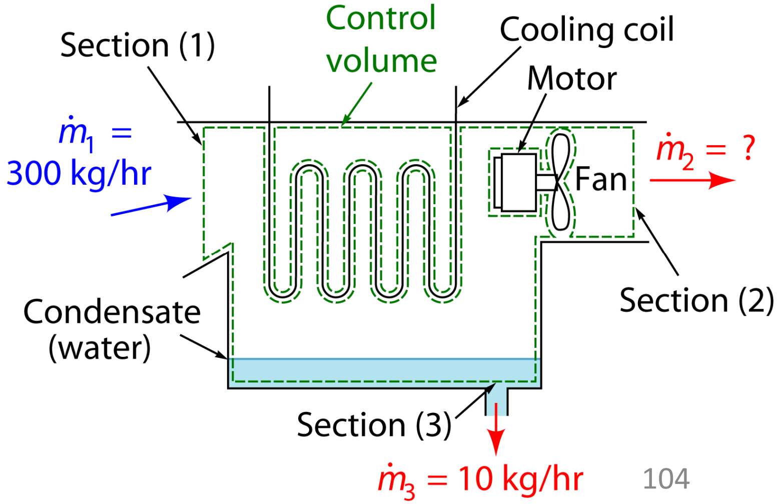
Conservation of Mass

- Example 7
  - Solution
    - ✓ CV includes air and water vapor mixture and condensate in dehumidifier
    - ✓ CV does not include fan, motor, condenser coils and refrigerant

$$-\dot{m}_1 + \dot{m}_2 + \dot{m}_3 = 0$$

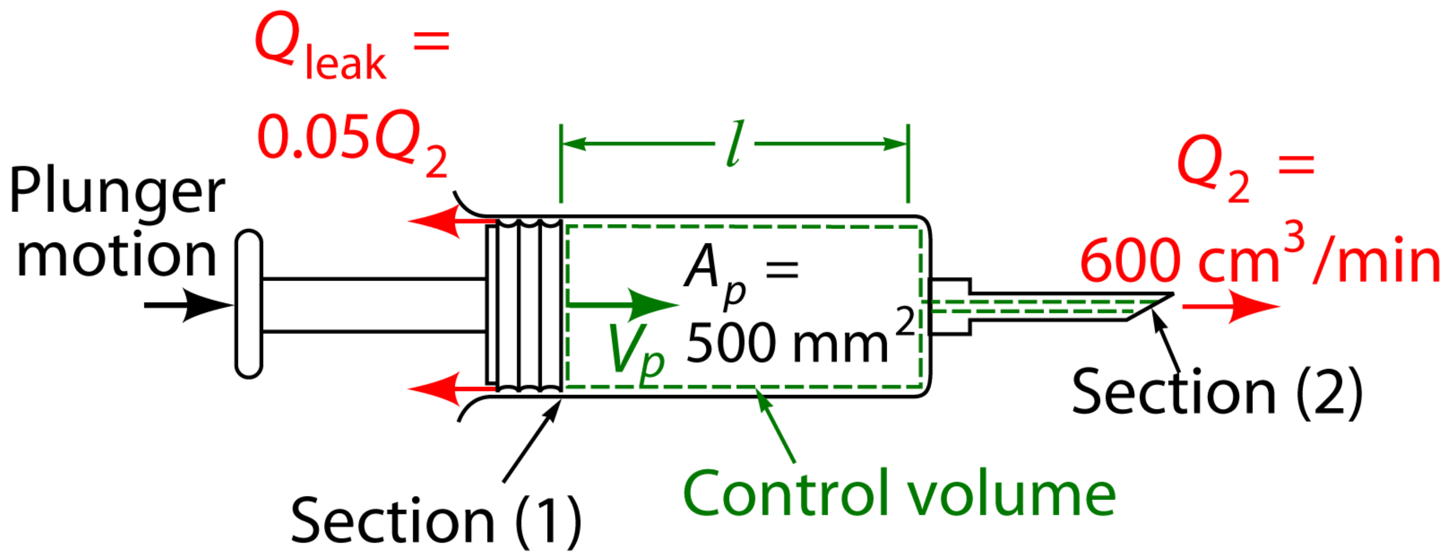
$$\dot{m}_2 = \dot{m}_1 - \dot{m}_3 = 300 - 10$$

$$\dot{m}_2 = 290 \text{ kg/hr}$$



Conservation of Mass

- Example 8
  - Question
    - ✓ Liquid in syringe to be injected steadily at rate of 600 cm^3/min
    - ✓ Face area of plunger = 500 mm^2
    - ✓ Leakage rate past plunger is 0.05 times volume flow rate out of needle
    - ✓ Determine plunger speed V_p



Conservation of Mass

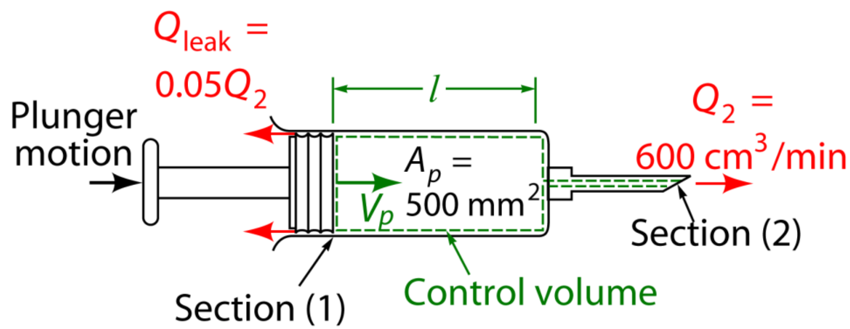
- Example 8
  - Solution

✓ Deformation CV: Section (1) of control surface moves with plunger

✓ Apply the continuity equation with constant density ρ :

$$\frac{d}{dt}\int_{CV}\rho d\Omega+\int_{CS}\rho\left(\vec{V}\cdot\hat{n}\right)dA=0$$

$$\frac{d}{dt}\int_{CV}d\Omega+Q_2+Q_{leak}=0$$



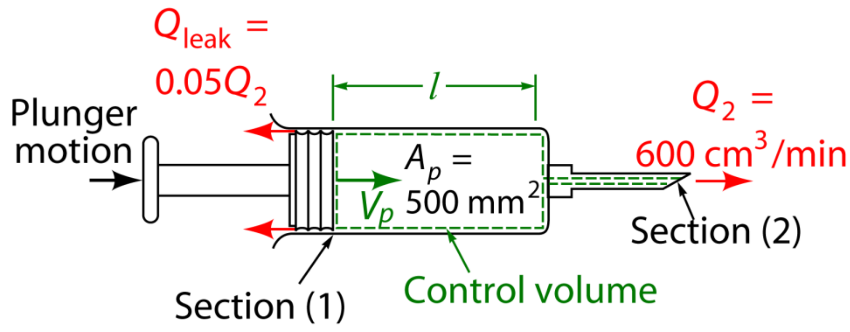
✓ Volume of CV

$$\int_{CV}d\Omega=lA_p+\Omega_{needle}$$

where l is the length of CV, which is varying with time.

Conservation of Mass

- Example 8
  - Solution



$$\frac{d}{dt}\int_{CV} d\Omega + Q_2 + Q_{leak} = 0$$

$$\frac{d}{dt}\int_{CV} d\Omega = A_p \frac{dl}{dt} = -V_p A_p$$

$$-V_p A_p + Q_2 + Q_{leak} = 0$$

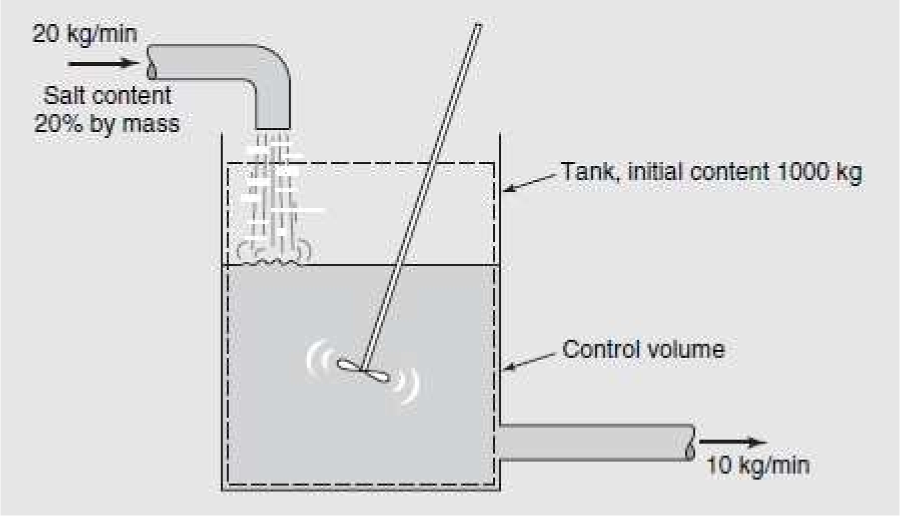
$$V_p = \frac{Q_2 + Q_{leak}}{A_p} = \frac{Q_2 + 0.05Q_2}{A_p} = \frac{1.05Q_2}{A_p}$$

$$V_p = \frac{(1.05)\left(600\times10^{-4}/60\right)}{500\times10^{-4}}$$

$$V_p = 0.021 \text{ m/s} = 2.1 \text{ cm/s}$$

Conservation of Mass

- Example 9
  - Question
    - ✓ A tank initially contains 1000 kg of brine containing 10% salt by mass. An inlet stream of brine containing 20% salt by mass flows into the tank at a rate of 20 kg/min. The mixture in the tank is kept uniform by stirring. Brine is removed from the tank via an outlet pipe at a rate of 10 kg/min. Find the amount of salt in the tank at any time t, and the elapsed time when the amount of salt in the tank is 200 kg.



Conservation of Mass

- Example 9
  - Solution
    - ✓ Assume the amounts of the brine and the salt at any time t are M_b and M_s, respectively. Initially, M_{b0} = 1000kg and M_{s0} = 1000×0.1 = 100kg
    - ✓ The mass balance of brine M_b :

$$\frac{d}{dt}\int_{CV} \rho d\Omega + \int_{CS} \rho\left(\vec{V}\cdot\hat{n}\right) dA = 0$$

$$\frac{d}{dt}\int_{M_{b0}}^{M_b} dM_b + \dot{m}_{b\_out} - \dot{m}_{b\_in} = 0$$

$$\frac{d}{dt}\left(M_b - 1000\right)\text{-}10 = 0$$

$$M_b = 1000 + 10t$$

Conservation of Mass

- Example 9
  - Solution
    - ✓ Salt concentration at any time t

$$\frac{M_s}{M_b} = \frac{M_s}{1000 + 10t}$$

    - ✓ The mass balance of salt M_s:

$$\frac{d}{dt}\int_{CV} \rho d\Omega + \int_{CS} \rho\left(\vec{V} \cdot \hat{n}\right) dA = 0$$

$$\frac{d}{dt}\int_{M_{s0}}^{M_s} dM_s + \dot{m}_{s\_out} - \dot{m}_{s\_in} = 0$$

$$\frac{d}{dt}\left(M_s - 100\right) + \frac{10M_s}{1000 + 10t} - 20 \times 0.2 = 0$$

Conservation of Mass

- Example 9
  - Solution

    ✓ The mass balance of salt M_s:

$$\frac{dM_s}{dt} + \frac{M_s}{100+t} = 4$$

$$M_s = \frac{2t\left(200+t\right)}{100+t} + \frac{C}{100+t}$$

✓ $M_{s0} = 100\text{kg}$ at $t = 0 \Rightarrow C = 10{,}000$

$$M_s = \frac{2t^2 + 400t + 10000}{100+t}$$

✓ $t = 36.6\text{min}$ for $M_s = 200\text{kg}$

Conservation of Mass

- Differential Form of Continuity Equation

$$\frac{d}{dt}\int_{CV}\rho d\Omega+\int_{CS}\rho\left(\vec{V}\cdot\hat{n}\right)dA=0$$

$\Downarrow$ Gauss Theory: the outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface

$$\frac{d}{dt}\int_{CV}\rho d\Omega+\int_{CV}div\left(\rho\vec{V}\right)d\Omega=0$$

$$\int_{CV}\frac{\partial\rho}{\partial t}d\Omega+\int_{CV}div\left(\rho\vec{V}\right)d\Omega=0$$

$$\int_{CV}\left[\frac{\partial\rho}{\partial t}+div\left(\rho\vec{V}\right)\right]d\Omega=0$$

$$\frac{\partial\rho}{\partial t}+div\left(\rho\vec{V}\right)=0 \qquad \frac{D\rho}{Dt}+\rho div\left(\vec{V}\right)=0$$

Conservation of Mass

- Differential Form of Continuity Equation

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

  - Steady Flow: $\partial\rho/\partial t = 0$

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

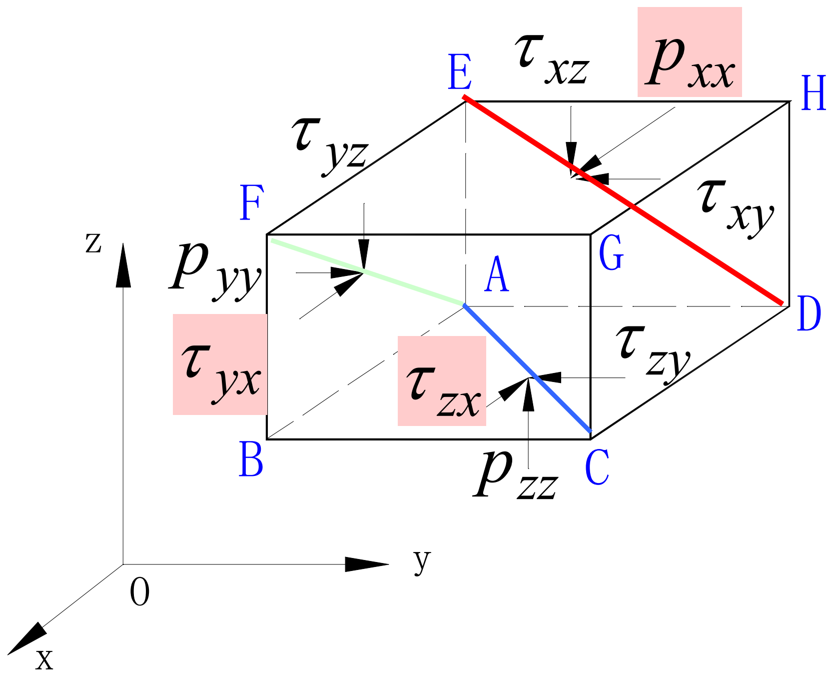
  - Incompressible flow: $\rho =$ constant

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Conservation of Momentum

- Stress on a fluid particle

$$\begin{bmatrix} p_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & p_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & p_{zz} \end{bmatrix}$$



Subscript: 1 surface normal
2 direction of stress

Assumption: Normal stress along positive direction
Shear stress along negative direction

Conservation of Momentum

- Force Balance
  - Newton's Second Law ***F*** = m***a***
    - ✓ In the x direction

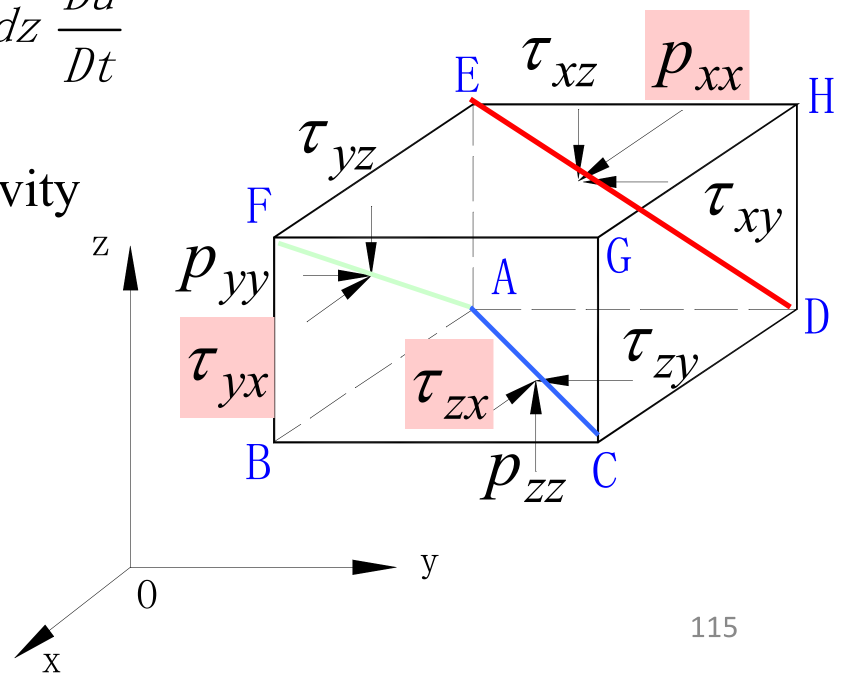
$$\rho dxdydzX + p_{xx}dydz - (p_{xx} + \frac{\partial p_{xx}}{\partial x}dx)dydz - \tau_{zx}dxdy + (\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}dz)dxdy$$

$$-\tau_{yx}dxdz + (\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}dy)dxdz = \rho dxdydz\frac{Du}{Dt}$$

X denotes external body force such as gravity

- ✓ Rearrange the above equation

$$X - \frac{1}{\rho}(\frac{\partial p_{xx}}{\partial x} - \frac{\partial \tau_{yx}}{\partial y} - \frac{\partial \tau_{zx}}{\partial z}) = \frac{Du}{Dt}$$



Conservation of Momentum

- Force Balance
  - Newton's Second Law ***F*** = m***a***
    - ✓ Summary in the all directions

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

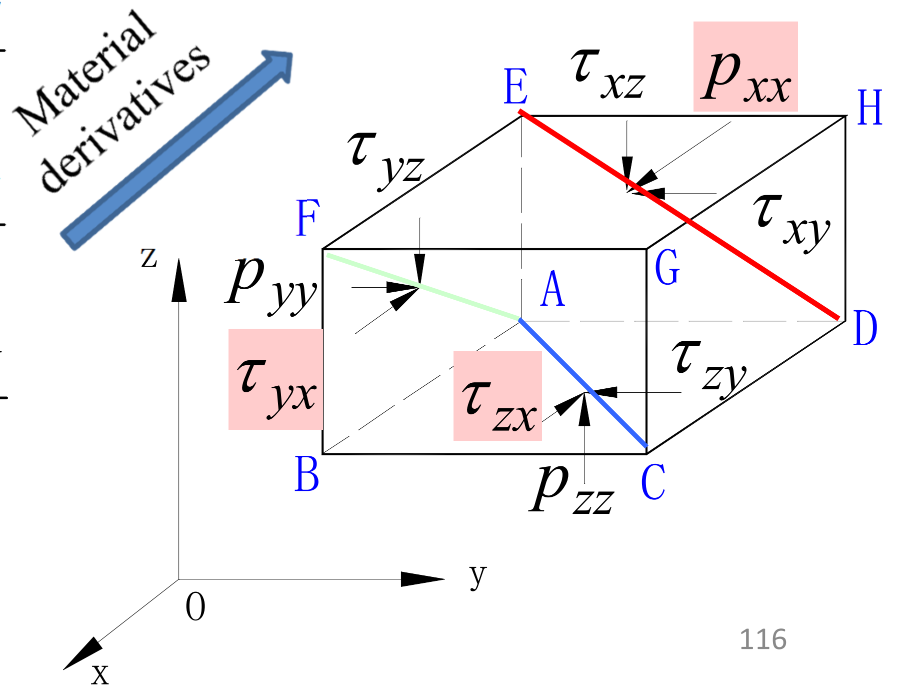
$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$X - \frac{1}{\rho}\left(\frac{\partial p_{xx}}{\partial x} - \frac{\partial \tau_{yx}}{\partial y} - \frac{\partial \tau_{zx}}{\partial z}\right) = \frac{Du}{Dt}$$

$$Y - \frac{1}{\rho}\left(\frac{\partial p_{yy}}{\partial y} - \frac{\partial \tau_{zy}}{\partial z} - \frac{\partial \tau_{xy}}{\partial x}\right) = \frac{Dv}{Dt}$$

$$Z - \frac{1}{\rho}\left(\frac{\partial p_{zz}}{\partial z} - \frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y}\right) = \frac{Dw}{Dt}$$



Conservation of Momentum

- Force Balance
  - Constitutive equations for stress

$$\left.\begin{aligned}
\tau_{xy} &= \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\
\tau_{yz} &= \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\
\tau_{zx} &= \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)
\end{aligned}\right\} (1)$$

$$\left.\begin{aligned}
p_{xx} &= -p + p'_{xx} = -p - \frac{2}{3}\mu div V + 2\mu\frac{\partial u}{\partial x} \\
p_{yy} &= -p + p'_{yy} = -p - \frac{2}{3}\mu div V + 2\mu\frac{\partial v}{\partial y} \\
p_{zz} &= -p + p'_{xx} = -p - \frac{2}{3}\mu div V + 2\mu\frac{\partial w}{\partial z}
\end{aligned}\right\} (2)$$

Conservation of Momentum

- Navier-Stokes Equations

$$X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \nu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \frac{Du}{Dt}$$

Hamiltonian $\quad \nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

Laplacian $\quad \nabla^2 = \nabla\bullet\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\frac{\partial u}{\partial t} + \left(\vec{V}\bullet\nabla\right)u = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u + \nu\frac{\partial}{\partial x}(\nabla\bullet\vec{V})$$

$$\frac{\partial v}{\partial t} + \left(\vec{V}\bullet\nabla\right)v = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v + \nu\frac{\partial}{\partial y}(\nabla\bullet\vec{V})$$

$$\frac{\partial w}{\partial t} + \left(\vec{V}\bullet\nabla\right)w = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w + \nu\frac{\partial}{\partial z}(\nabla\bullet\vec{V})$$

Conservation of Momentum

- Navier-Stokes Equations
  - Incompressible flow

ρ = constant $\xrightarrow[\text{Equation}]{\text{Continuity}}$ $\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0$ or $\nabla \bullet \vec{V} = 0$

$$\frac{\partial u}{\partial t} + \left(\vec{V} \bullet \nabla\right) u = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u$$

$$\frac{\partial v}{\partial t} + \left(\vec{V} \bullet \nabla\right) v = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \nabla^2 v$$

$$\frac{\partial w}{\partial t} + \left(\vec{V} \bullet \nabla\right) w = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 w$$

Conservation of Momentum

- Euler Equations (Inviscid flow $\nu = 0$)

$$\frac{\partial u}{\partial t} + \left(\vec{V} \bullet \nabla\right) u = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + \left(\vec{V} \bullet \nabla\right) v = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + \left(\vec{V} \bullet \nabla\right) w = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

- Fluid at Rest $u = v = w = 0$

$$X - \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

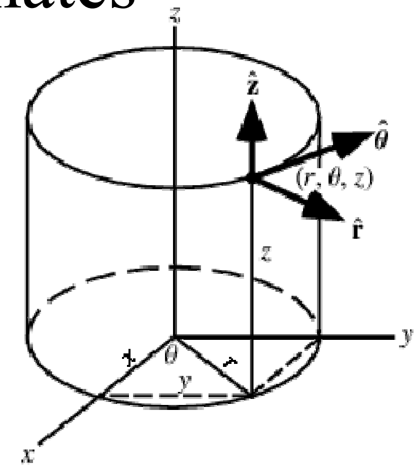
$$Y - \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

$$Z - \frac{1}{\rho}\frac{\partial p}{\partial z} = 0$$

Conservation of Momentum

- Incompressible Flow in Cylindrical Coordinates
  - Continuity Equations

$$\frac{1}{r}\frac{\partial (ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}=0$$



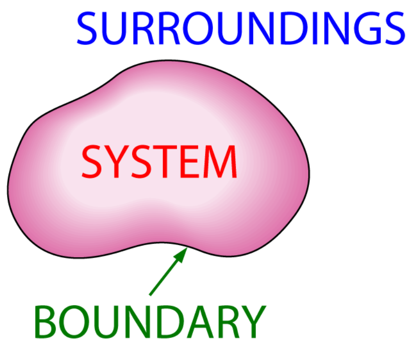
  - Navier-Stokes Equations

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}=X_r-\frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$+\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)-\frac{u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right]$$

$$\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}=X_\theta-\frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$
$$+\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right]$$

$$\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}=X_z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]$$

Review



- System, surrounding, boundary;
- Closed system: mass cannot cross boundary, energy can cross boundary
- Open system/control volume: both mass and energy can cross boundary
- Lagrangian description: tracks the motion of a generic individual fluid particle; observer moves with the fluid
- Eulerian description: individual fluid particles are not identified and tracked; field variables are defined
- Classification of fluid flows: viscous and inviscid flows; laminar and turbulent flows; compressible and incompressible flows; steady and unsteady flows; internal and external flows; one-, two- and three-dimensional flows

Review

- Material (substantial) derivative

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial x} + v\frac{\partial(\ )}{\partial y} + w\frac{\partial(\ )}{\partial z} = \frac{\partial(\ )}{\partial t} + (\vec{V} \bullet \nabla)(\ )$$

- Acceleration of fluid particle

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \underbrace{\frac{\partial w}{\partial t}}_{\text{local acceleration}} + \underbrace{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}}_{\text{convective acceleration}}$$

Review

- Streamlines are everywhere parallel to the local velocity: instantaneous flow pattern

$$\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

- A pathline is the actual path traced out by a fluid particle: flow over a period of time by a single particle

$$\vec{x}(t) = \vec{x}_0 + \int_{t_0}^{t} \vec{V} dt$$

- A streakline is the locus of all fluid particles that have passed through a point in the flow: flow pattern over a period of time by many particles
- Steady flow: streamlines = pathlines = streaklines

Review

- Mass and volume flow rates

$\rho = \text{constant}$

$$\dot{m} = \int_A \rho V dA \qquad \dot{m} = \rho A V \qquad Q = \int_A V dA \qquad Q = AV$$

- Average flow velocity

$$\bar{V} = \frac{Q}{A} = \frac{\int_A V dA}{A} \frac{\int_A V dA}{\int_A dA}$$

- Continuity equation

$$\frac{d}{dt}\int_{CV} \rho d\Omega + \int_{CS} \rho\left(\vec{V}\cdot\hat{n}\right) dA = 0$$

$$\frac{D\rho}{Dt} + \rho div\left(\vec{V}\right) = 0$$

$$\frac{\partial \rho}{\partial t} + div\left(\rho \vec{V}\right) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho u\right) + \frac{\partial}{\partial y}\left(\rho v\right) + \frac{\partial}{\partial z}\left(\rho w\right) = 0$$

Review

– Continuity equation for steady flow

$$\frac{d}{dt}\int_{CV} \rho d\Omega = 0$$

$$\sum_{in} \rho AV = \sum_{out} \rho AV$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\sum_{in} \dot{m} = \sum_{out} \dot{m}$$

– Continuity equation for incompressible steady flow

$$A_1 V_1 = A_2 V_2$$

$$Q_1 = Q_2$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Review

- Navier Stokes equations

$$\frac{\partial u}{\partial t} + \left(\vec{V} \bullet \nabla\right) u = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u + \nu \frac{\partial}{\partial x}(\nabla \bullet \vec{V})$$

$$\frac{\partial v}{\partial t} + \left(\vec{V} \bullet \nabla\right) v = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \nabla^2 v + \nu \frac{\partial}{\partial y}(\nabla \bullet \vec{V})$$

$$\frac{\partial w}{\partial t} + \left(\vec{V} \bullet \nabla\right) w = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 w + \nu \frac{\partial}{\partial z}(\nabla \bullet \vec{V})$$

- Navier Stokes equation for incompressible flow

$$\frac{\partial u}{\partial t} + \left(\vec{V} \bullet \nabla\right) u = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u$$

$$\frac{\partial v}{\partial t} + \left(\vec{V} \bullet \nabla\right) v = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \nabla^2 v$$

$$\frac{\partial w}{\partial t} + \left(\vec{V} \bullet \nabla\right) w = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 w$$

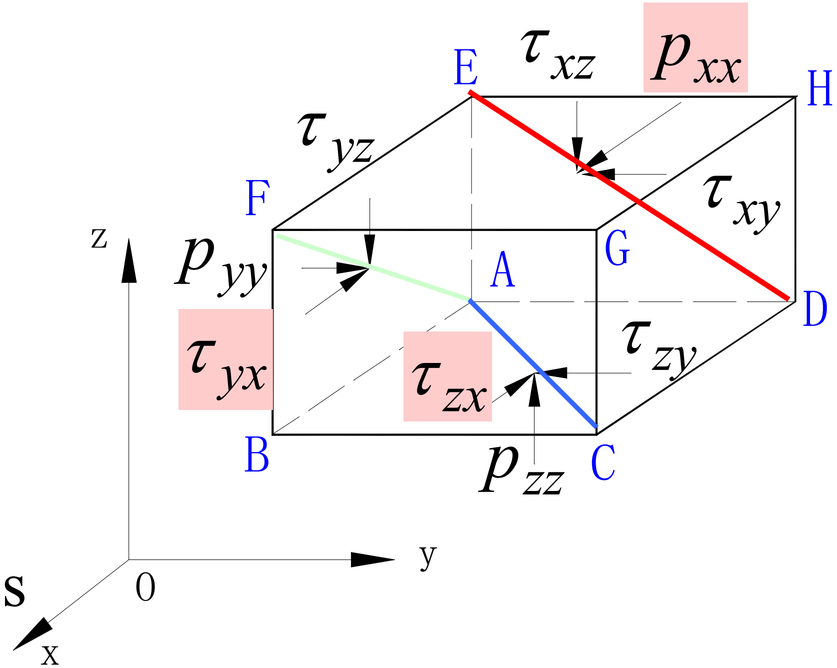
Review

– Euler equations

$$\frac{\partial u}{\partial t} + \left(\vec{V} \bullet \nabla\right) u = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + \left(\vec{V} \bullet \nabla\right) v = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + \left(\vec{V} \bullet \nabla\right) w = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$



– Constitutive equations for stress

$$\begin{bmatrix} p_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & p_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & p_{zz} \end{bmatrix}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \qquad p_{xx} = -p - \frac{2}{3}\mu divV + 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \qquad p_{yy} = -p - \frac{2}{3}\mu divV + 2\mu\frac{\partial v}{\partial y}$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad p_{zz} = -p - \frac{2}{3}\mu divV + 2\mu\frac{\partial w}{\partial z}$$

Conservation Law

- Example 10
  - Question: Does the following incompressible flow exist:

    $$u = x,\ v = y,\ w = z$$

  - Solution:
    - ✓ For incompressible flow, the continuity equation is

      $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

    - ✓ For the above mentioned flow:

      $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \neq 0$$

    - ✓ The flow does not exist because it does not satisfy continuity equation

Conservation Law

- Example 11
  - Question: Two dimensional steady incompressible flow; the velocity component in the x direction is given as

$$u = e^{-x} \cos y + 1$$

Determine the y-velocity component. (assume $v = 0$ at $y = 0$)

  - Solution:
    - ✓ For 2D incompressible flow, the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u = e^{-x} \cos y + 1$$

$$\Rightarrow \quad \frac{\partial \left(e^{-x} \cos y + 1 \right)}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Conservation Law

- Example 11
  - Solution:

$$\frac{\partial v}{\partial y} = e^{-x} \cos y$$

$$v = e^{-x} \sin y + C$$

✓ $v = 0$ at $y = 0$

$$C = 0$$

$$v = e^{-x} \sin y$$


Thank You for Your Attention!
Any Questions?