MAE309

Lecture 2

Fluid Statics

General Principle of Transport Phenomena

Learning Objectives

- To understand
  - The concept of pressure & how it varies in a fluid at rest
  - How to calculate & measure pressure with manometers
  - The concept of buoyancy
  - How to calculate forces on plane and curved surfaces, including buoyancy forces
  - How to calculate forces and pressures in many typical static fluid mechanics problems
  - How to calculate the stability of floating objects in simple flow configurations
  - How to calculate the shape of free surface for the moving fluid.

Introduction

- Fluid Statics: Fluids at Rest
  - Hydrostatics $\Rightarrow$ liquids ::: Aerostatics $\Rightarrow$ gases
  - no relative motion between adjacent fluid layers
  - no relative motion between fluid and solid surface
  - no shear (tangential) stresses
  - Recall: $\tau = \mu du/dy = 0 \Rightarrow u = 0$, or constant everywhere
  - Only normal stresses $\Rightarrow$ force exerted on fluid at rest is normal to surface at point of contact
  - The normal stress is the pressure, by convention
  - Fluid statics $\Rightarrow$ pressure variation only due to weight of fluid $\Rightarrow$ involves gravity fields and gravitational acceleration g

Introduction

- Applications / significance of fluid statics:
  - Pressure distribution in atmosphere and oceans
  - Design of manometer pressure measuring instruments
  - Forces on submerged plane (flat) and curved surfaces
  - Design of water dams, liquid storage tanks
  - Buoyancy forces acting on floating or submerged bodies
  - Stability analysis of floating and submerged bodies

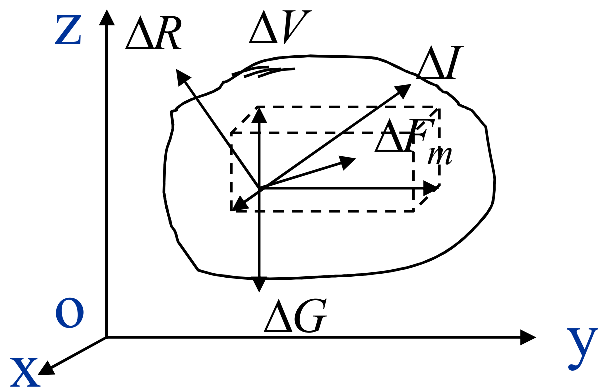
Forced on a Fluid

- Two types forces exist on a fluid particle/parcel: Surface Forces and Body Forces
  - Body Force: distributed over the entire mass or volume of the element. It is usually expressed per unit mass of the element or medium upon which the forces act. Example: Gravitational Force
  - Surface Force: Forces exerted on the fluid element by its surroundings through direct contact at the surface. Surface force has two components:
    - ✓ Normal Force: along the normal to the area
    - ✓ Shear Force: along the plane of the area.
    - ✓ The ratios of these forces and the elemental area in the limit of the area tending to zero are called the normal and shear stresses respectively.

Forces on a Fluid

- Body Forces
  - Gravity ΔG
  - Inertial Force ΔI
  - Inertial spin forces such as the Centrifugal force ΔR
  - These forces are proportional to the mass of fluid particle

$$\begin{cases} \Delta \boldsymbol{G} = \Delta M \cdot \boldsymbol{g} \\ \Delta \boldsymbol{I} = \Delta M \cdot \boldsymbol{a} \\ \Delta \boldsymbol{R} = \Delta M \cdot \boldsymbol{r}\omega^2 \end{cases}$$

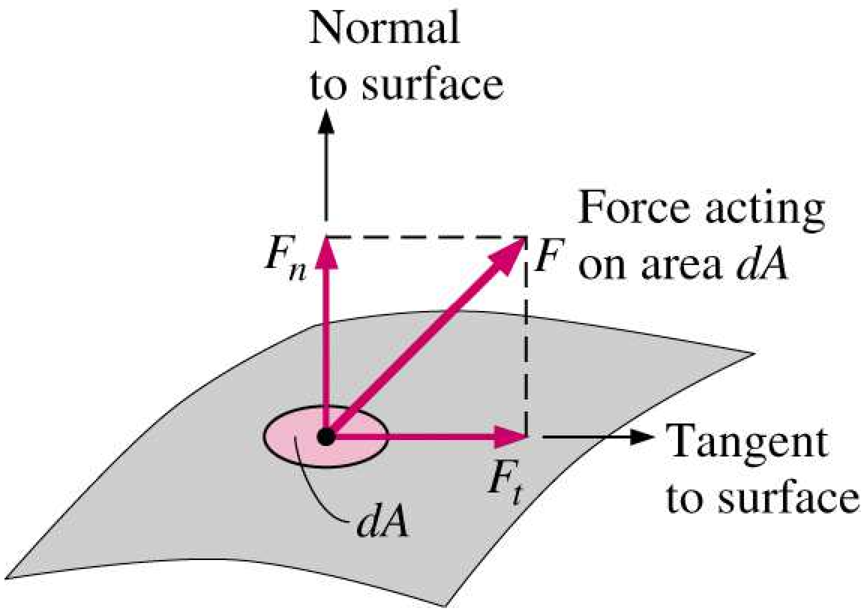


Forces on a Fluid

- Surface Forces
  - Surface force depends on the orientation of surface: Normal and Shear Forces

$$\tau_t = \lim_{\delta A \to 0} \frac{\delta F_t}{\delta A}$$

$$\sigma_n = \lim_{\delta A \to 0} \frac{\delta F_n}{\delta A}$$



Forces on a Fluid

- Surface Forces
  - Fluid continuously deforms under applied shear forces
  - When a fluid is at rest, neither shear forces nor shear stresses exist in it.
  - Fluid at rest only experience normal surface force or normal surface stress.

Pressure

- Pressure
  - Pressure is (-ive) normal force of a fluid, SI units: N/m^2 or Pa
  - Standard atmospheric pressure: 101.33 kPa

Blaise Pascal

1623-1662

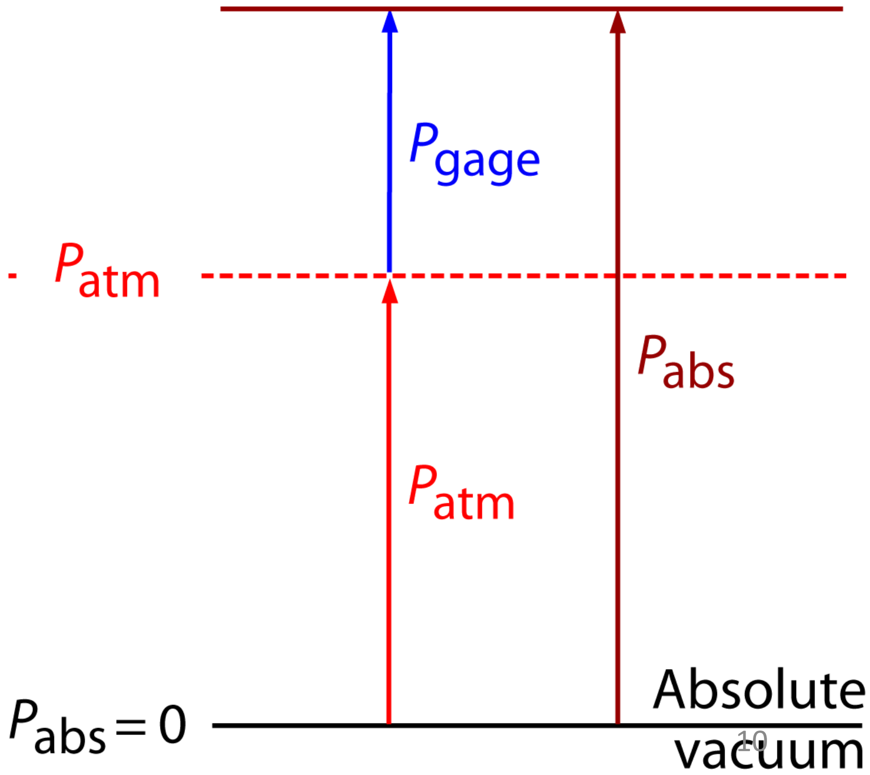
- a French mathematician & philosopher, did the early experiments with barometer, and based on these, suggested that the pressure remains constant at the same level throughout a static fluid, and independent of the shape or cross section of the container (Pascal Principle)
- Together with Fermat, Pascal also puts the theory of probability on firm foundation (Pascal's triangle)
- Unit of pressure is named after him: 1 Pa = $1N/m^2$

Pressure

- Absolute Pressure (P_{abs})
  - Actual pressure at a given point
  - Measured relative to absolute vacuum (absolute zero pressure)
  - Cannot be negative

- Gag Pressure (P_{gage})
  - Difference between absolute pressure and local atmospheric pressure

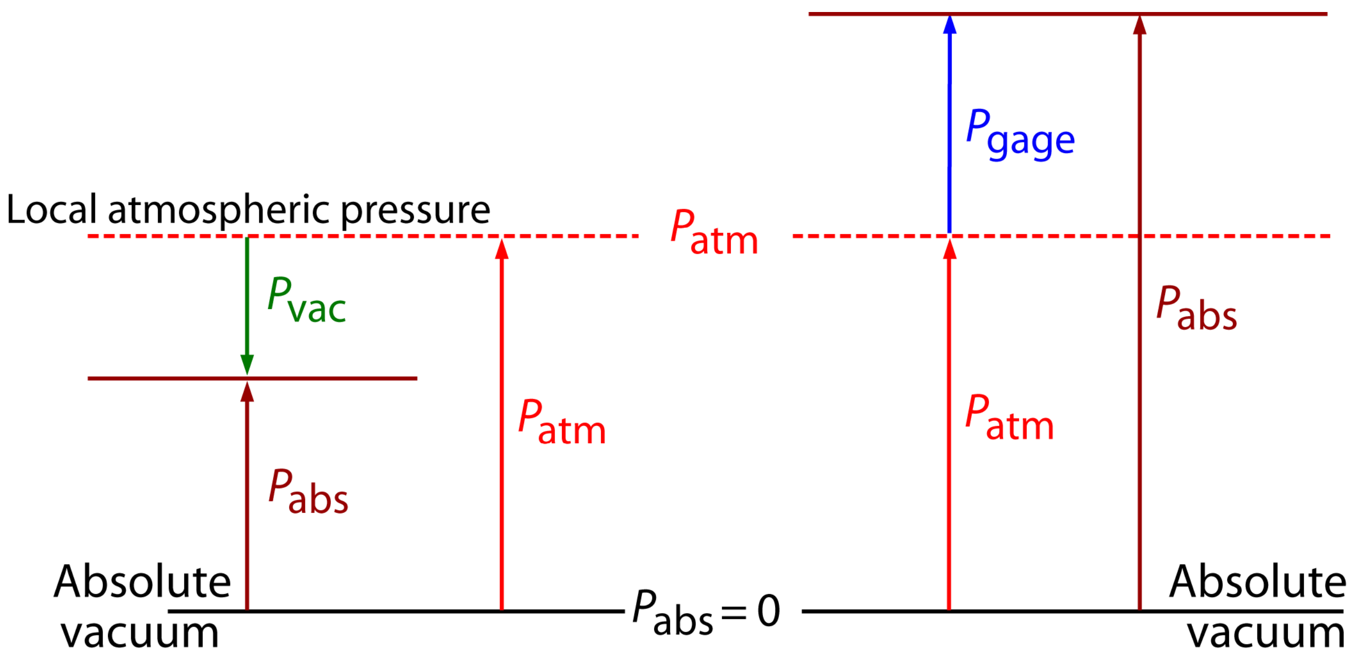
$$P_{gage} = P_{abs} - P_{atm}$$



Pressure

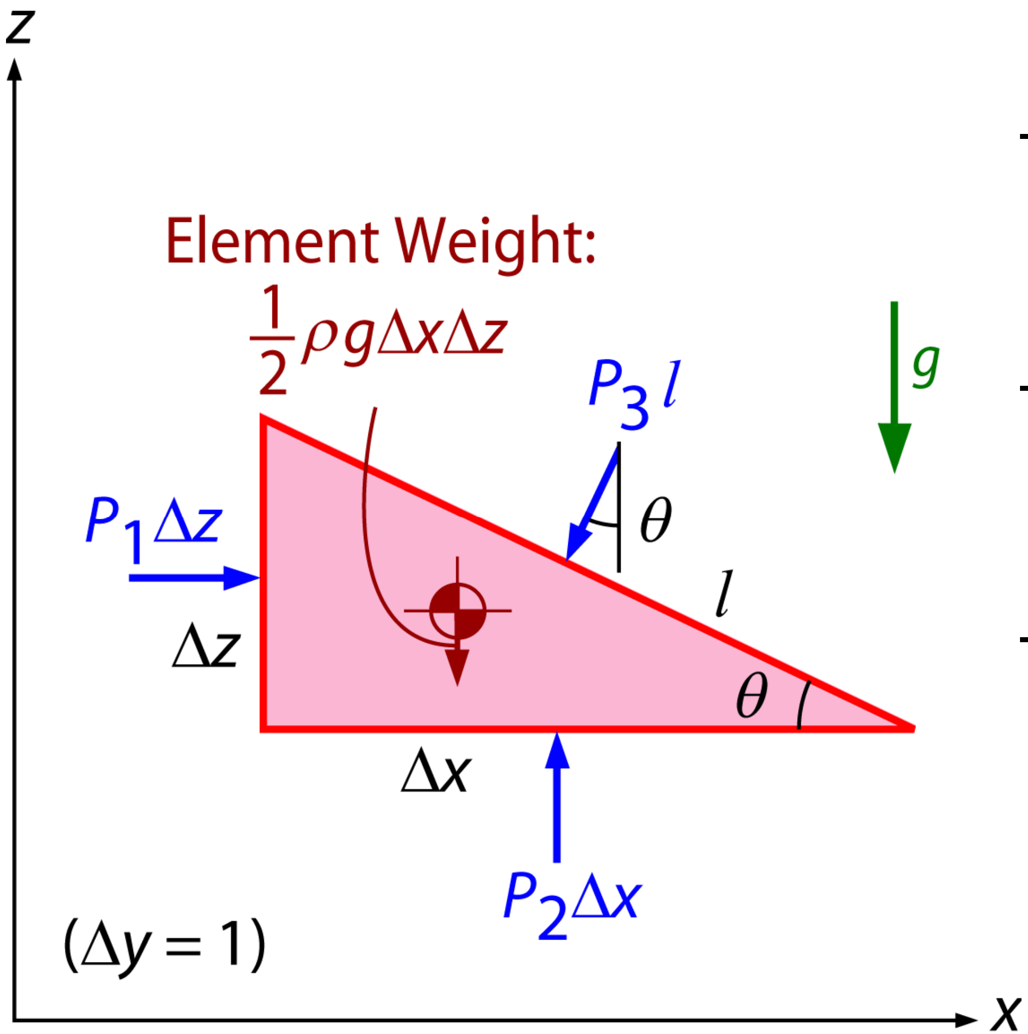
- Vacuum Pressure (P_{vac})
  - Used when absolute pressure falls below atmospheric pressure
  - Negative gage pressure

$$P_{\text{vac}} = P_{\text{atm}} - P_{\text{abs}}$$



Pressure

- Pressure at a Point



  - Pressure at any point in a fluid is the same in all directions
  - Pressure is a scalar quantity: it has a magnitude, but not a specific direction
  - Consider wedge-shaped fluid element of unit length (into page) in equilibrium

Pressure

- Pressure at a Point
  - Mean pressures at three surfaces are P_1, P_2 and P_3
  - Newton's second law $\Rightarrow$ force balance in x- and z-directions:

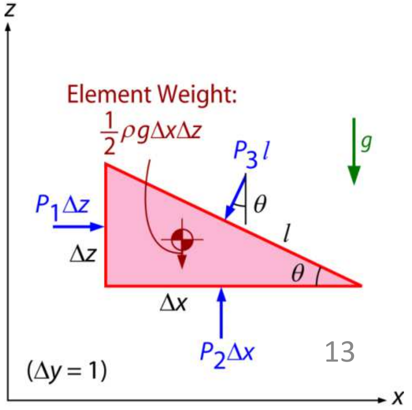
$$\sum F_x = ma_x = 0 \Rightarrow P_1 \Delta z - P_3 l \sin\theta = 0$$

$$\sum F_z = ma_z = 0 \Rightarrow P_2 \Delta x - P_3 l \cos\theta - \underbrace{\frac{1}{2}\rho g \Delta x \Delta z}_{\text{weight of fluid element}} = 0$$

  - From geometry

$$\Delta x = l\cos\theta$$

$$\Delta z = l\sin\theta$$



Pressure

- Pressure at a Point (cont'd)
  - Substituting the geometry equation to force balance equation

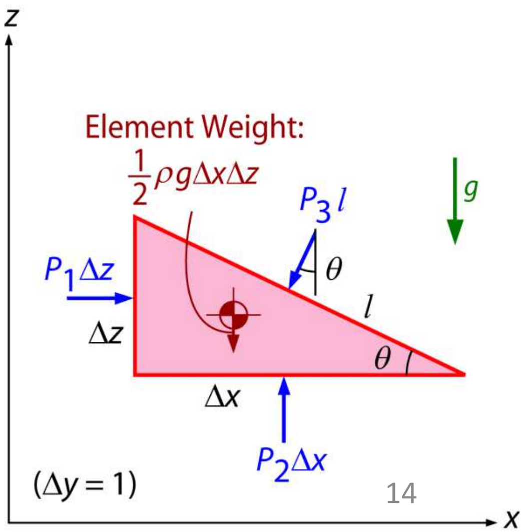
$$P_1 - P_3 = 0 \quad \text{and} \quad P_2 - P_3 - \frac{1}{2}\rho g \Delta z = 0$$

  - $\Delta z = 0 \Rightarrow$ last term the above equation goes to zero $\Rightarrow$ wedge becomes infinitesimal $\Rightarrow$ fluid element shrinks to a point
  - Combining the above results,

$$P_1 = P_2 = P = P_3$$

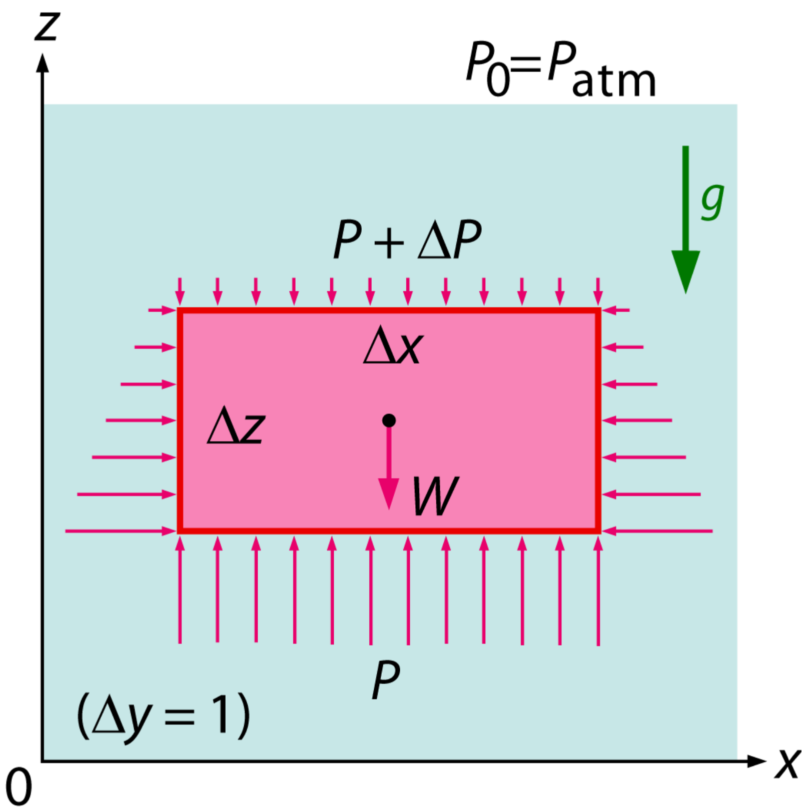
regardless of value of θ

  - Pressure at a point in a fluid has the same magnitude in all directions.



Pressure

- Variation of Pressure with Depth
  - Consider a rectangular fluid element of height Δz, length Δx, and unit depth (into the page) in equilibrium



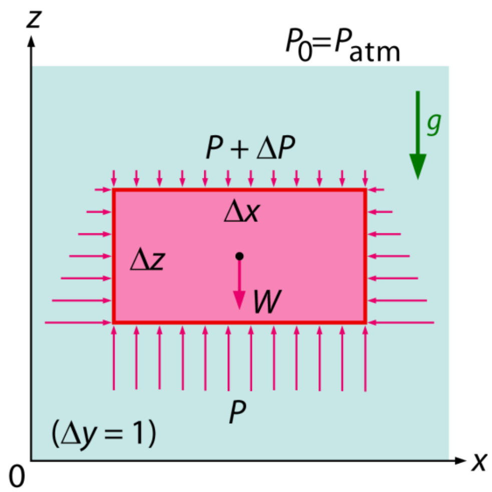
Pressure

- Variation of Pressure with Depth (cont'd)
  - Force balance in vertical z-direction:

$$\sum F_z = ma_z = 0 \;\Rightarrow\; P\Delta x - \left(P + \Delta P\right)\Delta x - \rho g \Delta x \Delta z = 0$$

$$-\Delta P \Delta x - \rho g \Delta x \Delta z = 0$$

$$\Delta P + \rho g \Delta z = 0$$



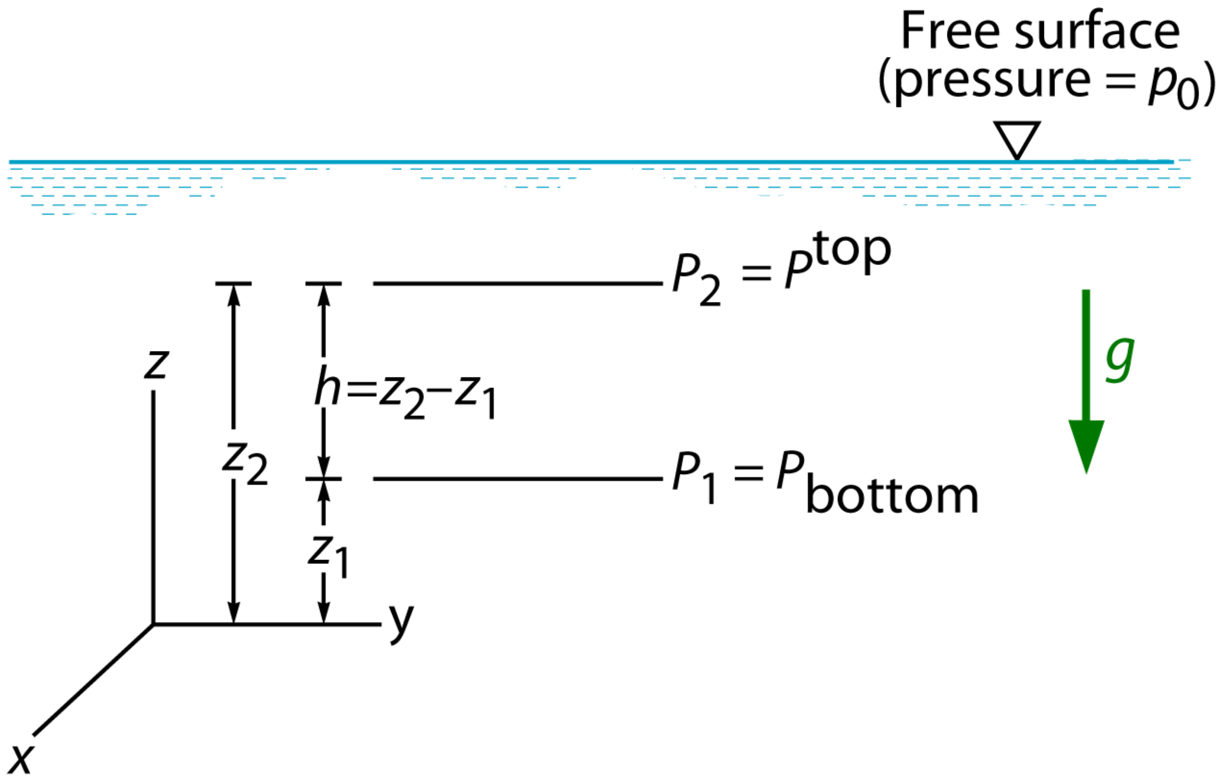
  - In the limit as $\Delta z \rightarrow 0$:

$$\frac{dP}{dz} = -\rho g$$

  - Negative sign $\Rightarrow$ pressure in a static fluid increases with depth

Pressure

- Hydrostatic Pressure in Liquids



Pressure

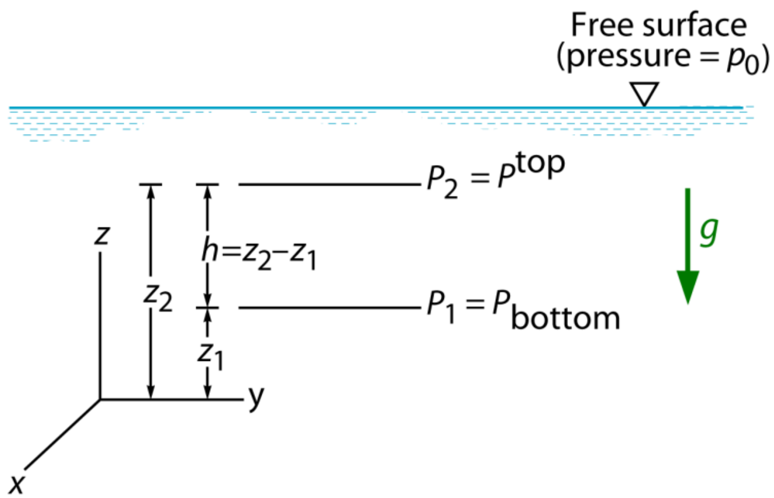
- Hydrostatic Pressure in Liquids
  - Assume incompressible fluid $\Rightarrow \rho =$ constant
  - Integrating the pressure gradient formulation between two points with elevations z_1 and z_2:

$$\int_{P_1}^{P_2} dP = -\rho g \int_{z_1}^{z_2} dz$$

$$P_2 - P_1 = -\rho g \left(z_2 - z_1 \right)$$

$$\Delta P = -\rho g \Delta z$$

$$P_{bottom} = P^{top} + \rho g \left| \Delta z \right|$$



where $|\Delta z|$ is the absolute difference (distance) in depth between the two points of interest

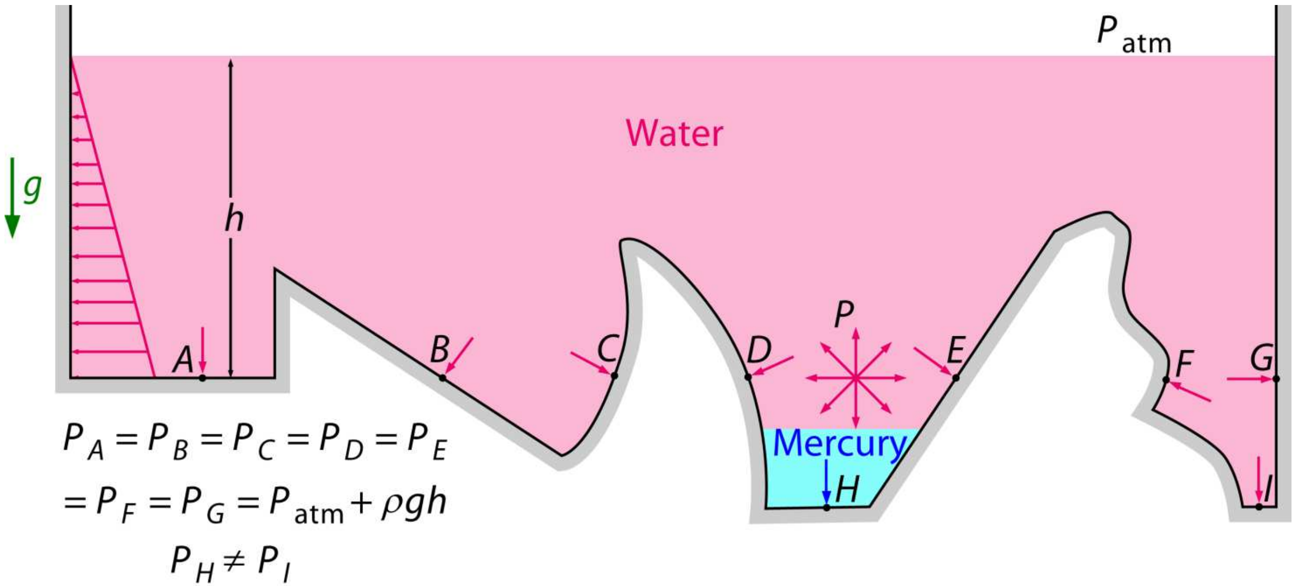
Pressure

- Hydrostatic Pressure in Liquids
  - Pressure in a fluid is independent of shape or cross section of container
    - ✓ Except for small diameter tubes where surface tension effects become significant
  - Pressure changes with vertical distance (depth), but remains constant in other directions
  - Pressure is the same at all points on a horizontal plane in a given fluid
  - Pascal's Law: if a continuous line can be drawn through the same fluid from point 1 to 2 then

$$P_1 = P_2 \text{ if } z_1 = z_2$$

Pressure

- Hydrostatic Pressure in Liquids

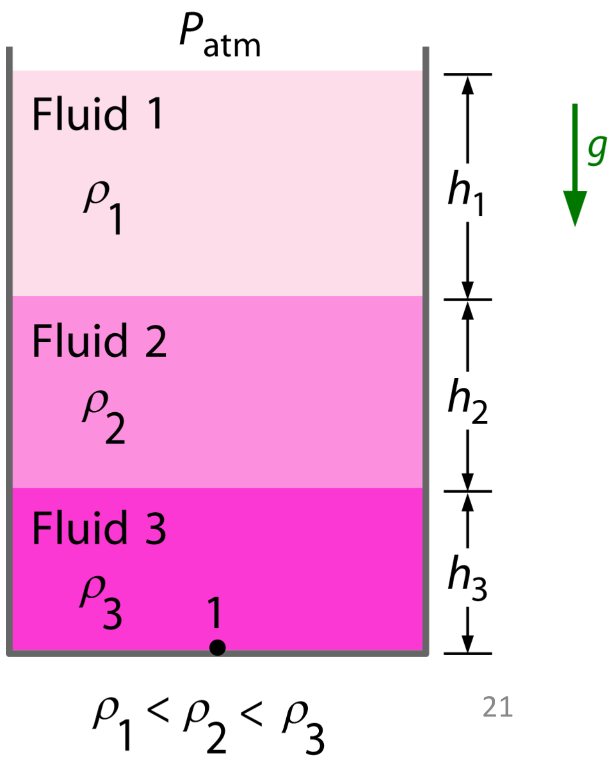


  - Same pressures at *A*, *B*, *C*, *D*, *E*, *F* and *G* since they are at the same depth and they are interconnected by same fluid
  - H and I : pressures different since these 2 points cannot be interconnected by the same fluid, even though they are at same depth

Pressure

- Hydrostatic Pressure in Liquids
  - Pressure force exerted by fluid always normal to surface at specified points
  - Multiple immiscible fluids of different densities stacked on top of one another

$$P_1 - P_{atm} = \left(P_2 - P_{atm}\right) + \left(P_3 - P_2\right) + \left(P_1 - P_3\right)$$
$$= \rho_1 g h_1 + \rho_2 g h_2 + \rho_3 g h_3$$



Pressure

- Hydrostatic Pressure in Liquids: Summary
  - Pressure change across a fluid column of height *h* is

$$\Delta P = \rho g h$$

  - Pressure increase downwards with depth in a given fluid

$$P_{bottom} = P^{top} + \rho g h$$

  - Pascal's Law: Two points at the same elevation in a continuous fluid at rest are at the same pressure
  - Pressure is constant across a flat fluid-fluid interface

Pressure

- Hydrostatic Pressure in Gases
  - Isothermal conditions: $T = T_0 = \text{constant}$

$$\frac{dP}{dz} = -\rho g \quad \text{and} \quad P = \rho RT \Rightarrow \frac{dP}{dz} = -\frac{gP}{RT}$$

Separating the variables:

$$\int_{P_1}^{P_2} \frac{dP}{P} = \ln\frac{P_2}{P_1} = -\frac{g}{R}\int_{z_1}^{z_2} \frac{dz}{T}$$

Integrating the above equation :

$$P_2 = P_1 \exp\left[-\frac{g(z_2 - z_1)}{RT_0}\right] \qquad \text{Eq. A}$$

Pressure

- Hydrostatic Pressure in Gases
  - Linear temperature distribution: $T = T_1 - \beta z$

$$\frac{dP}{dz} = -\frac{gP}{RT} \quad \text{and} \quad dT = -\beta dz$$

Eliminate *dz* from the above two equations:

$$-\beta \frac{dP}{dT} = -\frac{gP}{RT}$$

Separating the variables:

$$\frac{dP}{P} = \frac{g}{R\beta}\frac{dT}{T}$$

Pressure

- Hydrostatic Pressure in Gases
  - Linear temperature distribution (Cont'd)

Integrating:

$$\int_{P_1}^{P} \frac{dP}{P} = \int_{T_1}^{T} \frac{g}{R\beta}\frac{dT}{T}$$

$$\ln\frac{P}{P_1} = \frac{g}{R\beta}\ln\frac{T}{T_1} = \ln\left(\frac{T}{T_1}\right)^{\frac{g}{R\beta}}$$

$$\ln\frac{P}{P_1} = \ln\left(\frac{T_1 - \beta z}{T_1}\right)^{\frac{g}{R\beta}}$$

$$P = P_1\left(1 - \frac{\beta z}{T_1}\right)^{\frac{g}{R\beta}} \qquad \text{Eq. B}$$

Pressure

- Hydrostatic Pressure in Gases
  - Application in Earth's Atmosphere
    - In the stratosphere (from z = 11 km to z = 20.1 km), $T = T_0$ = constant = -56.5°C ⇒ Pressure distribution is given by Eq. A

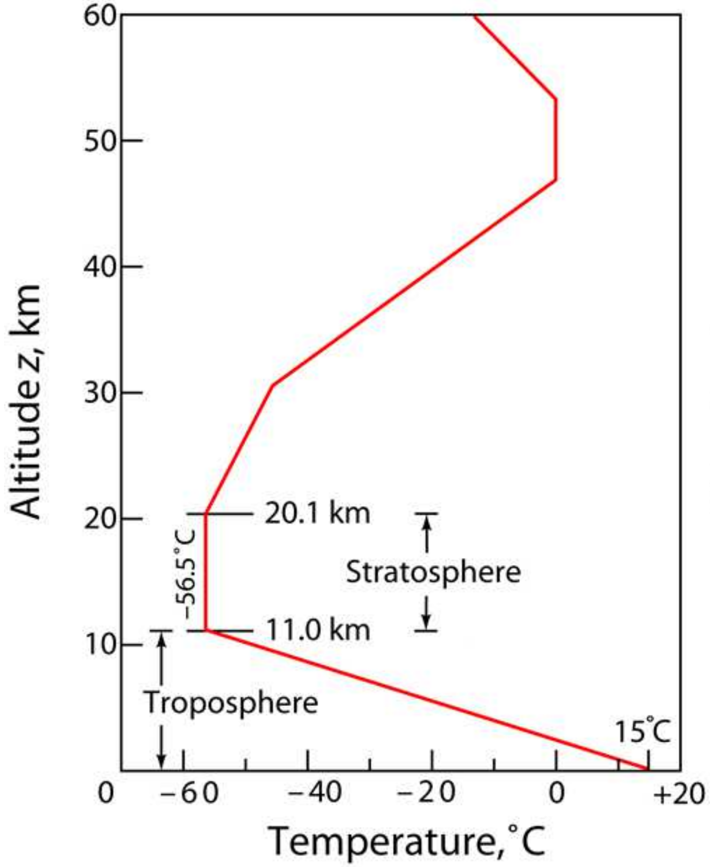
$$P_2 = P_1 \exp\left[-\frac{g(z_2 - z_1)}{RT_0}\right]$$

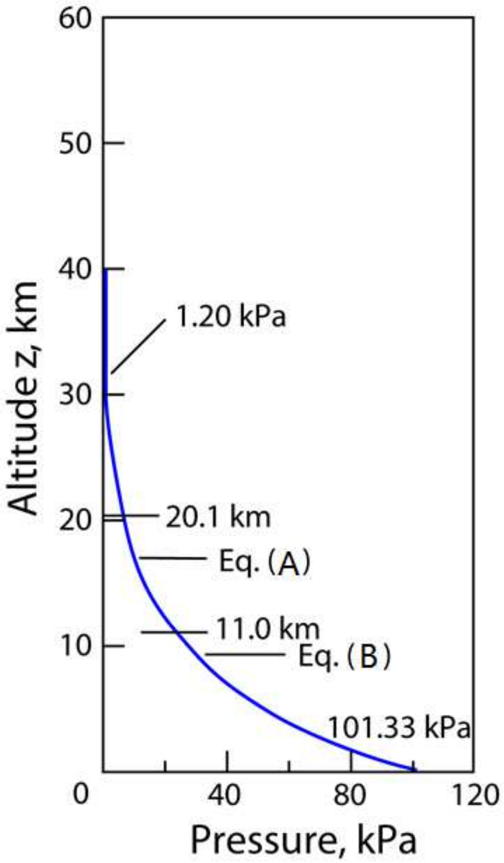
    - In the troposphere (from sea-level z = 0 to z =11 km), temperature variation is of the form $T = T_1 - \beta z$, where T_1 = 288.16 K = 15°C (temperature at sea-level) and β = 0.00650 K/m (lapse rate) ⇒ Pressure distribution is given by Eq. B

$$P = P_1\left(1 - \frac{\beta z}{T_1}\right)^{\frac{g}{R\beta}}$$

Pressure

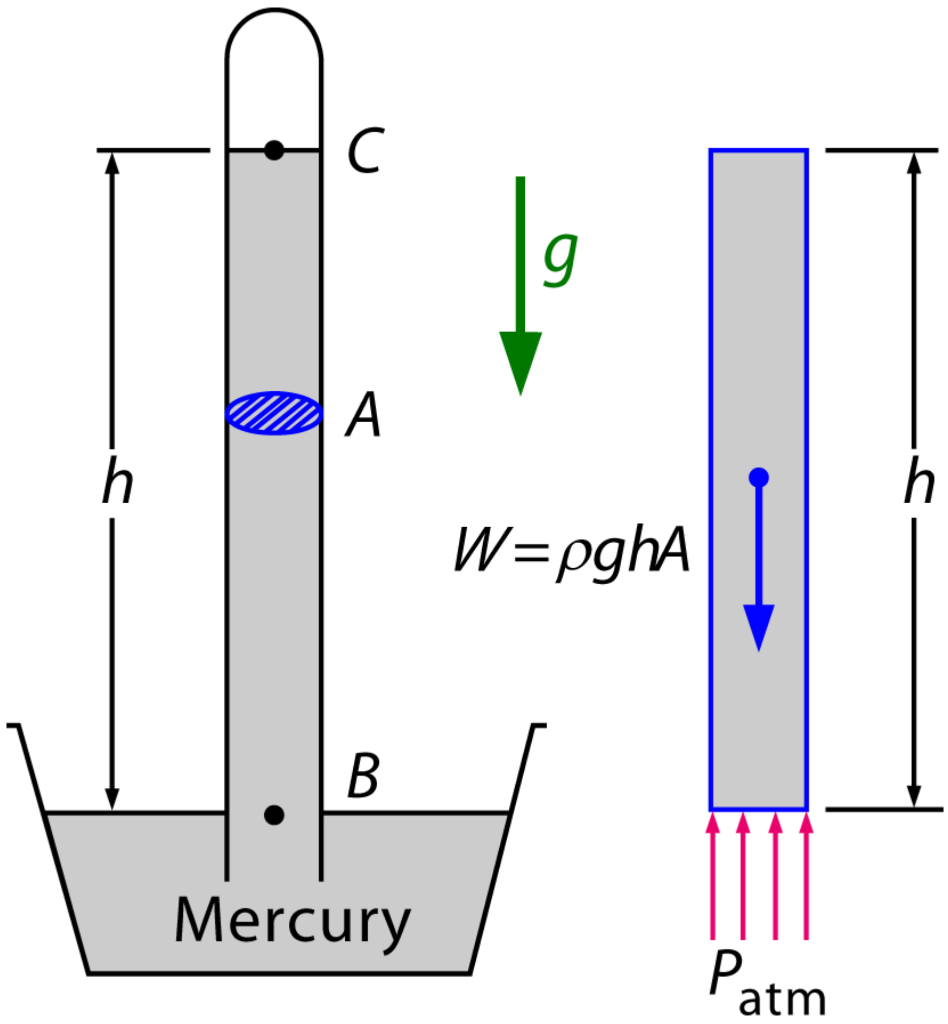
- Hydrostatic Pressure in Gases
  - Application in Earth's Atmosphere





Measurement of Pressure

- Barometer



- ✓ Barometer: used for measuring atmospheric pressure
- ✓ A tube is filled with mercury and inverted while submerged in a reservoir $P_B = P_{atm}$
- ✓ Mercury has a very low vapor pressure of 0.16 Pa at room temperature of 20 °C $\Rightarrow$ near vacuum in closed upper end $\Rightarrow P_C \approx 0$
- ✓ Force balance in vertical direction:

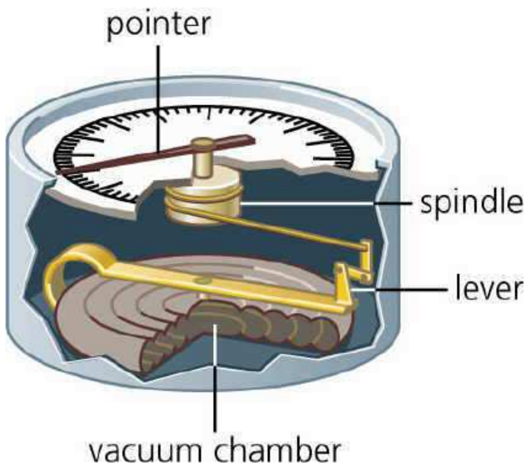
$$P_{atm} = \rho g h$$

Evangelista Torricelli
(1608-1647)

Measurement of Pressure

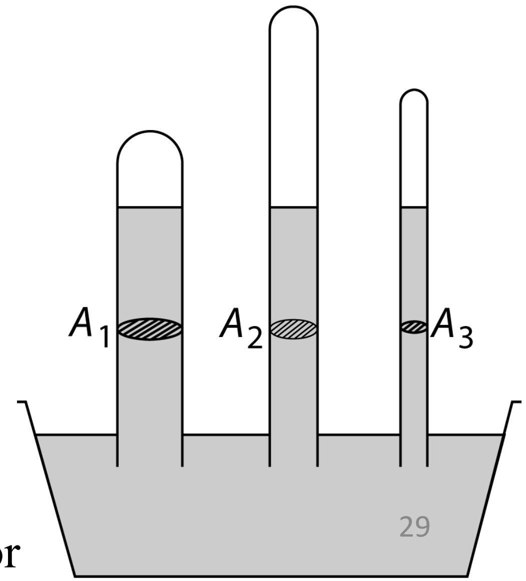
- Barometer
  - At sea-level, with P_{atm} = 101.3 kPa, and ρ_{Hg} = 13.6 ton/m^3, barometric height is h = 0.760 m.
  - A water barometer would be 10.3 m high.
  - Length and cross-sectional area of tube have no effect on h, provided tube diameter is sufficiently large to avoid surface tension (capillary) effects.

Mercury barometer



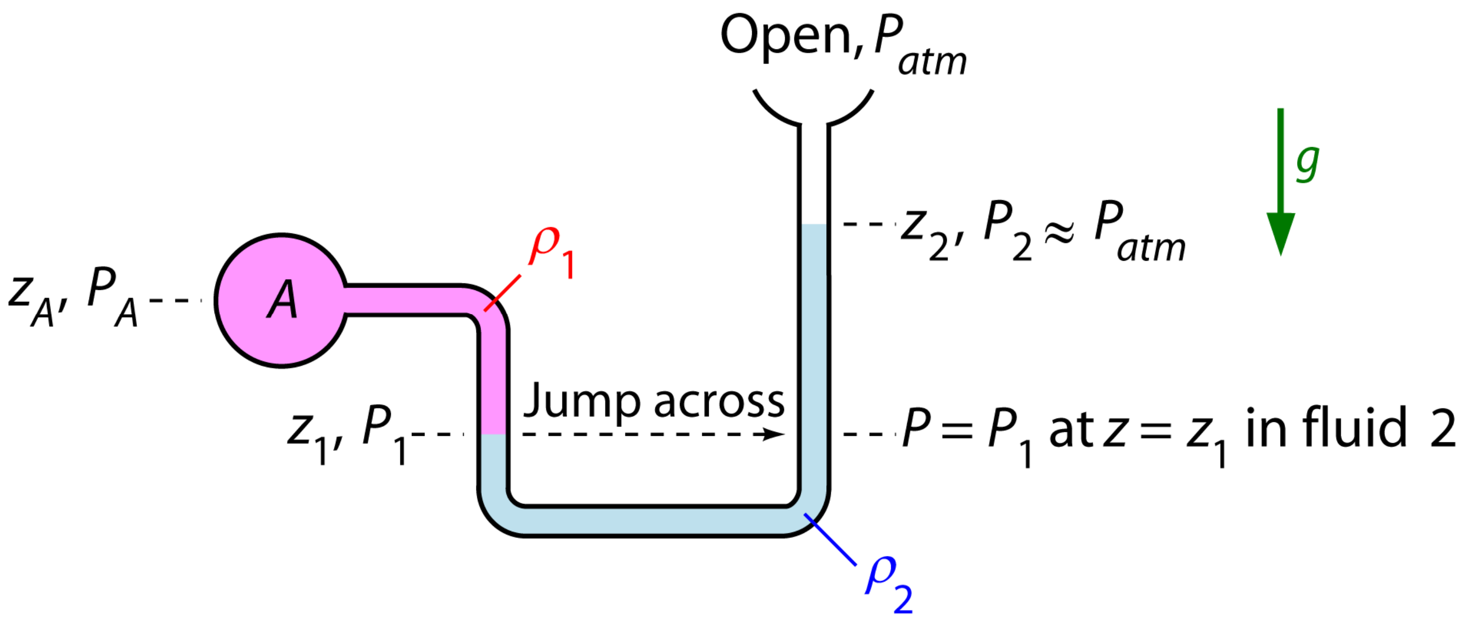
Aneroid barometer

iPhone 6 Barometer Sensor



Measurement of Pressure

- U-Tube Manometer
  - Manometers: vertical or inclined liquid columns for measuring pressure difference.
  - Simple open U-Tube manometer for measuring P_A in a closed chamber relative to atmospheric pressure P_{atm}, i.e. gage pressure.



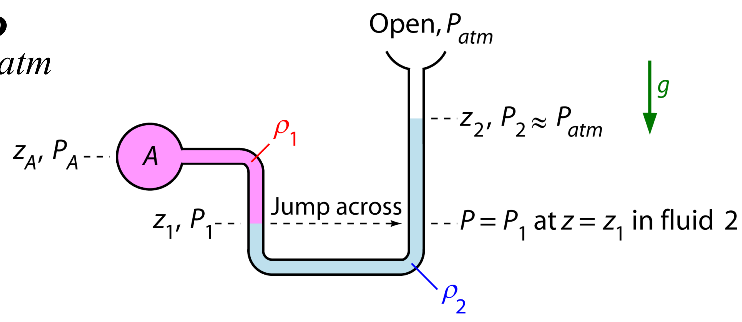
Measurement of Pressure

- U-Tube Manometer
  - Begin at $A \Rightarrow$ move down to level z_1 (add $\rho g|\Delta z|$) $\Rightarrow$ jump across fluid 2 to the same pressure $P_1 \Rightarrow$ move up to level z_2 (subtract $\rho g|\Delta z|$):

$$P_A + \rho_1 g|z_A - z_1| - \rho_2 g|z_1 - z_2| = P_2 \approx P_{atm}$$

$$P_A + \rho_1 g(z_A - z_1) - \rho_2 g(z_2 - z_1) = P_2$$

$$P_A - P_2 = -\rho_1 g(z_A - z_1) - \rho_2 g(z_1 - z_2)$$

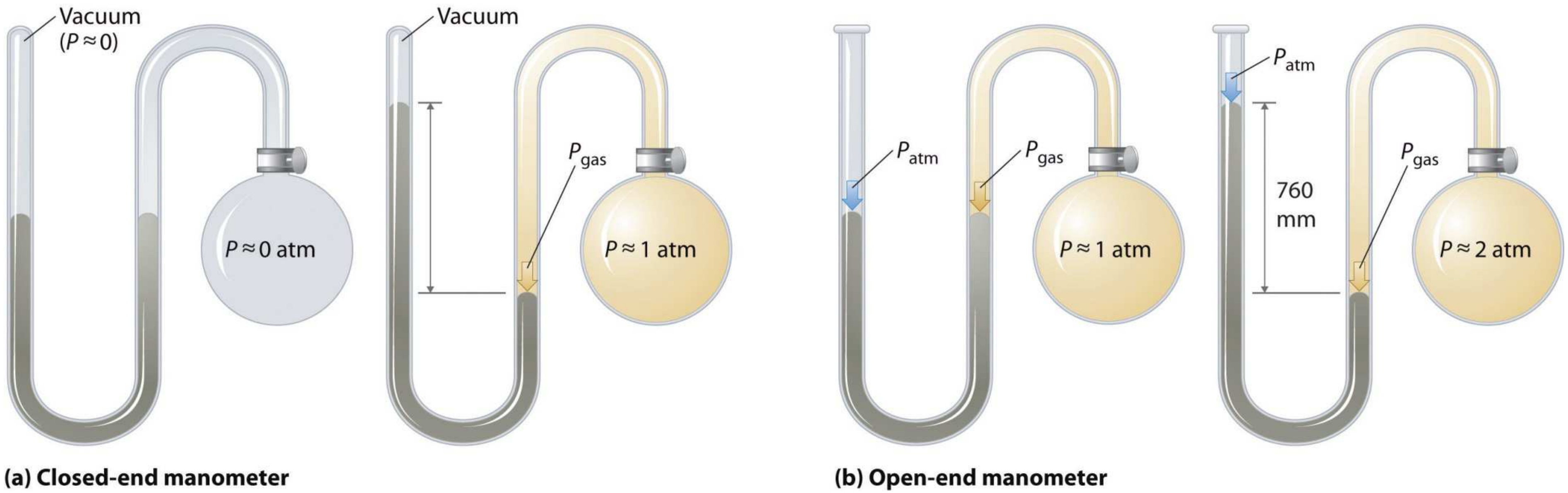


  - Another approach: Apply pressure difference equation repeatedly, jumping across at equal pressures when we come to a continuous column of same fluid:

$$P_A - P_2 = (P_A - P_1) + (P_1 - P_2)$$
$$= -\rho_1 g(z_A - z_1) - \rho_2 g(z_1 - z_2)$$

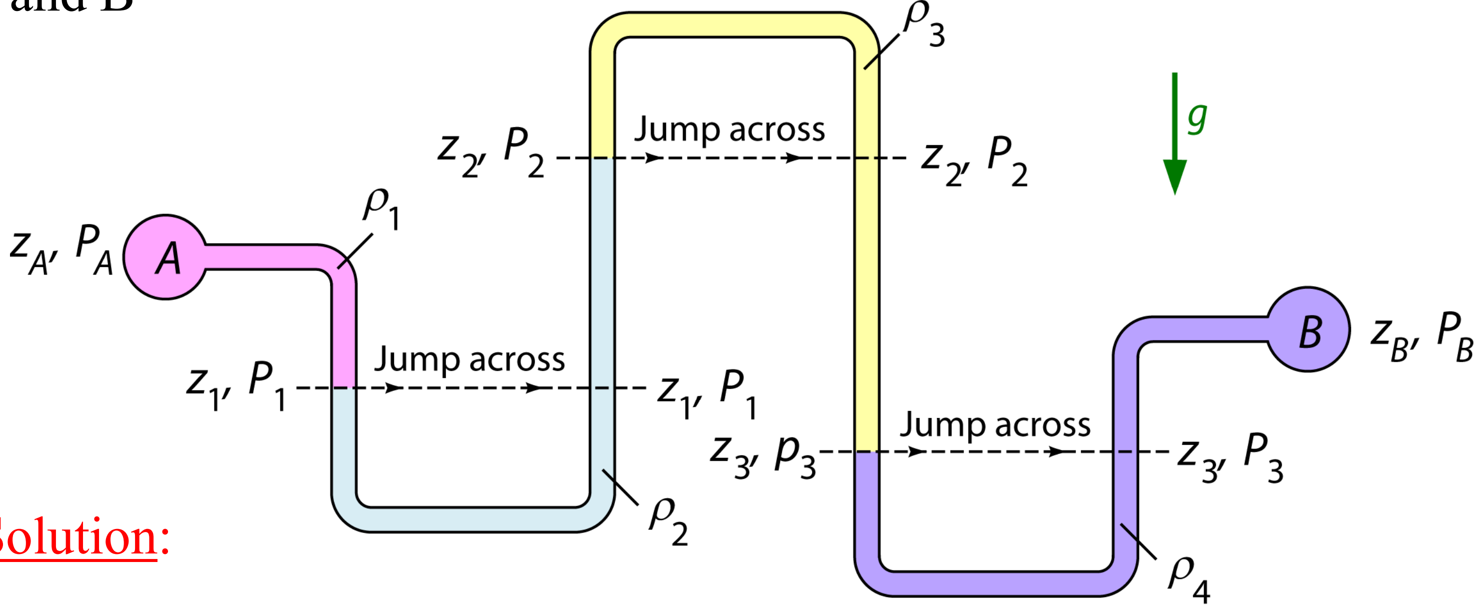
Measurement of Pressure

- U-Tube Manometer
  - Closed-end and open-end manometers:



Measurement of Pressure

- U-Tube Manometer
  - Multiple-fluid manometers: find pressure difference between chambers A and B

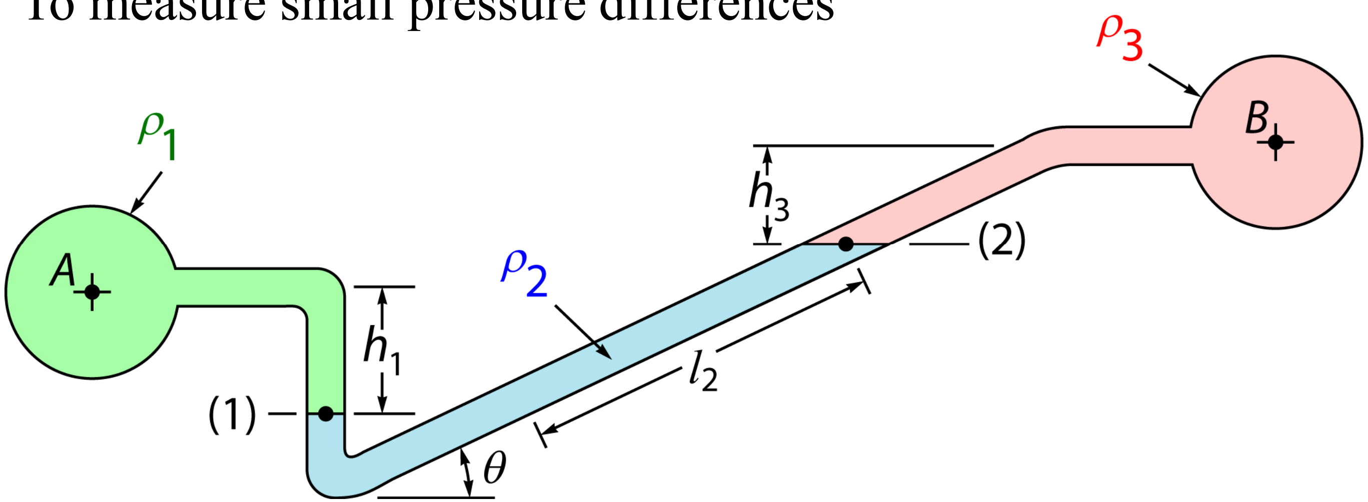


Solution:

$$
\begin{aligned}
&P_A - P_B \\
&= (P_A - P_1) + (P_1 - P_2) + (P_2 - P_3) + (P_3 - P_B) \\
&= -\rho_1 g(z_A - z_1) - \rho_2 g(z_1 - z_2) - \rho_3 g(z_2 - z_3) - \rho_4 g(z_3 - z_B)
\end{aligned}
$$

Measurement of Pressure

- Inclined-Tube Manometer
  - To measure small pressure differences



$$P_A - P_B = (P_A - P_1) + (P_1 - P_2) + (P_2 - P_B)$$

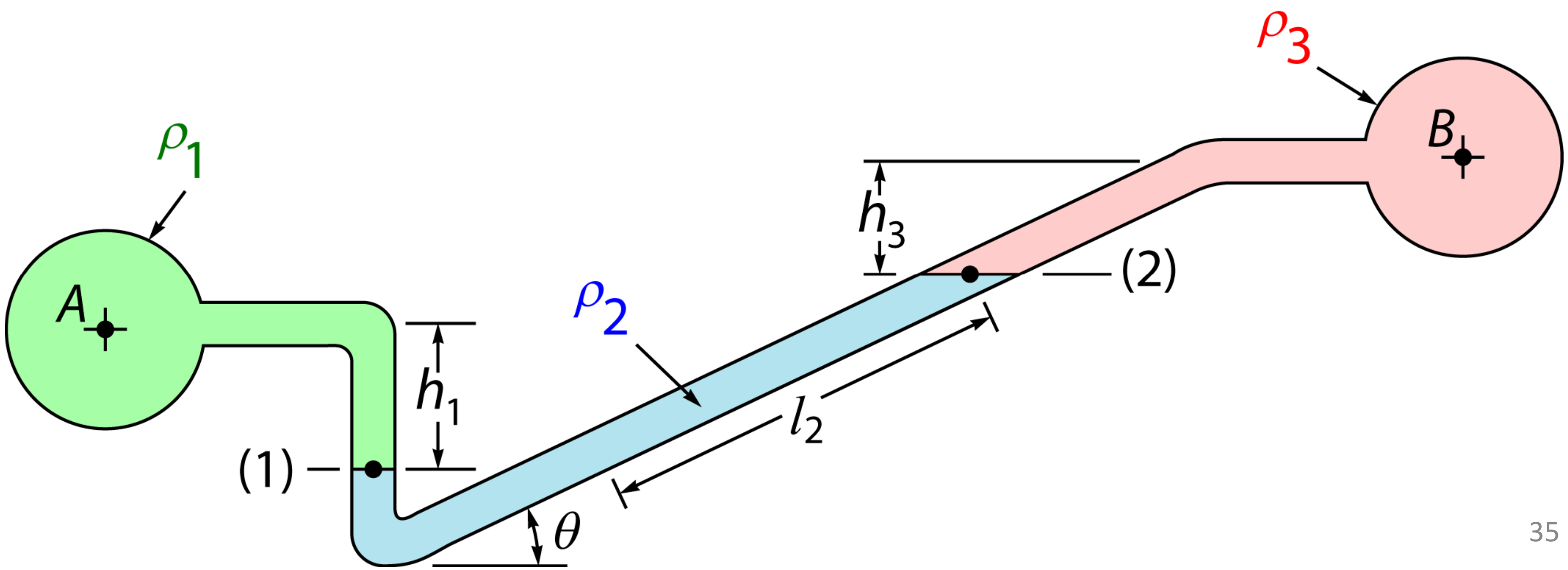
$$= -\rho_1 g h_1 + \rho_2 g l_2 \sin\theta + \rho_3 g h_3$$

$$P_A - P_B = \rho_2 g l_2 \sin\theta + \rho_3 g h_3 - \rho_1 g h_1$$

$$l_2 = \frac{P_A - P_B - \rho_3 g h_3 + \rho_1 g h_1}{\rho_2 g \sin\theta}$$

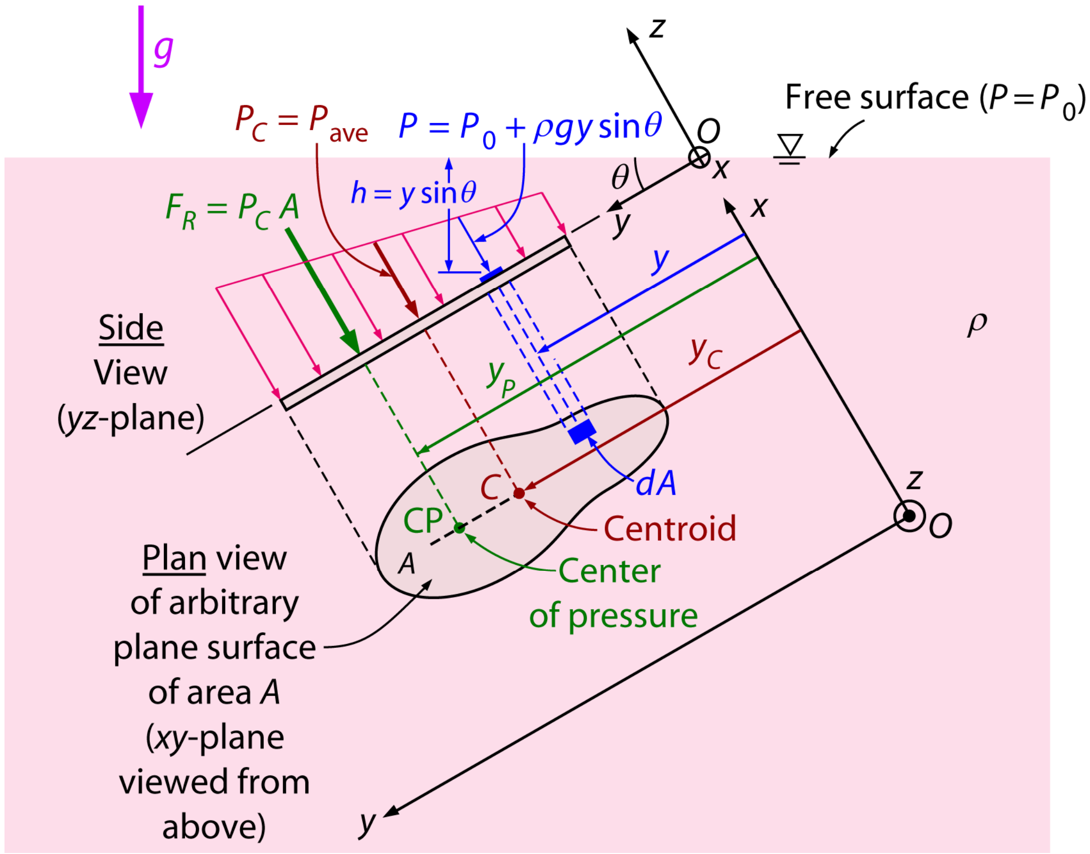
Measurement of Pressure

- Inclined-Tube Manometer
  - For a given pressure difference, differential reading l_2 of inclined-tube manometer can be increased over that obtained with conventional manometer by factor $1/\sin\theta$
  - Make θ small $\Rightarrow$ differential reading along inclined tube becomes large for small pressure differences



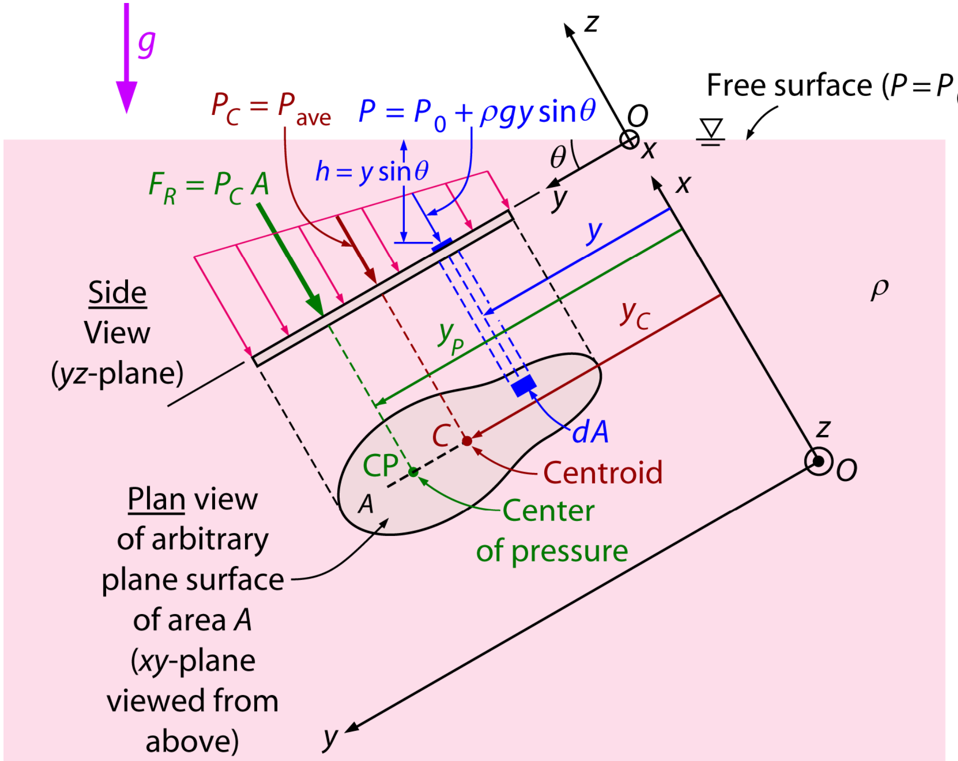
Hydrostatic Forces on Plane Submerged Surfaces

- Problem Definition
  - Consider the top flat, arbitrary shape surface, completely submerged in a liquid
  - Plane surface lies in *xy*-plane, making an angle of θ with the horizontal free surface
  - *x*-axis is the line of intersection of plane surface with horizontal free surface
  - *z*-axis passes through *O* and is normal to plane surface



Hydrostatic Forces on Plane Submerged Surfaces

- Problem Definition
  - On a plane surface, hydrostatic forces form a system of parallel forces need to determine
    - ✓ Magnitude of resultant hydrostatic force
    - ✓ Point of application of resultant hydrostatic force (center of pressure)



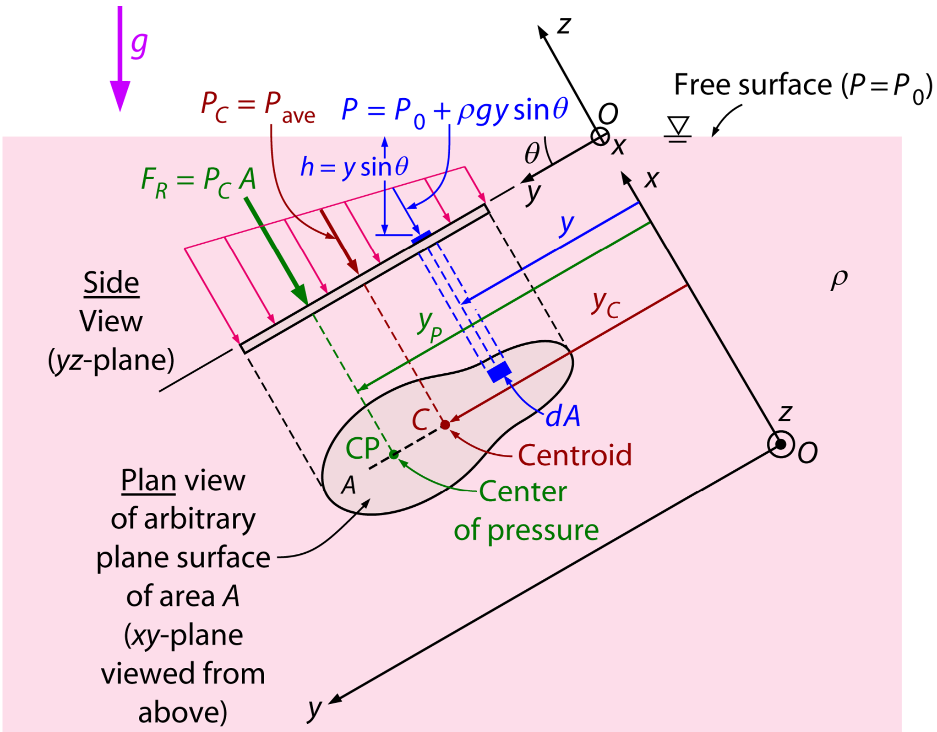
Aim: to find resultant force
and its line of action
STATICS: NO SHEAR STRESS

Hydrostatic Forces on Plane Submerged Surfaces

- Magnitude of Resultant Hydrostatic Force
  - Absolute pressure at any general point on the plate

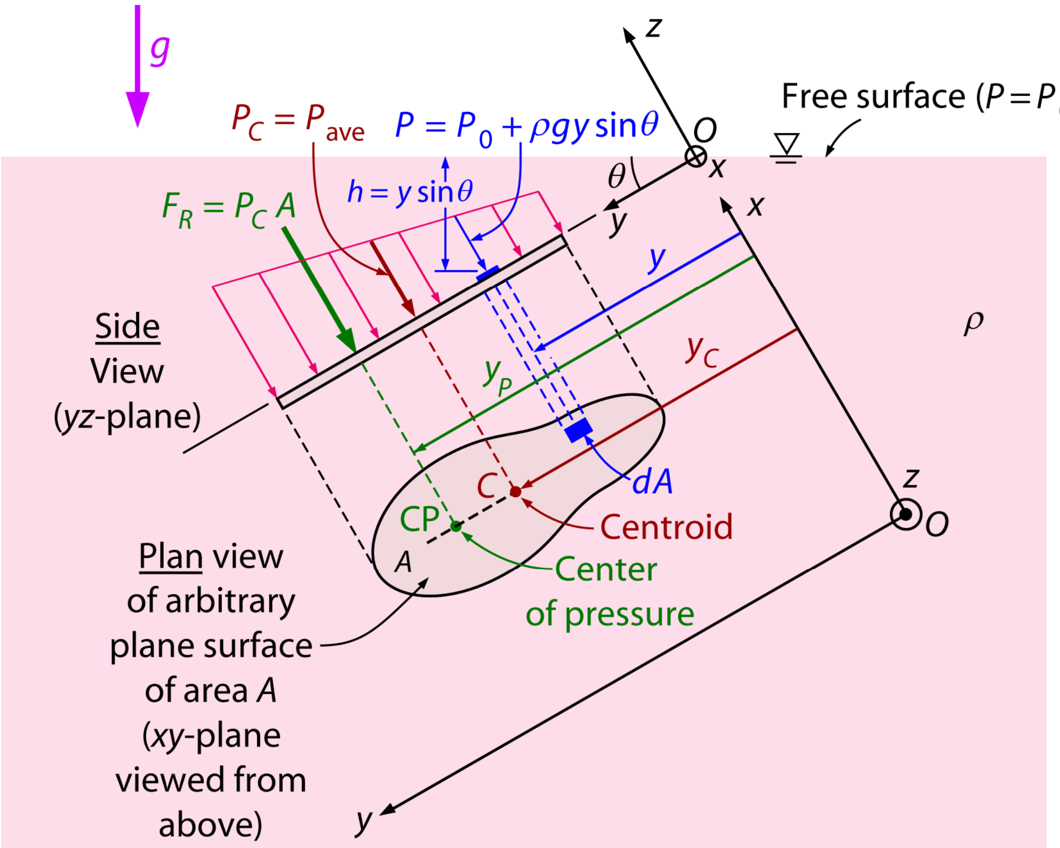
$$P = P_0 + \rho g h \qquad P = P_0 + \rho g y \sin\theta$$

where h: vertical distance of the point from free surface
y: distance of point from x-axis (from point O)



Hydrostatic Forces on Plane Submerged Surfaces

- Magnitude of Resultant Hydrostatic Force
  - Hydrostatic force acting on differential area dA: $dF = PdA$



Resultant hydrostatic force acting on surface:

$$dF = \left(P_0 + \rho g y \sin\theta\right) dA$$

$$F_R = \int_A dF = \int_A P dA$$

$$F_R = \int_A \left(P_0 + \rho g y \sin\theta\right) dA$$

$$F_R = P_0 A + \rho g \sin\theta \int_A y dA$$

Hydrostatic Forces on Plane Submerged Surfaces

- Magnitude of Resultant Hydrostatic Force
  - First moment of area

$$\int_A y dA$$

    - It is a measure of the distribution of the area of a shape in relation to an axis.
    - First moment of area is commonly used to determine the centroid of an area

$$\int_A y dA = \sum_{i=1}^{n} y_i A_i = y_C A$$

where y_C is the y-coordinate of the centroid (or geometric center) of the surface

Hydrostatic Forces on Plane Submerged Surfaces

- Magnitude of Resultant Hydrostatic Force
  - Geometric centre (centroid of the area, centroid of the volume)

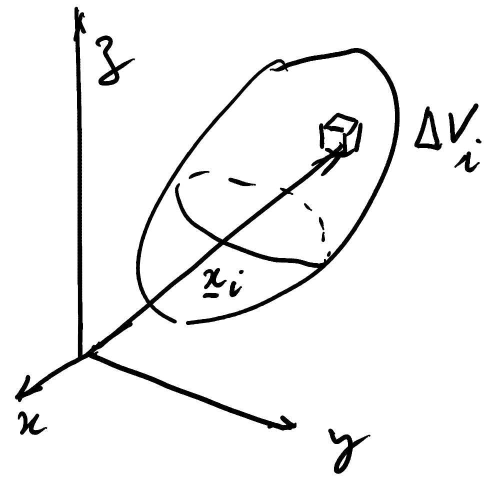
$$(2D)\quad \mathbf{x}_C = \frac{1}{A}\int_A \mathbf{x}\,dA \Rightarrow \left(x_C, y_C\right) = \frac{1}{A}\int_A \left(x, y\right) dA \approx \frac{1}{A}\sum_i \left(x_i, y_i\right)\Delta A_i$$

$$(3D)\quad \mathbf{x}_C = \frac{1}{V}\int_V \mathbf{x}\,dV \Rightarrow \left(x_C, y_C, z_c\right) = \frac{1}{V}\int_V \left(x, y, z\right) dV \approx \frac{1}{V}\sum_i \left(x_i, y_i, z_i\right)\Delta V_i$$

  - Mass centre (centre of gravity):

$$\mathbf{x}_M = \frac{1}{M}\int_A \mathbf{x}\,dm \therefore \left(x_C, y_C, z_C\right) = \frac{1}{M}\int_A \left(x, y, z\right)\rho\,dV$$

$$\approx \frac{1}{M}\sum_i \left(x_i, y_i, z_i\right)\Delta M_i = \frac{1}{M}\sum_i \left(x_i, y_i, z_i\right)\rho_i \Delta V_i$$



  - For homogeneous constant density body, mass centre = centroid

Hydrostatic Forces on Plane Submerged Surfaces

- Magnitude of Resultant Hydrostatic Force

$$F_R = P_0 A + \rho g \sin\theta \int_A y\,dA$$

$$\int_A y\,dA = \sum_{i=1}^{n} y_i A_i = y_C A$$

$$F_R = P_0 A + \rho g \sin\theta \left(y_C A \right)$$

$$F_R = \left(P_0 + \rho g y_C \sin\theta \right) A$$

$$F_R = \left(P_0 + \rho g h_C \right) A$$

$$F_R = P_C A$$

where

$$h_C = y_C \sin\theta$$

is the vertical distance of the centroid C from the free surface of the liquid and

$$P_C = P_0 + \rho g h_C$$

is the pressure at the centroid C of the surface, which is equivalent to the average pressure on the surface.

Hydrostatic Forces on Plane Submerged Surfaces

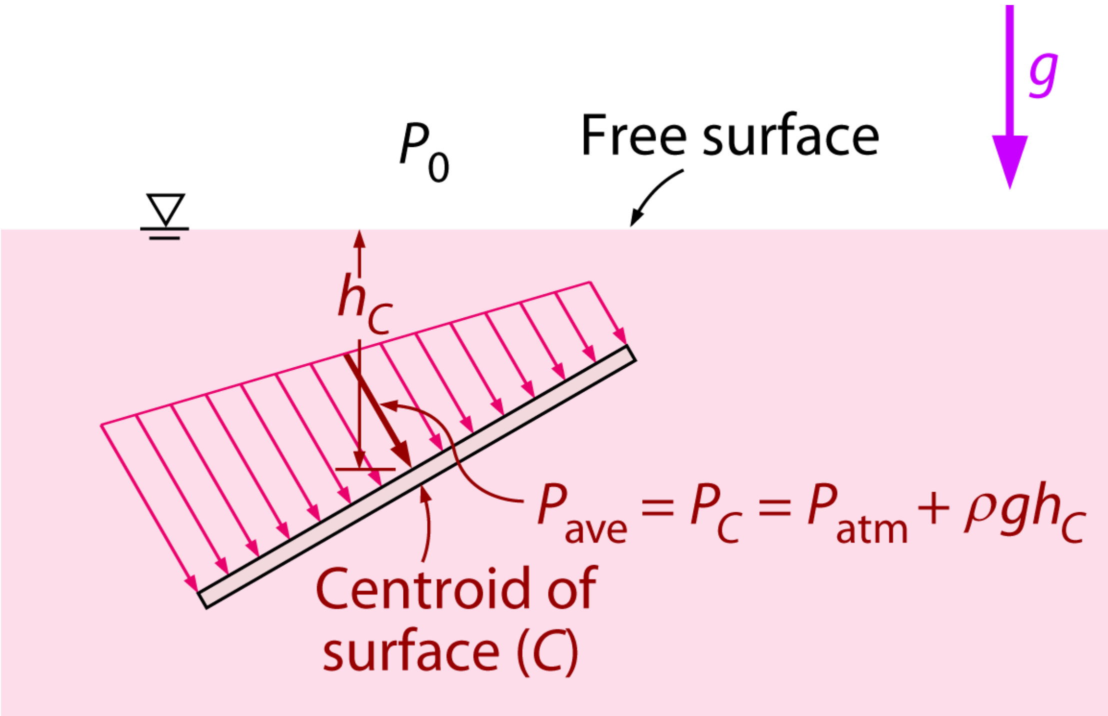
- Magnitude of Resultant Hydrostatic Force

$$F_R = P_C A = P_{ave} A$$

Note: The magnitude of the resultant force acting on a plane surface of a completely submerged plate in a homogeneous (constant density) fluid is equal to the product of the pressure P_C at the centroid of the surface and the area A of the surface

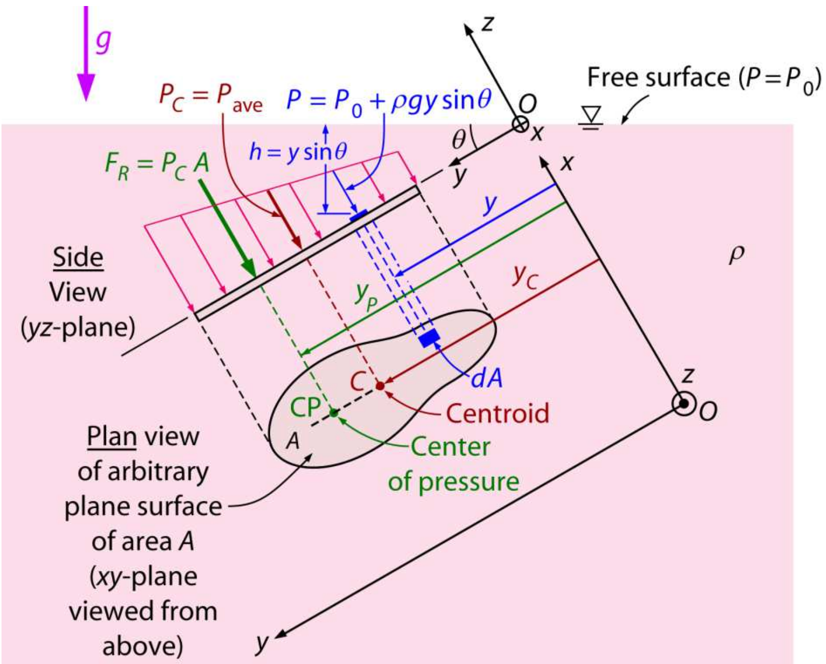
Hydrostatic Forces on Plane Submerged Surfaces

- Direction of Resultant Hydrostatic Force
  - Since all the differential forces that were summed to obtain F_R are perpendicular to the surface, the resultant F_R must also be perpendicular to the surface



Hydrostatic Forces on Plane Submerged Surfaces

- Line Action of Resultant Hydrostatic Force
  - Let line of action of resultant force F_R pass through center of pressure CP with coordinates (x_P, y_P). This point that the resultant force acts is determined by the moment condition



The line of action of a force F_R is a geometric representation of how the force is applied.

Hydrostatic Forces on Plane Submerged Surfaces

- Line Action of Resultant Hydrostatic Force
  - Determination of y_P
    - ✓ y_P is determined by equating moment of resultant force F_R about the x-axis to moment of distributed pressure force about the x-axis

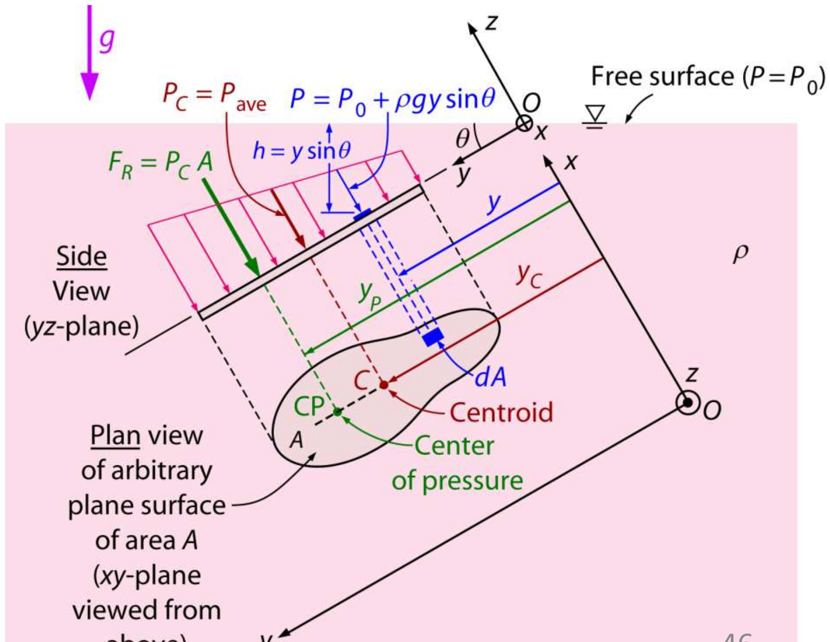
$$y_P F_R = \int_A y dF = \int_A y P dA$$

    - ✓ y_P is the distance of CP from x-axis

$$y_P F_R = \int_A y\left(P_0 + \rho g y \sin\theta\right) dA$$

$$y_P F_R = P_0 \int_A y dA + \rho g \sin\theta \int_A y^2 dA$$

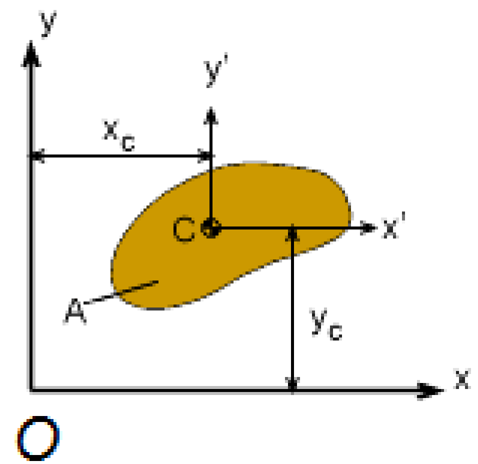
$$y_P F_R = P_0 y_C A + \rho g \sin\theta I_{xx,O}$$



Hydrostatic Forces on Plane Submerged Surfaces

- Line Action of Resultant Hydrostatic Force
  - Second moment of area
    - ✓ Second moment of area of plane surface about the x-axis passing through O:

$$I_{xx,O} = \int_A y^2 dA$$



    - ✓ Parallel axis theorem x-axis

$$I_{xx,O} = I_{xx,C} + y_C^2 A$$

    - ✓ $I_{xx,C}$ is the second moment of area of plane surface about an axis passing through the centroid and parallel to the x-axis
    - ✓ y_C (y-coordinate of centroid) is the distance between the two parallel axes

Hydrostatic Forces on Plane Submerged Surfaces

- Line Action of Resultant Hydrostatic Force
  - Determination of y_P

$$y_P F_R = P_0 y_C A + \rho g \sin\theta I_{xx,O}$$

$$F_R = \left(P_0 + \rho g y_C \sin\theta\right) A \qquad\qquad I_{xx,O} = I_{xx,C} + y_C^2 A$$

$$y_P\left(P_0 + \rho g y_C \sin\theta\right) A = P_0 y_C A + \rho g \sin\theta\left(I_{xx,C} + y_C^2 A\right)$$

$$y_P P_0 A - y_C P_0 A + y_P y_C \rho g A \sin\theta - y_C^2 \rho g A \sin\theta = \rho g \sin\theta I_{xx,C}$$

$$\left(y_P - y_C\right) P_0 A + \left(y_P - y_C\right) y_C \rho g A \sin\theta = \rho g \sin\theta I_{xx,C}$$

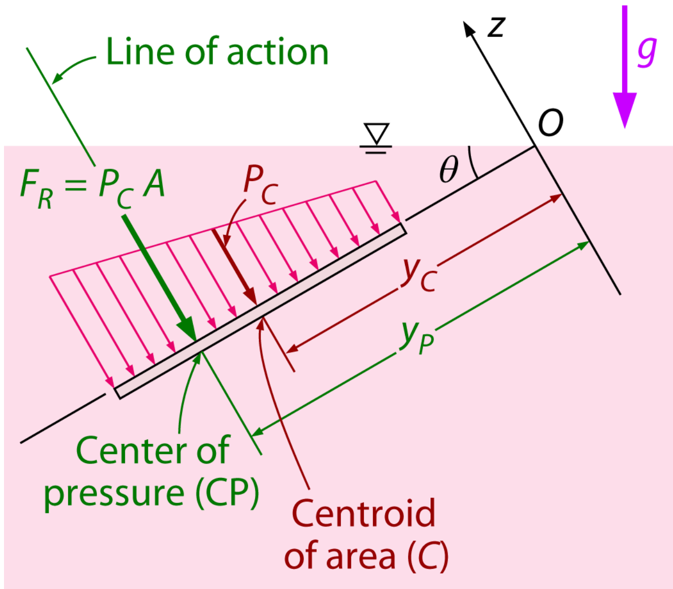
$$y_P - y_C = \frac{\rho g \sin\theta I_{xx,C}}{P_0 A + y_C \rho g A \sin\theta}$$

$$y_P = y_C + \frac{I_{xx,C}}{\left[P_0/\left(\rho g \sin\theta\right) + y_C\right] A}$$

Hydrostatic Forces on Plane Submerged Surfaces

- Line Action of Resultant Hydrostatic Force
  - Determination of y_P
    - ✓ If $P_0 = 0$ (considering gage pressures)

$$y_P = y_C + \frac{I_{xx,C}}{y_C A}$$

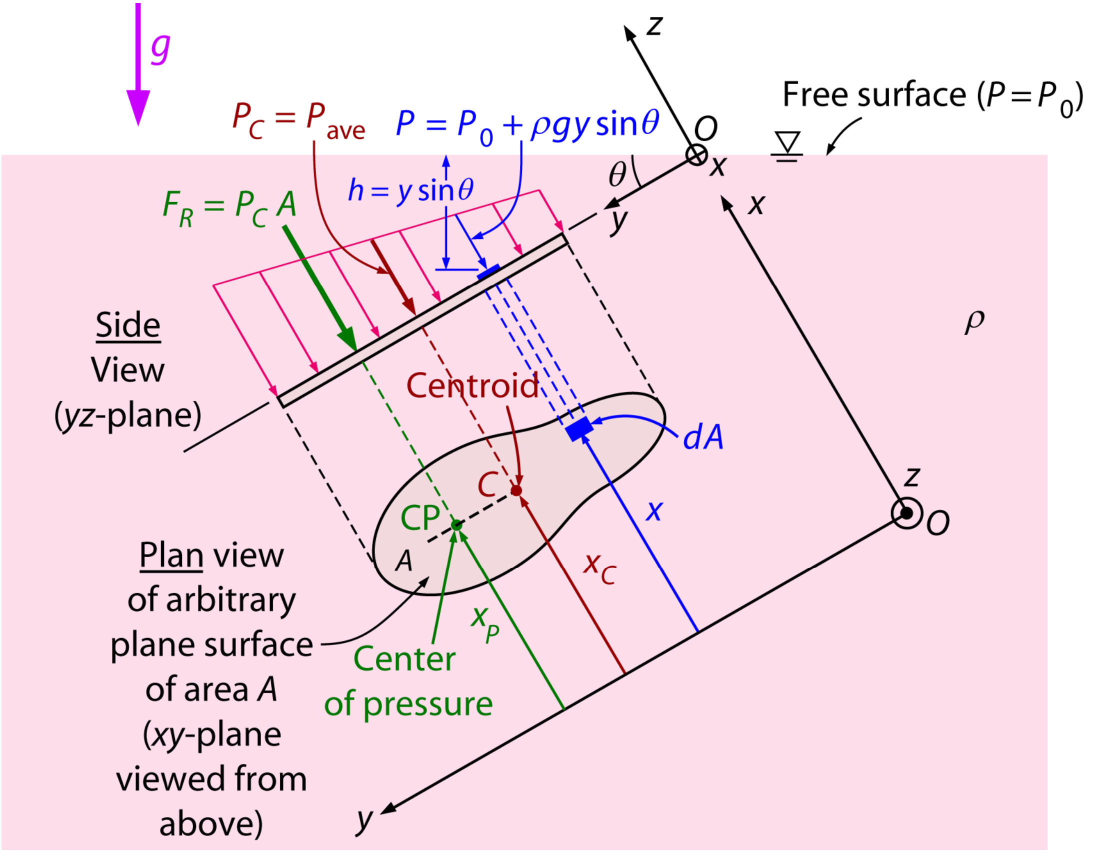


- ✓ Resultant for F_R does not pass through centroid C but pass through center of pressure CP

- ✓ Since $\frac{I_{xx,C}}{y_C A} > 0 \quad \Rightarrow y_p > y_c \Rightarrow CP$ lower than C (except when $\theta = 0°$)

Hydrostatic Forces on Plane Submerged Surfaces

- Line Action of Resultant Hydrostatic Force
  - Determination of x_P



Hydrostatic Forces on Plane Submerged Surfaces

- Line Action of Resultant Hydrostatic Force
  - Determination of x_P
    - ✓ Summing moments about the y-axis

$$x_P F_R = \int_A x dF = \int_A x P dA$$

$$x_P F_R = \int_A x\left(P_0 + \rho g y \sin\theta\right) dA$$

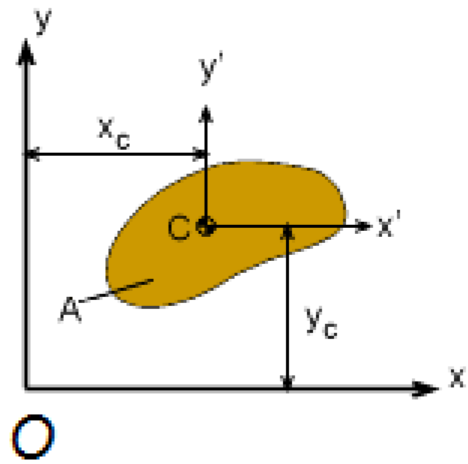
$$x_P F_R = P_0 \int_A x dA + \rho g \sin\theta \int_A xy dA$$

$$x_P F_R = P_0 x_C A + \rho g \sin\theta I_{xy,O}$$

Hydrostatic Forces on Plane Submerged Surfaces

- Line Action of Resultant Hydrostatic Force
  - Cross moment of area
    - ✓ Cross moment of area of plane surface about the *x*- and *y*-axes passing through O:

$$I_{xy,O} = \int_A xy\,dA$$



    - ✓ Parallel axis theorem:

$$I_{xy,O} = I_{xy,C} + x_C y_C A$$

    - ✓ $I_{xy,C}$ is the cross moment of area of plane surface about axes passing through the centroid and parallel to the *x*- and *y*-axes

Hydrostatic Forces on Plane Submerged Surfaces

- Line Action of Resultant Hydrostatic Force
  - Determination of x_P

$$x_P F_R = P_0 x_C A + \rho g \sin\theta I_{xy,O}$$

$$F_R = \left(P_0 + \rho g y_C \sin\theta\right) A \qquad\qquad I_{xy,O} = I_{xy,C} + x_C y_C A$$

$$x_P\left(P_0 + \rho g y_C \sin\theta\right) A = P_0 x_C A + \rho g \sin\theta\left(I_{xy,C} + x_C y_C A\right)$$

$$x_P P_0 A - x_C P_0 A + x_P y_C \rho g A \sin\theta - x_C y_C \rho g A \sin\theta = \rho g \sin\theta I_{xy,C}$$

$$\left(x_P - x_C\right) P_0 A + \left(x_P - x_C\right) y_C \rho g A \sin\theta = \rho g \sin\theta I_{xy,C}$$

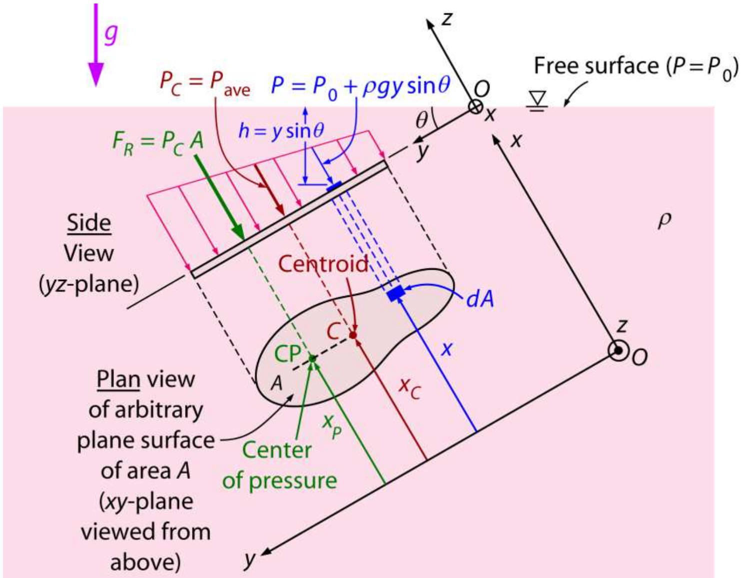
$$x_P - x_C = \frac{\rho g \sin\theta I_{xy,C}}{P_0 A + y_C \rho g A \sin\theta}$$

$$x_P = x_C + \frac{I_{xy,C}}{\left[P_0/\left(\rho g \sin\theta\right) + y_C\right] A}$$

Hydrostatic Forces on Plane Submerged Surfaces

- Line Action of Resultant Hydrostatic Force
  - Determination of x_P
    - ✓ If $P_0 = 0$ (considering gage pressures)

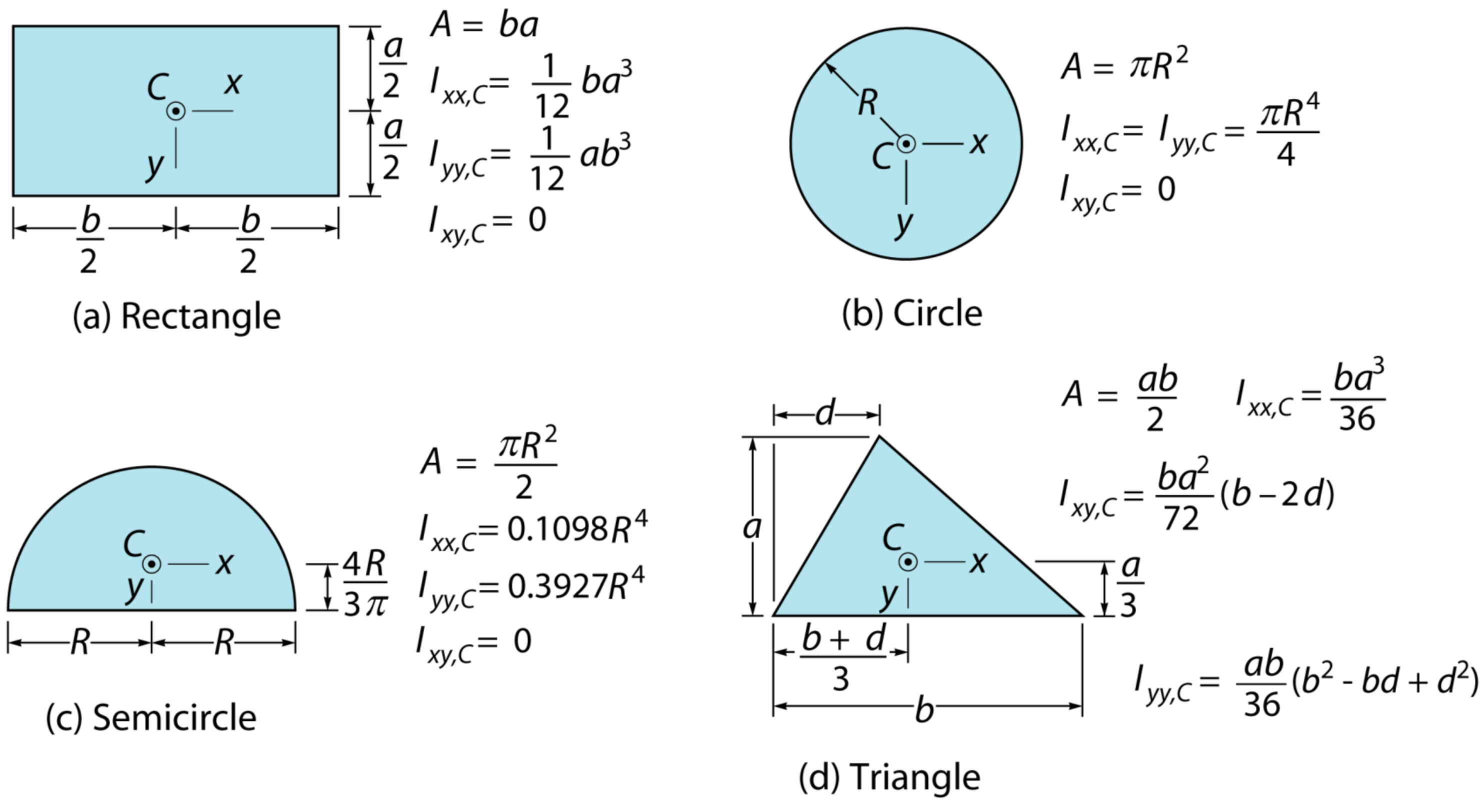
$$x_P = x_C + \frac{I_{xy,C}}{y_C A}$$



    - ✓ $I_{xy,C}$ can be positive, negative or zero
    - ✓ $I_{xy,C} = 0 \Rightarrow$ plane surface is symmetrical with respect to an axis passing through the centroid and parallel to either the x- or y-axes $\Rightarrow x_P = x_C \Rightarrow$ CP lies directly below C along the y-axis
    - ✓ Can assume $P_0 = 0$ if same ambient pressure acting on both sides of surface

Hydrostatic Forces on Plane Submerged Surfaces

- Line Action of Resultant Hydrostatic Force
  - Second moment of area
    - ✓ Centroidal coordinates and moments of area for some common areas



(a) Rectangle

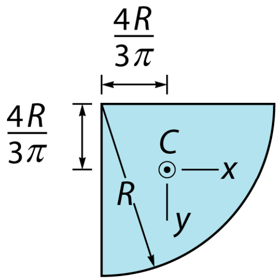
(b) Circle

(c) Semicircle

(d) Triangle

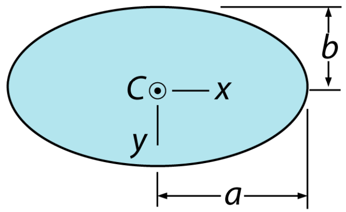
Hydrostatic Forces on Plane Submerged Surfaces

- Line Action of Resultant Hydrostatic Force
  - Second moment of area
    - ✓ Centroidal coordinates and moments of area for some common areas



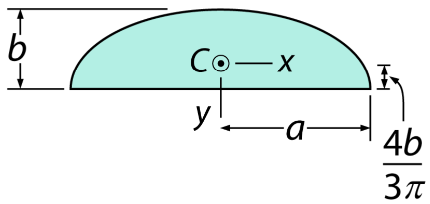
(e) Quadrant

$$A = \frac{\pi R^2}{4}$$
$$I_{xx,C} = I_{yy,C} = 0.05488\,R^4$$
$$I_{xy,C} = -0.01647\,R^4$$



(f) Ellipse

$$A = ab$$
$$I_{xx,C} = \frac{\pi ab^3}{4}$$
$$I_{yy,C} = \frac{\pi ba^3}{4}$$
$$I_{xy,C} = 0$$



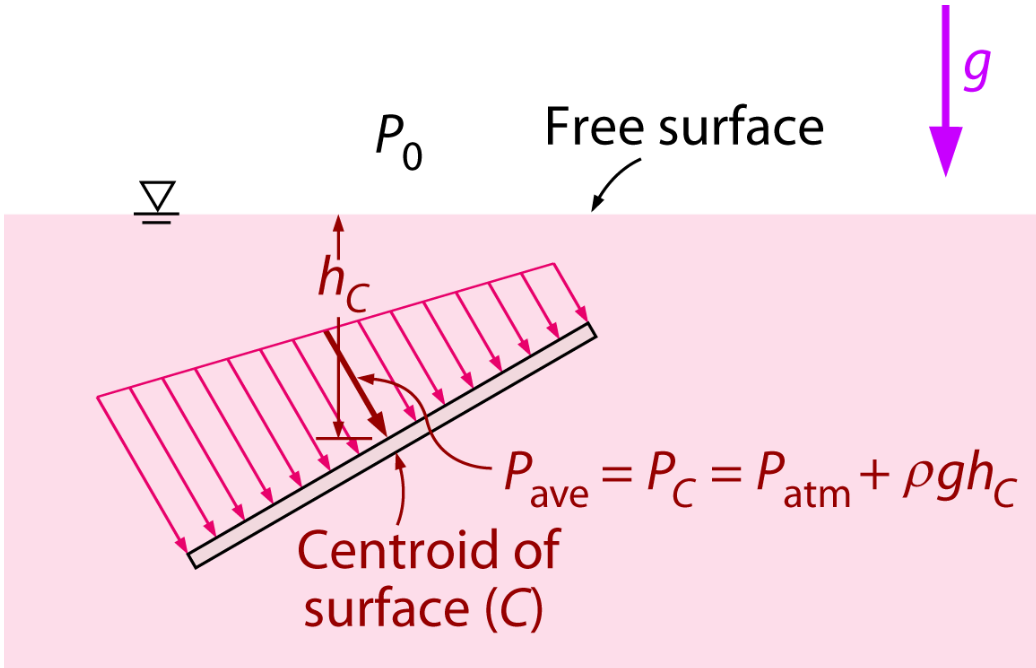
(g) Semiellipse

$$A = \frac{\pi ab}{2}$$
$$I_{xx,C} = 0.1098\,ab^3$$
$$I_{yy,C} = 0.3927\,ba^3$$
$$I_{xy,C} = 0$$

Hydrostatic Forces on Plane Submerged Surfaces

- Summary
  - The magnitude of the resultant force acting on a plane surface of a completely submerged plate in a homogeneous (constant density) fluid is equal to the product of the pressure P_C at the centroid of the surface and the area A of the surface

$$F_R = P_C A = P_{ave} A$$

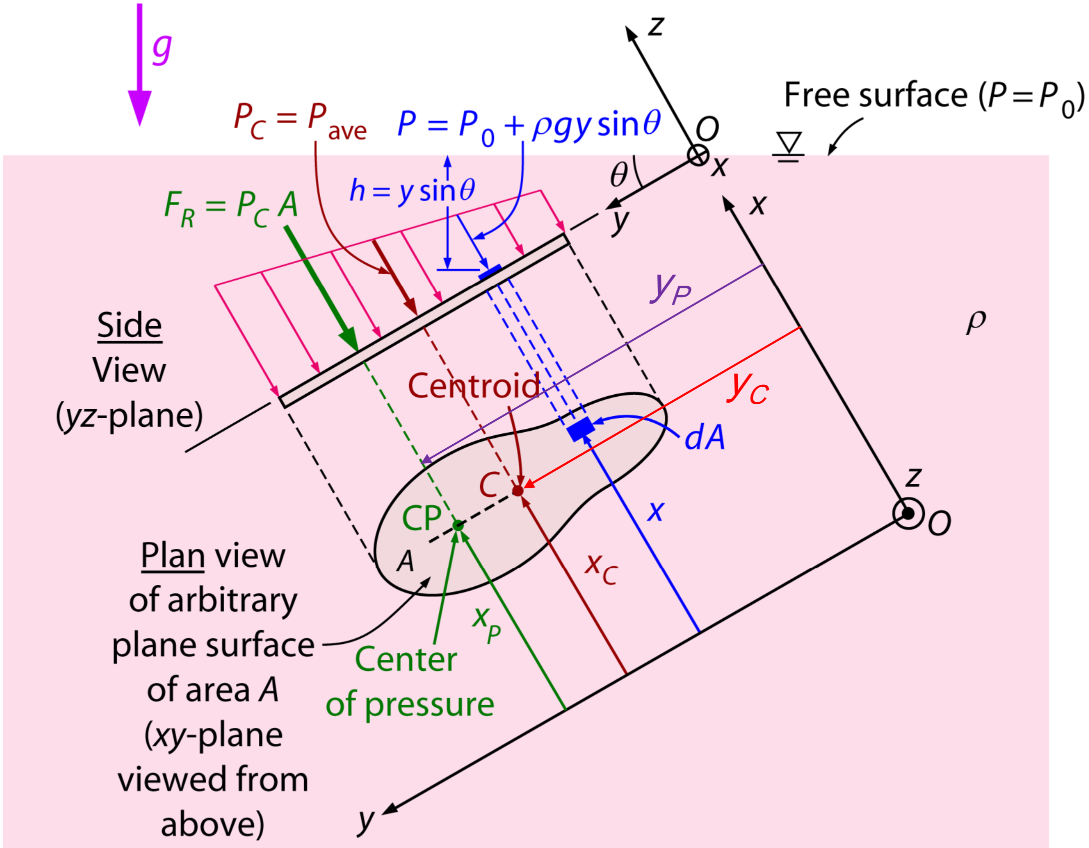


Hydrostatic Forces on Plane Submerged Surfaces

- Summary
  - In the case of gage pressure (or set $P_0 = 0$)

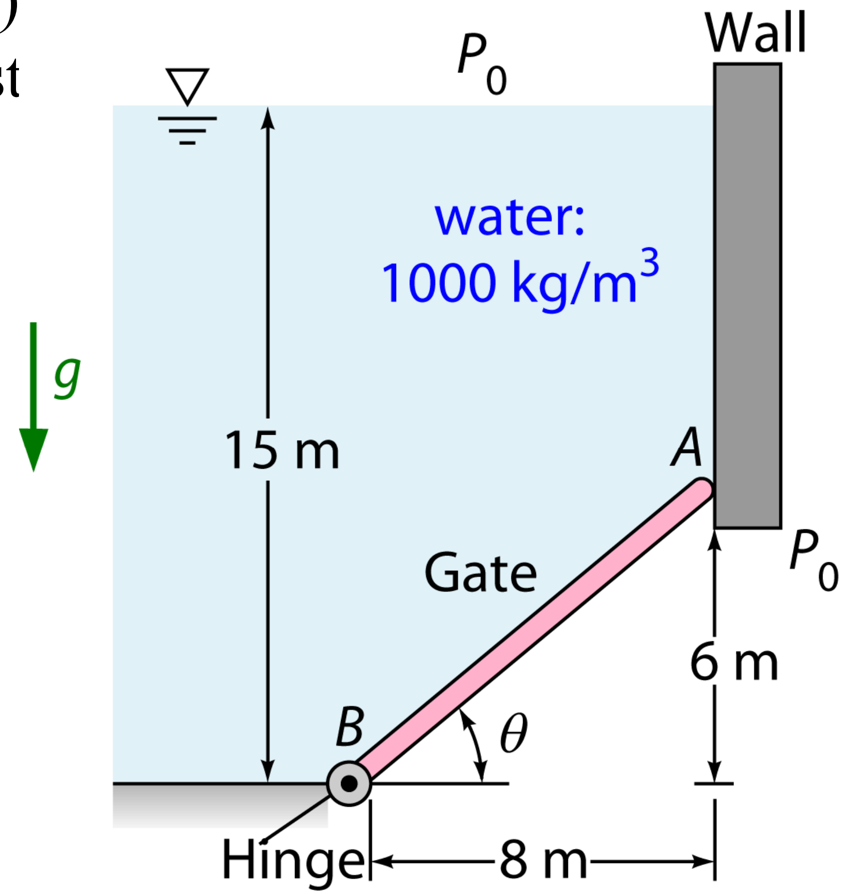
$$y_P = y_C + \frac{I_{xx,C}}{y_C A}$$

$$x_P = x_C + \frac{I_{xy,C}}{y_C A}$$



Hydrostatic Forces on Plane Submerged Surfaces

- Example 1
  - Gate (5 m wide and 10m long) is hinged at *B* and rests against smooth wall at *A*
  - Find:
    a) Force on gate due to water pressure
    b) Horizontal force *P* exerted by wall at *A*
    c) Reactions at hinge *B*



Hydrostatic Forces on Plane Submerged Surfaces

- Example 1
  - Solution for Question part (a) :
    - ✓ Gate is 10 m long from *A* to *B* ⇨ centroid (CG) is halfway between at elevation 3 m above *B*
    - ✓ Depth of centroid $h_C = 15 - 3 = 12$ m
    - ✓ Gate area = $10 \times 5 = 50$ m^2
    - ✓ P_0 acting on both sides of gate

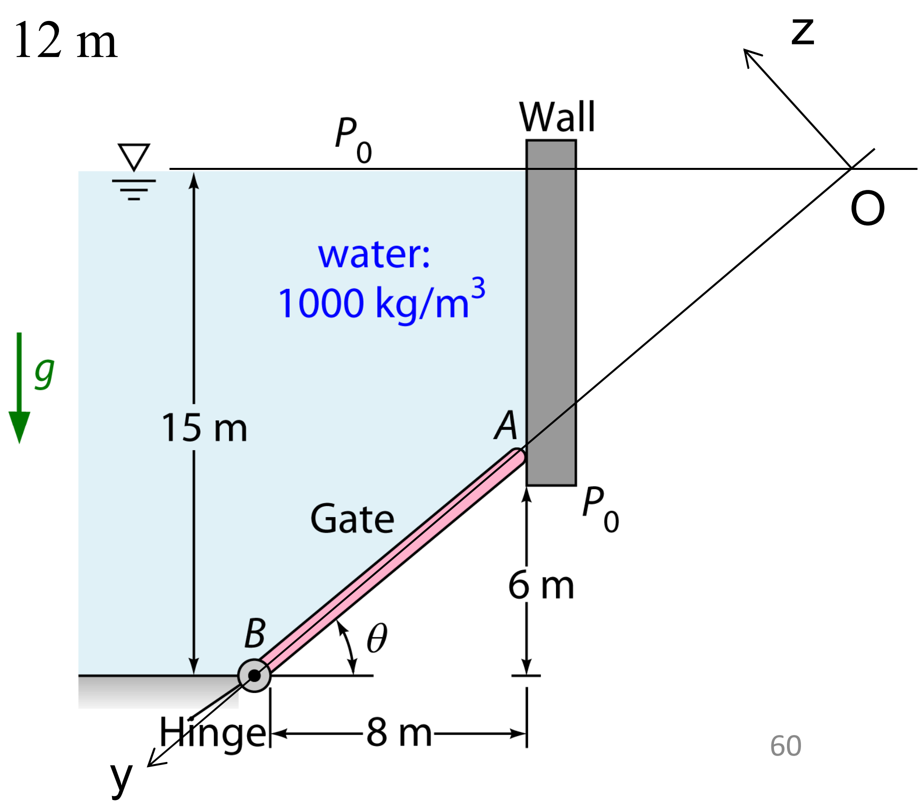
      $P_0 = 0$
    - ✓ Hydrostatic force on gate:

$$F_R = P_C A$$

$$F_R = \rho g h_C A$$

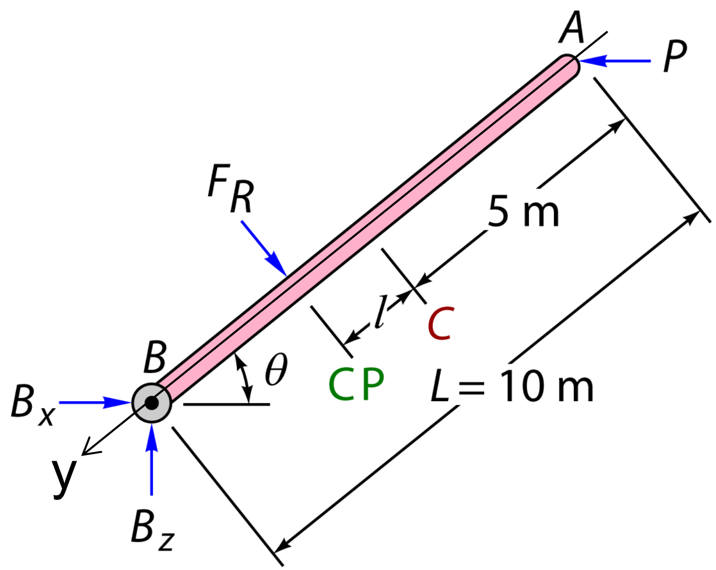
$$F_R = (1000)(9.81)(12)(50)$$

$$F_R = 5.886 \times 10^6 \text{ N}$$



Hydrostatic Forces on Plane Submerged Surfaces

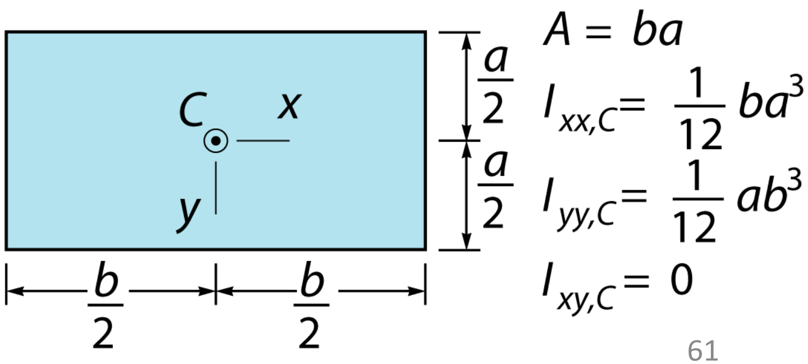
- Example 1
  - Solution for Question part (b) :
    - ✓ First find center of pressure of F_R
    - ✓ Gate is a rectangle:



$$I_{xy,C} = 0$$

$$I_{xx,C} = \frac{ba^3}{12} = \frac{(5)(10)^3}{12} = 417\ \text{m}^4$$

- ✓ Centroid (C)



$$h_C = y_C \sin\theta$$

$$y_C = \frac{h_C}{\sin\theta} = \frac{12}{(3/5)} = 20\ \text{m}$$

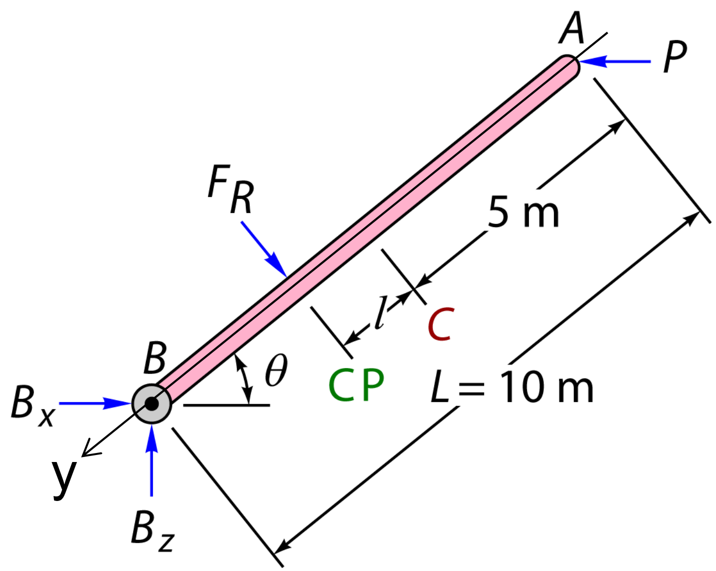
Hydrostatic Forces on Plane Submerged Surfaces

- Example 1
  - Solution for Question part (b) :
    - ✓ Center of Pressure (CP):

$$y_P = y_C + \frac{I_{xx,C}}{y_C A}, \quad x_P = x_C$$

$$l = y_P - y_C = \frac{I_{xx,C}}{y_C A}$$

$$l = \frac{417}{(20)(50)} = 0.417 \text{ m}$$



    - ✓ Distance of B to force $F_R = 10 - 1 - 5 = 4.583$ m
    - ✓ Taking moments counterclockwise about B:

$$PL\sin\theta - F_R(5 - l) = 0$$

$$P(10)(3/5) - (5.886 \times 10^6)(5 - 0.417) = 0$$

$$P = 4.496 \times 10^6 \text{ N}$$

Hydrostatic Forces on Plane Submerged Surfaces

- Example 1
  - Solution for Question part (c) :
    - ✓ Summing forces on gate:

$$\sum F_x = 0$$

$$B_x + F_R \sin\theta - P = 0$$

$$B_x + \left(5.886\times10^6\right)\left(3/5\right) - 4.496\times10^6 = 0$$

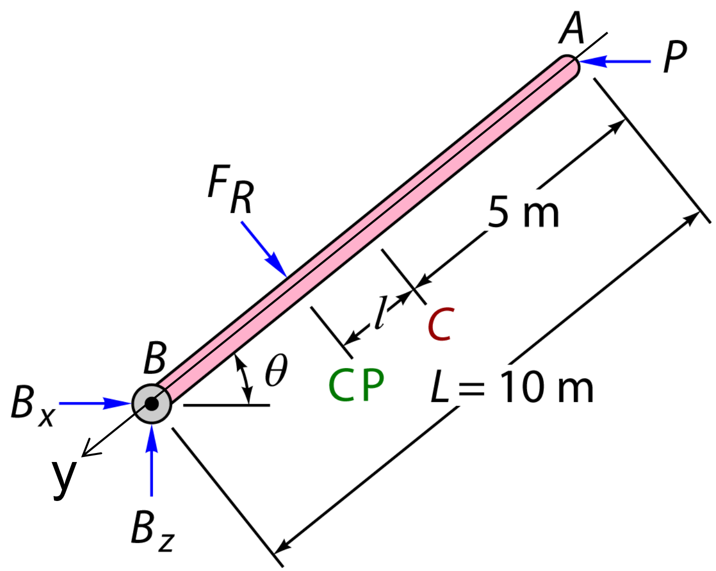
$$B_x = 0.964\times10^6 \text{ N}$$

$$\sum F_z = 0$$

$$B_z - F_R \cos\theta = 0$$

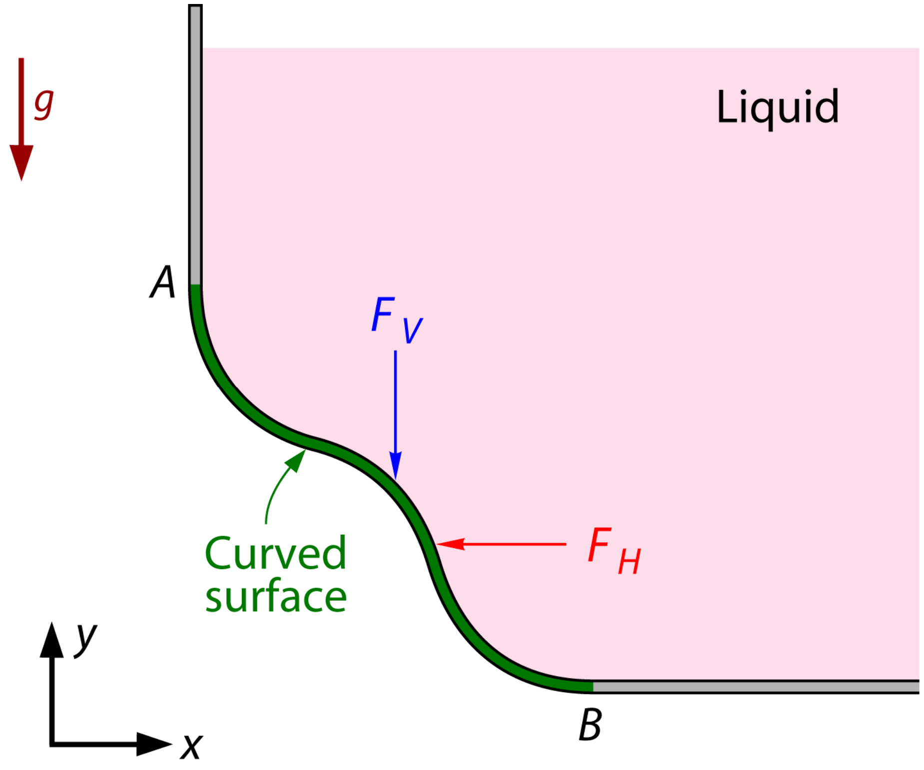
$$B_z - \left(5.886\times10^6\right)\left(4/5\right) = 0$$

$$B_z = 4.709\times10^6 \text{ N}$$



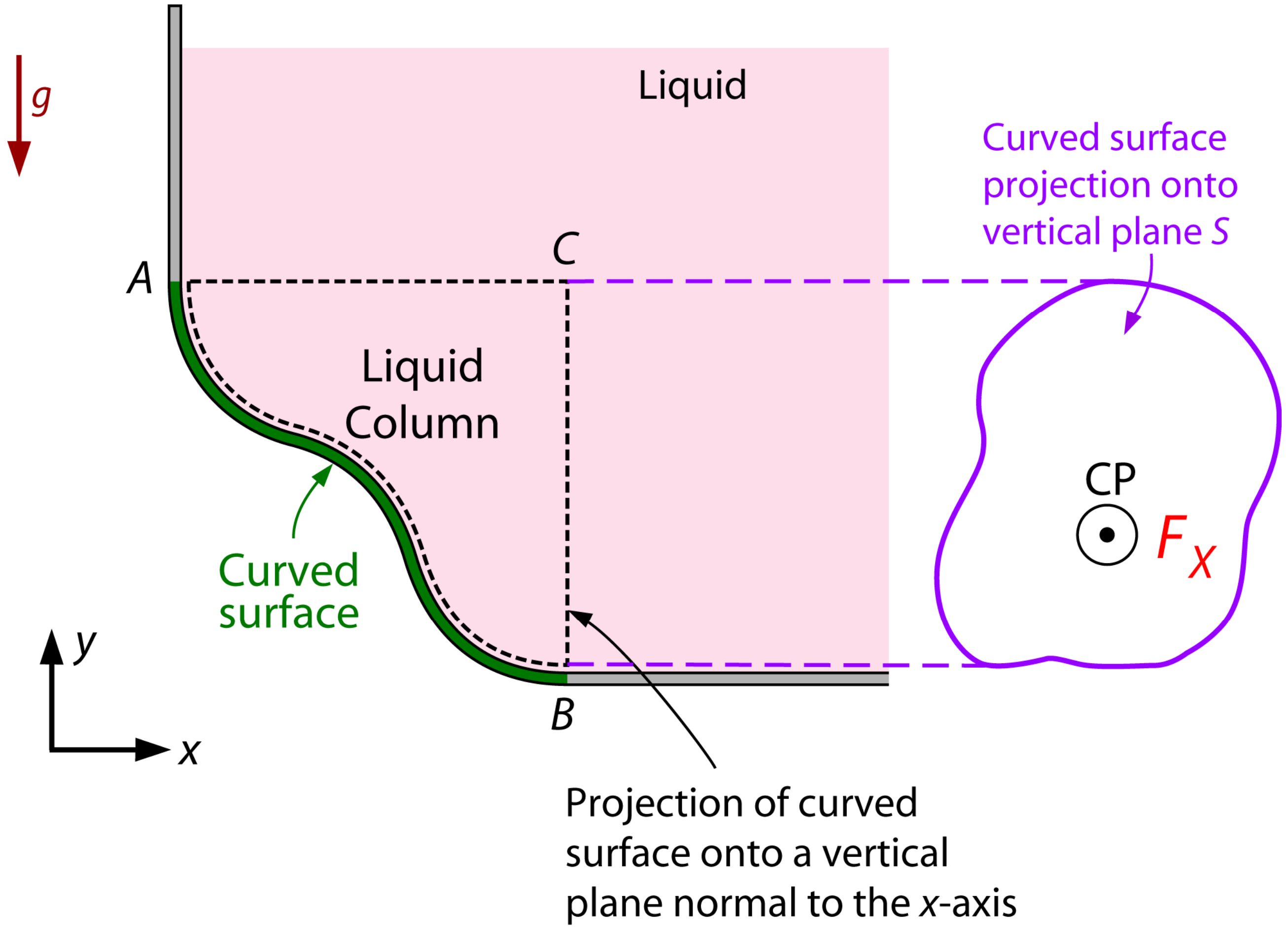
Hydrostatic Forces on Curved Submerged Surfaces

- Problem Definition
  - Consider arbitrary curved surface
  - Incremental pressure forces are normal to the local area element $\Rightarrow$ forces vary in direction along the surface $\Rightarrow$ cannot be added numerically
  - Separate into horizontal component F_H and vertical component F_V



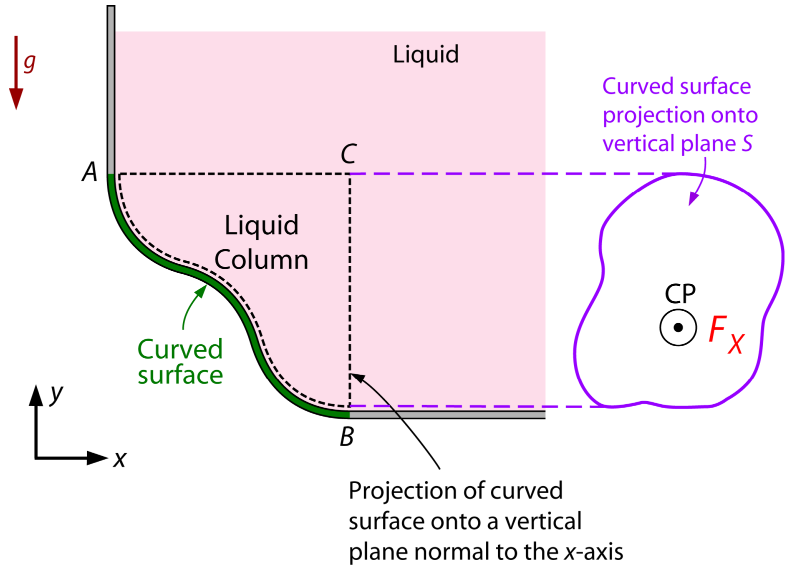
Hydrostatic Forces on Curved Submerged Surfaces

- Horizontal Component



Hydrostatic Forces on Curved Submerged Surfaces

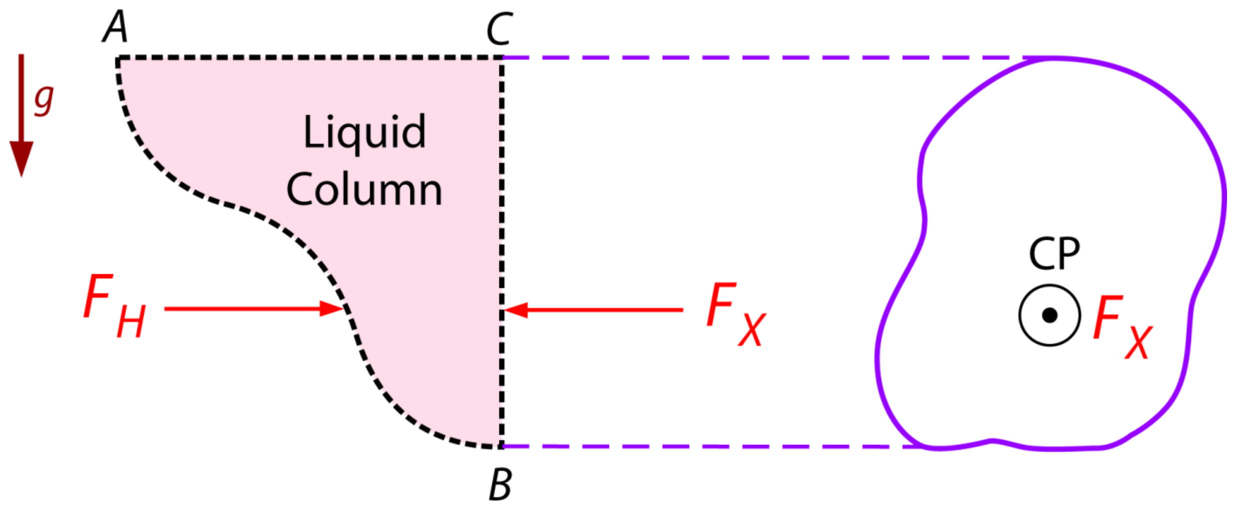
- Horizontal Component
  - Project curved surface AB horizontally (along x-axis) onto vertical plane $BC \Rightarrow$ get projected area S on vertical plane AB
  - Projected area S lies on a vertical plane ($\theta = 90°$)
    - ✓ determine centroid C and center of pressure CP
    - ✓ determine magnitude and line of action of resultant horizontal force due to hydrostatic pressure F_X



  - Consider column of fluid enclosed by curved surface AB and projected area S lying on vertical plane BC:

Hydrostatic Forces on Curved Submerged Surfaces

- Horizontal Component



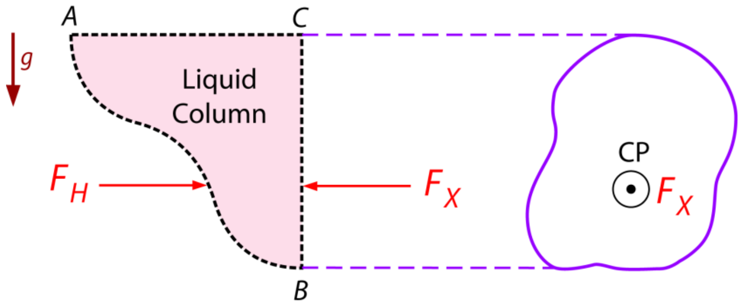
  - $F_H \leftarrow$ is the horizontal component of the force exerted by the fluid on the curved surface AB
  - By Newton's third law, $F_H \rightarrow$ is the horizontal component of the force exerted by the curved surface on the fluid (liquid column)

Hydrostatic Forces on Curved Submerged Surfaces

- Horizontal Component
  - Liquid column is in static equilibrium ⇒ horizontal forces must balance

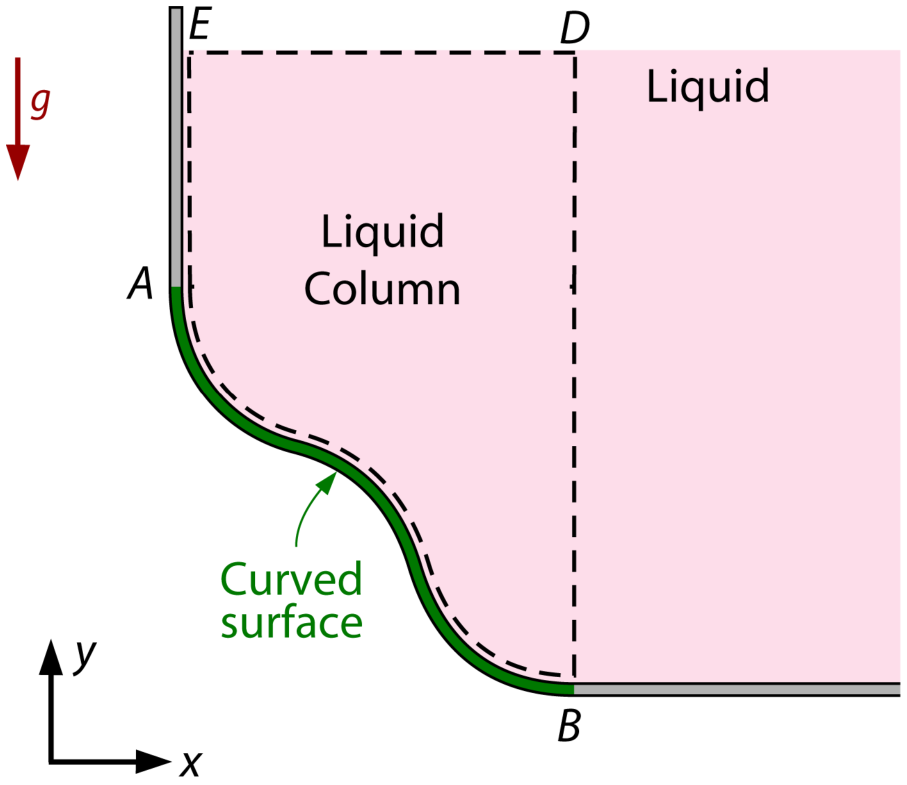
$$F_H = F_X$$

  - The horizontal component of hydrostatic force acting on a curved surface is equal to the force on the plane area formed by the projection of the curved surface onto a vertical plane normal to the component. It acts through the center of pressure (not centroid) of the projected area.



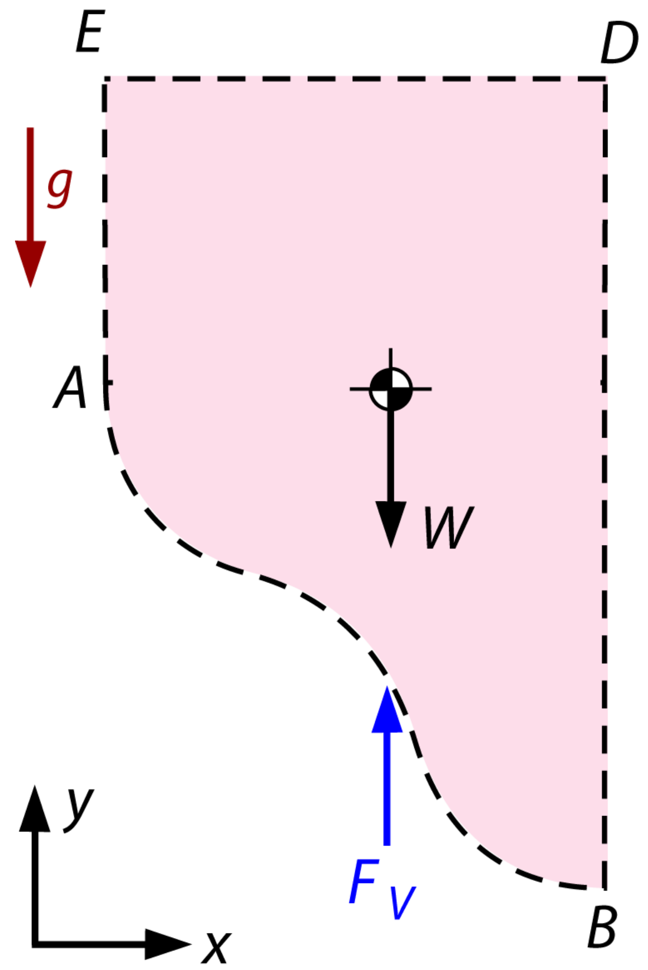
Hydrostatic Forces on Curved Submerged Surfaces

- Vertical Component
  - Consider free-body diagram of fluid column contained in vertical projection above curved surface *AB*:



Hydrostatic Forces on Curved Submerged Surfaces

- Vertical Component
  - $F_V \downarrow$ is the vertical component of the force exerted by the fluid on the curved surface AB
  - By Newton's third law, $F_V \uparrow$ is the vertical component of the force exerted by the curved surface on the fluid (liquid column)
  - W is the weight of the liquid column extending vertically from curved surface AB to horizontal free surface ED



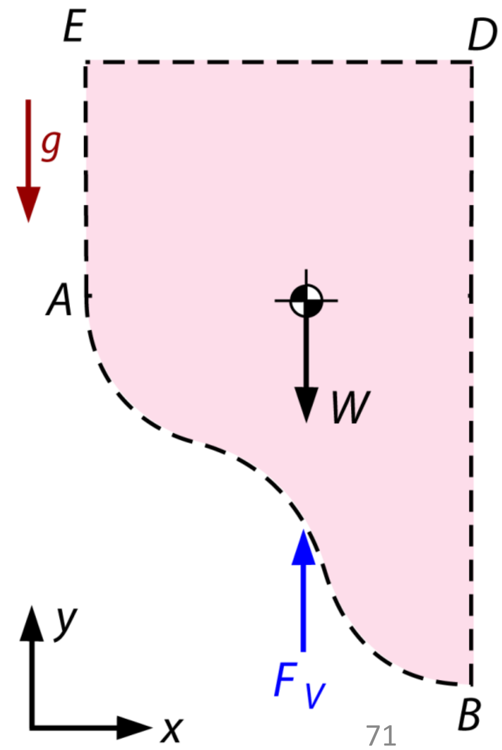
Hydrostatic Forces on Curved Submerged Surfaces

- Vertical Component
  - Assume $P_0 = 0$ (considering gage pressures)
  - Liquid column is in static equilibrium $\Rightarrow$ vertical forces must balance:

$$F_V = W$$

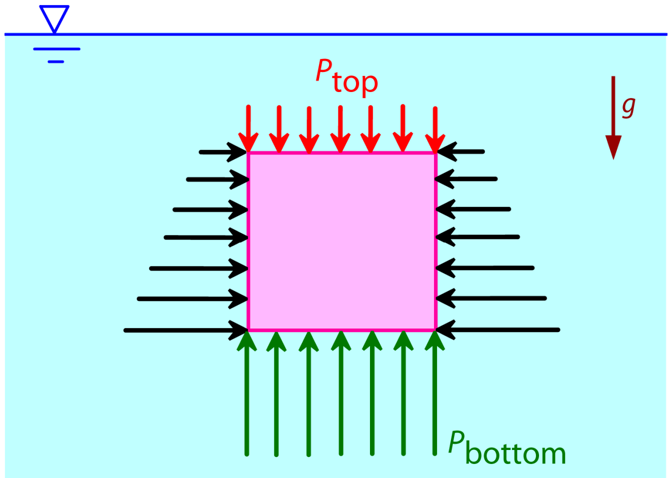
  - The vertical component of pressure force on a curved surface equals in both magnitude and direction to the weight of the entire fluid column above the curved surface, and acts through the center of gravity (centroid) of the fluid column

$$mx_c = \int xdm \quad \rho V x_c = \int \rho x_c dV \quad V x_c = \int x_c dV$$



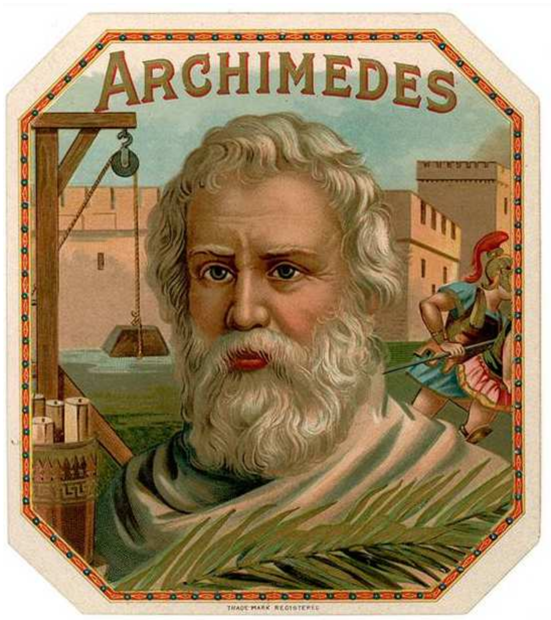
Buoyancy

- Physical Explanation for Origin of Buoyancy Force
  - Hydrostatic pressure in a constant density fluid increases linearly with depth
  - A net upward vertical force acts on body because pressure forces acting from below body are larger than the pressure forces acting from above body
  - Resultant upward vertical force due to unbalanced hydrostatic forces called buoyancy force or upthrust

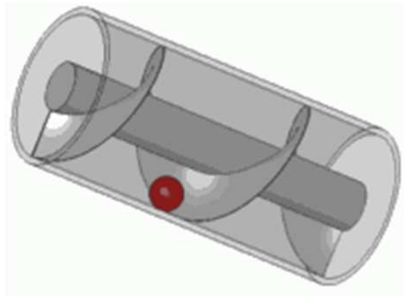


Buoyancy

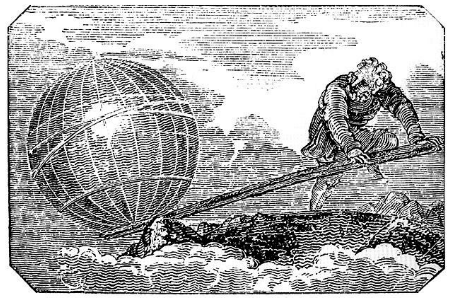
- Archimedes Principle
  - A body immersed in a fluid experiences a vertical buoyant (upthrust) force equal to the weight of the fluid it displaces



Archimedes
(287-212 BC)



Archimedes' screw: could be used to transfer water from a low-lying body of water into high canals. The Archimedes' screw is still in use today for pumping liquids and granulated solids such as coal and grain.

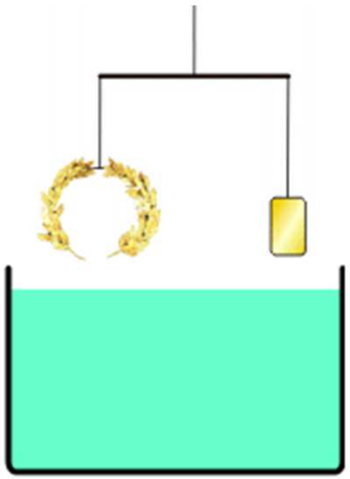


While Archimedes did not invent the lever, he gave an explanation of the principle involved in his work 'On the Equilibrium of Planes'.

Give me a place to stand on, and I will move the Earth

Buoyancy

- Archimedes Principle
  - A body immersed in a fluid experiences a vertical buoyant (upthrust) force equal to the weight of the fluid it displaces



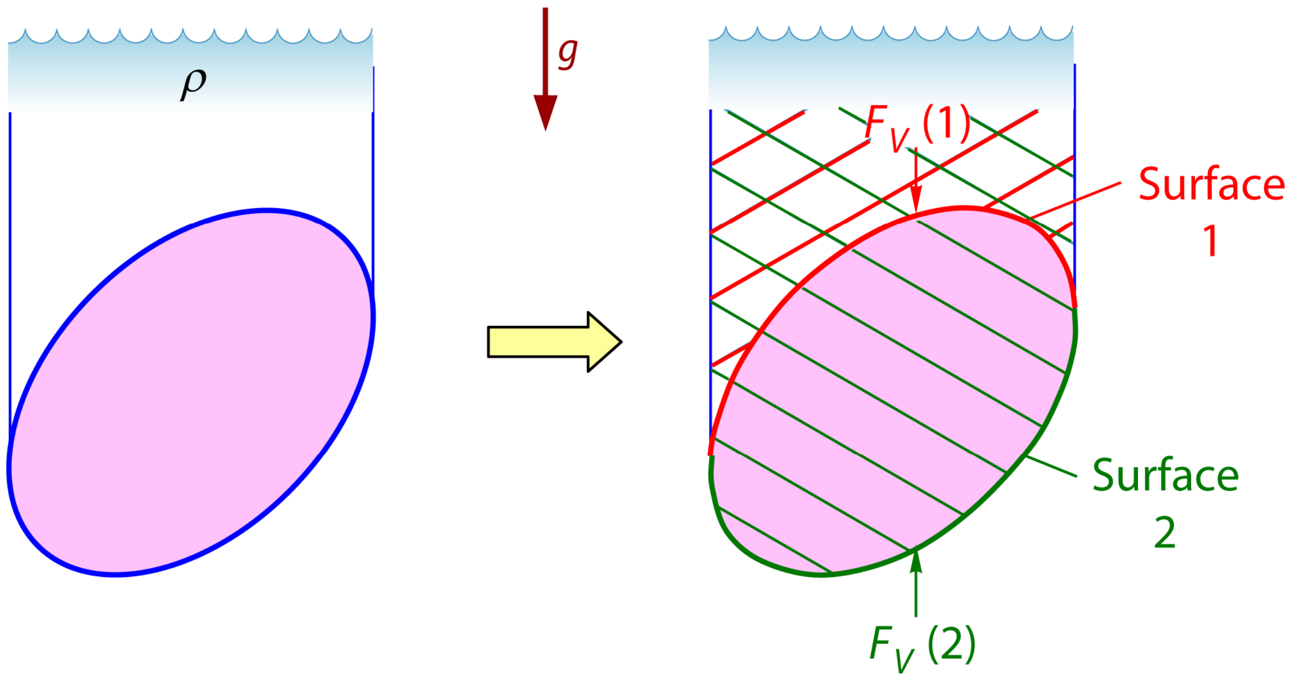
国王请金匠用纯金打造了一顶纯金王冠，做好了以后，国王怀疑金匠不老实，可能造假掺了“银”在里面，但是又不能把王冠毁坏来鉴定。怎样才能检验王冠是不是纯金的呢？阿基米德想了好久，一直没有好方法，吃不下饭也睡不好觉。有一天，他在洗澡的时候发现，当他坐在浴盆里时水位上升了，这使得他想到了：“上升了的水位正好应该等于王冠的体积，所以只要拿与王冠等重量的金子，放到水里，测出它的体积，看看它的体积是否与王冠的体积相同，如果王冠体积更大，这就表示其中造了假，掺了银。”阿基米德想到这里，不禁高兴的从浴盆跳了出来，光着身体就跑了出去，边跑还边喊著“εύρηκα！（我发现了！）”果然经过证明之后，王冠中确实含有其他杂质，阿基米德成功的揭穿了金匠的诡计，国王对他当然是更加的信服了。

实际上，因为王冠至少有头那么大，所用的容器也必然比王冠大，而金匠掺银的前提是不会使王冠颜色发生显著改变，所以也不会掺太多银，王冠比金块多出的体积也不会太多，所以即使王冠比金块多出的体积使水面上升，也不会十分显著，以阿基米德时代的测量技术，很难比较出王冠与金块的体积差异，即使有差异，也不能排除是实验中误差所致，一个更可能的方案是：阿基米德把王冠与金块放在天平两头，将天平置于有水的浴缸中，哪端更轻，则哪端体积更大。最终发现王冠体积更大。

后来阿基米德将这个发现进一步总结出浮力理论，为浮体学建立了基本的定理，并写在他的《浮体论》著作里，也就是：物体在浮体中所受的浮力，等于物体所排开的浮体的重量。

Buoyancy

- Archimedes Principle
  - Consider a submerged body which lies between an upper curved surface 1 and lower curved surface 2:



Buoyancy

- Archimedes Principle
  - Body experiences net upward buoyant or upthrust force

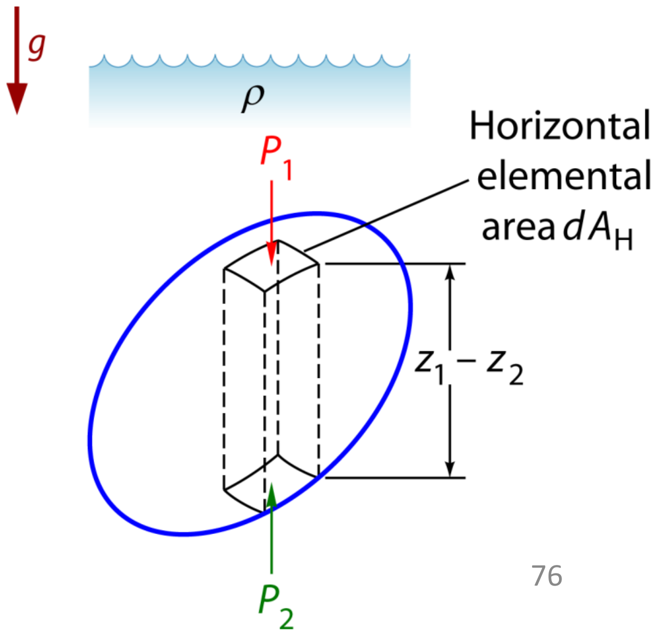
$$F_B = F_V(2) - F_V(1)$$

$= $ (fluid weight above 2) - (fluid weight above 1)

$= $ weight of fluid equivalent to body volume

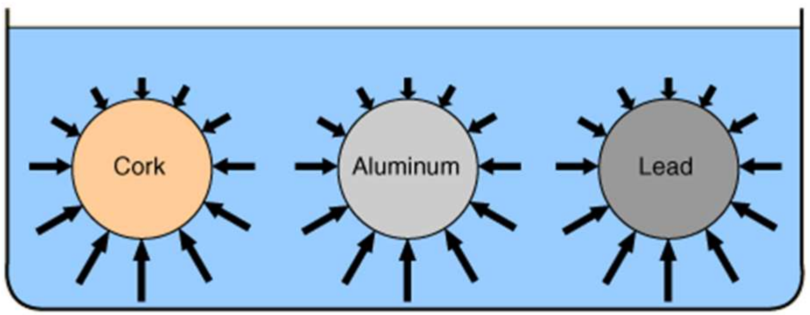
$$= -\rho g \int_{body} (z_2 - z_1) dA_H$$

$$= -\rho g (\text{body volume})$$



Buoyancy

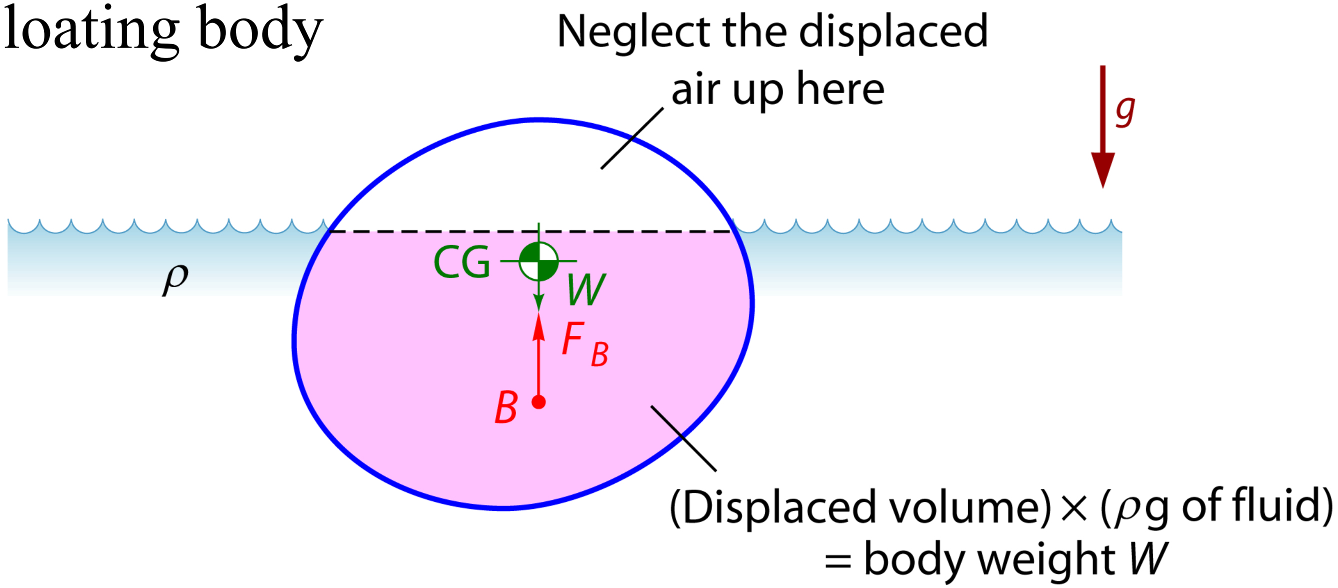
- Archimedes Principle
  - Note that the buoyant force does not care what's inside this volume (a brick, a gas, or vacuum): it depends only on the volume and the density of the outside gas (liquid).



Immersed Body

Buoyancy

- Archimedes Principle
  - Floating body



✓ Shaded portion of the body is the displaced volume

✓ Buoyancy force:

F_B = weight of fluid displaced $\Rightarrow F_B = \rho g$(displaced volume)

✓ Vertical equilibrium

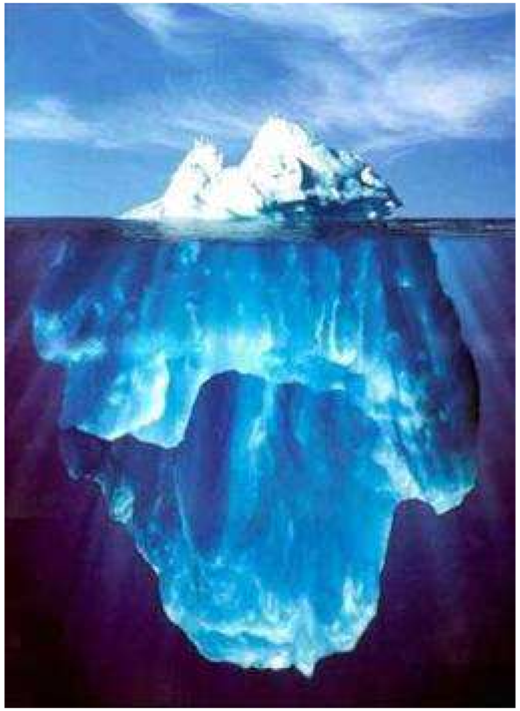
$$F_B = W$$

Buoyancy

- Archimedes Principle
  - Law of Flotation: Buoyancy force on an object equals to the displaced volume of fluid in which it floats

<u>Note</u>

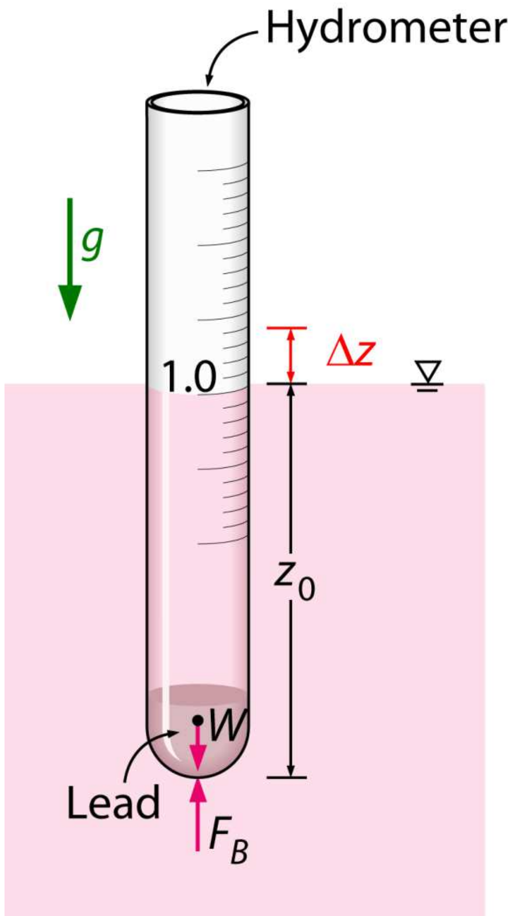
- Displaced volume = volume of submerged portion of floating body = V_{sub}
- Since there can be no net moments for static equilibrium, buoyant force F_B and body weight W are collinear



The tip of an iceberg

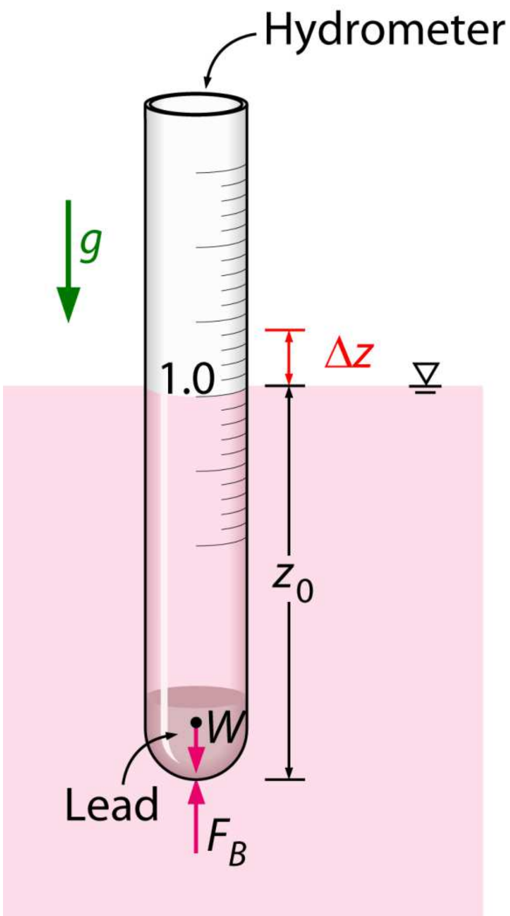
Buoyancy

- Example 2: Hydrometers
  - Hydrometers are devices to measure specific gravity of liquid ($\rho_{liquid}/\rho_{water}$)
  - Problem Statement
    - ✓ Hydrometer floats at level which is a measure of specific gravity of liquid
    - ✓ Top part of hydrometer extends above liquid surface
    - ✓ Divisions on hydrometer allow specific gravity to be read directly
    - ✓ Hydrometer calibrated such that in pure water it reads exactly 1.0 at air-water interface



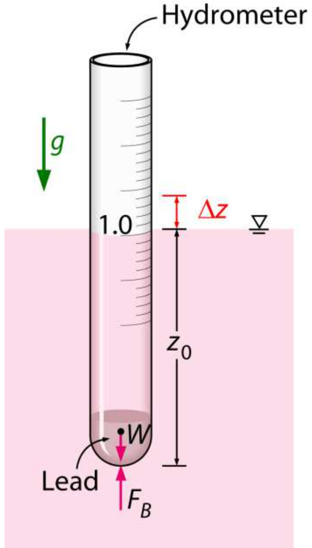
Buoyancy

- Example 2: Hydrometers
  - Questions
    a) Obtain relation for specific gravity of a liquid as a function of distance Δz from mark corresponding to pure water
    b) Determine mass of lead that must be poured into a 2-cm-diameter, 20-cm-long hydrometer if it is to float halfway (the 10-cm mark) in pure water



Buoyancy

- Example 2: Hydrometers
  - Solutions: Part a)



✓ Hydrometer in static equilibrium:

$$F_B = W = \rho_w g V_{sub} = \rho_w g A z_0$$

A: cross sectional area of tube; ρ_w: density of pure water

✓ In fluids less dense than water ($\rho_f < \rho_w$) $\Rightarrow$ hydrometer sinks deeper $\Rightarrow$ liquid level rises a distance Δz above z_0

$$F_B = W = \rho_f g V_{sub} = \rho_f g A\left(z_0 + \Delta z\right)$$

✓ Relation also valid for fluids denser than water ($\rho_f > \rho_w$) $\Rightarrow \Delta z < 0$

✓ Combine the above two equations

$$\rho_w g A z_0 = \rho_f g A\left(z_0 + \Delta z\right) \;\Rightarrow\; SG_f = \frac{\rho_f}{\rho_w} = \frac{z_0}{z_0 + \Delta z}$$

✓ z_0 is constant for a given hydrometer

Buoyancy

- Example 2: Hydrometers
  - Solutions: Part b)
    - ✓ Neglect weight of glass tube:

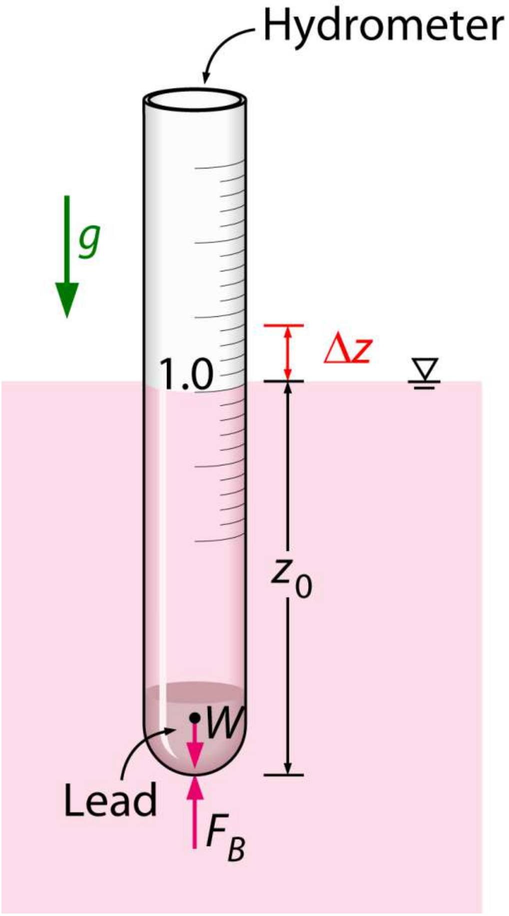
$$W = mg = F_B = \rho_w g V_{sub}$$

$$m = \rho_w V_{sub}$$

$$m = \rho_w \left(\pi R^2 h_{sub}\right)$$

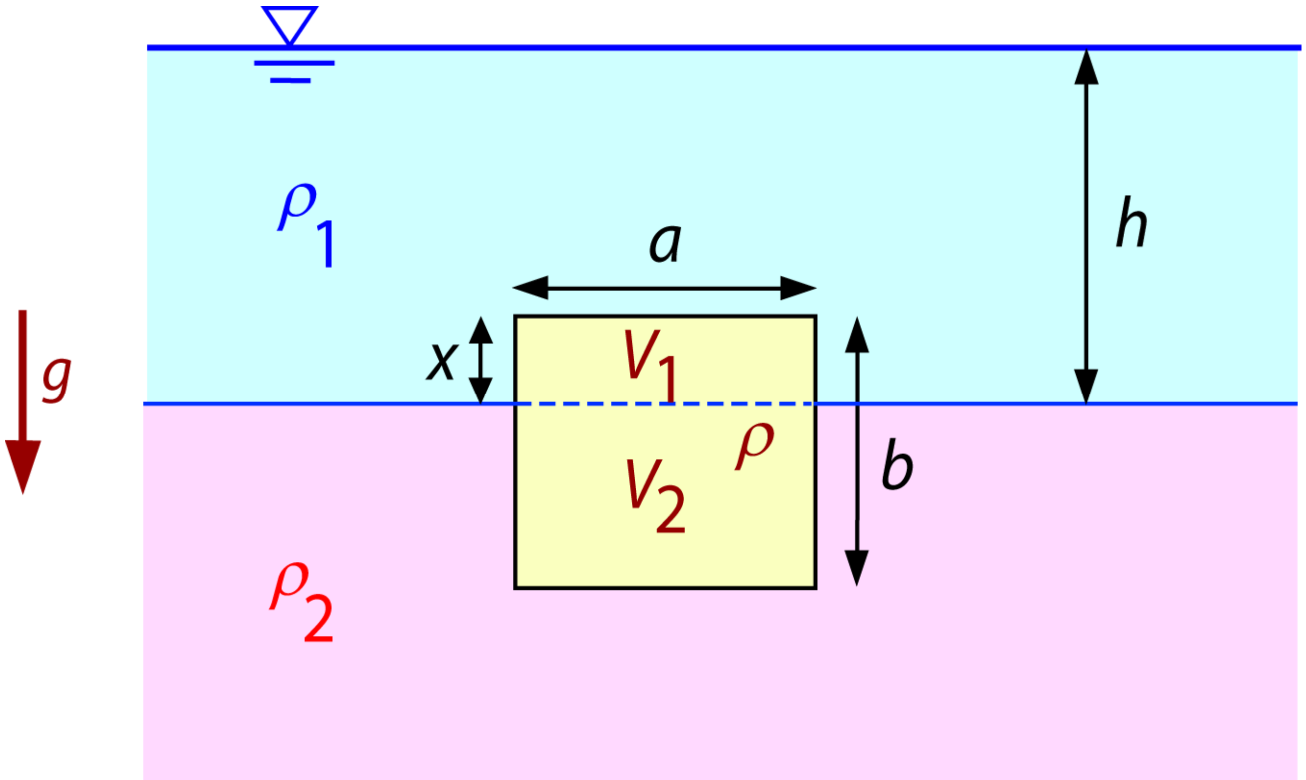
$$m = \left(1000 \times \pi \times 0.01^2 \times 0.1\right)$$

$$m = 0.0314 \text{ kg}$$



Buoyancy

- Example 3
  - Problem Statement
    - ✓ Body floats (dimensions: a, b, and L) in between 2 immiscible fluids
    - ✓ Evaluate x

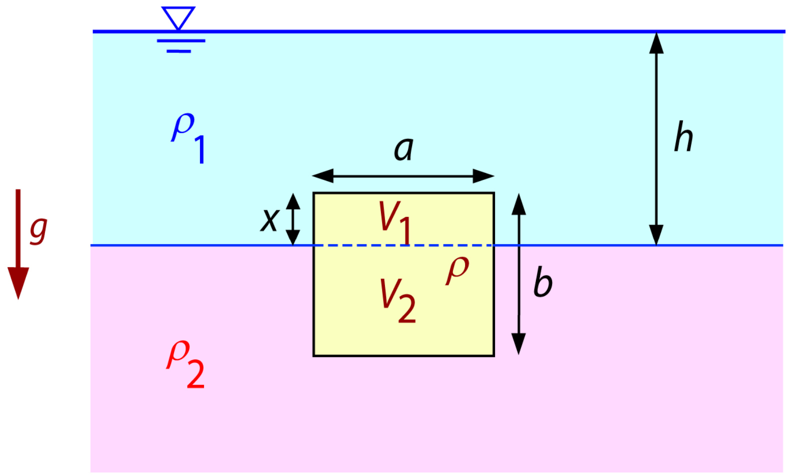


Buoyancy

- Example 3
  - Solution
    - ✓ Volumes of displaced fluids

$$V_1 = axL$$

$$V_2 = a\left(b - x\right)L$$



- ✓ Buoyancy force

$$F_{B1} = \rho_1 gaxL$$

$$F_{B2} = \rho_2 ga\left(b-x\right)L$$

$$F_B = \rho_1 gaxL + \rho_2 ga\left(b-x\right)L$$

- ✓ Weight of body

$$W = \rho gV = \rho gabL$$

Buoyancy

- Example 3
  - Solution
    - ✓ Vertical equilibrium

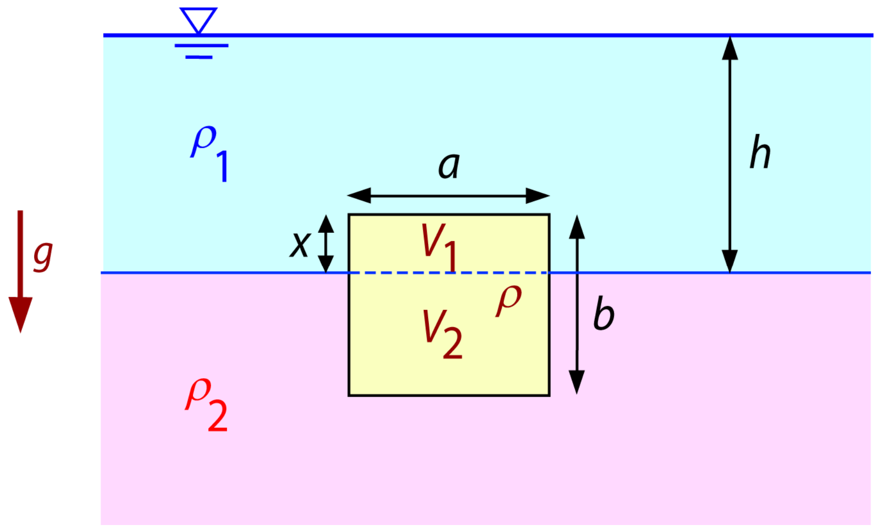
$$F_B = W$$

$$\rho_1 g a x L + \rho_2 g a (b - x) L = \rho g a b L$$

$$\rho_1 x + \rho_2 (b - x) = \rho b$$

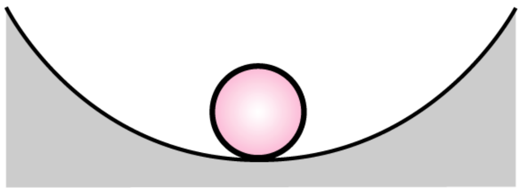
$$x = \frac{(\rho_2 - \rho) b}{\rho_2 - \rho_1}$$

✓ $0 \le x \le b \Rightarrow \rho_1 \le \rho \le \rho_2$

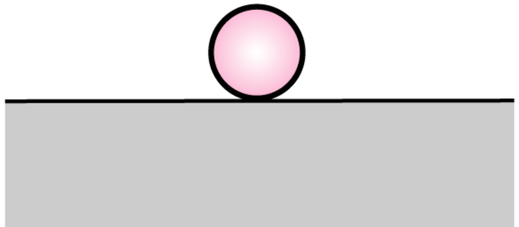


Stability of Submerged Bodies

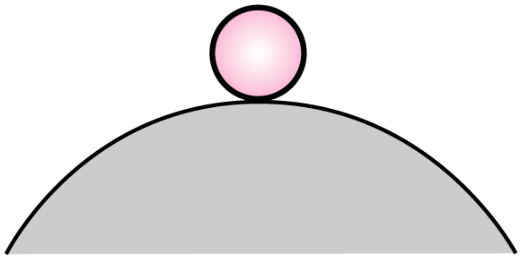
- Stability
  - Notion of stability by applying "ball on floor" analogy
    - ✓ Case (a) ⇒ **stable** ⇒ any small disturbance generates a restoring force (due to gravity) that returns body to its initial equilibrium position
    - ✓ Case (b) ⇒ **neutrally stable** ⇒ when displaced, body has no tendency to move back to its initial location, nor does it continue to move away
    - ✓ Case (c) ⇒ **unstable** ⇒ body may be in equilibrium instantaneously, but any infinitesimal disturbance causes body to roll off hill ⇒ body does not return to initial position but diverges from it



(a) Stable



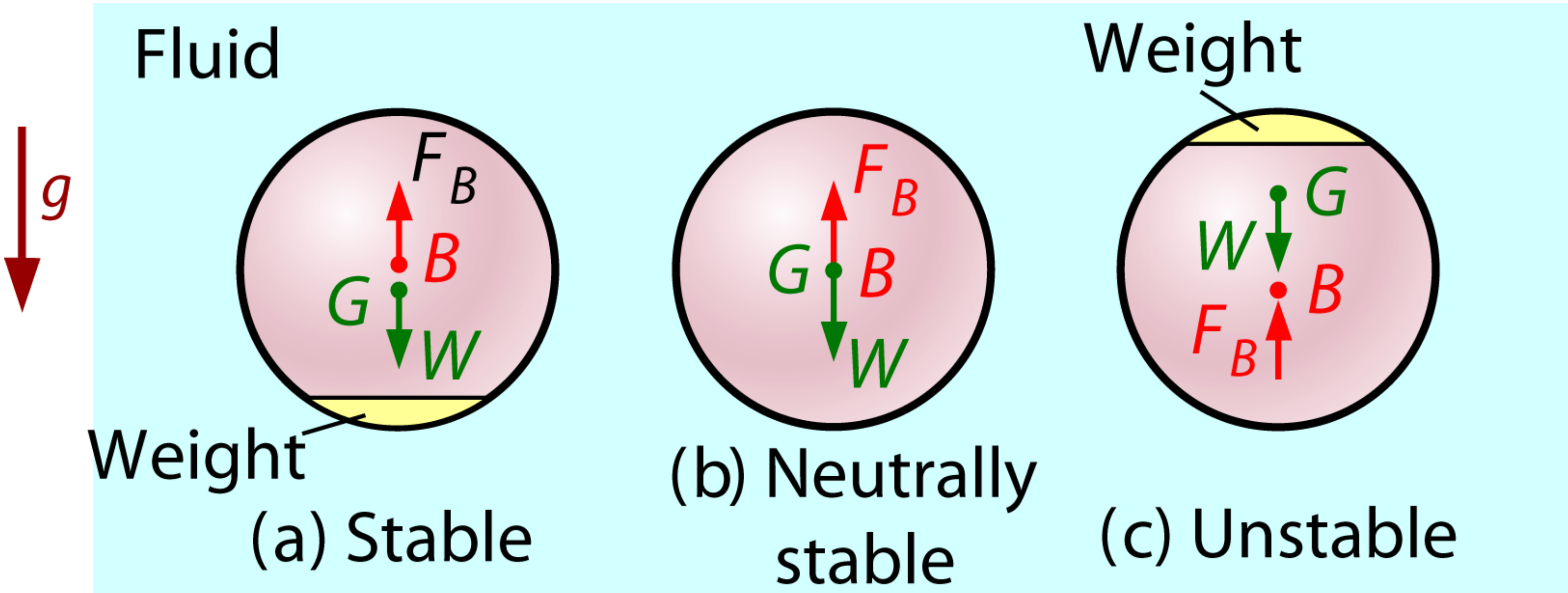
(b) Neutrally stable



(c) Unstable

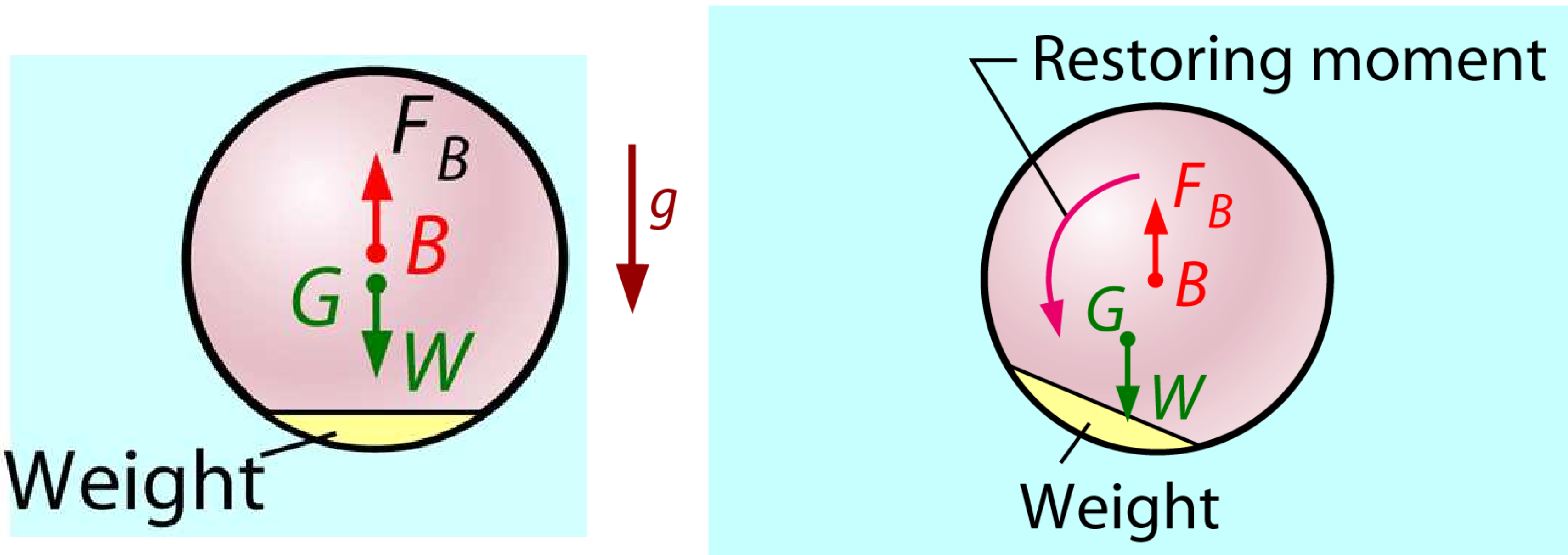
Stability of Submerged Bodies

- Stability of a submerged body depends on relative locations of
  - Center of gravity G of body
  - Center of buoyancy B (centroid of displaced volume)



Stability of Submerged Bodies

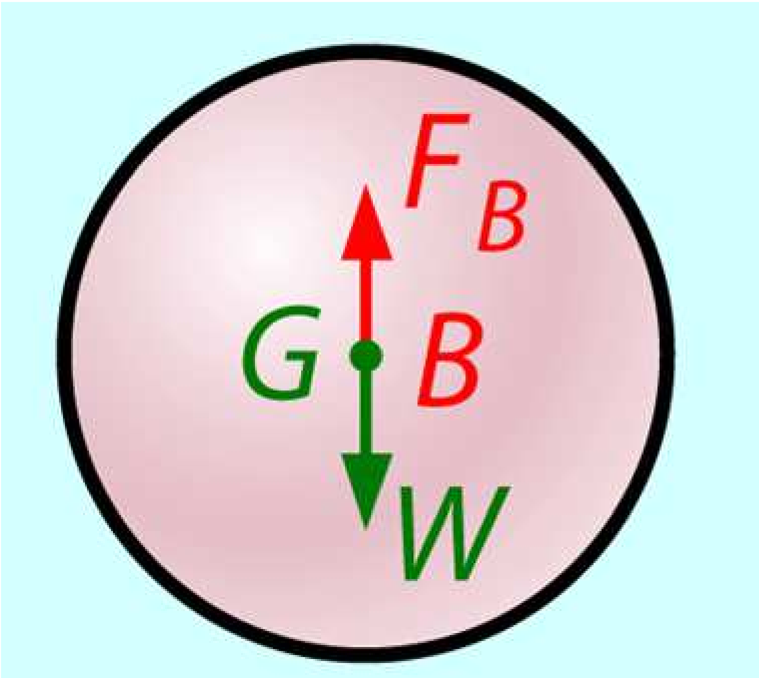
- Stable
  - *B* is above *G*

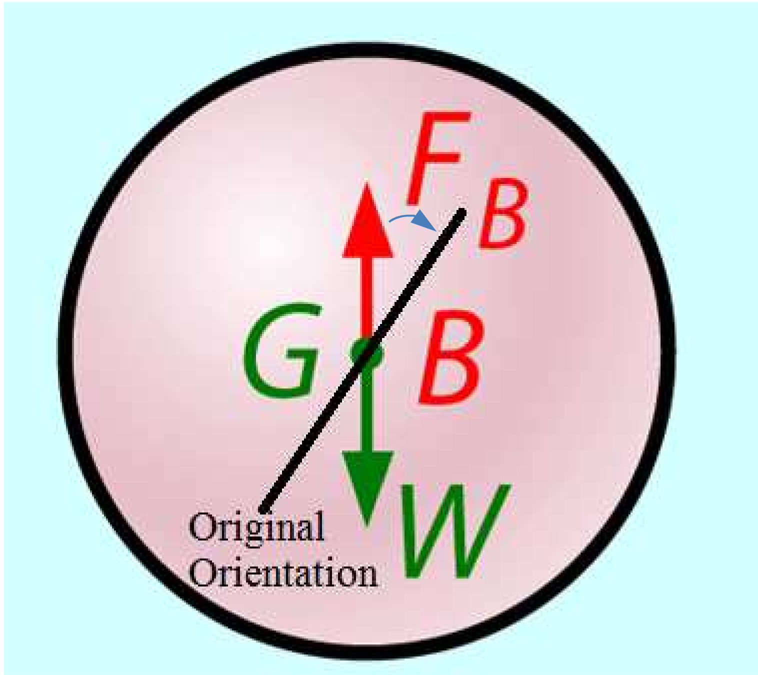


  - Disturbance of body produces a restoring moment to return body to its original stable position

Stability of Submerged Bodies

- Neutrally Stable
  - *B* and *G* coincide

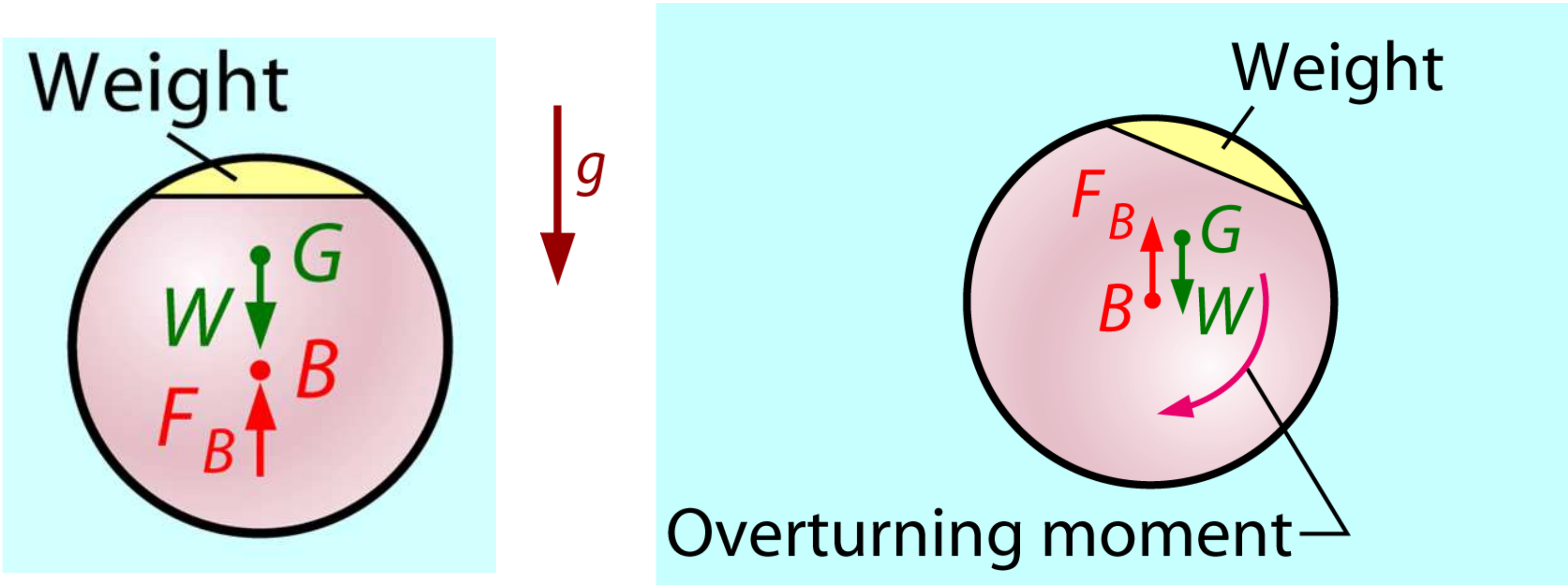




  - body has no tendency to overturn or right itself

Stability of Submerged Bodies

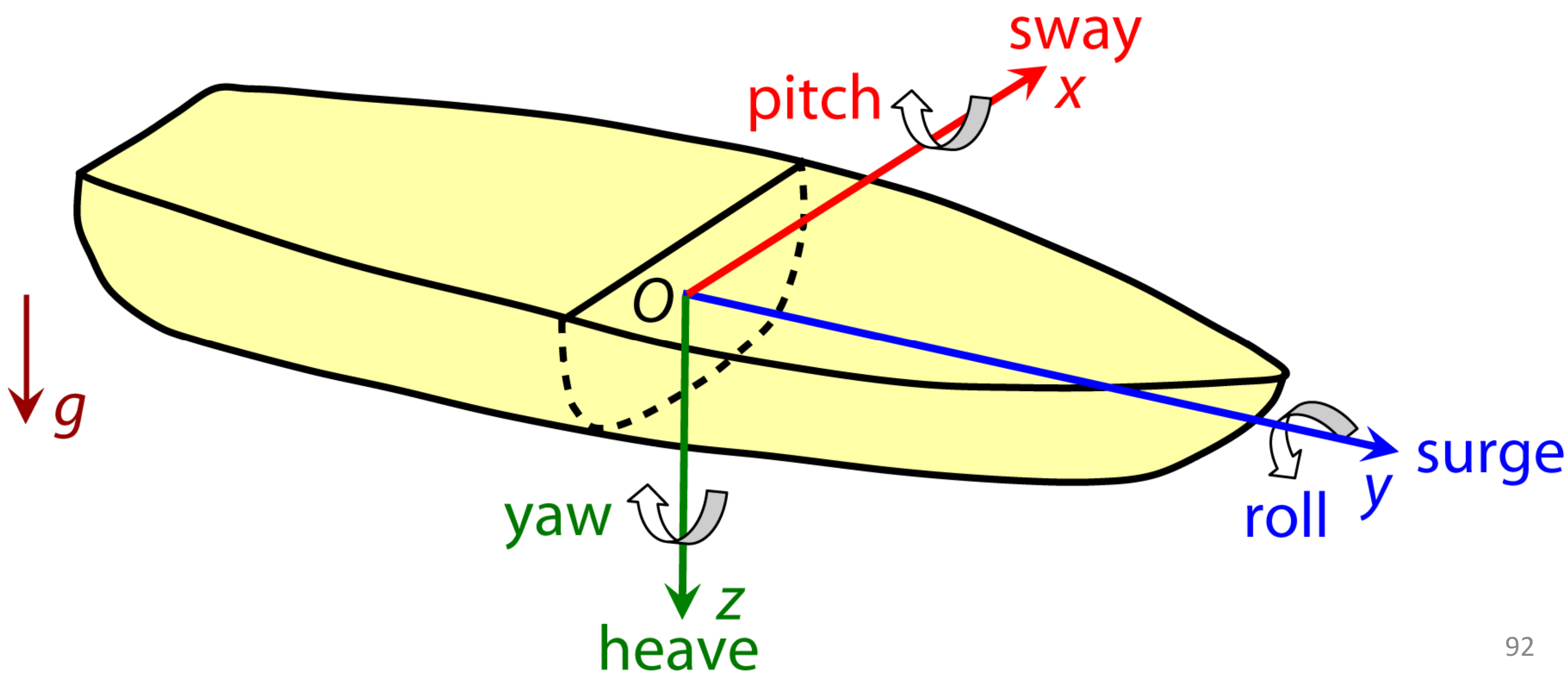
- Unstable
  - *B* is below *G*



  - Disturbance of body produces an overturning moment

Stability of Floating Bodies

- Degrees Freedom
  - A floating body has 6 degrees of freedom
    - Its motions are defined as translations (3 degrees of freedom) and rotations (3 degrees of freedom) about a set of orthogonal axes



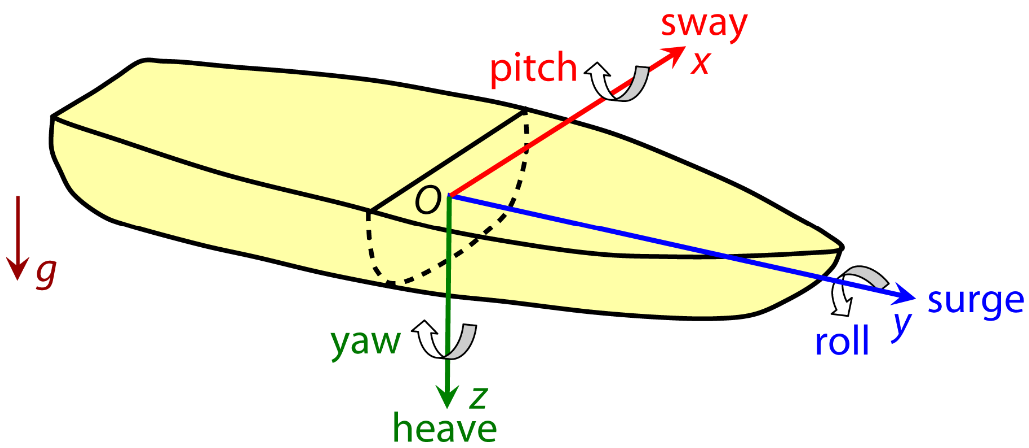
Stability of Floating Bodies

- Degrees Freedom
  - Along x-axis: Sway (starboard/port)
  - Along y-axis: Surge (forward/astern)
  - Along z-axis: Heave (up/down)

  $\}$ Translation

  - Along x-axis: Pitch (about sway axis)
  - Along y-axis: Roll (about surge axis)
  - Along z-axis: Yaw (about heave axis)

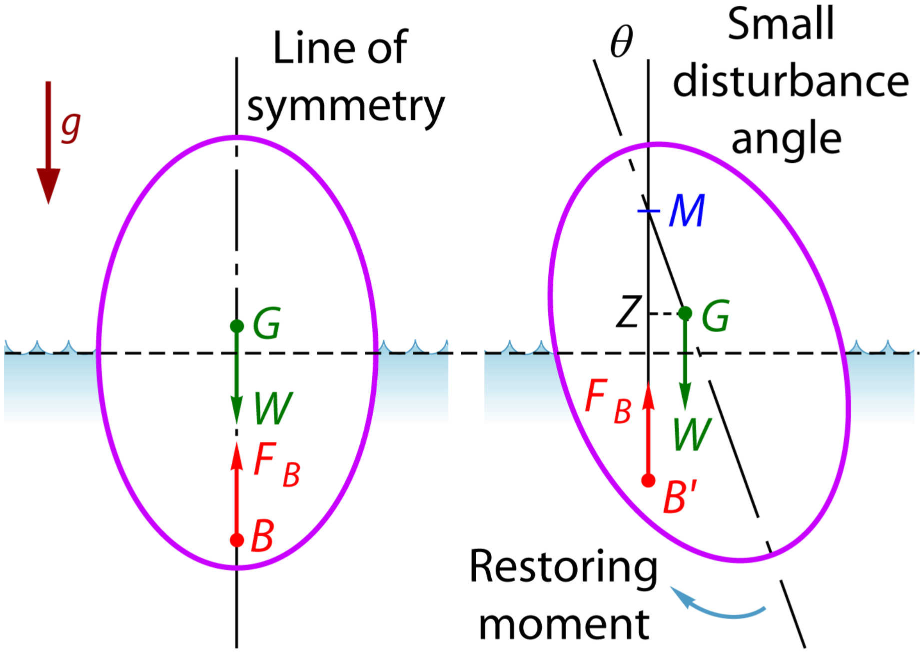
  $\}$ Rotation



➢ Roll and pitch are the dynamic equivalents of heel and trim, respectively

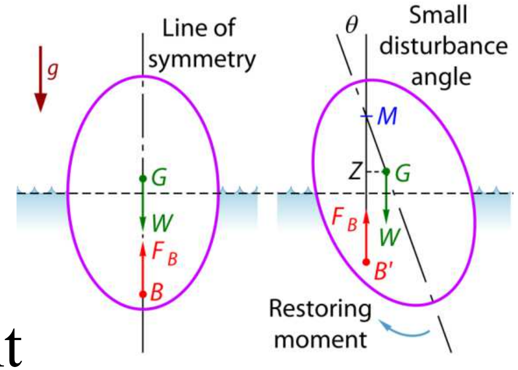
Stability of Floating Bodies

- Dynamics
  - As floating body rotates
    - ✓ location of the center of buoyancy B (which passes through centroid of the displaced volume) may change: B ⇨ B'
    - ✓ location of center of gravity G of body remains unchanged A



Stability of Floating Bodies

- Metacenter M
  - point of intersection of original vertical axis with line of action of buoyancy force after an angle of heel θ
- Metacentric height GM
  - determines stability of floating body
  - important parameter in design of floating bodies
  - need to determine GM_T (transverse metacentric height) corresponding to roll (angular displacement about y-axis) and GM_L (longitudinal metacentric height) corresponding to pitch (angular displacement about x-axis) for different water levels before construction of floating body

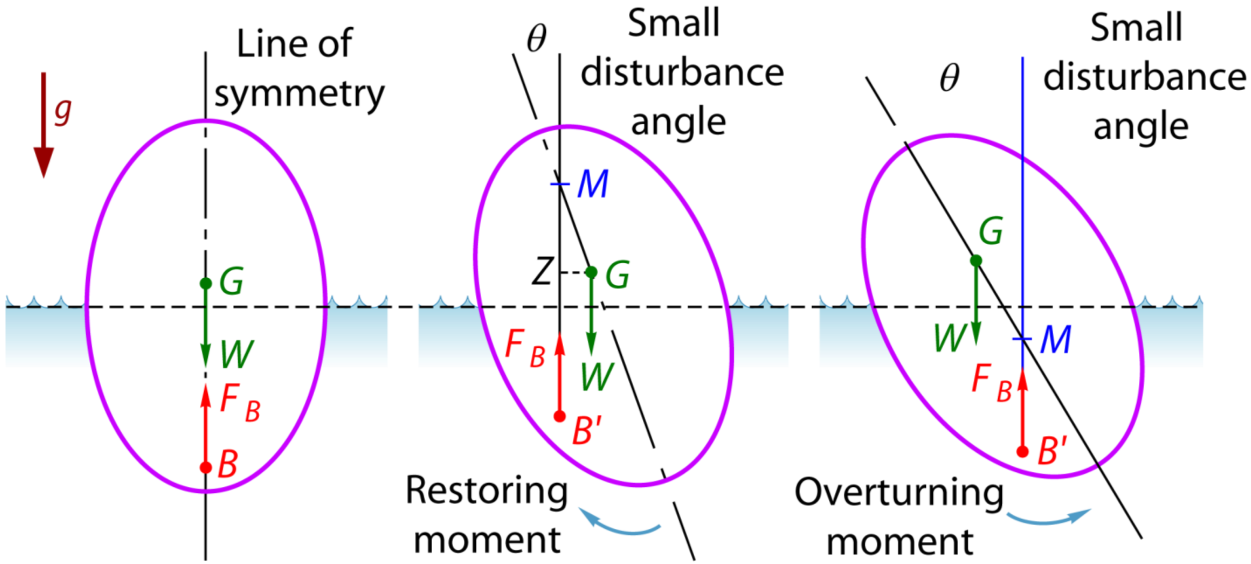


Stability of Floating Bodies

- Stable Equilibrium
  - M above $G \Rightarrow GM > 0$
  - Restoring couple acts on floating body in its displaced position tending to restore it to its original position

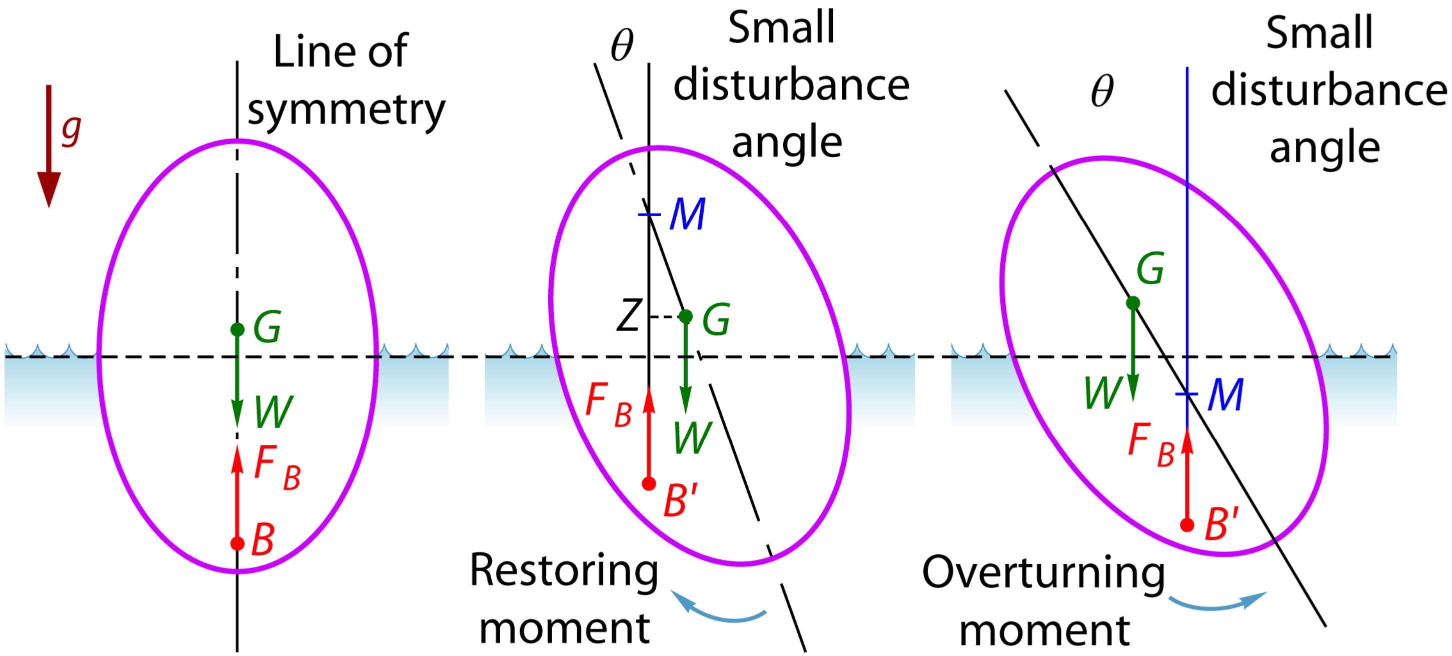
$$\text{Restoring couple} = W \cdot GM \sin\theta = W \cdot GZ$$

(GZ is called the righting arm)



Stability of Floating Bodies

- Unstable Equilibrium
  - M below $G \Rightarrow GM < 0$
  - Overturning couple acts on body



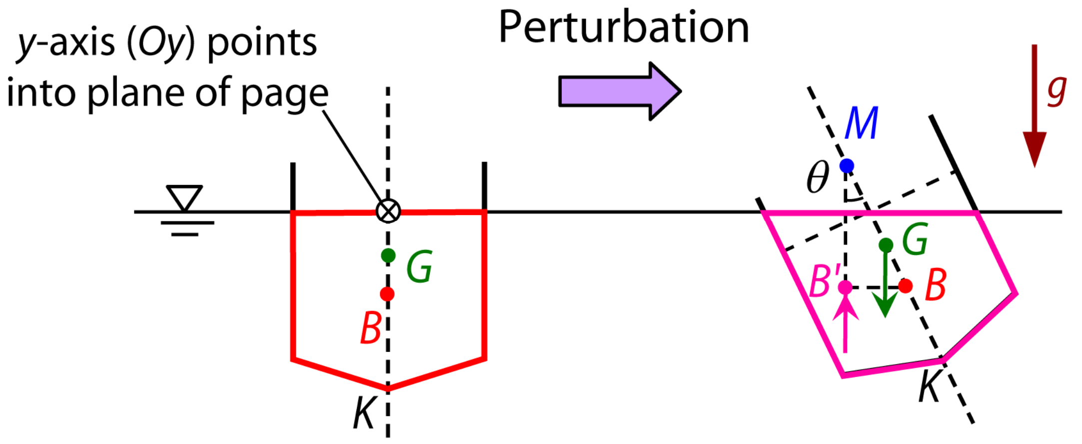
Stability of Floating Bodies

- Neutral Equilibrium
  - M coincides with $G \Rightarrow GM = 0$
  - Zero resultant couple $\Rightarrow$ body has no tendency to return to, nor move further away from original position

Note: Stability of floating body is not simply determined by relative positions of B and G, unlike submerged bodies

Stability of Floating Bodies

- Upright Vessel
  - For an upright vessel, point of buoyancy is at *B*
  - *B* is centroid of volume of fluid displaced by floating body (and is shape dependent)
  - Vessel is given a slight angular perturbation $\theta \Rightarrow$ center of buoyancy shifts: *B* ⇨ *B'*
  - *B* and *B'* are centroids of volume of displaced fluid before and after perturbations, respectively



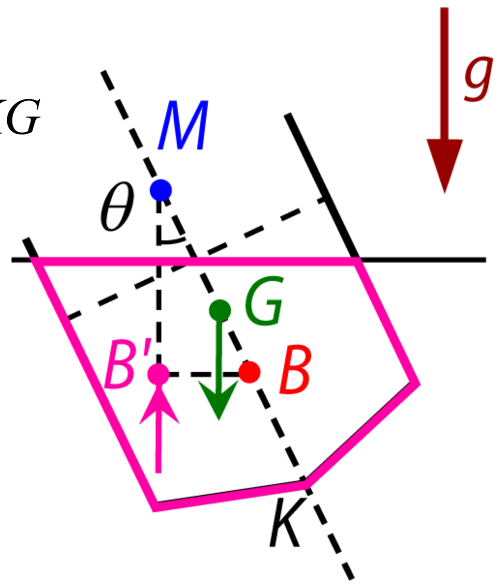
Stability of Floating Bodies

- Upright Vessel
  - Determination of GM
    - ✓ From geometry

$$KM = KG + GM = KB + BM \Rightarrow GM = KB + BM - KG$$

where KB and KG can be obtained from center of gravity and buoyancy calculations, and BM is known as the metacentric radius, which is given by

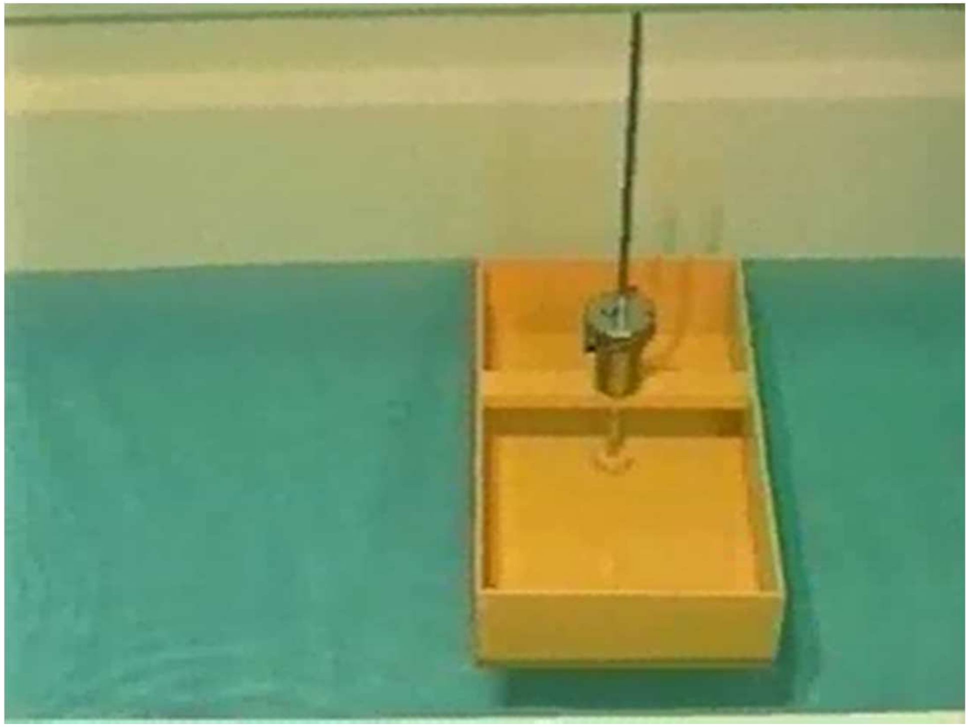
$$BM = \frac{I_{Oy}}{V_{sub}}$$



- ✓ $I_{Oy} \Rightarrow$ second moment of area of the plane of floatation (water line cross section) about the Oy-axis
- ✓ $V_{sub} \Rightarrow$ volume of submerged portion of floating body (displaced volume)
- ✓ Plane of flotation refers to water plane

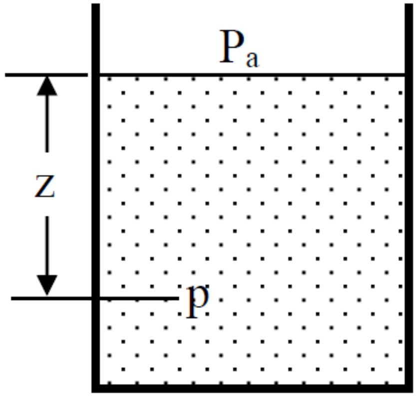
Stability of Floating Bodies

- Upright Vessel



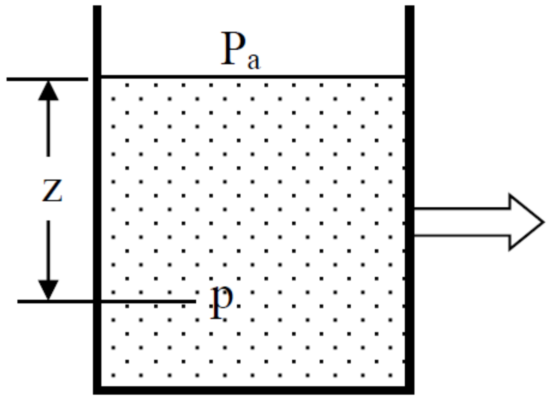
Equilibrium of Moving Fluids

- Statics of Moving System



Stationary

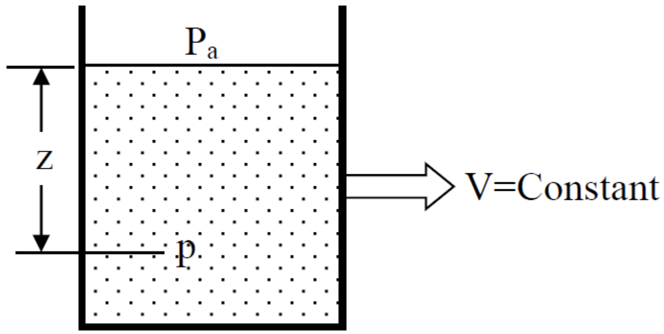
$p=p_a+\rho gz$



V=Constant ?

Equilibrium of Moving Fluids

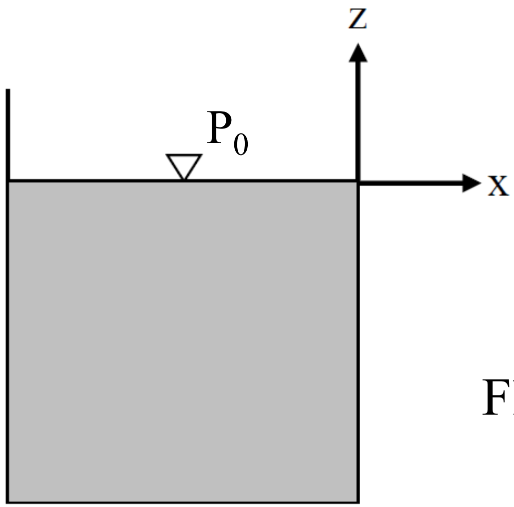
- Statics of Moving System
  - When the entire continuum is in uniform RECTILINER motion, the governing principle of the statics of a fluid in the gravity field remains the same. This follows from Newton second law, which states that force is a result of a change in motion. When the motion is uniform, there is no change in the motion, and accordingly there is no change in the force. Therefore, hydrostatic equation in a uniform velocity of translation remains the same.



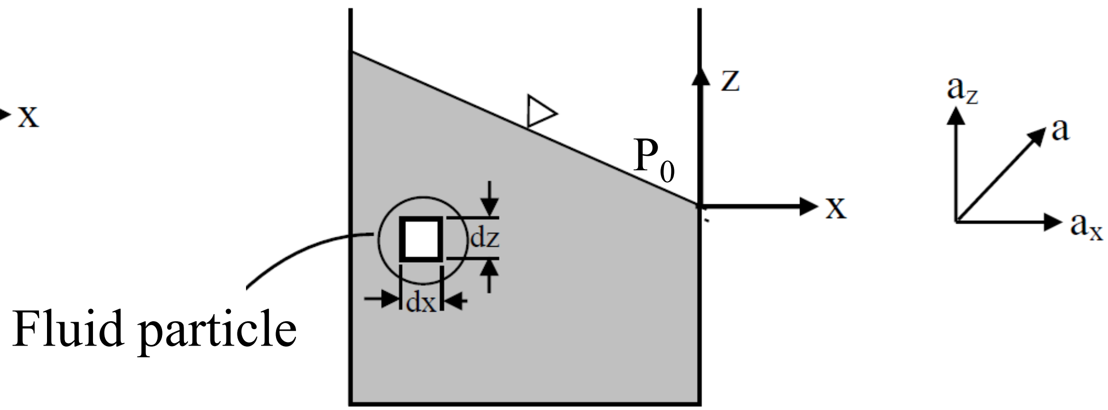
$$p = p_a + \rho g z$$

Equilibrium of Moving Fluids

- Uniform Rectilinear Acceleration
  - When a liquid in an open tank is subjected to a constant linear acceleration ***a***, the free surface of the fluid (which was horizontally at rest) is inclined at an angle θ to the direction of acceleration. The fluid must once and for all stay in that position for the given constant acceleration. Under this condition, the liquid is said to be in a state of relative rest.



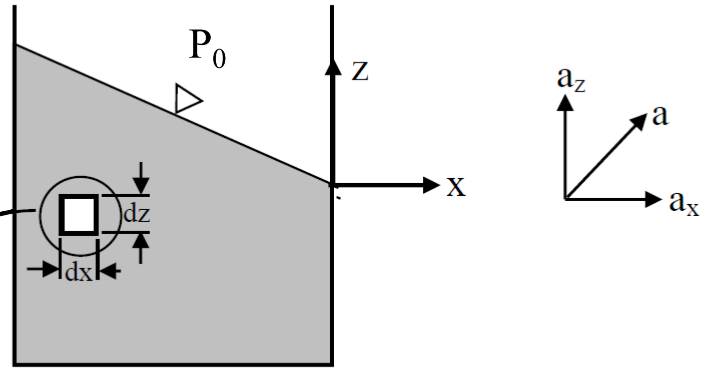
Liquid at rest



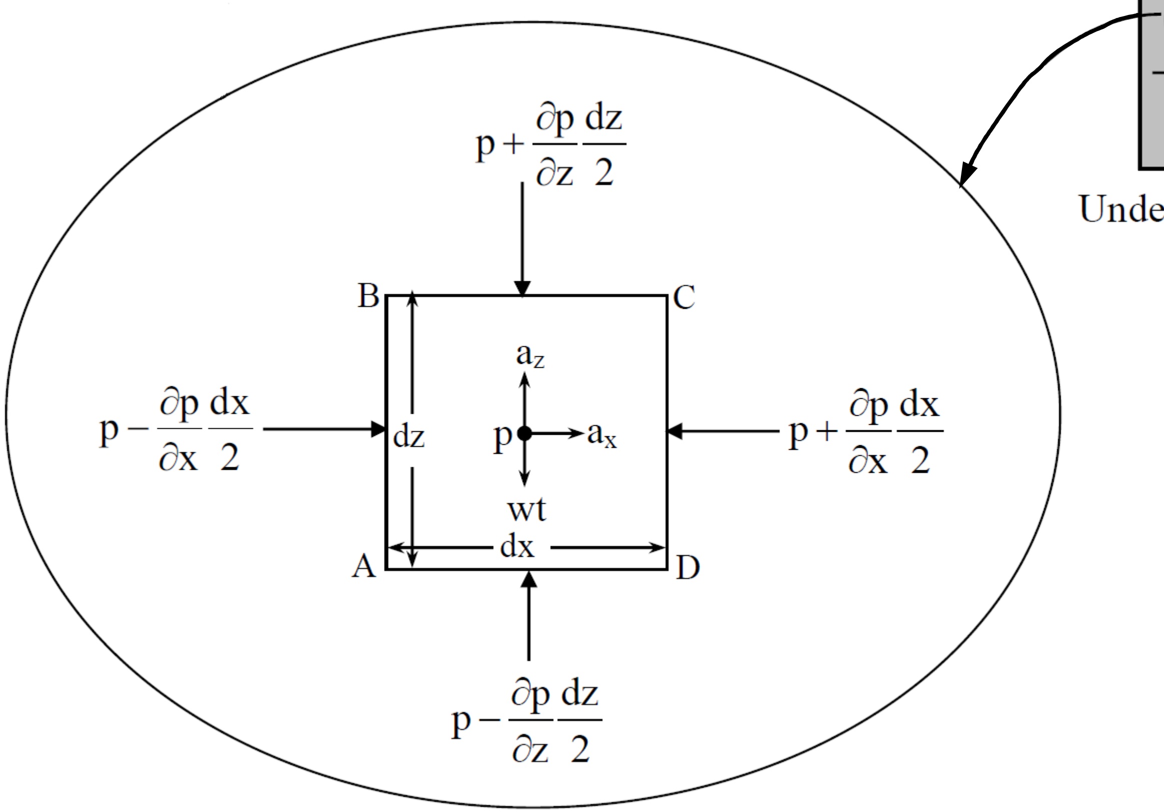
Under constant acceleration

Equilibrium of Moving Fluids

- Uniform Rectilinear Acceleration



Under constant acceleration



Equilibrium of Moving Fluids

- Uniform Rectilinear Acceleration
  - Consider a fluid particle of dimension $dx \cdot dz \cdot 1$(unit length)

Summary of the pressure and the force acting on each surface

Face	Area	Pressure	Force
AB	$dz \times 1$	$\left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right)$	$\left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right)dz.1$
CD	$dz \times 1$	$\left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)$	$\left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)dz.1$
AD	$dx \times 1$	$\left(p - \frac{\partial p}{\partial z}\frac{dz}{2}\right)$	$\left(p - \frac{\partial p}{\partial z}\frac{dz}{2}\right)dx.1$
BC	$dx \times 1$	$\left(p + \frac{\partial p}{\partial z}\frac{dz}{2}\right)$	$\left(p + \frac{\partial p}{\partial z}\frac{dz}{2}\right)dx.1$

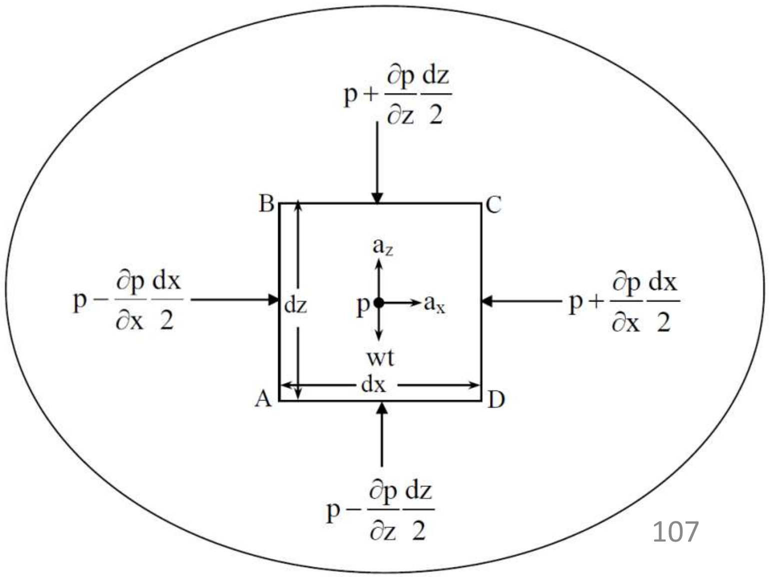
Equilibrium of Moving Fluids

- Uniform Rectilinear Acceleration
  - Force balance in the *x*-direction

$$\left(p-\frac{\partial p}{\partial x}\bullet\frac{dx}{2}\right)dz\bullet 1-\left(p+\frac{\partial p}{\partial x}\bullet\frac{dx}{2}\right)dz\bullet 1=\rho\left(dx\bullet dz\bullet 1\right)a_x$$

  - Simplify the equation:

$$\frac{\partial p}{\partial x}=-\rho a_x$$



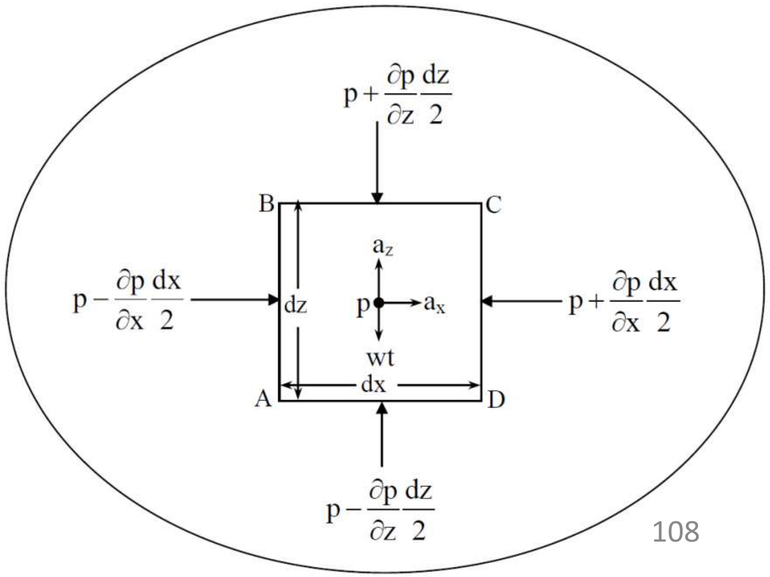
Equilibrium of Moving Fluids

- Uniform Rectilinear Acceleration
  - Force balance in the z-direction

$$\left(p-\frac{\partial p}{\partial z}\bullet\frac{dz}{2}\right)dx\bullet 1-\left(p+\frac{\partial p}{\partial z}\bullet\frac{dz}{2}\right)dx\bullet 1-\rho\left(dx\bullet dz\bullet 1\right)g=\rho\left(dx\bullet dz\bullet 1\right)a_x$$

  - Simplify the equation:

$$\frac{\partial p}{\partial z}=-\rho\left(a_z+g\right)$$

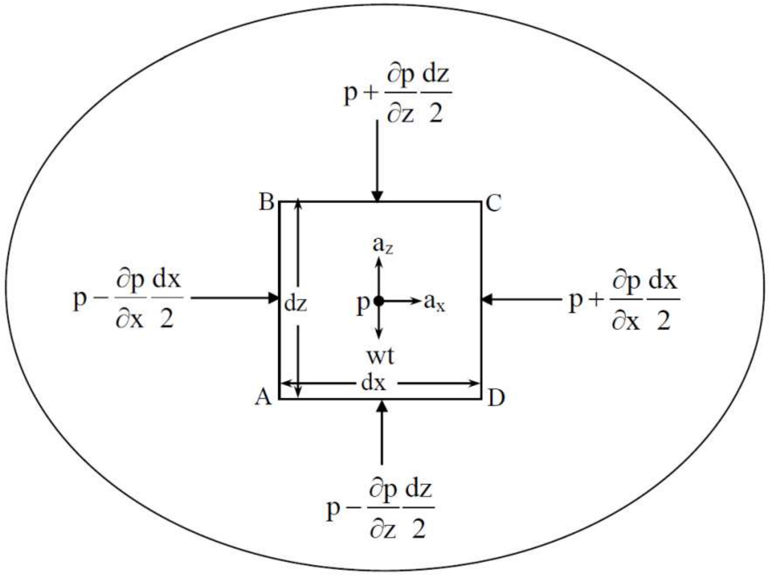


Equilibrium of Moving Fluids

- Uniform Rectilinear Acceleration
  - Pressure

$$\frac{\partial p}{\partial x} = -\rho a_x \quad \Rightarrow \quad p = -\rho a_x x + f(z) + c_1$$

$$\frac{\partial p}{\partial z} = -\rho (a_z + g) \quad \Rightarrow \quad p = -\rho (a_z + g) z + f(x) + c_2$$



$$p = -\rho \left[(a_z + g) z + a_x x \right] + c_3$$

at $x = 0, z = 0$

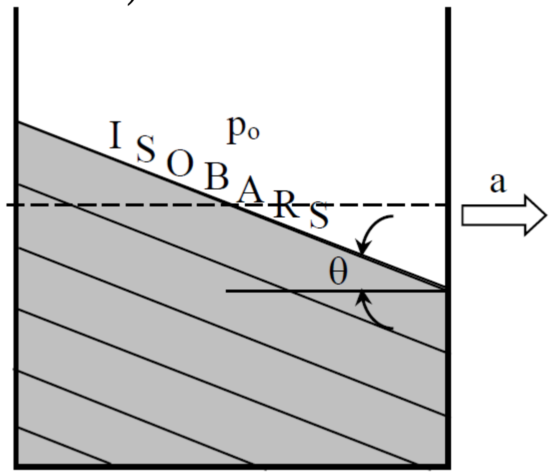
$p = p_0 = c_3$

$$p = -\rho \left[(a_z + g) z + a_x x \right] + p_0$$

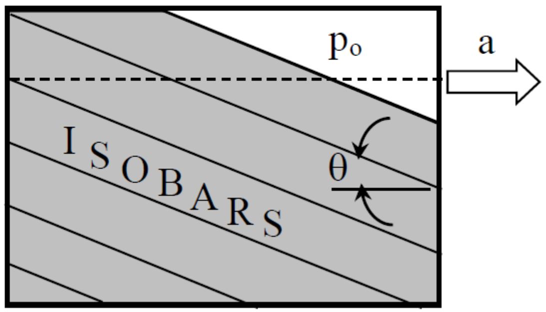
Equilibrium of Moving Fluids

- Uniform Rectilinear Acceleration
  - Pressure
    - ✓ To find the slop of the free surface, we substitute $p = p_0$ to the pressure equation, which leads to

$$\left(a_z + g\right) z + a_x x = 0 \Longrightarrow \tan\theta = \frac{dz}{dx} = \frac{a_x}{a_z + g}$$



A Large Open Cylinder



A Small Closed Cylinder

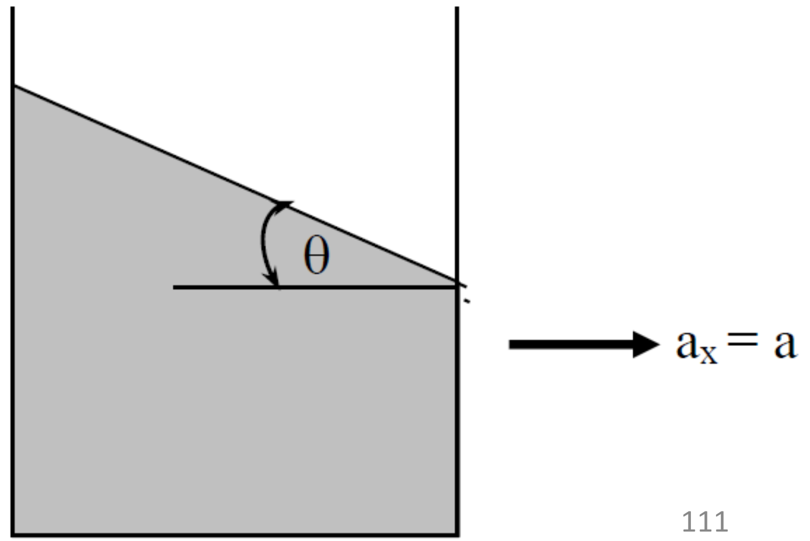
➢ The lines of constant pressure, which are also called **ISOBARS** , are parallel to the free surface

Equilibrium of Moving Fluids

- Uniform Rectilinear Acceleration
  - Simple case
    - ✓ The acceleration is along *x*-axis only

$$p = -\rho\left[gz + a_x x\right] + p_0$$

$$\tan\theta = \frac{dz}{dx} = \frac{a_x}{g}$$



Equilibrium of Moving Fluids

- Uniform Rectilinear Acceleration
  - Simple case
    - ✓ Application

Tank truck

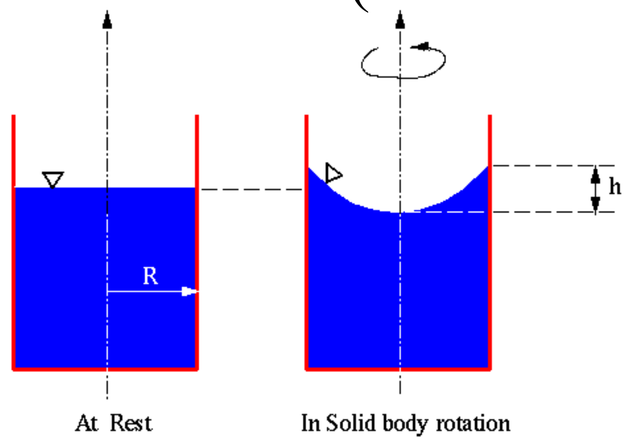
Equilibrium of Moving Fluids

- Uniform Rectilinear Acceleration
  - Simple case
    - ✓ Application

Tank car of railway

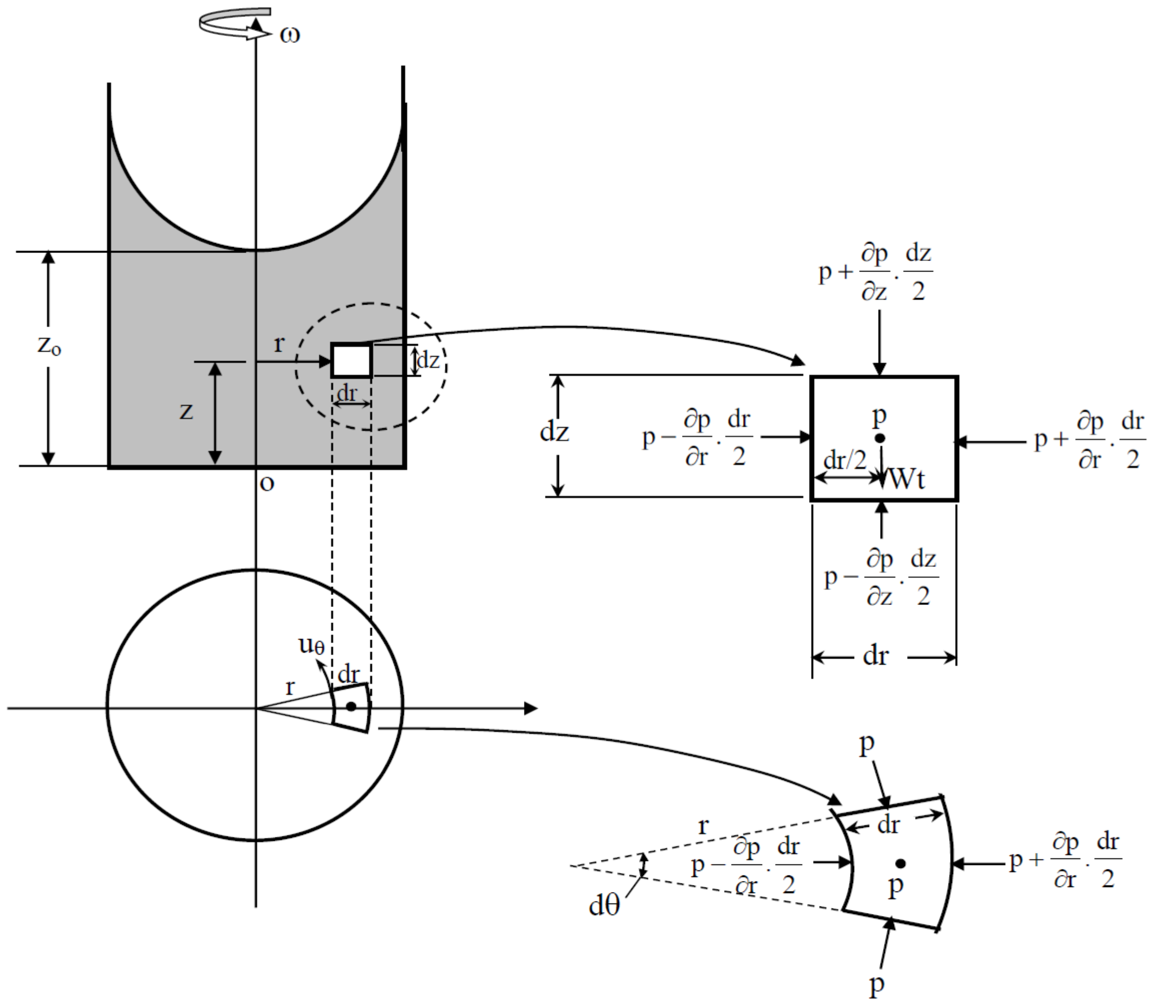
Equilibrium of Moving Fluids

- Uniform Spin of Liquid in a Container
  - When a body of fluid rotates uniformly without relative motion between different elements of the fluid in a container, each particle moves in a circle. Under this condition, the fluid is said to undergo a solid-body type rotation. Because an external torque is required to start the motion, the term "Forced Vortex" has also been used. Once steady conditions are established, there is no relative motion between fluid particles and thus no shear forces (i.e. frictional forces) exist, even in a real fluid.



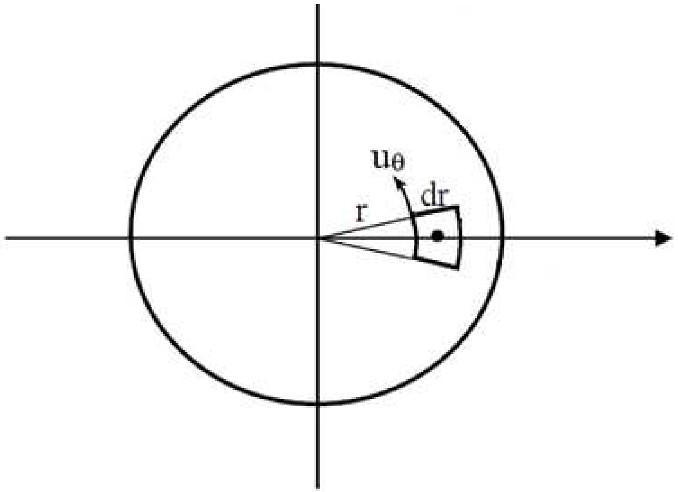
Equilibrium of Moving Fluids

- Uniform Spin of Liquid in a Container



Equilibrium of Moving Fluids

- Uniform Spin of Liquid in a Container
  - Consider a mass of liquid in a container subjected to a constant rotation ω.
  - The velocity of a typical element of dimension dr, $rd\theta$, dz at a radial distance r from the axis of rotation is $u = u_\theta = \omega r$.
  - The acceleration of the same element is given by $r\omega^2$ in a radially inward direction.



Equilibrium of Moving Fluids

- Uniform Spin of Liquid in a Container
  - Equation of motion of fluid particle in the radial *r* direction.

$$\left(p - \frac{\partial p}{\partial r}\frac{dr}{2}\right) rd\theta dz - \left(p + \frac{\partial p}{\partial r}\frac{dr}{2}\right)\left(r + dr\right) d\theta dz + 2pdr\frac{d\theta}{2}dz = -\omega^2 r\left(\rho rd\theta dzdr\right)$$

$$\left(p - \frac{\partial p}{\partial r}\frac{dr}{2}\right) r - \left(p + \frac{\partial p}{\partial r}\frac{dr}{2}\right)\left(r + dr\right) + pdr = -\rho\omega^2 r^2 dr$$

$$pr - \frac{\partial p}{\partial r}\frac{dr}{2} r - \left(pr + \frac{\partial p}{\partial r}\frac{dr}{2} r + pdr + \frac{\partial p}{\partial r}\frac{dr}{2} dr\right) + pdr = -\rho\omega^2 r^2 dr$$

$$-\frac{\partial p}{\partial r} rdr - \frac{\partial p}{\partial r}\frac{dr}{2} dr = -\rho\omega^2 r^2 dr$$

When $dr \to 0$, the above equation can be simplified as

$$\frac{\partial p}{\partial r} = \rho r\omega^2$$

Equilibrium of Moving Fluids

- Uniform Spin of Liquid in a Container
  - Equation of motion of fluid particle in the z direction.

$$\left(p - \frac{\partial p}{\partial z}\frac{dz}{2}\right) rd\theta \bullet dr - \left(p + \frac{\partial p}{\partial z}\frac{dz}{2}\right) rd\theta \bullet dr - \rho g\left(rd\theta dz dr\right) = 0$$

$$\frac{\partial p}{\partial z} = \rho g$$

  - Equation of motion of fluid particle in the θ direction.

Solid-body type rotation $\Longrightarrow$ $\dfrac{\partial p}{\partial \theta} = 0$

Equilibrium of Moving Fluids

- Uniform Spin of Liquid in a Container
  - Determine pressure.

$$\frac{\partial p}{\partial r} = \rho r \omega^2 \quad \Rightarrow \quad p = \frac{1}{2}\rho r^2 \omega^2 + f(z) + c_1$$

$$\frac{\partial p}{\partial z} = \rho g \quad \Rightarrow \quad p = -\rho g z + f(r) + c_2$$

$$p = -\rho g z + \frac{1}{2}\rho r^2 \omega^2 + c_3$$

$$\text{at } r = 0, z = z_0 : p = p_0$$

$$c_3 = p_0 + \rho g z_0$$

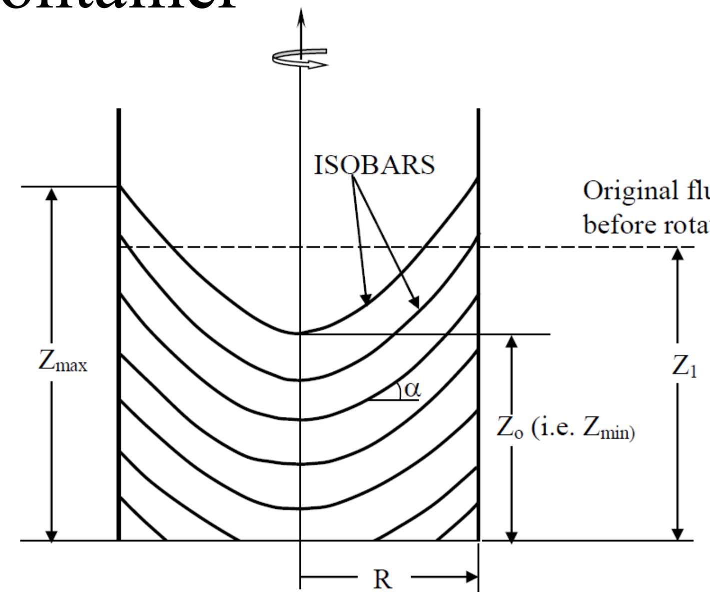
$$p = -\rho g (z - z_0) + \frac{1}{2}\rho r^2 \omega^2 + p_0$$

Equilibrium of Moving Fluids

- Uniform Spin of Liquid in a Container
  - Shape of free surface.

$$p = p_0 \quad \text{at the free surface}$$

$$0 = -\rho g \left(z_s - z_0 \right) + \frac{1}{2} \rho r^2 \omega^2$$



Paraboloid of Revolution

$$z_s = z_0 + \frac{1}{2g} r^2 \omega^2$$

The slope of the liquid level at any radius *r* is given by

$$\tan \alpha = \frac{dz}{dr} = \frac{2 r \omega^2}{2g} = \frac{r \omega^2}{g}$$

Equilibrium of Moving Fluids

- Uniform Spin of Liquid in a Container
  - More geometrical information.
    - ✓ Volume of liquid

$$V_c = \int_{r=0}^{r} 2\pi z_s r dr = 2\pi \int_{r=0}^{r} \left(z_0 + \frac{1}{2g} r^2 \omega^2 \right) r dr = \pi R^2 \left(\frac{\omega^2 R^2}{4g} + z_0 \right)$$

$$V_c == \pi R^2 z_1$$

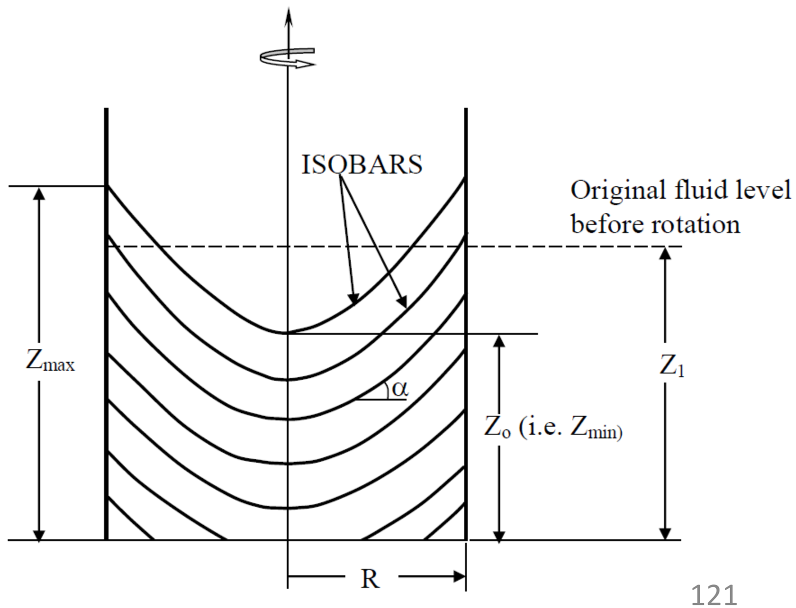
    - ✓ Determine z_0 and z_{max}

$$z_0 = z_1 - \frac{\omega^2 R^2}{4g}$$

$$z_s = z_1 - \frac{\omega^2}{4g} \left(R^2 - 2r^2 \right)$$

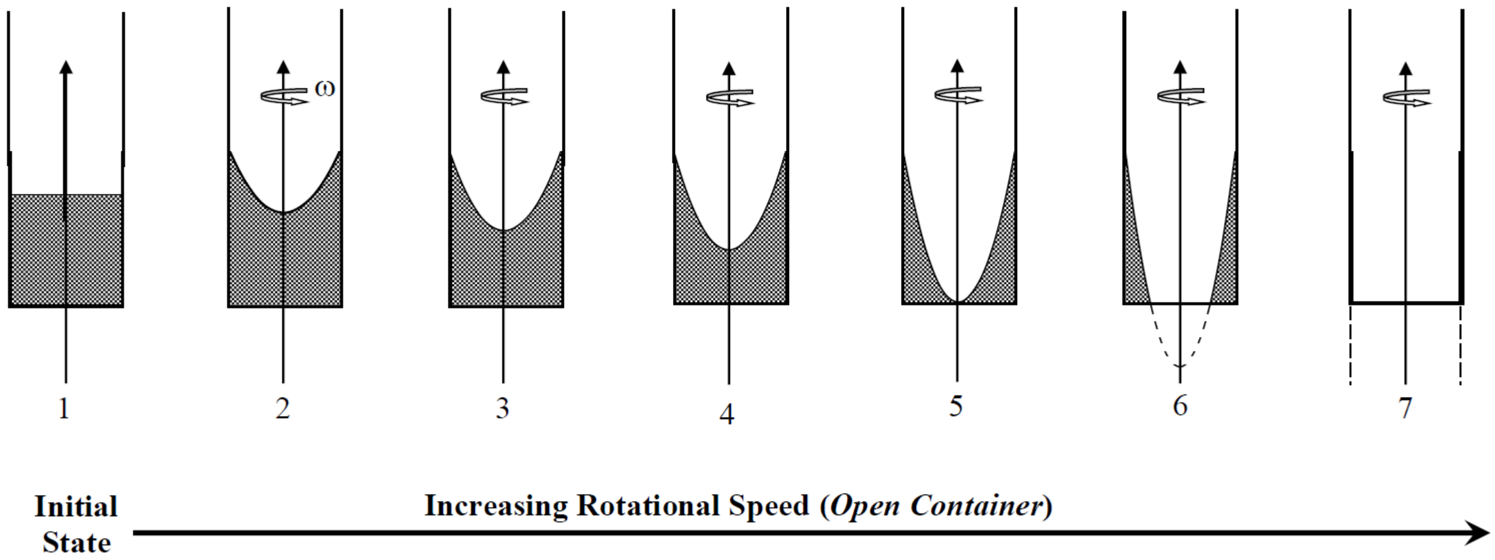
$$z_{\max} = z_1 + \frac{R^2 \omega^2}{4g}$$

$$h_f = z_{\max} - z_0 = \frac{R^2 \omega^2}{2g}$$



Equilibrium of Moving Fluids

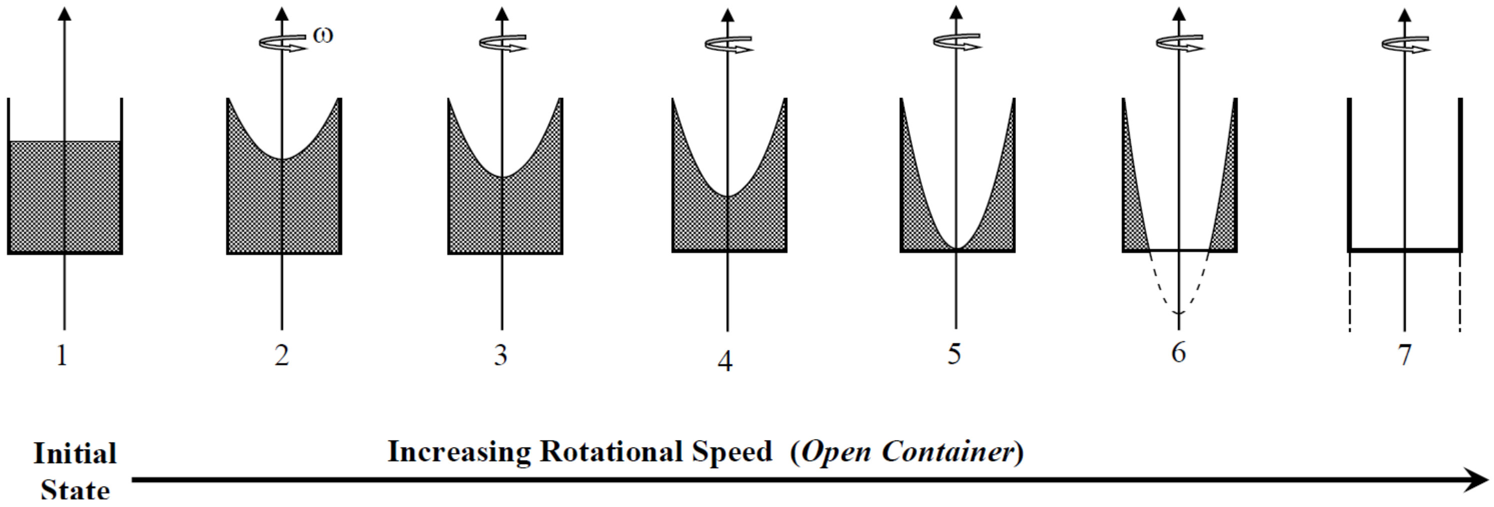
- Uniform Spin of Liquid in a Container



Tall Cylinder: No spillage of fluid

Equilibrium of Moving Fluids

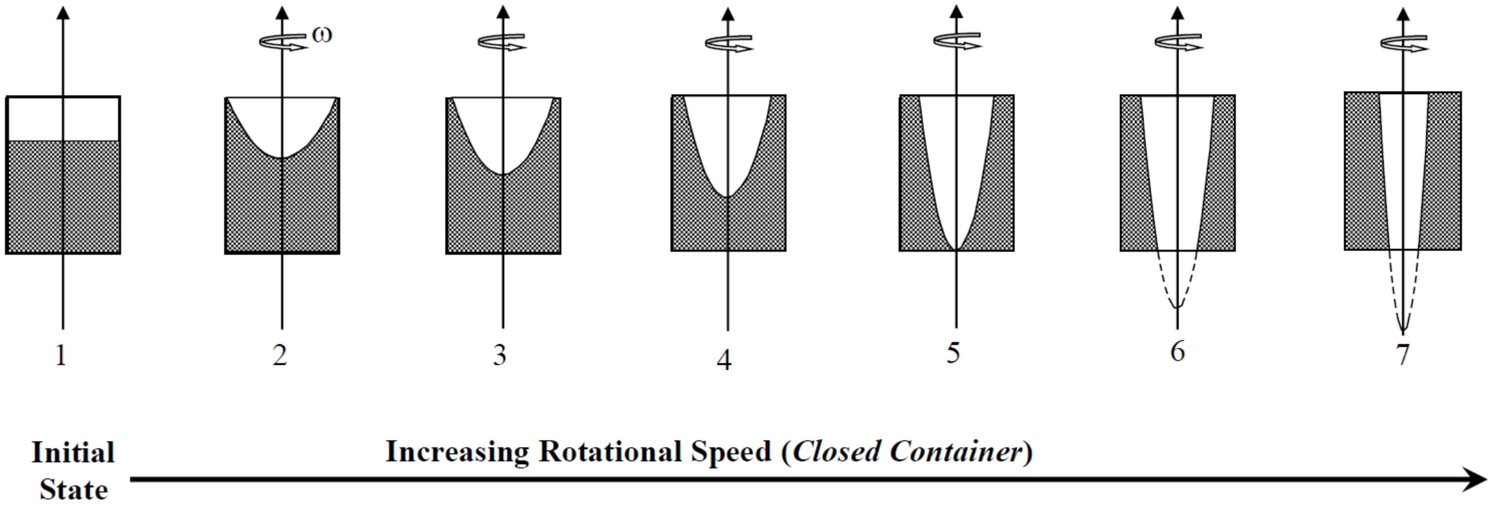
- Uniform Spin of Liquid in a Container



Short cylinder: Spillage of fluid

Equilibrium of Moving Fluids

- Uniform Spin of Liquid in a Container



Fluid Statics

- Reviewing
  - Pressure in a fluid is independent of shape or cross section of container
  - Pressure is the same at all points on a horizontal plane in a given fluid (No other acceleration except gravity)
  - Pressures changes with vertical distance (depth), but remains constant in other directions

$$\frac{dP}{dz} = -\rho g$$

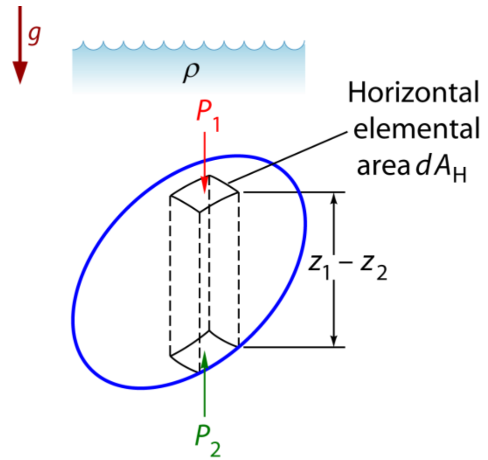
$$P_{bottom} = P^{top} + \rho g \left| \Delta z \right|$$

where $|\Delta z|$ is the absolute difference (distance) in depth between the two points of interest

Fluid Statics

- Reviewing
  - **Archimedes Principle**: A body immersed in a fluid experiences a vertical buoyant (upthrust) force equal to the weight of the fluid it displaces
  - Center of buoyancy B may or may not correspond to actual mass center of immersed body's own material
  - For stability the metacentre must be above the center of gravity or GM > 0

$$F_B = \rho g\left(\text{body volume}\right)$$

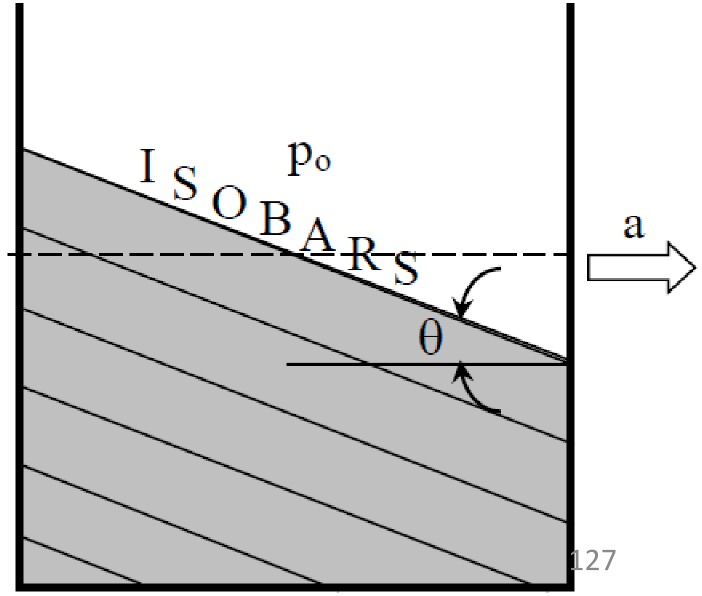


Fluid Statics

- Reviewing
  - Hydrostatic equation in a uniform velocity of translation remains the same
  - When a liquid in an open tank is subjected to a constant linear acceleration a, the free surface of the fluid is inclined at an angle θ to the direction of acceleration.

$$p = -\rho\left[\left(a_z + g\right) z + a_x x\right] + p_0$$

$$\tan\theta = \frac{dz}{dx} = \frac{a_x}{a_z + g}$$



Fluid Statics

- Reviewing
  - For a solid-body type rotation, the free surface is paraboloid shape

$$p = -\rho g\left(z - z_0\right) + \frac{1}{2}\rho r^2\omega^2 + p_0$$

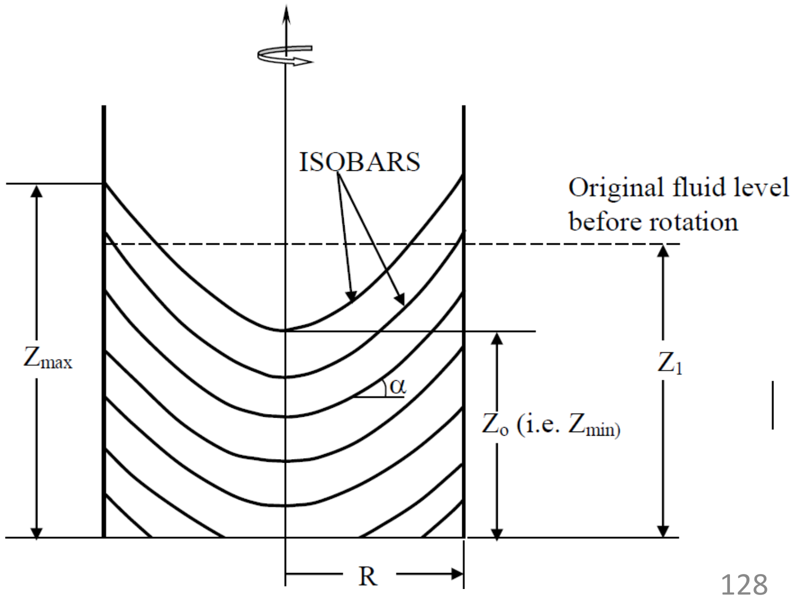
$$\tan\alpha = \frac{dz}{dr} = \frac{2r\omega^2}{2g} = \frac{r\omega^2}{g}$$

$$z_0 = z_1 - \frac{\omega^2 R^2}{4g}$$

$$z_s = z_1 - \frac{\omega^2}{4g}\left(R^2 - 2r^2\right)$$

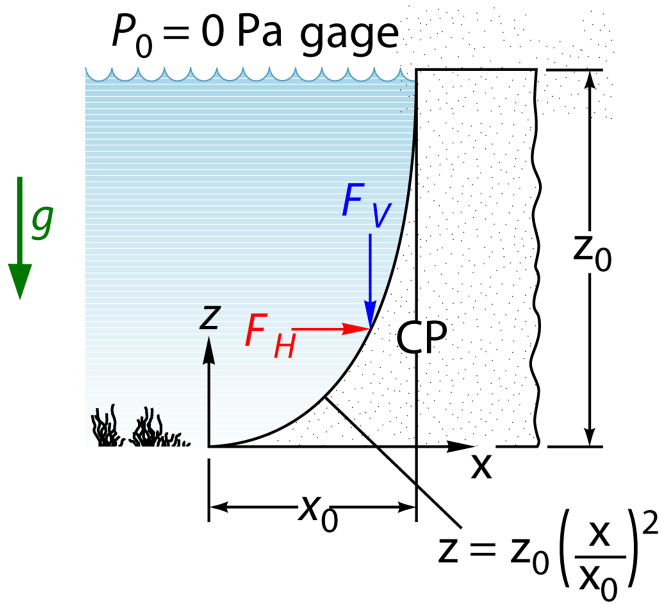
$$z_{\max} = z_1 + \frac{R^2\omega^2}{4g}$$

$$h_f = z_{\max} - z_0 = \frac{R^2\omega^2}{2g}$$



Fluid Statics

- Example 4
  - Problem Statement
    - ✓ Dam (width b = 100 m) with parabolic shape, x_0 = 10 m, z_0 = 24 m
    - ✓ Fluid: water (ρ = 1000 kg/m^3), omit atmospheric pressure (P_0 = 0 Pa)
  - Questions
    - ✓ Find F_H and F_V acting on dam and position CP where they act

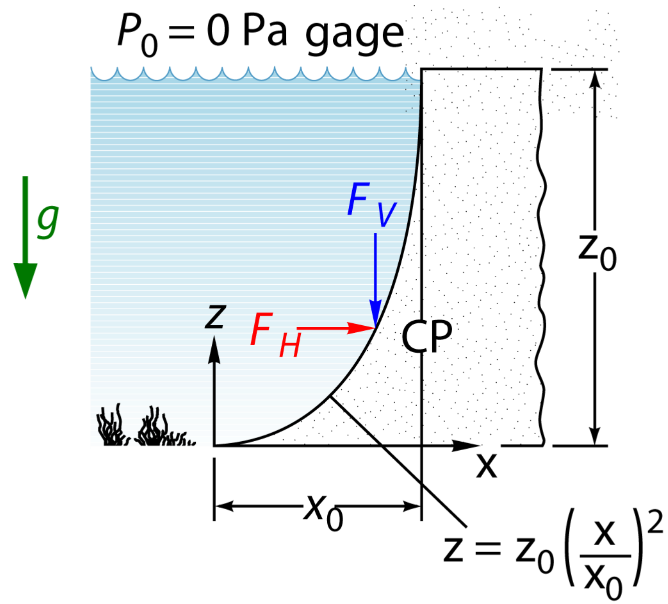


Fluid Statics

- Example 4
  - Solution:
    - ✓ Vertical projection of curved surface is a rectangle 24 m high and 100 m wide

$$h_C = y_C \sin\theta$$

$$h_C = y_C \sin 90° = y_C$$



    - ✓ Depth of centroid:

$$y_C = h_C = 12 \text{ m}$$

    - ✓ Horizontal component F_H

$$F_H = \rho g h_C A$$

$$F_H = 1000 \times 9.81 \times 12 \times 24 \times 100$$

$$F_H = 2.825 \times 10^8 \text{ N}$$

Fluid Statics

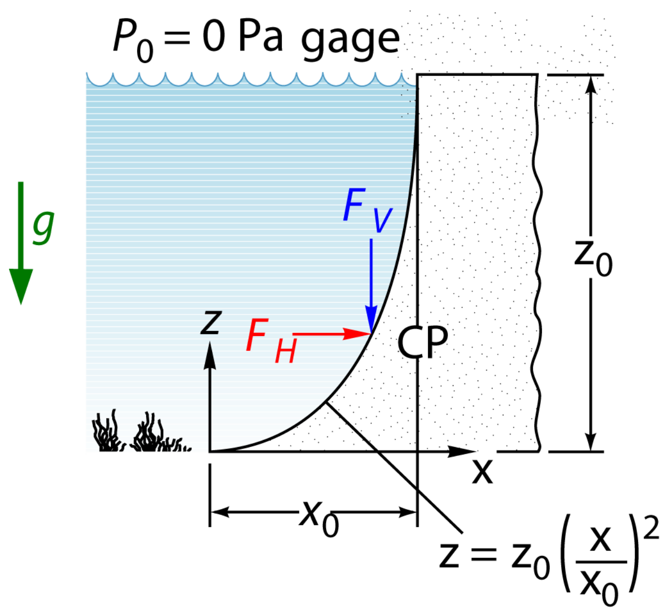
- Example 4
  - Solution:
    - ✓ Line of action of F_H below free surface:

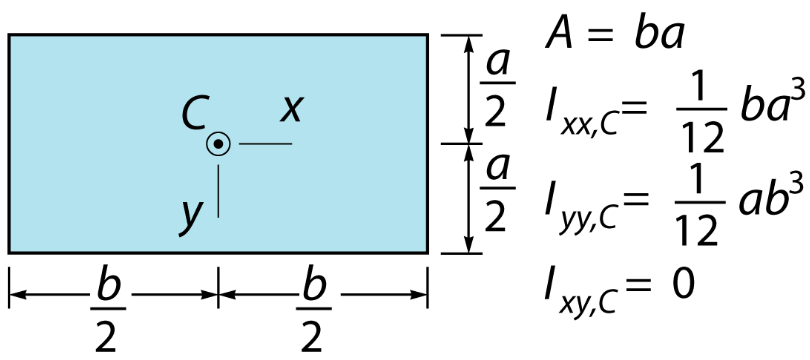
$$h_P = y_P = y_C + \frac{I_{xx,C}}{y_C A}$$

$$h_P = y_P = 12 + \frac{\frac{1}{12}(100)(24)^3}{(12)(24)(100)}$$

$$h_P = 16 \text{ m}$$

    - ✓ F_H acts 8 m from bottom.





Fluid Statics

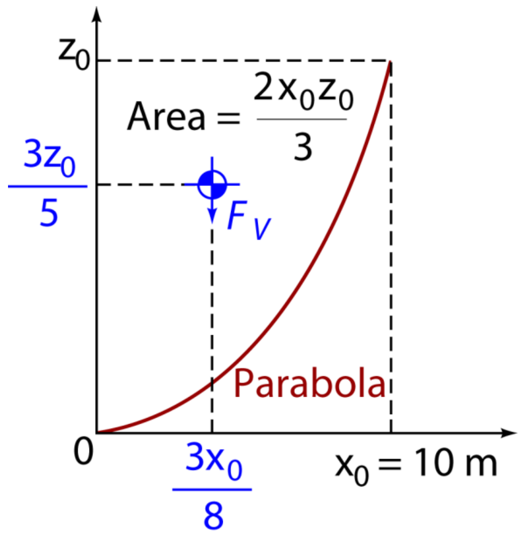
- Example 4
  - Solution:
    - ✓ Vertical component $F_V \Rightarrow$ weight of parabolic portion of fluid above curved surface

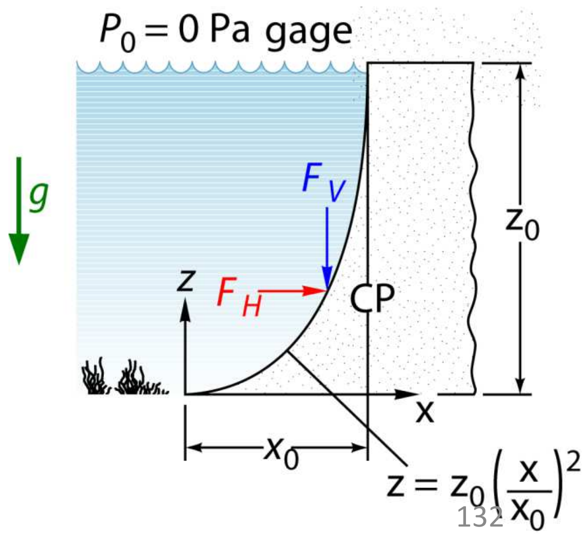
$$F_V = \rho g\left(\frac{2}{3}x_0 z_0 b\right)$$

$$F_V = (1000)(9.81)\left(\frac{2}{3}\right)(10)(24)(100)$$

$$F_V = 1.570\times 10^8 \text{ N}$$

- ✓ F_V acts downward on surface at $3x_0/8 = 3.75$ from origin





Fluid Statics

- Example 4
  - Solution:
    - ✓ Total resultant force on dam:

$$F = \sqrt{F_H^2 + F_V^2}$$

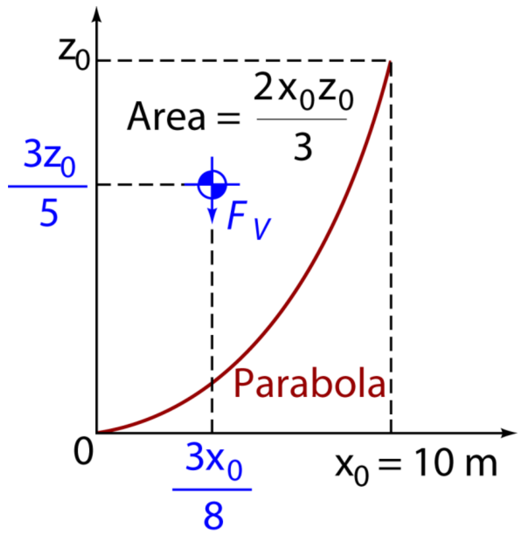
$$F = \sqrt{\left(2.825\times10^8\right)^2 + \left(1.570\times10^8\right)^2}$$

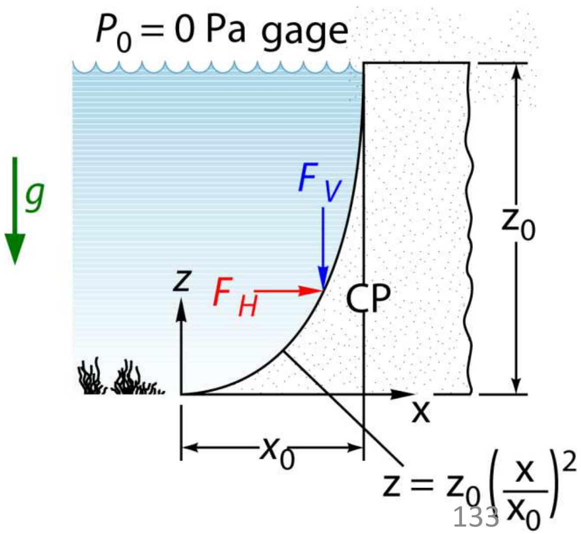
$$F = 3.232 \text{ x } 10^8 \text{ N}$$

    - ✓ *F* acts down and to the right at angle of

$$\tan^{-1}\left(\frac{1.570}{2.825}\right) = 29°$$

    - ✓ *F* passes through (3.75 m, 8 m)

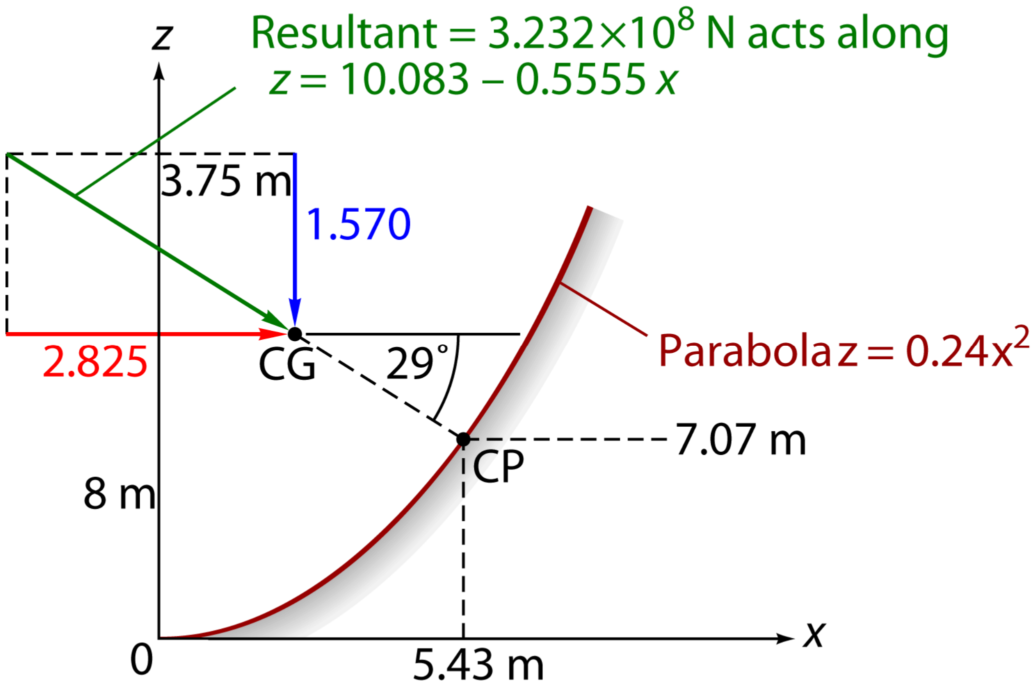




Fluid Statics

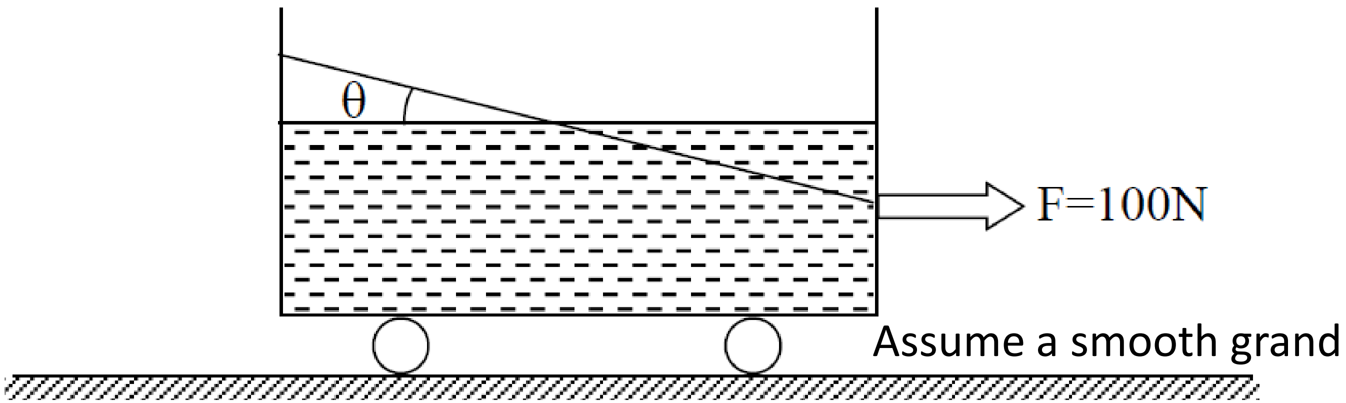
- Example 4
  - Solution:
    - ✓ Equivalent center of pressure CP: move down along 29º line until strike dam

$$x_{CP} = 5.43 \text{ m and } z_{CP} = 7.07 \text{ m}$$



Fluid Statics

- Example 5
  - Question:
    - ✓ A tank weighing 80 N and containing 0.35 m^3 of water is acted upon by a force F of 100 N. What is θ when the free surface of the water assumes a fixed orientation?



Fluid Statics

- Example 5
  - Solution:
    - ✓ From Newton's Second Law, the acceleration of the tank is given by

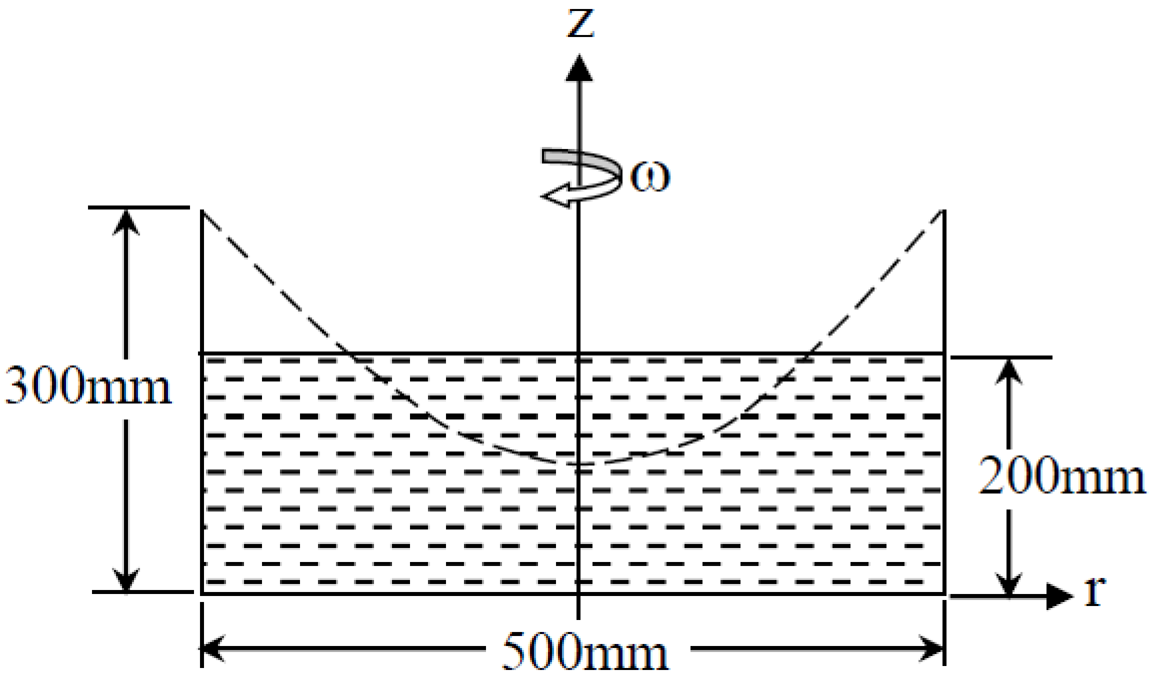
$$a_x = \frac{F}{M} = \frac{F}{M_{\text{tank}} + M_{\text{water}}} = \frac{100}{\frac{80}{9.8} + 1000 \times 0.35} = 0.279\text{m/s}^2$$

    - ✓ The angle:

$$\tan\theta = \frac{dz}{dx} = -\frac{a_x}{g} = \frac{0.279}{9.8} = -0.0284$$

$$\theta = -1.63°$$

- Example 6
  - Question:
    - ✓ A tank of water is rotating at an angular speed of ω radian/sec. At what speed must the cylinder be rotating before the water spills over the top?



Fluid Statics

- Example 6
  - Solution:

$$z_{max} = z_1 + \frac{R^2\omega^2}{4g}$$

$$\omega_{min} = \frac{2}{R}\sqrt{g\bullet\left(z_{max} - z_1\right)}$$

$$\omega_{min} = \frac{2}{0.25}\sqrt{9.8\bullet\left(0.3 - 0.2\right)}$$

$$\omega_{min} = 7.92 \text{ rad/sec}$$


Thank You for Your Attention!
Any Questions?