Chapter 14 - Problem Set 2

Calculus 3

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Section 5: The Chain Rule

3-7 (odd)

Use The Chain Rule to find $\dfrac{dz}{dt}$ or $\dfrac{dw}{dt}$.

3. $z = xy^3 - x^2y, \quad x = t^2 + 1, \quad y = t^2 - 1$

Solution

$$\frac{dz}{dt} = \frac{dz}{dx}\left(\frac{dx}{dt}\right) + \frac{dz}{dy}\left(\frac{dy}{dt}\right)$$

$$\frac{dz}{dt} = [y^3 - 2xy](2t) + [3xy^2 - x^2](2t)$$

$$\boxed{\frac{dz}{dt} = 2t[y^3 - 2xy + 3xy^2 - x^2]}$$

5. $z = \sin x \cos y, \quad x = \sqrt{t}, \quad y = 1/t$

Solution

$$\frac{dz}{dt} = \frac{dz}{dx}\left(\frac{dx}{dt}\right) + \frac{dz}{dy}\left(\frac{dy}{dt}\right)$$

$$\frac{dz}{dt} = [\cos x \cos y]\left(\frac{1}{2}t^{-\frac{1}{2}}\right) + [-\sin x \sin y]\left(-t^{-2}\right)$$

$$\boxed{\frac{dz}{dt} = \frac{1}{2\sqrt{t}}\cos x \cos y + \frac{1}{t^2}\sin x \sin y}$$

7. $w = xe^{y/z}, \quad x = t^2, \quad y = 1 - t, \quad z = 1 + 2t$

Solution

$$\frac{dw}{dt} = \frac{dw}{dx}\left(\frac{dx}{dt}\right) + \frac{dw}{dy}\left(\frac{dy}{dt}\right) + \frac{dw}{dz}\left(\frac{dz}{dt}\right)$$

$$\frac{dw}{dt} = \left[e^{y/z}\right](2t) + \left[\frac{x}{z}e^{y/z}\right](-1) + \left[-\frac{xy}{z^2}e^{y/z}\right](2)$$

$$\boxed{\frac{dw}{dt} = e^{y/z}\left[2t - \frac{x}{z} - \frac{2xy}{z^2}\right]}$$

11-15 (odd)

Use the Chain Rule to find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$

11. $z = (x-y)^5, \quad x = s^2t, \quad y = st^2$

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}\right)$$

$$\frac{\partial z}{\partial s} = \left[5(x-y)^4\right](2st) + \left[-5(x-y)^4\right](t^2)$$

$$\boxed{\frac{\partial z}{\partial s} = 5(x-y)^4\left[2st - t^2\right]}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial t}\right)$$

$$\frac{\partial z}{\partial t} = \left[5(x-y)^4\right](s^2) + \left[-5(x-y)^4\right](2st)$$

$$\boxed{\frac{\partial z}{\partial t} = 5(x-y)^4\left[s^2 - 2st\right]}$$

13. $z = \ln(3x+2y), \quad x = s\sin t, \quad y = t\cos s$

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}\right)$$

$$\frac{\partial z}{\partial s} = \left[\frac{3}{3x+2y}\right](\sin t) + \left[\frac{2}{3x+2y}\right](-t\sin s)$$

$$\boxed{\frac{\partial z}{\partial s} = \frac{3\sin t - 2t\sin s}{3x+2y}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial t}\right)$$

$$\frac{\partial z}{\partial t} = \left[\frac{3}{3x+2y}\right](s\cos t) + \left[\frac{2}{3x+2y}\right](\cos s)$$

$$\boxed{\frac{\partial z}{\partial t} = \frac{3s\cos t + 2\cos s}{3x+2y}}$$

15. $z = (\sin\theta)/r, \quad r = st, \quad \theta = s^2 + t^2$

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\left(\frac{\partial r}{\partial s}\right) + \frac{\partial z}{\partial \theta}\left(\frac{\partial \theta}{\partial s}\right)$$
$$\frac{\partial z}{\partial s} = \left[-\frac{\sin\theta}{r^2}\right](t) + \left[\frac{\cos\theta}{r}\right](2s)$$
$$\boxed{\frac{\partial z}{\partial s} = -\frac{t\sin\theta}{r^2} + \frac{2s\cos\theta}{r}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\left(\frac{\partial r}{\partial t}\right) + \frac{\partial z}{\partial \theta}\left(\frac{\partial \theta}{\partial t}\right)$$
$$\frac{\partial z}{\partial t} = \left[-\frac{\sin\theta}{r^2}\right](s) + \left[\frac{\cos\theta}{r}\right](2t)$$
$$\boxed{\frac{\partial z}{\partial t} = -\frac{s\sin\theta}{r^2} + \frac{2t\cos\theta}{r}}$$

25-29 (odd)

Use the Chain Rule to find the indicated partial derivatives.

25. $z = x^4 + x^2y, \quad x = s + 2t - u, \quad y = stu^2$**;**

$\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}, \frac{\partial z}{\partial u}$ **when** $s = 4, t = 2, u = 1$

Solution

When $s = 4$, $t = 2$, and $u = 1 \quad \Rightarrow \quad x = 7$, $y = 8$

$$\frac{\partial z}{\partial s} = \left[4x^3 + 2xy\right](1) + \left[x^2\right](tu^2) = 4x^3 + 2xy + x^2tu^2$$
$$\frac{\partial z}{\partial s} = 4(7)^3 + 2(7)(8) + (7)^2(2)(1)^2$$
$$\boxed{\frac{\partial z}{\partial s} = 1582}$$

$$\frac{\partial z}{\partial t} = \left[4x^3 + 2xy\right](2) + \left[x^2\right](su^2) = 8x^3 + 4xy + x^2su^2$$
$$\frac{\partial z}{\partial t} = 8(7)^3 + 4(7)(8) + (7)^2(4)(1)^2$$
$$\boxed{\frac{\partial z}{\partial t} = 3164}$$

$$\frac{\partial z}{\partial u} = \left[4x^3 + 2xy\right](-1) + \left[x^2\right](2stu) = -4x^3 - 2xy + 2x^2stu$$
$$\frac{\partial z}{\partial u} = -4(7)^3 - 2(7)(8) + 2(7)^2(4)(2)(1)$$
$$\boxed{\frac{\partial z}{\partial u} = -700}$$

27. $w = xy + yz + zx, \quad x = r\cos\theta, \quad y = r\sin\theta, \quad z = r\theta$**;**

$\dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial \theta}$ **when** $r = 2, \theta = \pi/2$

Solution

When $r = 2$ and $\theta = \pi/2 \quad \Rightarrow \quad x = 0,\ y = 2,\ z = \pi$

$$\frac{\partial w}{\partial r} = [y+z](\cos\theta) + [x+z](\sin\theta) + [x+y](\theta)$$
$$\frac{\partial w}{\partial r} = [2+\pi](0) + [0+\pi](1) + [0+2](\pi/2) = 0 + \pi + \pi$$
$$\boxed{\frac{\partial w}{\partial r} = 2\pi}$$

$$\frac{\partial w}{\partial \theta} = [y+z](-r\sin\theta) + [x+z](r\cos\theta) + [x+y](r)$$
$$\frac{\partial w}{\partial \theta} = [2+\pi](-2\cdot 1) + [0+\pi](0) + [0+2](2) = -4 - 2\pi + 0 + 4$$
$$\boxed{\frac{\partial w}{\partial \theta} = -2\pi}$$

29. $N = \dfrac{p+q}{p+r}, \quad p = u + vw, \quad q = v + uw, \quad r = w + uv$

$\dfrac{\partial N}{\partial u}, \dfrac{\partial N}{\partial v}, \dfrac{\partial N}{\partial w}$ **when** $u = 2, v = 3, w = 4$

Solution

When $u = 2$, $v = 3$, and $w = 4 \quad \Rightarrow \quad p = 14$, $q = 11$, $r = 10$

$$\frac{\partial N}{\partial u} = \left[\frac{(p+r)-(p+q)}{(p+r)^2}\right](1) + \left[\frac{1}{p+r}\right](w) + \left[-\frac{p+q}{(p+r)^2}\right](v)$$
$$\frac{\partial N}{\partial u} = \frac{r-q}{(p+r)^2} + \frac{w}{p+r} - \frac{v(p+q)}{(p+r)^2}$$
$$\frac{\partial N}{\partial u} = \frac{10-11}{(14+10)^2} + \frac{4}{14+10} - \frac{3(14+11)}{(14+10)^2}$$
$$\frac{\partial N}{\partial u} = \frac{-1}{576} + \frac{4}{24} - \frac{75}{576} = \frac{-1+96-75}{576} = \frac{20}{576}$$
$$\boxed{\frac{\partial N}{\partial u} = \frac{5}{144}}$$

$$\frac{\partial N}{\partial v} = \left[\frac{r-q}{(p+r)^2}\right](w) + \left[\frac{1}{p+r}\right](1) + \left[-\frac{p+q}{(p+r)^2}\right](u)$$
$$\frac{\partial N}{\partial v} = \frac{w(r-q)}{(p+r)^2} + \frac{1}{p+r} - \frac{u(p+q)}{(p+r)^2}$$
$$\frac{\partial N}{\partial v} = \frac{4(10-11)}{(14+10)^2} + \frac{1}{14+10} - \frac{2(14+11)}{(14+10)^2}$$
$$\frac{\partial N}{\partial v} = \frac{-4}{576} + \frac{1}{24} - \frac{50}{576} = \frac{-4+24-50}{576} = \frac{-30}{576}$$
$$\boxed{\frac{\partial N}{\partial v} = -\frac{5}{96}}$$

$$\frac{\partial N}{\partial w} = \left[\frac{r-q}{(p+r)^2}\right](v) + \left[\frac{1}{p+r}\right](u) + \left[-\frac{p+q}{(p+r)^2}\right](1)$$
$$\frac{\partial N}{\partial w} = \frac{v(r-q)}{(p+r)^2} + \frac{u}{p+r} - \frac{p+q}{(p+r)^2}$$
$$\frac{\partial N}{\partial w} = \frac{3(10-11)}{(14+10)^2} + \frac{2}{(14+10)} - \frac{14+11}{(14+10)^2}$$
$$\frac{\partial N}{\partial w} = \frac{-3}{576} + \frac{2}{24} - \frac{25}{576} = \frac{-3+48-25}{576} = \frac{20}{576}$$
$$\boxed{\frac{\partial N}{\partial w} = \frac{5}{144}}$$

Use Equation 5 to find $\frac{dy}{dx}$

31. $y\cos x = x^2 + y^2$

Solution

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Let $F = y\cos x - x^2 - y^2$

$$F_x = -y\sin x - 2x$$
$$F_y = \cos x - 2y$$
$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{(-y\sin x - 2x)}{\cos x - 2y} = \frac{2x + y\sin x}{\cos x - 2y}}$$

Use Equations 6 to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

35. $x^2 + 2y^2 + 3z^2 = 1$

Solution

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Let $F = x^2 + 2y^2 + 3z^2 - 1$

$$F_x = 2x \qquad F_y = 4y \qquad F_z = 6z$$
$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{6z} = -\frac{x}{3z}}$$
$$\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{4y}{6z} = -\frac{2y}{3z}}$$

Section 6: Directional Derivatives and the Gradient Vector

5, 7

Find the directional derivative of f at the given point in the direction indicated by the angle θ.

5. $f(x,y) = y\cos(xy)$, $(0,1)$, $\theta = \pi/4$

Solution

$$\begin{aligned}
D_u f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\
D_u f(x,y) &= \left[-y^2\sin(xy)\right]\cos\frac{\pi}{4} + \left[\cos(xy) - xy\sin(xy)\right]\sin\frac{\pi}{4} \\
D_u f(x,y) &= \frac{\sqrt{2}}{2}\left[-y^2\sin(xy) + \cos(xy) - xy\sin(xy)\right] \\
D_u f(0,1) &= \frac{\sqrt{2}}{2}\left[0 + 1 - 0\right] \\
&\boxed{D_u f(0,1) = \frac{\sqrt{2}}{2}}
\end{aligned}$$

7. $f(x,y) = arctan(xy)$, $(2,-3)$, $\theta = 3\pi/4$

Solution

$$\begin{aligned}
D_u f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\
D_u f(x,y) &= \left[\frac{y}{1+(xy)^2}\right]\cos\frac{3\pi}{4} + \left[\frac{x}{1+(xy)^2}\right]\sin\frac{3\pi}{4} \\
D_u f(x,y) &= \frac{\sqrt{2}}{2}\left[\frac{x}{1+(xy)^2} - \frac{y}{1+(xy)^2}\right] \\
D_u f(2,-3) &= \frac{\sqrt{2}}{2}\left[\frac{2}{1+(2(-3))^2} - \frac{-3}{1+(2(-3))^2}\right] \\
D_u f(2,-3) &= \frac{\sqrt{2}}{2}\left[\frac{5}{37}\right] \\
&\boxed{D_u f(2,-3) = \frac{5\sqrt{2}}{74}}
\end{aligned}$$

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$f(x, y) = x/y, \quad P(2, 1), \quad \vec{u} = \frac{3}{5}\,\hat{\mathbf{i}} + \frac{4}{5}\,\hat{\mathbf{j}}$

(a) Find the gradient of f

(b) Evaluate the gradient at the point P

(c) Find the rate of change of f at P in the direction of the vector $\vec{u}$

Solution

(a)

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle \quad \text{or} \quad \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}}$$

$$\boxed{\nabla f(x, y) = \left\langle \frac{1}{y}, -\frac{x}{y^2} \right\rangle}$$

(b)

$$\boxed{\nabla f(2, 1) = \left\langle \frac{1}{1}, -\frac{2}{1^2} \right\rangle = \langle 1, -2\rangle}$$

(c)

$$D_u f(x, y) = \nabla f(x, y) \cdot \vec{u}$$
$$D_u f(2, 1) = \nabla f(2, 1) \cdot \vec{u}$$
$$D_u f(2, 1) = \langle 1, -2\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$
$$D_u f(2, 1) = \frac{3}{5} - \frac{8}{5}$$
$$\boxed{D_u f(2, 1) = -1}$$

13, 15

Find the directional derivative of the function at the given point in the direction of the vector $\vec{v}$.

13. $f(x,y) = e^x \sin y, \quad (0, \pi/3), \quad \vec{v} = \langle -6, 8 \rangle$

Solution

$$\begin{aligned}
\vec{u} &= \frac{\vec{v}}{\|\vec{v}\|} = \frac{-6\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}}}{\sqrt{(6)^2 + (8)^2}} \\
\vec{u} &= -\frac{3}{5}\,\hat{\mathbf{i}} + \frac{4}{5}\,\hat{\mathbf{j}} \\
D_u f(x,y) &= \nabla f(x,y) \cdot \vec{u} \\
D_u f(x,y) &= (e^x \sin y\,\hat{\mathbf{i}} + e^x \cos y\,\hat{\mathbf{j}}) \cdot \left(-\frac{3}{5}\,\hat{\mathbf{i}} + \frac{4}{5}\,\hat{\mathbf{j}}\right) \\
D_u f(x,y) &= -\frac{3e^x \sin y}{5} + \frac{4e^x \cos y}{5} \\
D_u f(0, \frac{\pi}{3}) &= -\frac{3(\frac{\sqrt{3}}{2})}{5} + \frac{4(\frac{1}{2})}{5} \\
&\boxed{D_u f(0, \frac{\pi}{3}) = \frac{4 - 3\sqrt{3}}{10}}
\end{aligned}$$

15. $g(s,t) = s\sqrt{t}, \quad (2,4), \quad \vec{v} = 2\,\hat{\mathbf{i}} - \hat{\mathbf{j}}$

Solution

$$\begin{aligned}
\vec{u} &= \frac{\vec{v}}{\|\vec{v}\|} = \frac{2\,\hat{\mathbf{i}} - \hat{\mathbf{j}}}{\sqrt{(2)^2 + (-1)^2}} \\
\vec{u} &= \frac{2}{\sqrt{5}}\,\hat{\mathbf{i}} - \frac{1}{\sqrt{5}}\,\hat{\mathbf{j}} \\
D_u g(s,t) &= \nabla g(s,t) \cdot \vec{u} \\
D_u g(s,t) &= \left(\sqrt{t}\,\hat{\mathbf{i}} + \frac{s}{2\sqrt{t}}\right) \cdot \left(\frac{2}{\sqrt{5}}\,\hat{\mathbf{i}} - \frac{1}{\sqrt{5}}\,\hat{\mathbf{j}}\right) \\
D_u g(s,t) &= \frac{2\sqrt{t}}{\sqrt{5}} - \frac{s}{2\sqrt{t}\sqrt{5}} \\
D_u g(s,t) &= \frac{1}{\sqrt{5}}\left(2\sqrt{t} - \frac{s}{2\sqrt{t}}\right) \\
D_u g(2,4) &= \frac{1}{\sqrt{5}}\left(2\sqrt{4} - \frac{2}{2\sqrt{4}}\right) = \frac{1}{\sqrt{5}}\left(4 - \frac{1}{2}\right) \\
&\boxed{D_u g(2,4) = \frac{1}{\sqrt{5}}\left(\frac{8-1}{2}\right) = \frac{7}{2\sqrt{5}}}
\end{aligned}$$

21, 23

Find the directional derivative of the function at the point P in the direction of the point Q.

21. $f(x,y) = x^2y^2 - y^3, \quad P(1,2), \quad Q(-3,5)$

Solution

$$\begin{aligned}
\vec{v} &= \vec{PQ} = \langle -3-1, 5-2\rangle = \langle -4, 3\rangle \\
\vec{u} &= \frac{\vec{v}}{\|v\|} = \frac{\langle -4, 3\rangle}{\sqrt{16+9}} \\
\vec{u} &= \left\langle -\frac{4}{5}, \frac{3}{5}\right\rangle \\
D_uf(x,y) &= \nabla f(x,y)\cdot\vec{u} \\
D_uf(x,y) &= \left\langle 2xy^2, 2x^2y - 3y^2\right\rangle \cdot \left\langle -\frac{4}{5}, \frac{3}{5}\right\rangle \\
D_uf(x,y) &= -\frac{8xy^2}{5} + \frac{3(2x^2y-3y^2)}{5} = \frac{1}{5}(-8xy^2 + 6x^2y - 9y^2) \\
D_uf(1,2) &= \frac{1}{5}[-8(1)(4) + 6(1)(2) - 9(4)] = \frac{1}{5}[-32+12-36] \\
&\boxed{D_uf(1,2) = -\frac{56}{5}}
\end{aligned}$$

23. $f(x,y) = \sqrt{xy}, \quad P(2,8), \quad Q(5,4)$

Solution

$$\begin{aligned}
\vec{v} &= \vec{PQ} = (5-2)\,\hat{\mathbf{i}} + (4-8)\,\hat{\mathbf{j}} = 3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} \\
\vec{u} &= \frac{\vec{v}}{\|v\|} = \frac{3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}}{\sqrt{9+16}} \\
\vec{u} &= \frac{3}{5}\,\hat{\mathbf{i}} - \frac{4}{5}\,\hat{\mathbf{j}} \\
D_uf(x,y) &= \nabla f(x,y)\cdot\vec{u} \\
D_uf(x,y) &= \left(\frac{y}{2\sqrt{xy}}\,\hat{\mathbf{i}} + \frac{x}{2\sqrt{xy}}\,\hat{\mathbf{j}}\right)\cdot\left(\frac{3}{5}\,\hat{\mathbf{i}} - \frac{4}{5}\,\hat{\mathbf{j}}\right) \\
D_uf(x,y) &= \frac{3y}{10\sqrt{xy}} - \frac{4x}{10\sqrt{xy}} = \frac{3y-4x}{10\sqrt{xy}} \\
D_uf(2,8) &= \frac{3(8)-4(2)}{10\sqrt{2(8)}} = \frac{24-8}{10\sqrt{16}} = \frac{16}{40} \\
&\boxed{D_uf(x,y) = \frac{2}{5}}
\end{aligned}$$

27, 29

Find the maximum rate of change of f at the given point and the direction in which it occurs.

27. $f(x, y) = 5xy^2, \quad (3, -2)$

Solution

The maximum rate of change of f and its direction is given by the gradient vector

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \left\langle 5y^2, 10xy \right\rangle$$
$$\nabla f(3, -2) = \left\langle 5(-2)^2, 10(3)(-2) \right\rangle$$

Direction of the maximum rate of change,

$$\boxed{\nabla f(3, -2) = \langle 20, -60 \rangle = 20 \langle 1, -3 \rangle}$$

Maximum rate of change,

$$\boxed{\|\nabla f(3, -2)\| = 20\sqrt{1+9} = 20\sqrt{10}}$$

29. $f(x, y) = \sin(xy), \quad (1, 0)$

Solution

The maximum rate of change of f and its direction is given by the gradient vector

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} = y\cos(xy)\,\hat{\mathbf{i}} + x\cos(xy)\,\hat{\mathbf{j}}$$
$$\nabla f(1, 0) = 0 + \ \hat{\mathbf{j}}$$

Direction of the maximum rate of change,

$$\boxed{\nabla f(1, 0) = \ \hat{\mathbf{j}}}$$

Maximum rate of change,

$$\boxed{\|\nabla f(1, 0)\| = \sqrt{1} = 1}$$

37

The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is $120°$

(a) Find the rate of change of T at $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.

(b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points toward the origin

Solution

Since $T \propto \frac{1}{d}$, $T(x, y, z) = \dfrac{c}{\sqrt{x^2 + y^2 + z^2}}$, for some constant c, and Euclidean distance to describe the distance from the origin.

If $120 = T(1, 2, 2) = \dfrac{c}{\sqrt{1^2 + 2^2 + 2^2}} = \dfrac{c}{3}$, then $c = 360$

$\therefore$ Our function for temperature is,

$$T(x, y, z) = \frac{360}{\sqrt{x^2 + y^2 + z^2}}$$

(a)

$$\vec{v} = \langle 2 - 1, 1 - 2, 3 - 2 \rangle = \langle 1, -1, 1 \rangle$$
$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1, -1, 1 \rangle}{\sqrt{1 + 1 + 1}}$$
$$\vec{u} = \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

Getting our gradient vector,

$$\nabla T(x, y, z) = \langle T_x, T_y, T_z \rangle$$
$$\nabla T(x, y, z) = \Big\langle -180 \cdot 2x(x^2 + y^2 + z^2)^{-3/2}, -180 \cdot 2y(x^2 + y^2 + z^2)^{-3/2}, -180 \cdot 2z(x^2 + y^2 + z^2)^{-3/2}$$
$$\nabla T(x, y, z) = \frac{360}{(x^2 + y^2 + z^2)^{3/2}} \langle -x, -y, -z \rangle$$

so, our rate of change is

$$D_u T(1, 2, 2) = \nabla T(1, 2, 2) \cdot \vec{u}$$
$$D_u T(1, 2, 2) = \frac{360}{(1 + 4 + 4)^{3/2}} \langle -1, -2, -2 \rangle \cdot \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$
$$D_u T(1, 2, 2) = \frac{360}{27} \left(-\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}} \right) = \frac{360}{27} \left(-\frac{1}{\sqrt{3}} \right)$$
$$\boxed{D_u T(1, 2, 2) = \frac{40}{3} \left(-\frac{1}{\sqrt{3}} \right) = -\frac{40}{3\sqrt{3}}}$$

(b) The direction of greatest increase at any point is the direction of ∇T at that point. In part (a) we saw that $\nabla T(x, y, z)$ and $\langle -x, -y, -z \rangle$ have the same direction and the vector $\langle -x, -y, -z \rangle$ points towards the origin.

47-51 (odd)

Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

47. $2(x-2)^2+(y-1)^2+(z-3)^2=10, \quad (3,3,5)$

Solution

(a)

$$F(x,y,z)=2(x-2)^2+(y-1)^2+(z-3)^2-10$$

$$\begin{aligned}F_x(3,3,5)&=4(x-2)=4(1)=4\\F_y(3,3,5)&=2(y-1)=2(2)=4\\F_z(3,3,5)&=2(z-3)=2(2)=4\end{aligned}$$

The equation of the tangent plane is given by,

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)$$

$$\begin{gathered}4(x-3)+4(y-3)+4(z-5)=0\\x-3+y-3+z-5=0\\\boxed{x+y+z=11}\end{gathered}$$

(b) The equation of the normal line is given by,

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)}=\frac{y-y_0}{F_y(x_0,y_0,z_0)}=\frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

$$\begin{gathered}\frac{x-3}{4}=\frac{y-3}{4}=\frac{z-5}{4}\\\boxed{x-3=y-3=z-5}\end{gathered}$$

49. $xy^2z^3=8, \quad (2,2,1)$

Solution

(a)

$$F(x,y,z)=xy^2z^3-8$$

$$\begin{aligned}F_x(2,2,1)&=y^2z^3=(4)(1)=4\\F_y(2,2,1)&=2xyz^3=2(2)(2)(1)=8\\F_z(2,2,1)&=3xy^2z^2=3(2)(4)(1)=24\end{aligned}$$

The equation of the tangent plane is given by,

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)$$

$$\begin{gathered}4(x-2)+8(y-2)+24(z-1)=0\\x-2+2(y-2)+6(z-1)=0\\x-2+2y-4+6z-6=0\\\boxed{x+2y+6z=12}\end{gathered}$$

(b) The equation of the normal line is given by,

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

$$\frac{x-2}{4} = \frac{y-2}{8} = \frac{z-1}{24}$$
$$\boxed{x-2 = \frac{y-2}{2} = \frac{z-1}{6}}$$

51. $x+y+z = e^{xyz}, \quad (0,0,1)$

Solution

(a)

$$F(x,y,z) = x+y+z-e^{xyz}$$

$$F_x(0,0,1) = 1 - yze^{xyz} = 1-0 = 1$$
$$F_y(0,0,1) = 1 - xze^{xyz} = 1-0 = 1$$
$$F_z(0,0,1) = 1 - xye^{xyz} = 1-0 = 1$$

The equation of the tangent plane is given by,

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0)$$

$$1(x-0) + 1(y-0) + 1(z-1) = 0$$
$$\boxed{x+y+z=1}$$

(b) The equation of the normal line is given by,

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

$$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-1}{1}$$
$$\boxed{x=y=z-1}$$

Section 7: Maximum and Minimum Values

5-21 (odd)

Find the local maximum and minimum values and saddle point(s) of the function. You are encouraged to use a calculator or computer to graph the function with a domain and viewpoint that reveals all the important aspects of the function.

5. $f(x, y) = x^2 + xy + y^2 + y$

Solution

$$\nabla f(x, y) = \langle 2x + y,\ x + 2y + 1 \rangle$$

Finding critical values, (system of two equations)

$$\begin{aligned} 2x + y &= 0 \\ x + 2y &= -1 \end{aligned} \qquad\qquad \begin{aligned} 2x + y &= 0 \\ -2x - 4y &= 2 \\ -3y &= 2 \\ y &= -\frac{2}{3} \quad \Rightarrow \quad x = \frac{1}{3} \end{aligned}$$

Critical point at $\left(\frac{1}{3}, -\frac{2}{3}\right)$

Hessian matrix to determine maxima and minima,

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 1$$

$$H_f(x, y) = det \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \qquad\qquad H_f\left(\frac{1}{3}, \frac{-2}{3}\right) = det \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$H_f\left(\frac{1}{3}, -\frac{2}{3}\right) > 0$ and $f_{xx} = 2 > 0$

$$\boxed{\therefore f\left(\frac{1}{3}, -\frac{2}{3}\right) \text{ is a local minimum}}$$

7. $f(x,y) = 2x^2 - 8xy + y^4 - 4y^3$

Solution

$$\nabla f(x,y) = \left\langle 4x - 8y,\ -8x + 4y^3 - 12y^2 \right\rangle$$

Finding critical values,

$$\begin{aligned} 0 &= 4x - 8y \\ 2x &= 8y \\ x &= 2y \end{aligned} \qquad\qquad \begin{aligned} 0 &= -8x + 4x^3 - 12y^2 \\ 0 &= -8(2y) + 4y^3 - 12y^2 \\ 0 &= 4y^3 - 12y^2 - 16y \\ 0 &= 4y(y^2 - 3y - 4) \\ 0 &= 4y(y-4)(y+1) \\ y &= -1, 0, 4 \quad \Rightarrow \quad x = -2, 0, 8 \end{aligned}$$

Critical points at $(-2,-1), (0,0), (8,4)$

$$f_{xx} = 4, \quad f_{yy} = 12y^2 - 24y, \quad f_{xy} = -8$$

$$H_f(8,4) = det \begin{vmatrix} 4 & -8 \\ -8 & 96 \end{vmatrix} = 384 - 64 = 320 \qquad H_f(-2,-1) = det \begin{vmatrix} 4 & -8 \\ -8 & 36 \end{vmatrix} = 144 - 64 = 80$$

$$H_f(0,0) = det \begin{vmatrix} 4 & -8 \\ -8 & 0 \end{vmatrix} = 0 - 64 = -64$$

$H_f(8,4) > 0$ and $f_{xx}(8,4) = 4 > 0$ so f(8,4) is a local minima
$H_f(-2,-1) > 0$ and $f_{xx}(-2,-1) = 4 > 0$ so f(8,4) is a local minima
$H_f(0,0) < 0$ so f(8,4) is a saddle point

$\boxed{\text{Minima: } f(8,4) = -128, \quad f(-2,-1) = 3}$

$\boxed{\text{Saddle Point: } f(0,0)}$

9. $f(x,y) = (x-y)(1-xy)$

Solution

$$\nabla f(x,y) = \left\langle 1 - 2xy + y^2,\ -x^2 - 1 + 2xy \right\rangle$$

Finding critical values,

$$\begin{aligned} 0 &= 1 - 2xy + y^2 \\ 0 &= y^2 - 2xy + 1 \\ -1 &= y(y - 2x) \\ y &= -1, \quad y = 2x - 1 \end{aligned}$$

$$\begin{aligned} 0 &= -x^2 - 1 + 2xy \\ 0 &= -x^2 - 1 + 2x(2x-1) \\ 0 &= -x^2 - 1 + 4x^2 - 2x \\ 0 &= 3x^2 - 2x - 1 \\ 1 &= 3x^2 - 2x \\ 1 &= x(3x-2) \\ x &= 1 \quad \Rightarrow \quad 3x - 2 = 1 \quad \Rightarrow \quad x = 1 \\ \therefore\ x &= 1 \quad \Rightarrow \quad y = 2(1) - 1 = 1 \end{aligned}$$

If $y = -1$

$$\begin{aligned} 0 &= -1 - x^2 + 2x(-1) \\ 0 &= -x^2 - 2x - 1 \\ 0 &= x^2 + 2x + 1 \\ 0 &= (x+1)^2 \\ x &= -1 \end{aligned}$$

Critical points at (-1, -1), (1, 1)

$$f_{xx} = -2y, \quad f_{yy} = 2x, \quad f_{xy} = -2x + 2y$$

$$H_f(-1,-1) = det \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 - 0 = -4 \qquad H_f(1,1) = det \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -4 - 0 = -4$$

$H_f(-1,-1)$ and $H_f(1,1)$ are both < 0, so they are saddle points.

$\boxed{\text{Saddle Points: } f(-1,-1), \quad f(1,1)}$

11. $f(x,y) = y\sqrt{x} - y^2 - 2x + 7y$

Solution

$$\nabla f(x,y,z) = \left\langle \frac{y}{2\sqrt{x}} - 2,\ \sqrt{x} - 2y + 7 \right\rangle$$

Finding critical values,

$$\begin{aligned} 0 &= \frac{1}{2}yx^{-1/2} - 2 \\ 4 &= yx^{-1/2} \\ y &= 4x^{1/2} \end{aligned} \qquad\qquad \begin{aligned} x^{1/2} - 2y + 7 &= 0 \\ x^{1/2} - 2(4x^{1/2}) + 7 &= 0 \\ x^{1/2} - 8x^{1/2}) + 7 &= 0 \\ -7x^{1/2} &= -7 \\ x = \sqrt{1} &= 1 \quad \Rightarrow \quad y = 4\sqrt{1} = 4 \end{aligned}$$

Critical point at (1, 4)

$$f_{xx} = -\frac{y}{4x^{3/2}}, \quad f_{yy} = -2, \quad f_{xy} = \frac{1}{2\sqrt{x}}$$

$$H_f(1,4) = det\begin{vmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -2 \end{vmatrix} = 2 - \frac{1}{4} = \frac{7}{4}$$

$H_f(1,4) > 0$ and $f_{xx}(1,4) = -1 < 0$ so f(1,4) is a local maximum

$\boxed{\text{Maximum: } f(1,4) = 4}$

13. $f(x, y) = x^3 - 3x + 3xy^2$

Solution

$$\nabla f(x, y) = \left\langle 3x^2 - 3 + 3y^2,\ 6xy \right\rangle$$

Finding critical values,

$$\begin{aligned} 0 &= 3x^2 - 3 + 3y^2 \\ 3 &= 3x^2 + 3y^2 \\ 1 &= x^2 + y^2 \\ x &= 0, \pm 1 \quad \Rightarrow \quad y = \pm 1, 0 \end{aligned}$$

Critical points at (0, 1), (0, -1), (1, 0), (-1, 0)

$$f_{xx} = 6x, \quad f_{yy} = 6x, \quad f_{xy} = 6y$$

$$H_f(0, 1) = det \begin{vmatrix} 0 & 6 \\ 6 & 0 \end{vmatrix} = 0 - 36 = -36 \qquad H_f(0, -1) = det \begin{vmatrix} 0 & -6 \\ -6 & 0 \end{vmatrix} = 0 - 36 = -36$$

$$H_f(1, 0) = det \begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} = 36 - 0 = 36 \qquad H_f(-1, 0) = det \begin{vmatrix} -6 & 0 \\ 0 & -6 \end{vmatrix} = 36 - 0 = 36$$

$H_f(0, \pm 1) < 0$ so f(0, ±1) are saddle points
$H_f(1, 0) > 0$ and $f_{xx}(1, 0) = 6 > 0$ so f(1, 0) is a local minimum
$H_f(-1, 0) > 0$ and $f_{xx}(-1, 0) = -6 < 0$ so f(-1, 0) is a local maximum

$\boxed{\text{Maximum: } f(-1, 0) = 2}$

$\boxed{\text{Minimum: } f(1, 0) = -2}$

$\boxed{\text{Saddle Points: } f(0, \pm 1)}$

15. $f(x,y) = x^4 - 2x^2 + y^3 - 3y$

Solution

$$\nabla f(x,y) = \left\langle 4x^3 - 4x,\ 3y^2 - 3 \right\rangle$$

Finding critical values,

$$\begin{aligned} 0 &= 4x^3 - 4x \\ 0 &= 4x(x^2-1) \\ 0 &= 4x(x-1)(x+1) \\ x &= 0, \pm 1 \end{aligned} \qquad \begin{aligned} 0 &= 3y^2 - 3 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

Critical points at (0, ±1), (± 1, ±1)

$$f_{xx} = 12x^2 - 4, \quad f_{yy} = 6y, \quad f_{xy} = 0$$

$$H_f(0,\pm1) = det\begin{vmatrix} -4 & 0 \\ 0 & 6,-6 \end{vmatrix} = -24, \quad 24 \qquad H_f(1,\pm1) = det\begin{vmatrix} 8 & 0 \\ 0 & 6,-6 \end{vmatrix} = 48,, \quad -48$$

$$H_f(-1,\pm1) = det\begin{vmatrix} 8 & 0 \\ 0 & 6,-6 \end{vmatrix} = 48,, \quad -48$$

$H_f(0,1) < 0$ so f(0, 1) is a saddle point
$H_f(0,-1) > 0$ and $f_{xx}(0,-1) = -4 < 0$ so f(0, -1) is a local maximum
$H_f(1,1) > 0$ and $f_{xx}(1,1) = 8 > 0$ so f(1, 1) is a local minimum
$H_f(1,-1) < 0$ so f(1, -1) is a saddle point
$H_f(-1,1) > 0$ and $f_{xx}(-1,1) = 8 > 0$ so f(-1, 1) is a local minimum
$H_f(-1,-1) < 0$ so f(-1, -1) is a saddle point

$\boxed{\text{Maximum: } f(0,-1) = 2}$

$\boxed{\text{Minima: } f(\pm1,1) = -3}$

$\boxed{\text{Saddle Points: } f(0,1), \quad f(\pm1,-1)}$

17. $f(x, y) = xy - x^2y - xy^2$

Solution

$$\nabla f(x, y) = \langle y - 2xy - y^2,\ x - x^2 - 2xy \rangle$$

Finding critical values,

$$\begin{aligned} 0 &= y - 2xy - y^2 \\ 0 &= y(1 - 2x - y) \\ y &= 0, \quad 0 = 1 - 2x - y \\ y &= 1 - 2x \end{aligned} \qquad \begin{aligned} 0 &= x - x^2 - 2xy \\ 0 &= x - x^2 - 2x(1 - 2x) \\ 0 &= x - x^2 - 2x - 4x^2 \\ 0 &= 3x^2 - x \\ 0 &= x(3x - 1) \\ x &= 0, \quad 0 = 3x - 1 \\ x &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} y &= 1 - 2x = 1 - 2\left(\frac{1}{3}\right) = \frac{1}{3} \qquad \left[\text{when x} = \frac{1}{3}\right] \\ y &= 1 - 2x = 1 - 2\,(0) = 1 \qquad [\text{when x} = 0] \end{aligned}$$

Critical points at (0, 0), ($\frac{1}{3}$, $\frac{1}{3}$), (0, 1)

$$f_{xx} = -2y, \quad f_{yy} = -2x, \quad f_{xy} = 1 - 2x - 2y$$

$$H_f(0, 0) = det \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 \qquad H_f(0, 1) = det \begin{vmatrix} -2 & -1 \\ -1 & 0 \end{vmatrix} = 0 - 1 = -1$$

$$H_f\left(\frac{1}{3}, \frac{1}{3}\right) = det \begin{vmatrix} -\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{vmatrix} = \frac{4}{9} - \frac{1}{9} = \frac{3}{9}$$

$H_f(0, 0)$ and $H_f(0, 1) < 0$ so f(0, 0) and f(0, 1) are a saddle points
$H_f\left(\frac{1}{3}, \frac{1}{3}\right) > 0$ and $f_{xx}\left(\frac{1}{3}, \frac{1}{3}\right) = -\frac{2}{3} < 0$ so f$\left(\frac{1}{3}, \frac{1}{3}\right)$ is a local maximum

$$\boxed{\text{Maximum: } f\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27}}$$

$$\boxed{\text{Saddle Points: } f(0, 0), \quad f(0, 1)}$$

19. $f(x, y) = e^x \cos y$

Solution

$$\nabla f(x, y) = \langle e^x \cos y,\ -e^x \sin y \rangle$$

Finding critical values,

$$\begin{aligned} 0 &= e^x \cos y \\ 0 &= -e^x \sin y \end{aligned}$$

There is no value x that satisfies both $f_x = 0$ and $f_y = 0$,

$$\boxed{\therefore \text{There are no critical points or saddle points}}$$

21. $f(x,y) = y^2 - 2y\cos x, \quad -1 \le x \le 7$

Solution

$$\nabla f(x,y) = \langle 2y\sin x,\ 2y - 2\cos x\rangle$$

Finding critical values,

$$\begin{aligned} 0 &= 2y\sin x \\ y &= 0, \quad x = 0, \pi, 2\pi \end{aligned}$$

$$\begin{aligned} 0 &= 2y - 2\cos x \qquad [\text{y = 0}] \\ 0 &= -2\cos x \\ x &= \frac{\pi}{2}, \quad \frac{3\pi}{2} \end{aligned} \qquad \begin{aligned} 0 &= 2y - 2\cos x \qquad [\text{x = 0}] \\ 0 &= 2y - 2 \\ y &= 1 \end{aligned}$$

$$\begin{aligned} 0 &= 2y - 2\cos x \qquad [\text{x} = \pi] \\ 0 &= 2y + 2 \\ y &= -1 \end{aligned} \qquad \begin{aligned} 0 &= 2y - 2\cos x \qquad [\text{x} = 2\pi] \\ 0 &= 2y - 2 \\ y &= 1 \end{aligned}$$

Critical points at ($\frac{\pi}{2}$, 0), ($\frac{3\pi}{2}$, 0), (0, 1), (π, -1), (2π, 1)

$$f_{xx} = 2y\cos x, \quad f_{yy} = 2, \quad f_{xy} = 2\sin x$$

$$H_f\left(\frac{\pi}{2}, 0\right) = det\begin{vmatrix} 0 & 2 \\ 2 & 2 \end{vmatrix} = 0 - 4 = -4 \qquad H_f\left(\frac{3\pi}{2}, 0\right) = det\begin{vmatrix} 0 & -2 \\ -2 & 2 \end{vmatrix} = 0 - 4 = -4$$

$$H_f(0,1) = det\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 - 0 = 4 \qquad H_f(\pi, -1) = det\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 - 0 = 4$$

$$H_f(2\pi, 1) = det\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 - 0 = 4$$

$H_f(\frac{\pi}{2}, 0)$ and $H_f(\frac{3\pi}{2}, 0) < 0$ so f($\frac{\pi}{2}$, 0) and f($\frac{3\pi}{2}$, 0) are a saddle points
$H_f\,(0,1) > 0$ and $f_{xx}\,(0,1) = 2 > 0$ so f$(0,1)$ is a local minimum
$H_f\,(\pi,-1) > 0$ and $f_{xx}\,(\pi,-1) = 2 > 0$ so f$(\pi,-1)$ is a local minimum
$H_f\,(2\pi,1) > 0$ and $f_{xx}\,(2\pi,1) = 2 > 0$ so f$(2\pi,1)$ is a local minimum

$\boxed{\text{Minima } f(0,1) = f(\pi,-1) = f(2\pi,1) = -1}$

$\boxed{\text{Saddle Points: } f\left(\frac{\pi}{2}, 0\right), \quad f\left(\frac{3\pi}{2}, 0\right)}$

33-39 (odd)

Find the absolute maximum and minimum values of f on the set D.

33. $f(x, y) = x^2 + y^2 - 2x$,

D is the closed triangular region with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$

Solution

Since f is a polynomial, it is continuous on the bounded triangle D

$$\nabla f(x, y) = \langle 2x - 2,\ 2y \rangle$$

$$\begin{aligned} 0 &= 2x - 2 & \qquad 0 &= 2y \\ x &= 1 & \qquad y &= 0 \end{aligned}$$

Critical point at (1, 0)

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 0$$

$$H_f(1, 0) = det \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0, \quad f_{xx}(1, 0) = 2 > 0$$

$\therefore$ f(1, 0) is an absolute minimum

Extreme values of f(x,y) in closed region,

$$\begin{aligned} f(x, 2) &= x^2 + 4 - 2x = x^2 - 2x + 4 & x &= 0 & f(0, 2) &= 4 \\ f(x, -2) &= x^2 + 4 - 2x = x^2 - 2x + 4 & x &= 0 & f(0, -2) &= 4 \\ & & & & f(2, 0) &= 0 \end{aligned}$$

$\boxed{\text{Minimum: } f(1, 0) = -1}$

$\boxed{\text{Maximum: } f(0, \pm 2) = 4}$

35. $f(x,y) = x^2 + y^2 + x^2y + 4$,

$D = \{(x,y) \mid |x| \leq 1,\ |y| \leq 1\}$

Solution

$$\nabla f(x,y) = \left\langle 2x + 2xy,\ 2y + x^2 \right\rangle$$

$$\begin{aligned} 0 &= 2x + 2xy \\ 0 &= 2x(1+y) \\ x &= 0, \quad y = -1 \end{aligned} \qquad \begin{aligned} 0 &= 2y + x^2 \qquad [x = 0] \\ 0 &= 2y \\ y &= 0 \end{aligned}$$

$$\begin{aligned} 0 &= 2y + x^2 \qquad [y = -1] \\ 0 &= -2 + x^2 \\ x &= \sqrt{2} \qquad [x = |\sqrt{2}| > 1 \text{ so it is not in the set } D.] \end{aligned}$$

Critical point at (0, 0) which f(0, 0) = 4

According to extreme value theorem, since we are in a closed, bounded set D, the greatest and lowest values found are the absolute maximum and minimum, respectively. Thus, we do not need to test for maximum or minimum and only compute the value at each point.

Extreme values of f(x,y) in closed region,

$$\begin{aligned} f(x,1) &= x^2 + 1 + x^2 + 4 = 2x^2 + 5 && x \in [-1,1] \\ f(x,-1) &= 5 && x \in [-1,1] \end{aligned}$$

$$\begin{aligned} f(-1,1) &= 7 & f(-1,-1) &= 5 \\ f(1,1) &= 7 & f(1,-1) &= 5 \end{aligned}$$

$\boxed{\text{Minimum: } f(0,0) = 4}$

$\boxed{\text{Maximum: } f(\pm 1,1) = 7}$

37. $f(x, y) = x^2 + 2y^2 - 2x - 4y + 1,$
$D = \{(x, y) \mid 0 \le x \le 2,\ 0 \le y \le 3\}$

Solution

$$\nabla f(x, y) = \langle 2x - 2,\ 4y - 4\rangle$$

$$\begin{aligned} 0 &= 2x - 2 & 0 &= 4y - 4 \\ x &= 1 & y &= 1 \end{aligned}$$

Critical point at (1, 1) which f(1, 1) = -2

Extreme values of f(x, y) in closed region,

$$\begin{aligned} f(x, 0) &= x^2 - 2x + 7 & x &\in [0, 2] \\ f(x, 3) &= x^2 - 2x + 7 & x &\in [0, 2] \end{aligned}$$

$$\begin{aligned} f(0, 0) &= 1 & f(2, 0) &= 1 \\ f(0, 3) &= 7 & f(2, 3) &= 7 \end{aligned}$$

$\boxed{\text{Minimum: } f(1, 1) = -2}$
$\boxed{\text{Maximum: } f(0, 3) = f(2, 3) = 7}$

39. $f(x, y) = 2x^3 + y^4,$
$D = \{(x, y) \mid x^2 + y^2 \le 1\}$

Solution

$$\nabla f(x, y) = \left\langle 6x^2,\ 4y^3\right\rangle$$

No real critical points

Extreme values of f(x, y) in closed region, (we check vertices in this case)

$$\begin{aligned} f(1, 0) &= 2 & f(0, -1) &= 1 \\ f(-1, 0) &= -2 & f(0, 1) &= 1 \end{aligned}$$

$\boxed{\text{Minimum: } f(-1, 0) = -2}$
$\boxed{\text{Maximum: } f(1, 0) = 2}$

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Find the shortest distance from the point $(2, 0, -3)$ to the plane $x + y + z = 1$.

Solution

$$x + y + z = 1 \quad \Rightarrow \quad z = 1 - x - y$$

Let $d = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$

$$\begin{aligned} d &= \sqrt{(x-2)^2 + y^2 + (z+3)^2} \\ d &= \sqrt{(x-2)^2 + y^2 + [(1-x-y)+3]^2} \\ d &= \sqrt{(x-2)^2 + y^2 + (4-x-y)^2} \\ d^2 &= (x-2)^2 + y^2 + (4-x-y)^2 \end{aligned}$$

$$\begin{aligned} d_x &= 2(x-2) - 2(4-x-y) = 2x - 4 - 8 + 2x + 2y = \underline{4x + 2y - 12} \\ d_y &= 2y - 2(4-x-y) = 2y - 8 + 2x + 2y = \underline{2x + 4y - 8} \end{aligned}$$

$$\nabla d(x, y) = \langle 4x + 2y - 12,\ 2x + 4y - 8 \rangle = 0$$

$$\begin{aligned} 0 &= 4x + 2y - 12 \\ 12 &= 4x + 2y \\ 12 &= 4x + 2y \end{aligned} \qquad\qquad \begin{aligned} 0 &= 2x + 4y - 8 \\ 8 &= 2x + 4y \\ 8 &= 2x + 4y \end{aligned}$$

$$\begin{aligned} 12 &= 4x + 2y \quad \Rightarrow \\ 8 &= 2x + 4y \quad \Rightarrow \end{aligned} \qquad\qquad \begin{aligned} -24 &= -8x - 4y \\ \underline{8} &\underline{= 2x + 4y} \\ -16 &= -6x \\ x &= \frac{8}{3} \end{aligned}$$

$$\begin{aligned} 8 &= 2\left(\frac{8}{3}\right) + 4y \\ 4y &= 8 - \frac{16}{3} = \frac{24 - 16}{3} \\ 4y &= \frac{8}{3} \\ y &= \frac{8}{12} = \frac{2}{3} \end{aligned}$$

Critical point at $(\frac{8}{3}, \frac{2}{3})$

$$d_{xx} = 4, \quad d_{yy} = 4, \quad d_{xy} = 2$$

$$H_f(\frac{8}{3}, \frac{2}{3}) = det\begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} = 16 - 4 = 12 > 0, \quad f_{xx} = 4 > 0$$

$$\therefore \ \mathrm{f}\left(\frac{8}{3}, \frac{2}{3}\right) \text{ is an absolute minimum since this was our only critical value}$$

At this point, the distance is

$$f\left(\frac{8}{3},\frac{2}{3}\right) = \sqrt{\left(\frac{8}{3}-2\right)^2 + \left(\frac{2}{3}\right)^2 + \left(4-\frac{8}{3}-\frac{2}{3}\right)^2}$$
$$f\left(\frac{8}{3},\frac{2}{3}\right) \approx 1.15\ units \quad \text{or} \quad \frac{2}{\sqrt{3}}\ units$$

$\boxed{\therefore \text{ The shortest distance would be 1.15 units}}$

45

Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

Solution

Let $d = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$

$$d = \sqrt{(x-4)^2 + (y-2)^2 + z^2}$$
$$d = \sqrt{(x-4)^2 + (y-2)^2 + x^2 + y^2} \qquad [z^2 = x^2 + y^2]$$
$$d^2 = (x-4)^2 + (y-2)^2 + x^2 + y^2$$

$$d_x = 2(x-4) + 2x = 2x - 8 + 2x = \underline{4x-8}$$
$$d_y = 2(y-2) + 2y = 2y - 4 + 2y = \underline{4y-4}$$

$$\nabla f(x,y) = \langle 4x-8,\ 4y-4\rangle = 0$$

$$0 = 4x - 8 \qquad\qquad 0 = 4y - 4$$
$$x = 2 \qquad\qquad y = 1$$

Critical point at (2, 1)

$$f_{xx} = 4, \quad f_{yy} = 4, \quad f_{xy} = 0$$

$$H_f(2,1) = det\begin{vmatrix} 4 & 0 \\ 0 & 4 \end{vmatrix} = 16 - 0 = 16 > 0, \quad f_{xx} = 4 > 0$$

$\therefore$ f(2, 1) is an absolute minimum

At this point, the distance is

$$d(2,1) = \sqrt{(2-4)^2 + (1-2)^2 + 2^2 + 1^2}$$
$$d(2,1) = \sqrt{4+1+4+1} = \sqrt{10}$$

Now we find the values of z to complete the ordered pairs on the cone that are the closest to the given point

$$\sqrt{10} = \sqrt{(2-4)^2 + (1-2)^2 + z^2} = \sqrt{4+1+z^2}$$
$$10 = 5 + z^2$$
$$z = \pm\sqrt{5}$$

$\boxed{\therefore \text{ The points on the cone closest to the given point are } (2,1,\sqrt{5}) \text{ and } (2,1,-\sqrt{5})}$

47

Find three positive numbers whose sum is 100 and whose product is a maximum.

Solution

Let $f(x, y, z) = xyz, \quad g(x, y, z) = x + y + z - 100 \qquad g(x, y, z)$ *from our constraint,* $x + y + z = 100$

We need to introduce our constraint into our function.

$$x + y + z = 100 \quad \Rightarrow \quad z = 100 - x - y$$
$$f(x, y, z) = xyz = xy(100 - x - y)$$
$$f(x, y, z) = 100xy - x^2y - xy^2$$

$$\nabla f(x, y) = \left\langle 100y - 2xy - y^2,\ 100x - x^2 - 2xy \right\rangle = 0$$

$$\begin{aligned} 0 &= 100y - 2xy - y^2 \\ 0 &= y(100 - 2x - y) \\ y &= 0, \quad 100 = 2x + y \end{aligned} \qquad\qquad \begin{aligned} 0 &= 100x - x^2 - 2xy \\ 0 &= x(100 - x - 2y) \\ x &= 0, \quad 100 = x + 2y \end{aligned}$$

System of two equations,

$$\begin{aligned} 2x + y &= 100 \\ x + 2y &= 100 \end{aligned} \qquad\qquad \begin{aligned} 2x + y &= 100 \\ -2x - 4y &= -200 \\ -3y &= -100 \\ y &= \frac{100}{3} \end{aligned}$$

$$\begin{aligned} \text{so, } x &= 100 - 2\left(\frac{100}{3}\right) \\ x &= 100 - \left(\frac{200}{3}\right) \\ x &= \frac{100}{3} \end{aligned}$$

Critical point at $(\frac{100}{3}, \frac{100}{3})$

$$f_{xx} = -2y, \quad f_{yy} = -2x, \quad f_{xy} = 100 - 2x - 2y$$

$$H_f(\frac{100}{3}, \frac{100}{3}) = det \begin{vmatrix} -\frac{200}{3} & -\frac{100}{3} \\ -\frac{100}{3} & -\frac{200}{3} \end{vmatrix} = \frac{40000}{9} - \frac{10000}{9} = \frac{30000}{9} > 0, \quad f_{xx} = -\frac{200}{3} < 0$$

$$\therefore\ \text{f}\left(\frac{100}{3}, \frac{100}{3}\right) \text{ is indeed an absolute maximum}$$

Lastly, we plug values back into our constraint to find the third value

$$\begin{aligned} 100 &= \frac{100}{3} + \frac{100}{3} + z \\ z &= 100 - \frac{100}{3} - \frac{100}{3} \\ z &= 100 - \frac{200}{3} \\ z &= \frac{100}{3} \end{aligned}$$

$$\boxed{\therefore \text{ The three numbers that meet the conditions are } \frac{100}{3}, \frac{100}{3}, \text{ and } \frac{100}{3}}$$

55

A cardboard box without a lid is to have a volume of 32,000 cm^3. Find the dimensions that minimize the amount of cardboard used.

Solution

Let the width be **x**, the length be **y**, and the height be **z**.

$$\text{Front} + \text{Back} = 2yz \qquad \text{Constraint: } V = xyz = 32000\ cm^3$$

$$\text{Sides} = 2xz \qquad xyz = 32000$$

$$\text{Bottom} = xy \qquad z = \frac{32000}{xy}$$

$$f(x,y,z) = xy + 2xz + 2yz$$

We first introduce the constraint into our function.

$$f(x,y) = xy + 2x\left(\frac{32000}{xy}\right) + 2y\left(\frac{32000}{xy}\right)$$
$$f(x,y) = xy + \left(\frac{64000}{y}\right) + \left(\frac{64000}{x}\right)$$

$$\nabla f(x,y) = \left\langle y - \frac{64000}{x^2},\ x - \frac{64000}{y^2} \right\rangle = 0$$

$$0 = y - \frac{64000}{x^2} \qquad 0 = x - \frac{64000}{y^2}$$
$$y = \frac{64000}{x^2} \qquad x = \frac{64000}{y^2}$$

We see that $x = y$, so

$$x = \frac{64000}{x^2}$$
$$x^3 = 64000$$
$$x = 40 \quad \Rightarrow \quad y = 40$$

We go back to our constraint to find the third value

$$z = \frac{32000}{(40)(40)}$$
$$z = \frac{32000}{(1600)} = 20$$

$\boxed{\therefore \text{ The dimensions to minimize the amount of cardboard used are 40 cm x 40 cm x 20 cm}}$

Section 8: Lagrange Multipliers

3-13 (odd)

Each of these extreme value problems has a solution with both a maximum value and minimum value. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint.

3. $f(x, y) = x^2 - y^2, \quad x^2 + y^2 = 1$

Solution

$$\nabla f(x, y) = \langle 2x,\ -2y \rangle, \quad \nabla g(x, y) = \langle 2x,\ 2y \rangle$$

(1) $2x = \lambda 2x$

$$\lambda 2x - 2x \quad \Rightarrow \quad 2x(\lambda - 1) = 0$$

(2) $-2y = \lambda 2y$

$$\lambda 2y + 2y \quad \Rightarrow \quad 2y(\lambda + 1) = 0$$

$x = 0, \quad \lambda = 1 \quad \Rightarrow \quad y = 0, \quad \lambda = -1$

(3) $x^2 + y^2 = 1$

$$\begin{aligned} 0 + y^2 = 1 \quad &\Rightarrow \quad y = \pm 1 \qquad [x = 0] \\ x^2 + 0 = 1 \quad &\Rightarrow \quad x = \pm 1 \qquad [y = 0] \end{aligned}$$

Extreme values at (0, ±1), (±1, 0)

$$f(0, \pm 1) = -1 \qquad\qquad f(\pm 1, 0) = 1$$

$\boxed{\text{Minimum: } f(0, \pm 1) = -1}$

$\boxed{\text{Maximum: } f(\pm 1, 0) = 1}$

5. $f(x,y) = xy, \quad 4x^2 + y^2 = 8$

Solution

$$\nabla f(x,y) = \langle y,\ x\rangle\,, \quad \nabla g(x,y) = \langle 8x,\ 2y\rangle$$

(1) $y = \lambda 8x$

(2) $x = \lambda 2y$

$$x = \lambda 2(\lambda 8x) \quad \Rightarrow \quad x = \lambda^2 16x \quad \Rightarrow \quad x = 0, \quad \lambda = \pm\frac{1}{4}$$

(3) $4x^2 + y^2 = 8$

$$y^2 = 8 \qquad [x = 0]$$

$$y = \pm 2\sqrt{2}$$

$$y = \pm\frac{1}{4}(8x) = 2x, -2x \qquad \left[\lambda = \pm\frac{1}{4}\right]$$

$$4x^2 + (\pm 2x)^2 = 8$$

$$4x^2 + 4x^2 = 8$$

$$x = \pm 1 \quad \Rightarrow \quad y = \pm 2$$

Extreme values at $(0,\ \pm 2\sqrt{2})$, $(\pm 1, \pm 2)$, $(\pm 1, \mp 2)$

$$f(0, \pm 2\sqrt{2}) = 0 \qquad f(\pm 1, \pm 2) = 2$$

$$f(\pm 1, \mp 2) = -2$$

$\boxed{\text{Minimum: } f(\pm 1, \mp 2) = -2}$

$\boxed{\text{Maximum: } f(\pm 1, \pm 2) = 2}$

7. $f(x,y) = 2x^2 + 6y^2, \quad x^4 + 3y^4 = 1$

Solution

$$\nabla f(x,y) = \langle 4x,\ 12y \rangle, \quad \nabla g(x,y) = \left\langle 4x^3,\ 12y^3 \right\rangle$$

(1) $4x = \lambda 4x^3$

$$4\lambda x^3 - 4x = 4x(\lambda x^2 - 1) = 0 \quad \Rightarrow \quad x = 0, \quad \lambda x^2 - 1 = 0 \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{\lambda}}$$

(2) $12y = \lambda 12y^3$

$$12\lambda y^3 - 12y = 12y(\lambda y^2 - 1) = 0 \quad \Rightarrow \quad y = 0, \quad \lambda y^2 - 1 = 0 \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{\lambda}}$$

(3) $x^4 + 3y^4 = 1$

$$\begin{aligned} 0^4 + 3y^4 &= 1 \qquad [x=0] \\ y &= \pm\frac{1}{3^{1/4}} \end{aligned}$$

$$\begin{aligned} x^4 + 3(0)^4 &= 1 \qquad [y=0] \\ x &= \pm 1 \end{aligned}$$

$$\begin{aligned} \left(\frac{1}{\lambda^{1/2}}\right)^4 + 3\left(\frac{1}{\lambda^{1/2}}\right)^4 &= 1 \\ \frac{1}{\lambda^2} + \frac{3}{\lambda^2} &= 1 \\ \frac{1}{\lambda^2}(1+3) &= 1 \\ \lambda^2 = 4 \quad &\Rightarrow \quad \lambda = 2 \\ x &= \pm\frac{1}{\sqrt{2}} \\ y &= \pm\frac{1}{\sqrt{2}} \end{aligned}$$

Extreme values at $\left(0, \pm\frac{1}{3^{1/4}}\right), (\pm 1, 0), \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

$$f\left(0, \pm\frac{1}{3^{1/4}}\right) = \frac{6}{\sqrt{3}} \qquad\qquad f(\pm 1, 0) = 2$$

$$f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = 4$$

$$\boxed{\text{Minimum: } f(\pm 1, 0) = 2}$$

$$\boxed{\text{Maximum: } f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = 4}$$

9. $f(x,y,z) = 2x + 2y + z, \quad x^2 + y^2 + z^2 = 9$

Solution

$$\nabla f(x,y,z) = \langle 2,\ 2,\ 1 \rangle, \quad \nabla g(x,y,z) = \langle 2x,\ 2y,\ 2z \rangle$$

(1) $2 = \lambda 2x$

(2) $2 = \lambda 2y$

(1) and (2) tells us that $x = y$,

$$2 = \lambda 2x \quad \Rightarrow \quad x = y = \frac{1}{\lambda}$$

(3) $1 = \lambda 2z$

$$\frac{1}{\lambda} = 2z \quad \Rightarrow \quad x \textit{ or } y = 2z$$

(4) $x^2 + y^2 + z^2 = 9$

$$\begin{aligned}
&(2z)^2 + (2z)^2 + z^2 = 9 \\
&4x^2 + 4x^2 + z^2 = 9 \\
&9z^2 = 9 \\
&z = \pm 1 \quad \Rightarrow \quad x = y = \pm 2
\end{aligned}$$

Extreme values at $(\pm 2, \pm 2, \pm 1), (\mp 2, \mp 2, \pm 1)$

$$f(2,2,1) = 9 \qquad f(-2,-2,-1) = -9$$
$$f(-2,-2,1) = 7 \qquad f(2,2,-1) = -7$$

$\boxed{\text{Minimum: } f(-2,-2,-1) = -9}$

$\boxed{\text{Maximum: } f(2,2,1) = 9}$

11. $f(x,y,z) = xy^2z, \quad x^2+y^2+z^2=4$

Solution

$$\nabla f(x,y,z) = \left\langle y^2z,\ 2xyz,\ xy^2 \right\rangle, \quad \nabla g(x,y,z) = \langle 2x,\ 2y,\ 2z\rangle$$

(1) $y^2z = \lambda 2x$

We relate (1) and (3),

$$\begin{aligned} *x \to (1) : xy^2z &= \lambda 2x^2 \\ *z \to (3) : xy^2z &= \lambda 2z^2 \end{aligned} \qquad\qquad \begin{aligned} \lambda 2x^2 &= \lambda 2z^2 \\ x &= z \end{aligned}$$

(2) $2xyz = \lambda 2y$

Manipulate (2) to enable substitution,

$$\begin{aligned} 2xyz &= \lambda 2y \\ xyz &= \lambda y \end{aligned} \qquad\qquad \begin{aligned} *y \to (2) : xy^2z &= \lambda y^2 \\ \lambda 2x^2 &= \lambda y^2 \\ x^2 &= \frac{y^2}{2} \\ x &= \pm\frac{y}{\sqrt{2}} \quad \Rightarrow \quad z = \pm\frac{y}{\sqrt{2}} \end{aligned}$$

(3) $xy^2 = \lambda 2z$

(4) $x^2+y^2+z^2=4$

$$\begin{aligned} \left(\frac{y}{\sqrt{2}}\right)^2 + y^2 + \left(\frac{y}{\sqrt{2}}\right)^2 &= 4 \\ \frac{y^2}{2} + y^2 + \frac{y^2}{2} &= 4 \\ 2y^2 &= 4 \\ y &= \pm\sqrt{2} \\ so,\ x = z &= \pm 1 \end{aligned}$$

Extreme values at $(\pm1, \pm\sqrt{2}, \pm1), (\pm1, \mp\sqrt{2}, \pm1)$

$$\begin{aligned} f(1,\sqrt{2},1) &= 2 \\ f(1,-\sqrt{2},1) &= -2 \end{aligned} \qquad\qquad \begin{aligned} f(-1,-\sqrt{2},-1) &= -2 \\ f(-1,\sqrt{2},-1) &= 2 \end{aligned}$$

$\boxed{\text{Minimum: } f(\pm1, -\sqrt{2}, \pm1) = -2}$

$\boxed{\text{Maximum: } f(\pm1, \sqrt{2}, \pm1) = 2}$

13. $f(x, y, z) = x^2 + y^2 + z^2, \quad x^4 + y^4 + z^4 = 1$

Solution

$$\nabla f(x,y,z) = \langle 2x,\ 2y,\ 2z \rangle, \quad \nabla g(x,y,z) = \left\langle 4x^3,\ 4y^3,\ 4z^3 \right\rangle$$

(1) $2x = \lambda 4x^3$

(2) $2y = \lambda 4y^3$

(3) $2z = \lambda 4z^3$

We see that $x = y = z$,

(4) $x^2 + y^2 + z^2 = 1$

$$\begin{aligned} x^4 + x^4 + x^4 &= 1 \\ 3x^4 &= 1 \\ x^4 &= \frac{1}{3} \\ x &= \pm\frac{1}{3^{1/4}} \quad \Rightarrow \quad y = z = \pm\frac{1}{3^{1/4}} \end{aligned}$$

There are also additional values that satisfy the constraint and is in the domain of $f(x, y, z)$,

$$x^4 = 1 \qquad [y = z = 0] \qquad\qquad y^4 = 1 \qquad [x = z = 0]$$
$$z^4 = 1 \qquad [x = y = 0]$$

Extreme values at $(\pm1, 0, 0), (0, \pm1, 0), (0, 0, \pm1), \left(\pm\frac{1}{3^{1/4}}, \pm\frac{1}{3^{1/4}}, \pm\frac{1}{3^{1/4}}\right)$

$$f(\pm1, 0, 0) = 1 \qquad\qquad f(0, \pm1, 0) = 1$$
$$f(0, 0, \pm1) = 1 \qquad\qquad f\left(\pm\frac{1}{3^{1/4}}, \pm\frac{1}{3^{1/4}}, \pm\frac{1}{3^{1/4}}\right) = \sqrt{3}$$

$\boxed{\text{Minimum: } f(\pm1, 0, 0) = f(0, \pm1, 0) = f(0, 0, \pm1) = 1}$

$\boxed{\text{Maximum: } f\left(\pm\frac{1}{3^{1/4}}, \pm\frac{1}{3^{1/4}}, \pm\frac{1}{3^{1/4}}\right) = \sqrt{3}}$

23

The method of Lagrange multipliers assumes that the extreme values exist, but that is not always the case. Show that the problem of finding the minimum value of f subject to the given constraint can be solved using Lagrange multipliers, but f does not have a maximum value with that constraint.

$$f(x,y) = x^2 + y^2, \quad xy = 1$$

Solution

$$\nabla f(x,y) = \langle 2x,\ 2y \rangle, \quad \nabla g(x,y) = \langle y,\ x \rangle$$

(1) $2x = \lambda y$

$$2 = \frac{\lambda y}{x}$$

(2) $2y = \lambda x$

$$2 = \frac{\lambda x}{y}$$

$$x = \frac{1}{y} \ or \ y = \frac{1}{x}$$

(3) $xy = 1$

With this constraint, as x and y approach infinity, $f(x,y)$ goes to infinity as well, so there is no maximum value.

However, the smallest value x and y can take while meeting the restraint is

$$x = \pm 1,\ y = \frac{1}{\pm 1} \quad \Rightarrow \quad f(\pm 1, \pm 1) = 2$$

$\boxed{\therefore \text{ The minimum value would be } f(\pm 1, \pm 1) = 2}$

25

Use Lagrange multipliers to find the maximum value of f subject to the given constraint. Then show that f has no minimum value with that constraint.

$$f(x, y) = e^{xy}, \quad x^3 + y^3 = 16$$

Solution

$$\nabla f(x, y) = \langle ye^{xy},\ xe^{xy} \rangle, \quad \nabla g(x, y) = \left\langle 3x^2,\ 3y^2 \right\rangle$$

(1) $ye^{xy} = \lambda 3x^2$

$$e^{xy} = \frac{\lambda 3x^2}{y}$$

(2) $xe^{xy} = \lambda 3y^2$

$$\begin{aligned} e^{xy} &= \frac{\lambda 3y^2}{x} \\ \frac{x^2}{y} &= \frac{y^2}{x} \\ x^3 &= y^3 \quad \Rightarrow \quad x = y \end{aligned}$$

(3) $x^3 + y^3 = 16$

$$\begin{aligned} x^3 + x^3 &= 16 \\ 2x^3 &= 16 \\ x^3 &= 8 \\ x = y &= 2 \end{aligned}$$

$x = y = 2$ would be the only value that would satisfy the constraint.

$\boxed{\therefore \text{ The maximum value would be } f(2, 2) = e^4}$

27, 29

Find the extreme values of f on the region described by the inequality.

27. $f(x,y) = x^2 + y^2 + 4x - 4y, \quad x^2 + y^2 \leq 9$

Solution

Let $D = \{(x,y) \mid x^2 + y^2 \leq 9\}$

$$\nabla f(x,y) = \langle 2x+4,\ 2y-4 \rangle, \nabla g(x,y) = \langle 2x,\ 2y \rangle$$

$$\begin{aligned} 2x + 4 &= 0 \\ x &= -2 \end{aligned} \qquad\qquad \begin{aligned} 2y - 4 &= 0 \\ y &= 2 \end{aligned}$$

Critical point at (-2, 2)

(1) $2x + 4 = \lambda 2x$

$$\lambda = \frac{2x+4}{2x}$$

(2) $2y - 4 = \lambda 2y$

$$\begin{aligned} \lambda &= \frac{2y-4}{2y} \\ \frac{2x+4}{2x} &= \frac{2y-4}{2y} \\ 2y(2x+4) &= 2x(2y-4) \\ 4xy + 8y &= 4xy - 8x \\ 8y &= -8x \\ y &= -x \end{aligned}$$

(3) $x^2 + y^2 \leq 9$

$$\begin{aligned} x^2 + (-x)^2 &\leq 9 \\ x^2 + x^2 &\leq 9 \\ 2x^2 &\leq 9 \\ x &\leq \pm\frac{3}{\sqrt{2}} \quad \Rightarrow \quad y \leq \mp\frac{3}{\sqrt{2}} \end{aligned}$$

We compare the critical points in D with the extreme values of f on the boundary of D (*just evaluate at the critical points, extreme values and pick highest/lowest*).

$$f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = 12\sqrt{2} + 9 \qquad f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = -12\sqrt{2} + 9 \approx -7.97$$

$$f(-2,2) = -8$$

$$\boxed{\text{Minimum: } f(-2,-2) = -8}$$

$$\boxed{\text{Maximum: } f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = 12\sqrt{2} + 9}$$

29. $f(x,y) = e^{-xy}, \quad x^2 + 4y^2 \le 1$

Solution

Let $D = \{(x,y) \mid x^2 + 4y^2 \le 1\}$

$$\nabla f(x,y) = \left\langle -ye^{-xy},\ -xe^{-xy} \right\rangle, \nabla g(x,y) = \langle 2x,\ 8y \rangle$$

Critical point at (0, 0)

(1) $-ye^{-xy} = \lambda 2x$

$$\lambda = -\frac{y}{2xe^{xy}}$$

(2) $-xe^{-xy} = \lambda 8y$

$$\begin{aligned}
\lambda &= -\frac{x}{8ye^{xy}} \\
-\frac{y}{2xe^{xy}} &= -\frac{x}{8ye^{xy}} \\
\frac{y}{2x} &= \frac{x}{8y} \\
8y^2 &= 2x^2 \\
y^2 &= \frac{x^2}{4} \\
y &= \pm\frac{x}{2} \quad \Rightarrow \quad x \pm 2y
\end{aligned}$$

(3) $x^2 + 4y^2 \le 1$

$$\begin{aligned}
(2y)^2 + 4y^2 &\le 1 \\
4y^2 + 4y^2 &\le 1 \\
8y^2 &\le 1 \\
y &\le \frac{1}{\sqrt{8}} \\
y &\le \frac{1}{2\sqrt{2}} \quad \Rightarrow \quad x = 2\left(\frac{1}{2\sqrt{2}}\right) = \pm\frac{1}{\sqrt{2}}
\end{aligned}$$

Extreme values at $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}), (\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}})$

$$f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right) = e^{-\frac{1}{4}} \qquad\qquad f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right) = e^{\frac{1}{4}}$$

$$f(0,0) = 1$$

$$\boxed{\text{Minimum: } f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right) = e^{-1/4}}$$

$$\boxed{\text{Maximum: } f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right) = e^{1/4}}$$

31, 33

Find the extreme values of f subject to both constraints

31. $f(x, y, z) = x + y + z; \quad x^2 + z^2 = 2, \quad x + y = 1$

Solution

$$\nabla f(x, y, z) = \langle 1,\ 1,\ 1 \rangle, \quad \nabla g(x, y, z) = \langle 2x,\ 0,\ 2z \rangle, \quad \nabla h(x, y, z) = \langle 1,\ 1,\ 0 \rangle$$

(1) $1 = \lambda 2x + \mu$

$$1 = \lambda 2x + 1 \quad \Rightarrow \quad x = 0$$

(2) $1 = \mu$

(3) $1 = \lambda 2z$

(4) $x^2 + z^2 = 2$

(5) $x + y = 1$

$$\begin{aligned} 0^2 + z^2 = 2 \quad &\Rightarrow \quad z = \pm\sqrt{2} \quad [x = 0] \\ 0 + y = 1 \quad &\Rightarrow \quad y = 1 \quad [x = 0] \end{aligned}$$

Extreme values at $(0,\ 1,\ \pm\sqrt{2})$

$$f\left(0, 1\sqrt{2}\right) = 1 + \sqrt{2} \qquad\qquad f\left(0, 1 - \sqrt{2}\right) = 1 - \sqrt{2}$$

$\boxed{\text{Minimum: } f\left(0, 1, -\sqrt{2}\right) = 1 - \sqrt{2}}$

$\boxed{\text{Maximum: } f\left(0, 1, \sqrt{2}\right) = 1 + \sqrt{2}}$

33. $f(x,y,z) = yz + xy; \quad xy = 1, \quad y^2 + z^2 = 1$

Solution

$$\nabla f(x,y,z) = \langle y,\ z+x,\ y\rangle, \quad \nabla g(x,y,z) = \langle y,\ x,\ 0\rangle, \quad \nabla h(x,y,z) = \langle 0,\ 2y,\ 2z\rangle$$

(1) $y = \lambda y$

$$\lambda = 1$$

(2) $z + x = \lambda x + \mu 2y$

$$z + x = x + \mu 2y \quad \Rightarrow \quad \mu = \frac{z}{2y}$$

(3) $y = \mu 2z$

$$\begin{aligned} y &= \mu 2z \\ \mu &= \frac{y}{2z} \\ \frac{z}{2y} &= \frac{y}{2z} \quad \Rightarrow \quad 2z^2 = 2y^2 \quad \Rightarrow \quad z = y \end{aligned}$$

(4) $xy = 1$

(5) $y^2 + z^2 = 1$

$$\begin{aligned} y^2 + y^2 = 1 \quad &\Rightarrow \quad 2y^2 = 1 \\ y = \pm\frac{1}{\sqrt{2}} \quad &\Rightarrow \quad z = \pm\frac{1}{\sqrt{2}} \\ x\left(\pm\frac{1}{\sqrt{2}}\right) = 1 \quad &\Rightarrow \quad x = \pm\sqrt{2} \end{aligned}$$

Extreme values at $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right), \left(\mp\sqrt{2}, \mp\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

$$f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \frac{3}{2} \qquad\qquad f\left(\mp\sqrt{2}, \mp\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$$

$$\boxed{\text{Minimum: } f\left(\mp\sqrt{2}, \mp\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \frac{1}{2}}$$

$$\boxed{\text{Maximum: } f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \frac{3}{2}}$$