

Chapter 13 Section 3 & 4 Problem Set

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Section 3: Arc Length and Curvature

Problem 1a

Use Equation 2 to compute the length of the given line segment.

$$\vec{r}(t) = \langle 3 - t, 2t, 4t + 1 \rangle \quad 1 \leq t \leq 3$$

Solution

Let the length of the line segment be

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \Rightarrow L = \int_a^b \|\vec{r}'(t)\| dt$$

$$D : \{ t \mid 1 \leq t \leq 3 \}$$

$$\vec{r}'(t) = \langle -1, 2, 4 \rangle \Rightarrow L = \int_1^3 \sqrt{(-1)^2 + (2)^2 + (4)^2} dt = \int_1^3 \sqrt{21} dt = \sqrt{21}t \Big|_1^3 = \sqrt{21}(3) - \sqrt{21}(1) = 2\sqrt{21}$$

Problems 3-7 odd

Find the length of the curve.

3. $\vec{r}(t) = \langle t, 3 \cos t, 3 \sin t \rangle \quad 25 \leq t \leq 5$

Solution

$$\begin{aligned} \vec{r}'(t) = \langle 1, -3 \sin t, 3 \cos t \rangle &\Rightarrow L = \int_{-5}^5 \sqrt{1^2 + (-3 \sin t)^2 + (3 \cos t)^2} dt \\ &= \int_{-5}^5 \sqrt{1 + 9 \sin^2 t + 9 \cos^2 t} dt \\ &= \int_{-5}^5 \sqrt{1 + 9(1)} dt \\ &= \int_{-5}^5 \sqrt{10} dt \\ &= \sqrt{10}t \Big|_{-5}^5 = \sqrt{10}(5) - \sqrt{10}(-5) = 10\sqrt{10} \end{aligned}$$

5. $\vec{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle \quad 0 \leq t \leq 1$

Solution

$$\begin{aligned}\vec{r}'(t) &= \langle \sqrt{2}, e^t, -e^{-t} \rangle \Rightarrow L = \int_0^1 \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2} dt \\ &= \int_0^1 \sqrt{2 + e^{2t} + e^{-2t}} dt \\ &= \int_0^1 \sqrt{(e^t + e^{-t})^2} dt \\ &= \int_0^1 e^t + e^{-t} dt \\ &= e^t - e^{-t} \Big|_0^1 = (e^1 - \frac{1}{e^1}) - (e^0 - \frac{1}{e^0}) = e - \frac{1}{e} - 1 + 1 = e - \frac{1}{e}\end{aligned}$$

7. $\vec{r}(t) = \langle 1, t^2, t^3 \rangle \quad 0 \leq t \leq 1$

Solution

$$\begin{aligned}\vec{r}'(t) &= \langle 0, 2t, 3t^2 \rangle \Rightarrow L = \int_0^1 \sqrt{(0)^2 + (2t)^2 + (3t^2)^2} dt \\ &= \int_0^1 \sqrt{4t^2 + 9t^4} dt \\ &= \int_0^1 \sqrt{t^2(4 + 9t^2)} dt \\ &= \int_0^1 t\sqrt{4 + 9t^2} dt\end{aligned}$$

Using u-substitution,

$$u^2 = 4 + 9t^2$$

$$2u du = 18t dt$$

$$\frac{u du}{9} = t dt$$

$$\int_0^1 t\sqrt{4 + 9t^2} dt = \int_0^1 u \cdot \left(\frac{1}{9}u\right) du = \frac{1}{9} \int_0^1 u^2 du = \frac{1}{9} \left(\frac{1}{3}u^3\right) \Big|_2^{\sqrt{13}} = \frac{1}{27}(u^3) \Big|_2^{\sqrt{13}} = \frac{1}{27}(13^{\frac{3}{2}} - 2^3) = \frac{13\sqrt{13}}{27} - 3$$

Problems 19-23 odd

(a) Find the unit tangent and unit normal vectors $\vec{T}(t)$ and $\vec{N}(t)$.

(b) Use Formula 9 to find the curvature.

19. $\vec{r}(t) = \langle t^2, \sin t - t \cos t, \cos t + t \sin t \rangle, \quad t > 0$

Solution

a.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

$$\begin{aligned}\vec{r}'(t) &= \langle 2t, \cos t - \cos t + t \sin t, -\sin t + \sin t + t \cos t \rangle = \langle 2t, t \sin t, t \cos t \rangle \\ \|\vec{r}'(t)\| &= \sqrt{4t^2 + t^2 \sin^2 t + t^2 \cos^2 t} = \sqrt{4t^2 + t^2} = \sqrt{5t^2} = \sqrt{5}t \quad [\cos^2 t + \sin^2 t = 1] \\ \vec{T}(t) &= \frac{\langle 2t, t \sin t, t \cos t \rangle}{\sqrt{5}t} = \frac{1}{\sqrt{5}} \langle 2, \sin t, \cos t \rangle\end{aligned}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

$$\begin{aligned}\vec{T}'(t) &= \frac{1}{\sqrt{5}} \langle 0, \cos t, -\sin t \rangle \\ \|\vec{T}'(t)\| &= \frac{1}{\sqrt{5}} \sqrt{0^2 + \cos^2 t + \sin^2 t} = \sqrt{1} = \frac{1}{\sqrt{5}} \\ \vec{N}(t) &= \frac{\frac{1}{\sqrt{5}} \langle 0, \cos t, -\sin t \rangle}{\frac{1}{\sqrt{5}}} = \langle 0, \cos t, -\sin t \rangle\end{aligned}$$

b.

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\frac{1}{\sqrt{5}}}{\sqrt{5}t} = \frac{1}{5t}$$

21. $\vec{r}(t) = \langle t, t^2, 4 \rangle$

Solution

a.

$$\begin{aligned}\vec{r}'(t) &= \hat{\mathbf{i}} + 2t \hat{\mathbf{j}} \\ \|\vec{r}'(t)\| &= \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2} \\ \vec{T}(t) &= \frac{\hat{\mathbf{i}} + 2t \hat{\mathbf{j}}}{\sqrt{1 + 4t^2}} = \frac{1}{\sqrt{1 + 4t^2}} (\hat{\mathbf{i}} + 2t \hat{\mathbf{j}}) \\ \frac{d}{dt} [f(t)\vec{u}(t)] &= f'(t)\vec{u}(t) + f(t)\vec{u}'(t) \quad [\text{vector product rule}] \\ \vec{T}'(t) &= -\frac{4t}{(1 + 4t^2)^{\frac{3}{2}}} (\hat{\mathbf{i}} + 2t \hat{\mathbf{j}}) + \frac{1}{(1 + 4t^2)^{\frac{1}{2}}} (2 \hat{\mathbf{j}}) \\ &= \frac{1}{(1 + 4t^2)^{\frac{3}{2}}} (-4t(\hat{\mathbf{i}} + 2t \hat{\mathbf{j}}) + (1 + 4t^2)(2 \hat{\mathbf{j}})) = \frac{1}{(1 + 4t^2)^{\frac{3}{2}}} (-4t \hat{\mathbf{i}} - 8t^2 \hat{\mathbf{j}} + 2 \hat{\mathbf{j}} + 8t^2 \hat{\mathbf{j}}) \\ &= \frac{1}{(1 + 4t^2)^{\frac{3}{2}}} (-4t \hat{\mathbf{i}} + 2 \hat{\mathbf{j}}) \\ \|\vec{T}'(t)\| &= \frac{1}{(1 + 4t^2)^{\frac{3}{2}}} \sqrt{(-4t)^2 + 2^2} = \frac{1}{(1 + 4t^2)^{\frac{3}{2}}} \sqrt{16t^2 + 4} = \frac{1}{(1 + 4t^2)^{\frac{3}{2}}} \sqrt{4(4t^2 + 1)} = \frac{2}{(1 + 4t^2)^{\frac{3}{2}}} \sqrt{1 + 4t^2} \\ &= \frac{2}{1 + 4t^2} \\ \vec{N}(t) &= \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \frac{1}{(1 + 4t^2)^{\frac{3}{2}}} (-4t \hat{\mathbf{i}} + 2 \hat{\mathbf{j}}) \cdot \frac{1 + 4t^2}{2} = \frac{(1 + 4t^2)^1}{2(1 + 4t^2)^{\frac{3}{2}}} (-4t \hat{\mathbf{i}} + 2 \hat{\mathbf{j}}) = \frac{1}{\sqrt{1 + 4t^2}} (-2t \hat{\mathbf{i}} + \hat{\mathbf{j}})\end{aligned}$$

b.

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{2}{1 + 4t^2} \cdot \frac{1}{(1 + 4t^2)^{\frac{1}{2}}} = \frac{2}{(1 + 4t^2)^{\frac{3}{2}}}$$

23. $\vec{r}(t) = \langle t, \frac{1}{2}t^2, t^2 \rangle$

Solution

a.

$$\begin{aligned}
 \vec{r}'(t) &= \langle 1, t, 2t \rangle \\
 \|\vec{r}'(t)\| &= \sqrt{1^2 + t^2 + (2t)^2} = \sqrt{1 + 5t^2} \\
 \vec{T}(t) &= \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\langle 1, t, 2t \rangle}{\sqrt{1 + t^2 + 4t^2}} = \frac{1}{\sqrt{1 + 5t^2}} \langle 1, t, 2t \rangle \\
 \frac{d}{dt} &= [f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t) \quad [\text{vector product rule}] \\
 \vec{T}'(t) &= -\frac{5t}{(1 + 5t^2)^{\frac{3}{2}}} \langle 1, t, 2t \rangle + \frac{1}{(1 + 5t^2)^{\frac{1}{2}}} \langle 0, 1, 2 \rangle \\
 &= \frac{1}{(1 + 5t^2)^{\frac{3}{2}}} (-5t \langle 1, t, 2t \rangle + (1 + 5t^2) \langle 0, 1, 2 \rangle) \\
 &= \frac{1}{(1 + 5t^2)^{\frac{3}{2}}} (\langle -5t, -5t^2, -10t^2 \rangle + \langle 0, 1 + 5t^2, 2 + 10t^2 \rangle) = \frac{1}{(1 + 5t^2)^{\frac{3}{2}}} \langle -5t, 1, 2 \rangle \\
 \|\vec{T}'(t)\| &= \frac{1}{(1 + 5t^2)^{\frac{3}{2}}} \sqrt{(-5t)^2 + 1^2 + 2^2} = \frac{1}{(1 + 5t^2)^{\frac{3}{2}}} \sqrt{25t^2 + 5} = \frac{1}{(1 + 5t^2)^{\frac{3}{2}}} \sqrt{5(5t^2 + 1)} \\
 &= \frac{\sqrt{5}(1 + 5t^2)^{\frac{1}{2}}}{(1 + 5t^2)^{\frac{3}{2}}} = \frac{\sqrt{5}}{1 + 5t^2} \\
 \|\vec{N}(t)\| &= \frac{\|\vec{T}'(t)\|}{\|\vec{T}'(t)\|} = \frac{1}{(1 + 5t^2)^{\frac{3}{2}}} \langle -5t, 1, 2 \rangle \cdot \frac{1 + 5t^2}{\sqrt{5}} = \frac{1}{\sqrt{5}\sqrt{1 + 5t^2}} \langle -5t, 1, 2 \rangle = \frac{1}{\sqrt{5 + 25t^2}} \langle -5t, 1, 2 \rangle
 \end{aligned}$$

b.

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\sqrt{5}}{1 + 5t^2} \cdot \frac{1}{\sqrt{1 + 5t^2}} = \frac{\sqrt{5}}{(1 + 5t^2)^{\frac{3}{2}}}$$

Problem 27

Use Theorem 10 to find the curvature

$$\vec{r}(t) = \sqrt{6}t^2 \hat{i} + 2t \hat{j} + 2t^3 \hat{k}$$

Solution

Theorem 10 states that

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

$$\text{Let } \vec{r}'(t) = 2\sqrt{6}t \hat{i} + 2 + 6t^2 \hat{k}$$

$$\vec{r}''(t) = 2\sqrt{6} \hat{i} + 12t \hat{k}$$

$$\begin{aligned}
 \|\vec{r}'(t) \times \vec{r}''(t)\| &\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\sqrt{6}t & 2 & 6t^2 \\ 2\sqrt{6} & 0 & 12t \end{vmatrix} = \hat{i} \begin{vmatrix} 2t & 6t^2 \\ 0 & 12t \end{vmatrix} - \hat{j} \begin{vmatrix} 2\sqrt{6}t & 6t^2 \\ 2\sqrt{6} & 12t \end{vmatrix} + \hat{k} \begin{vmatrix} 2\sqrt{6}t & 2 \\ 2\sqrt{6} & 0 \end{vmatrix} \\
 &= (24t^2 - 0) \hat{i} - (24t^2\sqrt{6} - 12t^2\sqrt{6}) \hat{j} + (0 - 4\sqrt{6}) \hat{k} = 24t^2 \hat{i} - 12t^2\sqrt{6} \hat{j} - 4\sqrt{6} \hat{k}
 \end{aligned}$$

Problem 28

Find the curvature of $\vec{r}(t) = \langle t^2, \ln t, t \ln t \rangle$ at the point $(1, 0, 0)$.

Solution

Problem 31 & 33

Use Formula 11 to find the curvature.

31. $y = x^4$

Solution

33. $y = xe^x$

Solution

Problem 51

Find the vectors **T**, **N**, and **B** at the given point.

$$\vec{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle, \quad (1, \frac{2}{3}, 1)$$

Solution

Problem 53

Find equations of the normal plane and osculating plane of the curve at the given point.

$$x = \sin 2t, \quad y = -\cos 2t, \quad z = 4t; \quad (0, 1, 2\pi)$$

Solution

Problem 66

Use Formula 14 to find the torsion at the given value of t .

$$\vec{r}(t) = \langle \sin t, 3t, \cos t \rangle, \quad t = \frac{\pi}{2}$$

Solution

Problem 70

Use Theorem 15 to find the torsion of the given curve at a general point and at the point corresponding to $t = 0$

$$\vec{r}(t) = \langle \cos t, \sin t, \sin t \rangle$$

Solution

Section 4: Motion in Space - Velocity and Acceleration

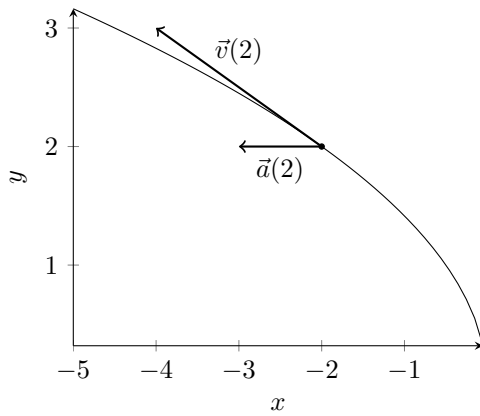
Problem 3-7 odd

Find the velocity, acceleration, and speed of a particle with the given position function. Sketch the path of the particle and draw the velocity and acceleration vectors for the specified value of t .

3. $\vec{r}(t) = \langle -\frac{1}{2}t^2, t \rangle, \quad t = 2$

Solution

$$\begin{aligned} \vec{r}'(t) = \vec{v}(t) &= \langle -t, 1 \rangle & \Rightarrow & \vec{v}(2) = \langle -2, 1 \rangle \\ \vec{r}''(t) = \vec{a}(t) &= \langle -1, 0 \rangle & \Rightarrow & \vec{a}(2) = \langle -1, 0 \rangle \end{aligned}$$



5. $\vec{r}(t) = 3 \cos t \hat{i} + 2 \sin t \hat{j} \quad t = \frac{\pi}{3}$

7. $\vec{r}(t) = t \hat{i} + t^2 \hat{j} + 2 \hat{k} \quad t = 1$

Problems 9-13 odd

Find the velocity, acceleration, and speed of a particle with the given position function.

9. $\vec{r}(t) = \langle t^2 + t, t^2 - t, t^3 \rangle$

11. $\vec{r}(t) = \sqrt{2}t \hat{i} + e^t \hat{j} + e^{-t} \hat{k}$

13. $\vec{r}(t) = e^t(\cos t \hat{i} + \sin t \hat{j} + t \hat{k})$

Problem 15

Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

$$a(t) = 2 \hat{i} + 2t \hat{k}, \quad v(0) = 3 \hat{i} - \hat{j}, \quad r(0) = \hat{j} + \hat{k}$$

Problem 17a

Find the position vector of a particle that has the given acceleration and the specified initial velocity and position.

$$a(t) = 2t \hat{i} + \sin t \hat{j} + \cos 2t \hat{k}, \quad v(0) = \hat{i}, \quad r(0) = \hat{j}$$

Problem 23

A projectile is fired with an initial speed of $200 \frac{m}{s}$ and angle of elevation 60° . Find (a) the range of the projectile, (b) the maximum height reached, and (c) the speed at impact.

Problem 26

A projectile is fired from a tank with initial speed $400 \frac{m}{s}$. Find two angles of elevation that can be used to hit a target $3000m$ away.

Problem 27

A rifle is fired with angle of elevation 36° . What is the initial speed if the maximum height of the bullet is $1600ft$?

Problem 37 & 39

Find the tangential and normal components of the acceleration vector.

37. $\vec{r}(t) = (t^2 + 1) \hat{\mathbf{i}} + t^3 \hat{\mathbf{j}}, \quad t \geq 0$

39. $\vec{r}(t) = \cos t \hat{\mathbf{i}} + \sin t \hat{\mathbf{j}} + t \hat{\mathbf{k}}$

Problem 41

Find the tangential and normal components of the acceleration vector at the given point.

$$\vec{r}(t) = \ln t \hat{\mathbf{i}} + (t^2 + 3t) \hat{\mathbf{j}} + 4\sqrt{t} \hat{\mathbf{k}}, \quad (0, 4, 4)$$