Chapter 14 - Problem Set 2

Calculus 3

# Section 5: The Chain Rule

3-7 (odd)

Use The Chain Rule to find  $\frac{dz}{dt}$  or  $\frac{dw}{dt}$ .

**3.**  $z = xy^3 - x^2y$ ,  $x = t^2 + 1$ ,  $y = t^2 - 1$ 

Solution

$$\frac{dz}{dt} = \frac{dz}{dx} \left(\frac{dx}{dt}\right) + \frac{dz}{dy} \left(\frac{dy}{dt}\right)$$
$$\frac{dz}{dt} = [y^3 - 2xy](2t) + [3xy^2 - x^2](2t)$$
$$\frac{dz}{dt} = 2t[y^3 - 2xy + 3xy^2 - x^2]$$

**5.**  $z = \sin x \cos y$ ,  $x = \sqrt{t}$ , y = 1/t

Solution

$$\frac{dz}{dt} = \frac{dz}{dx} \left(\frac{dx}{dt}\right) + \frac{dz}{dy} \left(\frac{dy}{dt}\right)$$
$$\frac{dz}{dt} = \left[\cos x \cos y\right] \left(\frac{1}{2}t^{-\frac{1}{2}}\right) + \left[-\sin x \sin y\right] \left(-t^{-2}\right)$$
$$\frac{dz}{dt} = \frac{1}{2\sqrt{t}} \cos x \cos y + \frac{1}{t^2} \sin x \sin y$$

7.  $w = xe^{y/z}$ ,  $x = t^2$ , y = 1 - t, z = 1 + 2t

$$\begin{split} \frac{dw}{dt} &= \frac{dw}{dx} \left( \frac{dx}{dt} \right) + \frac{dw}{dy} \left( \frac{dy}{dt} \right) + \frac{dw}{dz} \left( \frac{dz}{dt} \right) \\ \frac{dw}{dt} &= \left[ e^{y/z} \right] (2t) + \left[ \frac{x}{z} e^{y/z} \right] (-1) + \left[ -\frac{xy}{z^2} e^{y/z} \right] (2) \\ \hline \frac{dw}{dt} &= e^{y/z} \left[ 2t - \frac{x}{z} - \frac{2xy}{z^2} \right] \end{split}$$

Use the Chain Rule to find  $\frac{\partial z}{\partial s}$  and  $\frac{\partial z}{\partial t}$ 

**11.** 
$$z = (x - y)^5$$
,  $x = s^2 t$ ,  $y = st^2$ 

Solution

$$\begin{split} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \left( \frac{\partial x}{\partial s} \right) + \frac{\partial z}{\partial y} \left( \frac{\partial y}{\partial s} \right) \\ \frac{\partial z}{\partial s} &= \left[ 5(x - y)^4 \right] (2st) + \left[ -5(x - y)^4 \right] (t^2) \\ \boxed{\frac{\partial z}{\partial s}} &= 5(x - y)^4 \left[ 2st - t^2 \right] \end{split}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \left( \frac{\partial x}{\partial t} \right) + \frac{\partial z}{\partial y} \left( \frac{\partial y}{\partial t} \right)$$
$$\frac{\partial z}{\partial t} = \left[ 5(x - y)^4 \right] (s^2) + \left[ -5(x - y)^4 \right] (2st)$$
$$\frac{\partial z}{\partial t} = 5(x - y)^4 \left[ s^2 - 2st \right]$$

**13.**  $z = \ln(3x + 2y)$ ,  $x = s\sin t$ ,  $y = t\cos s$ 

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \left( \frac{\partial x}{\partial s} \right) + \frac{\partial z}{\partial y} \left( \frac{\partial y}{\partial s} \right)$$
$$\frac{\partial z}{\partial s} = \left[ \frac{3}{3x + 2y} \right] (\sin t) + \left[ \frac{2}{3x + 2y} \right] (-t \sin s)$$
$$\frac{\partial z}{\partial s} = \frac{3 \sin t - 2t \sin s}{3x + 2y}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \left( \frac{\partial x}{\partial t} \right) + \frac{\partial z}{\partial y} \left( \frac{\partial y}{\partial t} \right)$$
$$\frac{\partial z}{\partial t} = \left[ \frac{3}{3x + 2y} \right] (s \cos t) + \left[ \frac{2}{3x + 2y} \right] (\cos s)$$
$$\frac{\partial z}{\partial t} = \frac{3s \cos t + 2 \cos s}{3x + 2y}$$

**15.** 
$$z = (\sin \theta)/r$$
,  $r = st$ ,  $\theta = s^2 + t^2$ 

Solution
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r} \left( \frac{\partial r}{\partial s} \right) + \frac{\partial z}{\partial \theta} \left( \frac{\partial \theta}{\partial s} \right) \\
\frac{\partial z}{\partial s} = \left[ -\frac{\sin \theta}{r^2} \right] (t) + \left[ \frac{\cos \theta}{r} \right] (2s)$$

$$\frac{\partial z}{\partial s} = -\frac{t \sin \theta}{r^2} + \frac{2s \cos \theta}{r}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r} \left( \frac{\partial r}{\partial t} \right) + \frac{\partial z}{\partial \theta} \left( \frac{\partial \theta}{\partial t} \right)$$
$$\frac{\partial z}{\partial t} = \left[ -\frac{\sin \theta}{r^2} \right] (s) + \left[ \frac{\cos \theta}{r} \right] (2t)$$
$$\frac{\partial z}{\partial t} = -\frac{s \sin \theta}{r^2} + \frac{2t \cos \theta}{r}$$

### 25-29 (odd)

Use the Chain Rule to find the indicated partial derivatives.

**25.** 
$$z = x^4 + x^2y$$
,  $x = s + 2t - u$ ,  $y = stu^2$ ;

$$\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}, \frac{\partial z}{\partial u}$$
 when  $s = 4, t = 2, u = 1$ 

When 
$$s = 4$$
,  $t = 2$ , and  $u = 1 \implies x = 7$ ,  $y = 8$ 

$$\frac{\partial z}{\partial s} = \left[4x^3 + 2xy\right](1) + \left[x^2\right](tu^2) = 4x^3 + 2xy + x^2tu^2$$

$$\frac{\partial z}{\partial s} = 4(7)^3 + 2(7)(8) + (7)^2(2)(1)^2$$

$$\boxed{\frac{\partial z}{\partial s} = 1582}$$

$$\frac{\partial z}{\partial t} = \left[4x^3 + 2xy\right](2) + \left[x^2\right](su^2) = 8x^3 + 4xy + x^2su^2$$

$$\frac{\partial z}{\partial t} = 8(7)^3 + 4(7)(8) + (7)^2(4)(1)^2$$

$$\boxed{\frac{\partial z}{\partial t} = 3164}$$

$$\frac{\partial z}{\partial u} = [4x^3 + 2xy] (-1) + [x^2] (2stu) = -4x^3 - 2xy + 2x^2 stu$$

$$\frac{\partial z}{\partial u} = -4(7)^3 - 2(7)(8) + 2(7)^2 (4)(2)(1)$$

$$\frac{\partial z}{\partial u} = -700$$

**27.** 
$$w = xy + yz + zx$$
,  $x = r\cos\theta$ ,  $y = r\sin\theta$ ,  $z = r\theta$ ;

$$\frac{\partial w}{\partial r}, \frac{\partial w}{\partial \theta} \quad \text{when } r=2, \theta=\pi/2$$

When 
$$r=2$$
 and  $\theta=\pi/2$   $\Rightarrow$   $x=0,\,y=2,\,z=\pi$  
$$\frac{\partial w}{\partial r}=[y+z](\cos\theta)+[x+z](\sin\theta)+[x+y](\theta)$$
 
$$\frac{\partial w}{\partial r}=[2+\pi](0)+[0+\pi](1)+[0+2](\pi/2)=0+\pi+\pi$$
 
$$\boxed{\frac{\partial w}{\partial r}=2\pi}$$

$$\frac{\partial w}{\partial \theta} = [y+z](-r\sin\theta) + [x+z](r\cos\theta) + [x+y](r)$$

$$\frac{\partial w}{\partial \theta} = [2+\pi](-2\cdot1) + [0+\pi](0) + [0+2](2) = -4 - 2\pi + 0 + 4$$

$$\frac{\partial w}{\partial \theta} = -2\pi$$

**29.** 
$$N = \frac{p+q}{p+r}$$
,  $p = u + vw$ ,  $q = v + uw$ ,  $r = w + uv$   $\frac{\partial N}{\partial u}$ ,  $\frac{\partial N}{\partial v}$ ,  $\frac{\partial N}{\partial w}$  when  $u = 2$ ,  $v = 3$ ,  $w = 4$ 

When 
$$u = 2$$
,  $v = 3$ , and  $w = 4 \implies p = 14$ ,  $q = 11$ ,  $r = 10$ 

$$\frac{\partial N}{\partial u} = \left[ \frac{(p+r) - (p+q)}{(p+r)^2} \right] (1) + \left[ \frac{1}{p+r} \right] (w) + \left[ -\frac{p+q}{(p+r)^2} \right] (v)$$

$$\frac{\partial N}{\partial u} = \frac{r-q}{(p+r)^2} + \frac{w}{p+r} - \frac{v(p+q)}{(p+r)^2}$$

$$\frac{\partial N}{\partial u} = \frac{10 - 11}{(14 + 10)^2} + \frac{4}{14 + 10} - \frac{3(14 + 11)}{(14 + 10)^2}$$

$$\frac{\partial N}{\partial u} = \frac{-1}{576} + \frac{4}{24} - \frac{75}{576} = \frac{-1 + 96 - 75}{576} = \frac{20}{576}$$

$$\frac{\partial N}{\partial u} = \frac{5}{144}$$

$$\begin{split} \frac{\partial N}{\partial v} &= \left[\frac{r-q}{(p+r)^2}\right](w) + \left[\frac{1}{p+r}\right](1) + \left[-\frac{p+q}{(p+r)^2}\right](u) \\ \frac{\partial N}{\partial v} &= \frac{w(r-q)}{(p+r)^2} + \frac{1}{p+r} - \frac{u(p+q)}{(p+r)^2} \\ \frac{\partial N}{\partial v} &= \frac{4(10-11)}{(14+10)^2} + \frac{1}{14+10} - \frac{2(14+11)}{(14+10)^2} \\ \frac{\partial N}{\partial v} &= \frac{-4}{576} + \frac{1}{24} - \frac{50}{576} = \frac{-4+24-50}{576} = \frac{-30}{576} \\ \frac{\partial N}{\partial v} &= -\frac{5}{96} \end{split}$$

$$\frac{\partial N}{\partial w} = \left[\frac{r-q}{(p+r)^2}\right](v) + \left[\frac{1}{p+r}\right](u) + \left[-\frac{p+q}{(p+r)^2}\right](1)$$

$$\frac{\partial N}{\partial w} = \frac{v(r-q)}{(p+r)^2} + \frac{u}{p+r} - \frac{p+q}{(p+r)^2}$$

$$\frac{\partial N}{\partial w} = \frac{3(10-11)}{(14+10)^2} + \frac{2}{(14+10)} - \frac{14+11}{(14+10)^2}$$

$$\frac{\partial N}{\partial w} = \frac{-3}{576} + \frac{2}{24} - \frac{25}{576} = \frac{-3+48-25}{576} = \frac{20}{576}$$

$$\frac{\partial N}{\partial w} = \frac{5}{144}$$

Use Equation 5 to find  $\frac{dy}{dx}$ 

**31.**  $y \cos x = x^2 + y^2$ 

Solution

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Let  $F = y \cos x - x^2 - y^2$ 

$$F_x = -y\sin x - 2x$$
$$F_y = \cos x - 2y$$

$$F_y = \cos x - 2y$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{(-y\sin x - 2x)}{\cos x - 2y} = \frac{2x + y\sin x}{\cos x - 2y}$$

Use Equations 6 to find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$ 

**35.**  $x^2 + 2y^2 + 3z^2 = 1$ 

Solution

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$
  $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$ 

Let  $F = x^2 + 2y^2 + 3z^2 - 1$ 

$$F_x = 2x \qquad F_y = 4y \qquad F_z = 6z$$

$$F_x = 2x F_y = 4y F_z = 6z$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{6z} = -\frac{x}{3z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{4y}{6z} = -\frac{2y}{3z}$$

## Section 6: Directional Derivatives and the Gradient Vector

#### 5, 7

Find the directional derivative of f at the given point in the direction indicated by the angle  $\theta$ .

**5.** 
$$f(x,y) = y\cos(xy)$$
,  $(0,1)$ ,  $\theta = \pi/4$ 

Solution

$$D_u f(x,y) = f_x(x,y) \cos \theta + f_y(x,y) \sin \theta$$

$$D_u f(x,y) = \left[ -y^2 \sin(xy) \right] \cos \frac{\pi}{4} + \left[ \cos(xy) - xy \sin(xy) \right] \sin \frac{\pi}{4}$$

$$D_u f(x,y) = \frac{\sqrt{2}}{2} \left[ -y^2 \sin(xy) + \cos(xy) - xy \sin(xy) \right]$$

$$D_u f(0,1) = \frac{\sqrt{2}}{2} \left[ 0 + 1 - 0 \right]$$

$$D_u f(0,1) = \frac{\sqrt{2}}{2}$$

7. 
$$f(x,y) = \arctan(xy), \quad (2,-3), \quad \theta = 3\pi/4$$

$$D_u f(x,y) = f_x(x,y) \cos \theta + f_y(x,y) \sin \theta$$

$$D_u f(x,y) = \left[ \frac{y}{1 + (xy)^2} \right] \cos \frac{3\pi}{4} + \left[ \frac{x}{1 + (xy)^2} \right] \sin \frac{3\pi}{4}$$

$$D_u f(x,y) = \frac{\sqrt{2}}{2} \left[ \frac{x}{1 + (xy)^2} - \frac{y}{1 + (xy)^2} \right]$$

$$D_u f(2,-3) = \frac{\sqrt{2}}{2} \left[ \frac{2}{1 + (2(-3))^2} - \frac{-3}{1 + (2(-3))^2} \right]$$

$$D_u f(2,-3) = \frac{\sqrt{2}}{2} \left[ \frac{5}{37} \right]$$

$$D_u f(2,-3) = \frac{5\sqrt{2}}{74}$$

9

$$f(x,y) = x/y, \quad P(2,1), \quad \vec{u} = \frac{3}{5} \hat{\mathbf{i}} + \frac{4}{5} \hat{\mathbf{j}}$$

- (a) Find the gradient of f
- (b) Evaluate the gradient at the point P
- (c) Find the rate of change of f at P in the direction of the vector  $\vec{u}$

Solution

(a)

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$
 or  $\frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}}$ 

$$\nabla f(x,y) = \langle \frac{1}{y}, -\frac{x}{y^2} \rangle$$

(b)

$$\boxed{ \nabla f(2,1) = \langle \frac{1}{1}, -\frac{2}{1^2} \rangle = \langle 1, -2 \rangle \, }$$

(c)

$$D_u f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

$$D_u f(2,1) = \nabla f(2,1) \cdot \vec{u}$$

$$D_u f(2,1) = \langle 1, -2 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle$$

$$D_u f(2,1) = \frac{3}{5} - \frac{8}{5}$$

$$D_u f(2,1) = -1$$

Find the directional derivative of the function at the given point in the direction of the vector  $\vec{v}$ .

**13.** 
$$f(x,y) = e^x \sin y$$
,  $(0,\pi/3)$ ,  $\vec{v} = \langle -6, 8 \rangle$ 

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{-6\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}}}{\sqrt{(6)^2 + (8)^2}}$$

$$\vec{u} = -\frac{3}{5}\,\hat{\mathbf{i}} + \frac{4}{5}\,\hat{\mathbf{j}}$$

$$D_u f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

$$D_u f(x,y) = (e^x \sin y \,\hat{\mathbf{i}} + e^x \cos y \,\hat{\mathbf{j}}) \cdot \left(-\frac{3}{5}\,\hat{\mathbf{i}} + \frac{4}{5}\,\hat{\mathbf{j}}\right)$$

$$D_u f(x,y) = -\frac{3e^x \sin y}{5} + \frac{4e^x \cos y}{5}$$

$$D_u f(0, \frac{\pi}{3}) = -\frac{3(\frac{\sqrt{3}}{2})}{5} + \frac{4(\frac{1}{2})}{5}$$

$$D_u f(0, \frac{\pi}{3}) = \frac{4 - 3\sqrt{3}}{10}$$

**15.** 
$$g(s,t) = s\sqrt{t}$$
,  $(2,4)$ ,  $\vec{v} = 2\hat{i} - \hat{j}$ 

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{2 \hat{\mathbf{i}} - \hat{\mathbf{j}}}{\sqrt{(2)^2 + (-1)^2}}$$

$$\vec{u} = \frac{2}{\sqrt{5}} \hat{\mathbf{i}} - \frac{1}{\sqrt{5}} \hat{\mathbf{j}}$$

$$D_u g(s,t) = \nabla g(s,t) \cdot \vec{u}$$

$$D_u g(s,t) = \left(\sqrt{t} \hat{\mathbf{i}} + \frac{s}{2\sqrt{t}}\right) \cdot \left(\frac{2}{\sqrt{5}} \hat{\mathbf{i}} - \frac{1}{\sqrt{5}} \hat{\mathbf{j}}\right)$$

$$D_u g(s,t) = \frac{2\sqrt{t}}{\sqrt{5}} - \frac{s}{2\sqrt{t}\sqrt{5}}$$

$$D_u g(s,t) = \frac{1}{\sqrt{5}} \left(2\sqrt{t} - \frac{s}{2\sqrt{t}}\right)$$

$$D_u g(2,4) = \frac{1}{\sqrt{5}} \left(2\sqrt{4} - \frac{2}{2\sqrt{4}}\right) = \frac{1}{\sqrt{5}} \left(4 - \frac{1}{2}\right)$$

$$D_u g(2,4) = \frac{1}{\sqrt{5}} \left(\frac{8-1}{2}\right) = \frac{7}{2\sqrt{5}}$$

Find the directional derivative of the function at the point P in the direction of the point Q.

**21.** 
$$f(x,y) = x^2y^2 - y^3$$
,  $P(1,2)$ ,  $Q(-3,5)$ 

Solution

$$\vec{v} = P\vec{Q} = \langle -3 - 1, 5 - 2 \rangle = \langle -4, 3 \rangle$$

$$\vec{u} = \frac{\vec{v}}{\|v\|} = \frac{\langle -4, 3 \rangle}{\sqrt{16 + 9}}$$

$$\vec{u} = \langle -\frac{4}{5}, \frac{3}{5} \rangle$$

$$D_u f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

$$D_u f(x, y) = \langle 2xy^2, 2x^2y - 3y^2 \rangle \cdot \langle -\frac{4}{5}, \frac{3}{5} \rangle$$

$$D_u f(x, y) = -\frac{8xy^2}{5} + \frac{3(2x^2y - 3y^2)}{5} = \frac{1}{5}(-8xy^2 + 6x^2y - 9y^2)$$

$$D_u f(1, 2) = \frac{1}{5}[-8(1)(4) + 6(1)(2) - 9(4)] = \frac{1}{5}[-32 + 12 - 36]$$

$$D_u f(1, 2) = -\frac{56}{5}$$

**23.** 
$$f(x,y) = \sqrt{xy}$$
,  $P(2,8)$ ,  $Q(5,4)$ 

$$\vec{v} = P\vec{Q} = (5-2)\,\hat{\mathbf{i}} + (4-8)\,\hat{\mathbf{j}} = 3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}$$

$$\vec{u} = \frac{\vec{v}}{\|v\|} = \frac{3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}}{\sqrt{9+16}}$$

$$\vec{u} = \frac{3}{5}\,\hat{\mathbf{i}} - \frac{4}{5}\,\hat{\mathbf{j}}$$

$$D_u f(x,y) = \nabla f(x,y) \cdot \vec{u}$$

$$D_u f(x,y) = \left(\frac{y}{2\sqrt{xy}}\,\hat{\mathbf{i}} + \frac{x}{2\sqrt{xy}}\,\hat{\mathbf{j}}\right) \cdot \left(\frac{3}{5}\,\hat{\mathbf{i}} - \frac{4}{5}\,\hat{\mathbf{j}}\right)$$

$$D_u f(x,y) = \frac{3y}{10\sqrt{xy}} - \frac{4x}{10\sqrt{xy}} = \frac{3y - 4x}{10\sqrt{xy}}$$

$$D_u f(2,8) = \frac{3(8) - 4(2)}{10\sqrt{2(8)}} = \frac{24 - 8}{10\sqrt{16}} = \frac{16}{40}$$

$$D_u f(x,y) = \frac{2}{5}$$

Find the maximum rate of change of f at the given point and the direction in which it occurs.

**27.** 
$$f(x,y) = 5xy^2$$
,  $(3,-2)$ 

Solution

The maximum rate of change of f and its direction is given by the gradient vector

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 5y^2, 10xy \rangle$$
$$\nabla f(3,-2) = \langle 5(-2)^2, 10(3)(-2) \rangle$$

Direction of the maximum rate of change,

$$\nabla f(3, -2) = \langle 20, -60 \rangle = 20\langle 1, -3 \rangle$$

Maximum rate of change,

$$\|\nabla f(3,-2)\| = 20\sqrt{1+9} = 20\sqrt{10}$$

**29.** 
$$f(x,y) = \sin(xy)$$
,  $(1,0)$ 

Solution

The maximum rate of change of f and its direction is given by the gradient vector

$$\nabla f(x,y) = \frac{\partial f}{\partial x} \,\hat{\mathbf{i}} + \frac{\partial f}{\partial y} \,\hat{\mathbf{j}} = y \cos(xy) \,\hat{\mathbf{i}} + x \cos(xy) \,\hat{\mathbf{j}}$$
$$\nabla f(1,0) = 0 + \,\hat{\mathbf{j}}$$

Direction of the maximum rate of change,

$$\nabla f(1,0) = \hat{\mathbf{j}}$$

Maximum rate of change,

$$\|\nabla f(1,0)\| = \sqrt{1} = 1$$

37

The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point (1, 2, 2) is  $120^{\circ}$ 

- (a) Find the rate of change of T at (1,2,2) in the direction toward the point (2,1,3).
- (b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points toward the origin

Solution

Since  $T \propto \frac{1}{d}$ ,  $T(x,y,z) = \frac{c}{\sqrt{x^2 + y^2 + z^2}}$ , for some constant c, and Euclidean distance to describe the distance from the origin.

If 
$$120 = T(1,2,2) = \frac{c}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{c}{3}$$
, then  $c = 360$ 

... Our function for temperature is,

$$T(x, y, z) = \frac{360}{\sqrt{x^2 + y^2 + z^2}}$$

(a)

$$\begin{split} \vec{v} &= \langle 2-1, 1-2, 3-2 \rangle = \langle 1, -1, 1 \rangle \\ \vec{u} &= \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1, -1, 1 \rangle}{\sqrt{1+1+1}} \\ \vec{u} &= \langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle \end{split}$$

Getting our gradient vector,

$$\nabla T(x,y,z) = \langle T_x, T_y, T_z \rangle$$

$$\nabla T(x,y,z) = \langle -180 \cdot 2x(x^2 + y^2 + z^2)^{-3/2}, -180 \cdot 2y(x^2 + y^2 + z^2)^{-3/2}, -180 \cdot 2z(x^2 + y^2 + z^2)^{-3/2} \rangle$$

$$\nabla T(x,y,z) = \frac{360}{(x^2 + y^2 + z^2)^{3/2}} \langle -x, -y, -z \rangle$$

so, our rate of change is

$$D_u T(1,2,2) = \nabla T(1,2,2) \cdot \vec{u}$$

$$D_u T(1,2,2) = \frac{360}{(1+4+4)^{3/2}} \langle -1, -2, -2 \rangle \cdot \langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$$

$$D_u T(1,2,2) = \frac{360}{27} \left( -\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}} \right) = \frac{360}{27} \left( -\frac{1}{\sqrt{3}} \right)$$

$$D_u T(1,2,2) = \frac{40}{3} \left( -\frac{1}{\sqrt{3}} \right) = -\frac{40}{3\sqrt{3}}$$

(b) The direction of greatest increase at any point is the direction of  $\nabla T$  at that point. In part (a) we saw that  $\nabla T(x,y,z)$  and  $\langle -x,-y,-z\rangle$  have the same direction and the vector  $\langle -x,-y,-z\rangle$  points towards the origin.

Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

**47.** 
$$2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10$$
,  $(3,3,5)$ 

Solution

(a)

$$F(x, y, z) = 2(x - 2)^{2} + (y - 1)^{2} + (z - 3)^{2} - 10$$

$$F_{x}(3, 3, 5) = 4(x - 2) = 2(2) = 4$$

$$F_{y}(3, 3, 5) = 2(y - 1) = 2(2) = 4$$

$$F_{z}(3, 3, 5) = 2(z - 3) = 2(2) = 4$$

The equation of the tangent plane is given by,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0)$$

$$4(x-3) + 4(y-3) + 4(z-5) = 0$$
$$x-3+y-3+z-5 = 0$$
$$x+y+z = 11$$

(b) The equation of the normal line is given by,

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$
$$\frac{x - 3}{4} = \frac{y - 2}{4} = \frac{z - 3}{4}$$
$$x - 3 = y - 2 = z - 3$$

**49.** 
$$xy^2z^3=8$$
,  $(2,2,1)$ 

Solution

(a)

$$F(x, y, z) = xy^{2}z^{3} - 8$$

$$F_{x}(2, 2, 1) = y^{2}z^{3} = (4)(1) = 4$$

$$F_{y}(2, 2, 1) = 2xyz^{3} = 2(2)(2)(1) = 8$$

$$F_{z}(2, 2, 1) = 3xy^{2}z^{2} = 3(2)(4)(1) = 24$$

The equation of the tangent plane is given by,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0)$$

$$4(x - 2) + 8(y - 2) + 24(z - 1) = 0$$

$$x - 2 + 2(y - 2) + 6(z - 1) = 0$$

$$x - 2 + 2y - 4 + 6z - 6 = 0$$

$$\boxed{x + 2y + 6z = 12}$$

(b) The equation of the normal line is given by,

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$
$$\frac{x - 2}{4} = \frac{y - 2}{8} = \frac{z - 1}{24}$$
$$x - 2 = \frac{y - 2}{2} = \frac{z - 1}{6}$$

**51.** 
$$x + y + z = e^{xyz}$$
,  $(0,0,1)$ 

Solution

(a)

$$F(x, y, z) = x + y + z - e^{xyz}$$

$$F_x(0, 0, 1) = 1 - yze^{xyz} = 1 - 0 = 1$$

$$F_y(0, 0, 1) = 1 - xze^{xyz} = 1 - 0 = 1$$

$$F_z(0, 0, 1) = 1 - xye^{xyz} = 1 - 0 = 1$$

The equation of the tangent plane is given by,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0)$$
$$1(x - 0) + 1(y - 0) + 1(z - 1) = 0$$
$$\boxed{x + y + z = 1}$$

(b) The equation of the normal line is given by,

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$
$$\frac{x - 0}{1} = \frac{y - 0}{1} = \frac{z - 1}{1}$$
$$x = y = z - 1$$

# Section 7: Maximum and Minimum Values

## 5-21 (odd)

Find the local maximum and minimum values and saddle point(s) of the function. You are encouraged to use a calculator or computer to graph the function with a domain and viewpoint that reveals all the important aspects of the function.

5. 
$$f(x,y) = x^2 + xy + y^2 + y$$

7. 
$$f(x,y) = 2x^2 - 8xy + y^4 - 4y^3$$

**9.** 
$$f(x,y) = (x-y)(1-xy)$$

**11.** 
$$f(x,y) = y\sqrt{x} - y^2 - 2x + 7y$$

**13.** 
$$f(x,y) = x^3 - 3x + 3xy^2$$

**15.** 
$$f(x,y) = x^4 - 2x^2 + y^3 - 3y$$

17. 
$$f(x,y) = xy - x^2y - xy^2$$

**19.** 
$$f(x,y) = e^x \cos y$$

**21.** 
$$f(x,y) = y^2 - 2y \cos x$$
,  $-1 \le x \le 7$ 

### $33-39 \ (odd)$

Find the absolute maximum and minimum values of f on the set D.

**33.** 
$$f(x,y) = x^2 + y^2 - 2x$$
,

D is the closed triangular region with vertices (2,0), (0,2), and (0,-2)

**35.** 
$$f(x,y) = x^2 + y^2 + x^2y + 4$$
,

$$D = \{(x, y) \mid |x| \le 1, |y| \le 1\}$$

**37.** 
$$f(x,y) = x^2 + 2y^2 - 2x - 4y + 1$$
,

$$D = \{(x, y) \mid 0 \le x \le 2, 0 \le y \le 3\}$$

**39.** 
$$f(x,y) = 2x^3 + y^4$$
.

$$D = \{(x, y) \mid x^2 + y^2 \le 1\}$$

43

Find the shortest distance from the point (2,0,-3) to the plan x+y+z=1.

45

Find the points on the cone  $z^2 = x^2 + y^2$  that are closest to the point (4,2,0).

47

Find three positive numbers whose sum is 100 and whose product is a maximum.

*55* 

A cardboard box without a lid is to have a volume of  $32,000 \ cm^3$ . Find the dimensions that minimize the amount of cardboard used.