Chapter 14 - Problem Set 2

Calculus 3

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Section 5: The Chain Rule

3-7 (odd)

Use The Chain Rule to find $\dfrac{dz}{dt}$ or $\dfrac{dw}{dt}$.

3. $z = xy^3 - x^2y, \quad x = t^2 + 1, \quad y = t^2 - 1$

Solution

$$\frac{dz}{dt} = \frac{dz}{dx}\left(\frac{dx}{dt}\right) + \frac{dz}{dy}\left(\frac{dy}{dt}\right)$$
$$\frac{dz}{dt} = [y^3 - 2xy](2t) + [3xy^2 - x^2](2t)$$
$$\boxed{\frac{dz}{dt} = 2t[y^3 - 2xy + 3xy^2 - x^2]}$$

5. $z = \sin x \cos y, \quad x = \sqrt{t}, \quad y = 1/t$

Solution

$$\frac{dz}{dt} = \frac{dz}{dx}\left(\frac{dx}{dt}\right) + \frac{dz}{dy}\left(\frac{dy}{dt}\right)$$
$$\frac{dz}{dt} = [\cos x \cos y]\left(\frac{1}{2}t^{-\frac{1}{2}}\right) + [-\sin x \sin y]\left(-t^{-2}\right)$$
$$\boxed{\frac{dz}{dt} = \frac{1}{2\sqrt{t}}\cos x \cos y + \frac{1}{t^2}\sin x \sin y}$$

7. $w = xe^{y/z}, \quad x = t^2, \quad y = 1 - t, \quad z = 1 + 2t$

Solution

$$\frac{dw}{dt} = \frac{dw}{dx}\left(\frac{dx}{dt}\right) + \frac{dw}{dy}\left(\frac{dy}{dt}\right) + \frac{dw}{dz}\left(\frac{dz}{dt}\right)$$
$$\frac{dw}{dt} = \left[e^{y/z}\right](2t) + \left[\frac{x}{z}e^{y/z}\right](-1) + \left[-\frac{xy}{z^2}e^{y/z}\right](2)$$
$$\boxed{\frac{dw}{dt} = e^{y/z}\left[2t - \frac{x}{z} - \frac{2xy}{z^2}\right]}$$

11-15 (odd)

Use the Chain Rule to find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$

11. $z = (x-y)^5, \quad x = s^2 t, \quad y = st^2$

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}\right)$$
$$\frac{\partial z}{\partial s} = \left[5(x-y)^4\right](2st) + \left[-5(x-y)^4\right](t^2)$$
$$\boxed{\frac{\partial z}{\partial s} = 5(x-y)^4\left[2st - t^2\right]}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial t}\right)$$
$$\frac{\partial z}{\partial t} = \left[5(x-y)^4\right](s^2) + \left[-5(x-y)^4\right](2st)$$
$$\boxed{\frac{\partial z}{\partial t} = 5(x-y)^4\left[s^2 - 2st\right]}$$

13. $z = \ln(3x+2y), \quad x = s\sin t, \quad y = t\cos s$

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}\right)$$
$$\frac{\partial z}{\partial s} = \left[\frac{3}{3x+2y}\right](\sin t) + \left[\frac{2}{3x+2y}\right](-t\sin s)$$
$$\boxed{\frac{\partial z}{\partial s} = \frac{3\sin t - 2t\sin s}{3x+2y}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial t}\right)$$
$$\frac{\partial z}{\partial t} = \left[\frac{3}{3x+2y}\right](s\cos t) + \left[\frac{2}{3x+2y}\right](\cos s)$$
$$\boxed{\frac{\partial z}{\partial t} = \frac{3s\cos t + 2\cos s}{3x+2y}}$$

15. $z = (\sin\theta)/r, \quad r = st, \quad \theta = s^2 + t^2$

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\left(\frac{\partial r}{\partial s}\right) + \frac{\partial z}{\partial \theta}\left(\frac{\partial \theta}{\partial s}\right)$$
$$\frac{\partial z}{\partial s} = \left[-\frac{\sin\theta}{r^2}\right](t) + \left[\frac{\cos\theta}{r}\right](2s)$$
$$\boxed{\frac{\partial z}{\partial s} = -\frac{t\sin\theta}{r^2} + \frac{2s\cos\theta}{r}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\left(\frac{\partial r}{\partial t}\right) + \frac{\partial z}{\partial \theta}\left(\frac{\partial \theta}{\partial t}\right)$$
$$\frac{\partial z}{\partial t} = \left[-\frac{\sin\theta}{r^2}\right](s) + \left[\frac{\cos\theta}{r}\right](2t)$$
$$\boxed{\frac{\partial z}{\partial t} = -\frac{s\sin\theta}{r^2} + \frac{2t\cos\theta}{r}}$$

25-29 (odd)

Use the Chain Rule to find the indicated partial derivatives.

25. $z = x^4 + x^2y, \quad x = s + 2t - u, \quad y = stu^2$**;**

$\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}, \frac{\partial z}{\partial u}$ **when** $s = 4, t = 2, u = 1$

Solution

When $s = 4$, $t = 2$, and $u = 1 \quad \Rightarrow \quad x = 7$, $y = 8$

$$\frac{\partial z}{\partial s} = \left[4x^3 + 2xy\right](1) + \left[x^2\right](tu^2) = 4x^3 + 2xy + x^2tu^2$$
$$\frac{\partial z}{\partial s} = 4(7)^3 + 2(7)(8) + (7)^2(2)(1)^2$$
$$\boxed{\frac{\partial z}{\partial s} = 1582}$$

$$\frac{\partial z}{\partial t} = \left[4x^3 + 2xy\right](2) + \left[x^2\right](su^2) = 8x^3 + 4xy + x^2su^2$$
$$\frac{\partial z}{\partial t} = 8(7)^3 + 4(7)(8) + (7)^2(4)(1)^2$$
$$\boxed{\frac{\partial z}{\partial t} = 3164}$$

$$\frac{\partial z}{\partial u} = \left[4x^3 + 2xy\right](-1) + \left[x^2\right](2stu) = -4x^3 - 2xy + 2x^2stu$$
$$\frac{\partial z}{\partial u} = -4(7)^3 - 2(7)(8) + 2(7)^2(4)(2)(1)$$
$$\boxed{\frac{\partial z}{\partial u} = -700}$$

27. $w = xy + yz + zx, \quad x = r\cos\theta, \quad y = r\sin\theta, \quad z = r\theta$**;**

$\dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial \theta}$ **when** $r = 2, \theta = \pi/2$

Solution

When $r = 2$ and $\theta = \pi/2 \quad \Rightarrow \quad x = 0,\ y = 2,\ z = \pi$

$$\frac{\partial w}{\partial r} = [y+z](\cos\theta) + [x+z](\sin\theta) + [x+y](\theta)$$
$$\frac{\partial w}{\partial r} = [2+\pi](0) + [0+\pi](1) + [0+2](\pi/2) = 0 + \pi + \pi$$
$$\boxed{\frac{\partial w}{\partial r} = 2\pi}$$

$$\frac{\partial w}{\partial \theta} = [y+z](-r\sin\theta) + [x+z](r\cos\theta) + [x+y](r)$$
$$\frac{\partial w}{\partial \theta} = [2+\pi](-2\cdot 1) + [0+\pi](0) + [0+2](2) = -4 - 2\pi + 0 + 4$$
$$\boxed{\frac{\partial w}{\partial \theta} = -2\pi}$$

29. $N = \dfrac{p+q}{p+r}, \quad p = u + vw, \quad q = v + uw, \quad r = w + uv$

$\dfrac{\partial N}{\partial u}, \dfrac{\partial N}{\partial v}, \dfrac{\partial N}{\partial w}$ **when** $u = 2, v = 3, w = 4$

Solution

When $u = 2$, $v = 3$, and $w = 4 \quad \Rightarrow \quad p = 14$, $q = 11$, $r = 10$

$$\frac{\partial N}{\partial u} = \left[\frac{(p+r)-(p+q)}{(p+r)^2}\right](1) + \left[\frac{1}{p+r}\right](w) + \left[-\frac{p+q}{(p+r)^2}\right](v)$$
$$\frac{\partial N}{\partial u} = \frac{r-q}{(p+r)^2} + \frac{w}{p+r} - \frac{v(p+q)}{(p+r)^2}$$
$$\frac{\partial N}{\partial u} = \frac{10-11}{(14+10)^2} + \frac{4}{14+10} - \frac{3(14+11)}{(14+10)^2}$$
$$\frac{\partial N}{\partial u} = \frac{-1}{576} + \frac{4}{24} - \frac{75}{576} = \frac{-1+96-75}{576} = \frac{20}{576}$$
$$\boxed{\frac{\partial N}{\partial u} = \frac{5}{144}}$$

$$\frac{\partial N}{\partial v} = \left[\frac{r-q}{(p+r)^2}\right](w) + \left[\frac{1}{p+r}\right](1) + \left[-\frac{p+q}{(p+r)^2}\right](u)$$
$$\frac{\partial N}{\partial v} = \frac{w(r-q)}{(p+r)^2} + \frac{1}{p+r} - \frac{u(p+q)}{(p+r)^2}$$
$$\frac{\partial N}{\partial v} = \frac{4(10-11)}{(14+10)^2} + \frac{1}{14+10} - \frac{2(14+11)}{(14+10)^2}$$
$$\frac{\partial N}{\partial v} = \frac{-4}{576} + \frac{1}{24} - \frac{50}{576} = \frac{-4+24-50}{576} = \frac{-30}{576}$$
$$\boxed{\frac{\partial N}{\partial v} = -\frac{5}{96}}$$

$$\frac{\partial N}{\partial w} = \left[\frac{r-q}{(p+r)^2}\right](v) + \left[\frac{1}{p+r}\right](u) + \left[-\frac{p+q}{(p+r)^2}\right](1)$$
$$\frac{\partial N}{\partial w} = \frac{v(r-q)}{(p+r)^2} + \frac{u}{p+r} - \frac{p+q}{(p+r)^2}$$
$$\frac{\partial N}{\partial w} = \frac{3(10-11)}{(14+10)^2} + \frac{2}{(14+10)} - \frac{14+11}{(14+10)^2}$$
$$\frac{\partial N}{\partial w} = \frac{-3}{576} + \frac{2}{24} - \frac{25}{576} = \frac{-3+48-25}{576} = \frac{20}{576}$$
$$\boxed{\frac{\partial N}{\partial w} = \frac{5}{144}}$$

Use Equation 5 to find $\frac{dy}{dx}$

31. $y\cos x = x^2 + y^2$

Solution

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Let $F = y\cos x - x^2 - y^2$

$$F_x = -y\sin x - 2x$$
$$F_y = \cos x - 2y$$

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{(-y\sin x - 2x)}{\cos x - 2y} = \frac{2x + y\sin x}{\cos x - 2y}}$$

Use Equations 6 to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

35. $x^2 + 2y^2 + 3z^2 = 1$

Solution

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Let $F = x^2 + 2y^2 + 3z^2 - 1$

$$F_x = 2x \qquad F_y = 4y \qquad F_z = 6z$$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{6z} = -\frac{x}{3z}}$$

$$\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{4y}{6z} = -\frac{2y}{3z}}$$

Section 6: Directional Derivatives and the Gradient Vector

5, 7

Find the directional derivative of f at the given point in the direction indicated by the angle θ.

5. $f(x,y) = xy^3 - x^2, \quad (1,2), \quad \theta = \pi/3$

7. $f(x,y) = arctan(xy), \quad (2,-3), \quad \theta = 3\pi/4$

(a) Find the gradient of f

(b) Evaluate the gradient at the point P

(c) Find the rate of change of f at P in the direction of the vector $\vec{u}$

9. $f(x,y) = x/y, \quad P(2,1), \quad \vec{u} = \frac{3}{5}\,\hat{\mathbf{i}} + \frac{4}{5}\,\hat{\mathbf{j}}$

13, 15

Find the directional derivative of the function at the given point in the direction of the vector $\vec{v}$.

13. $f(x,y) = e^x \sin y, \quad (0,\pi/3), \quad \vec{v} = \langle -6, 8 \rangle$

15. $g(s,t) = s\sqrt{t}, \quad (2,4), \quad \vec{v} = 2\,\hat{\mathbf{i}} - \,\hat{\mathbf{j}}$

21, 23

Find the directional derivative of the function at the point P in the direction of the point Q.

21. $f(x,y) = x^2y^2 - y^3, \quad P(1,2), \quad Q(-3,5)$

23. $f(x,y) = \sqrt{xy}, \quad P(2,8), \quad Q(5,4)$

27, 29

Find the maximum rate of change of f at the given point and the direction in which it occurs.

27. $f(x,y) = 5xy^2, \quad (3,-2)$

29. $f(x,y) = \sin(xy), \quad (1,0)$

37

The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1,2,2)$ is $120°$

(a) Find the rate of change of T at $(1,2,2)$ in the direction toward the point $(2,1,3)$.

(b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points toward the origin

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47-51 (odd)

Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

47. $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10, \quad (3,3,5)$

49. $xy^2z^3 = 8, \quad (2,2,1)$

51. $x + y + z = e^{xyz}, \quad (0,0,1)$

Section 7: Maximum and Minimum Values

5-21 (odd)

Find the local maximum and minimum values and saddle point(s) of the function. You are encouraged to use a calculator or computer to graph the function with a domain and viewpoint that reveals all the important aspects of the function.

5. $f(x, y) = x^2 + xy + y^2 + y$

7. $f(x, y) = 2x^2 - 8xy + y^4 - 4y^3$

9. $f(x, y) = (x - y)(1 - xy)$

11. $f(x, y) = y\sqrt{x} - y^2 - 2x + 7y$

13. $f(x, y) = x^3 - 3x + 3xy^2$

15. $f(x, y) = x^4 - 2x^2 + y^3 - 3y$

17. $f(x, y) = xy - x^2y - xy^2$

19. $f(x, y) = e^x \cos y$

21. $f(x, y) = y^2 - 2y \cos x, \quad -1 \leq x \leq 7$

33-39 (odd)

Find the absolute maximum and minimum values of f on the set D.

33. $f(x, y) = x^2 + y^2 - 2x$,

D is the closed triangular region with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$

35. $f(x, y) = x^2 + y^2 + x^2y + 4$,

$D = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}$

37. $f(x, y) = x^2 + 2y^2 - 2x - 4y + 1$,

$D = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 3\}$

39. $f(x, y) = 2x^3 + y^4$,

$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$

43

Find the shortest distance from the point $(2, 0, -3)$ to the plan $x + y + z = 1$.

45

Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

47

Find three positive numbers whose sum is 100 and whose product is a maximum.

55

A cardboard box without a lid is to have a volume of 32,000 cm^3. Find the dimensions that minimize the amount of cardboard used.
