Chapter 13 Section 3 & 4 Problem Set

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Section 3: Arc Length and Curvature

Problem 1a

Use Equation 2 to compute the length of the given line segment.

$$\vec{r}(t) = \langle 3-t, 2t, 4t+1 \rangle \quad 1 \le t \le 3$$

Solution

Let the length of the line segment be

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\, dt \Rightarrow L = \int_a^b \|\vec{r}\,'(t)\|\, dt$$

$$D : \{\, t \mid 1 \le t \le 3 \,\}$$

$$\vec{r}\,'(t) = \langle -1, 2, 4 \rangle \Rightarrow L = \int_1^3 \sqrt{(-1)^2 + (2)^2 + (4)^2} dt \quad = \int_1^3 \sqrt{21} dt = \sqrt{21}t \,\Big|_1^3 = \sqrt{21}(3) - \sqrt{21}(1) = 2\sqrt{21}$$

Problems 3-7 odd

Find the length of the curve.

3. $\vec{r}\,(t) = \langle t, 3\cos t, 3\sin t \rangle \quad 25 \le t \le 5$

Solution

$$\begin{aligned} \vec{r}\,'(t) = \langle 1, -3\sin t, 3\cos t \rangle \Rightarrow L &= \int_{-5}^5 \sqrt{1^2 + (-3\sin t)^2 + (3\cos t)^2}\, dt \\ &= \int_{-5}^5 \sqrt{1 + 9\sin^2 t + 9\cos^2 t}\, dt \\ &= \int_{-5}^5 \sqrt{1 + 9(1)}\, dt \\ &= \int_{-5}^5 \sqrt{10}\, dt \\ &= \sqrt{10}t \,\Big|_{-5}^5 dt = \sqrt{10}(5) - \sqrt{10}(-5) = 10\sqrt{10} \end{aligned}$$

5. $\vec{r}\,(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle \quad 0 \le t \le 1$

Solution

$$\begin{aligned}
\vec{r}\,'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle \Rightarrow L &= \int_0^1 \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2}\; dt \\
&= \int_0^1 \sqrt{2 + e^{2t} + e^{-2t}}\; dt \\
&= \int_0^1 \sqrt{(e^t + e^{-t})^2}\; dt \\
&= \int_0^1 e^t + e^{-t}\; dt \\
&= e^t - e^{-t}\,\Big|_0^1 = (e^1 - \frac{1}{e^1}) - (e^0 - \frac{1}{e^0}) = e - \frac{1}{e} - 1 + 1 = e - \frac{1}{e}
\end{aligned}$$

7. $\vec{r}\,(t) = \langle 1, t^2, t^3 \rangle \quad 0 \le t \le 1$

Solution

$$\begin{aligned}
\vec{r}\,'(t) = \langle 0, 2t, 3t^2 \rangle \Rightarrow L &= \int_0^1 \sqrt{(0)^2 + (2t)^2 + (3t^2)^2}\; dt \\
&= \int_0^1 \sqrt{4t^2 + 9t^4}\; dt \\
&= \int_0^1 \sqrt{t^2(4 + 9t^2)}\; dt \\
&= \int_0^1 t\sqrt{4 + 9t^2}\; dt
\end{aligned}$$

Using u-substitution,

$$\begin{aligned}
u^2 &= 4 + 9t^2 \\
2udu &= 18t\; dt \\
\frac{u\; du}{9} &= t\; dt
\end{aligned}$$

$$\int_0^1 t\sqrt{4 + 9t^2}\; dt = \int_0^1 u \cdot (\frac{1}{9}u)\; du \quad = \frac{1}{9}\int_0^1 u^2\; du = \frac{1}{9}(\frac{1}{3}u^3)\Big|_2^{\sqrt{13}} \quad = \frac{1}{27}(u^3)\Big|_2^{\sqrt{13}} = \frac{1}{27}(13^{\frac{3}{2}} - 2^3) = \frac{13\sqrt{13}}{27} - 3$$

Problems 19-23 odd

(a) Find the unit tangent and unit normal vectors $\vec{T}(t)$ and $\vec{N}(t)$.
(b) Use Formula 9 to find the curvature.

19. $\vec{r}(t) = \langle t^2,\ \sin t - t\cos t,\ \cos t + t\sin t \rangle, \quad t > 0$

Solution

a.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

$$\begin{aligned}
\vec{r}\,'(t) &= \langle 2t, \cos t - \cos t + t\sin t, -\sin t + \sin t + t\cos t \rangle = \langle 2t, t\sin t, t\cos t \rangle \\
\|\vec{r}\,'(t)\| &= \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{4t^2 + t^2} = \sqrt{5t^2} = \sqrt{5}t \quad [\cos^2 t + \sin^2 t = 1] \\
\vec{T}(t) &= \frac{\langle 2t, t\sin t, t\cos t \rangle}{\sqrt{5}t} \quad = \frac{1}{\sqrt{5}}\langle 2, \sin t, \cos t \rangle
\end{aligned}$$

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$$

$$
\begin{aligned}
\vec{T}\,'(t) &= \frac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle \\
\|\vec{T}\,'(t)\| &= \frac{1}{\sqrt{5}}\sqrt{0^2+\cos^2 t+\sin^2 t} = \sqrt{1} = \frac{1}{\sqrt{5}} \\
\vec{N}(t) &= \frac{\frac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle}{\frac{1}{\sqrt{5}}} = \langle 0, \cos t, -\sin t\rangle
\end{aligned}
$$

b.

$$
\kappa(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{\frac{1}{\sqrt{5}}}{\sqrt{5}t} = \frac{1}{5t}
$$

21. $\vec{r}(t) = \langle t, t^2, 4\rangle$

Solution

a.

$$
\begin{aligned}
\vec{r}\,'(t) &= \hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}} \\
\|\vec{r}\,'(t)\| &= \sqrt{1^2+(2t)^2} = \sqrt{1+4t^2} \\
\vec{T}(t) &= \frac{\hat{\mathbf{i}}+2t\,\hat{\mathbf{j}}}{\sqrt{1+4t^2}} = \frac{1}{\sqrt{1+4t^2}}(\,\hat{\mathbf{i}}+2t\,\hat{\mathbf{j}}) \\
\frac{d}{dt} &= [f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t) \quad [\textit{vector product rule}] \\
\vec{T}\,'(t) &= -\frac{4t}{(1+4t^2)^{\frac{3}{2}}}(\,\hat{\mathbf{i}}+2t\,\hat{\mathbf{j}}) + \frac{1}{(1+4t^2)^{\frac{1}{2}}}(2\,\hat{\mathbf{j}}) \\
&= \frac{1}{(1+4t^2)^{\frac{3}{2}}}\left(-4t(\,\hat{\mathbf{i}}+2t\,\hat{\mathbf{j}}) + (1+4t^2)(2\,\hat{\mathbf{j}})\right) = \frac{1}{(1+4t^2)^{\frac{3}{2}}}(-4t\,\hat{\mathbf{i}} - 8t^2\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{j}} + 8t^2\,\hat{\mathbf{j}}) \\
&= \frac{1}{(1+4t^2)^{\frac{3}{2}}}(-4t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}}) \\
\|\vec{T}\,'(t)\| &= \frac{1}{(1+4t^2)^{\frac{3}{2}}}\sqrt{(-4t)^2+2^2} = \frac{1}{(1+4t^2)^{\frac{3}{2}}}\sqrt{16t^2+4} = \frac{1}{(1+4t^2)^{\frac{3}{2}}}\sqrt{4(4t^2+1)} = \frac{2}{(1+4t^2)^{\frac{3}{2}}}\sqrt{1+4t^2} \\
&= \frac{2}{1+4t^2} \\
\vec{N}(t) &= \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} = \frac{1}{(1+4t^2)^{\frac{3}{2}}}(-4t\,\hat{\mathbf{i}}+2\,\hat{\mathbf{j}}) \cdot \frac{1+4t^2}{2} = \frac{(1+4t^2)^1}{2(1+4t^2)^{\frac{3}{2}}}(-4t\,\hat{\mathbf{i}}+2\,\hat{\mathbf{j}}) = \frac{1}{\sqrt{1+4t^2}}(-2t\,\hat{\mathbf{i}}+\,\hat{\mathbf{j}})
\end{aligned}
$$

b.

$$
\kappa(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{2}{1+4t^2} \cdot \frac{1}{(1+4t^2)^{\frac{1}{2}}} = \frac{2}{(1+4t^2)^{\frac{3}{2}}}
$$

23. $\vec{r}(t) = \langle t, \frac{1}{2}t^2, t^2\rangle$

Solution

a.

$$\begin{aligned}
\vec{r}\,'(t) &= \langle 1, t, 2t\rangle \\
\|\vec{r}\,'(t)\| &= \sqrt{1^2 + t^2 + (2t)^2} = \sqrt{1+5t^2} \\
\vec{T}(t) &= \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\langle 1, t, 2t\rangle}{\sqrt{1+t^2+4t^2}} = \frac{1}{\sqrt{1+5t^2}}\langle 1, t, 2t\rangle \\
\frac{d}{dt} &= [f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t) \quad [\textit{vector product rule}] \\
\vec{T}\,'(t) &= -\frac{5t}{(1+5t^2)^{\frac{3}{2}}}\langle 1, t, 2t\rangle + \frac{1}{(1+5t^2)^{\frac{1}{2}}}\langle 0, 1, 2\rangle \\
&= \frac{1}{(1+5t^2)^{\frac{3}{2}}}\left(-5t\langle 1, t, 2t\rangle + (1+5t^2)\langle 0, 1, 2\rangle\right) \\
&= \frac{1}{(1+5t^2)^{\frac{3}{2}}}\left(\langle -5t, -5t^2, -10t^2\rangle + \langle 0, 1+5t^2, 2+10t^2\rangle\right) = \frac{1}{(1+5t^2)^{\frac{3}{2}}}\langle -5t, 1, 2\rangle \\
\|\vec{T}\,'(t)\| &= \frac{1}{(1+5t^2)^{\frac{3}{2}}}\sqrt{(-5t)^2+1^2+2^2} = \frac{1}{(1+5t^2)^{\frac{3}{2}}}\sqrt{25t^2+5} = \frac{1}{(1+5t^2)^{\frac{3}{2}}}\sqrt{5(5t^2+1)} \\
&= \frac{\sqrt{5}(1+5t^2)^{\frac{1}{2}}}{(1+5t^2)^{\frac{3}{2}}} = \frac{\sqrt{5}}{1+5t^2} \\
\|\vec{N}(t)\| &= \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} = \frac{1}{(1+5t^2)^{\frac{3}{2}}}\langle -5t, 1, 2\rangle \cdot \frac{1+5t^2}{\sqrt{5}} = \frac{1}{\sqrt{5}\sqrt{1+5t^2}}\langle -5t, 1, 2\rangle = \frac{1}{\sqrt{5+25t^2}}\langle -5t, 1, 2\rangle
\end{aligned}$$

b.

$$\kappa(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{\sqrt{5}}{1+5t^2} \cdot \frac{1}{\sqrt{1+5t^2}} = \frac{\sqrt{5}}{(1+5t^2)^{\frac{3}{2}}}$$

Problem 27

Use Theorem 10 to find the curvature

$$\vec{r}(t) = \sqrt{6}t^2\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}} + 2t^3\,\hat{\mathbf{k}}$$

Solution

Theorem 10 states that

$$\kappa(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$$

$$\begin{aligned}
\vec{r}\,'(t) &= 2\sqrt{6}t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 6t^2\,\hat{\mathbf{k}} \\
\vec{r}\,''(t) &= 2\sqrt{6}\,\hat{\mathbf{i}} + 12t\,\hat{\mathbf{k}} \\
\|\vec{r}\,'(t)\| &= \sqrt{(2\sqrt{6}t)^2 + 2^2 + (6t^2)^2} = \sqrt{24t^2+4+36t^4} = \sqrt{4(9t^4+6t^2+1)} = 2\sqrt{(3t^2+1)^2} = 2(3t^2+1)
\end{aligned}$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) \Rightarrow \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2\sqrt{6}t & 2 & 6t^2 \\ 2\sqrt{6} & 0 & 12t \end{vmatrix} = \hat{\mathbf{i}}\begin{vmatrix} 2 & 6t^2 \\ 0 & 12t \end{vmatrix} - \hat{\mathbf{j}}\begin{vmatrix} 2\sqrt{6}t & 6t^2 \\ 2\sqrt{6} & 12t \end{vmatrix} + \hat{\mathbf{k}}\begin{vmatrix} 2\sqrt{6}t & 2 \\ 2\sqrt{6} & 0 \end{vmatrix}$$
$$= (24t - 0)\,\hat{\mathbf{i}} - (24t^2\sqrt{6} - 12t^2\sqrt{6})\,\hat{\mathbf{j}} + (0 - 4\sqrt{6})\,\hat{\mathbf{k}} = 24t\,\hat{\mathbf{i}} - 12t^2\sqrt{6}\,\hat{\mathbf{j}} - 4\sqrt{6}\,\hat{\mathbf{k}}$$

$$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = \sqrt{(24t)^2 + (-12t^2\sqrt{6})^2 + (-4\sqrt{6})^2} = \sqrt{576t^2+864t^4+96} = \sqrt{96(9t^4+6t^2+1)} = \sqrt{16\cdot 6(3t^2+1)^2}$$
$$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = 4\sqrt{6}(3t^2+1)$$

$$\kappa(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \frac{4\sqrt{6}(3t^2+1)}{(2(3t^2+1))^3} = \frac{4\sqrt{6}(3t^2+1)}{8(3t^2+1)^3} = \frac{\sqrt{6}}{2(3t^2+1)^2}$$

Problem 28

Find the curvature of $\vec{r}(t) = \langle t^2, \ln t, t \ln t\rangle$ at the point $(1,0,0)$.

Solution

$$\begin{aligned} &If\ x = t^2 \Rightarrow 1 = t^2,\ then \\ &\quad t = 1? \Rightarrow\ \ln 1 \equiv 0\ ,\ 1 \ln 1 \equiv 0 \\ &\therefore t = 1 \end{aligned}$$

$$\begin{aligned} &\vec{r}\,'(t) = \langle 2t, \frac{1}{t}, \ln t + 1\rangle, \quad \vec{r}\,''(t) = \langle 2, -\frac{1}{t^2}, \frac{1}{t}\rangle \\ &\vec{r}\,'(1)j = \langle 2,1,1\rangle, \quad \|\vec{r}\,'(1)\| = \sqrt{2^2+1^2+1^2} = \sqrt{6}, \quad \vec{r}\,''(1) = \langle 2,-1,1\rangle \end{aligned}$$

$$\vec{r}\,'(1) \times \vec{r}\,''(1) \Rightarrow \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = \hat{\mathbf{i}} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \hat{\mathbf{k}} \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix}$$
$$= (1-(-1))\,\hat{\mathbf{i}} - (2-2)\,\hat{\mathbf{j}} + (-2-2)\,\hat{\mathbf{k}} = 2\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{k}} \Rightarrow \langle 2,0,-4\rangle$$

$$\kappa(1) = \frac{\|\langle 2,0,-4\rangle\|}{\sqrt{6}^3} = \frac{\sqrt{2^2+0^2+(-4)^2}}{6^{\frac{3}{2}}} = \frac{\sqrt{20}}{6\sqrt{6}} = \frac{2\sqrt{5}}{6\sqrt{6}} = \frac{\sqrt{5}}{3\sqrt{6}} = \frac{\sqrt{30}}{18}$$

Problem 31 & 33

Use Formula 11 to find the curvature.

31. $y = x^4$

Solution

33. $y = xe^x$

Solution

Problem 51

Find the vectors **T**, **N**, and **B** at the given point.

$$\vec{r}(t) = \langle t^2, \frac{2}{3}t^3, t\rangle, \quad (1, \frac{2}{3}, 1)$$

Solution

Problem 53

Find equations of the normal plane and osculating plane of the curve at the given point.

$$x = \sin 2t,\ y = -\cos 2t,\ z = 4t; \quad (0,1,2\pi)$$

Solution

Problem 66

Use Formula 14 to find the torsion at the given value of t.

$$\vec{r}(t) = \langle \sin t, 3t, \cos t \rangle, \quad t = \frac{\pi}{2}$$

Solution

Problem 70

Use Theorem 15 to find the torsion of the given curve at a general point and at the point corresponding to $t = 0$

$$\vec{r}(t) = \langle \cos t, \sin t, \sin t \rangle$$

Solution

Section 4: Motion in Space - Velocity and Acceleration

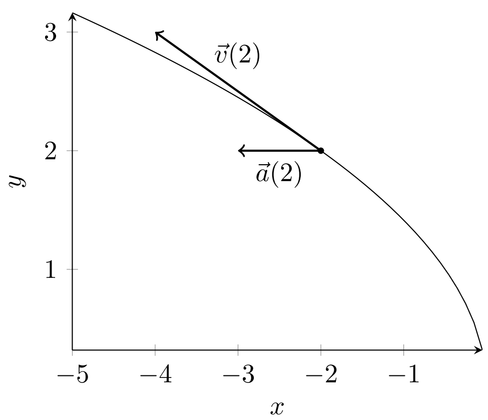
Problem 3-7 odd

Find the velocity, acceleration, and speed of a particle with the given position function. Sketch the path of the particle and draw the velocity and acceleration vectors for the specified value of t.

3. $\vec{r}(t) = \langle -\frac{1}{2}t^2, t \rangle, \quad t = 2$

Solution

$$\vec{r}\,'(t) = \vec{v}(t) = \langle -t, 1 \rangle \quad \Rightarrow \quad \vec{v}(2) = \langle -2, 1 \rangle$$
$$\vec{r}\,''(t) = \vec{a}(t) = \langle -1, 0 \rangle \quad \Rightarrow \quad \vec{a}(2) = \langle -1, 0 \rangle$$



5. $\vec{r}(t) = 3\cos t\ \hat{\mathbf{i}} + 2\sin t\ \hat{\mathbf{j}} \quad t = \frac{\pi}{3}$

7. $\vec{r}(t) = t\ \hat{\mathbf{i}} + t^2\ \hat{\mathbf{j}} + 2\ \hat{\mathbf{k}} \quad t = 1$

Problems 9-13 odd

Find the velocity, acceleration, and speed of a particle with the given position function.

9. $\vec{r}(t) = \langle t^2 + t, t^2 - t, t^3 \rangle$

11. $\vec{r}(t) = \sqrt{2}t\ \hat{\mathbf{i}} + e^t\ \hat{\mathbf{j}} + e^{-t}\ \hat{\mathbf{k}}$

13. $\vec{r}(t) = e^t(\cos t\ \hat{\mathbf{i}} + \sin t\ \hat{\mathbf{j}} + t\ \hat{\mathbf{k}})$

Problem 15

Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

$$a(t) = 2\ \hat{\mathbf{i}} + 2t\ \hat{\mathbf{k}}, \quad v(0) = 3\ \hat{\mathbf{i}} - \ \hat{\mathbf{j}}, \quad r(0) = \ \hat{\mathbf{j}} + \ \hat{\mathbf{k}}$$

Problem 17a

Find the position vector of a particle that has the given acceleration and the specified initial velocity and position.

$$a(t) = 2t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + \cos 2t\,\hat{\mathbf{k}}, \quad v(0) = \hat{\mathbf{i}}, \quad r(0) = \hat{\mathbf{j}}$$

Problem 23

A projectile is fired with an initial speed of $200\frac{m}{s}$ and angle of elevation $60°$. Find (a) the range of the projectile, (b) the maximum height reached, and (c) the speed at impact.

Problem 26

A projectile is fired from a tank with initial speed $400\frac{m}{s}$. Find two angles of elevation that can be used to hit a target $3000m$ away.

Problem 27

A rifle is fired with angle of elevation $36°$. What is the initial speed if the maximum height of the bullet is $1600ft$?

Problem 37 & 39

Find the tangential and normal components of the acceleration vector.

37. $\vec{r}(t) = (t^2+1)\,\hat{\mathbf{i}} + t^3\,\hat{\mathbf{j}}, \quad t \geq 0$

39. $\vec{r}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$

Problem 41

Find the tangential and normal components of the acceleration vector at the given point.

$$\vec{r}(t) = \ln t\,\hat{\mathbf{i}} + (t^2+3t)\,\hat{\mathbf{j}} + 4\sqrt{t}\,\hat{\mathbf{k}}, \quad (0,4,4)$$