Chapter 14 - Problem Set 2

Calculus 3

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Section 5: The Chain Rule

3-7 (odd)

Use The Chain Rule to find $\dfrac{dz}{dt}$ or $\dfrac{dw}{dt}$.

3. $z = xy^3 - x^2y, \quad x = t^2 + 1, \quad y = t^2 - 1$

Solution

$$\frac{dz}{dt} = \frac{dz}{dx}\left(\frac{dx}{dt}\right) + \frac{dz}{dy}\left(\frac{dy}{dt}\right)$$

$$\frac{dz}{dt} = [y^3 - 2xy](2t) + [3xy^2 - x^2](2t)$$

$$\boxed{\frac{dz}{dt} = 2t[y^3 - 2xy + 3xy^2 - x^2]}$$

5. $z = \sin x \cos y, \quad x = \sqrt{t}, \quad y = 1/t$

Solution

$$\frac{dz}{dt} = \frac{dz}{dx}\left(\frac{dx}{dt}\right) + \frac{dz}{dy}\left(\frac{dy}{dt}\right)$$

$$\frac{dz}{dt} = [\cos x \cos y]\left(\frac{1}{2}t^{-\frac{1}{2}}\right) + [-\sin x \sin y]\left(-t^{-2}\right)$$

$$\boxed{\frac{dz}{dt} = \frac{1}{2\sqrt{t}}\cos x \cos y + \frac{1}{t^2}\sin x \sin y}$$

7. $w = xe^{y/z}, \quad x = t^2, \quad y = 1 - t, \quad z = 1 + 2t$

Solution

$$\frac{dw}{dt} = \frac{dw}{dx}\left(\frac{dx}{dt}\right) + \frac{dw}{dy}\left(\frac{dy}{dt}\right) + \frac{dw}{dz}\left(\frac{dz}{dt}\right)$$

$$\frac{dw}{dt} = \left[e^{y/z}\right](2t) + \left[\frac{x}{z}e^{y/z}\right](-1) + \left[-\frac{xy}{z^2}e^{y/z}\right](2)$$

$$\boxed{\frac{dw}{dt} = e^{y/z}\left[2t - \frac{x}{z} - \frac{2xy}{z^2}\right]}$$

11-15 (odd)

Use the Chain Rule to find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$

11. $z = (x-y)^5, \quad x = s^2 t, \quad y = st^2$

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}\right)$$
$$\frac{\partial z}{\partial s} = \left[5(x-y)^4\right](2st) + \left[-5(x-y)^4\right](t^2)$$
$$\boxed{\frac{\partial z}{\partial s} = 5(x-y)^4\left[2st - t^2\right]}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial t}\right)$$
$$\frac{\partial z}{\partial t} = \left[5(x-y)^4\right](s^2) + \left[-5(x-y)^4\right](2st)$$
$$\boxed{\frac{\partial z}{\partial t} = 5(x-y)^4\left[s^2 - 2st\right]}$$

13. $z = \ln(3x+2y), \quad x = s\sin t, \quad y = t\cos s$

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}\right)$$
$$\frac{\partial z}{\partial s} = \left[\frac{3}{3x+2y}\right](\sin t) + \left[\frac{2}{3x+2y}\right](-t\sin s)$$
$$\boxed{\frac{\partial z}{\partial s} = \frac{3\sin t - 2t\sin s}{3x+2y}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial t}\right)$$
$$\frac{\partial z}{\partial t} = \left[\frac{3}{3x+2y}\right](s\cos t) + \left[\frac{2}{3x+2y}\right](\cos s)$$
$$\boxed{\frac{\partial z}{\partial t} = \frac{3s\cos t + 2\cos s}{3x+2y}}$$

15. $z = (\sin\theta)/r, \quad r = st, \quad \theta = s^2 + t^2$

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\left(\frac{\partial r}{\partial s}\right) + \frac{\partial z}{\partial \theta}\left(\frac{\partial \theta}{\partial s}\right)$$
$$\frac{\partial z}{\partial s} = \left[-\frac{\sin\theta}{r^2}\right](t) + \left[\frac{\cos\theta}{r}\right](2s)$$
$$\boxed{\frac{\partial z}{\partial s} = -\frac{t\sin\theta}{r^2} + \frac{2s\cos\theta}{r}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\left(\frac{\partial r}{\partial t}\right) + \frac{\partial z}{\partial \theta}\left(\frac{\partial \theta}{\partial t}\right)$$
$$\frac{\partial z}{\partial t} = \left[-\frac{\sin\theta}{r^2}\right](s) + \left[\frac{\cos\theta}{r}\right](2t)$$
$$\boxed{\frac{\partial z}{\partial t} = -\frac{s\sin\theta}{r^2} + \frac{2t\cos\theta}{r}}$$

25-29 (odd)

Use the Chain Rule to find the indicated partial derivatives.

25. $z = x^4 + x^2y, \quad x = s + 2t - u, \quad y = stu^2$**;**

$\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}, \frac{\partial z}{\partial u}$ **when** $s = 4, t = 2, u = 1$

Solution

When $s = 4$, $t = 2$, and $u = 1 \quad \Rightarrow \quad x = 7$, $y = 8$

$$\frac{\partial z}{\partial s} = \left[4x^3 + 2xy\right](1) + \left[x^2\right](tu^2) = 4x^3 + 2xy + x^2tu^2$$
$$\frac{\partial z}{\partial s} = 4(7)^3 + 2(7)(8) + (7)^2(2)(1)^2$$
$$\boxed{\frac{\partial z}{\partial s} = 1582}$$

$$\frac{\partial z}{\partial t} = \left[4x^3 + 2xy\right](2) + \left[x^2\right](su^2) = 8x^3 + 4xy + x^2su^2$$
$$\frac{\partial z}{\partial t} = 8(7)^3 + 4(7)(8) + (7)^2(4)(1)^2$$
$$\boxed{\frac{\partial z}{\partial t} = 3164}$$

$$\frac{\partial z}{\partial u} = \left[4x^3 + 2xy\right](-1) + \left[x^2\right](2stu) = -4x^3 - 2xy + 2x^2stu$$
$$\frac{\partial z}{\partial u} = -4(7)^3 - 2(7)(8) + 2(7)^2(4)(2)(1)$$
$$\boxed{\frac{\partial z}{\partial u} = -700}$$

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27. $w = xy + yz + zx, \quad x = r\cos\theta, \quad y = r\sin\theta, \quad z = r\theta$**;**

$\dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial \theta}$ **when** $r = 2, \theta = \pi/2$

Solution

When $r = 2$ and $\theta = \pi/2 \quad \Rightarrow \quad x = 0,\ y = 2,\ z = \pi$

$$\frac{\partial w}{\partial r} = [y+z](\cos\theta) + [x+z](\sin\theta) + [x+y](\theta)$$
$$\frac{\partial w}{\partial r} = [2+\pi](0) + [0+\pi](1) + [0+2](\pi/2) = 0 + \pi + \pi$$
$$\boxed{\frac{\partial w}{\partial r} = 2\pi}$$

$$\frac{\partial w}{\partial \theta} = [y+z](-r\sin\theta) + [x+z](r\cos\theta) + [x+y](r)$$
$$\frac{\partial w}{\partial \theta} = [2+\pi](-2\cdot 1) + [0+\pi](0) + [0+2](2) = -4 - 2\pi + 0 + 4$$
$$\boxed{\frac{\partial w}{\partial \theta} = -2\pi}$$

29. $N = \dfrac{p+q}{p+r}, \quad p = u + vw, \quad q = v + uw, \quad r = w + uv$

$\dfrac{\partial N}{\partial u}, \dfrac{\partial N}{\partial v}, \dfrac{\partial N}{\partial w}$ **when** $u = 2, v = 3, w = 4$

Solution

When $u = 2$, $v = 3$, and $w = 4 \quad \Rightarrow \quad p = 14$, $q = 11$, $r = 10$

$$\frac{\partial N}{\partial u} = \left[\frac{(p+r)-(p+q)}{(p+r)^2}\right](1) + \left[\frac{1}{p+r}\right](w) + \left[-\frac{p+q}{(p+r)^2}\right](v)$$
$$\frac{\partial N}{\partial u} = \frac{r-q}{(p+r)^2} + \frac{w}{p+r} - \frac{v(p+q)}{(p+r)^2}$$
$$\frac{\partial N}{\partial u} = \frac{10-11}{(14+10)^2} + \frac{4}{14+10} - \frac{3(14+11)}{(14+10)^2}$$
$$\frac{\partial N}{\partial u} = \frac{-1}{576} + \frac{4}{24} - \frac{75}{576} = \frac{-1+96-75}{576} = \frac{20}{576}$$
$$\boxed{\frac{\partial N}{\partial u} = \frac{5}{144}}$$

$$\frac{\partial N}{\partial v} = \left[\frac{r-q}{(p+r)^2}\right](w) + \left[\frac{1}{p+r}\right](1) + \left[-\frac{p+q}{(p+r)^2}\right](u)$$
$$\frac{\partial N}{\partial v} = \frac{w(r-q)}{(p+r)^2} + \frac{1}{p+r} - \frac{u(p+q)}{(p+r)^2}$$
$$\frac{\partial N}{\partial v} = \frac{4(10-11)}{(14+10)^2} + \frac{1}{14+10} - \frac{2(14+11)}{(14+10)^2}$$
$$\frac{\partial N}{\partial v} = \frac{-4}{576} + \frac{1}{24} - \frac{50}{576} = \frac{-4+24-50}{576} = \frac{-30}{576}$$
$$\boxed{\frac{\partial N}{\partial v} = -\frac{5}{96}}$$

$$\frac{\partial N}{\partial w} = \left[\frac{r-q}{(p+r)^2}\right](v) + \left[\frac{1}{p+r}\right](u) + \left[-\frac{p+q}{(p+r)^2}\right](1)$$
$$\frac{\partial N}{\partial w} = \frac{v(r-q)}{(p+r)^2} + \frac{u}{p+r} - \frac{p+q}{(p+r)^2}$$
$$\frac{\partial N}{\partial w} = \frac{3(10-11)}{(14+10)^2} + \frac{2}{(14+10)} - \frac{14+11}{(14+10)^2}$$
$$\frac{\partial N}{\partial w} = \frac{-3}{576} + \frac{2}{24} - \frac{25}{576} = \frac{-3+48-25}{576} = \frac{20}{576}$$
$$\boxed{\frac{\partial N}{\partial w} = \frac{5}{144}}$$

Use Equation 5 to find $\dfrac{dy}{dx}$

31. $y\cos x = x^2 + y^2$

Solution

$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Let $F = y\cos x - x^2 - y^2$

$$F_x = -y\sin x - 2x$$
$$F_y = \cos x - 2y$$
$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{(-y\sin x - 2x)}{\cos x - 2y} = \frac{2x + y\sin x}{\cos x - 2y}}$$

Use Equations 6 to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

35. $x^2 + 2y^2 + 3z^2 = 1$

Solution

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Let $F = x^2 + 2y^2 + 3z^2 - 1$

$$F_x = 2x \qquad F_y = 4y \qquad F_z = 6z$$
$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{6z} = -\frac{x}{3z}}$$
$$\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{4y}{6z} = -\frac{2y}{3z}}$$

Section 6: Directional Derivatives and the Gradient Vector

5, 7

Find the directional derivative of f at the given point in the direction indicated by the angle θ.

5. $f(x,y) = y\cos(xy), \quad (0,1), \quad \theta = \pi/4$

Solution

$$\begin{aligned}
D_u f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\
D_u f(x,y) &= \left[-y^2\sin(xy)\right]\cos\frac{\pi}{4} + \left[\cos(xy) - xy\sin(xy)\right]\sin\frac{\pi}{4} \\
D_u f(x,y) &= \frac{\sqrt{2}}{2}\left[-y^2\sin(xy) + \cos(xy) - xy\sin(xy)\right] \\
D_u f(0,1) &= \frac{\sqrt{2}}{2}\left[0 + 1 - 0\right] \\
&\boxed{D_u f(0,1) = \frac{\sqrt{2}}{2}}
\end{aligned}$$

7. $f(x,y) = arctan(xy), \quad (2,-3), \quad \theta = 3\pi/4$

Solution

$$\begin{aligned}
D_u f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\
D_u f(x,y) &= \left[\frac{y}{1+(xy)^2}\right]\cos\frac{3\pi}{4} + \left[\frac{x}{1+(xy)^2}\right]\sin\frac{3\pi}{4} \\
D_u f(x,y) &= \frac{\sqrt{2}}{2}\left[\frac{x}{1+(xy)^2} - \frac{y}{1+(xy)^2}\right] \\
D_u f(2,-3) &= \frac{\sqrt{2}}{2}\left[\frac{2}{1+(2(-3))^2} - \frac{-3}{1+(2(-3))^2}\right] \\
D_u f(2,-3) &= \frac{\sqrt{2}}{2}\left[\frac{5}{37}\right] \\
&\boxed{D_u f(2,-3) = \frac{5\sqrt{2}}{74}}
\end{aligned}$$

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$f(x,y) = x/y, \quad P(2,1), \quad \vec{u} = \frac{3}{5}\,\hat{\mathbf{i}} + \frac{4}{5}\,\hat{\mathbf{j}}$

(a) Find the gradient of f

(b) Evaluate the gradient at the point P

(c) Find the rate of change of f at P in the direction of the vector $\vec{u}$

Solution

(a)

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle \quad \text{or} \quad \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}}$$

$$\boxed{\nabla f(x,y) = \langle \frac{1}{y}, -\frac{x}{y^2}\rangle}$$

(b)

$$\boxed{\nabla f(2,1) = \langle \frac{1}{1}, -\frac{2}{1^2}\rangle = \langle 1, -2\rangle}$$

(c)

$$D_u f(x,y) = \nabla f(x,y)\cdot\vec{u}$$
$$D_u f(2,1) = \nabla f(2,1)\cdot\vec{u}$$
$$D_u f(2,1) = \langle 1,-2\rangle\cdot\langle \frac{3}{5}, \frac{4}{5}\rangle$$
$$D_u f(2,1) = \frac{3}{5} - \frac{8}{5}$$
$$\boxed{D_u f(2,1) = -1}$$

13, 15

Find the directional derivative of the function at the given point in the direction of the vector $\vec{v}$.

13. $f(x,y) = e^x \sin y, \quad (0, \pi/3), \quad \vec{v} = \langle -6, 8 \rangle$

Solution

$$\begin{aligned}
\vec{u} &= \frac{\vec{v}}{\|\vec{v}\|} = \frac{-6\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}}}{\sqrt{(6)^2 + (8)^2}} \\
\vec{u} &= -\frac{3}{5}\,\hat{\mathbf{i}} + \frac{4}{5}\,\hat{\mathbf{j}} \\
D_u f(x,y) &= \nabla f(x,y) \cdot \vec{u} \\
D_u f(x,y) &= (e^x \sin y\,\hat{\mathbf{i}} + e^x \cos y\,\hat{\mathbf{j}}) \cdot \left(-\frac{3}{5}\,\hat{\mathbf{i}} + \frac{4}{5}\,\hat{\mathbf{j}}\right) \\
D_u f(x,y) &= -\frac{3e^x \sin y}{5} + \frac{4e^x \cos y}{5} \\
D_u f(0, \frac{\pi}{3}) &= -\frac{3(\frac{\sqrt{3}}{2})}{5} + \frac{4(\frac{1}{2})}{5} \\
&\boxed{D_u f(0, \frac{\pi}{3}) = \frac{4 - 3\sqrt{3}}{10}}
\end{aligned}$$

15. $g(s,t) = s\sqrt{t}, \quad (2,4), \quad \vec{v} = 2\,\hat{\mathbf{i}} - \,\hat{\mathbf{j}}$

Solution

$$\begin{aligned}
\vec{u} &= \frac{\vec{v}}{\|\vec{v}\|} = \frac{2\,\hat{\mathbf{i}} - \,\hat{\mathbf{j}}}{\sqrt{(2)^2 + (-1)^2}} \\
\vec{u} &= \frac{2}{\sqrt{5}}\,\hat{\mathbf{i}} - \frac{1}{\sqrt{5}}\,\hat{\mathbf{j}} \\
D_u g(s,t) &= \nabla g(s,t) \cdot \vec{u} \\
D_u g(s,t) &= \left(\sqrt{t}\,\hat{\mathbf{i}} + \frac{s}{2\sqrt{t}}\right) \cdot \left(\frac{2}{\sqrt{5}}\,\hat{\mathbf{i}} - \frac{1}{\sqrt{5}}\,\hat{\mathbf{j}}\right) \\
D_u g(s,t) &= \frac{2\sqrt{t}}{\sqrt{5}} - \frac{s}{2\sqrt{t}\sqrt{5}} \\
D_u g(s,t) &= \frac{1}{\sqrt{5}}\left(2\sqrt{t} - \frac{s}{2\sqrt{t}}\right) \\
D_u g(2,4) &= \frac{1}{\sqrt{5}}\left(2\sqrt{4} - \frac{2}{2\sqrt{4}}\right) = \frac{1}{\sqrt{5}}\left(4 - \frac{1}{2}\right) \\
&\boxed{D_u g(2,4) = \frac{1}{\sqrt{5}}\left(\frac{8-1}{2}\right) = \frac{7}{2\sqrt{5}}}
\end{aligned}$$

21, 23

Find the directional derivative of the function at the point P in the direction of the point Q.

21. $f(x,y) = x^2y^2 - y^3, \quad P(1,2), \quad Q(-3,5)$

Solution

$$
\begin{aligned}
\vec{v} &= \vec{PQ} = \langle -3-1, 5-2 \rangle = \langle -4, 3 \rangle \\
\vec{u} &= \frac{\vec{v}}{\|v\|} = \frac{\langle -4, 3 \rangle}{\sqrt{16+9}} \\
\vec{u} &= \langle -\frac{4}{5}, \frac{3}{5} \rangle \\
D_u f(x,y) &= \nabla f(x,y) \cdot \vec{u} \\
D_u f(x,y) &= \langle 2xy^2, 2x^2y - 3y^2 \rangle \ \cdot \ \langle -\frac{4}{5}, \frac{3}{5} \rangle \\
D_u f(x,y) &= -\frac{8xy^2}{5} + \frac{3(2x^2y - 3y^2)}{5} = \frac{1}{5}(-8xy^2 + 6x^2y - 9y^2) \\
D_u f(1,2) &= \frac{1}{5}[-8(1)(4) + 6(1)(2) - 9(4)] = \frac{1}{5}[-32 + 12 - 36] \\
&\boxed{D_u f(1,2) = -\frac{56}{5}}
\end{aligned}
$$

23. $f(x,y) = \sqrt{xy}, \quad P(2,8), \quad Q(5,4)$

Solution

$$
\begin{aligned}
\vec{v} &= \vec{PQ} = (5-2)\,\hat{\mathbf{i}} + (4-8)\,\hat{\mathbf{j}} = 3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} \\
\vec{u} &= \frac{\vec{v}}{\|v\|} = \frac{3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}}{\sqrt{9+16}} \\
\vec{u} &= \frac{3}{5}\,\hat{\mathbf{i}} - \frac{4}{5}\,\hat{\mathbf{j}} \\
D_u f(x,y) &= \nabla f(x,y) \cdot \vec{u} \\
D_u f(x,y) &= \left(\frac{y}{2\sqrt{xy}}\,\hat{\mathbf{i}} + \frac{x}{2\sqrt{xy}}\,\hat{\mathbf{j}} \right) \cdot \left(\frac{3}{5}\,\hat{\mathbf{i}} - \frac{4}{5}\,\hat{\mathbf{j}} \right) \\
D_u f(x,y) &= \frac{3y}{10\sqrt{xy}} - \frac{4x}{10\sqrt{xy}} = \frac{3y - 4x}{10\sqrt{xy}} \\
D_u f(2,8) &= \frac{3(8) - 4(2)}{10\sqrt{2(8)}} = \frac{24-8}{10\sqrt{16}} = \frac{16}{40} \\
&\boxed{D_u f(x,y) = \frac{2}{5}}
\end{aligned}
$$

27, 29

Find the maximum rate of change of f at the given point and the direction in which it occurs.

27. $f(x, y) = 5xy^2, \quad (3, -2)$

Solution

The maximum rate of change of f and its direction is given by the gradient vector

$$\begin{aligned} \nabla f(x, y) &= \langle f_x, f_y \rangle = \langle 5y^2, 10xy \rangle \\ \nabla f(3, -2) &= \langle 5(-2)^2, 10(3)(-2) \rangle \end{aligned}$$

Direction of the maximum rate of change,

$$\boxed{\nabla f(3, -2) = \langle 20, -60 \rangle = 20\langle 1, -3 \rangle}$$

Maximum rate of change,

$$\boxed{\|\nabla f(3, -2)\| = 20\sqrt{1 + 9} = 20\sqrt{10}}$$

29. $f(x, y) = \sin(xy), \quad (1, 0)$

Solution

The maximum rate of change of f and its direction is given by the gradient vector

$$\begin{aligned} \nabla f(x, y) &= \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} = y\cos(xy)\,\hat{\mathbf{i}} + x\cos(xy)\,\hat{\mathbf{j}} \\ \nabla f(1, 0) &= 0 + \ \hat{\mathbf{j}} \end{aligned}$$

Direction of the maximum rate of change,

$$\boxed{\nabla f(1, 0) = \ \hat{\mathbf{j}}}$$

Maximum rate of change,

$$\boxed{\|\nabla f(1, 0)\| = \sqrt{1} = 1}$$

37

The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is $120°$

(a) Find the rate of change of T at $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.

(b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points toward the origin

Solution

Since $T \propto \frac{1}{d}$, $T(x, y, z) = \dfrac{c}{\sqrt{x^2 + y^2 + z^2}}$, for some constant c, and Euclidean distance to describe the distance from the origin.

If $120 = T(1, 2, 2) = \dfrac{c}{\sqrt{1^2 + 2^2 + 2^2}} = \dfrac{c}{3}$, then $c = 360$

$\therefore$ Our function for temperature is,

$$T(x, y, z) = \frac{360}{\sqrt{x^2 + y^2 + z^2}}$$

(a)

$$\begin{aligned}
\vec{v} &= \langle 2 - 1, 1 - 2, 3 - 2\rangle = \langle 1, -1, 1\rangle \\
\vec{u} &= \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1, -1, 1\rangle}{\sqrt{1 + 1 + 1}} \\
\vec{u} &= \langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle
\end{aligned}$$

Getting our gradient vector,

$$\begin{aligned}
\nabla T(x, y, z) &= \langle T_x, T_y, T_z\rangle \\
\nabla T(x, y, z) &= \langle -180 \cdot 2x(x^2 + y^2 + z^2)^{-3/2}, -180 \cdot 2y(x^2 + y^2 + z^2)^{-3/2}, -180 \cdot 2z(x^2 + y^2 + z^2)^{-3/2}\rangle \\
\nabla T(x, y, z) &= \frac{360}{(x^2 + y^2 + z^2)^{3/2}}\langle -x, -y, -z\rangle
\end{aligned}$$

so, our rate of change is

$$\begin{aligned}
D_uT(1, 2, 2) &= \nabla T(1, 2, 2) \cdot \vec{u} \\
D_uT(1, 2, 2) &= \frac{360}{(1 + 4 + 4)^{3/2}}\langle -1, -2, -2\rangle \cdot \langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle \\
D_uT(1, 2, 2) &= \frac{360}{27}\left(-\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}}\right) = \frac{360}{27}\left(-\frac{1}{\sqrt{3}}\right) \\
&\boxed{D_uT(1, 2, 2) = \frac{40}{3}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{40}{3\sqrt{3}}}
\end{aligned}$$

(b) The direction of greatest increase at any point is the direction of ∇T at that point. In part (a) we saw that $\nabla T(x, y, z)$ and $\langle -x, -y, -z\rangle$ have the same direction and the vector $\langle -x, -y, -z\rangle$ points towards the origin.

47-51 (odd)

Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

47. $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10, \quad (3,3,5)$

Solution

(a)

$$F(x,y,z) = 2(x-2)^2 + (y-1)^2 + (z-3)^2 - 10$$

$$\begin{aligned} F_x(3,3,5) &= 4(x-2) = 4(1) = 4 \\ F_y(3,3,5) &= 2(y-1) = 2(2) = 4 \\ F_z(3,3,5) &= 2(z-3) = 2(2) = 4 \end{aligned}$$

The equation of the tangent plane is given by,

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0)$$

$$\begin{gathered} 4(x-3) + 4(y-3) + 4(z-5) = 0 \\ x-3+y-3+z-5 = 0 \\ \boxed{x+y+z = 11} \end{gathered}$$

(b) The equation of the normal line is given by,

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

$$\begin{gathered} \frac{x-3}{4} = \frac{y-3}{4} = \frac{z-5}{4} \\ \boxed{x-3 = y-3 = z-5} \end{gathered}$$

49. $xy^2z^3 = 8, \quad (2,2,1)$

Solution

(a)

$$F(x,y,z) = xy^2z^3 - 8$$

$$\begin{aligned} F_x(2,2,1) &= y^2z^3 = (4)(1) = 4 \\ F_y(2,2,1) &= 2xyz^3 = 2(2)(2)(1) = 8 \\ F_z(2,2,1) &= 3xy^2z^2 = 3(2)(4)(1) = 24 \end{aligned}$$

The equation of the tangent plane is given by,

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0)$$

$$\begin{gathered} 4(x-2) + 8(y-2) + 24(z-1) = 0 \\ x-2+2(y-2)+6(z-1) = 0 \\ x-2+2y-4+6z-6 = 0 \\ \boxed{x+2y+6z = 12} \end{gathered}$$

(b) The equation of the normal line is given by,

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

$$\frac{x-2}{4} = \frac{y-2}{8} = \frac{z-1}{24}$$
$$\boxed{x-2 = \frac{y-2}{2} = \frac{z-1}{6}}$$

51. $x+y+z = e^{xyz}, \quad (0,0,1)$

Solution

(a)

$$F(x,y,z) = x+y+z-e^{xyz}$$

$$F_x(0,0,1) = 1 - yze^{xyz} = 1-0 = 1$$
$$F_y(0,0,1) = 1 - xze^{xyz} = 1-0 = 1$$
$$F_z(0,0,1) = 1 - xye^{xyz} = 1-0 = 1$$

The equation of the tangent plane is given by,

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0)$$

$$1(x-0) + 1(y-0) + 1(z-1) = 0$$
$$\boxed{x+y+z = 1}$$

(b) The equation of the normal line is given by,

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

$$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-1}{1}$$
$$\boxed{x = y = z-1}$$

Section 7: Maximum and Minimum Values

5-21 (odd)

Find the local maximum and minimum values and saddle point(s) of the function. You are encouraged to use a calculator or computer to graph the function with a domain and viewpoint that reveals all the important aspects of the function.

5. $f(x, y) = x^2 + xy + y^2 + y$

Solution

$$\nabla f(x, y) = \langle 2x + y,\ x + 2y + 1\rangle$$

Finding critical values, (system of two equations)

$$\begin{aligned} 2x + y &= 0 \\ x + 2y &= -1 \end{aligned}$$

$$\begin{aligned} 2x + y &= 0 \\ -2x - 4y &= 2 \\ -3y &= 2 \\ y &= -\frac{2}{3} \quad \Rightarrow \quad x = \frac{1}{3} \end{aligned}$$

Critical point at $\left(\frac{1}{3}, -\frac{2}{3}\right)$

Hessian matrix to determine maxima and minima,

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 1$$

$$\begin{aligned} H_f(x, y) &= det \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \\ H_f\left(\frac{1}{3}, \frac{-2}{3}\right) &= det \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 \\ H_f\left(\frac{1}{3}, -\frac{2}{3}\right) &> 0 \text{ and } f_{xx} = 2 > 0 \end{aligned}$$

$$\boxed{\therefore f\left(\frac{1}{3}, -\frac{2}{3}\right) \text{ is a local minimum}}$$

7. $f(x, y) = 2x^2 - 8xy + y^4 - 4y^3$

Solution

$$\nabla f(x, y) = \langle 4x - 8y,\ -8x + 4y^3 - 12y^2\rangle$$

Finding critical values,

$$
\begin{aligned}
0 &= = 4x - 8y \\
2x &= 8y \\
x &= 2y
\end{aligned}
$$

$$
\begin{aligned}
0 &= -8x + 4x^3 - 12y^2 \\
0 &= -8(2y) + 4y^3 - 12y^2 \\
0 &= 4y^3 - 12y^2 - 16y \\
0 &= 4y(y^2 - 3y - 4) \\
0 &= 4y(y-4)(y+1) \\
y = -1, 0, 4 \quad &\Rightarrow \quad x = -2, 0, 8
\end{aligned}
$$

Critical points at $(-2,-1), (0,0), (8,4)$

$$f_{xx} = 4, \quad f_{yy} = 12y^2 - 24y, \quad f_{xy} = -8$$

$$
\begin{aligned}
H_f(8,4) &= det \begin{vmatrix} 4 & -8 \\ \\ -8 & 96 \end{vmatrix} = 384 - 64 = 320 \\
H_f(-2,-1) &= det \begin{vmatrix} 4 & -8 \\ \\ -8 & 36 \end{vmatrix} = 144 - 64 = 80 \\
H_f(0,0) &= det \begin{vmatrix} 4 & -8 \\ \\ -8 & 0 \end{vmatrix} = 0 - 64 = -64
\end{aligned}
$$

$H_f(8,4) > 0$ and $f_{xx} = 4 > 0$ so f(8,4) is a local minima
$H_f(-2,-1) > 0$ and $f_{xx} = 4 > 0$ so f(8,4) is a local minima
$H_f(0,0)$ so f(8,4) is a saddle point

$\boxed{\text{Minima: } f(8, 4 = -128), \quad f(-2,-1) = 3}$

$\boxed{\text{Saddle Point: } f(0,0)}$

9. $f(x,y) = (x-y)(1-xy)$

Solution

$$\nabla f(x,y) = \langle 1 - 2xy + y^2,\ -x^2 - 1 + 2xy \rangle$$

Finding critical values,

$$\begin{aligned}
&0 = 1 - 2xy + y^2 \\
&0 = 1 - y(2x + y) \\
&1 = y(2x + y) \\
&y = 1, \quad 2x + y = 1 \quad \Rightarrow \quad y = 1 - 2x
\end{aligned}$$

$$\begin{aligned}
&0 = -x^2 - 1 + 2xy \\
&0 = -x^2 - 1 + 2x(1 - 2x) \\
&0 = -x^2 - 1 + 2x - 4x \\
&0 = -x^2 - 2x - 1 \\
&0 = x^2 + 2x + 1 \\
&0 = (x + 1)(x + 1) \\
&x = -1 \quad \Rightarrow \quad y = 1 - 2(-1) = -1 \quad \Rightarrow \quad (-1, -1)
\end{aligned}$$

If $y = 1$,

$$\begin{aligned}
&0 = -x^2 + 2x - 1 \\
&0 = x - 2x + 1 \\
&0 = (x - 1)(x - 1) \\
&x = 1 \quad \Rightarrow \quad (1, 1)
\end{aligned}$$

Critical points at (-1, -1), (1, 1)

$$f_{xx} = -2y, \quad f_{yy} = 2x, \quad f_{xy} = -2x + 2y$$

$$H_f(-1, -1) = det \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 - 0 = -4$$

$$H_f(1, 1) = det \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -4 - 0 = -4$$

$H_f(-1, -1)$ and $H_f(1, 1)$ are both < 0, so they are saddle points.

$\boxed{\text{Saddle Points: } f(-1, -1), f(1, 1)}$

11. $f(x, y) = y\sqrt{x} - y^2 - 2x + 7y$

Solution

13. $f(x, y) = x^3 - 3x + 3xy^2$

15. $f(x, y) = x^4 - 2x^2 + y^3 - 3y$

17. $f(x, y) = xy - x^2y - xy^2$

19. $f(x, y) = e^x \cos y$

21. $f(x, y) = y^2 - 2y \cos x, \quad -1 \leq x \leq 7$

33-39 (odd)

Find the absolute maximum and minimum values of f on the set D.

33. $f(x, y) = x^2 + y^2 - 2x$,

D is the closed triangular region with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$

35. $f(x, y) = x^2 + y^2 + x^2y + 4$,

$D = \{(x, y) \mid |x| \leq 1,\ |y| \leq 1\}$

37. $f(x, y) = x^2 + 2y^2 - 2x - 4y + 1$,

$D = \{(x, y) \mid 0 \leq x \leq 2,\ 0 \leq y \leq 3\}$

39. $f(x, y) = 2x^3 + y^4$,

$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$

43

Find the shortest distance from the point $(2, 0, -3)$ to the plane $x + y + z = 1$.

45

Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

47

Find three positive numbers whose sum is 100 and whose product is a maximum.

55

A cardboard box without a lid is to have a volume of 32,000 cm^3. Find the dimensions that minimize the amount of cardboard used.

Section 8: Lagrange Multipliers

3-13 (odd)

Each of these extreme value problems has a solution with both a maximum value and minimum value. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint.

3. $f(x, y) = x^2 - y^2, \quad x^2 + y^2 = 1$

5. $f(x, y) = xy, \quad 4x^2 + y^2 = 8$

7. $f(x, y) = 2x^2 + 6y^2, \quad x^4 + 3y^4 = 1$

9. $f(x, y, z) = 2x + 2y + z, \quad x^2 + y^2 + z^2 = 9$

11. $f(x, y, z) = xy^2z, \quad x^2 + y^2 + z^2 = 4$

13. $f(x, y, z) = x^2 + y^2 + z^2, \quad x^4 + y^4 + z^4 = 1$

23

The method of Lagrange multipliers assumes that the extreme values exist, but that is not always the case. Show that the problem of finding the minimum value of f subject to the given constraint can be solved using Lagrange multipliers, but f does not have a maximum value with that constraint.

$$f(x, y) = x^2 + y^2, \quad xy = 1$$

25

Use Lagrange multipliers to find the maximum value of f subject to the given constraint. Then show that f has no minimum value with that constraint.

$$f(x, y) = e^{xy}, \quad x^3 + y^3 = 16$$

27, 29

Find the extreme values of f on the region described by the inequality.

27. $f(x, y) = x^2 + y^2 + 4x - 4y, \quad x^2 + y^2 \leq 9$

29. $f(x, y) = e^{-xy}, \quad x^2 + 4y^2 \leq 1$

31, 33

Find the extreme values of f subject to both constraints

31. $f(x, y, z) = x + y + z; \quad x^2 + z^2 = 2, \quad x + y = 1$

33. $f(x, y, z) = yz + xy; \quad xy = 1, \quad y^2 + z^2 = 1$
