Ch 14 - Problem Set 1

Calculus 3

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Section 1: Functions of Several Variables

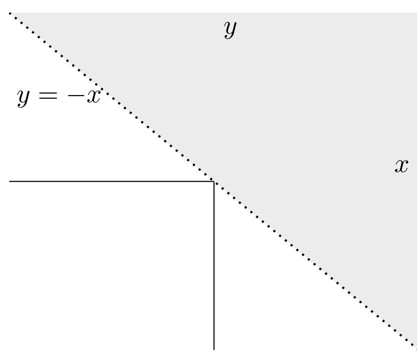
3. Let $g(x, y) = x^2 \ln(x + y)$
(*a*) Evaluate $g(3, 1)$**.**
(*b*) Find and sketch the domain of g**.**
(*c*) Find the range of g**.**

Solution

a) $9 \ln 4$

b) $D : \{(x, y) \mid y > -x\}$

Domain of g



c) $\mathbb{R}$

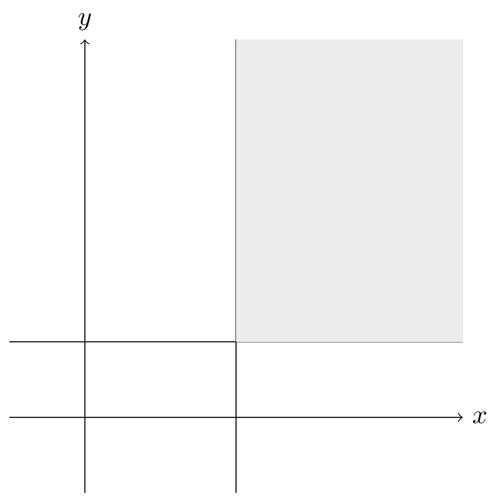
7 - 15 (odd)

Find and sketch the domain of the function.

7. $f(x, y) = \sqrt{x - 2} + \sqrt{y - 1}$

Solution

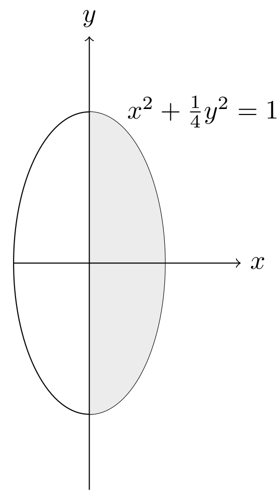
$D : \{(x, y) \mid x \geq 2, y \geq 1\}$



9. $q(x,y) = \sqrt{x} + \sqrt{4 - 4x^2 - y^2}$

Solution

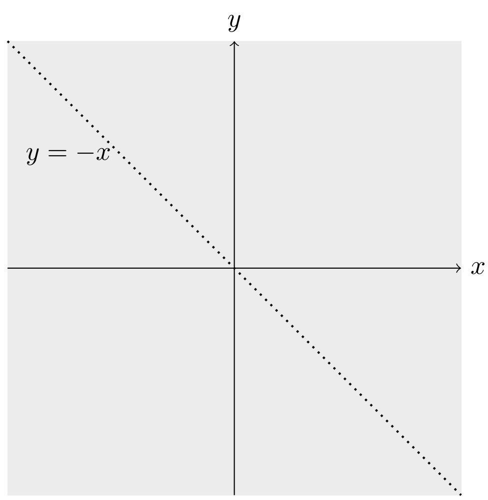
$D : \{(x,y) \mid x^2 + \frac{1}{4}y^2 \le 1, x \ge 0\}$



11. $g(x,y) = \dfrac{x-y}{x+y}$

Solution

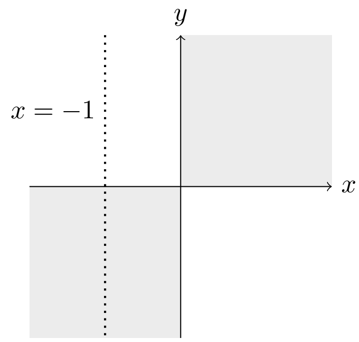
$D : \{(x,y) \mid y \ne -x\}$



13. $p(x,y) = \dfrac{\sqrt{xy}}{x+1}$

Solution

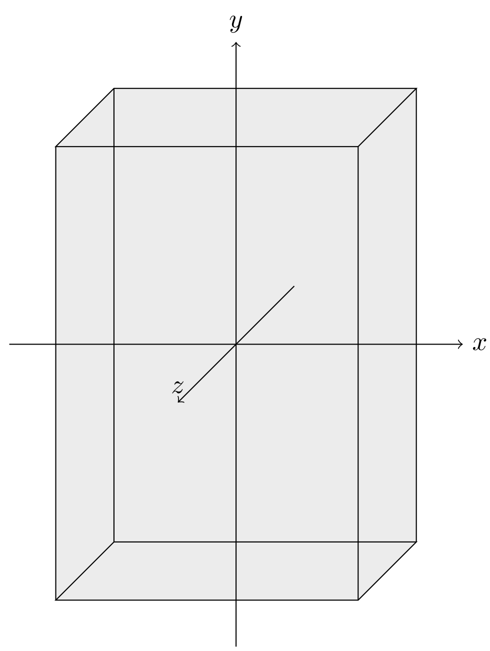
$D : \{(x, y) \mid x \neq 0, xy \geq 0\}$



15. $f(x, y, z) = \sqrt{4 - x^2} + \sqrt{9 - y^2} + \sqrt{1 - z^2}$

Solution

$D : \{(x, y, z) \mid -2 \leq x \leq 2, -3 \leq y \leq 3, -1 \leq z \leq 1\}$



17. A model for the surface area of a human body is given by the function

$$S = f(w, h) = 0.1091 w^{0.425} h^{0.725}$$

where w is the weight (in pounds), h is the height (in inches), and S is measured in square feet.
(a) Find $f(160, 70)$ and interpret it.
(b) What is your own surface area?

Solution

a)

$$f(160,70) = 0.1091(160^{0.425})(70^{0.725}) \approx 20.5$$

The surface area of a human body that weighs 160 pounds and is 70 inches tall is about 20.5 square feet.

b)

$$f(160,68) = 0.1091(160^{0.425})(70^{0.725}) \approx 20.5$$

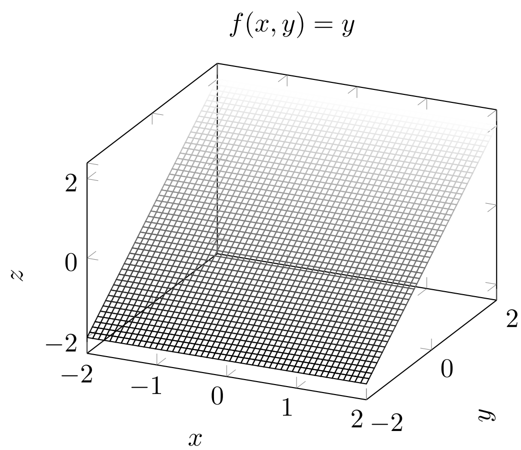
23 - 31 (odd)

Sketch the graph of the function

23. $f(x, y) = y$

Solution

This is an equation of the plane that goes through the origin and is parallel to the x-axis.



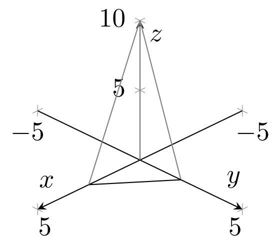
25. $f(x, y) = 10 - 4x - 5y$

Solution

Let $x = y = 0 \quad \Rightarrow \quad z = 10, \quad x = z = 0 \quad \Rightarrow \quad y = 2, \quad y = z = 0 \quad \Rightarrow \quad x = 2.5$

This is an equation of a plane that goes through the points $(0,0,10), (0,2,0), (2.5,0,0)$ [imagine it is shaded in].

$$f(x,y) = 10 - 4x - 5y$$

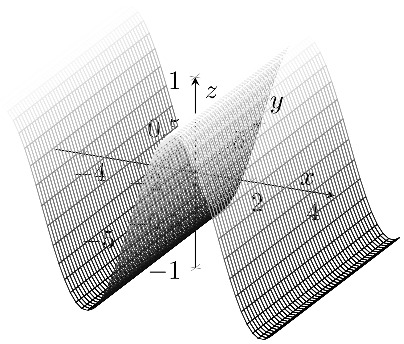


27. $f(x,y) = \sin x$

Solution

This is an equation of a cylinder that goes through the origin and is parallel to the x-axis

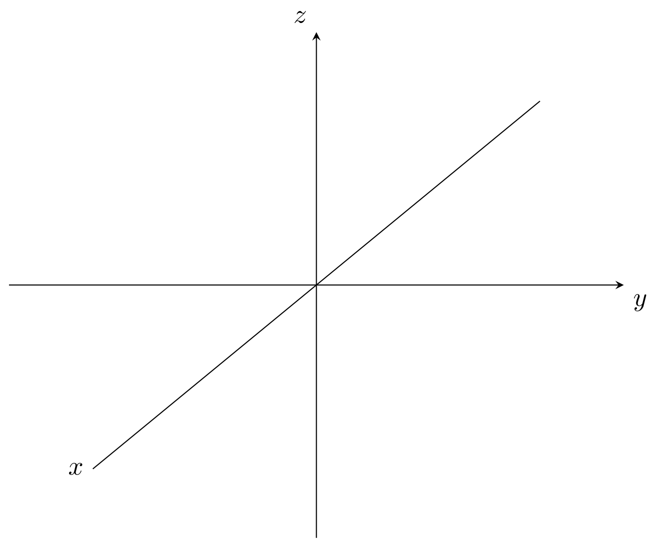
$$f(x,y) = \sin x$$



29. $f(x,y) = x^2 + 4y^2 + 1$

Solution

This is an equation of an elliptic paraboloid that goes through the origin and is parallel to the z-axis.

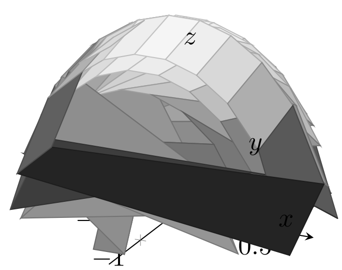


31. $f(x,y) = \sqrt{4 - 4x^2 - y^2}$

Solution

This is the top half of ellipsoid

$$f(x,y) = \sqrt{4 - 4x^2 - y^2}$$



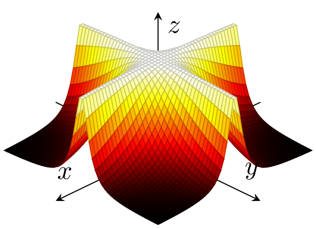
32. Match the function with its graph (labeled I–VI). Give reasons for your choices.

(***a***) $f(x,y) = \dfrac{1}{1+x^2+y^2}$ (***b***) $f(x,y) = \dfrac{1}{1+x^2y^2}$

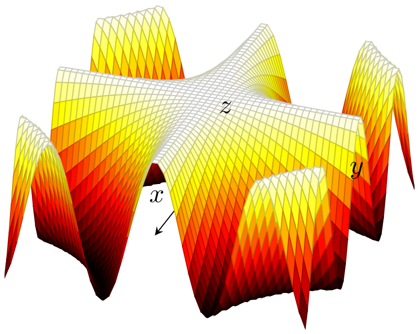
(***c***) $f(x,y) = \ln(x^2+y^2)$ (***d***) $f(x,y) = \cos\sqrt{x^2+y^2}$

(***e***) $f(x,y) = |xy|$ (***f***) $f(x,y) = \cos(xy)$

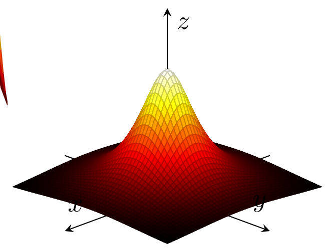
I



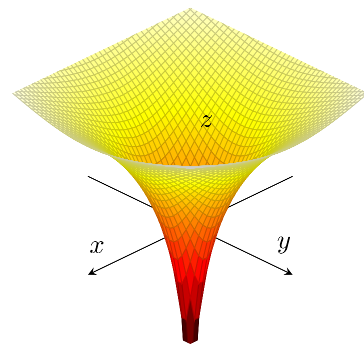
II



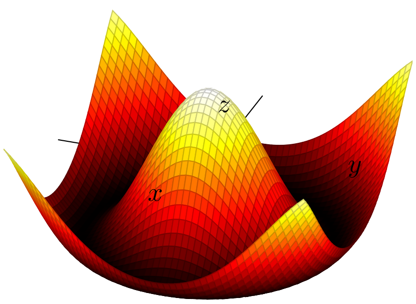
III



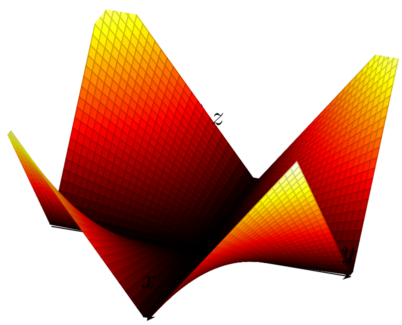
IV



V



VI



Solution

a)

The graph of $f(x,y) = \dfrac{1}{1+x^2+y^2}$ is III

When $x = y = 0 \quad \Rightarrow \quad z = 1$, so the graph intersects the z-axis at $(0,0,1)$.

If we solve for the zx and zy planes we get $z = \dfrac{1}{1+x^2}$ and $z = \dfrac{1}{1+y^2}$ respectively.

b)

The graph of $f(x,y) = \dfrac{1}{1+x^2y^2}$ is I

When $x = y = 0 \quad \Rightarrow \quad z = 1$, so the graph intersects the z-axis at $(0, 0, 1)$.

Let $x = 1$ and then we solve for $z = \lim_{y\to\infty} \dfrac{1}{1+y^2} = 0$. For graph I, if we gauge the $x = 1$ position and move up the y axis, we can see that z does indeed approach a value like 0.

c)

The graph of $f(x, y) = \ln(x^2 + y^2)$ is IV

When $x = y = 0$ z is undefined

The only graph that seems to have a hole at the origin is IV.

d)

The graph of $f(x, y) = \cos\sqrt{x^2 + y^2}$ is V

When $x = y = 0 \quad \Rightarrow \quad z = 1$, so the graph intersects the z-axis at $(0, 0, 1)$.

When $x = 0$ and $y = 0$ then $z = \cos y$ and $z = \cos x$ respectively.

The only graph that has a point at $(0, 0, 1)$ and has sinusoidal movement when $(0, (\text{y or x}) \to \infty, -1 \leq z \leq 1)$ is V.

e)

The graph of $f(x, y) = |xy|$ is VI

When $x = y = 0 \quad \Rightarrow \quad z = 0$, so the graph intersects the z-axis at $(0, 0, 0)$.

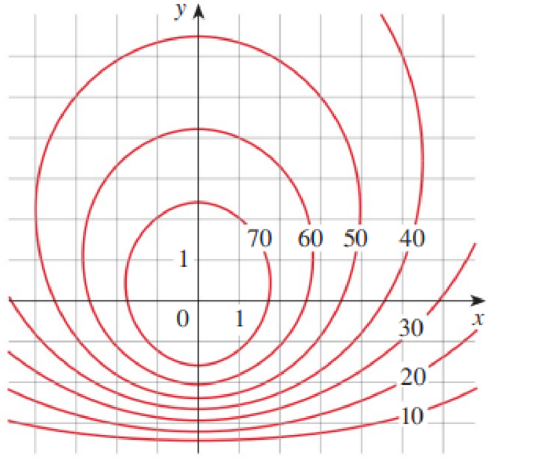
Out of the remaining graphs, the only graph that seems like it has an intersection at the origin is VI.

f)

The graph of $f(x, y) = \cos(xy)$ is II

Process of elimination :) (please don't dock me points for this)

33. A contour map for a function f is shown. Use it to estimate the values of $f(-3, 3)$ and $f(3, -2)$. What can you say about the shape of the graph?



Solution

Looking at the contour map, it seems that $f(-3, 3)$ is ≈ 56 because it is between the 50 and 60 but a little closer to the 60.

$f(3, -2)$ seems like it is ≈ 35 because it is in the middle of 40 and 35

The shape of the graph seems like a hill or the top half of an ellipsoid

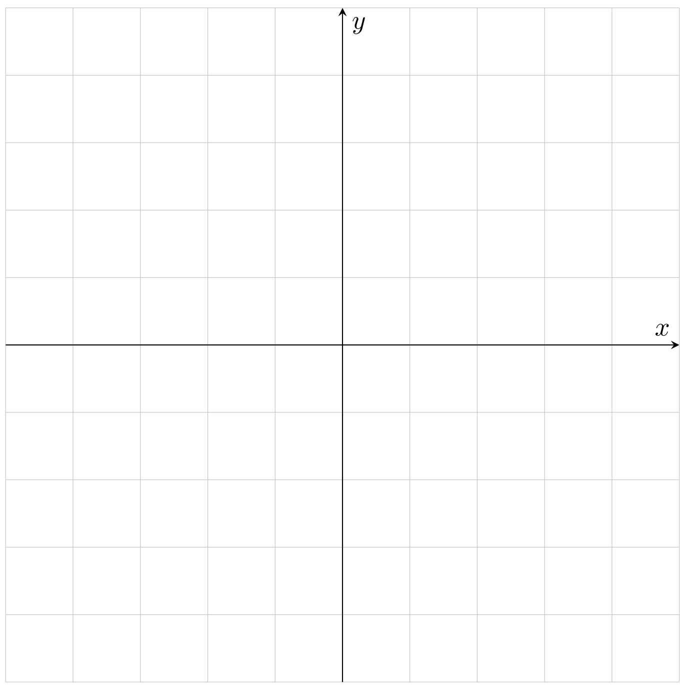
45, 47 & 51

Draw a contour map of the function showing several level curves.

45. $f(x,y) = x^2 - y^2$

Solution

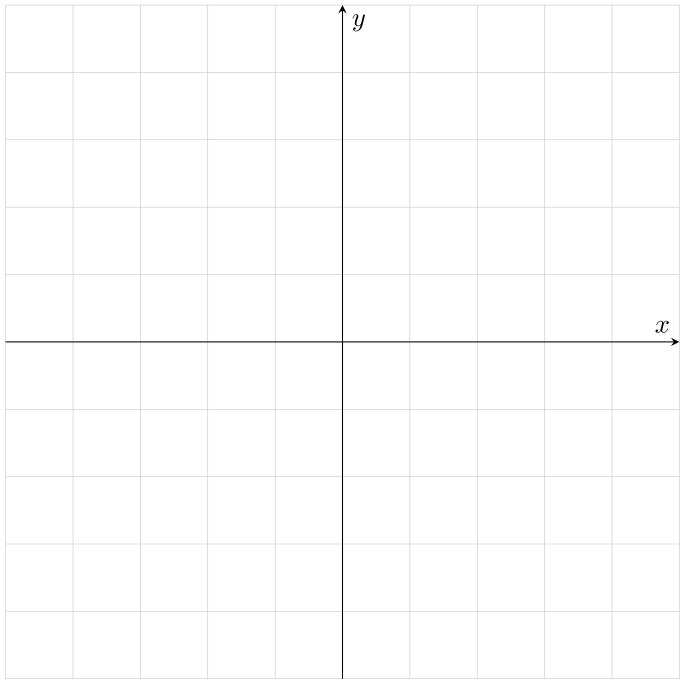
Contour map $f(x,y) = x^2 - y^2$



47. $f(x,y) = \sqrt{x} + y$

Solution

Contour map $f(x,y) = \sqrt{x} + y$



51. $f(x,y) = \sqrt{x^2 + y^2}$

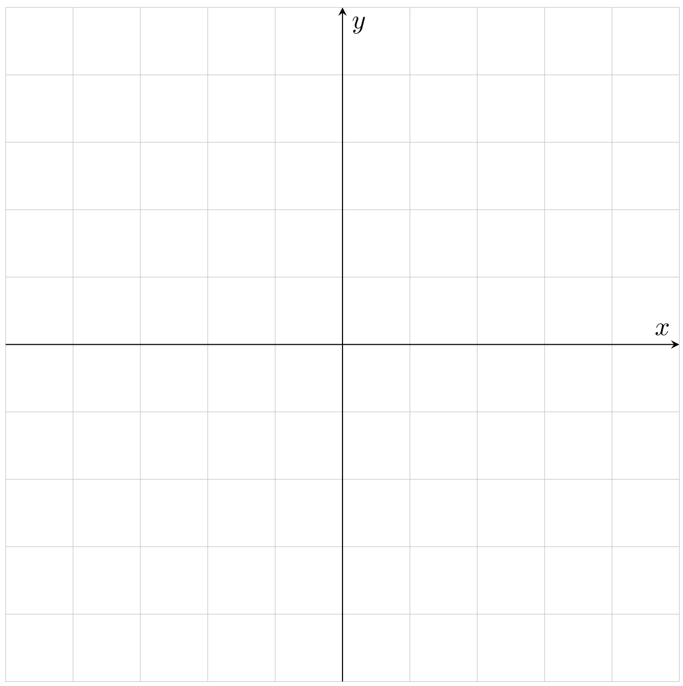
Solution

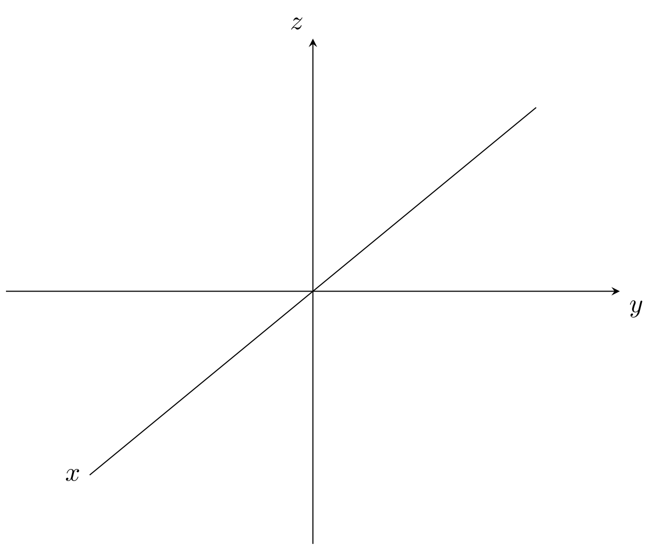
53. Sketch both a contour map and a graph of the given function and compare them.

$$f(x, y) = x^2 + 9y^2$$

Solution

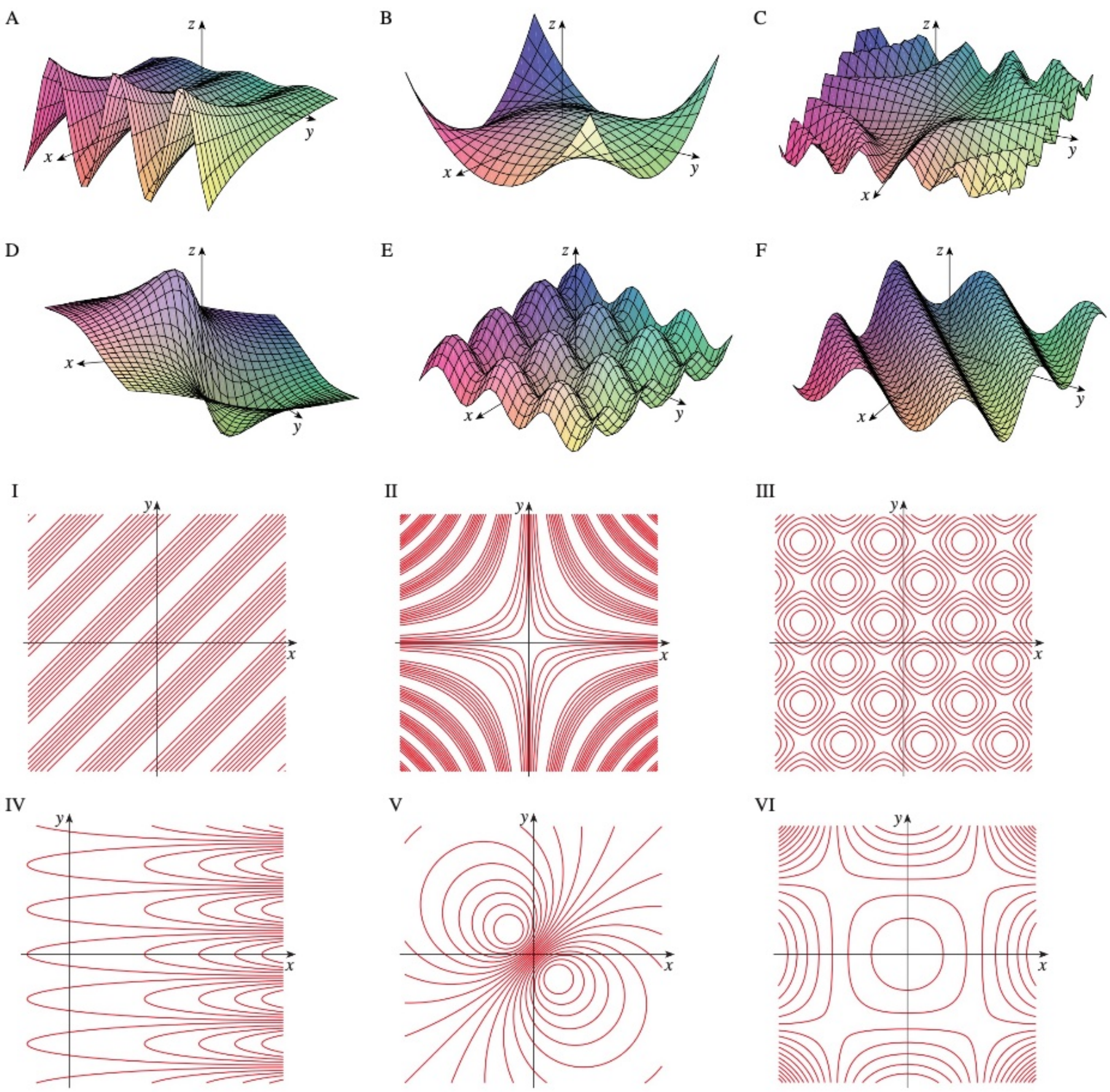
Contour map $f(x, y) = x^2 + 9y^2$;





61 - 66

Match the function (a) with its graph (labeled A–F below) and (b) with its contour map (labeled I–VI). Give reasons for your choices.



61. $z = \sin(xy)$

Solution

It seems like the graph of $z = \sin(xy)$ is C

When $x \to \infty$ and $y \to \infty$ then $-1 \leq z \leq 1$.

In other words, at $45°, 135°, 235°, 315°$ in terms of xy, z should be infinitely sinusoidal the farther you go out

Since the function of z is sin then the graph must intersect the z at origin

For the contour map, the graph that looks that follows this description is II

62. $z = e^x \cos y$

Solution

When $x = y = 0$ then $z = 1 \quad \Rightarrow \quad (0, 0, 1)$.

Setting $x = 0 \quad \Rightarrow \quad z = \cos y \quad \Rightarrow \quad (0, y \to \infty, -1 \leq z \leq 1)$ this just means that x is constant and as y increases/decreases towards either positive or negative infinity, z will be sinusoidal

Setting $y = 0 \quad \Rightarrow \quad z = e^x \quad \Rightarrow \quad (0, y \to \infty^+, \infty^+)(0, y \to \infty^-, 0)$ this just means that y is constant and depending if x is increasing or decreasing, z will increase exponentially to infinity or approach 0

$\therefore$ The graph that seems to follow this description is A and the associated contour map seems to be IV

63. $z = \sin(x - y)$

Solution

The graph would have an intersection at the origin $x = y = 0 \quad \Rightarrow \quad z = 0$

When $x = 0 \quad \Rightarrow \quad z = \sin(-y)$, so the function will first dip down to $z = -1$ in the zy-trace

When $y = 0 \quad \Rightarrow \quad z = \sin(x)$, so the function will first go up to $z = 1$ in the zx-trace

$\therefore$ The graph that matches this description looks like F and the associated contour map seems to be I

64. $z = \sin x - \sin y$

Solution

The graph would have an intersection at the origin $x = y = 0 \quad \Rightarrow \quad z = 0$

$z = 1 - 1 = 0 \quad \leftrightarrow \quad \text{x} = \text{y} = \text{n} \cdot \frac{\pi}{2}, \{n \in \mathbb{Z} \mid n = 2k - 1, k \in \mathbb{Z}\}$

$z = 0 - 0 = 0 \quad \leftrightarrow \quad \text{x} = \text{y} = \text{n} \cdot \frac{\pi}{2}, \{n \in \mathbb{Z} \mid n = 2k, k \in \mathbb{Z}\}$

This behavior appears symmetric with $z = 0$ appearing at areas where $x = y$.

$\therefore$ The graph that best matches this behavior is E and the associated contour map would be III

65. $z = (1 - x^2)(1 - y^2)$

Solution

The graph would have an intersection at $(0, 0, 1) \quad x = y = 0 \quad \Rightarrow \quad z = 1$

If we look at the zy and zx traces, we see that it is a parabola opening down to negative z

However, if we take $\lim_{(x,y)\to(\infty,\infty)}(1 - x^2)(1 - y^2), \quad \{(x, y) \mid x = y\}$ then we get ∞ where the graph, in this direction, would exponentially grow.

$\therefore$ The graph that best fits this description would be B and the associated contour map would be VI.

66. $z = \dfrac{x - y}{1 + x^2 + y^2}$

Solution

The graph would have an intersection at $(0, 0, 0) \quad x = y = 0 \quad \Rightarrow \quad z = 0$

Using process of elimination, the only graph left would be D and the associated contour map would be V.

We could note its behavior in the zy trace and see that $lim_{y\to\infty^+} \frac{-y}{1+y^2} = 0$ with z decreasing at first, vice versa with zx trace

To find the point at which z is at a minimum when $y \to \infty^+$,

$$\begin{aligned}
z &= -y(1 + y^2)^{-1} \\
\frac{dz}{dy} &= -\frac{1}{1 + y^2} + \frac{2y^2}{(1 + y^2)^2} \\
0 &= -\frac{1}{1 + y^2} + \frac{2y^2}{(1 + y^2)^2} \\
\frac{1}{1 + y^2} &= \frac{2y^2}{(1 + y^2)^2} \\
1 &= \frac{2y^2}{1 + y^2} \\
1 + y^2 &= 2y^2 \\
1 &= y^2 \\
y &= \pm 1 = +1
\end{aligned}$$

And it seems there would be a maximum at $y = -1$

67. Describe the level surfaces of the function.

$$f(x, y, z) = 2y - z + 1$$

Solution

If we rearrange the function, $x - 2y + z = 1$ and we see that this is an equation of a plane. If we substitute k for 1 and play around with its value (choosing 3-5 vals for k) then we see that no matter which value we pick, the planes will be parallel

Section 2: Limits and Continuity

5 - 11 (odd)

Find the limit

5. $\lim_{(x,y)\to(3,2)}(x^2y^3 - 4y^2)$

Solution

Using direct substitution,

$$(9)(8) - 4(4) = 72 - 16 = 56$$

7. $\lim_{(x,y)\to(-3,1)} \dfrac{x^2y - xy^3}{x - y + 2}$

Solution

Using direct substitution,

$$\frac{9(1) + 3(1^3)}{-3 - 1 + 2} = \frac{12}{-2} = -6$$

9. $\lim_{(x,y)\to(\pi,\pi/2)} y\sin(x - y)$

Solution

Using direct substitution,

$$\frac{\pi}{2}\sin(\pi - \frac{\pi}{2}) = \frac{\pi}{2}\sin(\frac{\pi}{2}) = \frac{\pi}{2}(1) = \frac{\pi}{2}$$

11. $\lim_{(x,y)\to(1,1)} \left(\dfrac{x^2y^3 - x^3y^2}{x^2 - y^2}\right)$

Solution

Let $x = 1$,

$$\lim_{x\to 1}\left(\frac{1^2y^3 - 1^3y^2}{1^2 - y^2}\right) = \lim_{x\to 1}\left(\frac{y^3 - y^2}{1 - y^2}\right) \overset{L'H}{=} \lim_{x\to 1}\frac{3y^2 - 2y}{-2y} = \frac{1}{-2}$$

Let $y = 1$,

$$\lim_{y\to 1}\left(\frac{x^2 1^3 - x^3 1^2}{1^2 - y^2}\right) = \lim_{y\to 1}\left(\frac{x^2 - x^3}{x^2 - 1}\right) \overset{L'H}{=} \lim_{x\to 1}\frac{2x - 3x^2}{2x} = \frac{-1}{2}$$

$\therefore$ The limit $= \frac{1}{2}$

13 - 17 (odd)

Show that the limit does not exist

13. $\lim_{(x,y)\to(0,0)} \dfrac{y^2}{x^2+y^2}$

Solution

Let $x = 0$,

$$\lim_{y\to 0} \frac{y^2}{y^2} = 1$$

Let $y = 0$,

$$\lim_{x\to 0} \frac{0}{x^2} = 0$$

$1 \neq 0 \quad \therefore$ The limit DNE

15. $\lim_{(x,y)\to(0,0)} \dfrac{(x+y)^2}{x^2+y^2}$

Solution

Let $y = x$,

$$\lim_{x\to 0} \frac{(x+x)^2}{x^2+x^2} = \lim_{x\to 0} \frac{4x^2}{2x^2} = 2$$

Let $y = -x$,

$$\lim_{x\to 0} \frac{(0)^2}{x^2+(-x)^2} = \lim_{x\to 0} \frac{0}{2x^2} = 0$$

$2 \neq 0 \therefore$ The limit DNE

17. $\lim_{(x,y)\to(0,0)} \dfrac{y^2 \sin^2 x}{x^4+y^4}$

Solution

Let $y = x$,

$$\lim_{x\to 0} \frac{x^2 \sin^2 x}{x^4+x^4} \quad \Rightarrow \quad \lim_{x\to 0} \frac{x^2 \sin^2 x}{2x^4} = \frac{\sin^2 x}{2x^2}$$
$$\overset{L'H}{=} \frac{2\sin x \cos x}{4x} = \frac{\sin(2x)}{4x} \overset{L'H}{=} \lim_{x\to 0} \frac{2\cos(2x)}{4} = \frac{1}{2}$$

Let y = 0,

$$\lim_{x\to 0} \frac{0 \cdot \sin^2 x}{x^4+0^4} = \frac{0}{x^4} = 0$$

The limit when $y = x \neq$ limit when $y = 0$

$\therefore$ The limit DNE

19 - 25 (odd)

Find the limit, if it exists, or show that the limit does not exist.

19. $\lim_{(x,y)\to(-1,-2)}(x^2y - xy^2 + 3)^3$

Solution

Using direct substitution

$$((-1)^2(-2) - (-1)(-2)^2 + 3)^3 = (5)^3 = 125$$

21. $\lim_{(x,y)\to(2,3)} \dfrac{3x - 2y}{4x^2 - y^2}$

Solution

Using direct substitution

$$\frac{3(2) - 2(3)}{4(2)^2 - (3)^2} = \frac{0}{7} = 0$$

23. $\lim_{(x,y)\to(0,0)} \dfrac{xy^2 \cos y}{x^2 + y^4}$

Solution

Let $x = y^2$

$$\lim_{y\to 0} \frac{y^2y^2 \cos y}{y^4 + y^4} = \frac{y^4 \cos y}{2y^4} = \frac{1}{2}$$

Let $x = 0$

$$\lim_{y\to 0} \frac{0}{y^4} = 0$$

$\therefore$ The limit DNE

25. $\lim_{(x,y)\to(0,0)} \dfrac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

Solution

Rationalizing the denominator,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} \cdot \frac{\sqrt{x^2 + y^2 + 1} + 1}{\sqrt{x^2 + y^2 + 1} + 1} = \frac{(x^2 + y^2)\sqrt{x^2 + y^2 + 1} + 1}{x^2 + y^2 + 1 - 1} = \sqrt{x^2 + y^2 + 1} + 1 = 2$$

31 & 33

Use the Squeeze Theorem to find the limit.

31. $\lim_{(x,y)\to(0,0)} xy\sin\dfrac{1}{x^2+y^2}$

Solution

$$\lim_{(x,y)\to(0,0)} xy\sin\frac{1}{x^2+y^2} \quad\Rightarrow\quad -xy \le xy\sin\frac{1}{x^2+y^2} \le xy$$
$$\lim_{(x,y)\to(0,0)} -xy \le \lim_{(x,y)\to(0,0)} xy\sin\frac{1}{x^2+y^2} \le \lim_{(x,y)\to(0,0)} xy \quad\Rightarrow\quad 0 \le \lim_{(x,y)\to(0,0)} xy\sin\frac{1}{x^2+y^2} \le 0$$

$\therefore$ The limit $= 0$

33. $\lim_{(x,y)\to(0,0)} \dfrac{xy^4}{x^4+y^4}$

Solution

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4} \quad\Rightarrow\quad [x=r\cos\theta, y=r\sin\theta]$$
$$\lim_{(x,y)\to(0,0)} \frac{r\cos\theta r^4\sin\theta}{r^4\cos^4\theta+r^4\sin^4\theta} = \frac{r^5\cos\theta\sin\theta}{r^4(1)} = r^4\cos\theta\sin\theta$$

Using Squeeze theorem,

$$-r^4 \le r^4\cos\theta\sin\theta \le r^4 \quad\Rightarrow\quad 0 \le r^4\cos\theta\sin\theta \le 0$$

$\therefore$ The limit $= 0$

41, 43 & 45

Determine the set of points at which the function is continuous.

41. $F(x,y) = \dfrac{xy}{1+e^{x-y}}$

Solution

We can see that $1+e^{x-y} \ne 0 \quad\Rightarrow\quad e^{x-y} \ne -1$ which can never happen

$\therefore$ the function is continuous in the set $\{(x,y) \mid x\in\mathbb{R}, y\in\mathbb{R}\}$

43. $F(x,y) = \dfrac{1+x^2+y^2}{1-x^2-y^2}$

Solution

$1-x^2-y^2 \ne 0 \quad\Rightarrow\quad x^2+y^2 \ne 1$ so the function is continuous in the set $\{(x,y) \mid x^2+y^2 \ne 1\}$

45. $G(x, y) = \sqrt{x} + \sqrt{1 - x^2 - y^2}$

Solution

We see that $x \geq 0$ and $1 - x^2 - y^2 \geq 0$

$\therefore$ the function is continuous in the set $\{(x, y) \mid x \geq 0, x^2 + y^2 \leq 1\}$

Section 3: Partial Derivatives

9 - 25 (odd)

Find the first partial derivatives of the function.

9. $f(x,y) = x^4 + 5xy^3$

Solution

$$\frac{\partial f}{\partial x} = 4x^3 + 5y^3$$
$$\frac{\partial f}{\partial y} = 15xy^2$$

11. $g(x,y) = x^3 \sin y$

Solution

$$\frac{\partial g}{\partial x} = 3x^2 \sin y$$
$$\frac{\partial g}{\partial y} = 3x^2 \cos y$$

13. $z = \ln(x + t^2)$

Solution

$$\frac{\partial z}{\partial x} = \frac{1}{x + t^2}$$
$$\frac{\partial z}{\partial t} = \frac{2t}{x + t^2}$$

15. $f(x,y) = ye^{xy}$

Solution

$$\frac{\partial f}{\partial x} = y^2 e^{xy}$$
$$\frac{\partial f}{\partial y} = e^{xy} + xye^{xy}$$

17. $g(x,y) = y(x + x^2y)^5$

Solution

$$\frac{\partial g}{\partial x} = 5y(x + x^2y)^4 \cdot (1 + 2xy) \quad \Rightarrow \quad 5y(1 + 2xy) + (x + x^2y)^4$$
$$\frac{\partial g}{\partial y} = (x + x^2y)^5 + 5y(x + x^2y)^4 \cdot x^2 \quad \Rightarrow \quad 5x^2y(x + x^2y)^4 + (x + x^2y)^5$$

19. $f(x,y) = \dfrac{ax + by}{cx + dy}$

Solution

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{c[ax+by]-a[cx+dy]}{[cx+dy]^2} \\
&= \frac{acx+bcy-acx-ady}{[cx+day]^2} = \frac{(ad-bc)y}{[cx+dy]^2} \\
\frac{\partial f}{\partial y} &= \frac{b[cx+dy]-d[ax+by]}{[cx+dy]^2} \\
&= \frac{bcx+bdy-adx-bdy}{[cx+dy]^2} = \frac{(bc-ad)x}{[cx+dy]^2}
\end{aligned}$$

21. $g(u,v) = (u^2v - v^3)^5$

Solution

$$\begin{aligned}
\frac{\partial g}{\partial u} &= 5 \cdot 2uv(u^2v-v^3)^4 = 10uv(u^2-v^3)^4 \\
\frac{\partial g}{\partial v} &= 5(u^2-3v^2)(u^2v-v^3)^4) = 5u^2 - 15v^2(u^2v-v^3)^4
\end{aligned}$$

23. $R(p,q) = \tan^{-1}(pq^2)$

Solution

$$\begin{aligned}
\frac{\partial R}{\partial p} &= \frac{q^2}{1+(pq^2)^2} = \frac{q^2}{1+p^2q^4} \\
\frac{\partial R}{\partial q} &= \frac{2pq}{1+p^2q^4}
\end{aligned}$$

25. $F(x,y) = \int_y^x \cos(e^t)dt$

Solution

First we need to resolve the integral,

$$\begin{aligned}
F(x,y) &= \int_y^x \cos(e^t)dt \\
&= \sin(e^x) - \sin(e^y)
\end{aligned}$$

So our rewritten function would be,

$$F(x,y) = \sin(e^x) - \sin(e^y)$$

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \cos(e^x) \\
\frac{\partial F}{\partial y} &= -\cos(e^y)
\end{aligned}$$

37. Find the indicated partial derivative.

$$R(s,t) = te^{s/t}; \qquad R_t(0,1)$$

Solution

$$\begin{aligned}
R_t(s,t) &= t \cdot \left(-\frac{s}{t^2}e^{s/t}\right) + e^{s/t} \\
&= -\frac{st}{t^2}e^{s/t} + e^{s/t} = e^{s/t}\left(1 - \frac{s}{t}\right) \\
R_t(0,1) &= e^{0/1}(1 - \frac{0}{1}) = 1(1) = 1
\end{aligned}$$

41 & 43

Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$

41. $x^2 + 2y^2 + 3z^2 = 1$

Solution

$$\frac{d}{dx}(x^2 + 2y^2 + 3z^2) = \frac{d}{dx}(1)$$
$$2xdx + 6zdz = 0 \quad \Rightarrow \quad \frac{dz}{dx} = -\frac{x}{3z}$$
$$\frac{d}{dy}(x^2 + 2y^2 + 3z^2) = \frac{d}{dy}(1)$$
$$4ydy + 6zdz = 0 \quad \Rightarrow \quad \frac{dz}{dy} = -\frac{2y}{3z}$$

43. $e^z = xyz$

Solution

$$\frac{d}{dx}e^z = \frac{d}{dx}(xyz) \quad \Rightarrow \quad e^z\frac{dz}{dx} = yz + xy\frac{dz}{dx}$$
$$(e^x - xy)\frac{dz}{dx} = yz$$
$$\frac{dz}{dx} = \frac{yz}{e^x - xy}$$

$$\frac{d}{dy}e^z = \frac{d}{dy}(xyz) \quad \Rightarrow \quad e^z\frac{dz}{dy} = xz + xy\frac{dz}{dy}$$
$$(e^x - xz)\frac{dz}{dy} = xz$$
$$\frac{dz}{dy} = \frac{xz}{e^x - xy}$$

45. Find $\partial z/\partial x$ and $\partial z/\partial y$.

$$(a)\ z = f(x) + g(y); \qquad (b)\ z = f(x+y)$$

Solution

a)

$$\frac{\partial z}{\partial x} = f'(x)$$
$$\frac{\partial z}{\partial y} = g'(y)$$

b)

$$\frac{\partial z}{\partial x} = f'(x+y)$$
$$\frac{\partial z}{\partial y} = f'(x+y)$$

47. Find all the second partial derivatives.

$$f(x, y) = x^4y - 2x^3y^2$$

Solution

$$f_x = 4x^3y - 6x^2y^2 \quad \Rightarrow \quad f_y = x^4 - 4x^3y$$

$$\begin{aligned} f_{xx} &= 12x^2y - 12xy^2, \qquad & f_{yy} &= -4x^3 \\ f_{xy} &= 4x^3 - 12x^2y, \qquad & f_{yx} &= 4x^3 - 12x^2y \end{aligned}$$

57 - 61 (odd)

Find the indicated partial derivative(s).

57. $f(x, y) = x^4y^2 - x^3y; \quad f_{xxx},\ f_{xyx}$

Solution

$$\begin{aligned} f_x &= 4x^3y^2 - 3x^2y \\ f_{xx} &= 12x^2y^2 - 6xy \\ f_{xxx} &= 24xy^2 - 6y \end{aligned}$$

$$\begin{aligned} f_{xy} &= 8x^3y - 3x^2 \\ f_{xyx} &= 24xy - 6x \end{aligned}$$

59. $f(x, y, z) = e^{xyz^2}; \quad f_{xyz}$

Solution

$$\begin{aligned} f_x &= yz^2e^{xyz^2} \\ f_{xy} &= z^2e^{xyz^2} + yz^2 \cdot xz^3r^{xyz^2} = z^2e^{xyz^2} + xyz^4e^{xyz^2} \\ f_{xyz} &= [2ze^{xyz^2} + 2xyz^3e^{xyz^2}] + [4xyz^3e^{xyz^2} + 2x^2y^2z^5e^{xyz^2}] \quad \text{[product rule x2]} \\ &= e^{xyz^2}(2z + 2xyz^3) + e^{xyz^2}(4xyz^3 + 2x^2y^2z^5) \\ &= e^{xyz^2}(2x^2y^2z^5 + 6xyz^3 + 2z) \end{aligned}$$

61. $W = \sqrt{u + v^2}; \quad \dfrac{\partial^3 W}{\partial u^2 \partial v}$

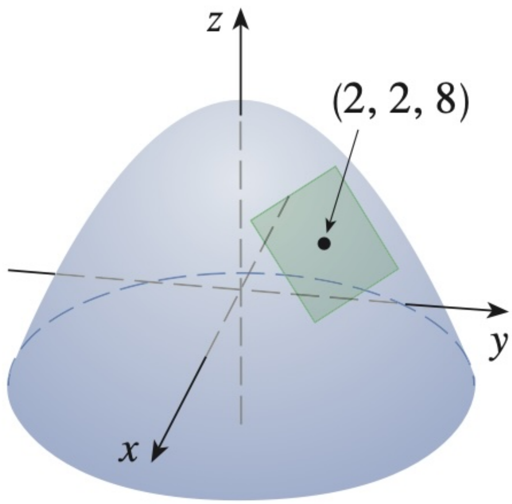
Solution

$$\begin{aligned} W_u &= \frac{1}{2}(u + v^2)^{-1/2} \\ W_{uu} &= -\frac{1}{4}(u + v^2)^{-3/2} \\ W_{uuv} &= \frac{3}{8}(u + v^2)^{-5/2} \cdot 2v = \frac{3}{4}v(u + v^2)^{-5/2} \end{aligned}$$

Section 4: Tangent Planes and Linear Approximations

1. The graph of a function f is shown. Find an equation of the tangent plane to the surface $z = f(x, y)$ at the specified point

$$f(x, y) = 16 - x^2 - y^2$$



$$z = 16 - x^2 - y^2$$

Solution

Let the graph of a surface $z = f(z, y)$ at the point $P(x_0, y_0, z_0)$ be

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Then, if $P(2, 2, 8)$

$$\begin{aligned} f_x = -2x \quad &\Rightarrow \quad f_x(x_0, y_0, z_0) = -4 \\ f_y = -2y \quad &\Rightarrow \quad f_y(x_0, y_0, z_0) = -4 \\ z - 8 &= -4(x - 2) - 4(y - 2) \\ z &= -4x - 4y + 24 \end{aligned}$$

3 - 9 (odd)

Find an equation of the tangent plane to the given surface at the specified point.

3. $z = 2x^2 + y^2 - 5y, \quad (1, 2, -4)$

Solution

Finding the partials,

$$\begin{aligned} f_x = 4x^2 \quad &\Rightarrow \quad f_x(1,2) = 4 \\ f_y = 2y - 5 \quad &\Rightarrow \quad f_y(1,2) = 4 - 5 = -1 \end{aligned}$$

Eq of tangent plane,

$$\begin{aligned} z - z_0 &= f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ z + 4 &= 4(x - 1) - (y - 2) \\ z &= 4x - y - 2 - 4 = 4x - y - 6 \end{aligned}$$

5. $z = e^{x-y}, \quad (2, 2, 1)$

Solution

Finding the partials,

$$\begin{aligned} f_x = e^{x-y} \quad &\Rightarrow \quad f_x(2,2) = e^0 = 1 \\ f_y = -e^{x-y} \quad &\Rightarrow \quad f_y(2,2) = -e^0 = -1 \end{aligned}$$

Eq of tangent plane,

$$\begin{aligned} z - z_0 &= f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ z - 1 &= (x - 2) - (y - 2) \\ z &= x - y + 1 \end{aligned}$$

7. $z = 2\sqrt{y}/x, \quad (-1, 1, -2)$

Solution

Finding the partials,

$$\begin{aligned} f_x = \frac{-2\sqrt{y}}{x^2} \quad &\Rightarrow \quad f_x(-1,1) = -2/1 = -2 \\ f_y = \frac{1}{\sqrt{y}} \cdot \frac{1}{x} \quad &\Rightarrow \quad f_y(-1,1) = 1/1 \cdot 1/-1 = -1 \end{aligned}$$

Eq of tangent plane,

$$\begin{aligned} z - z_0 &= f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ z + 2 &= -2(x + 1) - (y - 1) \\ z &= -2x - y - 1 - 2 = -2x - y - 3 \end{aligned}$$

9. $z = x\sin(x + y), \quad (-1, 1, 0)$

Solution

Finding the partials,

$$\begin{aligned} f_x = \sin(x+y) + x\cos(x+y) \quad &\Rightarrow \quad f_x(-1,1) = \sin 0 + -1\cos 0 = 0 - 1 = -1 \\ f_y = x\cos(x+y) \quad &\Rightarrow \quad f_y(-1,1) = -1\cos 0 = -1 \end{aligned}$$

Eq of tangent plane,

$$
\begin{aligned}
z - z_0 &= f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
z &= -1(x+1) - (-1)(y-1) \\
z &= -x - y
\end{aligned}
$$

15 - 19 odd

Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that point.

15. $f(x, y) = x^3 y^2, \quad (-2, 1)$

Solution

$$
\begin{aligned}
f_x &= 3x^2y^2 \text{ which is differentiable at (-2,1)} \\
f_y &= 2x^3y \text{ which is differentiable at (-2,1)}
\end{aligned}
$$

Linearization,

$$
\begin{aligned}
L(x, y) &\approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
L(-2, 1) &\approx -8 + 12(x+2) + (-16)(y-1) \\
L(-2, 1) &\approx -8 + 12x + 24 - 16y + 16 \\
L(-2, 1) &\approx 12x - 16y + 32
\end{aligned}
$$

17. $f(x, y) = 1 + x\ln(xy - 5), \quad (2, 3)$

Solution

$$
\begin{aligned}
f_x &= \ln(xy - 5) + \frac{xy}{xy - 5} \text{ which is differentiable at (2,3)} \\
f_y &= \frac{x^2}{xy - 3} \text{ which is differentiable at (2,3)}
\end{aligned}
$$

Linearization,

$$
\begin{aligned}
L(x, y) &\approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
L(2, 3) &\approx 1 + 6(x-2) + 4(y-3) = 1 + 6x - 12 + 4y \\
L(2, 3) &\approx 6x + 4y - 23
\end{aligned}
$$

19. $f(x, y) = x^2 e^y, \quad (1, 0)$

Solution

$$
\begin{aligned}
f_x &= 2xe^4 \quad \Rightarrow \quad f_x(1, 0) = 2(1)e^0 = 2 \\
f_y &= x^2e^4 \quad \Rightarrow \quad f_y(1, 0) = 1e^0 = 1
\end{aligned}
$$

Linearization,

$$
\begin{aligned}
L(x, y) &\approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
L(1, 0) &\approx 1 + 2(x-1) + 1(y-0) = 1 + 2x - 2 + y \\
&= 2x + y - 1
\end{aligned}
$$