Chapter 13 Section 3 & 4 Problem Set

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Section 3: Arc Length and Curvature

Problem 1a

Use Equation 2 to compute the length of the given line segment.

$$\vec{r}(t) = \langle 3-t, 2t, 4t+1\rangle \quad 1 \le t \le 3$$

Solution

Let the length of the line segment be

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\, dt \Rightarrow L = \int_a^b \|\vec{r}\,'(t)\|\, dt$$

$$D : \{\, t \mid 1 \le t \le 3\,\}$$

$$\vec{r}\,'(t) = \langle -1, 2, 4\rangle \Rightarrow L = \int_1^3 \sqrt{(-1)^2 + (2)^2 + (4)^2} dt \quad = \int_1^3 \sqrt{21} dt = \sqrt{21} t \,\Big|_1^3 = \sqrt{21}(3) - \sqrt{21}(1) = 2\sqrt{21}$$

Problems 3-7 odd

Find the length of the curve.

3. $\vec{r}\,(t) = \langle t, 3\cos t, 3\sin t\rangle \quad 25 \le t \le 5$

Solution

$$\begin{aligned}
\vec{r}\,'(t) = \langle 1, -3\sin t, 3\cos t\rangle \Rightarrow L &= \int_{-5}^{5} \sqrt{1^2 + (-3\sin t)^2 + (3\cos t)^2}\, dt \\
&= \int_{-5}^{5} \sqrt{1 + 9\sin^2 t + 9\cos^2 t}\, dt \\
&= \int_{-5}^{5} \sqrt{1 + 9(1)}\, dt \\
&= \int_{-5}^{5} \sqrt{10}\, dt \\
&= \sqrt{10} t \,\Big|_{-5}^{5} dt = \sqrt{10}(5) - \sqrt{10}(-5) = 10\sqrt{10}
\end{aligned}$$

5. $\vec{r}\,(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle \quad 0 \le t \le 1$

Solution

$$\begin{aligned}
\vec{r}\,'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle \Rightarrow L &= \int_0^1 \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2}\; dt \\
&= \int_0^1 \sqrt{2 + e^{2t} + e^{-2t}}\; dt \\
&= \int_0^1 \sqrt{(e^t + e^{-t})^2}\; dt \\
&= \int_0^1 e^t + e^{-t}\; dt \\
&= e^t - e^{-t}\,\Big|_0^1 = (e^1 - \frac{1}{e^1}) - (e^0 - \frac{1}{e^0}) = e - \frac{1}{e} - 1 + 1 = e - \frac{1}{e}
\end{aligned}$$

7. $\vec{r}\,(t) = \langle 1, t^2, t^3 \rangle \quad 0 \le t \le 1$

Solution

$$\begin{aligned}
\vec{r}\,'(t) = \langle 0, 2t, 3t^2 \rangle \Rightarrow L &= \int_0^1 \sqrt{(0)^2 + (2t)^2 + (3t^2)^2}\; dt \\
&= \int_0^1 \sqrt{4t^2 + 9t^4}\; dt \\
&= \int_0^1 \sqrt{t^2(4 + 9t^2)}\; dt \\
&= \int_0^1 t\sqrt{4 + 9t^2}\; dt
\end{aligned}$$

Using u-substitution,

$$\begin{aligned}
u^2 &= 4 + 9t^2 \\
2udu &= 18t\; dt \\
\frac{u\; du}{9} &= t\; dt
\end{aligned}$$

$$\int_0^1 t\sqrt{4 + 9t^2}\; dt = \int_0^1 u \cdot (\frac{1}{9}u)\; du \quad = \frac{1}{9}\int_0^1 u^2\; du = \frac{1}{9}(\frac{1}{3}u^3)\Big|_2^{\sqrt{13}} \quad = \frac{1}{27}(u^3)\Big|_2^{\sqrt{13}} = \frac{1}{27}(13^{\frac{3}{2}} - 2^3) = \frac{13\sqrt{13}}{27} - 3$$

Problems 19-23 odd

(a) Find the unit tangent and unit normal vectors $\vec{T}(t)$ and $\vec{N}(t)$.
(b) Use Formula 9 to find the curvature.

19. $\vec{r}(t) = \langle t^2,\ \sin t - t\cos t,\ \cos t + t\sin t \rangle, \quad t > 0$

Solution

a.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

$$\begin{aligned}
\vec{r}\,'(t) &= \langle 2t, \cos t - \cos t + t\sin t, -\sin t + \sin t + t\cos t \rangle = \langle 2t, t\sin t, t\cos t \rangle \\
\|\vec{r}\,'(t)\| &= \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{4t^2 + t^2} = \sqrt{5t^2} = \sqrt{5}t \quad [\cos^2 t + \sin^2 t = 1] \\
\vec{T}(t) &= \frac{\langle 2t, t\sin t, t\cos t \rangle}{\sqrt{5}t} = \frac{1}{\sqrt{5}}\langle 2, \sin t, \cos t \rangle
\end{aligned}$$

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$$

$$
\begin{aligned}
\vec{T}\,'(t) &= \frac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle \\
\|\vec{T}\,'(t)\| &= \frac{1}{\sqrt{5}}\sqrt{0^2 + \cos^2 t + \sin^2 t} = \sqrt{1} = \frac{1}{\sqrt{5}} \\
\vec{N}(t) &= \frac{\frac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle}{\frac{1}{\sqrt{5}}} = \langle 0, \cos t, -\sin t\rangle
\end{aligned}
$$

b.

$$\kappa(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{\frac{1}{\sqrt{5}}}{\sqrt{5}t} = \frac{1}{5t}$$

21. $\vec{r}(t) = \langle t, t^2, 4\rangle$

Solution

a.

$$
\begin{aligned}
\vec{r}\,'(t) &= \ \hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}} \\
\|\vec{r}\,'(t)\| &= \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2} \\
\vec{T}(t) &= \frac{\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}}}{\sqrt{1+4t^2}} = \frac{1}{\sqrt{1+4t^2}}(\ \hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}}) \\
\frac{d}{dt} &= [f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t) \quad [\textit{vector product rule}] \\
\vec{T}\,'(t) &= -\frac{4t}{(1+4t^2)^{\frac{3}{2}}}(\ \hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}}) + \frac{1}{(1+4t^2)^{\frac{1}{2}}}(2\,\hat{\mathbf{j}}) \\
&= \frac{1}{(1+4t^2)^{\frac{3}{2}}}\big(-4t(\ \hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}}) + (1+4t^2)(2\,\hat{\mathbf{j}})\big) = \frac{1}{(1+4t^2)^{\frac{3}{2}}}(-4t\,\hat{\mathbf{i}} - 8t^2\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{j}} + 8t^2\,\hat{\mathbf{j}}) \\
&= \frac{1}{(1+4t^2)^{\frac{3}{2}}}(-4t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}}) \\
\|\vec{T}\,'(t)\| &= \frac{1}{(1+4t^2)^{\frac{3}{2}}}\sqrt{(-4t)^2 + 2^2} = \frac{1}{(1+4t^2)^{\frac{3}{2}}}\sqrt{16t^2+4} = \frac{1}{(1+4t^2)^{\frac{3}{2}}}\sqrt{4(4t^2+1)} = \frac{2}{(1+4t^2)^{\frac{3}{2}}}\sqrt{1+4t^2} \\
&= \frac{2}{1+4t^2} \\
\vec{N}(t) &= \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} = \frac{1}{(1+4t^2)^{\frac{3}{2}}}(-4t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}}) \cdot \frac{1+4t^2}{2} = \frac{(1+4t^2)^1}{2(1+4t^2)^{\frac{3}{2}}}(-4t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}}) = \frac{1}{\sqrt{1+4t^2}}(-2t\,\hat{\mathbf{i}} + \ \hat{\mathbf{j}})
\end{aligned}
$$

b.

$$\kappa(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{2}{1+4t^2} \cdot \frac{1}{(1+4t^2)^{\frac{1}{2}}} = \frac{2}{(1+4t^2)^{\frac{3}{2}}}$$

23. $\vec{r}(t) = \langle t, \frac{1}{2}t^2, t^2\rangle$

Solution

a.

$$\begin{aligned}
\vec{r}\,'(t) &= \langle 1, t, 2t\rangle \\
\|\vec{r}\,'(t)\| &= \sqrt{1^2+t^2+(2t)^2} = \sqrt{1+5t^2} \\
\vec{T}(t) &= \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\langle 1,t,2t\rangle}{\sqrt{1+t^2+4t^2}} = \frac{1}{\sqrt{1+5t^2}}\langle 1,t,2t\rangle \\
\frac{d}{dt} &= [f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t) \quad [\textit{vector product rule}] \\
\vec{T}\,'(t) &= -\frac{5t}{(1+5t^2)^{\frac{3}{2}}}\langle 1,t,2t\rangle + \frac{1}{(1+5t^2)^{\frac{1}{2}}}\langle 0,1,2\rangle \\
&= \frac{1}{(1+5t^2)^{\frac{3}{2}}}\left(-5t\langle 1,t,2t\rangle + (1+5t^2)\langle 0,1,2\rangle\right) \\
&= \frac{1}{(1+5t^2)^{\frac{3}{2}}}\left(\langle -5t,-5t^2,-10t^2\rangle + \langle 0,1+5t^2,2+10t^2\rangle\right) = \frac{1}{(1+5t^2)^{\frac{3}{2}}}\langle -5t,1,2\rangle \\
\|\vec{T}\,'(t)\| &= \frac{1}{(1+5t^2)^{\frac{3}{2}}}\sqrt{(-5t)^2+1^2+2^2} = \frac{1}{(1+5t^2)^{\frac{3}{2}}}\sqrt{25t^2+5} = \frac{1}{(1+5t^2)^{\frac{3}{2}}}\sqrt{5(5t^2+1)} \\
&= \frac{\sqrt{5}(1+5t^2)^{\frac{1}{2}}}{(1+5t^2)^{\frac{3}{2}}} = \frac{\sqrt{5}}{1+5t^2} \\
\|\vec{N}(t)\| &= \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} = \frac{1}{(1+5t^2)^{\frac{3}{2}}}\langle -5t,1,2\rangle \cdot \frac{1+5t^2}{\sqrt{5}} = \frac{1}{\sqrt{5}\sqrt{1+5t^2}}\langle -5t,1,2\rangle = \frac{1}{\sqrt{5+25t^2}}\langle -5t,1,2\rangle
\end{aligned}$$

b.

$$\kappa(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{\sqrt{5}}{1+5t^2}\cdot\frac{1}{\sqrt{1+5t^2}} = \frac{\sqrt{5}}{(1+5t^2)^{\frac{3}{2}}}$$

Problem 27

Use Theorem 10 to find the curvature

$$\vec{r}(t) = \sqrt{6}t^2\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}} + 2t^3\,\hat{\mathbf{k}}$$

Solution

Theorem 10 states that

$$\kappa(t) = \frac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$$

$$\begin{aligned}
\vec{r}\,'(t) &= 2\sqrt{6}t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 6t^2\,\hat{\mathbf{k}} \\
\vec{r}\,''(t) &= 2\sqrt{6}\,\hat{\mathbf{i}} + 12t\,\hat{\mathbf{k}} \\
\|\vec{r}\,'(t)\| &= \sqrt{(2\sqrt{6}t)^2+2^2+(6t^2)^2} = \sqrt{24t^2+4+36t^4} = \sqrt{4(9t^4+6t^2+1)} = 2\sqrt{(3t^2+1)^2} = 2(3t^2+1)
\end{aligned}$$

$$\vec{r}\,'(t)\times\vec{r}\,''(t) \Rightarrow \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2\sqrt{6}t & 2 & 6t^2 \\ 2\sqrt{6} & 0 & 12t \end{vmatrix} = \hat{\mathbf{i}}\begin{vmatrix} 2 & 6t^2 \\ 0 & 12t \end{vmatrix} - \hat{\mathbf{j}}\begin{vmatrix} 2\sqrt{6}t & 6t^2 \\ 2\sqrt{6} & 12t \end{vmatrix} + \hat{\mathbf{k}}\begin{vmatrix} 2\sqrt{6}t & 2 \\ 2\sqrt{6} & 0 \end{vmatrix}$$

$$= (24t-0)\,\hat{\mathbf{i}} - (24t^2\sqrt{6} - 12t^2\sqrt{6})\,\hat{\mathbf{j}} + (0-4\sqrt{6})\,\hat{\mathbf{k}} = 24t\,\hat{\mathbf{i}} - 12t^2\sqrt{6}\,\hat{\mathbf{j}} - 4\sqrt{6}\,\hat{\mathbf{k}}$$

$$\|\vec{r}\,'(t)\times\vec{r}\,''(t)\| = \sqrt{(24t)^2+(-12t^2\sqrt{6})^2+(-4\sqrt{6})^2} = \sqrt{576t^2+864t^4+96} = \sqrt{96(9t^4+6t^2+1)} = \sqrt{16\cdot 6(3t^2+1)^2}$$

$$\|\vec{r}\,'(t)\times\vec{r}\,''(t)\| = 4\sqrt{6}(3t^2+1)$$

$$\kappa(t) = \frac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \frac{4\sqrt{6}(3t^2+1)}{(2(3t^2+1))^3} = \frac{4\sqrt{6}(3t^2+1)}{8(3t^2+1)^3} = \frac{\sqrt{6}}{2(3t^2+1)^2}$$

Problem 28

Find the curvature of $\vec{r}(t) = \langle t^2, \ln t, t \ln t \rangle$ at the point $(1, 0, 0)$.

Solution

$$\begin{aligned}
&If\ x = t^2 \Rightarrow 1 = t^2,\ then \\
&\quad t = 1? \Rightarrow\ \ln 1 \equiv 0\ ,\ 1 \ln 1 \equiv 0 \\
&\therefore t = 1
\end{aligned}$$

$$\begin{aligned}
&\vec{r}\,'(t) = \langle 2t, \frac{1}{t}, \ln t + 1 \rangle, \quad \vec{r}\,''(t) = \langle 2, -\frac{1}{t^2}, \frac{1}{t} \rangle \\
&\vec{r}\,'(1) = \langle 2, 1, 1 \rangle, \quad \|\vec{r}\,'(1)\| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}, \quad \vec{r}\,''(1) = \langle 2, -1, 1 \rangle
\end{aligned}$$

$$\vec{r}\,'(1) \times \vec{r}\,''(1) \Rightarrow \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = \hat{\mathbf{i}} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} - \hat{\mathbf{j}} \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \hat{\mathbf{k}} \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix}$$
$$= (1 - (-1))\, \hat{\mathbf{i}} - (2 - 2)\, \hat{\mathbf{j}} + (-2 - 2)\, \hat{\mathbf{k}} = 2\, \hat{\mathbf{i}} - 4\, \hat{\mathbf{k}} \Rightarrow \langle 2, 0, -4 \rangle$$

$$\kappa(1) = \frac{\|\langle 2, 0, -4 \rangle\|}{\sqrt{6}^3} = \frac{\sqrt{2^2 + 0^2 + (-4)^2}}{6^{\frac{3}{2}}} = \frac{\sqrt{20}}{6\sqrt{6}} = \frac{2\sqrt{5}}{6\sqrt{6}} = \frac{\sqrt{5}}{3\sqrt{6}} = \frac{\sqrt{30}}{18}$$

Problem 31 & 33

Use Formula 11 to find the curvature.

31. $y = x^4$

Solution

Formula 11 states that

$$\kappa = \frac{|f''(x)|}{[1 + (f'(x)^2)]^{\frac{3}{2}}}$$

$$\begin{aligned}
&y = x^4 \Rightarrow y' = 4x^3 \Rightarrow y'' = 12x^2 \\
&\kappa = \frac{|12x^2|}{[1 + (4x^3)^2]^{\frac{3}{2}}} = \frac{12x^2}{(1 + 16x^6)^{\frac{3}{2}}}
\end{aligned}$$

33. $y = xe^x$

Solution

Formula 11 states that

$$\kappa = \frac{|f''(x)|}{[1 + (f'(x)^2)]^{\frac{3}{2}}}$$

$$\begin{aligned}
&y' = e^x + xe^x \Rightarrow y'' = e^x + e^x + xe^x = 2e^x + xe^x \\
&\kappa = \frac{|2e^x + xe^x|}{[1 + (e^x + xe^x)^2]^{\frac{3}{2}}} = \frac{e^x(2 + x)}{[1 + (e^x + xe^x)^2]^{\frac{3}{2}}}
\end{aligned}$$

Problem 51

Find the vectors **T**, **N**, and **B** at the given point.

$$\vec{r}(t) = \langle t^2, \frac{2}{3}t^3, t\rangle, \quad (1, \frac{2}{3}, 1)$$

Solution

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}, \quad \vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}, \quad \vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

$$z = t \Rightarrow t = 1?, \quad 1^2 \equiv 1, \ \frac{2}{3}1^3 \equiv \frac{2}{3}, \ 1 \equiv 1 \quad \therefore t = 1$$

$$\vec{r}\,'(t) = \langle 2t, 2t^2, 1\rangle, \quad \|\vec{r}\,'(t)\| = \sqrt{(2t)^2 + (2t^2)^2 + (1)^2} = \sqrt{4t^2 + 4t^4 + 1} = \sqrt{(2t^2+1)^2} = 2t^2 + 1$$

$$\vec{T}(t) = \frac{\langle 2t, 2t^2, 1\rangle}{2t^2+1} \Rightarrow \vec{T}(1) = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle, \quad f(t) = \frac{1}{2t^2+1}, \ \vec{u} = \langle 2t, 2t^2, 1\rangle \Rightarrow \vec{T}\,'(t) = f\,'(t)\,\vec{u} + f(t)\,\vec{u}\,'$$

$$\begin{aligned}\vec{T}\,'(t) = -4t(2t^2+1)^{-2}\langle 2t, 2t^2, 1\rangle + (2t^2+1)^{-1}\langle 2, 4t, 0\rangle &= (2t^2+1)^{-2}(-4t\langle 2t, 2t^2, 1\rangle + (2t^2+1)\langle 2, 4t, 0\rangle) \\ &= (2t^2+1)^{-2}(\langle -8t^2, -8t^3, -4t\rangle + \langle 4t^2+2, 8t^3+4t, 0\rangle) \\ &= (2t^2+1)^{-2}\langle -4t^2+2, 4t, -4t\rangle \\ &= 2(2t^2+1)^{-2}\langle -2t^2+1, 2t, -2t\rangle\end{aligned}$$

$$\vec{T}\,'(1) = 2(2(1)^2+1)^{-2}\langle -2(1)^2+1, 2(1), -2(1)\rangle = 2(2+1)^{-2}\langle -2+1, 2, -2\rangle = \frac{2}{9}\langle -1, 2, -2\rangle = \langle -\frac{2}{9}, \frac{4}{9}, -\frac{4}{9}\rangle$$

$$\vec{N}(1) = \frac{\langle -\frac{2}{9}, \frac{4}{9}, -\frac{4}{9}\rangle}{\sqrt{\left(-\frac{2}{9}\right)^2 + \left(\frac{4}{9}\right)^2 + \left(-\frac{4}{9}\right)^2}} = \frac{\langle -\frac{2}{9}, \frac{4}{9}, -\frac{4}{9}\rangle}{\sqrt{\frac{36}{81}}} = \frac{9}{6}\langle -\frac{2}{9}, \frac{4}{9}, -\frac{4}{9}\rangle = \langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\rangle$$

$$\vec{B}(1) = \vec{T}(1) \times \vec{N}(1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{vmatrix} = (-\frac{4}{9} - \frac{2}{9})\,\hat{\mathbf{i}} + (-\frac{1}{9} + \frac{4}{9})\,\hat{\mathbf{j}} + (\frac{4}{9} - (-\frac{2}{9}))\,\hat{\mathbf{k}} = \langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\rangle$$

Problem 53

Find equations of the normal plane and osculating plane of the curve at the given point.

$$x = \sin 2t, \ y = -\cos 2t, \ z = 4t; \quad (0, 1, 2\pi)$$

Solution

If $z = 4t$ and $z = 2\pi$, then $t = \frac{2\pi}{4} = \frac{\pi}{2}$
$0 \equiv \sin 2(\frac{\pi}{2}), \quad 1 \equiv -\cos(2(\frac{\pi}{2})), \quad 2\pi \equiv 4(\frac{\pi}{2}) \Rightarrow \quad \therefore t = \frac{\pi}{2}$

The point $(0, 1, 2\pi)$ corresponds to $t = \frac{\pi}{2}$

$$\begin{aligned}\text{Let } \vec{r}(t) &= \langle \sin 2t, -\cos 2t, 4t\rangle \\ \vec{r}\,'(t) &= \langle 2\cos 2t, 2\sin 2t, 4\rangle \\ \vec{r}\,'(\frac{\pi}{2}) &= \langle -2, 0, 4\rangle\end{aligned}$$

So, the normal plane has normal vector $\vec{r}\,'(\frac{\pi}{2}) = \langle -2, 0, 4\rangle$

$\therefore$ The equation of the normal plane is

$$-2(x-0)+0(y-1)+4(z-2\pi)=0 \quad \Rightarrow \quad -2x+4z-8\pi=0 \quad or \quad 4z-x=4\pi$$

To find the osculating plane at $(0,1,2\pi)$ we need vectors $\vec{T}(t)$ and $\vec{N}(t)$

$$\vec{T}(t)=\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}=\frac{\langle 2\cos 2t, 2\sin 2t, 4\rangle}{\sqrt{4\cos^2 2t+4\sin^2 2t+16}}=\frac{\langle 2\cos 2t, 2\sin 2t, 4\rangle}{\sqrt{20}}=\frac{1}{2\sqrt{5}}\langle 2\cos 2t, 2\sin 2t, 2\rangle$$
$$=\frac{1}{\sqrt{5}}\langle \cos 2t, \sin 2t, 2\rangle$$

$$\vec{T}(\frac{\pi}{2})=\frac{1}{\sqrt{5}}\langle -1,0,2\rangle \qquad \vec{T}\,'(t)=\frac{1}{\sqrt{5}}\langle -2\sin 2t, 2\cos 2t, 0\rangle \qquad \|\vec{T}\,'(t)\|=\sqrt{4\sin^2 2t+4\cos^2 2t}=\frac{2}{\sqrt{5}}$$

$$\vec{N}(\frac{\pi}{2})=\frac{\frac{1}{\sqrt{5}}\langle 0,-2,0\rangle}{\frac{2}{\sqrt{5}}}=\frac{1}{2}\langle 0,-2,0\rangle=\langle 0.-1.0\rangle$$

A vector normal to the osculating plane would be $\vec{B}(t)=\vec{T}(t)\times\vec{N}(t) \quad \Rightarrow \quad \vec{B}(\frac{\pi}{2})=\vec{T}(\frac{\pi}{2})\times\vec{N}(\frac{\pi}{2})$

$$=\frac{1}{\sqrt{5}}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 0 & 2 \\ 0 & -1 & 0 \end{vmatrix}=\frac{1}{\sqrt{5}}\langle 0-(-2), 0-0, 1-0\rangle=\frac{1}{\sqrt{5}}\langle 2,0,1\rangle$$

$\therefore$ The equation of the osculating plane is

$$2(x-0)+0(y-1)+1(z-2\pi)=0 \quad \Rightarrow \quad 2x+z-2\pi=0 \quad or \quad 2x+z=2\pi$$

Problem 66

Use Formula 14 to find the torsion at the given value of t.

$$\vec{r}(t)=\langle \sin t, 3t, \cos t\rangle, \quad t=\frac{\pi}{2}$$

Solution

The torsion of a curve with the paramater t is defined as

$$\tau=-\frac{\vec{B}\,'(t)\cdot\vec{N}(t)}{\|\vec{r}\,'(t)\|}$$

We need to find $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$

$$\vec{r}(t)=\langle \sin t, 3t, t\rangle \quad \Rightarrow \quad \vec{r}\,'(t)=\langle \cos t, 3, -\sin t\rangle \quad \Rightarrow \quad \|\vec{r}\,'(t)\|=\sqrt{\cos^2 t+9+\sin^2 t}=\sqrt{10}$$

$$\vec{T}(t)=\frac{\vec{r}\,'(t)}{\|\vec{r}(t)\|}=\frac{\langle \cos t, 3, -\sin t\rangle}{\sqrt{10}}=\frac{1}{\sqrt{10}}\langle \cos t, 3, -\sin t\rangle$$

$$\vec{T}\,'(t)=\frac{1}{\sqrt{10}}\langle -\sin t, 0, -\cos t\rangle \quad \Rightarrow \quad \|\vec{T}\,'(t)\|=\frac{1}{\sqrt{10}}\sqrt{\sin^2 t+\cos^2 t}=\frac{1}{\sqrt{10}}$$
$$\vec{N}(t)=\frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}=\frac{1}{1}\langle -\sin t, 0, -\cos t\rangle=\langle -\sin t, 0, -\cos t\rangle$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \tfrac{1}{\sqrt{10}} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos t & 3 & -\sin t \\ -\sin t & 0 & -\cos t \end{vmatrix} = \tfrac{1}{\sqrt{10}} \langle -3\cos t - 0, \sin^2 t - (-\cos^2 t), 0 - (-3\sin t) \rangle$$

$$= \tfrac{1}{\sqrt{10}} \langle -3\cos t, 1, 3\sin t \rangle$$

$$\vec{B}\,'(t) = \frac{1}{\sqrt{10}} \langle 3\sin t, 0, 3\cos t \rangle \quad \Rightarrow \quad \vec{B}\,'(t) \cdot \vec{N}(t) = \frac{1}{\sqrt{10}} \langle 3\sin t, 0, 3\cos t \rangle \cdot \langle -\sin t, 0, -\cos t \rangle$$

$$= \frac{1}{\sqrt{10}} (-3\sin^2 t - 3\cos^2 t) = \frac{-3}{\sqrt{10}}$$

$\therefore$ The torsion of the curve at $t = \frac{\pi}{2}$ is

$$\tau = -\frac{-\frac{3}{\sqrt{10}}}{\sqrt{10}} = \frac{3}{10}$$

Problem 70

Use Theorem 15 to find the torsion of the given curve at a general point and at the point corresponding to $t = 0$

$$\vec{r}(t) = \langle \cos t, \sin t, \sin t \rangle$$

Solution

Theorem 15 states that

$$\tau(t) = \frac{[\vec{r}\,'(t) \times \vec{r}\,''(t)] \cdot \vec{r}\,'''(t)}{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|^2}$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, \cos t \rangle \qquad \vec{r}\,''(t) = \langle -\cos t, -\sin t, -\sin t \rangle \qquad \vec{r}\,'''(t) = \langle \sin t, -\cos t, -\cos t \rangle$$

$$\vec{r}\,'(0) = \langle 0, 1, 1 \rangle \qquad \vec{r}\,''(0) = \langle -1, 0, 0 \rangle \qquad \vec{r}\,'''(0) = \langle 0, -1, -1 \rangle$$

$$[\vec{r}\,'(0) \times \vec{r}\,''(0)] \cdot \vec{r}\,'''(0) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{vmatrix} = \langle 0 - 0, -1 - 0, 0 - (-1) \rangle = \langle 0, -1, 1 \rangle \cdot \langle 0, -1, -1 \rangle$$

$$= 0 + 1 - 1 = 0$$

$\therefore$ The torsion of the curve at $t = 0$ is

$$\tau(0) = \frac{0}{\sqrt{2}^2} = 0$$

Section 4: Motion in Space - Velocity and Acceleration

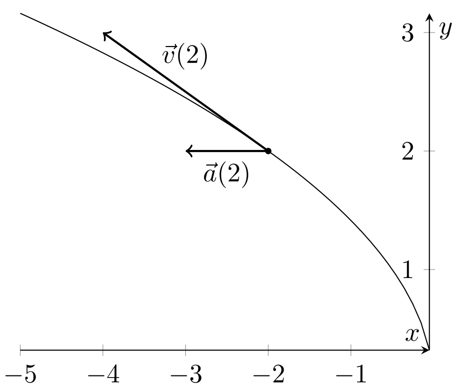
Problem 3-7 odd

Find the velocity, acceleration, and speed of a particle with the given position function. Sketch the path of the particle and draw the velocity and acceleration vectors for the specified value of t.

3. $\vec{r}(t) = \langle -\frac{1}{2}t^2, t\rangle, \quad t = 2$

Solution

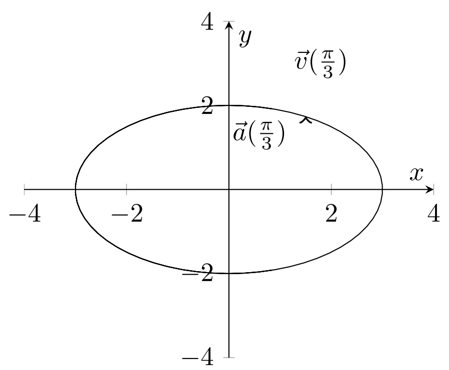
$$\begin{aligned} \vec{r}\,'(t) &= \vec{v}(t) = \langle -t, 1\rangle \quad \Rightarrow \quad \vec{v}(2) = \langle -2, 1\rangle \\ \vec{r}\,''(t) &= \vec{a}(t) = \langle -1, 0\rangle \quad \Rightarrow \quad \vec{a}(2) = \langle -1, 0\rangle \\ \|\vec{v}(t)\| &= \sqrt{t^2+1} \end{aligned}$$



5. $\vec{r}(t) = 3\cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} \quad t = \frac{\pi}{3}$

Solution

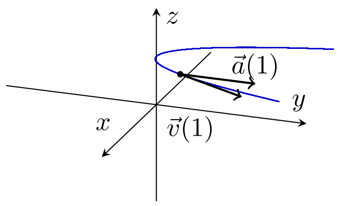
$$\begin{aligned} \vec{r}\,'(t) &= \vec{v}(t) = \langle -3\sin t, 2\cos t\rangle \quad \Rightarrow \quad \vec{v}(\frac{\pi}{3}) = \langle -3(\frac{\sqrt{3}}{2}), 2(\frac{1}{2})\rangle = \langle -\frac{3\sqrt{3}}{2}, 1\rangle \\ \vec{r}\,''(t) &= \vec{a}(t) = \langle -3\cos t, -2\sin t\rangle \quad \Rightarrow \quad \vec{a}(\frac{\pi}{2}) = \langle -3(\frac{1}{2}), -2(\frac{\sqrt{3}}{2})\rangle = \langle \frac{-3}{2}, -\sqrt{3}\rangle \\ \|\vec{v}(t)\| &= \sqrt{(-3\sin t)^2 + (2\cos t)^2} = \sqrt{9\sin^2 t + 4\cos^2 t} = \sqrt{9(1-\cos^2 t) + 4\cos^2 t} = \sqrt{9 - 5\cos^2 t} \end{aligned}$$



7. $\vec{r}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}} \quad t = 1$

Solution

$$\begin{aligned} \vec{r}\,'(t) &= \vec{v}(t) = \langle 1, 2t, 0\rangle \quad \Rightarrow \quad \vec{v}(1) = \langle 1, 2, 0\rangle \\ \vec{r}\,''(t) &= \vec{a}(t) = \langle 0, 2, 0\rangle \quad \Rightarrow \quad \vec{a}(1) = \langle 0, 2, 0\rangle \\ \|\vec{v}(t)\| &= \sqrt{1^2 + (2t)^2} = \sqrt{1+4t^2} \end{aligned}$$



Problems 9-13 odd

Find the velocity, acceleration, and speed of a particle with the given position function.

9. $\vec{r}(t) = \langle t^2 + t, t^2 - t, t^3 \rangle$

Solution

$$\begin{aligned}
\vec{v}(t) &= \langle 2t+1, 2t-1, 3t^2 \rangle \\
\vec{a}(t) &= \langle 2, 2, 6t \rangle \\
\|\vec{v}(t)\| &= \sqrt{(2t+1)^2 + (2t-1)^2 + (3t^2)^2} \\
&= \sqrt{(4t^2+4t+1) + (4t^2-4t+1) + 9t^4} \\
&= \sqrt{9t^4 + 8t^2 + 2}
\end{aligned}$$

11. $\vec{r}(t) = \sqrt{2}t\,\hat{\mathbf{i}} + e^t\,\hat{\mathbf{j}} + e^{-t}\,\hat{\mathbf{k}}$

Solution

$$\begin{aligned}
\vec{v}(t) &= \sqrt{2}\,\hat{\mathbf{i}} + e^t\,\hat{\mathbf{j}} - e^{-t}\,\hat{\mathbf{k}} \\
\vec{a}(t) &= e^t\,\hat{\mathbf{j}} + e^{-t}\,\hat{\mathbf{k}} \\
\|\vec{v}(t)\| &= \sqrt{\left(\sqrt{2}\right)^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{(e^t + e^{-t})^2} \\
&= e^t + e^{-t}
\end{aligned}$$

13. $\vec{r}(t) = e^t(\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}})$

Solution

$$\begin{aligned}
\vec{v}(t) &= e^t(\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}) + e^t(-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} + \ \hat{\mathbf{k}}) \\
&= e^t\left[(\cos t - \sin t)\,\hat{\mathbf{i}} + (\sin t + \cos t)\,\hat{\mathbf{j}} + (t+1)\,\hat{\mathbf{k}}\right] \qquad \text{[vector product rule]} \\
\vec{a}(t) &= e^t(-\cos t\,\hat{\mathbf{i}} - \sin t\,\hat{\mathbf{j}}) \\
\|\vec{v}(t)\| &= \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}
\end{aligned}$$

Problem 15

Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

$$a(t) = 2\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{k}}, \quad v(0) = 3\,\hat{\mathbf{i}} - \ \hat{\mathbf{j}}, \quad r(0) = \ \hat{\mathbf{j}} + \ \hat{\mathbf{k}}$$

Solution

Problem 17a

Find the position vector of a particle that has the given acceleration and the specified initial velocity and position.

$$a(t) = 2t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + \cos 2t\,\hat{\mathbf{k}}, \quad v(0) = \ \hat{\mathbf{i}}, \quad r(0) = \ \hat{\mathbf{j}}$$

Solution

Problem 23

A projectile is fired with an initial speed of $200\frac{m}{s}$ and angle of elevation $60°$. Find (a) the range of the projectile, (b) the maximum height reached, and (c) the speed at impact.

Solution

Problem 26

A projectile is fired from a tank with initial speed $400\frac{m}{s}$. Find two angles of elevation that can be used to hit a target $3000m$ away.

Solution

Problem 27

A rifle is fired with angle of elevation $36°$. What is the initial speed if the maximum height of the bullet is $1600ft$?

Solution

Problem 37 & 39

Find the tangential and normal components of the acceleration vector.

37. $\vec{r}(t) = (t^2+1)\,\hat{\mathbf{i}} + t^3\,\hat{\mathbf{j}}, \quad t \geq 0$

Solution

39. $\vec{r}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$

Solution

Problem 41

Find the tangential and normal components of the acceleration vector at the given point.

$$\vec{r}(t) = \ln t\,\hat{\mathbf{i}} + (t^2+3t)\,\hat{\mathbf{j}} + 4\sqrt{t}\,\hat{\mathbf{k}}, \quad (0,4,4)$$

Solution