

Chapter 14 - Problem Set 2

Calculus 3

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Section 5: The Chain Rule

3-7 (odd)

Use The Chain Rule to find $\frac{dz}{dt}$ or $\frac{dw}{dt}$.

3. $z = xy^3 - x^2y, \quad x = t^2 + 1, \quad y = t^2 - 1$

Solution

$$\begin{aligned}\frac{dz}{dt} &= \frac{dz}{dx} \left(\frac{dx}{dt} \right) + \frac{dz}{dy} \left(\frac{dy}{dt} \right) \\ \frac{dz}{dt} &= [y^3 - 2xy](2t) + [3xy^2 - x^2](2t) \\ \boxed{\frac{dz}{dt} &= 2t[y^3 - 2xy + 3xy^2 - x^2]}\end{aligned}$$

5. $z = \sin x \cos y, \quad x = \sqrt{t}, \quad y = 1/t$

Solution

$$\begin{aligned}\frac{dz}{dt} &= \frac{dz}{dx} \left(\frac{dx}{dt} \right) + \frac{dz}{dy} \left(\frac{dy}{dt} \right) \\ \frac{dz}{dt} &= [\cos x \cos y] \left(\frac{1}{2} t^{-\frac{1}{2}} \right) + [-\sin x \sin y] (-t^{-2}) \\ \boxed{\frac{dz}{dt} &= \frac{1}{2\sqrt{t}} \cos x \cos y + \frac{1}{t^2} \sin x \sin y}\end{aligned}$$

7. $w = xe^{y/z}, \quad x = t^2, \quad y = 1 - t, \quad z = 1 + 2t$

Solution

$$\begin{aligned}\frac{dw}{dt} &= \frac{dw}{dx} \left(\frac{dx}{dt} \right) + \frac{dw}{dy} \left(\frac{dy}{dt} \right) + \frac{dw}{dz} \left(\frac{dz}{dt} \right) \\ \frac{dw}{dt} &= [e^{y/z}] (2t) + \left[\frac{x}{z} e^{y/z} \right] (-1) + \left[-\frac{xy}{z^2} e^{y/z} \right] (2) \\ \boxed{\frac{dw}{dt} &= e^{y/z} \left[2t - \frac{x}{z} - \frac{2xy}{z^2} \right]}\end{aligned}$$

11-15 (odd)

Use the Chain Rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

11. $z = (x - y)^5, \quad x = s^2t, \quad y = st^2$

Solution

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \left(\frac{\partial x}{\partial s} \right) + \frac{\partial z}{\partial y} \left(\frac{\partial y}{\partial s} \right) \\ \frac{\partial z}{\partial s} &= [5(x - y)^4] (2st) + [-5(x - y)^4] (t^2)\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial s} = 5(x - y)^4 [2st - t^2]}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \left(\frac{\partial x}{\partial t} \right) + \frac{\partial z}{\partial y} \left(\frac{\partial y}{\partial t} \right) \\ \frac{\partial z}{\partial t} &= [5(x - y)^4] (s^2) + [-5(x - y)^4] (2st)\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial t} = 5(x - y)^4 [s^2 - 2st]}$$

13. $z = \ln(3x + 2y), \quad x = s \sin t, \quad y = t \cos s$

Solution

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \left(\frac{\partial x}{\partial s} \right) + \frac{\partial z}{\partial y} \left(\frac{\partial y}{\partial s} \right) \\ \frac{\partial z}{\partial s} &= \left[\frac{3}{3x + 2y} \right] (\sin t) + \left[\frac{2}{3x + 2y} \right] (-t \sin s)\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial s} = \frac{3 \sin t - 2t \sin s}{3x + 2y}}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \left(\frac{\partial x}{\partial t} \right) + \frac{\partial z}{\partial y} \left(\frac{\partial y}{\partial t} \right) \\ \frac{\partial z}{\partial t} &= \left[\frac{3}{3x + 2y} \right] (s \cos t) + \left[\frac{2}{3x + 2y} \right] (\cos s)\end{aligned}$$

$$\boxed{\frac{\partial z}{\partial t} = \frac{3s \cos t + 2 \cos s}{3x + 2y}}$$

15. $z = (\sin \theta)/r, \quad r = st, \quad \theta = s^2 + t^2$

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r} \left(\frac{\partial r}{\partial s} \right) + \frac{\partial z}{\partial \theta} \left(\frac{\partial \theta}{\partial s} \right)$$

$$\frac{\partial z}{\partial s} = \left[-\frac{\sin \theta}{r^2} \right] (t) + \left[\frac{\cos \theta}{r} \right] (2s)$$

$$\boxed{\frac{\partial z}{\partial s} = -\frac{t \sin \theta}{r^2} + \frac{2s \cos \theta}{r}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r} \left(\frac{\partial r}{\partial t} \right) + \frac{\partial z}{\partial \theta} \left(\frac{\partial \theta}{\partial t} \right)$$

$$\frac{\partial z}{\partial t} = \left[-\frac{\sin \theta}{r^2} \right] (s) + \left[\frac{\cos \theta}{r} \right] (2t)$$

$$\boxed{\frac{\partial z}{\partial t} = -\frac{s \sin \theta}{r^2} + \frac{2t \cos \theta}{r}}$$

25-29 (odd)

Use the Chain Rule to find the indicated partial derivatives.

25. $z = x^4 + x^2y, \quad x = s + 2t - u, \quad y = stu^2;$

$\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}, \frac{\partial z}{\partial u}$ when $s = 4, t = 2, u = 1$

Solution

When $s = 4, t = 2$, and $u = 1 \Rightarrow x = 7, y = 8$

$$\frac{\partial z}{\partial s} = [4x^3 + 2xy] (1) + [x^2] (tu^2) = 4x^3 + 2xy + x^2tu^2$$

$$\frac{\partial z}{\partial s} = 4(7)^3 + 2(7)(8) + (7)^2(2)(1)^2$$

$$\boxed{\frac{\partial z}{\partial s} = 1582}$$

$$\frac{\partial z}{\partial t} = [4x^3 + 2xy] (2) + [x^2] (su^2) = 8x^3 + 4xy + x^2su^2$$

$$\frac{\partial z}{\partial t} = 8(7)^3 + 4(7)(8) + (7)^2(4)(1)^2$$

$$\boxed{\frac{\partial z}{\partial t} = 3164}$$

$$\frac{\partial z}{\partial u} = [4x^3 + 2xy] (-1) + [x^2] (2stu) = -4x^3 - 2xy + 2x^2stu$$

$$\frac{\partial z}{\partial u} = -4(7)^3 - 2(7)(8) + 2(7)^2(4)(2)(1)$$

$$\boxed{\frac{\partial z}{\partial u} = -700}$$

27. $w = xy + yz + zx, \quad x = r \cos \theta, \quad y = r \sin \theta, \quad z = r\theta;$

$$\frac{\partial w}{\partial r}, \frac{\partial w}{\partial \theta} \quad \text{when } r = 2, \theta = \pi/2$$

Solution

When $r = 2$ and $\theta = \pi/2 \Rightarrow x = 0, y = 2, z = \pi$

$$\frac{\partial w}{\partial r} = [y + z](\cos \theta) + [x + z](\sin \theta) + [x + y](\theta)$$

$$\frac{\partial w}{\partial r} = [2 + \pi](0) + [0 + \pi](1) + [0 + 2](\pi/2) = 0 + \pi + \pi$$

$$\boxed{\frac{\partial w}{\partial r} = 2\pi}$$

$$\frac{\partial w}{\partial \theta} = [y + z](-r \sin \theta) + [x + z](r \cos \theta) + [x + y](r)$$

$$\frac{\partial w}{\partial \theta} = [2 + \pi](-2 \cdot 1) + [0 + \pi](0) + [0 + 2](2) = -4 - 2\pi + 0 + 4$$

$$\boxed{\frac{\partial w}{\partial \theta} = -2\pi}$$

$$29. N = \frac{p+q}{p+r}, \quad p = u + vw, \quad q = v + uw, \quad r = w + uv$$

$$\frac{\partial N}{\partial u}, \frac{\partial N}{\partial v}, \frac{\partial N}{\partial w} \quad \text{when } u = 2, v = 3, w = 4$$

Solution

$$\text{When } u = 2, v = 3, \text{ and } w = 4 \Rightarrow p = 14, q = 11, r = 10$$

$$\frac{\partial N}{\partial u} = \left[\frac{(p+r) - (p+q)}{(p+r)^2} \right] (1) + \left[\frac{1}{p+r} \right] (w) + \left[-\frac{p+q}{(p+r)^2} \right] (v)$$

$$\frac{\partial N}{\partial u} = \frac{r-q}{(p+r)^2} + \frac{w}{p+r} - \frac{v(p+q)}{(p+r)^2}$$

$$\frac{\partial N}{\partial u} = \frac{10-11}{(14+10)^2} + \frac{4}{14+10} - \frac{3(14+11)}{(14+10)^2}$$

$$\frac{\partial N}{\partial u} = \frac{-1}{576} + \frac{4}{24} - \frac{75}{576} = \frac{-1+96-75}{576} = \frac{20}{576}$$

$$\boxed{\frac{\partial N}{\partial u} = \frac{5}{144}}$$

$$\frac{\partial N}{\partial v} = \left[\frac{r-q}{(p+r)^2} \right] (w) + \left[\frac{1}{p+r} \right] (1) + \left[-\frac{p+q}{(p+r)^2} \right] (u)$$

$$\frac{\partial N}{\partial v} = \frac{w(r-q)}{(p+r)^2} + \frac{1}{p+r} - \frac{u(p+q)}{(p+r)^2}$$

$$\frac{\partial N}{\partial v} = \frac{4(10-11)}{(14+10)^2} + \frac{1}{14+10} - \frac{2(14+11)}{(14+10)^2}$$

$$\frac{\partial N}{\partial v} = \frac{-4}{576} + \frac{1}{24} - \frac{50}{576} = \frac{-4+24-50}{576} = \frac{-30}{576}$$

$$\boxed{\frac{\partial N}{\partial v} = -\frac{5}{96}}$$

$$\frac{\partial N}{\partial w} = \left[\frac{r-q}{(p+r)^2} \right] (v) + \left[\frac{1}{p+r} \right] (u) + \left[-\frac{p+q}{(p+r)^2} \right] (1)$$

$$\frac{\partial N}{\partial w} = \frac{v(r-q)}{(p+r)^2} + \frac{u}{p+r} - \frac{p+q}{(p+r)^2}$$

$$\frac{\partial N}{\partial w} = \frac{3(10-11)}{(14+10)^2} + \frac{2}{(14+10)} - \frac{14+11}{(14+10)^2}$$

$$\frac{\partial N}{\partial w} = \frac{-3}{576} + \frac{2}{24} - \frac{25}{576} = \frac{-3+48-25}{576} = \frac{20}{576}$$

$$\boxed{\frac{\partial N}{\partial w} = \frac{5}{144}}$$

Use Equation 5 to find $\frac{dy}{dx}$

31. $y \cos x = x^2 + y^2$

Solution

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Let $F = y \cos x - x^2 - y^2$

$$F_x = -y \sin x - 2x$$

$$F_y = \cos x - 2y$$

$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{(-y \sin x - 2x)}{\cos x - 2y} = \frac{2x + y \sin x}{\cos x - 2y}$
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Use Equations 6 to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

35. $x^2 + 2y^2 + 3z^2 = 1$

Solution

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Let $F = x^2 + 2y^2 + 3z^2 - 1$

$$F_x = 2x \quad F_y = 4y \quad F_z = 6z$$

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{6z} = -\frac{x}{3z}$

$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{4y}{6z} = -\frac{2y}{3z}$
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Section 6: Directional Derivatives and the Gradient Vector

5, 7

Find the directional derivative of f at the given point in the direction indicated by the angle θ .

5. $f(x, y) = y \cos(xy)$, $(0, 1)$, $\theta = \pi/4$

Solution

$$D_u f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta$$

$$D_u f(x, y) = [-y^2 \sin(xy)] \cos \frac{\pi}{4} + [\cos(xy) - xy \sin(xy)] \sin \frac{\pi}{4}$$

$$D_u f(x, y) = \frac{\sqrt{2}}{2} [-y^2 \sin(xy) + \cos(xy) - xy \sin(xy)]$$

$$D_u f(0, 1) = \frac{\sqrt{2}}{2} [0 + 1 - 0]$$

$$D_u f(0, 1) = \frac{\sqrt{2}}{2}$$

7. $f(x, y) = \arctan(xy)$, $(2, -3)$, $\theta = 3\pi/4$

Solution

$$D_u f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta$$

$$D_u f(x, y) = \left[\frac{y}{1 + (xy)^2} \right] \cos \frac{3\pi}{4} + \left[\frac{x}{1 + (xy)^2} \right] \sin \frac{3\pi}{4}$$

$$D_u f(x, y) = \frac{\sqrt{2}}{2} \left[\frac{x}{1 + (xy)^2} - \frac{y}{1 + (xy)^2} \right]$$

$$D_u f(2, -3) = \frac{\sqrt{2}}{2} \left[\frac{2}{1 + (2(-3))^2} - \frac{-3}{1 + (2(-3))^2} \right]$$

$$D_u f(2, -3) = \frac{\sqrt{2}}{2} \left[\frac{5}{37} \right]$$

$$D_u f(2, -3) = \frac{5\sqrt{2}}{74}$$

9

$$f(x, y) = x/y, \quad P(2, 1), \quad \vec{u} = \frac{3}{5} \hat{\mathbf{i}} + \frac{4}{5} \hat{\mathbf{j}}$$

- (a) Find the gradient of f
(b) Evaluate the gradient at the point P
(c) Find the rate of change of f at P in the direction of the vector \vec{u}

Solution

(a)

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle \quad \text{or} \quad \frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}}$$

$$\boxed{\nabla f(x, y) = \left\langle \frac{1}{y}, -\frac{x}{y^2} \right\rangle}$$

(b)

$$\boxed{\nabla f(2, 1) = \left\langle \frac{1}{1}, -\frac{2}{1^2} \right\rangle = \langle 1, -2 \rangle}$$

(c)

$$D_u f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

$$D_u f(2, 1) = \nabla f(2, 1) \cdot \vec{u}$$

$$D_u f(2, 1) = \langle 1, -2 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

$$D_u f(2, 1) = \frac{3}{5} - \frac{8}{5}$$

$$\boxed{D_u f(2, 1) = -1}$$

13, 15

Find the directional derivative of the function at the given point in the direction of the vector \vec{v} .

13. $f(x, y) = e^x \sin y, \quad (0, \pi/3), \quad \vec{v} = \langle -6, 8 \rangle$

Solution

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{-6\hat{\mathbf{i}} + 8\hat{\mathbf{j}}}{\sqrt{(6)^2 + (8)^2}}$$

$$\vec{u} = -\frac{3}{5}\hat{\mathbf{i}} + \frac{4}{5}\hat{\mathbf{j}}$$

$$D_{\vec{u}}f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

$$D_{\vec{u}}f(x, y) = (e^x \sin y \hat{\mathbf{i}} + e^x \cos y \hat{\mathbf{j}}) \cdot \left(-\frac{3}{5}\hat{\mathbf{i}} + \frac{4}{5}\hat{\mathbf{j}}\right)$$

$$D_{\vec{u}}f(x, y) = -\frac{3e^x \sin y}{5} + \frac{4e^x \cos y}{5}$$

$$D_{\vec{u}}f(0, \frac{\pi}{3}) = -\frac{3(\frac{\sqrt{3}}{2})}{5} + \frac{4(\frac{1}{2})}{5}$$

$$\boxed{D_{\vec{u}}f(0, \frac{\pi}{3}) = \frac{4 - 3\sqrt{3}}{10}}$$

15. $g(s, t) = s\sqrt{t}, \quad (2, 4), \quad \vec{v} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}}$

Solution

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{2\hat{\mathbf{i}} - \hat{\mathbf{j}}}{\sqrt{(2)^2 + (-1)^2}}$$

$$\vec{u} = \frac{2}{\sqrt{5}}\hat{\mathbf{i}} - \frac{1}{\sqrt{5}}\hat{\mathbf{j}}$$

$$D_{\vec{u}}g(s, t) = \nabla g(s, t) \cdot \vec{u}$$

$$D_{\vec{u}}g(s, t) = \left(\sqrt{t}\hat{\mathbf{i}} + \frac{s}{2\sqrt{t}}\hat{\mathbf{j}}\right) \cdot \left(\frac{2}{\sqrt{5}}\hat{\mathbf{i}} - \frac{1}{\sqrt{5}}\hat{\mathbf{j}}\right)$$

$$D_{\vec{u}}g(s, t) = \frac{2\sqrt{t}}{\sqrt{5}} - \frac{s}{2\sqrt{t}\sqrt{5}}$$

$$D_{\vec{u}}g(s, t) = \frac{1}{\sqrt{5}} \left(2\sqrt{t} - \frac{s}{2\sqrt{t}}\right)$$

$$D_{\vec{u}}g(2, 4) = \frac{1}{\sqrt{5}} \left(2\sqrt{4} - \frac{2}{2\sqrt{4}}\right) = \frac{1}{\sqrt{5}} \left(4 - \frac{1}{2}\right)$$

$$\boxed{D_{\vec{u}}g(2, 4) = \frac{1}{\sqrt{5}} \left(\frac{8-1}{2}\right) = \frac{7}{2\sqrt{5}}}$$

21, 23

Find the directional derivative of the function at the point P in the direction of the point Q .

21. $f(x, y) = x^2y^2 - y^3$, $P(1, 2)$, $Q(-3, 5)$

Solution

$$\vec{v} = \vec{PQ} = \langle -3 - 1, 5 - 2 \rangle = \langle -4, 3 \rangle$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle -4, 3 \rangle}{\sqrt{16 + 9}}$$

$$\vec{u} = \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle$$

$$D_u f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

$$D_u f(x, y) = \langle 2xy^2, 2x^2y - 3y^2 \rangle \cdot \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle$$

$$D_u f(x, y) = -\frac{8xy^2}{5} + \frac{3(2x^2y - 3y^2)}{5} = \frac{1}{5}(-8xy^2 + 6x^2y - 9y^2)$$

$$D_u f(1, 2) = \frac{1}{5}[-8(1)(4) + 6(1)(2) - 9(4)] = \frac{1}{5}[-32 + 12 - 36]$$

$$\boxed{D_u f(1, 2) = -\frac{56}{5}}$$

23. $f(x, y) = \sqrt{xy}$, $P(2, 8)$, $Q(5, 4)$

Solution

$$\vec{v} = \vec{PQ} = (5 - 2)\hat{\mathbf{i}} + (4 - 8)\hat{\mathbf{j}} = 3\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{3\hat{\mathbf{i}} - 4\hat{\mathbf{j}}}{\sqrt{9 + 16}}$$

$$\vec{u} = \frac{3}{5}\hat{\mathbf{i}} - \frac{4}{5}\hat{\mathbf{j}}$$

$$D_u f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

$$D_u f(x, y) = \left(\frac{y}{2\sqrt{xy}}\hat{\mathbf{i}} + \frac{x}{2\sqrt{xy}}\hat{\mathbf{j}} \right) \cdot \left(\frac{3}{5}\hat{\mathbf{i}} - \frac{4}{5}\hat{\mathbf{j}} \right)$$

$$D_u f(x, y) = \frac{3y}{10\sqrt{xy}} - \frac{4x}{10\sqrt{xy}} = \frac{3y - 4x}{10\sqrt{xy}}$$

$$D_u f(2, 8) = \frac{3(8) - 4(2)}{10\sqrt{2(8)}} = \frac{24 - 8}{10\sqrt{16}} = \frac{16}{40}$$

$$\boxed{D_u f(x, y) = \frac{2}{5}}$$

27, 29

Find the maximum rate of change of f at the given point and the direction in which it occurs.

27. $f(x, y) = 5xy^2, \quad (3, -2)$

Solution

The maximum rate of change of f and its direction is given by the gradient vector

$$\begin{aligned}\nabla f(x, y) &= \langle f_x, f_y \rangle = \langle 5y^2, 10xy \rangle \\ \nabla f(3, -2) &= \langle 5(-2)^2, 10(3)(-2) \rangle\end{aligned}$$

Direction of the maximum rate of change,

$$\boxed{\nabla f(3, -2) = \langle 20, -60 \rangle = 20\langle 1, -3 \rangle}$$

Maximum rate of change,

$$\boxed{\|\nabla f(3, -2)\| = 20\sqrt{1+9} = 20\sqrt{10}}$$

29. $f(x, y) = \sin(xy), \quad (1, 0)$

Solution

The maximum rate of change of f and its direction is given by the gradient vector

$$\begin{aligned}\nabla f(x, y) &= \frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}} = y \cos(xy) \hat{\mathbf{i}} + x \cos(xy) \hat{\mathbf{j}} \\ \nabla f(1, 0) &= 0 + \hat{\mathbf{j}}\end{aligned}$$

Direction of the maximum rate of change,

$$\boxed{\nabla f(1, 0) = \hat{\mathbf{j}}}$$

Maximum rate of change,

$$\boxed{\|\nabla f(1, 0)\| = \sqrt{1} = 1}$$

37

The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is 120°

- (a) Find the rate of change of T at $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.
- (b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points toward the origin

Solution

Since $T \propto \frac{1}{d}$, $T(x, y, z) = \frac{c}{\sqrt{x^2 + y^2 + z^2}}$, for some constant c , and Euclidean distance to describe the distance from the origin.

If $120 = T(1, 2, 2) = \frac{c}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{c}{3}$, then $c = 360$

\therefore Our function for temperature is,

$$T(x, y, z) = \frac{360}{\sqrt{x^2 + y^2 + z^2}}$$

(a)

$$\begin{aligned}\vec{v} &= \langle 2 - 1, 1 - 2, 3 - 2 \rangle = \langle 1, -1, 1 \rangle \\ \vec{u} &= \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1, -1, 1 \rangle}{\sqrt{1 + 1 + 1}} \\ \vec{u} &= \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle\end{aligned}$$

Getting our gradient vector,

$$\begin{aligned}\nabla T(x, y, z) &= \langle T_x, T_y, T_z \rangle \\ \nabla T(x, y, z) &= \langle -180 \cdot 2x(x^2 + y^2 + z^2)^{-3/2}, -180 \cdot 2y(x^2 + y^2 + z^2)^{-3/2}, -180 \cdot 2z(x^2 + y^2 + z^2)^{-3/2} \rangle \\ \nabla T(x, y, z) &= \frac{360}{(x^2 + y^2 + z^2)^{3/2}} \langle -x, -y, -z \rangle\end{aligned}$$

so, our rate of change is

$$\begin{aligned}D_u T(1, 2, 2) &= \nabla T(1, 2, 2) \cdot \vec{u} \\ D_u T(1, 2, 2) &= \frac{360}{(1 + 4 + 4)^{3/2}} \langle -1, -2, -2 \rangle \cdot \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \\ D_u T(1, 2, 2) &= \frac{360}{27} \left(-\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}} \right) = \frac{360}{27} \left(-\frac{1}{\sqrt{3}} \right) \\ \boxed{D_u T(1, 2, 2) &= \frac{40}{3} \left(-\frac{1}{\sqrt{3}} \right) = -\frac{40}{3\sqrt{3}}}\end{aligned}$$

(b) The direction of greatest increase at any point is the direction of ∇T at that point. In part (a) we saw that $\nabla T(x, y, z)$ and $\langle -x, -y, -z \rangle$ have the same direction and the vector $\langle -x, -y, -z \rangle$ points towards the origin.

47-51 (odd)

Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

47. $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10, \quad (3, 3, 5)$

Solution

(a)

$$F(x, y, z) = 2(x-2)^2 + (y-1)^2 + (z-3)^2 - 10$$

$$F_x(3, 3, 5) = 4(x-2) = 4(1) = 4$$

$$F_y(3, 3, 5) = 2(y-1) = 2(2) = 4$$

$$F_z(3, 3, 5) = 2(z-3) = 2(2) = 4$$

The equation of the tangent plane is given by,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0)$$

$$4(x-3) + 4(y-3) + 4(z-5) = 0$$

$$x-3 + y-3 + z-5 = 0$$

$$\boxed{x + y + z = 11}$$

(b) The equation of the normal line is given by,

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

$$\frac{x-3}{4} = \frac{y-3}{4} = \frac{z-5}{4}$$

$$\boxed{x-3 = y-3 = z-5}$$

49. $xy^2z^3 = 8, \quad (2, 2, 1)$

Solution

(a)

$$F(x, y, z) = xy^2z^3 - 8$$

$$F_x(2, 2, 1) = y^2z^3 = (4)(1) = 4$$

$$F_y(2, 2, 1) = 2xy^2z^3 = 2(2)(2)(1) = 8$$

$$F_z(2, 2, 1) = 3xy^2z^2 = 3(2)(4)(1) = 24$$

The equation of the tangent plane is given by,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0)$$

$$4(x-2) + 8(y-2) + 24(z-1) = 0$$

$$x-2 + 2(y-2) + 6(z-1) = 0$$

$$x-2 + 2y-4 + 6z-6 = 0$$

$$\boxed{x + 2y + 6z = 12}$$

(b) The equation of the normal line is given by,

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

$$\frac{x - 2}{4} = \frac{y - 2}{8} = \frac{z - 1}{24}$$

$$\boxed{x - 2 = \frac{y - 2}{2} = \frac{z - 1}{6}}$$

51. $x + y + z = e^{xyz}$, $(0, 0, 1)$

Solution

(a)

$$F(x, y, z) = x + y + z - e^{xyz}$$

$$F_x(0, 0, 1) = 1 - yze^{xyz} = 1 - 0 = 1$$

$$F_y(0, 0, 1) = 1 - xze^{xyz} = 1 - 0 = 1$$

$$F_z(0, 0, 1) = 1 - xye^{xyz} = 1 - 0 = 1$$

The equation of the tangent plane is given by,

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0)$$

$$1(x - 0) + 1(y - 0) + 1(z - 1) = 0$$

$$\boxed{x + y + z = 1}$$

(b) The equation of the normal line is given by,

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

$$\frac{x - 0}{1} = \frac{y - 0}{1} = \frac{z - 1}{1}$$

$$\boxed{x = y = z - 1}$$

Section 7: Maximum and Minimum Values

5-21 (odd)

Find the local maximum and minimum values and saddle point(s) of the function. You are encouraged to use a calculator or computer to graph the function with a domain and viewpoint that reveals all the important aspects of the function.

5. $f(x, y) = x^2 + xy + y^2 + y$

Solution

$$\nabla f(x, y) = \langle 2x + y, x + 2y + 1 \rangle$$

Finding critical values, (system of two equations)

$$2x + y = 0$$

$$x + 2y = -1$$

$$2x + y = 0$$

$$-2x - 4y = 2$$

$$-3y = 2$$

$$y = -\frac{2}{3} \Rightarrow x = \frac{1}{3}$$

Critical point at $\left(\frac{1}{3}, -\frac{2}{3}\right)$

Hessian matrix to determine maxima and minima,

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 1$$

$$H_f(x, y) = \det \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$H_f\left(\frac{1}{3}, -\frac{2}{3}\right) = \det \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$$H_f\left(\frac{1}{3}, -\frac{2}{3}\right) > 0 \text{ and } f_{xx} = 2 > 0$$

$\therefore f\left(\frac{1}{3}, -\frac{2}{3}\right) \text{ is a local minimum}$

7. $f(x, y) = 2x^2 - 8xy + y^4 - 4y^3$

Solution

$$\nabla f(x, y) = \langle 4x - 8y, -8x + 4y^3 - 12y^2 \rangle$$

Finding critical values,

$$0 = 4x - 8y$$

$$2x = 8y$$

$$x = 2y$$

$$0 = -8x + 4x^3 - 12y^2$$

$$0 = -8(2y) + 4y^3 - 12y^2$$

$$0 = 4y^3 - 12y^2 - 16y$$

$$0 = 4y(y^2 - 3y - 4)$$

$$0 = 4y(y - 4)(y + 1)$$

$$y = -1, 0, 4 \Rightarrow x = -2, 0, 8$$

Critical points at $(-2, -1), (0, 0), (8, 4)$

$$f_{xx} = 4, \quad f_{yy} = 12y^2 - 24y, \quad f_{xy} = -8$$

$$H_f(8, 4) = \det \begin{vmatrix} 4 & -8 \\ -8 & 96 \end{vmatrix} = 384 - 64 = 320$$

$$H_f(-2, -1) = \det \begin{vmatrix} 4 & -8 \\ -8 & 36 \end{vmatrix} = 144 - 64 = 80$$

$$H_f(0, 0) = \det \begin{vmatrix} 4 & -8 \\ -8 & 0 \end{vmatrix} = 0 - 64 = -64$$

$H_f(8, 4) > 0$ and $f_{xx}(8, 4) = 4 > 0$ so $f(8, 4)$ is a local minima

$H_f(-2, -1) > 0$ and $f_{xx}(-2, -1) = 4 > 0$ so $f(-2, -1)$ is a local minima

$H_f(0, 0) < 0$ so $f(0, 0)$ is a saddle point

Minima: $f(8, 4) = -128, \quad f(-2, -1) = 3$

Saddle Point: $f(0, 0)$

9. $f(x, y) = (x - y)(1 - xy)$

Solution

$$\nabla f(x, y) = \langle 1 - 2xy + y^2, -x^2 - 1 + 2xy \rangle$$

Finding critical values,

$$\begin{aligned}
 0 &= 1 - 2xy + y^2 \\
 0 &= 1 - y(2x + y) \\
 1 &= y(2x + y) \\
 y = 1, \quad 2x + y &= 1 \quad \Rightarrow \quad y = 1 - 2x \\
 \\
 0 &= -x^2 - 1 + 2xy \\
 0 &= -x^2 - 1 + 2x(1 - 2x) \\
 0 &= -x^2 - 1 + 2x - 4x^2 \\
 0 &= -x^2 - 2x - 1 \\
 0 &= x^2 + 2x + 1 \\
 0 &= (x + 1)(x + 1) \\
 x = -1 \quad \Rightarrow \quad y &= 1 - 2(-1) = -1 \quad \Rightarrow \quad (-1, -1)
 \end{aligned}$$

If $y = 1$,

$$\begin{aligned}
 0 &= -x^2 + 2x - 1 \\
 0 &= x - 2x + 1 \\
 0 &= (x - 1)(x - 1) \\
 x = 1 \quad \Rightarrow \quad (1, 1)
 \end{aligned}$$

Critical points at $(-1, -1)$, $(1, 1)$

$$\begin{aligned}
 f_{xx} &= -2y, \quad f_{yy} = 2x, \quad f_{xy} = -2x + 2y \\
 H_f(-1, -1) &= \det \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 - 0 = -4 \\
 H_f(1, 1) &= \det \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -4 - 0 = -4
 \end{aligned}$$

$H_f(-1, -1)$ and $H_f(1, 1)$ are both < 0 , so they are saddle points.

Saddle Points: $f(-1, -1)$, $f(1, 1)$

11. $f(x, y) = y\sqrt{x} - y^2 - 2x + 7y$

Solution

$$\nabla f(x, y, z) = \left\langle \frac{y}{2\sqrt{x}} - 2, \sqrt{x} - 2y + 7 \right\rangle$$

Finding critical values,

$$\begin{aligned}
 0 &= \frac{1}{2}x^{-1/2} - 2 \\
 y &= 2(2x^{1/2}) \\
 y &= 4x^{1/2}
 \end{aligned}$$

$$\begin{aligned}
 x^{1/2} - 2y + 7 &= 0 \\
 x^{1/2} - 2(4x^{1/2}) + 7 &= 0 \\
 x^{1/2} - 8x^{1/2} + 7 &= 0 \\
 -7x^{1/2} &= -7 \\
 x = \sqrt{1} = 1 &\Rightarrow y = 4\sqrt{1} = 4
 \end{aligned}$$

Critical point at (1, 4)

$$f_{xx} = -\frac{y}{4x^{3/2}}, \quad f_{yy} = -2, \quad f_{xy} = \frac{1}{2\sqrt{x}}$$

$$H_f(1, 4) = \det \begin{vmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -2 \end{vmatrix} = 2 - \frac{1}{4} = \frac{7}{4}$$

$H_f(1, 4) > 0$ and $f_{xx}(1, 4) = -1 < 0$ so $f(1, 4)$ is a local maximum

Maximum: $f(1, 4) = 4$

13. $f(x, y) = x^3 - 3x + 3xy^2$

Solution

$$\nabla f(x, y) = \langle 3x^2 - 3 + 3y^2, 6xy \rangle$$

Finding critical values,

$$\begin{aligned}
 0 &= 3x^2 - 3 + 3y^2 \\
 3 &= 3x^2 + 3y^2 \\
 1 &= x^2 + y^2 \\
 x = 0, \pm 1 &\Rightarrow y = \pm 1, 0
 \end{aligned}$$

Critical points at (0, 1), (0, -1), (1, 0), (-1, 0)

$$f_{xx} = 6x, \quad f_{yy} = 6x, \quad f_{xy} = 6y$$

$$\begin{aligned}
 H_f(0, 1) &= \det \begin{vmatrix} 0 & 6 \\ 6 & 0 \end{vmatrix} = 0 - 36 = -36 \\
 H_f(0, -1) &= \det \begin{vmatrix} 0 & -6 \\ -6 & 0 \end{vmatrix} = 0 - 36 = -36 \\
 H_f(1, 0) &= \det \begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} = 36 - 0 = 36 \\
 H_f(-1, 0) &= \det \begin{vmatrix} -6 & 0 \\ 0 & -6 \end{vmatrix} = 36 - 0 = 36
 \end{aligned}$$

$H_f(0, \pm 1) < 0$ so $f(0, \pm 1)$ are saddle points

$H_f(1, 0) > 0$ and $f_{xx}(1, 0) = 6 > 0$ so $f(1, 0)$ is a local minimum

$H_f(-1, 0) > 0$ and $f_{xx}(-1, 0) = -6 < 0$ so $f(-1, 0)$ is a local maximum

Maximum: $f(-1, 0) = 2$
Minimum: $f(1, 0) = -2$
Saddle Points: $f(0, \pm 1)$

15. $f(x, y) = x^4 - 2x^2 + y^3 - 3y$

17. $f(x, y) = xy - x^2y - xy^2$

19. $f(x, y) = e^x \cos y$

21. $f(x, y) = y^2 - 2y \cos x, \quad -1 \leq x \leq 7$

33-39 (odd)

Find the absolute maximum and minimum values of f on the set D .

33. $f(x, y) = x^2 + y^2 - 2x,$

D is the closed triangular region with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$

35. $f(x, y) = x^2 + y^2 + x^2y + 4,$

$D = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}$

37. $f(x, y) = x^2 + 2y^2 - 2x - 4y + 1,$

$D = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 3\}$

39. $f(x, y) = 2x^3 + y^4,$

$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$

43

Find the shortest distance from the point $(2, 0, -3)$ to the plane $x + y + z = 1$.

45

Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4, 2, 0)$.

47

Find three positive numbers whose sum is 100 and whose product is a maximum.

55

A cardboard box without a lid is to have a volume of $32,000 \text{ cm}^3$. Find the dimensions that minimize the amount of cardboard used.

Section 8: Lagrange Multipliers

3-13 (odd)

Each of these extreme value problems has a solution with both a maximum value and minimum value. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint.

3. $f(x, y) = x^2 - y^2, \quad x^2 + y^2 = 1$

5. $f(x, y) = xy, \quad 4x^2 + y^2 = 8$

7. $f(x, y) = 2x^2 + 6y^2, \quad x^4 + 3y^4 = 1$

9. $f(x, y, z) = 2x + 2y + z, \quad x^2 + y^2 + z^2 = 9$

11. $f(x, y, z) = xy^2z, \quad x^2 + y^2 + z^2 = 4$

13. $f(x, y, z) = x^2 + y^2 + z^2, \quad x^4 + y^4 + z^4 = 1$

23

The method of Lagrange multipliers assumes that the extreme values exist, but that is not always the case. Show that the problem of finding the minimum value of f subject to the given constraint can be solved using Lagrange multipliers, but f does not have a maximum value with that constraint.

$$f(x, y) = x^2 + y^2, \quad xy = 1$$

25

Use Lagrange multipliers to find the maximum value of f subject to the given constraint. Then show that f has no minimum value with that constraint.

$$f(x, y) = e^{xy}, \quad x^3 + y^3 = 16$$

27, 29

Find the extreme values of f on the region described by the inequality.

27. $f(x, y) = x^2 + y^2 + 4x - 4y, \quad x^2 + y^2 \leq 9$

29. $f(x, y) = e^{-xy}, \quad x^2 + 4y^2 \leq 1$

31, 33

Find the extreme values of f subject to both constraints

31. $f(x, y, z) = x + y + z; \quad x^2 + z^2 = 2, \quad x + y = 1$

33. $f(x, y, z) = yz + xy; \quad xy = 1, \quad y^2 + z^2 = 1$