

Chapter 13 Section 3 & 4 Problem Set

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October 9, 2023

Section 3: Arc Length and Curvature

Problem 1a

Use Equation 2 to compute the length of the given line segment.

$$\vec{r}(t) = \langle 3 - t, 2t, 4t + 1 \rangle \quad 1 \leq t \leq 3$$

Solution

Let the length of the line segment be

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \Rightarrow L = \int_a^b \|\vec{r}'(t)\| dt$$

$$D : \{ t \mid 1 \leq t \leq 3 \}$$

$$\vec{r}'(t) = \langle -1, 2, 4 \rangle \Rightarrow L = \int_1^3 \sqrt{(-1)^2 + (2)^2 + (4)^2} dt = \int_1^3 \sqrt{21} dt = \sqrt{21}t \Big|_1^3 = \sqrt{21}(3) - \sqrt{21}(1) = 2\sqrt{21}$$

Problems 3-7 odd

Find the length of the curve.

3. $\vec{r}(t) = \langle t, 3 \cos t, 3 \sin t \rangle \quad 25 \leq t \leq 5$

Solution

$$\begin{aligned} \vec{r}'(t) = \langle 1, -3 \sin t, 3 \cos t \rangle &\Rightarrow L = \int_{-5}^5 \sqrt{1^2 + (-3 \sin t)^2 + (3 \cos t)^2} dt \\ &= \int_{-5}^5 \sqrt{1 + 9 \sin^2 t + 9 \cos^2 t} dt \\ &= \int_{-5}^5 \sqrt{1 + 9(1)} dt \\ &= \int_{-5}^5 \sqrt{10} dt \\ &= \sqrt{10}t \Big|_{-5}^5 = \sqrt{10}(5) - \sqrt{10}(-5) = 10\sqrt{10} \end{aligned}$$

5. $\vec{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle \quad 0 \leq t \leq 1$

Solution

$$\begin{aligned}\vec{r}'(t) &= \langle \sqrt{2}, e^t, -e^{-t} \rangle \Rightarrow L = \int_0^1 \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2} dt \\ &= \int_0^1 \sqrt{2 + e^{2t} + e^{-2t}} dt \\ &= \int_0^1 \sqrt{(e^t + e^{-t})^2} dt \\ &= \int_0^1 e^t + e^{-t} dt \\ &= e^t - e^{-t} \Big|_0^1 = (e^1 - \frac{1}{e^1}) - (e^0 - \frac{1}{e^0}) = e - \frac{1}{e} - 1 + 1 = e - \frac{1}{e}\end{aligned}$$

7. $\vec{r}(t) = \langle 1, t^2, t^3 \rangle \quad 0 \leq t \leq 1$

Solution

$$\begin{aligned}\vec{r}'(t) &= \langle 0, 2t, 3t^2 \rangle \Rightarrow L = \int_0^1 \sqrt{(0)^2 + (2t)^2 + (3t^2)^2} dt \\ &= \int_0^1 \sqrt{4t^2 + 9t^4} dt \\ &= \int_0^1 \sqrt{t^2(4 + 9t^2)} dt \\ &= \int_0^1 t\sqrt{4 + 9t^2} dt\end{aligned}$$

Using u-substitution,

$$u^2 = 4 + 9t^2$$

$$2u du = 18t dt$$

$$\frac{u du}{9} = t dt$$

$$\int_0^1 t\sqrt{4 + 9t^2} dt = \int_0^1 u \cdot \left(\frac{1}{9}u\right) du = \frac{1}{9} \int_0^1 u^2 du = \frac{1}{9} \left(\frac{1}{3}u^3\right) \Big|_2^{\sqrt{13}} = \frac{1}{27}(u^3) \Big|_2^{\sqrt{13}} = \frac{1}{27}(13^{\frac{3}{2}} - 2^3) = \frac{13\sqrt{13}}{27} - 3$$

Problems 19-23 odd

(a) Find the unit tangent and unit normal vectors $\vec{T}(t)$ and $\vec{N}(t)$.

(b) Use Formula 9 to find the curvature.

19. $\vec{r}(t) = \langle t^2, \sin t - t \cos t, \cos t + t \sin t \rangle, \quad t > 0$

Solution

a.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

$$\begin{aligned}\vec{r}'(t) &= \langle 2t, \cos t - \cos t + t \sin t, -\sin t + \sin t + t \cos t \rangle = \langle 2t, t \sin t, t \cos t \rangle \\ \|\vec{r}'(t)\| &= \sqrt{4t^2 + t^2 \sin^2 t + t^2 \cos^2 t} = \sqrt{4t^2 + t^2} = \sqrt{5t^2} = \sqrt{5}t \quad [\cos^2 t + \sin^2 t = 1] \\ \vec{T}(t) &= \frac{\langle 2t, t \sin t, t \cos t \rangle}{\sqrt{5}t} = \frac{1}{\sqrt{5}} \langle 2, \sin t, \cos t \rangle\end{aligned}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

$$\begin{aligned}\vec{T}'(t) &= \frac{1}{\sqrt{5}} \langle 0, \cos t, -\sin t \rangle \\ \|\vec{T}'(t)\| &= \frac{1}{\sqrt{5}} \sqrt{0^2 + \cos^2 t + \sin^2 t} = \sqrt{1} = \frac{1}{\sqrt{5}} \\ \vec{N}(t) &= \frac{\frac{1}{\sqrt{5}} \langle 0, \cos t, -\sin t \rangle}{\frac{1}{\sqrt{5}}} = \langle 0, \cos t, -\sin t \rangle\end{aligned}$$

b.

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\frac{1}{\sqrt{5}}}{\sqrt{5}t} = \frac{1}{5t}$$

21. $\vec{r}(t) = \langle t, t^2, 4 \rangle$

Solution

a.

$$\begin{aligned}\vec{r}'(t) &= \hat{\mathbf{i}} + 2t \hat{\mathbf{j}} \\ \|\vec{r}'(t)\| &= \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2} \\ \vec{T}(t) &= \frac{\hat{\mathbf{i}} + 2t \hat{\mathbf{j}}}{\sqrt{1 + 4t^2}} = \frac{1}{\sqrt{1 + 4t^2}} (\hat{\mathbf{i}} + 2t \hat{\mathbf{j}}) \\ \frac{d}{dt} [f(t)\vec{u}(t)] &= f'(t)\vec{u}(t) + f(t)\vec{u}'(t) \quad [\text{vector product rule}] \\ \vec{T}'(t) &= -\frac{4t}{(1 + 4t^2)^{\frac{3}{2}}} (\hat{\mathbf{i}} + 2t \hat{\mathbf{j}}) + \frac{1}{(1 + 4t^2)^{\frac{1}{2}}} (2 \hat{\mathbf{j}}) \\ &= \frac{1}{(1 + 4t^2)^{\frac{3}{2}}} (-4t(\hat{\mathbf{i}} + 2t \hat{\mathbf{j}}) + (1 + 4t^2)(2 \hat{\mathbf{j}})) = \frac{1}{(1 + 4t^2)^{\frac{3}{2}}} (-4t \hat{\mathbf{i}} - 8t^2 \hat{\mathbf{j}} + 2 \hat{\mathbf{j}} + 8t^2 \hat{\mathbf{j}}) \\ &= \frac{1}{(1 + 4t^2)^{\frac{3}{2}}} (-4t \hat{\mathbf{i}} + 2 \hat{\mathbf{j}}) \\ \|\vec{T}'(t)\| &= \frac{1}{(1 + 4t^2)^{\frac{3}{2}}} \sqrt{(-4t)^2 + 2^2} = \frac{1}{(1 + 4t^2)^{\frac{3}{2}}} \sqrt{16t^2 + 4} = \frac{1}{(1 + 4t^2)^{\frac{3}{2}}} \sqrt{4(4t^2 + 1)} = \frac{2}{(1 + 4t^2)^{\frac{3}{2}}} \sqrt{1 + 4t^2} \\ &= \frac{2}{1 + 4t^2} \\ \vec{N}(t) &= \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \frac{1}{(1 + 4t^2)^{\frac{3}{2}}} (-4t \hat{\mathbf{i}} + 2 \hat{\mathbf{j}}) \cdot \frac{1 + 4t^2}{2} = \frac{(1 + 4t^2)^1}{2(1 + 4t^2)^{\frac{3}{2}}} (-4t \hat{\mathbf{i}} + 2 \hat{\mathbf{j}}) = \frac{1}{\sqrt{1 + 4t^2}} (-2t \hat{\mathbf{i}} + \hat{\mathbf{j}})\end{aligned}$$

b.

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{2}{1 + 4t^2} \cdot \frac{1}{(1 + 4t^2)^{\frac{1}{2}}} = \frac{2}{(1 + 4t^2)^{\frac{3}{2}}}$$

23. $\vec{r}(t) = \langle t, \frac{1}{2}t^2, t^2 \rangle$

Solution

a.

$$\begin{aligned}
 \vec{r}'(t) &= \langle 1, t, 2t \rangle \\
 \|\vec{r}'(t)\| &= \sqrt{1^2 + t^2 + (2t)^2} = \sqrt{1 + 5t^2} \\
 \vec{T}(t) &= \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\langle 1, t, 2t \rangle}{\sqrt{1 + t^2 + 4t^2}} = \frac{1}{\sqrt{1 + 5t^2}} \langle 1, t, 2t \rangle \\
 \frac{d}{dt} &= [f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t) \quad [\text{vector product rule}] \\
 \vec{T}'(t) &= -\frac{5t}{(1 + 5t^2)^{\frac{3}{2}}} \langle 1, t, 2t \rangle + \frac{1}{(1 + 5t^2)^{\frac{1}{2}}} \langle 0, 1, 2 \rangle \\
 &= \frac{1}{(1 + 5t^2)^{\frac{3}{2}}} (-5t \langle 1, t, 2t \rangle + (1 + 5t^2) \langle 0, 1, 2 \rangle) \\
 &= \frac{1}{(1 + 5t^2)^{\frac{3}{2}}} (\langle -5t, -5t^2, -10t^2 \rangle + \langle 0, 1 + 5t^2, 2 + 10t^2 \rangle) = \frac{1}{(1 + 5t^2)^{\frac{3}{2}}} \langle -5t, 1, 2 \rangle \\
 \|\vec{T}'(t)\| &= \frac{1}{(1 + 5t^2)^{\frac{3}{2}}} \sqrt{(-5t)^2 + 1^2 + 2^2} = \frac{1}{(1 + 5t^2)^{\frac{3}{2}}} \sqrt{25t^2 + 5} = \frac{1}{(1 + 5t^2)^{\frac{3}{2}}} \sqrt{5(5t^2 + 1)} \\
 &= \frac{\sqrt{5}(1 + 5t^2)^{\frac{1}{2}}}{(1 + 5t^2)^{\frac{3}{2}}} = \frac{\sqrt{5}}{1 + 5t^2} \\
 \|\vec{N}(t)\| &= \frac{\|\vec{T}'(t)\|}{\|\vec{T}'(t)\|} = \frac{1}{(1 + 5t^2)^{\frac{3}{2}}} \langle -5t, 1, 2 \rangle \cdot \frac{1 + 5t^2}{\sqrt{5}} = \frac{1}{\sqrt{5}\sqrt{1 + 5t^2}} \langle -5t, 1, 2 \rangle = \frac{1}{\sqrt{5 + 25t^2}} \langle -5t, 1, 2 \rangle
 \end{aligned}$$

b.

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\sqrt{5}}{1 + 5t^2} \cdot \frac{1}{\sqrt{1 + 5t^2}} = \frac{\sqrt{5}}{(1 + 5t^2)^{\frac{3}{2}}}$$

Problem 27

Use Theorem 10 to find the curvature

$$\vec{r}(t) = \sqrt{6}t^2 \hat{i} + 2t \hat{j} + 2t^3 \hat{k}$$

Solution

Theorem 10 states that

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

$$\vec{r}'(t) = 2\sqrt{6}t \hat{i} + 2 \hat{j} + 6t^2 \hat{k}$$

$$\vec{r}''(t) = 2\sqrt{6} \hat{i} + 12t \hat{k}$$

$$\|\vec{r}'(t)\| = \sqrt{(2\sqrt{6}t)^2 + 2^2 + (6t^2)^2} = \sqrt{24t^2 + 4 + 36t^4} = \sqrt{4(9t^4 + 6t^2 + 1)} = 2\sqrt{(3t^2 + 1)^2} = 2(3t^2 + 1)$$

$$\begin{aligned}
 \vec{r}'(t) \times \vec{r}''(t) &\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2\sqrt{6}t & 2 & 6t^2 \\ 2\sqrt{6} & 0 & 12t \end{vmatrix} = \hat{i} \begin{vmatrix} 2 & 6t^2 \\ 0 & 12t \end{vmatrix} - \hat{j} \begin{vmatrix} 2\sqrt{6}t & 6t^2 \\ 2\sqrt{6} & 12t \end{vmatrix} + \hat{k} \begin{vmatrix} 2\sqrt{6}t & 2 \\ 2\sqrt{6} & 0 \end{vmatrix} \\
 &= (24t - 0) \hat{i} - (24t^2\sqrt{6} - 12t^2\sqrt{6}) \hat{j} + (0 - 4\sqrt{6}) \hat{k} = 24t \hat{i} - 12t^2\sqrt{6} \hat{j} - 4\sqrt{6} \hat{k}
 \end{aligned}$$

$$\begin{aligned}
 \|\vec{r}'(t) \times \vec{r}''(t)\| &= \sqrt{(24t)^2 + (-12t^2\sqrt{6})^2 + (-4\sqrt{6})^2} = \sqrt{576t^2 + 864t^4 + 96} = \sqrt{96(9t^4 + 6t^2 + 1)} = \sqrt{16 \cdot 6(3t^2 + 1)^2} \\
 \|\vec{r}'(t) \times \vec{r}''(t)\| &= 4\sqrt{6}(3t^2 + 1)
 \end{aligned}$$

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{4\sqrt{6}(3t^2+1)}{(2(3t^2+1))^3} = \frac{4\sqrt{6}(3t^2+1)}{8(3t^2+1)^3} = \frac{\sqrt{6}}{2(3t^2+1)^2}$$

Problem 28

Find the curvature of $\vec{r}(t) = \langle t^2, \ln t, t \ln t \rangle$ at the point $(1, 0, 0)$.

Solution

If $x = t^2 \Rightarrow 1 = t^2$, then

$$t = 1? \Rightarrow \ln 1 \equiv 0, \quad 1 \ln 1 \equiv 0$$

$$\therefore t = 1$$

$$\vec{r}'(t) = \langle 2t, \frac{1}{t}, \ln t + 1 \rangle, \quad \vec{r}''(t) = \langle 2, -\frac{1}{t^2}, \frac{1}{t} \rangle$$

$$\vec{r}'(1) = \langle 2, 1, 1 \rangle, \quad \|\vec{r}'(1)\| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}, \quad \vec{r}''(1) = \langle 2, -1, 1 \rangle$$

$$\begin{aligned} \vec{r}'(1) \times \vec{r}''(1) &\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = \hat{i} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} \\ &= (1 - (-1))\hat{i} - (2 - 2)\hat{j} + (-2 - 2)\hat{k} = 2\hat{i} - 4\hat{k} \Rightarrow \langle 2, 0, -4 \rangle \end{aligned}$$

$$\kappa(1) = \frac{\|\langle 2, 0, -4 \rangle\|}{\sqrt{6}^3} = \frac{\sqrt{2^2 + 0^2 + (-4)^2}}{6^{\frac{3}{2}}} = \frac{\sqrt{20}}{6\sqrt{6}} = \frac{2\sqrt{5}}{6\sqrt{6}} = \frac{\sqrt{5}}{3\sqrt{6}} = \frac{\sqrt{30}}{18}$$

Problem 31 & 33

Use Formula 11 to find the curvature.

31. $y = x^4$

Formula 11 states that

$$\kappa = \frac{|f''(x)|}{[1 + (f'(x)^2)]^{\frac{3}{2}}}$$

$$y = x^4 \Rightarrow y' = 4x^3 \Rightarrow y'' = 12x^2$$

$$\kappa = \frac{|12x^2|}{[1 + (4x^3)^2]^{\frac{3}{2}}} = \frac{12x^2}{(1 + 16x^6)^{\frac{3}{2}}}$$

Solution

33. $y = xe^x$

Solution

Formula 11 states that

$$\kappa = \frac{|f''(x)|}{[1 + (f'(x)^2)]^{\frac{3}{2}}}$$

$$y' = e^x + xe^x \Rightarrow y'' = e^x + e^x + xe^x = 2e^x + xe^x$$

$$\kappa = \frac{|2e^x + xe^x|}{[1 + (e^x + xe^x)^2]^{\frac{3}{2}}} = \frac{e^x(2 + x)}{[1 + (e^x + xe^x)^2]^{\frac{3}{2}}}$$

Problem 51

Find the vectors \mathbf{T} , \mathbf{N} , and \mathbf{B} at the given point.

$$\vec{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle, \quad (1, \frac{2}{3}, 1)$$

Solution

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}, \quad \vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}, \quad \vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

$$z = t \Rightarrow t = 1?, \quad 1^2 \equiv 1, \quad \frac{2}{3}1^3 \equiv \frac{2}{3}, \quad 1 \equiv 1 \quad \therefore t = 1$$

$$\vec{r}'(t) = \langle 2t, 2t^2, 1 \rangle, \quad \|\vec{r}'(t)\| = \sqrt{(2t)^2 + (2t^2)^2 + (1)^2} = \sqrt{4t^2 + 4t^4 + 1} = \sqrt{(2t^2 + 1)^2} = 2t^2 + 1$$

$$\vec{T}(t) = \frac{\langle 2t, 2t^2, 1 \rangle}{2t^2 + 1} \Rightarrow \vec{T}(1) = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle, \quad f(t) = \frac{1}{2t^2 + 1}, \quad \vec{u} = \langle 2t, 2t^2, 1 \rangle \Rightarrow \vec{T}'(t) = f'(t) \vec{u} + f(t) \vec{u}'$$

$$\begin{aligned} \vec{T}'(t) &= -4t(2t^2 + 1)^{-2} \langle 2t, 2t^2, 1 \rangle + (2t^2 + 1)^{-1} \langle 2, 4t, 0 \rangle = (2t^2 + 1)^{-2} (-4t \langle 2t, 2t^2, 1 \rangle + (2t^2 + 1) \langle 2, 4t, 0 \rangle) \\ &= (2t^2 + 1)^{-2} (\langle -8t^2, -8t^3, -4t \rangle + \langle 4t^2 + 2, 8t^3 + 4t, 0 \rangle) \\ &= (2t^2 + 1)^{-2} \langle -4t^2 + 2, 4t, -4t \rangle \\ &= 2(2t^2 + 1)^{-2} \langle -2t^2 + 1, 2t, -2t \rangle \end{aligned}$$

$$\vec{T}'(1) = 2(2(1)^2 + 1)^{-2} \langle -2(1)^2 + 1, 2(1), -2(1) \rangle = 2(2 + 1)^{-2} \langle -2 + 1, 2, -2 \rangle = \frac{2}{9} \langle -1, 2, -2 \rangle = \langle -\frac{2}{9}, \frac{4}{9}, -\frac{4}{9} \rangle$$

$$\vec{N}(1) = \frac{\langle -\frac{2}{9}, \frac{4}{9}, -\frac{4}{9} \rangle}{\sqrt{(-\frac{2}{9})^2 + (\frac{4}{9})^2 + (-\frac{4}{9})^2}} = \frac{\langle -\frac{2}{9}, \frac{4}{9}, -\frac{4}{9} \rangle}{\sqrt{\frac{36}{81}}} = \frac{9}{6} \langle -\frac{2}{9}, \frac{4}{9}, -\frac{4}{9} \rangle = \langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \rangle$$

$$\vec{B}(1) = \vec{T}(1) \times \vec{N}(1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{vmatrix} = (-\frac{4}{9} - \frac{2}{9}) \hat{\mathbf{i}} + (-\frac{1}{9} + \frac{4}{9}) \hat{\mathbf{j}} + (\frac{4}{9} - (-\frac{2}{9})) \hat{\mathbf{k}} = \langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \rangle$$

Problem 53

Find equations of the normal plane and osculating plane of the curve at the given point.

$$x = \sin 2t, \quad y = -\cos 2t, \quad z = 4t; \quad (0, 1, 2\pi)$$

Solution

If $z = 4t$ and $z = 2\pi$, then $t = \frac{2\pi}{4} = \frac{\pi}{2}$

$$0 \equiv \sin 2(\frac{\pi}{2}), \quad 1 \equiv -\cos(2(\frac{\pi}{2})), \quad 2\pi \equiv 4(\frac{\pi}{2}) \Rightarrow \therefore t = \frac{\pi}{2}$$

The point $(0, 1, 2\pi)$ corresponds to $t = \frac{\pi}{2}$

$$\text{Let } \vec{r}(t) = \langle \sin 2t, -\cos 2t, 4t \rangle$$

$$\vec{r}'(t) = \langle 2 \cos 2t, 2 \sin 2t, 4 \rangle$$

$$\vec{r}'(\frac{\pi}{2}) = \langle -2, 0, 4 \rangle$$

So, the normal plane has normal vector $\vec{r}'(\frac{\pi}{2}) = \langle -2, 0, 4 \rangle$

\therefore The equation of the normal plane is

$$-2(x - 0) + 0(y - 1) + 4(z - 2\pi) = 0 \Rightarrow -2x + 4z - 8\pi = 0 \quad \text{or} \quad 4z - x = 4\pi$$

To find the osculating plane at $(0, 1, 2\pi)$ we need vectors $\vec{T}(t)$ and $\vec{N}(t)$

$$\begin{aligned} \vec{T}(t) &= \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\langle 2 \cos 2t, 2 \sin 2t, 4 \rangle}{\sqrt{4 \cos^2 2t + 4 \sin^2 2t + 16}} = \frac{\langle 2 \cos 2t, 2 \sin 2t, 4 \rangle}{\sqrt{20}} = \frac{1}{2\sqrt{5}} \langle 2 \cos 2t, 2 \sin 2t, 2 \rangle \\ &= \frac{1}{\sqrt{5}} \langle \cos 2t, \sin 2t, 2 \rangle \end{aligned}$$

$$\vec{T}\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{5}} \langle -1, 0, 2 \rangle \quad \vec{T}'(t) = \frac{1}{\sqrt{5}} \langle -2 \sin 2t, 2 \cos 2t, 0 \rangle \quad \|\vec{T}'(t)\| = \sqrt{4 \sin^2 2t + 4 \cos^2 2t} = \frac{2}{\sqrt{5}}$$

$$\vec{N}\left(\frac{\pi}{2}\right) = \frac{\frac{1}{\sqrt{5}} \langle 0, -2, 0 \rangle}{\frac{2}{\sqrt{5}}} = \frac{1}{2} \langle 0, -2, 0 \rangle = \langle 0, -1, 0 \rangle$$

A vector normal to the osculating plane would be $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) \Rightarrow \vec{B}\left(\frac{\pi}{2}\right) = \vec{T}\left(\frac{\pi}{2}\right) \times \vec{N}\left(\frac{\pi}{2}\right)$

$$= \frac{1}{\sqrt{5}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 2 \\ 0 & -1 & 0 \end{vmatrix} = \frac{1}{\sqrt{5}} \langle 0 - (-2), 0 - 0, 1 - 0 \rangle = \frac{1}{\sqrt{5}} \langle 2, 0, 1 \rangle$$

\therefore The equation of the osculating plane is

$$2(x - 0) + 0(y - 1) + 1(z - 2\pi) = 0 \Rightarrow 2x + z - 2\pi = 0 \quad \text{or} \quad 2x + z = 2\pi$$

Problem 66

Use Formula 14 to find the torsion at the given value of t .

$$\vec{r}(t) = \langle \sin t, 3t, \cos t \rangle, \quad t = \frac{\pi}{2}$$

Solution

The torsion of a curve with the parameter t is defined as

$$\tau = -\frac{\vec{B}'(t) \cdot \vec{N}(t)}{\|\vec{r}'(t)\|}$$

We need to find $\vec{T}(t)$, $\vec{N}(t)$, and $\vec{B}(t)$

$$\vec{r}(t) = \langle \sin t, 3t, \cos t \rangle \Rightarrow \vec{r}'(t) = \langle \cos t, 3, -\sin t \rangle \Rightarrow \|\vec{r}'(t)\| = \sqrt{\cos^2 t + 9 + \sin^2 t} = \sqrt{10}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\langle \cos t, 3, -\sin t \rangle}{\sqrt{10}} = \frac{1}{\sqrt{10}} \langle \cos t, 3, -\sin t \rangle$$

$$\vec{T}'(t) = \frac{1}{\sqrt{10}} \langle -\sin t, 0, -\cos t \rangle \Rightarrow \|\vec{T}'(t)\| = \frac{1}{\sqrt{10}} \sqrt{\sin^2 t + \cos^2 t} = \frac{1}{\sqrt{10}}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \frac{1}{1} \langle -\sin t, 0, -\cos t \rangle = \langle -\sin t, 0, -\cos t \rangle$$

$$\begin{aligned}\vec{B}(t) &= \vec{T}(t) \times \vec{N}(t) = \frac{1}{\sqrt{10}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & 3 & -\sin t \\ -\sin t & 0 & -\cos t \end{vmatrix} = \frac{1}{\sqrt{10}} \langle -3 \cos t - 0, \sin^2 t - (-\cos^2 t), 0 - (-3 \sin t) \rangle \\ &= \frac{1}{\sqrt{10}} \langle -3 \cos t, 1, 3 \sin t \rangle\end{aligned}$$

$$\begin{aligned}\vec{B}'(t) &= \frac{1}{\sqrt{10}} \langle 3 \sin t, 0, 3 \cos t \rangle \Rightarrow \vec{B}'(t) \cdot \vec{N}(t) = \frac{1}{\sqrt{10}} \langle 3 \sin t, 0, 3 \cos t \rangle \cdot \langle -\sin t, 0, -\cos t \rangle \\ &= \frac{1}{\sqrt{10}} (-3 \sin^2 t - 3 \cos^2 t) = \frac{-3}{\sqrt{10}}\end{aligned}$$

\therefore The torsion of the curve at $t = \frac{\pi}{2}$ is

$$\tau = -\frac{-\frac{3}{\sqrt{10}}}{\sqrt{10}} = \frac{3}{10}$$

Problem 70

Use Theorem 15 to find the torsion of the given curve at a general point and at the point corresponding to $t = 0$

$$\vec{r}(t) = \langle \cos t, \sin t, \sin t \rangle$$

Solution

Theorem 15 states that

$$\tau(t) = \frac{[\vec{r}'(t) \times \vec{r}''(t)] \cdot \vec{r}'''(t)}{\|\vec{r}'(t) \times \vec{r}''(t)\|^2}$$

$$\vec{r}'(t) = \langle -\sin t, \cos t, \cos t \rangle \quad \vec{r}''(t) = \langle -\cos t, -\sin t, -\sin t \rangle \quad \vec{r}'''(t) = \langle \sin t, -\cos t, -\cos t \rangle$$

$$\vec{r}'(0) = \langle 0, 1, 1 \rangle \quad \vec{r}''(0) = \langle -1, 0, 0 \rangle \quad \vec{r}'''(0) = \langle 0, -1, -1 \rangle$$

$$\begin{aligned}[\vec{r}'(0) \times \vec{r}''(0)] \cdot \vec{r}'''(0) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{vmatrix} = \langle 0 - 0, -1 - 0, 0 - (-1) \rangle = \langle 0, -1, 1 \rangle \cdot \langle 0, -1, -1 \rangle \\ &= 0 + 1 - 1 = 0\end{aligned}$$

\therefore The torsion of the curve at $t = 0$ is

$$\tau(0) = \frac{0}{\sqrt{2}^2} = 0$$

Section 4: Motion in Space - Velocity and Acceleration

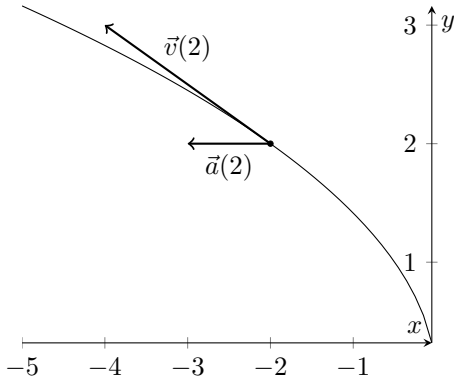
Problem 3-7 odd

Find the velocity, acceleration, and speed of a particle with the given position function. Sketch the path of the particle and draw the velocity and acceleration vectors for the specified value of t .

3. $\vec{r}(t) = \langle -\frac{1}{2}t^2, t \rangle, \quad t = 2$

Solution

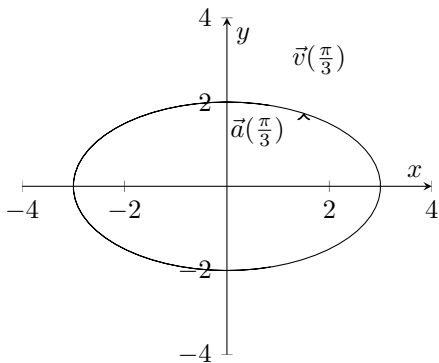
$$\begin{aligned}\vec{r}'(t) = \vec{v}(t) &= \langle -t, 1 \rangle \Rightarrow \vec{v}(2) = \langle -2, 1 \rangle \\ \vec{r}''(t) = \vec{a}(t) &= \langle -1, 0 \rangle \Rightarrow \vec{a}(2) = \langle -1, 0 \rangle \\ \|\vec{v}(t)\| &= \sqrt{t^2 + 1}\end{aligned}$$



5. $\vec{r}(t) = 3 \cos t \hat{i} + 2 \sin t \hat{j} \quad t = \frac{\pi}{3}$

Solution

$$\begin{aligned}\vec{r}'(t) = \vec{v}(t) &= \langle -3 \sin t, 2 \cos t \rangle \Rightarrow \vec{v}\left(\frac{\pi}{3}\right) = \left\langle -3\left(\frac{\sqrt{3}}{2}\right), 2\left(\frac{1}{2}\right) \right\rangle = \left\langle -\frac{3\sqrt{3}}{2}, 1 \right\rangle \\ \vec{r}''(t) = \vec{a}(t) &= \langle -3 \cos t, -2 \sin t \rangle \Rightarrow \vec{a}\left(\frac{\pi}{3}\right) = \left\langle -3\left(\frac{1}{2}\right), -2\left(\frac{\sqrt{3}}{2}\right) \right\rangle = \left\langle -\frac{3}{2}, -\sqrt{3} \right\rangle \\ \|\vec{v}(t)\| &= \sqrt{(-3 \sin t)^2 + (2 \cos t)^2} = \sqrt{9 \sin^2 t + 4 \cos^2 t} = \sqrt{9(1 - \cos^2 t) + 4 \cos^2 t} = \sqrt{9 - 5 \cos^2 t}\end{aligned}$$



7. $\vec{r}(t) = t \hat{i} + t^2 \hat{j} + 2 \hat{k} \quad t = 1$

Problems 9-13 odd

Find the velocity, acceleration, and speed of a particle with the given position function.

9. $\vec{r}(t) = \langle t^2 + t, t^2 - t, t^3 \rangle$

11. $\vec{r}(t) = \sqrt{2}t \hat{i} + e^t \hat{j} + e^{-t} \hat{k}$

13. $\vec{r}(t) = e^t(\cos t \hat{i} + \sin t \hat{j} + t \hat{k})$

Problem 15

Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

$$a(t) = 2 \hat{i} + 2t \hat{k}, \quad v(0) = 3 \hat{i} - \hat{j}, \quad r(0) = \hat{j} + \hat{k}$$

Problem 17a

Find the position vector of a particle that has the given acceleration and the specified initial velocity and position.

$$a(t) = 2t \hat{\mathbf{i}} + \sin t \hat{\mathbf{j}} + \cos 2t \hat{\mathbf{k}}, \quad v(0) = \hat{\mathbf{i}}, \quad r(0) = \hat{\mathbf{j}}$$

Problem 23

A projectile is fired with an initial speed of $200 \frac{m}{s}$ and angle of elevation 60° . Find (a) the range of the projectile, (b) the maximum height reached, and (c) the speed at impact.

Problem 26

A projectile is fired from a tank with initial speed $400 \frac{m}{s}$. Find two angles of elevation that can be used to hit a target $3000m$ away.

Problem 27

A rifle is fired with angle of elevation 36° . What is the initial speed if the maximum height of the bullet is $1600ft$?

Problem 37 & 39

Find the tangential and normal components of the acceleration vector.

37. $\vec{r}(t) = (t^2 + 1) \hat{\mathbf{i}} + t^3 \hat{\mathbf{j}}, \quad t \geq 0$

39. $\vec{r}(t) = \cos t \hat{\mathbf{i}} + \sin t \hat{\mathbf{j}} + t \hat{\mathbf{k}}$

Problem 41

Find the tangential and normal components of the acceleration vector at the given point.

$$\vec{r}(t) = \ln t \hat{\mathbf{i}} + (t^2 + 3t) \hat{\mathbf{j}} + 4\sqrt{t} \hat{\mathbf{k}}, \quad (0, 4, 4)$$