

Ch 14 - Problem Set 1

Calculus 3

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Section 1: Functions of Several Variables

3. Let $g(x, y) = x^2 \ln(x + y)$

(a) Evaluate $g(3, 1)$.

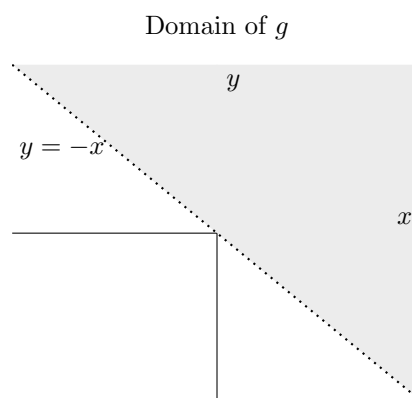
(b) Find and sketch the domain of g .

(c) Find the range of g .

Solution

a) $9 \ln 4$

b) $D : \{(x, y) \mid y > -x\}$



c) \mathbb{R}

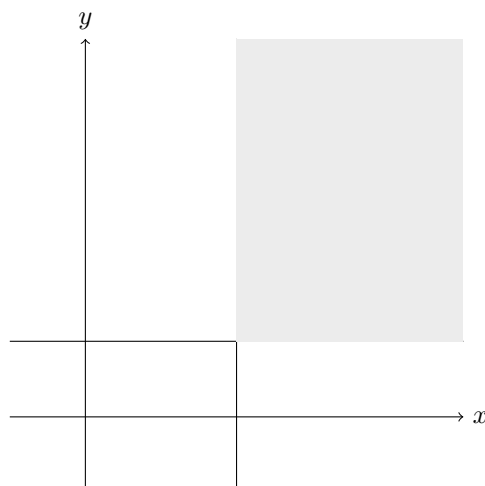
7 - 15 (odd)

Find and sketch the domain of the function.

7. $f(x, y) = \sqrt{x - 2} + \sqrt{y - 1}$

Solution

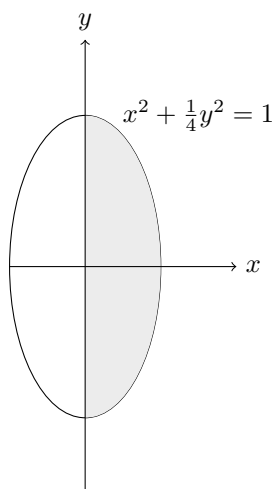
$D : \{(x, y) \mid x \geq 2, y \geq 1\}$



9. $q(x, y) = \sqrt{x} + \sqrt{4 - 4x^2 - y^2}$

Solution

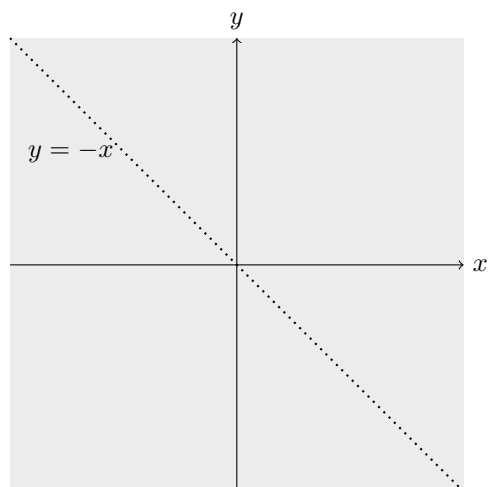
$D : \{(x, y) \mid x^2 + \frac{1}{4}y^2 \leq 1, x \geq 0\}$



11. $g(x, y) = \frac{x - y}{x + y}$

Solution

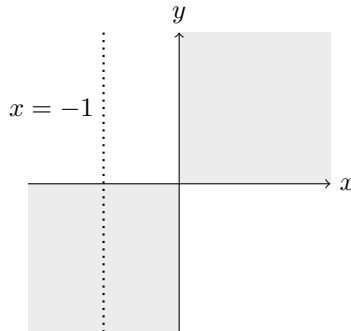
$D : \{(x, y) \mid y \neq -x\}$



13. $p(x, y) = \frac{\sqrt{xy}}{x + 1}$

Solution

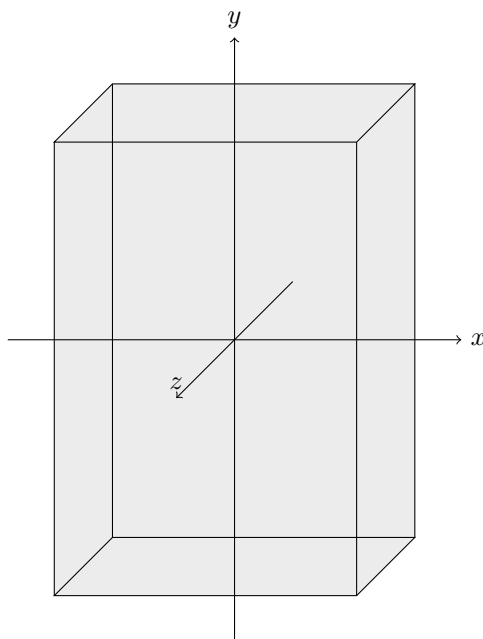
$$D : \{(x, y) \mid x \neq 0, xy \geq 0\}$$



$$15. f(x, y, z) = \sqrt{4 - x^2} + \sqrt{9 - y^2} + \sqrt{1 - z^2}$$

Solution

$$D : \{(x, y, z) \mid -2 \leq x \leq 2, -3 \leq y \leq 3, -1 \leq z \leq 1\}$$



17. A model for the surface area of a human body is given by the function

$$S = f(w, h) = 0.1091w^{0.425}h^{0.725}$$

where w is the weight (in pounds), h is the height (in inches), and S is measured in square feet.

(a) Find $f(160, 70)$ and interpret it.

(b) What is your own surface area?

Solution

a)

$$f(160, 70) = 0.1091(160^{0.425})(70^{0.725}) \approx 20.5$$

The surface area of a human body that weighs 160 pounds and is 70 inches tall is about 20.5 square feet.

23 - 31 (odd)

Sketch the graph of the function

23. $f(x, y) = y$

Solution

This is an equation of the plane that goes through the origin and is parallel to the x -axis.

25. $f(x, y) = 10 - 4x - 5y$

Solution

$$\text{Let } x = y = 0 \Rightarrow z = 10, \quad x = z = 0 \Rightarrow y = 2, \quad y = z = 0 \Rightarrow x = 2.5$$

This is an equation of a plane that goes through the points $(0, 0, 10)$, $(0, 2, 0)$, $(2.5, 0, 0)$ [imagine it is shaded in].

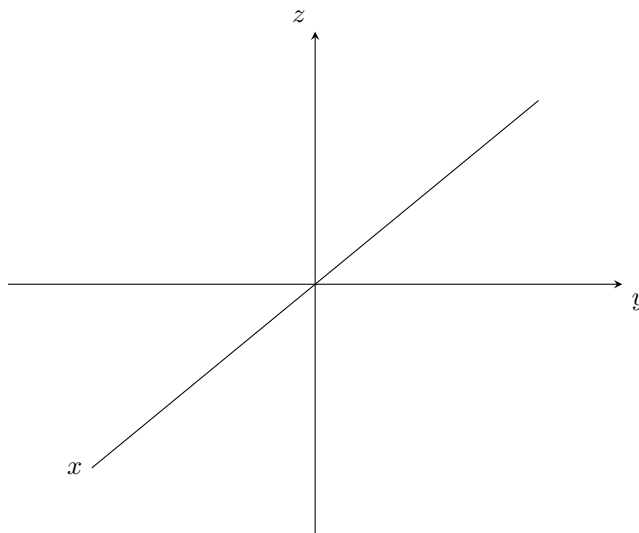
27. $f(x, y) = \sin x$

Solution

29. $f(x, y) = x^2 + 4y^2 + 1$

Solution

This is an equation of an elliptic paraboloid that goes through the origin and is parallel to the z -axis.



31. $f(x, y) = \sqrt{4 - 4x^2 - y^2}$

Solution

This is the top half of ellipsoid

32. Match the function with its graph (labeled I–VI). Give reasons for your choices.

(a) $f(x, y) = \frac{1}{1 + x^2 + y^2}$

(b) $f(x, y) = \frac{1}{1 + x^2 y^2}$

(c) $f(x, y) = \ln(x^2 + y^2)$

(d) $f(x, y) = \cos \sqrt{x^2 + y^2}$

(e) $f(x, y) = |xy|$

(f) $f(x, y) = \cos(xy)$

Solution

a)

The graph of $f(x, y) = \frac{1}{1 + x^2 + y^2}$ is III

When $x = y = 0 \Rightarrow z = 1$, so the graph intersects the z -axis at $(0, 0, 1)$.

If we solve for the zx and zy planes we get $z = \frac{1}{1 + x^2}$ and $z = \frac{1}{1 + y^2}$ respectively.

b)

The graph of $f(x, y) = \frac{1}{1 + x^2 y^2}$ is I

When $x = y = 0 \Rightarrow z = 1$, so the graph intersects the z -axis at $(0, 0, 1)$.

Let $x = 1$ and then we solve for $z = \lim_{y \rightarrow \infty} \frac{1}{1 + y^2} = 0$. For graph I, if we gauge the $x = 1$ position and move up the y axis, we can see that z does indeed approach a value like 0.

c)

The graph of $f(x, y) = \ln(x^2 + y^2)$ is IV

When $x = y = 0$ z is undefined

The only graph that seems to have a hole at the origin is IV.

d)

The graph of $f(x, y) = \cos \sqrt{x^2 + y^2}$ is V

When $x = y = 0 \Rightarrow z = 1$, so the graph intersects the z -axis at $(0, 0, 1)$.

When $x = 0$ and $y = 0$ then $z = \cos y$ and $z = \cos x$ respectively.

The only graph that has a point at $(0, 0, 1)$ and has sinusoidal movement when $(0, (y \text{ or } x) \rightarrow \infty, -1 \leq z \leq 1)$ is V.

e)

The graph of $f(x, y) = |xy|$ is VI

When $x = y = 0 \Rightarrow z = 0$, so the graph intersects the z -axis at $(0, 0, 0)$.

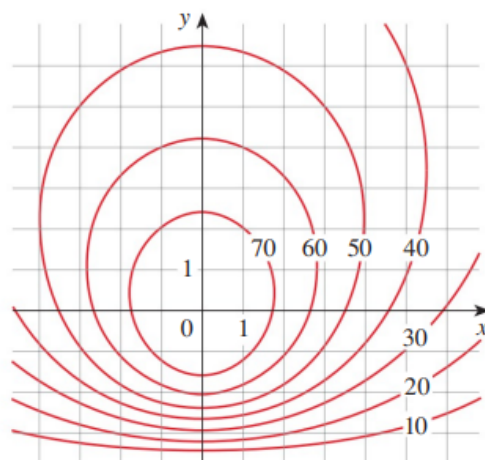
Out of the remaining graphs, the only graph that seems like it has an intersection at the origin is VI.

f)

The graph of $f(x, y) = \cos(xy)$ is II

Process of elimination :) (please don't dock me points for this)

33. A contour map for a function f is shown. Use it to estimate the values of $f(-3, 3)$ and $f(3, -2)$. What can you say about the shape of the graph?



Solution

Looking at the contour map, it seems that $f(-3, 3)$ is ≈ 56 because it is between the 50 and 60 but a little closer to the 60.

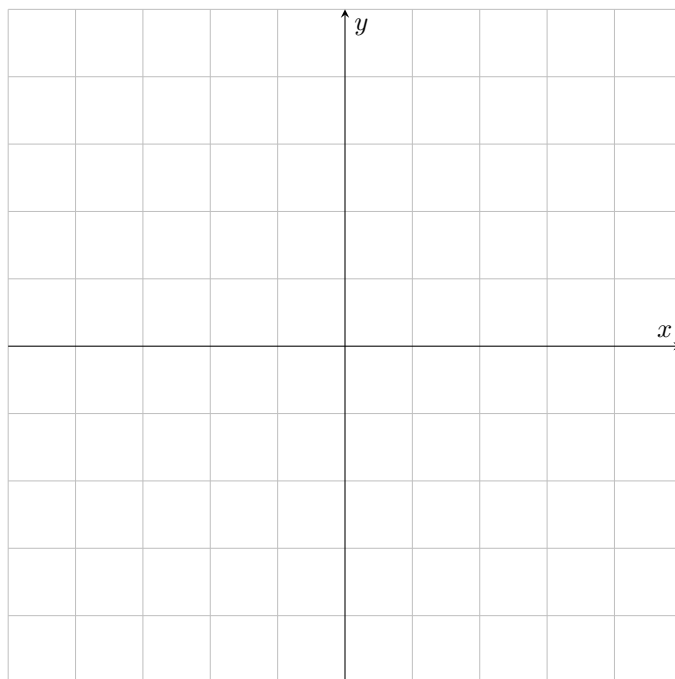
$f(3, -2)$ seems like it is ≈ 35 because it is in the middle of 40 and 30.

The shape of the graph seems like a hill or the top half of an ellipsoid.

45, 47 & 51

Draw a contour map of the function showing several level curves.

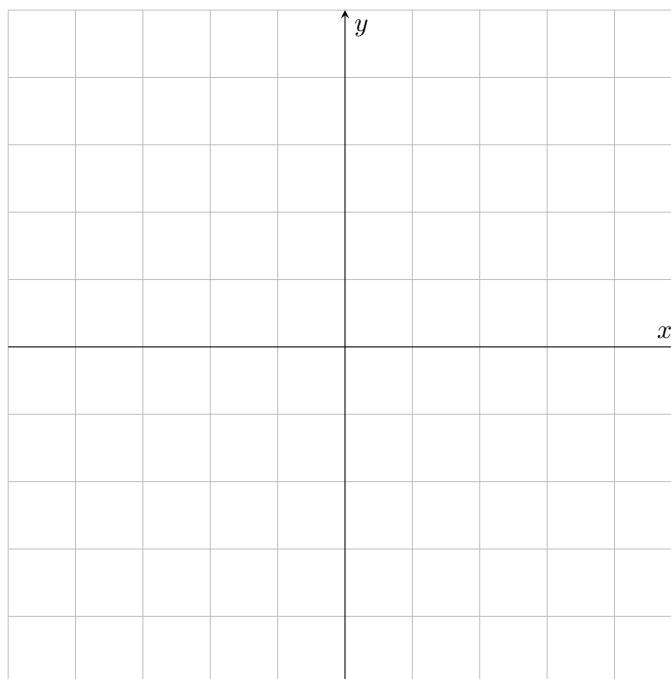
45. $f(x, y) = x^2 - y^2$

SolutionContour map $f(x, y) = x^2 - y^2$ 

47. $f(x, y) = \sqrt{x} + y$

Solution

Contour map $f(x, y) = \sqrt{x} + y$



51. $f(x, y) = \sqrt{x^2 + y^2}$

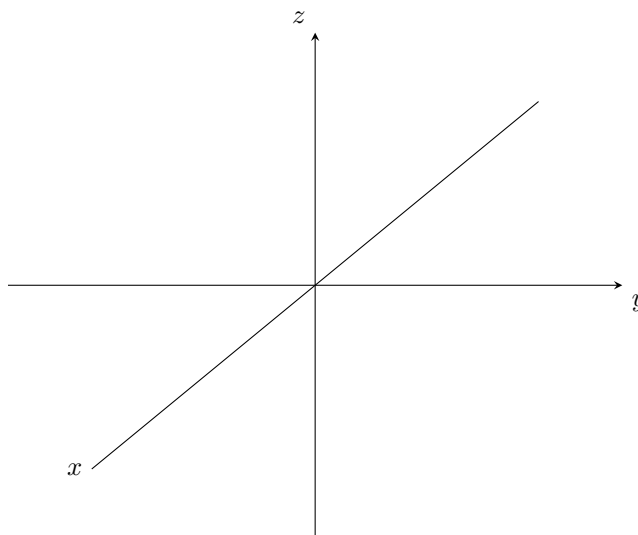
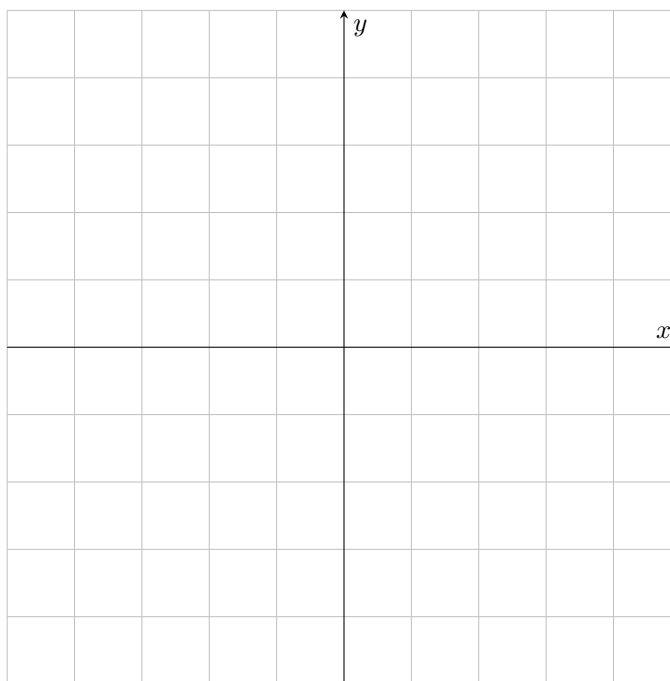
Solution

53. Sketch both a contour map and a graph of the given function and compare them.

$$f(x, y) = x^2 + 9y^2$$

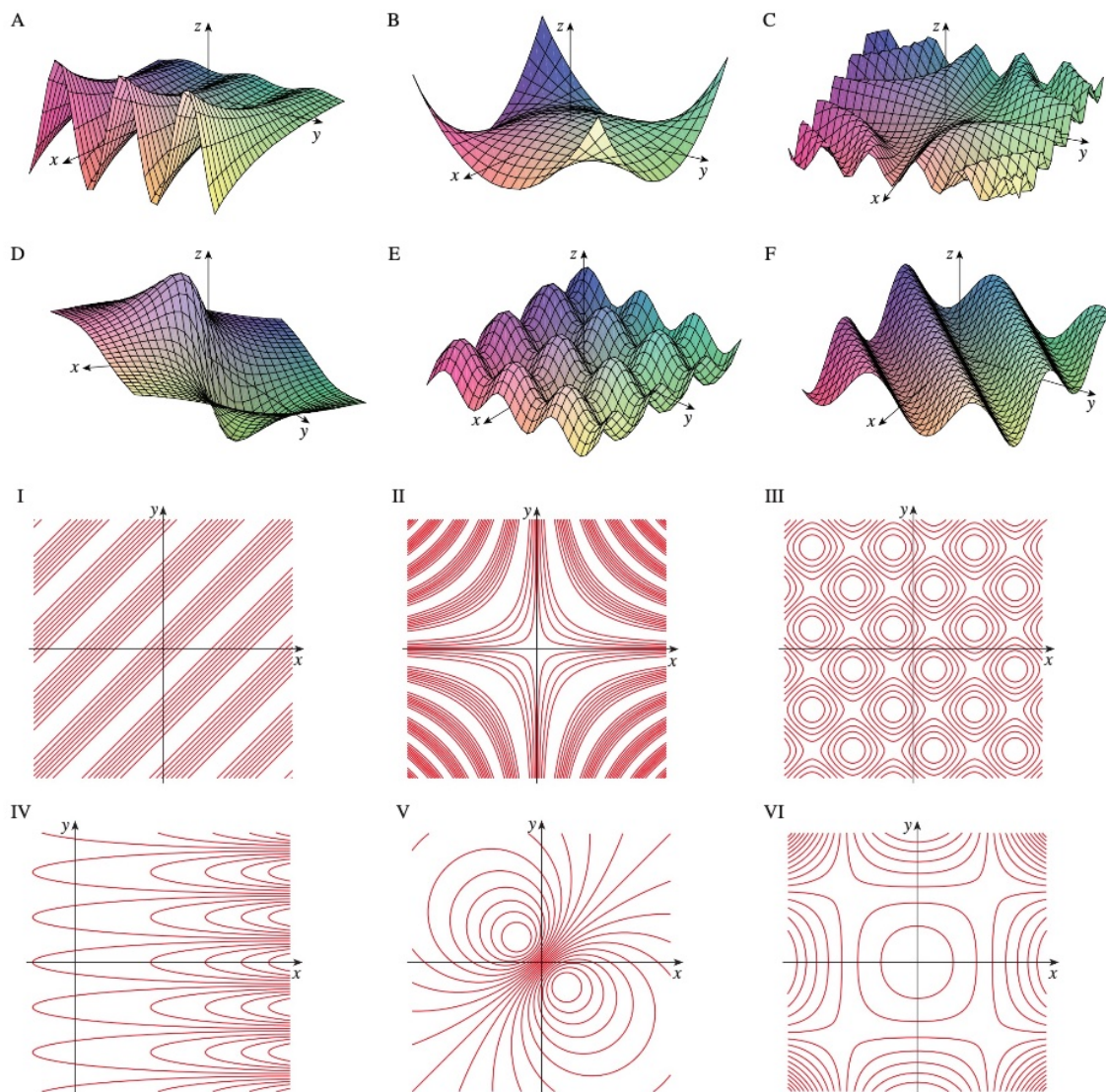
Solution

Contour map $f(x, y) = x^2 + 9y^2$;



61 - 66

Match the function (a) with its graph (labeled A–F below) and (b) with its contour map (labeled I–VI). Give reasons for your choices.



61. $z = \sin(xy)$

Solution

It seems like the graph of $z = \sin(xy)$ is C

When $x \rightarrow \infty$ and $y \rightarrow \infty$ then $-1 \leq z \leq 1$.

In other words, at $45^\circ, 135^\circ, 235^\circ, 315^\circ$ in terms of xy , z should be infinitely sinusoidal the farther you go out

Since the function of z is \sin then the graph must intersect the z at origin

For the contour map, the graph that looks that follows this description is II

62. $z = e^x \cos y$

Solution

When $x = y = 0$ then $z = 1 \Rightarrow (0, 0, 1)$.

Setting $x = 0 \Rightarrow z = \cos y \Rightarrow (0, y \rightarrow \infty, -1 \leq z \leq 1)$ this just means that x is constant and as y increases/decreases towards either positive or negative infinity, z will be sinusoidal

Setting $y = 0 \Rightarrow z = e^x \Rightarrow (0, y \rightarrow \infty^+, \infty^+)(0, y \rightarrow \infty^-, 0)$ this just means that y is constant and depending if x is increasing or decreasing, z will increase exponentially to infinity or approach 0

\therefore The graph that seems to follow this description is A and the associated contour map seems to be IV

63. $z = \sin(x - y)$

Solution

The graph would have an intersection at the origin $x = y = 0 \Rightarrow z = 0$

When $x = 0 \Rightarrow z = \sin(-y)$, so the function will first dip down to $z = -1$ in the zy -trace

When $y = 0 \Rightarrow z = \sin(x)$, so the function will first go up to $z = 1$ in the zx -trace

\therefore The graph that matches this description looks like F and the associated contour map seems to be I

64. $z = \sin x - \sin y$

Solution

The graph would have an intersection at the origin $x = y = 0 \Rightarrow z = 0$

$$z = 1 - 1 = 0 \Leftrightarrow x = y = n \cdot \frac{\pi}{2}, \{n \in \mathbb{Z} \mid n = 2k - 1, k \in \mathbb{Z}\}$$

$$z = 0 - 0 = 0 \Leftrightarrow x = y = n \cdot \frac{\pi}{2}, \{n \in \mathbb{Z} \mid n = 2k, k \in \mathbb{Z}\}$$

This behavior appears symmetric with $z = 0$ appearing at areas where $x = y$.

\therefore The graph that best matches this behavior is E and the associated contour map would be III

65. $z = (1 - x^2)(1 - y^2)$

Solution

The graph would have an intersection at $(0, 0, 1)$ $x = y = 0 \Rightarrow z = 1$

If we look at the zy and zx traces, we see that it is a parabola opening down to negative z

However, if we take $\lim_{(x,y) \rightarrow (\infty, \infty)} (1 - x^2)(1 - y^2)$, $\{(x, y) \mid x = y\}$ then we get ∞ where the graph, in this direction, would exponentially grow.

\therefore The graph that best fits this description would be B and the associated contour map would be VI.

66. $z = \frac{x - y}{1 + x^2 + y^2}$

Solution

The graph would have an intersection at $(0, 0, 0)$ $x = y = 0 \Rightarrow z = 0$

Using process of elimination, the only graph left would be D and the associated contour map would be V.

We could note its behavior in the zy trace and see that $\lim_{y \rightarrow \infty} \frac{-y}{1+y^2} = 0$ with z decreasing at first, vice versa with zx trace

To find the point at which z is at a minimum when $y \rightarrow \infty^+$,

$$\begin{aligned} z &= -y(1 + y^2)^{-1} \\ \frac{dz}{dy} &= -\frac{1}{1 + y^2} + \frac{2y^2}{(1 + y^2)^2} \\ 0 &= -\frac{1}{1 + y^2} + \frac{2y^2}{(1 + y^2)^2} \\ \frac{1}{1 + y^2} &= \frac{2y^2}{(1 + y^2)^2} \\ 1 &= \frac{2y^2}{1 + y^2} \\ 1 + y^2 &= 2y^2 \\ 1 &= y^2 \\ y &= \pm 1 = +1 \end{aligned}$$

And it seems there would be a maximum at $y = -1$

67. Describe the level surfaces of the function.

$$f(x, y, z) = 2y - z + 1$$

Solution

If we rearrange the function, $x - 2y + z = 1$ and we see that this is an equation of a plane. If we substitute k for 1 and play around with its value (choosing 3-5 vals for k) then we see that no matter which value we pick, the planes will be parallel

Section 2: Limits and Continuity

5 - 11 (odd)

Find the limit

5. $\lim_{(x,y) \rightarrow (3,2)} (x^2y^3 - 4y^2)$

Solution

Using direct substitution,

$$(9)(8) - 4(4) = 72 - 16 = 56$$

7. $\lim_{(x,y) \rightarrow (-3,1)} \frac{x^2y - xy^3}{x - y + 2}$

Solution

Using direct substitution,

$$\frac{9(1) + 3(1^3)}{-3 - 1 + 2} = \frac{12}{-2} = -6$$

9. $\lim_{(x,y) \rightarrow (\pi, \pi/2)} y \sin(x - y)$

Solution

Using direct substitution,

$$\frac{\pi}{2} \sin\left(\pi - \frac{\pi}{2}\right) = \frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}(1) = \frac{\pi}{2}$$

11. $\lim_{(x,y) \rightarrow (1,1)} \left(\frac{x^2y^3 - x^3y^2}{x^2 - y^2} \right)$

Solution

Let $x = 1$,

$$\lim_{x \rightarrow 1} \left(\frac{1^2y^3 - 1^3y^2}{1^2 - y^2} \right) = \lim_{x \rightarrow 1} \left(\frac{y^3 - y^2}{1 - y^2} \right) \stackrel{L'H}{=} \lim_{x \rightarrow 1} \frac{3y^2 - 2y}{-2y} = \frac{1}{-2}$$

Let $y = 1$,

$$\lim_{y \rightarrow 1} \left(\frac{x^21^3 - x^31^2}{1^2 - y^2} \right) = \lim_{y \rightarrow 1} \left(\frac{x^2 - x^3}{x^2 - 1} \right) \stackrel{L'H}{=} \lim_{x \rightarrow 1} \frac{2x - 3x^2}{2x} = \frac{-1}{2}$$

\therefore The limit $= \frac{1}{2}$

13 - 17 (odd)

Show that the limit does not exist

13. $\lim_{(x,y) \rightarrow (0,0)} \frac{y^2}{x^2 + y^2}$

Solution

Let $x = 0$,

$$\lim_{y \rightarrow 0} \frac{y^2}{y^2} = 1$$

Let $y = 0$,

$$\lim_{x \rightarrow 0} \frac{0}{x^2} = 0$$

$1 \neq 0 \quad \therefore$ The limit DNE

15. $\lim_{(x,y) \rightarrow (0,0)} \frac{(x+y)^2}{x^2 + y^2}$

Solution

Let $y = x$, $\lim_{x \rightarrow 0} \frac{(x+x)^2}{x^2 + x^2} = \lim_{x \rightarrow 0} \frac{4x^2}{2x^2} = 2$ Let $y = -x$, $\lim_{x \rightarrow 0} \frac{(0)^2}{x^2 + (-x)^2} = \lim_{x \rightarrow 0} \frac{0}{2x^2} = 0$
 \therefore The limit DNE

*****17.** $\lim_{(x,y) \rightarrow (0,0)} \frac{y^2 \sin^2 x}{x^4 + y^4}$

Solution

skip

19 - 25 (odd)

Find the limit, if it exists, or show that the limit does not exist.

19. $\lim_{(x,y) \rightarrow (-1,-2)} (x^2 y - xy^2 + 3)^3$

Solution

Using direct substitution

$$((-1)^2(-2) - (-1)(-2)^2 + 3)^3 = (5)^3 = 125$$

21. $\lim_{(x,y) \rightarrow (2,3)} \frac{3x - 2y}{4x^2 - y^2}$

Solution

Using direct substitution

$$\frac{3(2) - 2(3)}{4(2)^2 - (3)^2} = \frac{0}{7} = 0$$

*****23.** $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2 \cos y}{x^2 + y^4}$

Solution

skip

25. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

Solution

Rationalizing the denominator,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} \cdot \frac{\sqrt{x^2 + y^2 + 1} + 1}{\sqrt{x^2 + y^2 + 1} + 1} = \frac{(x^2 + y^2)\sqrt{x^2 + y^2 + 1} + 1}{x^2 + y^2 + 1 - 1} = \sqrt{x^2 + y^2 + 1} + 1 = 2$$

31 & 33

Use the Squeeze Theorem to find the limit.

31. $\lim_{(x,y) \rightarrow (0,0)} xy \sin \frac{1}{x^2 + y^2}$

Solution

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} xy \sin \frac{1}{x^2 + y^2} &\Rightarrow -xy \leq xy \sin \frac{1}{x^2 + y^2} \leq xy \\ \lim_{(x,y) \rightarrow (0,0)} -xy &\leq \lim_{(x,y) \rightarrow (0,0)} xy \sin \frac{1}{x^2 + y^2} \leq \lim_{(x,y) \rightarrow (0,0)} xy \Rightarrow 0 \leq \lim_{(x,y) \rightarrow (0,0)} xy \sin \frac{1}{x^2 + y^2} \leq 0 \end{aligned}$$

\therefore The limit = 0

33. $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^4}{x^4 + y^4}$

Solution

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{xy^4}{x^4 + y^4} &\Rightarrow [x = r \cos \theta, y = r \sin \theta] \\ \lim_{(x,y) \rightarrow (0,0)} \frac{r \cos \theta r^4 \sin^4 \theta}{r^4 \cos^4 \theta + r^4 \sin^4 \theta} &= \frac{r^5 \cos \theta \sin^4 \theta}{r^4(1)} = r \cos \theta \sin^4 \theta \end{aligned}$$

Using Squeeze theorem,

$$-r^4 \leq r^4 \cos \theta \sin^4 \theta \leq r^4 \Rightarrow 0 \leq r^4 \cos \theta \sin^4 \theta \leq 0$$

\therefore The limit = 0

41, 43 & 45

Determine the set of points at which the function is continuous.

41. $F(x, y) = \frac{xy}{1 + e^{x-y}}$

Solution

We can see that $1 + e^{x-y} \neq 0 \Rightarrow e^{x-y} \neq -1$ which can never happen

\therefore the function is continuous in the set $\{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$

43. $F(x, y) = \frac{1 + x^2 + y^2}{1 - x^2 - y^2}$

Solution

$1 - x^2 - y^2 \neq 0 \Rightarrow x^2 + y^2 \neq 1$ so the function is continuous in the set $\{(x, y) \mid x^2 + y^2 \neq 1\}$

45. $G(x, y) = \sqrt{x} + \sqrt{1 - x^2 - y^2}$

Solution

We see that $x \geq 0$ and $1 - x^2 - y^2 \geq 0$

\therefore the function is continuous in the set $\{(x, y) \mid x \geq 0, x^2 + y^2 \leq 1\}$

Section 3: Partial Derivatives

9 - 25 (odd)

Find the first partial derivatives of the function.

9. $f(x, y) = x^4 + 5xy^3$

Solution

$$\frac{\partial f}{\partial x} = 4x^3 + 5y^3$$
$$\frac{\partial f}{\partial y} = 15xy^2$$

11. $g(x, y) = x^3 \sin y$

Solution

$$\frac{\partial g}{\partial x} = 3x^2 \sin y$$
$$\frac{\partial g}{\partial y} = 3x^2 \cos y$$

13. $z = \ln(x + t^2)$

Solution

$$\frac{\partial z}{\partial x} = \frac{1}{x + t^2}$$
$$\frac{\partial z}{\partial t} = \frac{2t}{x + t^2}$$

15. $f(x, y) = ye^{xy}$

Solution

$$\frac{\partial f}{\partial x} = y^2 e^{xy}$$
$$\frac{\partial f}{\partial y} = e^{xy} + xye^{xy}$$

17. $g(x, y) = y(x + x^2y)^5$

Solution

$$\frac{\partial g}{\partial x} = 5y(x + x^2y)^4 \cdot (1 + 2xy) \Rightarrow 5y(1 + 2xy) + (x + x^2y)^4$$
$$\frac{\partial g}{\partial y} = (x + x^2y)^5 + 5y(x + x^2y)^4 \cdot x^2 \Rightarrow 5x^2y(x + x^2y)^4 + (x + x^2y)^5$$

19. $f(x, y) = \frac{ax + by}{cx + dy}$

Solution

$$\frac{\partial f}{\partial x} =$$
$$\frac{\partial f}{\partial y} =$$

21. $g(u, v) = (u^2v - v^3)^5$

Solution

$$\frac{\partial g}{\partial u} =$$

$$\frac{\partial g}{\partial v} =$$

23. $R(p, q) = \tan^{-1}(pq^2)$

Solution

$$\frac{\partial R}{\partial p} =$$

$$\frac{\partial R}{\partial q} =$$

25. $F(x, y) = \int_y^x \cos(e^t) dt$

Solution

$$\frac{\partial F}{\partial x} =$$

$$\frac{\partial F}{\partial y} =$$

37. Find the indicated partial derivative.

$$R(s, t) = te^{s/t}; \quad R_t(0, 1)$$

41 & 43

Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$

41. $x^2 + 2y^2 + 3x^2 = 1$

43. $e^z = xyz$

45. Find $\partial z/\partial x$ and $\partial z/\partial y$.

$$(a) \ z = f(x) + g(y); \quad (b) \ z = f(x + y)$$

47. Find all the second partial derivatives.

$$f(x, y) = x^4y - 2x^3y^2$$

57 - 61 (odd)

Find the indicated partial derivative(s).

57. $f(x, y) = x^4y^2 - x^3y; \quad f_{xxx}, f_{xyx}$

59. $f(x, y, z) = e^{xyz^2}; \quad f_{xyz}$

61. $W = \sqrt{u + v^2}; \quad \frac{\partial^3 W}{\partial u^2 \partial v}$

Section 4: Tangent Planes and Linear Approximations

1. The graph of a function f is shown. Find an equation of the tangent plane to the surface $z = f(x, y)$ at the specified point

3 - 9 (odd)

Find an equation of the tangent plane to the given surface at the specified point.

3. $z = 2x^2 + y^5 - 5y, \quad (1, 2, -4)$

5. $z = e^{x-y}, \quad (2, 2, 1)$

7. $z = 2\sqrt{y}/x, \quad (-1, 1, -2)$

9. $z = x \sin(x + y), \quad (-1, 1, 0)$

15 - 19 odd

Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that point.

15. $f(x, y) = x^3y^2, \quad (-2, 1)$

17. $f(x, y) = 1 + x \ln(xy - 5), \quad (2, 3)$

19. $f(x, y) = x^2e^y, \quad (1, 0)$