Chapter 14 - Problem Set 2

Calculus 3

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Section 5: The Chain Rule

3-7 (odd)

Use The Chain Rule to find $\dfrac{dz}{dt}$ or $\dfrac{dw}{dt}$.

3. $z = xy^3 - x^2y, \quad x = t^2 + 1, \quad y = t^2 - 1$

Solution

$$\frac{dz}{dt} = \frac{dz}{dx}\left(\frac{dx}{dt}\right) + \frac{dz}{dy}\left(\frac{dy}{dt}\right)$$
$$\frac{dz}{dt} = [y^3 - 2xy](2t) + [3xy^2 - x^2](2t)$$
$$\boxed{\frac{dz}{dt} = 2t[y^3 - 2xy + 3xy^2 - x^2]}$$

5. $z = \sin x \cos y, \quad x = \sqrt{t}, \quad y = 1/t$

Solution

$$\frac{dz}{dt} = \frac{dz}{dx}\left(\frac{dx}{dt}\right) + \frac{dz}{dy}\left(\frac{dy}{dt}\right)$$
$$\frac{dz}{dt} = [\cos x \cos y]\left(\frac{1}{2}t^{-\frac{1}{2}}\right) + [-\sin x \sin y]\left(-t^{-2}\right)$$
$$\boxed{\frac{dz}{dt} = \frac{1}{2\sqrt{t}}\cos x \cos y + \frac{1}{t^2}\sin x \sin y}$$

7. $w = xe^{y/z}, \quad x = t^2, \quad y = 1 - t, \quad z = 1 + 2t$

Solution

$$\frac{dw}{dt} = \frac{dw}{dx}\left(\frac{dx}{dt}\right) + \frac{dw}{dy}\left(\frac{dy}{dt}\right) + \frac{dw}{dz}\left(\frac{dz}{dt}\right)$$
$$\frac{dw}{dt} = \left[e^{y/z}\right](2t) + \left[\frac{x}{z}e^{y/z}\right](-1) + \left[-\frac{xy}{z^2}e^{y/z}\right](2)$$
$$\boxed{\frac{dw}{dt} = e^{y/z}\left[2t - \frac{x}{z} - \frac{2xy}{z^2}\right]}$$

11-15 (odd)

Use the Chain Rule to find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$

11. $z = (x-y)^5, \quad x = s^2t, \quad y = st^2$

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}\right)$$
$$\frac{\partial z}{\partial s} = \left[5(x-y)^4\right](2st) + \left[-5(x-y)^4\right](t^2)$$
$$\boxed{\frac{\partial z}{\partial s} = 5(x-y)^4\left[2st - t^2\right]}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial t}\right)$$
$$\frac{\partial z}{\partial t} = \left[5(x-y)^4\right](s^2) + \left[-5(x-y)^4\right](2st)$$
$$\boxed{\frac{\partial z}{\partial t} = 5(x-y)^4\left[s^2 - 2st\right]}$$

13. $z = \ln(3x+2y), \quad x = s\sin t, \quad y = t\cos s$

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}\right)$$
$$\frac{\partial z}{\partial s} = \left[\frac{3}{3x+2y}\right](\sin t) + \left[\frac{2}{3x+2y}\right](-t\sin s)$$
$$\boxed{\frac{\partial z}{\partial s} = \frac{3\sin t - 2t\sin s}{3x+2y}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial t}\right)$$
$$\frac{\partial z}{\partial t} = \left[\frac{3}{3x+2y}\right](s\cos t) + \left[\frac{2}{3x+2y}\right](\cos s)$$
$$\boxed{\frac{\partial z}{\partial t} = \frac{3s\cos t + 2\cos s}{3x+2y}}$$

15. $z = (\sin\theta)/r, \quad r = st, \quad \theta = s^2 + t^2$

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\left(\frac{\partial r}{\partial s}\right) + \frac{\partial z}{\partial \theta}\left(\frac{\partial \theta}{\partial s}\right)$$
$$\frac{\partial z}{\partial s} = \left[-\frac{\sin\theta}{r^2}\right](t) + \left[\frac{\cos\theta}{r}\right](2s)$$
$$\boxed{\frac{\partial z}{\partial s} = -\frac{t\sin\theta}{r^2} + \frac{2s\cos\theta}{r}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\left(\frac{\partial r}{\partial t}\right) + \frac{\partial z}{\partial \theta}\left(\frac{\partial \theta}{\partial t}\right)$$
$$\frac{\partial z}{\partial t} = \left[-\frac{\sin\theta}{r^2}\right](s) + \left[\frac{\cos\theta}{r}\right](2t)$$
$$\boxed{\frac{\partial z}{\partial t} = -\frac{s\sin\theta}{r^2} + \frac{2t\cos\theta}{r}}$$

25-29 (odd)

Use the Chain Rule to find the indicated partial derivatives.

25. $z = x^4 + x^2 y, \quad x = s + 2t - u, \quad y = stu^2$**;**

$\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}, \frac{\partial z}{\partial u}$ **when** $s = 4, t = 2, u = 1$

Solution

When $s = 4$, $t = 2$, and $u = 1 \quad \Rightarrow \quad x = 7$, $y = 8$

$$\frac{\partial z}{\partial s} = \left[4x^3 + 2xy\right](1) + \left[x^2\right](tu^2) = 4x^3 + 2xy + x^2tu^2$$
$$\frac{\partial z}{\partial s} = 4(7)^3 + 2(7)(8) + (7)^2(2)(1)^2$$
$$\boxed{\frac{\partial z}{\partial s} = 1582}$$

$$\frac{\partial z}{\partial t} = \left[4x^3 + 2xy\right](2) + \left[x^2\right](su^2) = 8x^3 + 4xy + x^2su^2$$
$$\frac{\partial z}{\partial t} = 8(7)^3 + 4(7)(8) + (7)^2(4)(1)^2$$
$$\boxed{\frac{\partial z}{\partial t} = 3164}$$

$$\frac{\partial z}{\partial u} = \left[4x^3 + 2xy\right](-1) + \left[x^2\right](2stu) = -4x^3 - 2xy + 2x^2stu$$
$$\frac{\partial z}{\partial u} = -4(7)^3 - 2(7)(8) + 2(7)^2(4)(2)(1)$$
$$\boxed{\frac{\partial z}{\partial u} = -700}$$

27. $w = xy + yz + zx, \quad x = r\cos\theta, \quad y = r\sin\theta, \quad z = r\theta$**;**

$\dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial \theta}$ **when** $r = 2, \theta = \pi/2$

Solution

When $r = 2$ and $\theta = \pi/2 \quad \Rightarrow \quad x = 0,\ y = 2,\ z = \pi$

$$\frac{\partial w}{\partial r} = [y+z](\cos\theta) + [x+z](\sin\theta) + [x+y](\theta)$$
$$\frac{\partial w}{\partial r} = [2+\pi](0) + [0+\pi](1) + [0+2](\pi/2) = 0 + \pi + \pi$$
$$\boxed{\frac{\partial w}{\partial r} = 2\pi}$$

$$\frac{\partial w}{\partial \theta} = [y+z](-r\sin\theta) + [x+z](r\cos\theta) + [x+y](r)$$
$$\frac{\partial w}{\partial \theta} = [2+\pi](-2\cdot 1) + [0+\pi](0) + [0+2](2) = -4 - 2\pi + 0 + 4$$
$$\boxed{\frac{\partial w}{\partial \theta} = -2\pi}$$

29. $N = \dfrac{p+q}{p+r}, \quad p = u + vw, \quad q = v + uw, \quad r = w + uv$

$\dfrac{\partial N}{\partial u}, \dfrac{\partial N}{\partial v}, \dfrac{\partial N}{\partial w}$ **when** $u = 2, v = 3, w = 4$

Solution

When $u = 2$, $v = 3$, and $w = 4 \quad \Rightarrow \quad p = 14$, $q = 11$, $r = 10$

$$\frac{\partial N}{\partial u} = \left[\frac{(p+r)-(p+q)}{(p+r)^2}\right](1) + \left[\frac{1}{p+r}\right](w) + \left[-\frac{p+q}{(p+r)^2}\right](v)$$
$$\frac{\partial N}{\partial u} = \frac{r-q}{(p+r)^2} + \frac{w}{p+r} - \frac{v(p+q)}{(p+r)^2}$$
$$\frac{\partial N}{\partial u} = \frac{10-11}{(14+10)^2} + \frac{4}{14+10} - \frac{3(14+11)}{(14+10)^2}$$
$$\frac{\partial N}{\partial u} = \frac{-1}{576} + \frac{4}{24} - \frac{75}{576} = \frac{-1+96-75}{576} = \frac{20}{576}$$
$$\boxed{\frac{\partial N}{\partial u} = \frac{5}{144}}$$

$$\frac{\partial N}{\partial v} = \left[\frac{r-q}{(p+r)^2}\right](w) + \left[\frac{1}{p+r}\right](1) + \left[-\frac{p+q}{(p+r)^2}\right](u)$$
$$\frac{\partial N}{\partial v} = \frac{w(r-q)}{(p+r)^2} + \frac{1}{p+r} - \frac{u(p+q)}{(p+r)^2}$$
$$\frac{\partial N}{\partial v} = \frac{4(10-11)}{(14+10)^2} + \frac{1}{14+10} - \frac{2(14+11)}{(14+10)^2}$$
$$\frac{\partial N}{\partial v} = \frac{-4}{576} + \frac{1}{24} - \frac{50}{576} = \frac{-4+24-50}{576} = \frac{-30}{576}$$
$$\boxed{\frac{\partial N}{\partial v} = -\frac{5}{96}}$$

$$\frac{\partial N}{\partial w} = \left[\frac{r-q}{(p+r)^2}\right](v) + \left[\frac{1}{p+r}\right](u) + \left[-\frac{p+q}{(p+r)^2}\right](1)$$
$$\frac{\partial N}{\partial w} = \frac{v(r-q)}{(p+r)^2} + \frac{u}{p+r} - \frac{p+q}{(p+r)^2}$$
$$\frac{\partial N}{\partial w} = \frac{3(10-11)}{(14+10)^2} + \frac{2}{(14+10)} - \frac{14+11}{(14+10)^2}$$
$$\frac{\partial N}{\partial w} = \frac{-3}{576} + \frac{2}{24} - \frac{25}{576} = \frac{-3+48-25}{576} = \frac{20}{576}$$
$$\boxed{\frac{\partial N}{\partial w} = \frac{5}{144}}$$

Use Equation 5 to find $\dfrac{dy}{dx}$

31. $y\cos x = x^2 + y^2$

Solution

$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Let $F = y\cos x - x^2 - y^2$

$$F_x = -y\sin x - 2x$$
$$F_y = \cos x - 2y$$

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{(-y\sin x - 2x)}{\cos x - 2y} = \frac{2x + y\sin x}{\cos x - 2y}}$$

Use Equations 6 to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

35. $x^2 + 2y^2 + 3z^2 = 1$

Solution

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Let $F = x^2 + 2y^2 + 3z^2 - 1$

$$F_x = 2x \qquad F_y = 4y \qquad F_z = 6z$$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x}{6z} = -\frac{x}{3z}}$$

$$\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{4y}{6z} = -\frac{2y}{3z}}$$

Section 6: Directional Derivatives and the Gradient Vector

5, 7

Find the directional derivative of f at the given point in the direction indicated by the angle θ.

5. $f(x, y) = y\cos(xy), \quad (0, 1), \quad \theta = \pi/4$

Solution

$$\begin{aligned}
D_u f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\
D_u f(x,y) &= \left[-y^2\sin(xy)\right]\cos\frac{\pi}{4} + \left[\cos(xy) - xy\sin(xy)\right]\sin\frac{\pi}{4} \\
D_u f(x,y) &= \frac{\sqrt{2}}{2}\left[-y^2\sin(xy) + \cos(xy) - xy\sin(xy)\right] \\
D_u f(0,1) &= \frac{\sqrt{2}}{2}\left[0 + 1 - 0\right] \\
&\boxed{D_u f(0,1) = \frac{\sqrt{2}}{2}}
\end{aligned}$$

7. $f(x, y) = arctan(xy), \quad (2, -3), \quad \theta = 3\pi/4$

Solution

$$\begin{aligned}
D_u f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\
D_u f(x,y) &= \left[\frac{y}{1+(xy)^2}\right]\cos\frac{3\pi}{4} + \left[\frac{x}{1+(xy)^2}\right]\sin\frac{3\pi}{4} \\
D_u f(x,y) &= \frac{\sqrt{2}}{2}\left[\frac{x}{1+(xy)^2} - \frac{y}{1+(xy)^2}\right] \\
D_u f(2,-3) &= \frac{\sqrt{2}}{2}\left[\frac{2}{1+(2(-3))^2} - \frac{-3}{1+(2(-3))^2}\right] \\
D_u f(2,-3) &= \frac{\sqrt{2}}{2}\left[\frac{5}{37}\right] \\
&\boxed{D_u f(2,-3) = \frac{5\sqrt{2}}{74}}
\end{aligned}$$

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$f(x, y) = x/y, \quad P(2, 1), \quad \vec{u} = \frac{3}{5}\,\hat{\mathbf{i}} + \frac{4}{5}\,\hat{\mathbf{j}}$

(a) Find the gradient of f

(b) Evaluate the gradient at the point P

(c) Find the rate of change of f at P in the direction of the vector $\vec{u}$

Solution

(a)

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle \quad \text{or} \quad \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}}$$

$$\boxed{\nabla f(x, y) = \left\langle \frac{1}{y}, -\frac{x}{y^2} \right\rangle}$$

(b)

$$\boxed{\nabla f(2, 1) = \left\langle \frac{1}{1}, -\frac{2}{1^2} \right\rangle = \langle 1, -2\rangle}$$

(c)

$$D_u f(x, y) = \nabla f(x, y) \cdot \vec{u}$$
$$D_u f(2, 1) = \nabla f(2, 1) \cdot \vec{u}$$
$$D_u f(2, 1) = \langle 1, -2\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$
$$D_u f(2, 1) = \frac{3}{5} - \frac{8}{5}$$
$$\boxed{D_u f(2, 1) = -1}$$

13, 15

Find the directional derivative of the function at the given point in the direction of the vector $\vec{v}$.

13. $f(x, y) = e^x \sin y, \quad (0, \pi/3), \quad \vec{v} = \langle -6, 8 \rangle$

Solution

$$\begin{aligned}
\vec{u} &= \frac{\vec{v}}{\|\vec{v}\|} = \frac{-6\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}}}{\sqrt{(6)^2 + (8)^2}} \\
\vec{u} &= -\frac{3}{5}\,\hat{\mathbf{i}} + \frac{4}{5}\,\hat{\mathbf{j}} \\
D_u f(x, y) &= \nabla f(x, y) \cdot \vec{u} \\
D_u f(x, y) &= (e^x \sin y\,\hat{\mathbf{i}} + e^x \cos y\,\hat{\mathbf{j}}) \cdot \left(-\frac{3}{5}\,\hat{\mathbf{i}} + \frac{4}{5}\,\hat{\mathbf{j}}\right) \\
D_u f(x, y) &= -\frac{3e^x \sin y}{5} + \frac{4e^x \cos y}{5} \\
D_u f(0, \frac{\pi}{3}) &= -\frac{3(\frac{\sqrt{3}}{2})}{5} + \frac{4(\frac{1}{2})}{5} \\
&\boxed{D_u f(0, \frac{\pi}{3}) = \frac{4 - 3\sqrt{3}}{10}}
\end{aligned}$$

15. $g(s, t) = s\sqrt{t}, \quad (2, 4), \quad \vec{v} = 2\,\hat{\mathbf{i}} - \,\hat{\mathbf{j}}$

Solution

$$\begin{aligned}
\vec{u} &= \frac{\vec{v}}{\|\vec{v}\|} = \frac{2\,\hat{\mathbf{i}} - \,\hat{\mathbf{j}}}{\sqrt{(2)^2 + (-1)^2}} \\
\vec{u} &= \frac{2}{\sqrt{5}}\,\hat{\mathbf{i}} - \frac{1}{\sqrt{5}}\,\hat{\mathbf{j}} \\
D_u g(s, t) &= \nabla g(s, t) \cdot \vec{u} \\
D_u g(s, t) &= \left(\sqrt{t}\,\hat{\mathbf{i}} + \frac{s}{2\sqrt{t}}\right) \cdot \left(\frac{2}{\sqrt{5}}\,\hat{\mathbf{i}} - \frac{1}{\sqrt{5}}\,\hat{\mathbf{j}}\right) \\
D_u g(s, t) &= \frac{2\sqrt{t}}{\sqrt{5}} - \frac{s}{2\sqrt{t}\sqrt{5}} \\
D_u g(s, t) &= \frac{1}{\sqrt{5}}\left(2\sqrt{t} - \frac{s}{2\sqrt{t}}\right) \\
D_u g(2, 4) &= \frac{1}{\sqrt{5}}\left(2\sqrt{4} - \frac{2}{2\sqrt{4}}\right) = \frac{1}{\sqrt{5}}\left(4 - \frac{1}{2}\right) \\
&\boxed{D_u g(2, 4) = \frac{1}{\sqrt{5}}\left(\frac{8 - 1}{2}\right) = \frac{7}{2\sqrt{5}}}
\end{aligned}$$

21, 23

Find the directional derivative of the function at the point P in the direction of the point Q.

21. $f(x,y) = x^2y^2 - y^3, \quad P(1,2), \quad Q(-3,5)$

Solution

$$\begin{aligned}
\vec{v} &= \vec{PQ} = \langle -3-1, 5-2 \rangle = \langle -4, 3 \rangle \\
\vec{u} &= \frac{\vec{v}}{\|v\|} = \frac{\langle -4, 3 \rangle}{\sqrt{16+9}} \\
\vec{u} &= \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle \\
D_u f(x,y) &= \nabla f(x,y) \cdot \vec{u} \\
D_u f(x,y) &= \left\langle 2xy^2, 2x^2y - 3y^2 \right\rangle \ \cdot \ \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle \\
D_u f(x,y) &= -\frac{8xy^2}{5} + \frac{3(2x^2y - 3y^2)}{5} = \frac{1}{5}(-8xy^2 + 6x^2y - 9y^2) \\
D_u f(1,2) &= \frac{1}{5}[-8(1)(4) + 6(1)(2) - 9(4)] = \frac{1}{5}[-32 + 12 - 36] \\
&\boxed{D_u f(1,2) = -\frac{56}{5}}
\end{aligned}$$

23. $f(x,y) = \sqrt{xy}, \quad P(2,8), \quad Q(5,4)$

Solution

$$\begin{aligned}
\vec{v} &= \vec{PQ} = (5-2)\,\hat{\mathbf{i}} + (4-8)\,\hat{\mathbf{j}} = 3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} \\
\vec{u} &= \frac{\vec{v}}{\|v\|} = \frac{3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}}}{\sqrt{9+16}} \\
\vec{u} &= \frac{3}{5}\,\hat{\mathbf{i}} - \frac{4}{5}\,\hat{\mathbf{j}} \\
D_u f(x,y) &= \nabla f(x,y) \cdot \vec{u} \\
D_u f(x,y) &= \left(\frac{y}{2\sqrt{xy}}\,\hat{\mathbf{i}} + \frac{x}{2\sqrt{xy}}\,\hat{\mathbf{j}} \right) \cdot \left(\frac{3}{5}\,\hat{\mathbf{i}} - \frac{4}{5}\,\hat{\mathbf{j}} \right) \\
D_u f(x,y) &= \frac{3y}{10\sqrt{xy}} - \frac{4x}{10\sqrt{xy}} = \frac{3y - 4x}{10\sqrt{xy}} \\
D_u f(2,8) &= \frac{3(8) - 4(2)}{10\sqrt{2(8)}} = \frac{24 - 8}{10\sqrt{16}} = \frac{16}{40} \\
&\boxed{D_u f(x,y) = \frac{2}{5}}
\end{aligned}$$

27, 29

Find the maximum rate of change of f at the given point and the direction in which it occurs.

27. $f(x,y) = 5xy^2, \quad (3,-2)$

Solution

The maximum rate of change of f and its direction is given by the gradient vector

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \left\langle 5y^2, 10xy \right\rangle$$
$$\nabla f(3,-2) = \left\langle 5(-2)^2, 10(3)(-2) \right\rangle$$

Direction of the maximum rate of change,

$$\boxed{\nabla f(3,-2) = \langle 20, -60 \rangle = 20 \langle 1, -3 \rangle}$$

Maximum rate of change,

$$\boxed{\|\nabla f(3,-2)\| = 20\sqrt{1+9} = 20\sqrt{10}}$$

29. $f(x,y) = \sin(xy), \quad (1,0)$

Solution

The maximum rate of change of f and its direction is given by the gradient vector

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} = y\cos(xy)\,\hat{\mathbf{i}} + x\cos(xy)\,\hat{\mathbf{j}}$$
$$\nabla f(1,0) = 0 + \ \hat{\mathbf{j}}$$

Direction of the maximum rate of change,

$$\boxed{\nabla f(1,0) = \ \hat{\mathbf{j}}}$$

Maximum rate of change,

$$\boxed{\|\nabla f(1,0)\| = \sqrt{1} = 1}$$

37

The temperature T in a metal ball is inversely proportional to the distance from the center of the ball, which we take to be the origin. The temperature at the point $(1, 2, 2)$ is $120°$

(a) Find the rate of change of T at $(1, 2, 2)$ in the direction toward the point $(2, 1, 3)$.

(b) Show that at any point in the ball the direction of greatest increase in temperature is given by a vector that points toward the origin

Solution

Since $T \propto \frac{1}{d}$, $T(x, y, z) = \dfrac{c}{\sqrt{x^2 + y^2 + z^2}}$, for some constant c, and Euclidean distance to describe the distance from the origin.

If $120 = T(1, 2, 2) = \dfrac{c}{\sqrt{1^2 + 2^2 + 2^2}} = \dfrac{c}{3}$, then $c = 360$

$\therefore$ Our function for temperature is,

$$T(x, y, z) = \frac{360}{\sqrt{x^2 + y^2 + z^2}}$$

(a)

$$\begin{aligned}
\vec{v} &= \langle 2 - 1, 1 - 2, 3 - 2 \rangle = \langle 1, -1, 1 \rangle \\
\vec{u} &= \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1, -1, 1 \rangle}{\sqrt{1 + 1 + 1}} \\
\vec{u} &= \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle
\end{aligned}$$

Getting our gradient vector,

$$\nabla T(x, y, z) = \langle T_x, T_y, T_z \rangle$$

$$\nabla T(x, y, z) = \Big\langle -180 \cdot 2x(x^2 + y^2 + z^2)^{-3/2}, -180 \cdot 2y(x^2 + y^2 + z^2)^{-3/2}, -180 \cdot 2z(x^2 + y^2 + z^2)^{-3/2}$$

$$\nabla T(x, y, z) = \frac{360}{(x^2 + y^2 + z^2)^{3/2}} \langle -x, -y, -z \rangle$$

so, our rate of change is

$$\begin{aligned}
D_u T(1, 2, 2) &= \nabla T(1, 2, 2) \cdot \vec{u} \\
D_u T(1, 2, 2) &= \frac{360}{(1 + 4 + 4)^{3/2}} \langle -1, -2, -2 \rangle \cdot \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \\
D_u T(1, 2, 2) &= \frac{360}{27} \left(-\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}} \right) = \frac{360}{27} \left(-\frac{1}{\sqrt{3}} \right) \\
&\boxed{D_u T(1, 2, 2) = \frac{40}{3} \left(-\frac{1}{\sqrt{3}} \right) = -\frac{40}{3\sqrt{3}}}
\end{aligned}$$

(b) The direction of greatest increase at any point is the direction of ∇T at that point. In part (a) we saw that $\nabla T(x, y, z)$ and $\langle -x, -y, -z \rangle$ have the same direction and the vector $\langle -x, -y, -z \rangle$ points towards the origin.

47-51 (odd)

Find equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

47. $2(x-2)^2+(y-1)^2+(z-3)^2=10, \quad (3,3,5)$

Solution

(a)

$$F(x,y,z)=2(x-2)^2+(y-1)^2+(z-3)^2-10$$

$$\begin{aligned}F_x(3,3,5)&=4(x-2)=4(1)=4\\F_y(3,3,5)&=2(y-1)=2(2)=4\\F_z(3,3,5)&=2(z-3)=2(2)=4\end{aligned}$$

The equation of the tangent plane is given by,

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)$$

$$\begin{aligned}4(x-3)+4(y-3)+4(z-5)&=0\\x-3+y-3+z-5&=0\\\boxed{x+y+z=11}\end{aligned}$$

(b) The equation of the normal line is given by,

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)}=\frac{y-y_0}{F_y(x_0,y_0,z_0)}=\frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

$$\frac{x-3}{4}=\frac{y-3}{4}=\frac{z-5}{4}$$

$$\boxed{x-3=y-3=z-5}$$

49. $xy^2z^3=8, \quad (2,2,1)$

Solution

(a)

$$F(x,y,z)=xy^2z^3-8$$

$$\begin{aligned}F_x(2,2,1)&=y^2z^3=(4)(1)=4\\F_y(2,2,1)&=2xyz^3=2(2)(2)(1)=8\\F_z(2,2,1)&=3xy^2z^2=3(2)(4)(1)=24\end{aligned}$$

The equation of the tangent plane is given by,

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)$$

$$\begin{aligned}4(x-2)+8(y-2)+24(z-1)&=0\\x-2+2(y-2)+6(z-1)&=0\\x-2+2y-4+6z-6&=0\\\boxed{x+2y+6z=12}\end{aligned}$$

(b) The equation of the normal line is given by,

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)}=\frac{y-y_0}{F_y(x_0,y_0,z_0)}=\frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

$$\frac{x-2}{4}=\frac{y-2}{8}=\frac{z-1}{24}$$
$$\boxed{x-2=\frac{y-2}{2}=\frac{z-1}{6}}$$

51. $x+y+z=e^{xyz}, \quad (0,0,1)$

Solution

(a)

$$F(x,y,z)=x+y+z-e^{xyz}$$

$$F_x(0,0,1)=1-yze^{xyz}=1-0=1$$
$$F_y(0,0,1)=1-xze^{xyz}=1-0=1$$
$$F_z(0,0,1)=1-xye^{xyz}=1-0=1$$

The equation of the tangent plane is given by,

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)$$

$$1(x-0)+1(y-0)+1(z-1)=0$$
$$\boxed{x+y+z=1}$$

(b) The equation of the normal line is given by,

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)}=\frac{y-y_0}{F_y(x_0,y_0,z_0)}=\frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

$$\frac{x-0}{1}=\frac{y-0}{1}=\frac{z-1}{1}$$
$$\boxed{x=y=z-1}$$

Section 7: Maximum and Minimum Values

5-21 (odd)

Find the local maximum and minimum values and saddle point(s) of the function. You are encouraged to use a calculator or computer to graph the function with a domain and viewpoint that reveals all the important aspects of the function.

5. $f(x, y) = x^2 + xy + y^2 + y$

Solution

$$\nabla f(x, y) = \langle 2x + y,\ x + 2y + 1\rangle$$

Finding critical values, (system of two equations)

$$\begin{aligned} 2x + y &= 0 \\ x + 2y &= -1 \end{aligned}$$

$$\begin{aligned} 2x + y &= 0 \\ -2x - 4y &= 2 \\ -3y &= 2 \\ y &= -\frac{2}{3} \quad \Rightarrow \quad x = \frac{1}{3} \end{aligned}$$

Critical point at $\left(\frac{1}{3}, -\frac{2}{3}\right)$

Hessian matrix to determine maxima and minima,

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 1$$

$$\begin{aligned} H_f(x, y) &= det \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \\ H_f\left(\frac{1}{3}, \frac{-2}{3}\right) &= det \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 \\ H_f\left(\frac{1}{3}, -\frac{2}{3}\right) &> 0 \text{ and } f_{xx} = 2 > 0 \end{aligned}$$

$$\boxed{\therefore f\left(\frac{1}{3}, -\frac{2}{3}\right) \text{ is a local minimum}}$$

7. $f(x, y) = 2x^2 - 8xy + y^4 - 4y^3$

Solution

$$\nabla f(x, y) = \left\langle 4x - 8y,\ -8x + 4y^3 - 12y^2\right\rangle$$

Finding critical values,

$$\begin{aligned} 0 &= = 4x - 8y \\ 2x &= 8y \\ x &= 2y \end{aligned}$$

$$\begin{aligned} 0 &= -8x + 4x^3 - 12y^2 \\ 0 &= -8(2y) + 4y^3 - 12y^2 \\ 0 &= 4y^3 - 12y^2 - 16y \\ 0 &= 4y(y^2 - 3y - 4) \\ 0 &= 4y(y-4)(y+1) \\ y = -1, 0, 4 \quad &\Rightarrow \quad x = -2, 0, 8 \end{aligned}$$

Critical points at $(-2,-1), (0,0), (8,4)$

$$f_{xx} = 4, \quad f_{yy} = 12y^2 - 24y, \quad f_{xy} = -8$$

$$\begin{aligned} H_f(8,4) &= det \begin{vmatrix} 4 & -8 \\ -8 & 96 \end{vmatrix} = 384 - 64 = 320 \\ H_f(-2,-1) &= det \begin{vmatrix} 4 & -8 \\ -8 & 36 \end{vmatrix} = 144 - 64 = 80 \\ H_f(0,0) &= det \begin{vmatrix} 4 & -8 \\ -8 & 0 \end{vmatrix} = 0 - 64 = -64 \end{aligned}$$

$H_f(8,4) > 0$ and $f_{xx}(8,4) = 4 > 0$ so f(8,4) is a local minima
$H_f(-2,-1) > 0$ and $f_{xx}(-2,-1) = 4 > 0$ so f(8,4) is a local minima
$H_f(0,0) < 0$ so f(8,4) is a saddle point

$\boxed{\text{Minima: } f(8,4) = -128, \quad f(-2,-1) = 3}$

$\boxed{\text{Saddle Point: } f(0,0)}$

9. $f(x,y) = (x-y)(1-xy)$

Solution

$$\nabla f(x,y) = \left\langle 1 - 2xy + y^2,\ -x^2 - 1 + 2xy \right\rangle$$

Finding critical values,

$$
\begin{aligned}
&0 = 1 - 2xy + y^2\\
&0 = 1 - y(2x + y)\\
&1 = y(2x + y)\\
&y = 1, \quad 2x + y = 1 \quad \Rightarrow \quad y = 1 - 2x
\end{aligned}
$$

$$
\begin{aligned}
&0 = -x^2 - 1 + 2xy\\
&0 = -x^2 - 1 + 2x(1 - 2x)\\
&0 = -x^2 - 1 + 2x - 4x\\
&0 = -x^2 - 2x - 1\\
&0 = x^2 + 2x + 1\\
&0 = (x + 1)(x + 1)\\
&x = -1 \quad \Rightarrow \quad y = 1 - 2(-1) = -1 \quad \Rightarrow \quad (-1, -1)
\end{aligned}
$$

If $y = 1$,

$$
\begin{aligned}
&0 = -x^2 + 2x - 1\\
&0 = x - 2x + 1\\
&0 = (x - 1)(x - 1)\\
&x = 1 \quad \Rightarrow \quad (1, 1)
\end{aligned}
$$

Critical points at (-1, -1), (1, 1)

$$f_{xx} = -2y, \quad f_{yy} = 2x, \quad f_{xy} = -2x + 2y$$

$$
\begin{aligned}
H_f(-1, -1) &= det \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 - 0 = -4\\
H_f(1, 1) &= det \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -4 - 0 = -4
\end{aligned}
$$

$H_f(-1, -1)$ and $H_f(1, 1)$ are both < 0, so they are saddle points.

$\boxed{\text{Saddle Points: } f(-1, -1), \quad f(1, 1)}$

11. $f(x, y) = y\sqrt{x} - y^2 - 2x + 7y$

Solution

$$\nabla f(x, y, z) = \left\langle \frac{y}{2\sqrt{x}} - 2,\ \sqrt{x} - 2y + 7 \right\rangle$$

Finding critical values,

$$
\begin{aligned}
&0 = \frac{1}{2}x^{-1/2} - 2\\
&y = 2(2x^{1/2})\\
&y = 4x^{1/2}
\end{aligned}
$$

$$\begin{aligned} x^{1/2} - 2y + 7 &= 0 \\ x^{1/2} - 2(4x^{1/2}) + 7 &= 0 \\ x^{1/2} - 8x^{1/2}) + 7 &= 0 \\ -7x^{1/2} &= -7 \\ x = \sqrt{1} &= 1 \quad \Rightarrow \quad y = 4\sqrt{1} = 4 \end{aligned}$$

Critical point at (1, 4)

$$f_{xx} = -\frac{y}{4x^{3/2}}, \quad f_{yy} = -2, \quad f_{xy} = \frac{1}{2\sqrt{x}}$$

$$H_f(1,4) = det\begin{vmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -2 \end{vmatrix} = 2 - \frac{1}{4} = \frac{7}{4}$$

$H_f(1,4) > 0$ and $f_{xx}(1,4) = -1 < 0$ so f(1,4) is a local maximum

$\boxed{\text{Maximum: } f(1,4) = 4}$

13. $f(x,y) = x^3 - 3x + 3xy^2$

Solution

$$\nabla f(x,y) = \left\langle 3x^2 - 3 + 3y^2,\ 6xy \right\rangle$$

Finding critical values,

$$\begin{aligned} 0 &= 3x^2 - 3 + 3y^2 \\ 3 &= 3x^2 + 3y^2 \\ 1 &= x^2 + y^2 \\ x &= 0, \pm 1 \quad \Rightarrow \quad y = \pm 1, 0 \end{aligned}$$

Critical points at (0, 1), (0, -1), (1, 0), (-1, 0)

$$f_{xx} = 6x, \quad f_{yy} = 6x, \quad f_{xy} = 6y$$

$$\begin{aligned} H_f(0,1) &= det\begin{vmatrix} 0 & 6 \\ 6 & 0 \end{vmatrix} = 0 - 36 = -36 \\ H_f(0,-1) &= det\begin{vmatrix} 0 & -6 \\ -6 & 0 \end{vmatrix} = 0 - 36 = -36 \\ H_f(1,0) &= det\begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} = 36 - 0 = 36 \\ H_f(-1,0) &= det\begin{vmatrix} -6 & 0 \\ 0 & -6 \end{vmatrix} = 36 - 0 = 36 \end{aligned}$$

$H_f(0, \pm 1) < 0$ so f(0, ± 1) are saddle points
$H_f(1,0) > 0$ and $f_{xx}(1,0) = 6 > 0$ so f(1, 0) is a local minimum

$H_f(-1,0) > 0$ and $f_{xx}(-1,0) = -6 < 0$ so f(-1, 0) is a local maximum

$\boxed{\text{Maximum: } f(-1,0) = 2}$

$\boxed{\text{Minimum: } f(1,0) = -2}$

$\boxed{\text{Saddle Points: } f(0,\pm 1)}$

15. $f(x,y) = x^4 - 2x^2 + y^3 - 3y$

Solution

$$\nabla f(x,y) = \left\langle 4x^3 - 4x,\ 3y^2 - 3 \right\rangle$$

Finding critical values,

$$\begin{aligned} 0 &= 4x^3 - 4x \\ 0 &= 4x(x^2 - 1) \\ 0 &= 4x(x-1)(x+1) \\ x &= 0, \pm 1 \end{aligned}$$

$$\begin{aligned} 0 &= 3y^2 - 3 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

Critical points at (0, ±1), (± 1, ±1)

$$f_{xx} = 12x^2 - 4, \quad f_{yy} = 6y, \quad f_{xy} = 0$$

$$\begin{aligned} H_f(0,\pm 1) &= det \begin{vmatrix} -4 & 0 \\ 0 & 6,-6 \end{vmatrix} = -24 - 0 = -24, \qquad 24 - 0 = 24 \\ H_f(1,\pm 1) &= det \begin{vmatrix} 8 & 0 \\ 0 & 6,-6 \end{vmatrix} = 48 - 0 = 48, \qquad -48 - 0 = -48 \\ H_f(-1,\pm 1) &= det \begin{vmatrix} 8 & 0 \\ 0 & 6,-6 \end{vmatrix} = 48 - 0 = 48, \qquad -48 - 0 = -48 \end{aligned}$$

$H_f(0,1) < 0$ so f(0, 1) is a saddle point
$H_f(0,-1) > 0$ and $f_{xx}(0,-1) = -4 < 0$ so f(0, -1) is a local maximum
$H_f(1,1) > 0$ and $f_{xx}(1,1) = 8 > 0$ so f(1, 1) is a local minimum
$H_f(1,-1) < 0$ so f(1, -1) is a saddle point
$H_f(-1,1) > 0$ and $f_{xx}(-1,1) = 8 > 0$ so f(-1, 1) is a local minimum
$H_f(-1,-1) < 0$ so f(-1, -1) is a saddle point

$\boxed{\text{Maximum: } f(0,-1) = 2}$

$\boxed{\text{Minima: } f(\pm 1,1) = -3}$

$\boxed{\text{Saddle Points: } f(0,1), \quad f(\pm 1,-1)}$

17. $f(x,y) = xy - x^2y - xy^2$

Solution

$$\nabla f(x,y) = \langle y - 2xy - y^2,\ x - x^2 - 2xy \rangle$$

Finding critical values,

$$\begin{aligned}
0 &= y - 2xy - y^2 \\
0 &= y(1 - 2x - y) \\
y &= 0, \quad 0 = 1 - 2x - y \\
y &= 1 - 2x
\end{aligned}$$

$$\begin{aligned}
0 &= x - x^2 - 2xy \\
0 &= x - x^2 - 2x(1 - 2x) \\
0 &= x - x^2 - 2x - 4x^2 \\
0 &= 3x^2 - x \\
0 &= x(3x - 1) \\
x &= 0, \quad 0 = 3x - 1 \\
x &= \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
y &= 1 - 2x = 1 - 2\left(\frac{1}{3}\right) = \frac{1}{3} \qquad \left[\text{when x} = \frac{1}{3}\right] \\
y &= 1 - 2x = 1 - 2\,(0) = 1 \qquad [\text{when x} = 0]
\end{aligned}$$

Critical points at (0, 0), $(\frac{1}{3}, \frac{1}{3})$, (0, 1)

$$f_{xx} = -2y, \quad f_{yy} = -2x, \quad f_{xy} = 1 - 2x - 2y$$

$$\begin{aligned}
H_f(0,0) &= det \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 \\
H_f(0,1) &= det \begin{vmatrix} -2 & -1 \\ -1 & 0 \end{vmatrix} = 0 - 1 = -1 \\
H_f\left(\frac{1}{3}, \frac{1}{3}\right) &= det \begin{vmatrix} -\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{vmatrix} = \frac{4}{9} - \frac{1}{9} = \frac{3}{9}
\end{aligned}$$

$H_f(0,0)$ and $H_f(0,1) < 0$ so f(0, 0) and f(0, 1) are a saddle points
$H_f\left(\frac{1}{3}, \frac{1}{3}\right) > 0$ and $f_{xx}\left(\frac{1}{3}, \frac{1}{3}\right) = -\frac{2}{3} < 0$ so f$(\frac{1}{3}, \frac{1}{3})$ is a local maximum

$\boxed{\text{Maximum: } f\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27}}$

$\boxed{\text{Saddle Points: } f(0,0), \quad f(0,1)}$

19. $f(x, y) = e^x \cos y$

Solution

$$\nabla f(x, y) = \langle e^x \cos y,\ -e^x \sin y \rangle$$

Finding critical values,

$$0 = e^x \cos y$$
$$0 = -e^x \sin y$$

There is no value x that satisfies both $f_x = 0$ and $f_y = 0$,

$\boxed{\therefore \text{There are no critical points or saddle points}}$

21. $f(x, y) = y^2 - 2y \cos x, \quad -1 \leq x \leq 7$

Solution

$$\nabla f(x, y) = \langle 2y \sin x,\ 2y - 2 \cos x \rangle$$

Finding critical values,

$$0 = 2y \sin x$$
$$y = 0, \quad x = 0, \pi, 2\pi$$

$$0 = 2y - 2\cos x \qquad \text{[y = 0]}$$
$$0 = -2\cos x$$
$$x = \frac{\pi}{2}, \quad \frac{3\pi}{2}$$

$$0 = 2y - 2\cos x \qquad \text{[x = 0]}$$
$$0 = 2y - 2$$
$$y = 1$$

$$0 = 2y - 2\cos x \qquad [\text{x} = \pi]$$
$$0 = 2y + 2$$
$$y = -1$$

$$0 = 2y - 2\cos x \qquad [\text{x} = 2\pi]$$
$$0 = 2y - 2$$
$$y = 1$$

Critical points at ($\frac{\pi}{2}$, 0), ($\frac{3\pi}{2}$, 0), (0, 1), (π, -1), (2π, 1)

$$f_{xx} = 2y \cos x, \quad f_{yy} = 2, \quad f_{xy} = 2 \sin x$$

$$H_f\left(\frac{\pi}{2},0\right) = det\begin{vmatrix} 0 & 2 \\ 2 & 2 \end{vmatrix} = 0-4=-4$$
$$H_f\left(\frac{3\pi}{2},0\right) = det\begin{vmatrix} 0 & -2 \\ -2 & 2 \end{vmatrix} = 0-4=-4$$
$$H_f(0,1) = det\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4-0=4$$
$$H_f\left(\pi,-1\right) = det\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4-0=4$$
$$H_f\left(2\pi,1\right) = det\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4-0=4$$

$H_f(\frac{\pi}{2},0)$ and $H_f(\frac{3\pi}{2},0) < 0$ so f($\frac{\pi}{2}$, 0) and f($\frac{3\pi}{2}$, 0) are a saddle points
$H_f\left(0,1\right) > 0$ and $f_{xx}\left(0,1\right) = 2 > 0$ so f$(0,1)$ is a local minimum
$H_f\left(\pi,-1\right) > 0$ and $f_{xx}\left(\pi,-1\right) = 2 > 0$ so f$(\pi,-1)$ is a local minimum
$H_f\left(2\pi,1\right) > 0$ and $f_{xx}\left(2\pi,1\right) = 2 > 0$ so f$(2\pi,1)$ is a local minimum

$\boxed{\text{Minima } f(0,1) = f(\pi,-1) = f(2\pi,1) = -1}$

$\boxed{\text{Saddle Points: } f\left(\frac{\pi}{2},0\right), \quad f\left(\frac{3\pi}{2},0\right)}$

33-39 (odd)

Find the absolute maximum and minimum values of f on the set D.

33. $f(x,y) = x^2 + y^2 - 2x$,

D is the closed triangular region with vertices $(2,0)$, $(0,2)$, and $(0,-2)$

Solution

Since f is a polynomial, it is continuous on the bounded triangle D

$$\nabla f(x,y) = \langle 2x-2,\ 2y\rangle$$

$$0 = 2x-2, \quad 0 = 2y$$
$$x = 1, \quad y = 0$$

Critical point at (1, 0)

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 0$$

$$H_f(1,0) = det\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0, \quad f_{xx}(1,0) = 2 > 0$$

$\therefore$ f(1, 0) is an absolute minimum

Value of f(x,y) in closed region,

$$f(x,2) = x^2 + 4 - 2x = x^2 - 2x + 4 \qquad x \in [0,2]$$
$$f(x,-2) = x^2 + 4 - 2x = x^2 - 2x + 4 \qquad x \in [0,2]$$

$$f(0,2) = 4$$
$$f(0,-2) = 4$$

$\boxed{\text{Minimum: } f(1,0) = -1}$

$\boxed{\text{Maximum: } f(0,\pm 2) = 4}$

35. $f(x,y) = x^2 + y^2 + x^2y + 4,$

$D = \{(x,y) \mid |x| \leq 1,\ |y| \leq 1\}$

Solution

$$\nabla f(x,y) = \left\langle 2x + 2xy,\ 2y + x^2 \right\rangle$$

37. $f(x,y) = x^2 + 2y^2 - 2x - 4y + 1,$

$D = \{(x,y) \mid 0 \leq x \leq 2,\ 0 \leq y \leq 3\}$

39. $f(x,y) = 2x^3 + y^4,$

$D = \{(x,y) \mid x^2 + y^2 \leq 1\}$

43

Find the shortest distance from the point $(2,0,-3)$ to the plane $x + y + z = 1$.

45

Find the points on the cone $z^2 = x^2 + y^2$ that are closest to the point $(4,2,0)$.

47

Find three positive numbers whose sum is 100 and whose product is a maximum.

55

A cardboard box without a lid is to have a volume of 32,000 cm^3. Find the dimensions that minimize the amount of cardboard used.

Section 8: Lagrange Multipliers

3-13 (odd)

Each of these extreme value problems has a solution with both a maximum value and minimum value. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint.

3. $f(x, y) = x^2 - y^2, \quad x^2 + y^2 = 1$

5. $f(x, y) = xy, \quad 4x^2 + y^2 = 8$

7. $f(x, y) = 2x^2 + 6y^2, \quad x^4 + 3y^4 = 1$

9. $f(x, y, z) = 2x + 2y + z, \quad x^2 + y^2 + z^2 = 9$

11. $f(x, y, z) = xy^2z, \quad x^2 + y^2 + z^2 = 4$

13. $f(x, y, z) = x^2 + y^2 + z^2, \quad x^4 + y^4 + z^4 = 1$

23

The method of Lagrange multipliers assumes that the extreme values exist, but that is not always the case. Show that the problem of finding the minimum value of f subject to the given constraint can be solved using Lagrange multipliers, but f does not have a maximum value with that constraint.

$$f(x, y) = x^2 + y^2, \quad xy = 1$$

25

Use Lagrange multipliers to find the maximum value of f subject to the given constraint. Then show that f has no minimum value with that constraint.

$$f(x, y) = e^{xy}, \quad x^3 + y^3 = 16$$

27, 29

Find the extreme values of f on the region described by the inequality.

27. $f(x, y) = x^2 + y^2 + 4x - 4y, \quad x^2 + y^2 \leq 9$

29. $f(x, y) = e^{-xy}, \quad x^2 + 4y^2 \leq 1$

31, 33

Find the extreme values of f subject to both constraints

31. $f(x, y, z) = x + y + z; \quad x^2 + z^2 = 2, \quad x + y = 1$

33. $f(x, y, z) = yz + xy; \quad xy = 1, \quad y^2 + z^2 = 1$
