

# Modern Physics Final Study Guide

PHYS-122

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# 1 Special Relativity

## 1.1 Postulates of Special Relativity

### Postulates:

- The laws of physics are the same in all inertial frames.
- The speed of light is constant in a vacuum for all observers, regardless of their motion relative to the source.

## 1.2 Relativity of Simultaneity

**Relativity of Simultaneity:** Two events that appear simultaneous to one observer may not appear simultaneous to another observer moving relative to the first. This is a direct consequence of the constancy of the speed of light and the Lorentz transformations.

## 1.3 Lorentz Transformations

**Lorentz Transformations:** The Lorentz transformations relate spacetime coordinates of an event in two inertial reference frames moving at a constant velocity relative to each other. These transformations preserve the invariant interval ( $\Delta s^2$ ), a cornerstone of special relativity.

### Transformations:

- For a frame  $S'$  moving with velocity  $v$  along the  $x$ -axis relative to  $S$ :

- Time:  $t' = \gamma \left( t - \frac{vx}{c^2} \right)$
- Space ( $x$ -direction):  $x' = \gamma (x - vt)$
- Space ( $y$ - and  $z$ -directions):  $y' = y, \quad z' = z$

- The Lorentz factor is  $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ .

**Lorentz Transformation Matrix:** The Lorentz transformation can be expressed using matrix multiplication:

$$x' = \Lambda x,$$

where  $x = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$  and  $x'$  are the spacetime coordinates in  $S$  and  $S'$ , respectively. For a boost along the  $x$ -axis, the Lorentz transformation matrix  $\Lambda$  is:

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where  $\beta = v/c$ .

### Key Properties:

- The inverse transformation is obtained by negating  $v$ , flipping the sign of  $\beta$ :

$$\Lambda^{-1} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- The Lorentz transformations preserve the invariant interval:

$$\Delta s^2 = x \cdot x = x' \cdot x'.$$

**Applications:**

- Lorentz transformations explain phenomena like time dilation, length contraction, and the relativity of simultaneity.
- They form the basis for defining and transforming four-vectors, such as four-momentum and four-velocity.

**1.4 The Relativistic Invariant Interval**

**Invariant Interval:** The relativistic invariant interval is a spacetime quantity that remains unchanged under Lorentz transformations.

**Invariant Interval:**

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

**Classifications of the Invariant Interval:**

- **Time-like:**  $\Delta s^2 > 0$ , related to proper time:

$$\Delta s^2 = c^2 \Delta \tau^2$$

- **Space-like:**  $\Delta s^2 < 0$ , related to proper length:

$$-\Delta s^2 = L^2$$

- **Light-like:**  $\Delta s^2 = 0$ , events are connected by light:

$$c^2 \Delta t^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

## 1.5 Relativistic Velocity Addition

**Relativistic Velocity Addition:** When combining velocities in different inertial frames, the relativistic formula must be used. Typically, these problems involve 3 objects and the following reference frames and velocities are defined:

- $S$ : The stationary reference frame.
- $S'$ : The moving reference frame.
- $u$ : The velocity of an object relative to  $S$ .
- $v$ : The velocity of  $S'$  relative to  $S$ .
- $u'$ : The velocity of the object relative to  $S'$ .

**Relativistic Velocity Addition Formula:**

$$u' = \frac{u + v}{1 + \frac{uv}{c^2}}$$

Here,  $u'$  is the velocity of the object as measured in  $S'$ , given the velocity  $u$  of the object relative to  $S$  and the velocity  $v$  of  $S'$  relative to  $S$ .

## 1.6 Four-Vectors in Special Relativity

**Four-Vectors:** Physical quantities in relativity are represented as four-vectors that transform under Lorentz transformations.

**General Form of a Four-Vector:**

$$a^\mu = (a^0, \vec{a}) = (a^0, a^1, a^2, a^3), \quad a^0 = c\Delta t$$

**Dot Product of Four-Vectors:** The invariant dot product between two four-vectors  $a^\mu$  and  $b^\mu$  is:

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}$$

This quantity is Lorentz invariant (unchanged under Lorentz transformations).

## 1.7 Examples of Four-Vectors

### Common Four-Vectors:

- **Four-Position:**

$$x^\mu = (ct, x, y, z)$$

- **Four-Velocity:**

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \vec{v}), \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

- **Four-Momentum:**

$$p^\mu = mu^\mu = \gamma m(c, \vec{v})$$

For massless particles, like photons:

$$p \cdot p = 0, \quad E = |\vec{p}|c$$

## 1.8 Relativistic Conservation Laws

**Relativistic Conservation Laws:** The principles of conservation of energy and momentum extend to special relativity, incorporating the total energy (rest energy + kinetic energy) and relativistic momentum.

### Relativistic Total Energy:

- The total energy  $E$  includes both the rest energy and the relativistic kinetic energy:

$$E = \gamma m_0 c^2.$$

- The rest energy is given by  $E_0 = m_0 c^2$ , where  $m_0$  is the rest mass.
- In any physical process, the total energy is conserved.

**Relativistic Momentum:**

- The relativistic momentum is defined as:

$$\vec{p} = \gamma m_0 \vec{v},$$

where  $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ .

- Both the magnitude and direction of total momentum in an isolated system are conserved.

**Four-Momentum Conservation:**

- Energy and momentum form components of the four-momentum vector:

$$p^\mu = \left( \frac{E}{c}, \vec{p} \right).$$

- In all inertial frames, the conservation of four-momentum holds:

$$\sum p_{\text{initial}}^\mu = \sum p_{\text{final}}^\mu.$$

**Relativistic Collisions:**

- Total energy and total momentum are conserved in collisions, including elastic and inelastic cases.
- Unlike classical mechanics, the concept of conserved "mass" is not generally applicable. For example:
  - In pair annihilation, rest mass converts to energy.
  - In particle creation, energy converts into rest mass.



**Energy-Momentum Relation:**

$$E^2 = (pc)^2 + (m_0c^2)^2$$

This relation connects the total energy  $E$ , the momentum  $p$ , and the rest mass  $m_0$  of a particle.

**Invariant Four-Momentum Magnitude:**

$$p^\mu \cdot p_\mu = m_0^2 c^2$$

This invariant magnitude holds in all inertial frames, ensuring the consistency of relativistic conservation laws.

## 2 Quantum Mechanics

### 2.1 Quantum Theory of Light

#### 2.1.1 Blackbody Radiation and Planck Hypothesis

**Blackbody Radiation:** Classical physics predicted the "ultraviolet catastrophe," where energy radiated at high frequencies diverged. Planck resolved this by quantizing energy:

$$E = nhf, \quad n = 1, 2, 3, \dots$$

**Planck's Law:**

$$I(\nu, T) = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1},$$

where  $I(\nu, T)$  is the intensity of radiation at frequency  $\nu$ .

#### 2.1.2 Photoelectric Effect

**Photoelectric Effect:** Electrons are emitted from a material when light shines on it, with the energy of the electrons depending on the frequency of light.

$$hf = \phi + KE_{max}, \quad hf_c = \phi, \quad KE_{max} = eV_s$$

**Threshold Frequency:** No electrons are emitted if  $f < f_c$ .

### 2.1.3 Compton Scattering

**Compton Scattering:** The scattering of X-rays by electrons demonstrates particle-like behavior of light. The wavelength shift is given by:

$$\lambda' - \lambda = \frac{h}{m_e c} (1 - \cos \theta).$$

**Energy Relation:**

$$\frac{1}{E'} - \frac{1}{E} = \frac{1 - \cos \theta}{m_e c^2}.$$

### 2.1.4 Wave-Particle Duality

**Wave-Particle Duality:** Light and matter exhibit dual behavior, displaying characteristics of both waves and particles depending on the experimental conditions.

**Key Features:**

- Travels as a wave, interacting with itself, and demonstrates phenomena such as interference and diffraction.
- Interacts as a particle, transferring discrete packets of energy, known as photons or quanta.
- Confirmed by experiments:
  - **Double-Slit Experiment:** Demonstrates interference patterns for light and electrons, showing wave-like behavior.
  - **Photoelectric Effect:** Demonstrates the particle nature of light, as photons eject electrons from a material.
  - **Blackbody Radiation:** Explained by Planck's quantization hypothesis, resolving the ultraviolet catastrophe.
  - **Compton Scattering:** Demonstrates photon momentum through scattering with electrons.

**De Broglie Wavelength:**  $\lambda = \frac{h}{p} = \frac{h}{mv}$

**Wave Number:**  $k = \frac{2\pi}{\lambda}$

### 2.1.5 Heisenberg Uncertainty Principle

**Heisenberg Uncertainty Principle:** A fundamental principle of quantum mechanics stating that certain pairs of physical observables cannot be simultaneously measured with arbitrary precision.

**Key Relations:**

- **Position-Momentum Uncertainty:**

$$\Delta x \Delta p \geq \frac{\hbar}{2},$$

where  $\Delta x$  and  $\Delta p$  are the standard deviations of position and momentum, respectively.

- **Energy-Time Uncertainty:**

$$\Delta E \Delta t \geq \frac{\hbar}{2},$$

where  $\Delta E$  and  $\Delta t$  are the uncertainties in energy and time, respectively.

**Variance-Squared Average Relationship:** The uncertainty in a measurable quantity  $A$  is defined as:

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2},$$

where  $\langle A \rangle$  is the expectation value, and  $\langle A^2 \rangle$  is the expectation value of  $A^2$ .

**Implications:**

- Establishes the probabilistic nature of quantum mechanics, replacing deterministic classical physics.
- Demonstrates the impossibility of assigning definite trajectories to particles as in classical mechanics.

### 2.1.6 Group and Phase Velocities

**Group and Phase Velocities:** Velocities associated with wave propagation.

- Group velocity:

$$v_g = \frac{d\omega}{dk},$$

representing the velocity of the wave packet **and the velocity of the particle**

- Phase velocity:

$$v_p = \frac{\omega}{k},$$

representing the velocity of individual wave crests.

## 2.2 Schrödinger Equation

### 2.2.1 Overview

**Schrödinger Equation:** Governs the quantum mechanical behavior of particles. It describes how the wave function of a particle evolves under the influence of potential energy.

**General Forms:**

- **1D Time-Independent Schrödinger Equation:**

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x).$$

- **3D Schrödinger Equation (spherical coordinates):**

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(r, \theta, \phi) + V(r)\psi(r, \theta, \phi) = E\psi(r, \theta, \phi).$$

The solution is separable:

$$\psi(r, \theta, \phi) = R(r)Y(\theta, \phi),$$

where  $Y(\theta, \phi)$  are spherical harmonics.

### 2.2.2 Different Potentials

**Free Particle:**

- For  $V(x) = 0$ , solutions are plane waves:

$$\psi(x) = Ae^{ikx} + Be^{-ikx},$$

where  $k = \frac{p}{\hbar}$ ,  $\omega = \frac{E}{\hbar}$ , and  $p = \hbar k = \sqrt{2mE}$ .

- $e^{ikx}$  is right moving plane wave and  $e^{-ikx}$  is left moving plane wave.
- For full time dependent,  $\Psi(x, t) = \psi(x)e^{-i\omega t}$
- Wave function is not normalizable!

**Particle in a Box/Infinite Square Well Potential:**

- For  $V(x) = 0$  inside the box and  $V(x) = \infty$  outside:

- General Solution 1D:

$$\psi_n(x) = A \sin\left(\frac{n\pi x}{L}\right)$$

- General Solution 3D:

$$\psi_{n_1, n_2, n_3} = A \sin\left(\frac{n_1\pi x}{L}\right) \sin\left(\frac{n_2\pi y}{L}\right) \sin\left(\frac{n_3\pi z}{L}\right), \quad E_n = \frac{\hbar^2\pi^2}{2mL^2}(n_1^2 + n_2^2 + n_3^2)$$

- Particular Solution 1D:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2\hbar^2}{8mL^2}.$$

**Step Potential:**

$$V(x) = \begin{cases} 0, & x < 0 \\ U_0, & x \geq 0 \end{cases}$$

The particle's behavior depends on whether its energy  $E$  is greater than or less than  $U_0$ .

**Case 1:  $E > U_0$** **Wave Function Solutions:**

- $x < 0$ :  $\psi(x) = Ae^{ik_0x} + Be^{-ik_0x}$ ,  $k_0 = \sqrt{\frac{2mE}{\hbar^2}}$
- $x \geq 0$ :  $\psi(x) = Ce^{ik_1x}$ ,  $k_1 = \sqrt{\frac{2m(E-U_0)}{\hbar^2}}$

**Reflection and Transmission Coefficients:**

$$R = \left(\frac{k_0 - k_1}{k_0 + k_1}\right)^2, \quad T = \frac{4k_0k_1}{(k_0 + k_1)^2}, \quad R + T = 1$$



**Case 2:**  $E < U_0$

**Wave Function Solutions:**

- $x < 0$ :  $\psi(x) = Ae^{ik_0x} + Be^{-ik_0x}$ ,  $k_0 = \sqrt{\frac{2mE}{\hbar^2}}$
- $x \geq 0$ :  $\psi(x) = De^{-\kappa x}$ ,  $\kappa = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}}$

**Tunneling Insight:** The particle cannot propagate in the  $x \geq 0$  region but has a finite probability of being found in the barrier due to tunneling.

**Key Insights:**

- For  $E > U_0$ , the particle has probabilities of transmission and reflection at the step.
- For  $E < U_0$ , the particle exhibits tunneling with an exponentially decaying wave function in the barrier.

## 2.3 Hydrogen Atom

### 2.3.1 Wave Functions

**Wave Functions:**

$$\psi(r, \theta, \phi) = R_{nl}(r)Y_l^m(\theta, \phi),$$

where  $R_{nl}(r)$  is the radial wave function, and  $Y_l^m(\theta, \phi)$  are spherical harmonics.

**Radial Probability Distribution:** The probability density  $P(r)$  is given by:

$$P(r) = r^2 |R_{nl}(r)|^2.$$

### 2.3.2 Quantum Numbers

#### Quantum Numbers:

- **Principal quantum number ( $n$ ):** Determines energy level ( $n = 1, 2, 3, \dots$ ).

- Tells us the **energy** of the Hydrogen atom

$$E(\psi_{n,\ell,m}) = -\frac{13.6 \text{ eV}}{n^2}$$

- **Orbital angular momentum quantum number ( $\ell$ ):** Determines the shape of the orbital ( $\ell = 0, 1, \dots, n-1$ ).

- Tells us the **total angular momentum** of the Hydrogen atom

$$L(\psi_{n,\ell,m}) = \hbar\sqrt{\ell(\ell+1)}$$

- **Magnetic quantum number ( $m$ ):** Determines the orientation ( $m = -\ell, -\ell+1, \dots, \ell$ ).

- Tells us the **z-component of angular momentum** of the Hydrogen atom

$$L_z(\psi_{n,\ell,m}) = \hbar m$$

- **Spin quantum number ( $m_s$ ):** Intrinsic angular momentum ( $m_s = \pm\frac{1}{2}$ ).

- Has to be rotated **720 degrees** for it to return to original state
- Fermions have half integer spin (electrons, muons, protons, neutrons, etc.)
- Bosons have integer spin (photons, W and Z bosons, Higgs boson, etc.)
- Stern-Gerlach Experiment

### 2.3.3 Multi-Electron Atoms and the Pauli Exclusion Principle

**Multi-Electron Atoms:** In atoms with more than one electron, the arrangement of electrons is determined by their energies and quantum states:

- Electrons occupy orbitals starting from the lowest energy levels, but no more than two electrons can share the same orbital.
- Inner electrons shield outer electrons from the full nuclear charge, affecting orbital energies.
- Example: In Potassium ( $Z = 19$ ), the electron configuration is  $1s^2 2s^2 2p^6 3s^2 3p^6 4s^1$ . The last electron occupies the  $4s$  orbital instead of  $3d$  due to shielding.

**Pauli Exclusion Principle:** No two electrons in an atom can occupy the same quantum state. This means:

- No two electrons can have identical quantum numbers  $(n, l, m, m_s)$ .
- This principle determines the filling of electron orbitals and gives rise to electron configurations.

**Electron Configuration:** The distribution of electrons in orbitals follows these rules:

- Electrons are added one at a time to the lowest energy subshell available.
- The periodic table reflects the outer electron configurations, which dictate chemical properties.

### 3 Statistical Mechanics

#### 3.1 Key Concepts

- Microstates and macrostates: Connection between microscopic and macroscopic descriptions.
- Boltzmann distribution:  $P_i = \frac{e^{-E_i/k_B T}}{Z}$ .
- Partition function: Summation of states to understand thermodynamic properties,  $Z = \sum_i e^{-E_i/k_B T}$ .
- Occupation numbers: Average number of particles in a state,  $n_i = \frac{1}{e^{(E_i - \mu)/k_B T} \pm 1}$  (for fermions and bosons).
- Degeneracy: The number of microstates corresponding to a single energy level.
- Distinguishable vs. indistinguishable particles: Boltzmann distribution for distinguishable particles; Fermi-Dirac and Bose-Einstein distributions for indistinguishable particles.

#### 3.2 Essential Equations

$$\text{Boltzmann distribution: } P_i = \frac{e^{-E_i/k_B T}}{Z}$$

$$\text{Partition function: } Z = \sum_i g_i e^{-E_i/k_B T}$$

$$\text{Fermi-Dirac distribution: } n_i = \frac{1}{e^{(E_i - \mu)/k_B T} + 1}$$

$$\text{Bose-Einstein distribution: } n_i = \frac{1}{e^{(E_i - \mu)/k_B T} - 1}$$

$$\text{Average energy: } \langle E \rangle = \sum_i P_i E_i = -\frac{\partial \ln Z}{\partial \beta}$$

$$\text{Probability of a state: } P(E) = \frac{g_i e^{-E/k_B T}}{Z}$$

### 3.3 Units and Unit Conversions

- Energy:  $1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$ .
- Temperature: Convert between Kelvin and energy using  $k_B = 1.38 \times 10^{-23} \text{ J/K}$ .
- Wavelength and frequency:  $E = hf$ ,  $\lambda = \frac{c}{f}$ , with  $hc = 1240 \text{ eV} \cdot \text{nm}$ .

## 4 Quick Reference

### 4.1 Mathematical Tools

- Trigonometric Identities:

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\sin(2x) = 2 \sin x \cos x$$

- Series:

- Taylor Series Expansion:  $\sum_{n=0}^{\infty} C_n (x-a)^n$  where  $C_n = \frac{f^{(n)}(a)}{n!}$

- Binomial Series Expansion:  $(1 \pm x)^r = 1 \pm rx \pm \dots$

- Integration Tips:

- For even functions:  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

- For odd functions:  $\int_{-a}^a f(x) dx = 0$

- Gaussian integrals often appear in wave packets

- Complex Numbers Review

- Complex conjugate:  $z^* = a - bi$  for  $z = a + bi$
- Modulus:  $|z| = \sqrt{a^2 + b^2}$
- Euler's formula:  $e^{i\theta} = \cos \theta + i \sin \theta$
- Argument:  $\arg(z) = \tan^{-1} \left( \frac{y}{x} \right)$

- Second Order Differential Equations

- If ODE has this form:  $\frac{d^2 f}{dx^2} = k^2 f$  where  $k$  is a constant

The solution is:

$$f(x) = Ae^{kx} + Be^{-kx}$$

## 4.2 Important Constants

- Stefan-Boltzmann constant:  $\sigma = 5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4$
- Planck's constant:  $h = 6.626 \times 10^{-34} \text{ J} \cdot \text{s} = 4.136 \times 10^{-15} \text{ eV} \cdot \text{s}$
- Reduced Planck's constant:  $\hbar = 1.055 \times 10^{-34} \text{ J} \cdot \text{s}$
- Electron mass:  $m_e = 9.11 \times 10^{-31} \text{ kg}$
- Speed of light:  $c = 3.0 \times 10^8 \text{ m/s}$
- Bohr radius:  $a_0 = 0.0529 \times 10^{-9} \text{ m}$
- Compton wavelength of the electron:  $\frac{h}{m_e c} = 0.002426 \text{ nm}$

## 4.3 Unit Conversions

- $1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$
- $hc = 1240 \text{ eV} \cdot \text{nm}$
- $\hbar c = \frac{1240}{2\pi} \text{ eV} \cdot \text{nm}$