Modern Physics Final Study Guide

PHYS-122

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Contents

1 Special Relativity

1.1 Postulates of Special Relativity

Postulates:

- The laws of physics are the same in all inertial frames.
- The speed of light is constant in a vacuum for all observers, regardless of their motion relative to the source.

1.2 Relativity of Simultaneity

Relativity of Simultaneity: Two events that appear simultaneous to one observer may not appear simultaneous to another observer moving relative to the first. This is a direct consequence of the constancy of the speed of light and the Lorentz transformations.

1.3 Lorentz Transformations

Lorentz Transformations: The Lorentz transformations relate spacetime coordinates of an event in two inertial reference frames moving at a constant velocity relative to each other. These transformations preserve the invariant interval (Δs^2), a cornerstone of special relativity.

Transformations:

- For a frame S' moving with velocity v along the x-axis relative to S:
  - Time: $t' = \gamma\left(t - \frac{vx}{c^2}\right)$
  - Space (x-direction): $x' = \gamma\,(x - vt)$
  - Space (y- and z-directions): $y' = y, \quad z' = z$
- The Lorentz factor is $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

Lorentz Transformation Matrix: The Lorentz transformation can be expressed using matrix multiplication:

$$x' = \Lambda x,$$

where $x = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$ and x' are the spacetime coordinates in S and S', respectively. For a boost along the x-axis, the Lorentz transformation matrix Λ is:

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\beta = v/c$.

Key Properties:

- The inverse transformation is obtained by negating v, flipping the sign of β:

$$\Lambda^{-1} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- The Lorentz transformations preserve the invariant interval:

$$\Delta s^2 = x \cdot x = x' \cdot x'.$$

Applications:

- Lorentz transformations explain phenomena like time dilation, length contraction, and the relativity of simultaneity.
- They form the basis for defining and transforming four-vectors, such as four-momentum and four-velocity.

1.4 The Relativistic Invariant Interval

Invariant Interval: The relativistic invariant interval is a spacetime quantity that remains unchanged under Lorentz transformations.

Invariant Interval:

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

Classifications of the Invariant Interval:

- **Time-like:** $\Delta s^2 > 0$, related to proper time:

$$\Delta s^2 = c^2 \Delta \tau^2$$

- **Space-like:** $\Delta s^2 < 0$, related to proper length:

$$-\Delta s^2 = L^2$$

- **Light-like:** $\Delta s^2 = 0$, events are connected by light:

$$c^2 \Delta t^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

1.5 Relativistic Velocity Addition

Relativistic Velocity Addition: When combining velocities in different inertial frames, the relativistic formula must be used. Typically, these problems involve 3 objects and the following reference frames and velocities are defined:

- S: The stationary reference frame.
- S': The moving reference frame.
- u: The velocity of an object relative to S.
- v: The velocity of S' relative to S.
- u': The velocity of the object relative to S'.

Relativistic Velocity Addition Formula:

$$u' = \frac{u+v}{1+\frac{uv}{c^2}}$$

Here, u' is the velocity of the object as measured in S', given the velocity u of the object relative to S and the velocity v of S' relative to S.

1.6 Four-Vectors in Special Relativity

Four-Vectors: Physical quantities in relativity are represented as four-vectors that transform under Lorentz transformations.

General Form of a Four-Vector:

$$a^\mu = (a^0, \vec{a}) = (a^0, a^1, a^2, a^3), \quad a^0 = c\Delta t$$

Dot Product of Four-Vectors: The invariant dot product between two four-vectors a^μ and b^μ is:

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}$$

This quantity is Lorentz invariant (unchanged under Lorentz transformations).

1.7 Examples of Four-Vectors

Common Four-Vectors:

- **Four-Position:**

$$x^\mu = (ct, x, y, z)$$

- **Four-Velocity:**

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \vec{v}), \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

- **Four-Momentum:**

$$p^\mu = mu^\mu = \gamma m(c, \vec{v})$$

For massless particles, like photons:

$$p \cdot p = 0, \quad E = |\vec{p}|c$$

1.8 Relativistic Conservation Laws

Relativistic Conservation Laws: The principles of conservation of energy and momentum extend to special relativity, incorporating the total energy (rest energy + kinetic energy) and relativistic momentum.

Relativistic Total Energy:

- The total energy E includes both the rest energy and the relativistic kinetic energy:

$$E = \gamma m_0 c^2.$$

- The rest energy is given by $E_0 = m_0 c^2$, where m_0 is the rest mass.
- In any physical process, the total energy is conserved.

Relativistic Momentum:

- The relativistic momentum is defined as:

$$\vec{p} = \gamma m_0 \vec{v},$$

where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

- Both the magnitude and direction of total momentum in an isolated system are conserved.

Four-Momentum Conservation:

- Energy and momentum form components of the four-momentum vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right).$$

- In all inertial frames, the conservation of four-momentum holds:

$$\sum p^\mu_{\text{initial}} = \sum p^\mu_{\text{final}}.$$

Relativistic Collisions:

- Total energy and total momentum are conserved in collisions, including elastic and inelastic cases.
- Unlike classical mechanics, the concept of conserved "mass" is not generally applicable. For example:
  - In pair annihilation, rest mass converts to energy.
  - In particle creation, energy converts into rest mass.

Energy-Momentum Relation:

$$E^2 = (pc)^2 + (m_0 c^2)^2$$

This relation connects the total energy E, the momentum p, and the rest mass m_0 of a particle.

Invariant Four-Momentum Magnitude:

$$p^\mu \cdot p^\mu = m_0^2 c^2$$

This invariant magnitude holds in all inertial frames, ensuring the consistency of relativistic conservation laws.

2 Quantum Mechanics

2.1 Quantum Theory of Light

2.1.1 Blackbody Radiation and Planck Hypothesis

Blackbody Radiation: Classical physics predicted the "ultraviolet catastrophe," where energy radiated at high frequencies diverged. Planck resolved this by quantizing energy:

$$E = nhf, \quad n = 1, 2, 3, \dots$$

Planck's Law:

$$I(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1},$$

where $I(\nu, T)$ is the intensity of radiation at frequency ν.

2.1.2 Photoelectric Effect

Photoelectric Effect: Electrons are emitted from a material when light shines on it, with the energy of the electrons depending on the frequency of light.

$$hf = \phi + KE_{max}, \quad hf_c = \phi, \quad KE_{max} = eV_s$$

Threshold Frequency: No electrons are emitted if $f < f_c$.

2.1.3 Compton Scattering

Compton Scattering: The scattering of X-rays by electrons demonstrates particle-like behavior of light. The wavelength shift is given by:

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta).$$

Energy Relation:

$$\frac{1}{E'} - \frac{1}{E} = \frac{1 - \cos\theta}{m_e c^2}.$$

2.1.4 Wave-Particle Duality

Wave-Particle Duality: Light and matter exhibit dual behavior, displaying characteristics of both waves and particles depending on the experimental conditions.

Key Features:

- Travels as a wave, interacting with itself, and demonstrates phenomena such as interference and diffraction.
- Interacts as a particle, transferring discrete packets of energy, known as photons or quanta.
- Confirmed by experiments:
  - **Double-Slit Experiment:** Demonstrates interference patterns for light and electrons, showing wave-like behavior.
  - **Photoelectric Effect:** Demonstrates the particle nature of light, as photons eject electrons from a material.
  - **Blackbody Radiation:** Explained by Planck's quantization hypothesis, resolving the ultraviolet catastrophe.
  - **Compton Scattering:** Demonstrates photon momentum through scattering with electrons.

De Broglie Wavelength: $\lambda = \frac{h}{p} = \frac{h}{mv}$

Wave Number: $k = \frac{2\pi}{\lambda}$

2.1.5 Heisenberg Uncertainty Principle

Heisenberg Uncertainty Principle: A fundamental principle of quantum mechanics stating that certain pairs of physical observables cannot be simultaneously measured with arbitrary precision.

Key Relations:

- **Position-Momentum Uncertainty:**

$$\Delta x \Delta p \geq \frac{\hbar}{2},$$

where Δx and Δp are the standard deviations of position and momentum, respectively.

- **Energy-Time Uncertainty:**

$$\Delta E \Delta t \geq \frac{\hbar}{2},$$

where ΔE and Δt are the uncertainties in energy and time, respectively.

Variance-Squared Average Relationship: The uncertainty in a measurable quantity A is defined as:

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2},$$

where $\langle A \rangle$ is the expectation value, and $\langle A^2 \rangle$ is the expectation value of A^2.

Implications:

- Establishes the probabilistic nature of quantum mechanics, replacing deterministic classical physics.
- Demonstrates the impossibility of assigning definite trajectories to particles as in classical mechanics.

2.1.6 Group and Phase Velocities

Group and Phase Velocities: Velocities associated with wave propagation.

- Group velocity:

$$\boxed{v_g = \frac{d\omega}{dk},}$$

  representing the velocity of the wave packet **and the velocity of the particle**

- Phase velocity:

$$\boxed{v_p = \frac{\omega}{k},}$$

  representing the velocity of individual wave crests.

2.2 Schrödinger Equation

2.2.1 Overview

Schrödinger Equation: Governs the quantum mechanical behavior of particles. It describes how the wave function of a particle evolves under the influence of potential energy.

General Forms:

- **1D Time-Independent Schrödinger Equation:**

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x).$$

- **3D Schrödinger Equation (spherical coordinates):**

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r,\theta,\phi) + V(r)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi).$$

  The solution is separable:

$$\psi(r,\theta,\phi) = R(r)Y(\theta,\phi),$$

  where $Y(\theta,\phi)$ are spherical harmonics.

2.2.2 Different Potentials

Free Particle:

- For $V(x) = 0$, solutions are plane waves:

$$\psi(x) = Ae^{ikx} + Be^{-ikx},$$

  where $k = \frac{p}{\hbar}$, $\omega = \frac{E}{\hbar}$, and $p = \hbar k = \sqrt{2mE}$.

- e^{ikx} is right moving plane wave and e^{-ikx} is left moving plane wave.
- For full time dependent, $\Psi(x,t) = \psi(x)e^{-i\omega t}$
- Wave function is not normalizable!

Particle in a Box/Infinite Square Well Potential:

- For $V(x) = 0$ inside the box and $V(x) = \infty$ outside:
  - General Solution 1D:

$$\psi_n(x) = A \sin\left(\frac{n\pi x}{L}\right)$$

  - General Solution 3D:

$$\psi_{n_1,n_2,n_3} = A \sin\left(\frac{n_1\pi x}{L}\right) \sin\left(\frac{n_2\pi y}{L}\right) \sin\left(\frac{n_3\pi z}{L}\right)$$

  - Particular Solution 1D:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2h^2}{8mL^2}$$

  - Particular Solution 3D:

$$\psi_{n_1,n_2,n_3} = \left(\frac{2}{L}\right)^{3/2} \sin\left(\frac{n_1\pi x}{L}\right) \sin\left(\frac{n_2\pi y}{L}\right) \sin\left(\frac{n_3\pi z}{L}\right)$$

$$E_n = \frac{\hbar^2\pi^2}{2mL^2}(n_1^2 + n_2^2 + n_3^2)$$

Step Potential:

$$V(x) = \begin{cases} 0, & x < 0 \\ U_0, & x \geq 0 \end{cases}$$

The particle's behavior depends on whether its energy E is greater than or less than U_0.

Case 1: $E > U_0$

Wave Function Solutions:

- $x < 0$: $\psi(x) = Ae^{ik_0x} + Be^{-ik_0x}, \quad k_0 = \sqrt{\frac{2mE}{\hbar^2}}$
- $x \geq 0$: $\psi(x) = Ce^{ik_1x}, \quad k_1 = \sqrt{\frac{2m(E-U_0)}{\hbar^2}}$

Reflection and Transmission Coefficients:

$$R = \left(\frac{k_0 - k_1}{k_0 + k_1}\right)^2, \quad T = \frac{4k_0k_1}{(k_0 + k_1)^2}, \quad R + T = 1$$

Case 2: $E < U_0$

Wave Function Solutions:

- $x < 0$: $\psi(x) = Ae^{ik_0x} + Be^{-ik_0x}, \quad k_0 = \sqrt{\frac{2mE}{\hbar^2}}$
- $x \geq 0$: $\psi(x) = De^{-\kappa x}, \quad \kappa = \sqrt{\frac{2m(U_0-E)}{\hbar^2}}$

Tunneling Insight: The particle cannot propagate in the $x \geq 0$ region but has a finite probability of being found in the barrier due to tunneling.

Key Insights:

- For $E > U_0$, the particle has probabilities of transmission and reflection at the step.
- For $E < U_0$, the particle exhibits tunneling with an exponentially decaying wave function in the barrier.

2.3 Hydrogen Atom

2.3.1 Wave Functions

Wave Functions:

$$\psi(r, \theta, \phi) = R_{nl}(r) Y_l^m(\theta, \phi),$$

where $R_{nl}(r)$ is the radial wave function, and $Y_l^m(\theta, \phi)$ are spherical harmonics.
Radial Probability Distribution: The probability density $P(r)$ is given by:

$$P(r) = r^2 |R_{nl}(r)|^2.$$

2.3.2 Quantum Numbers

Quantum Numbers:

- **Principal quantum number (n):** Determines energy level ($n = 1, 2, 3, \dots$).
  - Tells us the **energy** of the Hydrogen atom

$$E(\psi_{n,\ell,m}) = -\frac{13.6\text{ eV}}{n^2}$$

- **Orbital angular momentum quantum number (ℓ):** Determines the shape of the orbital ($\ell = 0, 1, \dots, n-1$).
  - Tells us the **total angular momentum** of the Hydrogen atom

$$L(\psi_{n,\ell,m}) = \hbar\sqrt{\ell(\ell+1)}$$

- **Magnetic quantum number (m):** Determines the orientation ($m = -\ell, -\ell+1, \dots, \ell$).
  - Tells us the **z-component of angular momentum** of the Hydrogen atom

$$L_z(\psi_{n,\ell,m}) = \hbar m$$

- **Spin quantum number (m_s):** Intrinsic angular momentum ($m_s = \pm\frac{1}{2}$).
  - Has to be rotated **720 degrees** for it to return to original state
  - Fermions have half integer spin (electrons, muons, protons, neutrons, etc.)
  - Bosons have integer spin (photons, W and Z bosons, Higgs boson, etc.)
  - Stern-Gerlach Experiment

2.3.3 Multi-Electron Atoms and the Pauli Exclusion Principle

Multi-Electron Atoms: In atoms with more than one electron, the arrangement of electrons is determined by their energies and quantum states:

- Electrons occupy orbitals starting from the lowest energy levels, but no more than two electrons can share the same orbital.
- Inner electrons shield outer electrons from the full nuclear charge, affecting orbital energies.
- Example: In Potassium ($Z = 19$), the electron configuration is $1s^2 2s^2 2p^6 3s^2 3p^6 4s^1$. The last electron occupies the $4s$ orbital instead of $3d$ due to shielding.

Pauli Exclusion Principle: No two electrons in an atom can occupy the same quantum state. This means:

- No two electrons can have identical quantum numbers (n, l, m, m_s).
- This principle determines the filling of electron orbitals and gives rise to electron configurations.

Electron Configuration: The distribution of electrons in orbitals follows these rules:

- Electrons are added one at a time to the lowest energy subshell available.
- The periodic table reflects the outer electron configurations, which dictate chemical properties.

3 Statistical Mechanics

3.1 Microscopic vs. Macroscopic States

Microscopic States: The state of a system described by the position and momentum of every particle. For N particles, this requires $6N$ values (3 spatial coordinates and 3 momentum components per particle).
Macroscopic States: When N is large (e.g., Avogadro's number), it is impractical to track each particle. Instead, systems are described using averaged quantities like energy, volume, and temperature.
Example: For 10 coins:

- Total **microstates:** $2^{10} = 1024$.
- Total **macrostates:** 11 (e.g., all heads, 1 tail and 9 heads, etc.).

3.2 Multiplicity and Degeneracy

Multiplicity (g): The number of microstates corresponding to a macrostate. Calculated as:

$$g = \binom{N}{k} = \frac{N!}{k!(N-k)!},$$

where k is the number of particles in a specific state.
Example: For 10 coins with 3 tails and 7 heads:

$$g = \binom{10}{3} = \frac{10!}{3!7!} = 120.$$

Degeneracy: The number of quantum states associated with a given energy level.
Microcanonical Ensemble: A set of all microstates with the same energy, where each is equally probable.

3.3 Statistical Distributions

Probability Distributions: The likelihood of finding a particle in a state with energy E_i depends on temperature (T) and energy. The three primary distributions are:

- **Maxwell-Boltzmann (Distinguishable Particles):** Applicable when $T \gg T_c$ (thermal energy kT is much larger than quantum energy level spacing) and particle density is low.

$$f(E) = f_B(E) = Ae^{-E/kT}$$

- **Bose-Einstein (Indistinguishable Bosons):** Below the critical temperature (T_c), a Bose-Einstein condensate forms as many bosons occupy the ground state.

$$f(E) = f_{BE}(E) = \frac{1}{e^{\alpha}e^{E/k_BT} - 1}.$$

- **Fermi-Dirac (Indistinguishable Fermions):** Governed by the Pauli exclusion principle, where no two fermions occupy the same quantum state.

$$f(E) = f_{FD}(E) = \frac{1}{e^{\alpha}e^{E/k_BT} + 1}.$$

Occupation Number:

- The **occupation number**, $n(E)dE$, represents the average number of particles in a quantum state with energy between E and $E + dE$.
- It is calculated as:

$$n(E)dE = f(E)g(E)dE,$$

  where:

  - $f(E)$ is the probability distribution (Maxwell-Boltzmann, Bose-Einstein, or Fermi-Dirac).
  - $g(E)dE$ is the degeneracy, or the number of states available in the energy range E to $E + dE$.

3.4 Thermal Probability and Average Quantities

Thermal Probability: The probability of finding a particle in a specific energy state depends on the Boltzmann factor:

$$P(E) = \frac{e^{-E/k_B T}}{Z},$$

where $Z = \sum e^{-E_i/k_B T}$ is the partition function.

Average Energy:

$$\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta}, \quad \text{where } \beta = \frac{1}{k_B T}.$$

Normalization Conditions:

- Total number of particles:

$$\int f(E) g(E) dE = N.$$

- Total energy of the system:

$$\int f(E) g(E) E \, dE = E_{\text{tot}}.$$

Energy Density for Photon Gas:

$$u(E)dE = \frac{8\pi}{(hc)^3} E^3 \frac{1}{e^{E/k_B T} - 1} dE.$$

Degeneracy Pressure: Arises due to the Pauli exclusion principle and prevents fermions from collapsing under compression, as seen in white dwarfs and neutron stars.

3.5 Bose-Einstein Condensation

Bose-Einstein Condensation: Occurs when a system of bosons is cooled below a critical temperature (T_c), causing many particles to occupy the ground state.
Key Relations:

- Critical temperature:

$$T_c = \left(\frac{h^2}{2\pi mk}\right)\left(\frac{N}{V\zeta(3/2)}\right)^{2/3}.$$

where $\zeta(3/2) = 2.612$

- Fraction in the ground state:

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}.$$

3.6 Degeneracy Pressure

Degeneracy Pressure: Prevents the collapse of fermionic systems due to quantum mechanical effects, critical in white dwarfs and neutron stars.
Key Relation:

$$P \propto \left(\frac{\hbar^2}{m}\right)\left(\frac{N}{V}\right)^{5/3}.$$

4 Quick Reference

4.1 Mathematical Tools

- Trigonometric Identities:

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$
$$\sin^2 x = \frac{1-\cos(2x)}{2}$$
$$\sin(2x) = 2\sin x\cos x$$

- Series:
  - Taylor Series Expansion: $\sum_{n=0}^{\infty} C_n(x-a)^n$ where $C_n = \frac{f^{(n)}(a)}{n!}$
  - Binomial Series Expansion: $(1 \pm x)^r = 1 \pm rx \pm ...$

- Integration Tips:
  - For even functions: $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$
  - For odd functions: $\int_{-a}^{a} f(x)dx = 0$
  - Gaussian integrals often appear in wave packets

- Complex Numbers Review
  - Complex conjugate: $z^* = a - bi$ for $z = a + bi$
  - Modulus: $|z| = \sqrt{a^2 + b^2}$
  - Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$
  - Argument: $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

- Second Order Differential Equations
  - If ODE has this form: $\frac{d^2 f}{dx^2} = k^2 f$ where k is a constant

    The solution is:
    $$f(x) = Ae^{kx} + Be^{-kx}$$

4.2 Important Constants

- Stefan-Boltzmann constant: $\sigma = 5.67 \times 10^{-8}\ \mathrm{W/m^2} \cdot K^4$
- Planck's constant: $h = 6.626 \times 10^{-34}\,\mathrm{J \cdot s} = 4.136 \times 10^{-15}\ \mathrm{eV \cdot s}$
- Reduced Planck's constant: $\hbar = 1.055 \times 10^{-34}\,\mathrm{J \cdot s}$
- Electron mass: $m_e = 9.11 \times 10^{-31}\,\mathrm{kg}$
- Speed of light: $c = 3.0 \times 10^{8}\,\mathrm{m/s}$
- Bohr radius: $a_0 = 0.0529 \times 10^{-9}\,\mathrm{m}$
- Compton wavelength of the electron: $\frac{h}{m_e c} = 0.002426$ nm

4.3 Unit Conversions

- $1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$
- $hc = 1240\ \mathrm{eV \cdot nm}$
- $\hbar c = \frac{1240}{2\pi}\ \mathrm{eV \cdot nm}$
- Temperature: Convert between Kelvin and energy using $k_B = 1.38 \times 10^{-23}\ \mathrm{J/K}$.