Modern Physics Final Study Guide

PHYS-122

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Contents

1 Special Relativity

1.1 Key Concepts

Postulates:

- Laws of physics are the same in all inertial frames.
- The speed of light is constant in a vacuum for all observers, regardless of their motion relative to the source.

Relativity of Simultaneity: Two events that appear simultaneous to one observer may not appear simultaneous to another observer moving relative to the first.

Relativistic Velocity Addition: These problems typically involve three objects:

- S: Stationary reference frame.
- S': Reference frame of the second moving object.
- u: Velocity of the first object relative to the stationary frame.
- v: Velocity of the second object relative to the stationary frame.
- u': Velocity of the first object relative to the second moving object.

The Relativistic Invariant Interval: A spacetime quantity that remains unchanged under Lorentz transformations.

- **Properties:**
  - Negative spatial components distinguish it from a Euclidean interval.
  - It can be positive, negative, or zero:
    * **Time-like ($s^2 > 0$):** Events can be connected by a physical particle moving slower than the speed of light.
    * **Space-like ($s^2 < 0$):** Events are separated by more distance than time; no signal can connect them.
    * **Light-like ($s^2 = 0$):** Events are connected by a light signal.
- **Relationship to Proper Time and Proper Length:**
  - **Proper Time ($\Delta\tau$):** In the frame where the particle is stationary, the interval relates to proper time.
  - **Proper Length (ΔL):** In the frame where events are simultaneous, the interval relates to proper length.

Four-Vectors

- **Four-Position:** Represents the spacetime coordinates of an event:

$$x^\mu = (ct, \vec{x}) = (ct, x, y, z)$$

  where ct is the time component, and $\vec{x} = (x, y, z)$ is the spatial position in 3D space.

- **Four-Velocity:** Describes the rate of change of four-position with respect to proper time τ:

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \vec{v})$$

  where $\vec{v} = (v_x, v_y, v_z)$ is the 3-velocity and $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

  - The invariant magnitude of four-velocity is $u^\mu u_\mu = c^2$.

- **Four-Momentum:** Represents the energy and momentum of a particle:

$$p^\mu = m_0 u^\mu = (\frac{E}{c}, \vec{p})$$

  where $\vec{p} = \gamma m_0 \vec{v}$ is the 3-momentum, and $E = \gamma m_0 c^2$ is the total energy of the particle.

  - The invariant magnitude of four-momentum is $p^\mu p_\mu = m_0^2 c^2$, related to the rest energy.

- **Dot Products of Four-Vectors:**

  - Inner products between four-vectors are invariant under Lorentz transformations.
  - Example: For two four-momentum vectors p^μ and q^μ, the dot product:

$$p^\mu q_\mu = \frac{E_p E_q}{c^2} - \vec{p} \cdot \vec{q}$$

    is invariant and depends on the relative energies and momenta in 3D space.

- **Spacetime Diagrams:** Four-vectors can be visualized in spacetime diagrams:

  - Time-like vectors point mostly along the time axis.
  - Space-like vectors point mostly along spatial axes.
  - Light-like vectors lie on the cone separating time-like and space-like regions.

Relativistic Conservation Laws: In an isolated system of particles:

- The relativistic total energy (kinetic energy plus rest energy) remains constant.
- The total linear momentum remains constant.

1.2 Essential Equations

Time Dilation: $\Delta t = \gamma \Delta t_0$, where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

Length Contraction: $L = L_0 \gamma^{-1} = L_0 \sqrt{1 - v^2/c^2}$.

Relativistic Addition of Velocities: $u' = \dfrac{u+v}{1+uv/c^2}$.

Lorentz Transformations:

- **Time:** $t' = \gamma(t - \frac{vx}{c^2})$.
- **Space:** $x' = \gamma(x - vt)$.

Relativistic Dynamics:

- **Kinetic Energy:** $KE = (\gamma - 1)m_0 c^2$.
- **Total Energy:** $E = \gamma m_0 c^2$.

Energy-Momentum Relation: $E^2 = (pc)^2 + (m_0 c^2)^2$.

Four-Position:

- Defined as $x^\mu = (ct, x, y, z)$, where x^μ is the spacetime coordinate.
- Invariant interval: $s^2 = (ct)^2 - x^2 - y^2 - z^2$.

Four-Velocity:

- Defined as $u^\mu = \frac{dx^\mu}{d\tau}$, where τ is proper time.
- Invariant dot product: $u^\mu u_\mu = c^2$.

Four-Momentum:

- Defined as $p^\mu = m_0 u^\mu$, where m_0 is the rest mass.
- Invariant: $p^\mu p_\mu = -m_0^2 c^2$.

2 Early Quantum Theory and Quantum Mechanics

2.1 Key Concepts

- Blackbody radiation: Explained using Planck's quantum hypothesis $E = nhf$.
- Photoelectric effect: Energy of emitted electrons depends on frequency, not intensity.
- Wave-particle duality: Matter exhibits both particle and wave properties.
- Schrödinger equation: Governs quantum mechanical behavior of particles.

2.2 Essential Equations

- Planck's law: $E = hf$
- Photoelectric equation: $KE_{\text{max}} = hf - \phi$
- De Broglie wavelength: $\lambda = \frac{h}{p}$
- Schrödinger equation (time-independent):

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

- 3D Schrödinger equation (spherical coordinates):

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(r)\psi = E\psi$$

where
$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

- General solution to the wave function:

$$\psi(r,\theta,\phi) = R(r)Y(\theta,\phi)$$

where:

  - $R(r)$: Radial part of the wave function, satisfying a differential equation dependent on the potential $V(r)$.
  - $Y(\theta,\phi)$: Angular part, given by spherical harmonics:

$$Y_l^m(\theta,\phi) = (-1)^m\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(\cos\theta)e^{im\phi}$$

where P_l^m are associated Legendre polynomials, and l, m are quantum numbers.

3 Statistical Mechanics

3.1 Key Concepts

- Microstates and macrostates: Connection between microscopic and macroscopic descriptions.
- Boltzmann distribution: $P_i = \frac{e^{-E_i/k_BT}}{Z}$.
- Partition function: Summation of states to understand thermodynamic properties, $Z = \sum_i e^{-E_i/k_BT}$.
- Occupation numbers: Average number of particles in a state, $n_i = \frac{1}{e^{(E_i-\mu)/k_BT} \pm 1}$ (for fermions and bosons).
- Degeneracy: The number of microstates corresponding to a single energy level.
- Distinguishable vs. indistinguishable particles: Boltzmann distribution for distinguishable particles; Fermi-Dirac and Bose-Einstein distributions for indistinguishable particles.

3.2 Essential Equations

$$\text{Boltzmann distribution: } P_i = \frac{e^{-E_i/k_BT}}{Z}$$
$$\text{Partition function: } Z = \sum_i g_i e^{-E_i/k_BT}$$
$$\text{Fermi-Dirac distribution: } n_i = \frac{1}{e^{(E_i-\mu)/k_BT} + 1}$$
$$\text{Bose-Einstein distribution: } n_i = \frac{1}{e^{(E_i-\mu)/k_BT} - 1}$$
$$\text{Average energy: } \langle E \rangle = \sum_i P_i E_i = -\frac{\partial \ln Z}{\partial \beta}$$
$$\text{Probability of a state: } P(E) = \frac{g_i e^{-E/k_BT}}{Z}$$

3.3 Units and Unit Conversions

- Energy: 1 eV $= 1.602 \times 10^{-19}$ J.
- Temperature: Convert between Kelvin and energy using $k_B = 1.38 \times 10^{-23}$ J/K.
- Wavelength and frequency: $E = hf$, $\lambda = \frac{c}{f}$, with $hc = 1240$ eV $\cdot$ nm.

4 Quick Reference

4.1 Mathematical Tools

- Trigonometric Identities:

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$
$$\sin^2 x = \frac{1-\cos(2x)}{2}$$
$$\sin(2x) = 2\sin x \cos x$$

- Series:
  - Taylor Series Expansion: $\sum_{n=0}^{\infty} C_n (x-a)^n$ where $C_n = \frac{f^{(n)}(a)}{n!}$
  - Binomial Series Expansion: $(1 \pm x)^r = 1 \pm rx \pm ...$

- Integration Tips:
  - For even functions: $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$
  - For odd functions: $\int_{-a}^{a} f(x)dx = 0$
  - Gaussian integrals often appear in wave packets

- Complex Numbers Review
  - Complex conjugate: $z^* = a - bi$ for $z = a + bi$
  - Modulus: $|z| = \sqrt{a^2 + b^2}$
  - Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$
  - Argument: $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

- Second Order Differential Equations
  - If ODE has this form: $\frac{d^2 f}{dx^2} = k^2 f$ where k is a constant

    The solution is:

$$f(x) = Ae^{kx} + Be^{-kx}$$

4.2 Important Constants

- Stefan-Boltzmann constant: $\sigma = 5.67 \times 10^{-8}\ \mathrm{W/m^2} \cdot K^4$
- Planck's constant: $h = 6.626 \times 10^{-34}\,\mathrm{J \cdot s} = 4.136 \times 10^{-15}\ \mathrm{eV \cdot s}$
- Reduced Planck's constant: $\hbar = 1.055 \times 10^{-34}\,\mathrm{J \cdot s}$
- Electron mass: $m_e = 9.11 \times 10^{-31}\,\mathrm{kg}$
- Speed of light: $c = 3.0 \times 10^8\,\mathrm{m/s}$
- Bohr radius: $a_0 = 0.0529 \times 10^{-9}\,\mathrm{m}$
- Compton wavelength of the electron: $\frac{h}{m_e c} = 0.002426$ nm

4.3 Unit Conversions

- 1 eV $= 1.602 \times 10^{-19}$ J
- $hc = 1240$ eV $\cdot$ nm
- $\hbar c = \frac{1240}{2\pi}$ eV $\cdot$ nm