# Modern Physics Final Study Guide

PHYS-122

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# 1 Special Relativity

## 1.1 Postulates of Special Relativity

**Postulates:**

- The laws of physics are the same in all inertial frames.
- The speed of light is constant in a vacuum for all observers, regardless of their motion relative to the source.

## 1.2 Relativity of Simultaneity

**Relativity of Simultaneity:** Two events that appear simultaneous to one observer may not appear simultaneous to another observer moving relative to the first. This is a direct consequence of the constancy of the speed of light and the Lorentz transformations.

## 1.3 Lorentz Transformations

**Lorentz Transformations:** The Lorentz transformations relate spacetime coordinates of an event in two inertial reference frames moving at a constant velocity relative to each other. These transformations preserve the invariant interval ($\Delta s^2$), a cornerstone of special relativity.

**Transformations:**

- For a frame $S'$ moving with velocity $v$ along the $x$-axis relative to $S$:
  - Time: $t' = \gamma\left(t - \frac{vx}{c^2}\right)$
  - Space ($x$-direction): $x' = \gamma\left(x - vt\right)$
  - Space ($y$- and $z$-directions): $y' = y, \quad z' = z$
- The Lorentz factor is $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

**Lorentz Transformation Matrix:** The Lorentz transformation can be expressed using matrix multiplication:

$$x' = \Lambda x,$$

where $x = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$ and $x'$ are the spacetime coordinates in $S$ and $S'$, respectively. For a boost along the $x$-axis, the Lorentz transformation matrix $\Lambda$ is:

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\beta = v/c$.

**Key Properties:**

- The inverse transformation is obtained by negating $v$, flipping the sign of $\beta$:

$$\Lambda^{-1} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- The Lorentz transformations preserve the invariant interval:

$$\Delta s^2 = x \cdot x = x' \cdot x'.$$

**Applications:**

- Lorentz transformations explain phenomena like time dilation, length contraction, and the relativity of simultaneity.
- They form the basis for defining and transforming four-vectors, such as four-momentum and four-velocity.

## 1.4 The Relativistic Invariant Interval

**Invariant Interval:** The relativistic invariant interval is a spacetime quantity that remains unchanged under Lorentz transformations.

**Invariant Interval:**

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

**Classifications of the Invariant Interval:**

- **Time-like:** $\Delta s^2 > 0$, related to proper time:

$$\Delta s^2 = c^2 \Delta \tau^2$$

- **Space-like:** $\Delta s^2 < 0$, related to proper length:

$$-\Delta s^2 = L^2$$

- **Light-like:** $\Delta s^2 = 0$, events are connected by light:

$$c^2 \Delta t^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

## 1.5 Relativistic Velocity Addition

**Relativistic Velocity Addition:** When combining velocities in different inertial frames, the relativistic formula must be used. Typically, these problems involve 3 objects and the following reference frames and velocities are defined:

- $S$: The stationary reference frame.
- $S'$: The moving reference frame.
- $u$: The velocity of an object relative to $S$.
- $v$: The velocity of $S'$ relative to $S$.
- $u'$: The velocity of the object relative to $S'$.

**Relativistic Velocity Addition Formula:**

$$u' = \frac{u+v}{1+\frac{uv}{c^2}}$$

Here, $u'$ is the velocity of the object as measured in $S'$, given the velocity $u$ of the object relative to $S$ and the velocity $v$ of $S'$ relative to $S$.

## 1.6 Four-Vectors in Special Relativity

**Four-Vectors:** Physical quantities in relativity are represented as four-vectors that transform under Lorentz transformations.

**General Form of a Four-Vector:**

$$a^\mu = (a^0, \vec{a}) = (a^0, a^1, a^2, a^3), \quad a^0 = c\Delta t$$

**Dot Product of Four-Vectors:** The invariant dot product between two four-vectors $a^\mu$ and $b^\mu$ is:

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}$$

This quantity is Lorentz invariant (unchanged under Lorentz transformations).

## 1.7 Examples of Four-Vectors

**Common Four-Vectors:**

- **Four-Position:**

$$x^\mu = (ct, x, y, z)$$

- **Four-Velocity:**

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \vec{v}), \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

- **Four-Momentum:**

$$p^\mu = mu^\mu = \gamma m(c, \vec{v})$$

For massless particles, like photons:

$$p \cdot p = 0, \quad E = |\vec{p}|c$$

## 1.8 Relativistic Conservation Laws

**Relativistic Conservation Laws:** The principles of conservation of energy and momentum extend to special relativity, incorporating the total energy (rest energy + kinetic energy) and relativistic momentum.

**Relativistic Total Energy:**

- The total energy $E$ includes both the rest energy and the relativistic kinetic energy:

$$E = \gamma m_0 c^2.$$

- The rest energy is given by $E_0 = m_0 c^2$, where $m_0$ is the rest mass.
- In any physical process, the total energy is conserved.

**Relativistic Momentum:**

- The relativistic momentum is defined as:

$$\vec{p} = \gamma m_0 \vec{v},$$

  where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

- Both the magnitude and direction of total momentum in an isolated system are conserved.

**Four-Momentum Conservation:**

- Energy and momentum form components of the four-momentum vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right).$$

- In all inertial frames, the conservation of four-momentum holds:

$$\sum p^\mu_{\text{initial}} = \sum p^\mu_{\text{final}}.$$

**Relativistic Collisions:**

- Total energy and total momentum are conserved in collisions, including elastic and inelastic cases.

- Unlike classical mechanics, the concept of conserved "mass" is not generally applicable. For example:

  - In pair annihilation, rest mass converts to energy.
  - In particle creation, energy converts into rest mass.

**Energy-Momentum Relation:**

$$E^2 = (pc)^2 + (m_0c^2)^2$$

This relation connects the total energy $E$, the momentum $p$, and the rest mass $m_0$ of a particle.

**Invariant Four-Momentum Magnitude:**

$$p^\mu \cdot p^\mu = m_0^2c^2$$

This invariant magnitude holds in all inertial frames, ensuring the consistency of relativistic conservation laws.

# 2 Quantum Mechanics

## 2.1 Quantum Theory of Light

### 2.1.1 Blackbody Radiation and Planck Hypothesis

**Blackbody Radiation:** Classical physics predicted the "ultraviolet catastrophe," where energy radiated at high frequencies diverged. Planck resolved this by quantizing energy:

$$E = nhf, \quad n = 1, 2, 3, \ldots$$

**Planck's Law:**

$$I(\nu, T) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_B T} - 1},$$

where $I(\nu, T)$ is the intensity of radiation at frequency $\nu$.

### 2.1.2 Photoelectric Effect

**Photoelectric Effect:** Electrons are emitted from a material when light shines on it, with the energy of the electrons depending on the frequency of light.

$$hf = \phi + KE_{max}, \quad hf_c = \phi, \quad KE_{max} = eV_s$$

**Threshold Frequency:** No electrons are emitted if $f < f_c$.

### 2.1.3 Compton Scattering

**Compton Scattering:** The scattering of X-rays by electrons demonstrates particle-like behavior of light. The wavelength shift is given by:

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta).$$

**Energy Relation:**

$$\frac{1}{E'} - \frac{1}{E} = \frac{1 - \cos\theta}{m_e c^2}.$$

### 2.1.4 Wave-Particle Duality

**Wave-Particle Duality:** Light and matter exhibit dual behavior, displaying characteristics of both waves and particles depending on the experimental conditions.

**Key Features:**

- Travels as a wave, interacting with itself, and demonstrates phenomena such as interference and diffraction.
- Interacts as a particle, transferring discrete packets of energy, known as photons or quanta.
- Confirmed by experiments:
  - **Double-Slit Experiment:** Demonstrates interference patterns for light and electrons, showing wave-like behavior.
  - **Photoelectric Effect:** Demonstrates the particle nature of light, as photons eject electrons from a material.
  - **Blackbody Radiation:** Explained by Planck's quantization hypothesis, resolving the ultraviolet catastrophe.
  - **Compton Scattering:** Demonstrates photon momentum through scattering with electrons.

**De Broglie Wavelength:** $\lambda = \frac{h}{p} = \frac{h}{mv}$

**Wave Number:** $k = \frac{2\pi}{\lambda}$

### 2.1.5 Heisenberg Uncertainty Principle

**Heisenberg Uncertainty Principle:** A fundamental principle of quantum mechanics stating that certain pairs of physical observables cannot be simultaneously measured with arbitrary precision.

**Key Relations:**

- **Position-Momentum Uncertainty:**

$$\Delta x \Delta p \geq \frac{\hbar}{2},$$

where $\Delta x$ and $\Delta p$ are the standard deviations of position and momentum, respectively.

- **Energy-Time Uncertainty:**

$$\Delta E \Delta t \geq \frac{\hbar}{2},$$

where $\Delta E$ and $\Delta t$ are the uncertainties in energy and time, respectively.

**Variance-Squared Average Relationship:** The uncertainty in a measurable quantity $A$ is defined as:

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2},$$

where $\langle A \rangle$ is the expectation value, and $\langle A^2 \rangle$ is the expectation value of $A^2$.

**Implications:**

- Establishes the probabilistic nature of quantum mechanics, replacing deterministic classical physics.
- Demonstrates the impossibility of assigning definite trajectories to particles as in classical mechanics.

### 2.1.6 Group and Phase Velocities

**Group and Phase Velocities:** Velocities associated with wave propagation.

- Group velocity:

$$v_g = \frac{d\omega}{dk},$$

representing the velocity of the wave packet **and the velocity of the particle**

- Phase velocity:

$$v_p = \frac{\omega}{k},$$

representing the velocity of individual wave crests.

## 2.2 Schrödinger Equation

### 2.2.1 Overview

**Schrödinger Equation:** Governs the quantum mechanical behavior of particles. It describes how the wave function of a particle evolves under the influence of potential energy.

**General Forms:**

- **1D Time-Independent Schrödinger Equation:**

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x).$$

- **3D Schrödinger Equation (spherical coordinates):**

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r,\theta,\phi) + V(r)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi).$$

  The solution is separable:

$$\psi(r,\theta,\phi) = R(r)Y(\theta,\phi),$$

  where $Y(\theta,\phi)$ are spherical harmonics.

### 2.2.2 Different Potentials

**Free Particle:**

- For $V(x) = 0$, solutions are plane waves:

$$\psi(x) = Ae^{i(kx-\omega t)},$$

  where $k = \frac{p}{\hbar}$ and $\omega = \frac{E}{\hbar}$.

**Particle in a Box:**

- For $V(x) = 0$ inside the box and $V(x) = \infty$ outside:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}.$$

**Step Potential:**

- $E < U$: Tunneling occurs. The wave function decays exponentially in the classically forbidden region.
- $E > U$: The wave function is continuous, with partial reflection and transmission.

## 2.3 Hydrogen Atom

### 2.3.1 Quantized Energy Levels

**Hydrogen Atom:** The Schrödinger equation in spherical coordinates yields quantized energy levels and quantum numbers.
**Energy Levels:**

$$E_n = -\frac{13.6\,\mathrm{eV}}{n^2}.$$

### 2.3.2 Quantum Numbers

**Quantum Numbers:**

- **Principal quantum number ($n$):** Determines energy level ($n = 1, 2, 3, \ldots$).
- **Orbital angular momentum quantum number ($l$):** Determines the shape of the orbital ($l = 0, 1, \ldots, n-1$).
- **Magnetic quantum number ($m_l$):** Determines the orientation ($m_l = -l, -l+1, \ldots, l$).
- **Spin quantum number ($m_s$):** Intrinsic angular momentum ($m_s = \pm\frac{1}{2}$).

#### 2.3.3 Wave Functions

**Wave Functions:**

$$\psi(r,\theta,\phi) = R_{nl}(r)Y_l^m(\theta,\phi),$$

where $R_{nl}(r)$ is the radial wave function, and $Y_l^m(\theta,\phi)$ are spherical harmonics.
**Radial Probability Distribution:** The probability density $P(r)$ is given by:

$$P(r) = r^2|R_{nl}(r)|^2.$$

# 3 Statistical Mechanics

## 3.1 Key Concepts

- Microstates and macrostates: Connection between microscopic and macroscopic descriptions.
- Boltzmann distribution: $P_i = \frac{e^{-E_i/k_BT}}{Z}$.
- Partition function: Summation of states to understand thermodynamic properties, $Z = \sum_i e^{-E_i/k_BT}$.
- Occupation numbers: Average number of particles in a state, $n_i = \frac{1}{e^{(E_i-\mu)/k_BT} \pm 1}$ (for fermions and bosons).
- Degeneracy: The number of microstates corresponding to a single energy level.
- Distinguishable vs. indistinguishable particles: Boltzmann distribution for distinguishable particles; Fermi-Dirac and Bose-Einstein distributions for indistinguishable particles.

### 3.2 Essential Equations

$$\text{Boltzmann distribution: } P_i = \frac{e^{-E_i/k_B T}}{Z}$$
$$\text{Partition function: } Z = \sum_i g_i e^{-E_i/k_B T}$$
$$\text{Fermi-Dirac distribution: } n_i = \frac{1}{e^{(E_i-\mu)/k_B T} + 1}$$
$$\text{Bose-Einstein distribution: } n_i = \frac{1}{e^{(E_i-\mu)/k_B T} - 1}$$
$$\text{Average energy: } \langle E \rangle = \sum_i P_i E_i = -\frac{\partial \ln Z}{\partial \beta}$$
$$\text{Probability of a state: } P(E) = \frac{g_i e^{-E/k_B T}}{Z}$$

### 3.3 Units and Unit Conversions

- Energy: 1 eV $= 1.602 \times 10^{-19}$ J.
- Temperature: Convert between Kelvin and energy using $k_B = 1.38 \times 10^{-23}$ J/K.
- Wavelength and frequency: $E = hf$, $\lambda = \frac{c}{f}$, with $hc = 1240$ eV $\cdot$ nm.

# 4 Quick Reference

### 4.1 Mathematical Tools

- Trigonometric Identities:

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$
$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$
$$\sin(2x) = 2 \sin x \cos x$$

- Series:
  - Taylor Series Expansion: $\sum_{n=0}^{\infty} C_n(x-a)^n$ where $C_n = \frac{f^{(n)}(a)}{n!}$
  - Binomial Series Expansion: $(1 \pm x)^r = 1 \pm rx \pm ...$

- Integration Tips:
  - For even functions: $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$
  - For odd functions: $\int_{-a}^{a} f(x)dx = 0$
  - Gaussian integrals often appear in wave packets

- Complex Numbers Review
  - Complex conjugate: $z^* = a - bi$ for $z = a + bi$
  - Modulus: $|z| = \sqrt{a^2 + b^2}$
  - Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$
  - Argument: $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

- Second Order Differential Equations
  - If ODE has this form: $\frac{d^2 f}{dx^2} = k^2 f$ where k is a constant

    The solution is:

$$f(x) = Ae^{kx} + Be^{-kx}$$

## 4.2 Important Constants

- Stefan-Boltzmann constant: $\sigma = 5.67 \times 10^{-8}\ \mathrm{W/m}^2 \cdot K^4$
- Planck's constant: $h = 6.626 \times 10^{-34}\,\mathrm{J \cdot s} = 4.136 \times 10^{-15}\ \mathrm{eV \cdot s}$
- Reduced Planck's constant: $\hbar = 1.055 \times 10^{-34}\,\mathrm{J \cdot s}$
- Electron mass: $m_e = 9.11 \times 10^{-31}\,\mathrm{kg}$
- Speed of light: $c = 3.0 \times 10^8\,\mathrm{m/s}$
- Bohr radius: $a_0 = 0.0529 \times 10^{-9}\,\mathrm{m}$
- Compton wavelength of the electron: $\dfrac{h}{m_e c} = 0.002426\ \mathrm{nm}$

## 4.3 Unit Conversions

- $1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$
- $hc = 1240\ \mathrm{eV \cdot nm}$
- $\hbar c = \dfrac{1240}{2\pi}\ \mathrm{eV \cdot nm}$