Modern Physics Final Study Guide

PHYS-122

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Contents

1 Special Relativity

1.1 Key Concepts

Postulates:

- Laws of physics are the same in all inertial frames.
- The speed of light is constant in a vacuum for all observers, regardless of their motion relative to the source.

Relativity of Simultaneity: Two events that appear simultaneous to one observer may not appear simultaneous to another observer moving relative to the first.

Lorentz Transformations: The Lorentz transformations relate the spacetime coordinates of an event as measured in two inertial reference frames moving at a constant velocity relative to each other. These transformations preserve the invariant interval (Δs^2).

Transformations:

- For a frame S' moving with velocity v along the x-axis relative to S:
  - Time: $t' = \gamma \left(t - \frac{vx}{c^2}\right)$
  - Space (x-direction): $x' = \gamma (x - vt)$
  - Space (y- and z-directions): $y' = y, \quad z' = z$
- $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ is the Lorentz factor.

Lorentz Transformation Matrix: The Lorentz transformation can be expressed using matrix multiplication:

$$x' = \Lambda x,$$

where Λ is the Lorentz transformation matrix. For a boost along the x-axis, Λ is:

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with $\beta = v/c$.

Key Properties:

- The determinant of Λ is $+1$, ensuring the transformations preserve spacetime orientation.
- The inverse transformation is obtained by negating v, which flips the sign of β:

$$\Lambda^{-1} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- The Lorentz transformations preserve the invariant interval:

$$\Delta s^2 = x \cdot x = x' \cdot x'.$$

Applications:

- Lorentz transformations explain time dilation, length contraction, and the relativity of simultaneity.
- They are fundamental in deriving the transformations for four-vectors such as four-momentum and four-velocity.

Relativistic Velocity Addition: These problems typically involve three objects:

- S: Stationary reference frame.
- S': Reference frame of the second moving object.
- u: Velocity of the first object relative to the stationary frame.
- v: Velocity of the second object relative to the stationary frame.
- u': Velocity of the first object relative to the second moving object.

The Relativistic Invariant Interval: A spacetime quantity that remains unchanged under Lorentz transformations.

- **Properties:**
  - Negative spatial components distinguish it from a Euclidean interval.
  - It can be positive, negative, or zero:
    - **Time-like ($\Delta s^2 > 0$):** Events can be connected by a physical particle moving slower than the speed of light.
    - **Space-like ($\Delta s^2 < 0$):** Events are separated by more distance than time; no signal can connect them.
    - **Light-like ($\Delta s^2 = 0$):** Events are connected by a light signal.
- **Relationship to Proper Time and Proper Length:**
  - **Proper Time ($\Delta\tau$):** In the frame where the particle is stationary, the interval relates to proper time.
  - **Proper Length (ΔL):** In the frame where events are simultaneous, the interval relates to proper length.

Four-Vectors: A four-vector in special relativity is a quantity defined in four-dimensional spacetime that transforms under Lorentz transformations. It has the general form:

$$a = (a^0, \vec{a}),$$

where a^0 is the time component ($c\Delta t$), and $\vec{a} = (a^1, a^2, a^3)$ is the spatial 3-vector component.
Key Properties:

- The **invariant dot product** of two four-vectors a and b is:

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}.$$

  This quantity is Lorentz invariant (unchanged under Lorentz transformations).

- The invariant magnitude of a four-vector is a special case of the dot product:

$$a \cdot a = (a^0)^2 - |\vec{a}|^2.$$

- The magnitude classifies four-vectors as:
  - **Time-like:** $a \cdot a > 0$ then $\Delta s^2 = c^2\tau^2$.
  - **Space-like:** $a \cdot a < 0$ then $-\Delta s^2 = L$.
  - **Light-like:** $a \cdot a = 0$ then $\Delta s^2 = 0$.

Common Examples:

- **Four-Position:** $x = (ct, \vec{x})$, where ct represents the time component and $\vec{x} = (x, y, z)$ is the spatial position.
- **Four-Velocity:** $u = \dfrac{dx}{d\tau} = \gamma(c, \vec{v})$, where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$, c is the speed of light, and $\vec{v}$ is the 3-velocity.
- **Four-Momentum:** $p = mu = \gamma m(c, \vec{v}) = \left(\frac{E}{c}, \vec{p}\right)$, where E is the total energy and $\vec{p}$ is the 3-momentum.

Applications of the Invariant Dot Product:

- **Four-Velocity:** The invariant magnitude of u is always positive and equal to c^2:

$$u \cdot u = c^2.$$

- **Four-Momentum:** The invariant magnitude of p relates to the particle's rest mass m:

$$p \cdot p = m^2 c^2.$$

  For massless particles, like photons, $p \cdot p = 0$ and $E = |\vec{p}|c$.

Relativistic Conservation Laws: The principles of conservation of energy and momentum extend to special relativity with some key differences from classical mechanics.

- **Relativistic Total Energy:**
  - Total energy includes kinetic energy and rest energy: $E = \gamma m_0 c^2$, where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.
  - Rest energy ($m_0 c^2$) accounts for the energy associated with the rest mass of a particle.
  - Total energy is conserved in all relativistic processes, including collisions.
- **Relativistic Momentum:**
  - Momentum in special relativity is defined as $\vec{p} = \gamma m_0 \vec{v}$.
  - The magnitude and direction of the total momentum in an isolated system are conserved.
- **Relativistic Collisions:**
  - Unlike classical mechanics, all forms of energy (including rest energy) contribute to the total energy of the system.
  - The distinction between elastic and inelastic collisions in classical mechanics is less useful in relativity. Instead, we track the conservation of total energy and momentum.
  - Mass is not generally conserved in relativistic collisions. For example, mass can be converted into energy (e.g., pair annihilation) or vice versa (e.g., particle creation).
- **Implications:**
  - Energy and momentum form components of the four-momentum vector: $p^\mu = \left(\frac{E}{c}, \vec{p}\right)$.
  - Conservation laws apply to the four-momentum vector in all inertial frames:

$$\sum p^\mu_{\text{initial}} = \sum p^\mu_{\text{final}}$$

  - These conservation laws ensure the invariance of physical processes across reference frames.

1.2 Essential Equations

Time Dilation: $\Delta t = \gamma \Delta t_0$, where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

Length Contraction: $L = L_0 \gamma^{-1} = L_0 \sqrt{1 - v^2/c^2}$.

Relativistic Addition of Velocities: $u' = \dfrac{u + v}{1 + uv/c^2}$.

Lorentz Transformations:

- **Time:** $t' = \gamma(t - \frac{vx}{c^2})$.
- **Space:** $x' = \gamma(x - vt)$.

Relativistic Dynamics:

- **Kinetic Energy:** $KE = (\gamma - 1)m_0 c^2$.
- **Total Energy:** $E = \gamma m_0 c^2$.
- **Rest Mass:** $E = m_0 c^2$

Energy-Momentum Relation: $E^2 = (pc)^2 + (m_0 c^2)^2$.

Four-Position:

- Defined as $x = (c\Delta t, x, y, z)$, where x is the spacetime coordinate.
- Invariant interval: $\Delta s^2 = (c\Delta t)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$.

Four-Velocity:

- Defined as $u^\mu = \frac{dx^\mu}{d\tau}$, where τ is proper time.
- Invariant dot product: $u^\mu u_\mu = c^2$.

Four-Momentum:

- Defined as $p = m_0 u$, where m_0 is the rest mass.
- Invariant: $p \cdot p = m_0^2 c^2$.

2 Early Quantum Theory and Quantum Mechanics

Quantum Theory of Light:

- **Blackbody Problem and Ultraviolet Catastrophe:** Classical physics predicted infinite energy at short wavelengths (ultraviolet catastrophe). Planck resolved this by introducing quantized energy levels $E = nhf$, leading to Planck's radiation law.
- **Photoelectric Effect:** Light behaves as particles (photons). The energy of emitted electrons depends on the frequency, not intensity, of the incident light:
$$E = hf - \phi,$$
where ϕ is the work function.
- **Compton Scattering:** Photons scatter off electrons, resulting in a wavelength shift:
$$\Delta\lambda = \frac{h}{m_e c}(1 - \cos\theta),$$
confirming the particle nature of light.
- **Wave-Particle Duality:** Light and matter exhibit both wave and particle properties. De Broglie hypothesized the wavelength of matter waves:
$$\lambda = \frac{h}{p}.$$
- **Emission and Absorption Spectra:** Atoms absorb/emits photons corresponding to energy level transitions:
$$\Delta E = hf.$$
- **Bohr Model:** Electrons occupy quantized orbits, with energy levels:
$$E_n = -\frac{13.6}{n^2} \text{ eV for hydrogen.}$$
- **Franck-Hertz Experiment:** Confirmed quantized energy levels by measuring discrete energy loss during electron collisions with atoms.
- **Heisenberg Uncertainty Principle:** Fundamental limits on measurement precision:
$$\Delta x \Delta p \geq \frac{\hbar}{2}, \quad \Delta E \Delta t \geq \frac{\hbar}{2}.$$
- **Group and Phase Velocities:**
$$v_g = \frac{d\omega}{dk}, \quad v_p = \frac{\omega}{k}.$$
Group velocity corresponds to the velocity of the particle, while phase velocity is associated with the wave motion.

Schrödinger Equation: Governs the quantum mechanical behavior of particles.

- **1D Time-Independent Schrödinger Equation:**

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x).$$

- **3D Schrödinger Equation (spherical coordinates):**

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r,\theta,\phi) + V(r)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi).$$

  The solution is separable:

$$\psi(r,\theta,\phi) = R(r)Y(\theta,\phi),$$

  where $Y(\theta,\phi)$ are spherical harmonics.

- **Free Particle:** For $V(x) = 0$, solutions are plane waves:

$$\psi(x) = Ae^{i(kx-\omega t)}.$$

- **Particle in a Box:** For $V(x) = 0$ inside the box and $V(x) = \infty$ outside:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right), \quad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}.$$

- **Step Potential:**
  - $E < U$: Tunneling occurs. The wave function decays exponentially in the classically forbidden region.
  - $E > U$: The wave function is continuous, with partial reflection and transmission.

Hydrogen Atom: Solutions to the Schrödinger equation in spherical coordinates yield quantized energy levels and quantum numbers.

- **Energy Levels:**
$$E_n = -\frac{13.6\,\text{eV}}{n^2}.$$
- **Quantum Numbers:**
  - Principal quantum number n: Determines energy level ($n = 1, 2, 3, \ldots$).
  - Orbital angular momentum quantum number l: Determines the shape of the orbital ($l = 0, 1, \ldots, n-1$).
  - Magnetic quantum number m_l: Determines the orientation ($m_l = -l, -l+1, \ldots, l$).
  - Spin quantum number m_s: Intrinsic angular momentum ($m_s = \pm\frac{1}{2}$).
- **Wave Functions:**
$$\psi(r, \theta, \phi) = R_{nl}(r) Y_l^m(\theta, \phi),$$
where $R_{nl}(r)$ is the radial wave function, and $Y_l^m(\theta, \phi)$ are spherical harmonics.
- **Radial Probability Distribution:** The probability density $P(r)$ is given by:
$$P(r) = r^2 |R_{nl}(r)|^2.$$

2.1 Essential Equations

Planck's Hypothesis: $E = nhf$.

Photoelectric Effect: $KE_{\max} = hf - \phi$.

Compton Scattering: $\Delta\lambda = \frac{h}{m_e c}(1 - \cos\theta)$.

De Broglie Wavelength: $\lambda = \frac{h}{p}$.

Heisenberg Uncertainty Principles: $\Delta x \Delta p \geq \frac{\hbar}{2}, \quad \Delta E \Delta t \geq \frac{\hbar}{2}$.

Particle in a Box: $E_n = \dfrac{n^2\pi^2\hbar^2}{2mL^2}, \quad \psi_n(x) = \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{n\pi x}{L}\right).$

Radial Probability Distribution: $P(r) = r^2|R_{nl}(r)|^2$.

3 Statistical Mechanics

3.1 Key Concepts

- Microstates and macrostates: Connection between microscopic and macroscopic descriptions.
- Boltzmann distribution: $P_i = \frac{e^{-E_i/k_BT}}{Z}$.
- Partition function: Summation of states to understand thermodynamic properties, $Z = \sum_i e^{-E_i/k_BT}$.
- Occupation numbers: Average number of particles in a state, $n_i = \frac{1}{e^{(E_i-\mu)/k_BT}\pm 1}$ (for fermions and bosons).
- Degeneracy: The number of microstates corresponding to a single energy level.
- Distinguishable vs. indistinguishable particles: Boltzmann distribution for distinguishable particles; Fermi-Dirac and Bose-Einstein distributions for indistinguishable particles.

3.2 Essential Equations

$$\text{Boltzmann distribution: } P_i = \frac{e^{-E_i/k_B T}}{Z}$$
$$\text{Partition function: } Z = \sum_i g_i e^{-E_i/k_B T}$$
$$\text{Fermi-Dirac distribution: } n_i = \frac{1}{e^{(E_i-\mu)/k_B T} + 1}$$
$$\text{Bose-Einstein distribution: } n_i = \frac{1}{e^{(E_i-\mu)/k_B T} - 1}$$
$$\text{Average energy: } \langle E \rangle = \sum_i P_i E_i = -\frac{\partial \ln Z}{\partial \beta}$$
$$\text{Probability of a state: } P(E) = \frac{g_i e^{-E/k_B T}}{Z}$$

3.3 Units and Unit Conversions

- Energy: 1 eV $= 1.602 \times 10^{-19}$ J.
- Temperature: Convert between Kelvin and energy using $k_B = 1.38 \times 10^{-23}$ J/K.
- Wavelength and frequency: $E = hf$, $\lambda = \frac{c}{f}$, with $hc = 1240$ eV $\cdot$ nm.

4 Quick Reference

4.1 Mathematical Tools

- Trigonometric Identities:

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$
$$\sin^2 x = \frac{1-\cos(2x)}{2}$$
$$\sin(2x) = 2\sin x \cos x$$

- Series:
  - Taylor Series Expansion: $\sum_{n=0}^{\infty} C_n(x-a)^n$ where $C_n = \frac{f^{(n)}(a)}{n!}$
  - Binomial Series Expansion: $(1 \pm x)^r = 1 \pm rx \pm ...$

- Integration Tips:
  - For even functions: $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$
  - For odd functions: $\int_{-a}^{a} f(x)dx = 0$
  - Gaussian integrals often appear in wave packets

- Complex Numbers Review
  - Complex conjugate: $z^* = a - bi$ for $z = a + bi$
  - Modulus: $|z| = \sqrt{a^2 + b^2}$
  - Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$
  - Argument: $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

- Second Order Differential Equations
  - If ODE has this form: $\frac{d^2 f}{dx^2} = k^2 f$ where k is a constant

    The solution is:

    $$f(x) = Ae^{kx} + Be^{-kx}$$

4.2 Important Constants

- Stefan-Boltzmann constant: $\sigma = 5.67 \times 10^{-8}$ W/m$^2 \cdot K^4$
- Planck's constant: $h = 6.626 \times 10^{-34}\,\mathrm{J \cdot s} = 4.136 \times 10^{-15}$ eV $\cdot$ s
- Reduced Planck's constant: $\hbar = 1.055 \times 10^{-34}\,\mathrm{J \cdot s}$
- Electron mass: $m_e = 9.11 \times 10^{-31}\,\mathrm{kg}$
- Speed of light: $c = 3.0 \times 10^{8}\,\mathrm{m/s}$
- Bohr radius: $a_0 = 0.0529 \times 10^{-9}\,\mathrm{m}$
- Compton wavelength of the electron: $\dfrac{h}{m_e c} = 0.002426$ nm

4.3 Unit Conversions

- 1 eV $= 1.602 \times 10^{-19}$ J
- $hc = 1240$ eV $\cdot$ nm
- $\hbar c = \dfrac{1240}{2\pi}$ eV $\cdot$ nm