Modern Physics Final Study Guide

PHYS-122

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Contents

1 Special Relativity

1.1 Postulates of Special Relativity

Postulates:

- The laws of physics are the same in all inertial frames.
- The speed of light is constant in a vacuum for all observers, regardless of their motion relative to the source.

1.2 Relativity of Simultaneity

Relativity of Simultaneity: Two events that appear simultaneous to one observer may not appear simultaneous to another observer moving relative to the first. This is a direct consequence of the constancy of the speed of light and the Lorentz transformations.

1.3 Lorentz Transformations

Lorentz Transformations: The Lorentz transformations relate spacetime coordinates of an event in two inertial reference frames moving at a constant velocity relative to each other. These transformations preserve the invariant interval (Δs^2), a cornerstone of special relativity.

Transformations:

- For a frame S' moving with velocity v along the x-axis relative to S:
  - Time: $t' = \gamma\left(t - \frac{vx}{c^2}\right)$
  - Space (x-direction): $x' = \gamma\,(x - vt)$
  - Space (y- and z-directions): $y' = y, \quad z' = z$
- The Lorentz factor is $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

Lorentz Transformation Matrix: The Lorentz transformation can be expressed using matrix multiplication:

$$x' = \Lambda x,$$

where $x = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$ and x' are the spacetime coordinates in S and S', respectively. For a boost along the x-axis, the Lorentz transformation matrix Λ is:

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\beta = v/c$.

Key Properties:

- The inverse transformation is obtained by negating v, flipping the sign of β:

$$\Lambda^{-1} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- The Lorentz transformations preserve the invariant interval:

$$\Delta s^2 = x \cdot x = x' \cdot x'.$$

Applications:

- Lorentz transformations explain phenomena like time dilation, length contraction, and the relativity of simultaneity.
- They form the basis for defining and transforming four-vectors, such as four-momentum and four-velocity.

1.4 The Relativistic Invariant Interval

Invariant Interval: The relativistic invariant interval is a spacetime quantity that remains unchanged under Lorentz transformations.

Invariant Interval:

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

Classifications of the Invariant Interval:

- **Time-like:** $\Delta s^2 > 0$, related to proper time:

$$\Delta s^2 = c^2 \Delta \tau^2$$

- **Space-like:** $\Delta s^2 < 0$, related to proper length:

$$-\Delta s^2 = L^2$$

- **Light-like:** $\Delta s^2 = 0$, events are connected by light:

$$c^2 \Delta t^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

1.5 Relativistic Velocity Addition

Relativistic Velocity Addition: When combining velocities in different inertial frames, the relativistic formula must be used. Typically, these problems involve 3 objects and the following reference frames and velocities are defined:

- S: The stationary reference frame.
- S': The moving reference frame.
- u: The velocity of an object relative to S.
- v: The velocity of S' relative to S.
- u': The velocity of the object relative to S'.

Relativistic Velocity Addition Formula:

$$u' = \frac{u + v}{1 + \frac{uv}{c^2}}$$

Here, u' is the velocity of the object as measured in S', given the velocity u of the object relative to S and the velocity v of S' relative to S.

1.6 Four-Vectors in Special Relativity

Four-Vectors: Physical quantities in relativity are represented as four-vectors that transform under Lorentz transformations.

General Form of a Four-Vector:

$$a^\mu = (a^0, \vec{a}) = (a^0, a^1, a^2, a^3), \quad a^0 = c\Delta t$$

Dot Product of Four-Vectors: The invariant dot product between two four-vectors a^μ and b^μ is:

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}$$

This quantity is Lorentz invariant (unchanged under Lorentz transformations).

1.7 Examples of Four-Vectors

Common Four-Vectors:

- **Four-Position:**

$$x^\mu = (ct, x, y, z)$$

- **Four-Velocity:**

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \vec{v}), \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

- **Four-Momentum:**

$$p^\mu = mu^\mu = \gamma m(c, \vec{v})$$

For massless particles, like photons:

$$p \cdot p = 0, \quad E = |\vec{p}|c$$

1.8 Relativistic Conservation Laws

Relativistic Conservation Laws: The principles of conservation of energy and momentum extend to special relativity, incorporating the total energy (rest energy + kinetic energy) and relativistic momentum.

Relativistic Total Energy:

- The total energy E includes both the rest energy and the relativistic kinetic energy:

$$E = \gamma m_0 c^2.$$

- The rest energy is given by $E_0 = m_0 c^2$, where m_0 is the rest mass.
- In any physical process, the total energy is conserved.

Relativistic Momentum:

- The relativistic momentum is defined as:

$$\vec{p} = \gamma m_0 \vec{v},$$

  where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$.

- Both the magnitude and direction of total momentum in an isolated system are conserved.

Four-Momentum Conservation:

- Energy and momentum form components of the four-momentum vector:

$$p^\mu = \left(\frac{E}{c}, \vec{p}\right).$$

- In all inertial frames, the conservation of four-momentum holds:

$$\sum p^\mu_{\text{initial}} = \sum p^\mu_{\text{final}}.$$

Relativistic Collisions:

- Total energy and total momentum are conserved in collisions, including elastic and inelastic cases.

- Unlike classical mechanics, the concept of conserved "mass" is not generally applicable. For example:

  - In pair annihilation, rest mass converts to energy.
  - In particle creation, energy converts into rest mass.

Energy-Momentum Relation:

$$E^2 = (pc)^2 + (m_0 c^2)^2$$

This relation connects the total energy E, the momentum p, and the rest mass m_0 of a particle.

Invariant Four-Momentum Magnitude:

$$p^\mu \cdot p^\mu = m_0^2 c^2$$

This invariant magnitude holds in all inertial frames, ensuring the consistency of relativistic conservation laws.

2 Quantum Mechanics

2.1 Quantum Theory of Light

2.1.1 Blackbody Radiation and Planck Hypothesis

Blackbody Radiation: Classical physics predicted the "ultraviolet catastrophe," where energy radiated at high frequencies diverged. Planck resolved this by quantizing energy:

$$E = nhf, \quad n = 1, 2, 3, \dots$$

Planck's Law:

$$I(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1},$$

where $I(\nu, T)$ is the intensity of radiation at frequency ν.

2.1.2 Photoelectric Effect

Photoelectric Effect: Electrons are emitted from a material when light shines on it, with the energy of the electrons depending on the frequency of light.

$$hf = \phi + KE_{max}, \quad hf_c = \phi, \quad KE_{max} = eV_s$$

Threshold Frequency: No electrons are emitted if $f < f_c$.

2.1.3 Compton Scattering

Compton Scattering: The scattering of X-rays by electrons demonstrates particle-like behavior of light. The wavelength shift is given by:

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta).$$

Energy Relation:

$$\frac{1}{E'} - \frac{1}{E} = \frac{1 - \cos\theta}{m_e c^2}.$$

2.1.4 Wave-Particle Duality

Wave-Particle Duality: Light and matter exhibit dual behavior, displaying characteristics of both waves and particles depending on the experimental conditions.

Key Features:

- Travels as a wave, interacting with itself, and demonstrates phenomena such as interference and diffraction.
- Interacts as a particle, transferring discrete packets of energy, known as photons or quanta.
- Confirmed by experiments:
  - **Double-Slit Experiment:** Demonstrates interference patterns for light and electrons, showing wave-like behavior.
  - **Photoelectric Effect:** Demonstrates the particle nature of light, as photons eject electrons from a material.
  - **Blackbody Radiation:** Explained by Planck's quantization hypothesis, resolving the ultraviolet catastrophe.
  - **Compton Scattering:** Demonstrates photon momentum through scattering with electrons.

De Broglie Wavelength: $\lambda = \frac{h}{p} = \frac{h}{mv}$

Wave Number: $k = \frac{2\pi}{\lambda}$

2.1.5 Heisenberg Uncertainty Principle

Heisenberg Uncertainty Principle: A fundamental principle of quantum mechanics stating that certain pairs of physical observables cannot be simultaneously measured with arbitrary precision.

Key Relations:

- **Position-Momentum Uncertainty:**

$$\Delta x \Delta p \geq \frac{\hbar}{2},$$

where Δx and Δp are the standard deviations of position and momentum, respectively.

- **Energy-Time Uncertainty:**

$$\Delta E \Delta t \geq \frac{\hbar}{2},$$

where ΔE and Δt are the uncertainties in energy and time, respectively.

Variance-Squared Average Relationship: The uncertainty in a measurable quantity A is defined as:

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2},$$

where $\langle A \rangle$ is the expectation value, and $\langle A^2 \rangle$ is the expectation value of A^2.

Implications:

- Establishes the probabilistic nature of quantum mechanics, replacing deterministic classical physics.
- Demonstrates the impossibility of assigning definite trajectories to particles as in classical mechanics.

2.1.6 Group and Phase Velocities

Group and Phase Velocities: Velocities associated with wave propagation.

- Group velocity:

$$v_g = \frac{d\omega}{dk},$$

  representing the velocity of the wave packet **and the velocity of the particle**

- Phase velocity:

$$v_p = \frac{\omega}{k},$$

  representing the velocity of individual wave crests.

2.2 Schrödinger Equation

2.2.1 Overview

Schrödinger Equation: Governs the quantum mechanical behavior of particles. It describes how the wave function of a particle evolves under the influence of potential energy.

General Forms:

- **1D Time-Independent Schrödinger Equation:**

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x).$$

- **3D Schrödinger Equation (spherical coordinates):**

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r,\theta,\phi) + V(r)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi).$$

  The solution is separable:

$$\psi(r,\theta,\phi) = R(r)Y(\theta,\phi),$$

  where $Y(\theta,\phi)$ are spherical harmonics.

2.2.2 Different Potentials

Free Particle:

- For $V(x) = 0$, solutions are plane waves:

$$\psi(x) = Ae^{i(kx-\omega t)},$$

  where $k = \frac{p}{\hbar}$ and $\omega = \frac{E}{\hbar}$.

Particle in a Box:

- For $V(x) = 0$ inside the box and $V(x) = \infty$ outside:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}.$$

Step Potential:

- $E < U$: Tunneling occurs. The wave function decays exponentially in the classically forbidden region.
- $E > U$: The wave function is continuous, with partial reflection and transmission.

2.3 Hydrogen Atom

2.3.1 Quantized Energy Levels

Hydrogen Atom: The Schrödinger equation in spherical coordinates yields quantized energy levels and quantum numbers.
Energy Levels:

$$E_n = -\frac{13.6\,\mathrm{eV}}{n^2}.$$

2.3.2 Quantum Numbers

Quantum Numbers:

- **Principal quantum number (n):** Determines energy level ($n = 1, 2, 3, \ldots$).
- **Orbital angular momentum quantum number (l):** Determines the shape of the orbital ($l = 0, 1, \ldots, n-1$).
- **Magnetic quantum number (m_l):** Determines the orientation ($m_l = -l, -l+1, \ldots, l$).
- **Spin quantum number (m_s):** Intrinsic angular momentum ($m_s = \pm\frac{1}{2}$).

2.3.3 Wave Functions

Wave Functions:

$$\psi(r, \theta, \phi) = R_{nl}(r) Y_l^m(\theta, \phi),$$

where $R_{nl}(r)$ is the radial wave function, and $Y_l^m(\theta, \phi)$ are spherical harmonics.
Radial Probability Distribution: The probability density $P(r)$ is given by:

$$P(r) = r^2 |R_{nl}(r)|^2.$$

3 Statistical Mechanics

3.1 Key Concepts

- Microstates and macrostates: Connection between microscopic and macroscopic descriptions.
- Boltzmann distribution: $P_i = \frac{e^{-E_i/k_B T}}{Z}$.
- Partition function: Summation of states to understand thermodynamic properties, $Z = \sum_i e^{-E_i/k_B T}$.
- Occupation numbers: Average number of particles in a state, $n_i = \frac{1}{e^{(E_i - \mu)/k_B T} \pm 1}$ (for fermions and bosons).
- Degeneracy: The number of microstates corresponding to a single energy level.
- Distinguishable vs. indistinguishable particles: Boltzmann distribution for distinguishable particles; Fermi-Dirac and Bose-Einstein distributions for indistinguishable particles.

3.2 Essential Equations

Boltzmann distribution: $P_i = \frac{e^{-E_i/k_B T}}{Z}$

Partition function: $Z = \sum_i g_i e^{-E_i/k_B T}$

Fermi-Dirac distribution: $n_i = \frac{1}{e^{(E_i-\mu)/k_B T} + 1}$

Bose-Einstein distribution: $n_i = \frac{1}{e^{(E_i-\mu)/k_B T} - 1}$

Average energy: $\langle E \rangle = \sum_i P_i E_i = -\frac{\partial \ln Z}{\partial \beta}$

Probability of a state: $P(E) = \frac{g_i e^{-E/k_B T}}{Z}$

3.3 Units and Unit Conversions

- Energy: 1 eV $= 1.602 \times 10^{-19}$ J.
- Temperature: Convert between Kelvin and energy using $k_B = 1.38 \times 10^{-23}$ J/K.
- Wavelength and frequency: $E = hf$, $\lambda = \frac{c}{f}$, with $hc = 1240$ eV $\cdot$ nm.

4 Quick Reference

4.1 Mathematical Tools

- Trigonometric Identities:

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$
$$\sin^2 x = \frac{1-\cos(2x)}{2}$$
$$\sin(2x) = 2\sin x \cos x$$

- Series:
  - Taylor Series Expansion: $\sum_{n=0}^{\infty} C_n(x-a)^n$ where $C_n = \frac{f^{(n)}(a)}{n!}$
  - Binomial Series Expansion: $(1 \pm x)^r = 1 \pm rx \pm ...$

- Integration Tips:
  - For even functions: $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$
  - For odd functions: $\int_{-a}^{a} f(x)dx = 0$
  - Gaussian integrals often appear in wave packets

- Complex Numbers Review
  - Complex conjugate: $z^* = a - bi$ for $z = a + bi$
  - Modulus: $|z| = \sqrt{a^2 + b^2}$
  - Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$
  - Argument: $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

- Second Order Differential Equations
  - If ODE has this form: $\frac{d^2 f}{dx^2} = k^2 f$ where k is a constant

    The solution is:

$$f(x) = Ae^{kx} + Be^{-kx}$$

4.2 Important Constants

- Stefan-Boltzmann constant: $\sigma = 5.67 \times 10^{-8}$ W/m$^2 \cdot K^4$
- Planck's constant: $h = 6.626 \times 10^{-34}\,\mathrm{J \cdot s} = 4.136 \times 10^{-15}$ eV $\cdot$ s
- Reduced Planck's constant: $\hbar = 1.055 \times 10^{-34}\,\mathrm{J \cdot s}$
- Electron mass: $m_e = 9.11 \times 10^{-31}\,\mathrm{kg}$
- Speed of light: $c = 3.0 \times 10^8\,\mathrm{m/s}$
- Bohr radius: $a_0 = 0.0529 \times 10^{-9}\,\mathrm{m}$
- Compton wavelength of the electron: $\dfrac{h}{m_e c} = 0.002426$ nm

4.3 Unit Conversions

- $1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{J}$
- $hc = 1240\ \mathrm{eV \cdot nm}$
- $\hbar c = \dfrac{1240}{2\pi}\ \mathrm{eV \cdot nm}$