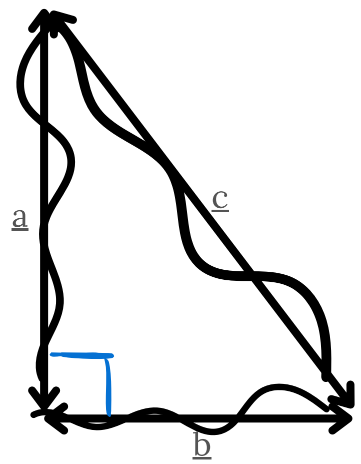
Reframing of Pythagoras theorem

Bithagoras theorem states that the sum of squared norms of a set of vectors is equal to the average squared length of all their signed combinations.

In order to intuitively derive the relationship, we first consider the simplest case of right angled triangles.



If I were to randomly infinitesimally travel a displacement of C, what else did I do?

Any displacement corresponds to its horizontal and vertical component, **VICE VERSA.**
By randomly travelling C, I also travelled Vertical displacement A and horizontal displacement B.
Question is, did I do both randomly as well?
By randomly travelling C, you are killing two birds with one stone, or at least implicitly and **indistinguishably**. This is only possible because of the **interchangeability** of C and (A,B)

Without randomness: Travel displacement A + B = same outcome but different process.

With randomness: same outcome, indistinguishable process. When visualizing, you cannot visualize A then B, they are concurrent and independent. (not fixed points either)

We can derive a relationship from randomness using variance.
Let X= unit length of a completed random path x.

$$Var(aX) + Var(bX) = Var(cX)$$

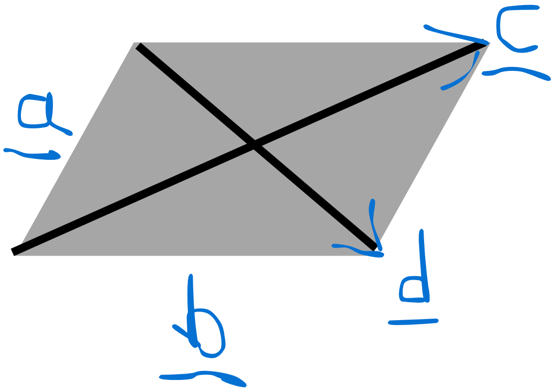
$$a^2 Var(X) + b^2 Var(X) = c^2 Var(X)$$

$$a^2 + b^2 = c^2 .$$

Intuitively, going in every possible direction requires AREA.
By proportionality, $a^2 + b^2 = c^2$.

Random paths is incomplete for non right-angled triangles.
This is because of Ambiguity.
We often imagine a vector to go 1 direction, the one we want.
In an equation, we can preserve the orientation of vectors relative to each other. However, we must allow the vectors to go both directions, because they are equally valid in the same equation.

A + B or A - B, it does not affect right-angled triangles because all 4 combinations result in the same displacement C.
However, in other triangles, going in opposite directions can lead to 2 different displacements, it is no longer one-to-one.
You need to include both in order to achieve interchangeability (respecting the equal sign)



We must treat the two displacements as equally probable since each have 2 combinations. To derive a relationship, we simply compare the variances once more.

2 Var(aX) + 2 Var(bX) = Var(cX) + var(dX)

$2a^2\mathrm{Var}(X) + 2b^2\mathrm{Var}(X) = c^2\mathrm{Var}(X) + d^2\mathrm{Var}(X)$

Parallelogram Law: $2(a^2 + b^2) = c^2 + d^2$

The Bithagoras Theorem is simply an extension to N amount of vectors.

Let $a_1, a_2, \ldots, a_n$ be vectors of a vector space.
For each element $\epsilon := (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \{-1, 1\}^n$ we define

$$v_\epsilon = \sum_{i=1}^{n} \epsilon_i \, a_i.$$

Then we get

$$\sum_{i=1}^{n} |a_i|^2 = \sum_{\epsilon \in \{-1,1\}^n} |v_\epsilon|^2 / 2^n$$