Florida Tech

CSE 5255

INTRODUCTION TO COMPUTER GRAPHICS CLASS NOTES

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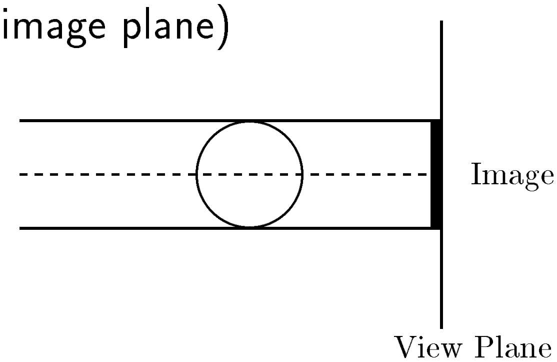
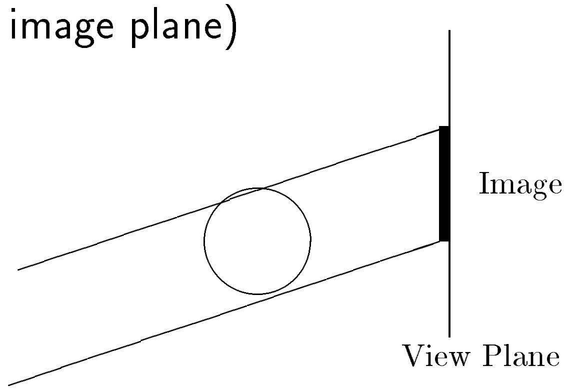
- Given an object in 3D view coordinates, project it onto a 2D space
- Two basic types of projections exist: parallel and perspective
- Parallel projections — useful in blueprints, schematic diagrams, etc.
- Parallel projections are linear transforms (implemented with a matrix)
- Perspective projections — useful in archtectural rendering, realistic views, etc.
- Perspective projections are non-linear transforms
  - They can be implemented with a matrix in projective space followed by a divide by the homogeneous coordinate

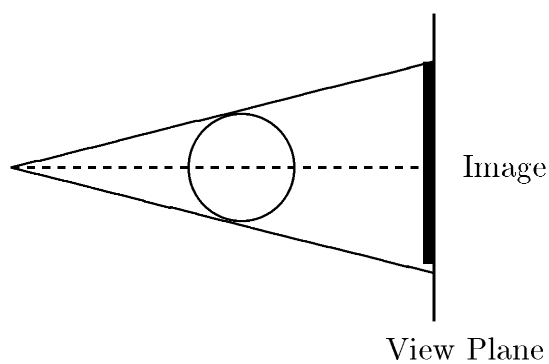
- Parallel projections
  - Points on an object are projected to the view plane along parallel lines (projectors)
  - Preserves relative dimensions (angles and line lengths)
  - Projection can be *orthographic* (perpendicular to image plane)



  - Projection can be *oblique* (not perpendicular to image plane)



- Perspective projections
  - Points on an object are projected to the view plane along lines (projectors) that converge to a center of projection
  - Produces realistic views, but does not preserve relative dimensions
  - Distant lines are foreshortened
  - Projection can be orthographic or oblique



- Parallel projection determined by (1) view plane and (2) direction of projection (projector)
  - Assume view plane (projection plane) is given by

$$ax + by + cz + d = 0$$

  - Assume direction of projection (projector) is given by

$$\langle q,\, p,\, r \rangle$$

- Normal vector to projection plane is $\langle a,\, b,\, c \rangle$
- Assume normal vector and projector have unit length
- Orthographic projection — direction of projection is perpendicular (orthogonal) to projection plane, i.e.,

$$\langle q,\, p,\, r \rangle = \langle a,\, b,\, c \rangle$$

- Oblique projection — direction of projection is <u>not</u> perpendicular to projection plane, i.e.,

$$\langle q,\, p,\, r \rangle \neq \langle a,\, b,\, c \rangle$$

- Let $ax + by + cz + d = 0$ be the projection plane
- Let $\langle q,\, p,\, r\rangle$ be the direction of projection
- Let $(x_o,\, y_o,\, z_o)$ be a point on the object to be projected
- Start at $(x_o,\, y_o,\, z_o)$ and travel along the line in direction $\langle q,\, p,\, r\rangle$ until plane $ax + by + cz + d = 0$ is hit
- It's easiest to use the parametric equation of the line

$$\begin{aligned} x &= x_o + qt \\ y &= y_o + pt \\ z &= z_o + rt \end{aligned}$$

- At some value of t, when the plane equation is satisfied, we are on the projection plane

$$\begin{aligned} 0 &= ax + by + cz + d \\ &= a(x_o + qt) + b(y_o + pt) + c(z_o + rt) + d \\ &= ax_o + by_o + cz_o + t(aq + bp + cr) + d \end{aligned}$$

- Solving for the unknown parameter value

$$t = -\left[\frac{ax_o + by_o + cz_o + d}{aq + bp + cr}\right]$$

provided $aq + bp + cr \neq 0$ (what does this mean?)

- Substituting this value of t into the previous line equation for x, y, and z gives an expression for the projected point (x_p, y_p, z_p)

$$\begin{aligned} x_p &= x_o - q\left[\frac{ax_o + by_o + cz_o + d}{aq + bp + cr}\right] \\ y_p &= y_o - p\left[\frac{ax_o + by_o + cz_o + d}{aq + bp + cr}\right] \\ z_p &= z_o - r\left[\frac{ax_o + by_o + cz_o + d}{aq + bp + cr}\right] \end{aligned}$$

- With some manipulation we can write this as a matrix equation

$$\begin{bmatrix} x_o & y_o & z_o & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 \\ m_{21} & m_{22} & m_{23} & 0 \\ m_{31} & m_{32} & m_{33} & 0 \\ m_{41} & m_{42} & m_{43} & 1 \end{bmatrix} = \begin{bmatrix} x_p & y_p & z_p & 1 \end{bmatrix}$$

- $m_{11} = (bp + cr)/(aq + bp + cr)$,
  $m_{21} = (-bq)/(aq + bp + cr)$,
  $m_{31} = (-cq)/(aq + bp + cr)$,
  $m_{41} = (-dq)/(aq + bp + cr)$
- $m_{12} = (-ap)/(aq + bp + cr)$,
  $m_{22} = (aq + cr)/(aq + bp + cr)$,
  $m_{32} = (-cp)/(aq + bp + cr)$,
  $m_{42} = (-dp)/(aq + bp + cr)$
- $m_{13} = (-ar)/(aq + bp + cr)$,
  $m_{23} = (-br)/(aq + bp + cr)$,
  $m_{33} = (aq + bp)/(aq + bp + cr)$,
  $m_{43} = (-dr)/(aq + bp + cr)$

- Orthographic projection onto the projection plane $z = 0$ is performed by the projection matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Find a matrix for parallel projections onto the plane

$$3x + y + 4z + 1 = 0$$

when

(a) an orthographic projection is used

(b) an oblique projection in direction $\langle 1,\, 1,\, 1 \rangle$ is used

- For blueprints, part schematics, etc., we often use
  - elevations – front, side, rear
  - plan – top
- *Axonometric* projections allow many sides to be seen
  - Axonometric parallel projection has projectors orthogonal to view plane, but not parallel to any principal axis
  - Isometric – projection plane normal (projector) makes equal angles with each principal axis (all three axes are equally foreshortened)
  - Dimetric – projection plane normal (projector) makes equal angles with two of three principal axes (two of three axes are equally foreshortened)
  - Trimetric – projection plane normal (projector) makes unequal angles with each principal axis (all three axes are unequally foreshortened)

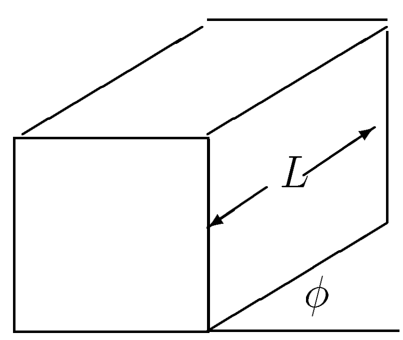
- Let $z = 0$ be the projection plane with projector $\langle q,\, p,\, r\rangle$
- Form the line equation

$$\begin{aligned} x &= x_o + qt \\ y &= y_o + pt \\ z &= z_o + rt \end{aligned}$$

- Find $t = -z_o/r$, so that the projection matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -q/r & -p/r & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- When the p and q values are equal, say both 1, a $\phi = 45^\circ$ projection results
- It can be shown that $-q/r = L\cos\phi$, $-p/r = L\sin\phi$



- Two common types of oblique parallel projections: cavalier and cabinet
- Cavalier — lines perpendicular to projection plane are preserved in length, that is, $L = 1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \cos\phi & \sin\phi & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Note the vector $\langle 0,\, 0,\, 1\rangle$ maps to $\langle \cos\phi,\, \sin\phi,\, 0\rangle$ which has length 1
- Cabinet — lines perpendicular to projection plane are 1/2 their true length, that is, $L = 1/2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ (\cos\phi)/2 & (\sin\phi)/2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Note the vector $\langle 0,\, 0,\, 1\rangle$ maps to $\langle (\cos\phi)/2,\, (\sin\phi)/2,\, 0\rangle$ which has length 1/2

- Project point along projection lines that meet at a center of projection $(x_c,\, y_c,\, z_c)$
- Produce an image similar to camera (or cyclops)
- Straight lines do not remain straight
- Can <u>not</u> represent with a matrix, except by fudging
- Assume $Ax + By + Cz + D = 0$ is projection plane
- Assume $(x_o,\, y_o,\, z_o)$ is point to be projected
- Parametric equation of the line from $(x_o,\, y_o,\, z_o)$ to $(x_c,\, y_c,\, z_c)$

$$\begin{aligned} x &= x_o + (x_c - x_o)t \\ y &= y_o + (y_c - y_o)t \\ z &= z_o + (z_c - z_o)t \end{aligned}$$

- Substitute line into plane and solve for t

$$Ax + By + Cz + D = 0$$

$$A(x_o+(x_c-x_o)t)+B(y_o+(y_c-y_o)t)+C(z_o+(z_c-z_o)t)+D = 0$$

$$Ax_o+By_o+Cz_o+t[A(x_c-x_o)+B(y_c-y_o)+C(z_c-z_o)]+D = 0$$

$$t = -\frac{Ax_o + By_o + Cz_o + D}{A(x_c - x_o) + B(y_c - y_o) + C(z_c - z_o)}$$

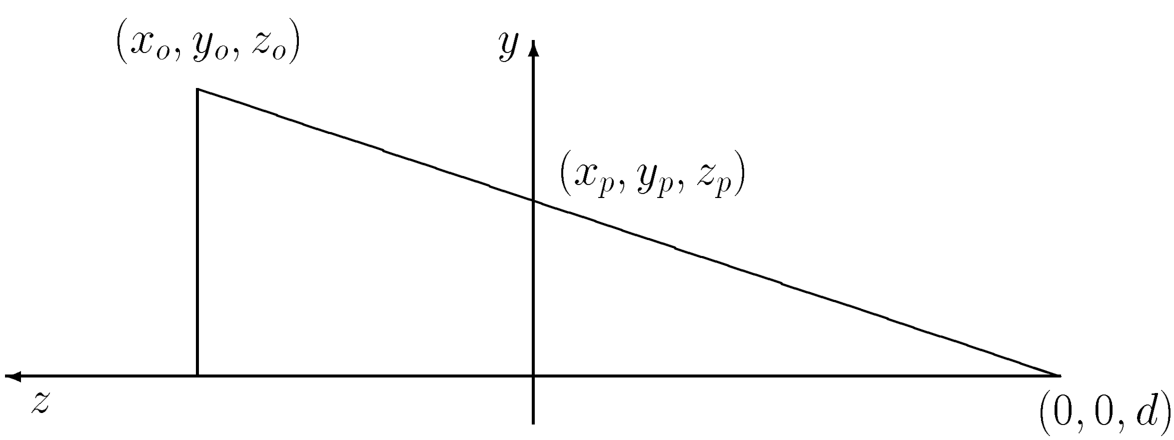
- Note denominator for t depends on object point (x_o, y_o, z_o)

- Assume $z = 0$ is projection plane
- Assume $(0,\,0,\,d)$ is center of projection
- From previous equations, $t = -z_o/(d - z_o)$, and

$$\begin{aligned}
x_p &= x_o + (0 - x_o)t = \frac{dx_o}{d - z_o} \\
y_p &= y_o + (0 - y_o)t = \frac{dy_o}{d - z_o} \\
z_p &= z_o + (d - z_o)t = 0
\end{aligned}$$

- Or by similar triangles

$$\begin{aligned}
\frac{x_p}{-d} &= \frac{x_o}{z_o - d} \\
\frac{y_p}{-d} &= \frac{y_o}{z_o - d} \\
z_p &= 0
\end{aligned}$$



- Use homogeneous coordinates to express transformation in two steps (1) a matrix multiply, (2) a division by homogeneous coordinate
- Step (1)

$$\left[\begin{array}{cccc} x & y & z & w \end{array}\right] = \left[\begin{array}{cccc} x_o & y_o & z_o & 1 \end{array}\right] \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/d \\ 0 & 0 & 0 & 1 \end{array}\right]$$

where

$$w = 1 - z_o/d$$

- Step (2) divide each coordinate in $(x\,y\,z\,w)$ by w to make the homogeneous coordinate become 1

$$\begin{aligned} x_p &= \frac{x_o}{1 - z_o/d} \\ y_p &= \frac{y_o}{1 - z_o/d} \\ z_p &= 0 \end{aligned}$$

Perspective Projection Example

- Let the object point be $(x_o, y_o, z_o) = (2, 7, 8)$
- Project onto the $z = 0$ plane
- Let center of projection be located at $(0, 0, -50)$
- By similar triangles

$$
\begin{aligned}
\frac{x_p}{50} &= \frac{2}{8 - (-50)} \\
\frac{y_p}{50} &= \frac{7}{8 - (-50)}
\end{aligned}
$$

and thus

$$
\begin{aligned}
x_p &= 50/29 \\
y_p &= 350/58
\end{aligned}
$$

- Parallel lines appear to converge to a vanishing point (provided the lines are not parallel to the projection plane)
- Lines parallel to principal axes converge to a principal vanishing point
- A perspective projection can have 1, 2, or 3 principal vanishing points
- If the projection plane is parallel to 2 principal axes, a one-point projection occurs
- If the projection plane is parallel to 1 principal axis, a two-point projection occurs
- If the projection plane is not parallel to any principal axis, a three-point projection occurs

- Normalized device coordinates are defined by a unit cube

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad 0 \leq z \leq 1$$

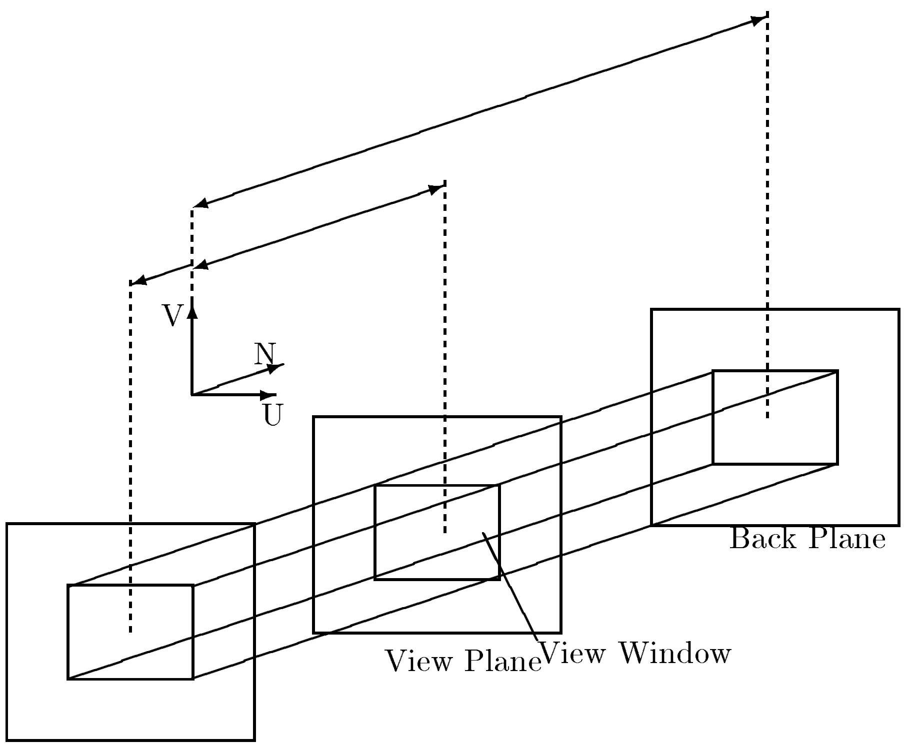
- Some systems support other ranges for NDC
- We need to maps from view coordinates to normalized device coordinates
- One map implements a *parallel view*
  - For orthographic parallel views the map, a translate and scale are used
  - For oblique parallel views the map, a shear, translate, and scale are used
- Another map implements a *perspective view*
- In either case, a *view volume* (which encloses what can be seen) is mapped to the unit cube in NDC

Orthographic Parallel View to NDC Map

- Assume the view volume, in view coordinates, we want to map to NDC is specified by

$$x_l \leq v_x \leq x_r, \quad y_b \leq v_y \leq y_t, \quad z_n \leq v_z \leq z_f$$

- x_l and x_r stand for the *left* and *right* sides of a box
- y_b and y_t stand for the *bottom* and *top* sides of a box
- z_n and z_f stand for the *near* and *far* sides of a box
- The near side is also called the *front* or *hither* plane
- The far side is also called the *back* or *yon* plane
- Only objects inside the box can be seen



Orthographic Parallel View to NDC Map

- The map transforms the ranges to $[0, 1]$
- For x,
$$x = \frac{1}{x_r - x_l}(v_x - x_l)$$
- For y,
$$y = \frac{1}{y_t - y_b}(v_y - y_b)$$
- For z,
$$z = \frac{1}{z_f - z_n}(v_z - z_n)$$
- This implements an orthographic view of the scene
- The general map would allow oblique views as well

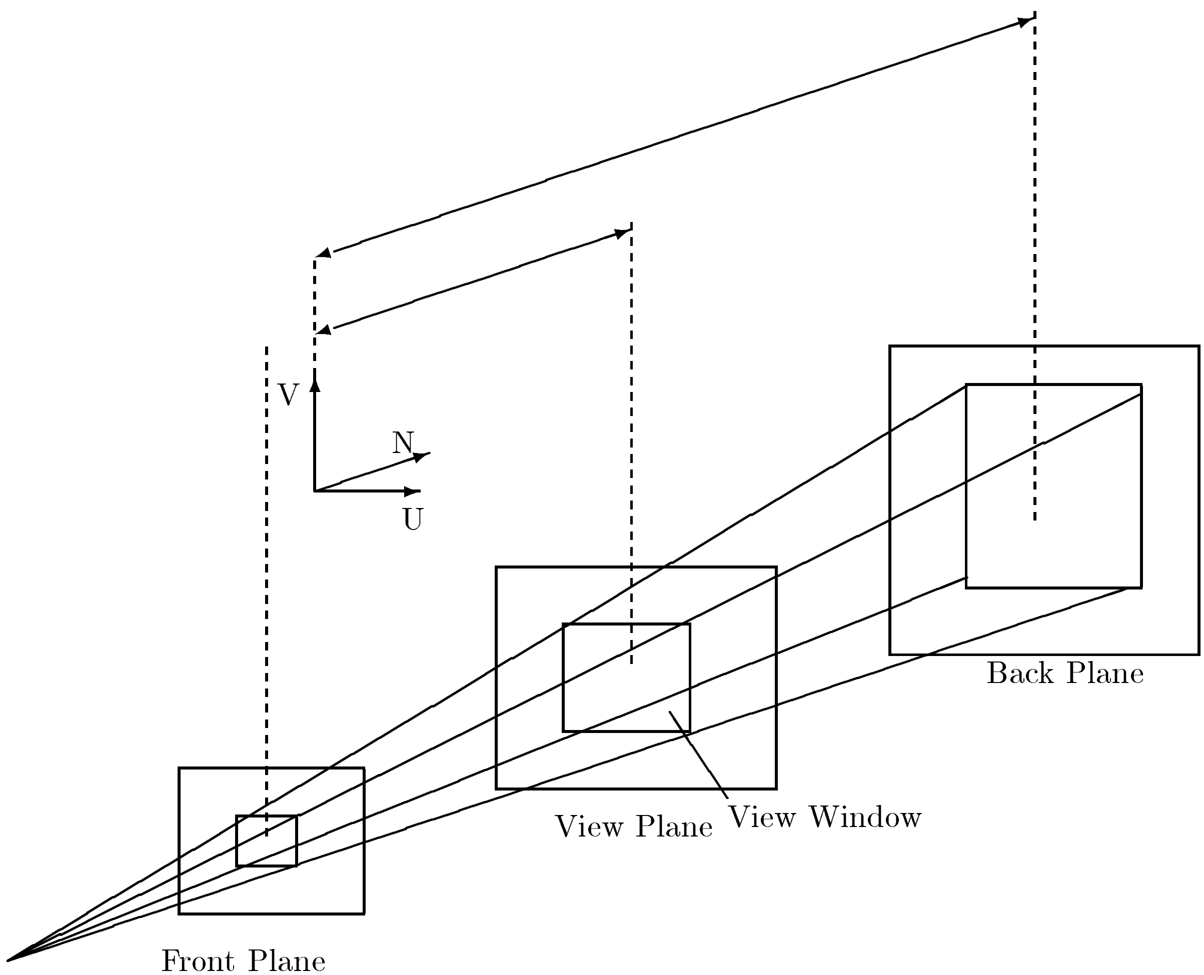
- Assume the view volume, in view coordinates, we want to map is specified by view angle

$$0 < \alpha < \pi$$

and near and far clipping planes

$$v_z = z_n \quad \text{and} \quad v_z = z_f$$

- This defines a view *frustrum* (a truncated pyramid)



- We'll first map this frustrum onto the region

$$-1 \le x \le 1,\ -1 \le y \le 1,\ 0 \le z \le 1$$

- Consider the slice of this frustrum in the $v_x v_z$ plane
- We find that

$$\tan(\frac{\alpha}{2}) = \frac{x_n}{z_n} = \frac{x_f}{z_f}$$
$$\cot(\frac{\alpha}{2}) = \frac{z_n}{x_n} = \frac{z_f}{x_f}$$

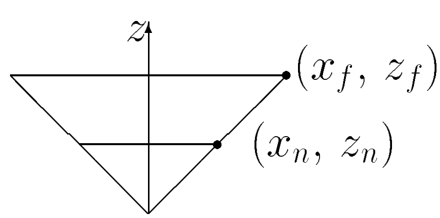
- The (un-normalize) homogeneous points

$$(x_n \cos(\frac{\alpha}{2}),\ *,\ *,\ z_n \sin(\frac{\alpha}{2}))$$
$$(x_f \cos(\frac{\alpha}{2}),\ *,\ *,\ z_f \sin(\frac{\alpha}{2}))$$

both normalize to

$$(1,\ *,\ *,\ 1)$$

- The points $(-x_n,\ z_n)$ and $(-x_f,\ z_f)$ map to $(-1,\ *,\ *,\ 1)$



- The same mapping works for the y coordinate
- This yields the matrix transformation

$$\begin{bmatrix} \cos(\frac{\alpha}{2}) & 0 & * & 0 \\ 0 & \cos(\frac{\alpha}{2}) & * & 0 \\ 0 & 0 & * & \sin(\frac{\alpha}{2}) \\ 0 & 0 & * & 0 \end{bmatrix}$$

- We want z_n to map to 0 This can be accomplished by the column

$$\begin{bmatrix} 0 \\ 0 \\ Q \\ -Qz_n \end{bmatrix}$$

  for any constant Q

- We want z_f to map to $1.$ The (un-normalized) homogeneous point

$$(*, *, (z_f - z_n)Q, z_f \sin(\frac{\alpha}{2}))$$

  should normalize to

$$(*, *, 1, 1)$$

- Thus

$$Q = \frac{z_f \sin(\frac{\alpha}{2})}{z_f - z_n}$$

Orthographic Perspective View to NDC Map

- The perspective view onto $-1 \le x \le 1$, $-1 \le y \le 1$, $0 \le z \le 1$ is given by the matrix

$$\begin{bmatrix} \cos(\frac{\alpha}{2}) & 0 & 0 & 0 \\ 0 & \cos(\frac{\alpha}{2}) & 0 & 0 \\ 0 & 0 & Q & \sin(\frac{\alpha}{2}) \\ 0 & 0 & -Qz_n & 0 \end{bmatrix}$$

- To map into normalized device coordinates:
  - Translate the x values by 1 and scale x by 1/2
  - Translate the y values by 1 and scale y by 1/2
- The map into NDC space is given by

$$\begin{bmatrix} \cos(\frac{\alpha}{2}) & 0 & 0 & 0 \\ 0 & \cos(\frac{\alpha}{2}) & 0 & 0 \\ 0 & 0 & Q & \sin(\frac{\alpha}{2}) \\ 0 & 0 & -Qz_n & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/2 & 1/2 & 0 & 1 \end{bmatrix}$$

- We'll define screen coordinates as the region

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1$$

- The map from NDC space to screen coordinates is obtained by simply dropping the z coordinate

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- We assume the *device driver* has been written to take points in this range and produce values that the hardware can draw

- Note — your system may require a different range, for example some systems assume integer coordinates in the range $[0, 32767]$ You will need to supply the appropriate map

- The region in the view coordinates being mapped to NDC space if frequently called the *window* (an over-used term)
- The window defines our view of the world
- All of NDC space will be displayed, after it is projected to screen space and mapped to device coordinates
- Often we want to want to project different world views to different regions, called *viewports*, of NDC space
- Explain how you would use a parallel view to map a view box to the viewport

  $$0 \le x \le 0.5, \quad 0 \le y \le 0.5$$

  the lower left corner of the screen
- Explain how you would use a perspective view to map a view frustrum to the viewport

  $$0 \le x \le 0.5, \quad 0 \le y \le 0.5$$

  the lower left corner of the screen

Problems

1. Find the projection operator that performs an orthographic projection onto the plane $2x + 7y + z + 8 = 0$.
2. Find the matrix that performs an orthographic parallel projection onto the plane $2x + 1y + 4z = 0$.
3. Find the projection operator that performs an oblique projection onto the plane $2x + 7y + z + 8 = 0$ with direction of projection $(3, 1, 4)$.
4. Find the matrix that performs an oblique parallel projection onto the plane $2x + 1y + 4z = 0$ with projectors given by $\langle 1, 0, 1 \rangle$.
5. Show the steps in computing a perspective projection onto the $z = 0$ plane with a center of projection at $(0, 0, 40)$.
6. Define *anoxometric projection*.
7. Find the matrix that performs an cavalier projection onto the plane $z = 0$ with a $30°$ angle between the x-axis and lines parallel to the z-axis.
8. Find the matrix that performs an cabinet projection onto the plane $z = 0$ with a $30°$ angle between the x-axis and lines parallel to the z-axis.
9. Find the matrix that performs an cavalier projection onto the plane $z = 0$ with angle θ angle between the x-axis and lines parallel to the z-axis.
10. Find the matrix that performs an cabinet projection onto the plane $z = 0$ with angle θ angle between the x-axis and lines parallel to the z-axis.
11. Find the transformation that performs a perspective projection onto the plane $z = 0$ where the center of projection is at $d = -10$.
12. Find the transformation that performs a perspective projection onto the plane $z = 0$ where the center of projection is at $d = -100$.
13. Find the transformation that performs a perspective projection onto the plane $z = 0$ where the center of projection is at $(1, 0, -10)$.
14. Find the transformation that performs a perspective projection onto the plane $x + z = 0$ where the center of projection is at $d = -10$.

Problems

15. Find the transformation that performs a perspective projection onto the plane $ax + by + cz + d = 0$ where the center of projection is at $(c_x, , c_y, c_z)$.
16. What are 1, 2, and 3 point perspective projections?
17. How many vanishing points are there for a perspective projection?
18. What is the difference between orthographic and oblique perspective projections?
19. Find the transformation that performs a parallel map of the following regions into NDC space.
    - $x_l = -10,\ x_r = 10,\ y_b = 0,\ y_t = 20,\ z_n = 0,\ z_f = 50.$
    - $x_l = 20,\ x_r = 50,\ y_b = 10,\ y_t = 15,\ z_n = 10,\ z_f = 20.$
    - $x_l = -10,\ x_r = 20,\ y_b = 0,\ y_t = 20,\ z_n = -10,\ z_f = 10.$
    - $x_l = -10,\ x_r = 50,\ y_b = 10,\ y_t = 20,\ z_n = -20,\ z_f = 50.$
20. Show how to construct a parallel transformation into NDC space where the boundaries are defined by $0 \leq x \leq 1023,\ 0 \leq y \leq 767,\ 0 \leq z \leq 4095.$
21. Find the transformation that performs a perspective map of the following regions into NDC space.
    - $\alpha = 90°,\ z_n = 0,\ z_f = 10.$
    - $\alpha = 120°,\ z_n = 10,\ z_f = 50.$
    - $\alpha = 60°,\ z_n = 0,\ z_f = 60.$
    - $\alpha = 45°(),\ z_n = 0,\ z_f = \infty.$
22. Show how to construct a perspective transformation into NDC space where the boundaries are defined by $0 \leq x \leq 1023,\ 0 \leq y \leq 767,\ 0 \leq z \leq 4095.$