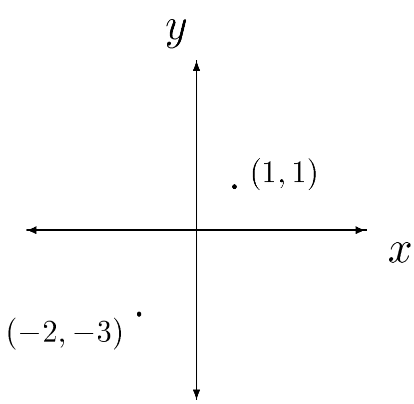
- Points in 2-dimensional Cartesian coordinate space

$$(x,\, y)$$

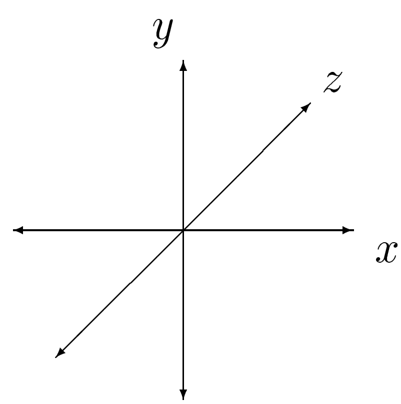
- Points in 3-dimensional Cartesian coordinate space

$$(x,\, y,\, z)$$

- Right-handed versus left-handed spaces
  - Right-handed: x in direction of thumb, y in direction of fingers, z in direction of palm
  - Left-handed: x in direction of thumb, y in direction of fingers, z in direction of palm

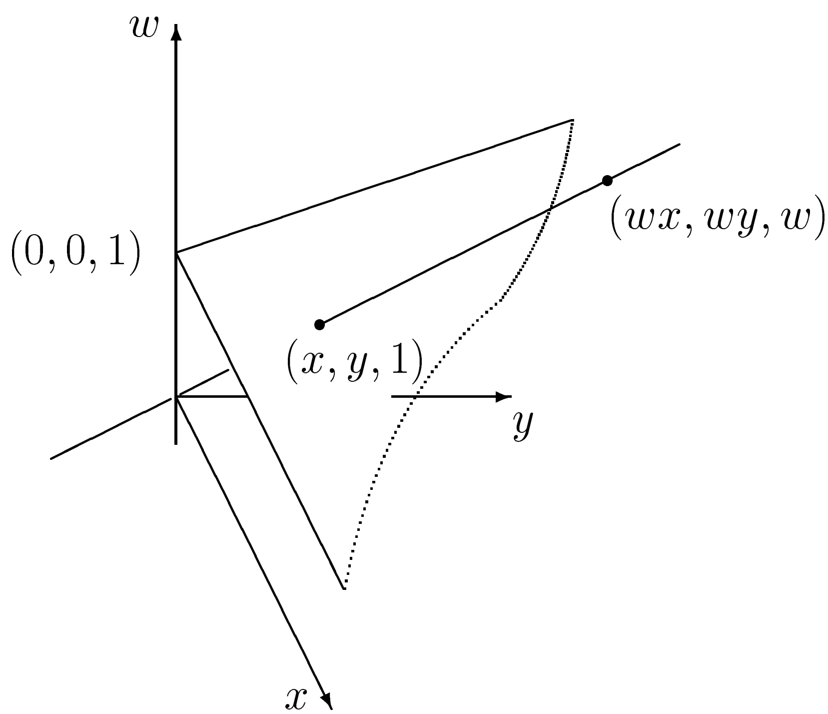


2D Cartesian Coordinates



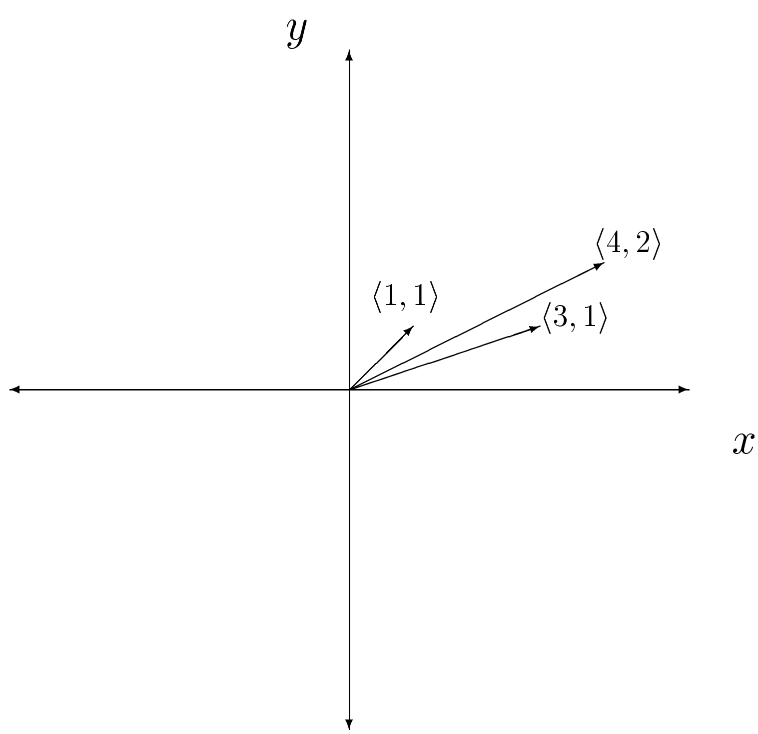
3D Cartesian Coordinates
(Left-handed)

- 2 dimensional projective space
  - Each point (x, y) in 2D is represented by the "punctured" line (wx, wy, w), $w \neq 0$
  - $(2, 3, 6)$, $(4, 6, 12)$, and $(\frac{1}{3}, \frac{1}{2}, 1)$ represent the same point in 2-dimensional projective space point
  - The "normalized" homogeneous point $(x, y, 1)$ is most used
  - A projective point $(x, y, 0)$ is considered a "point at infinity"
  - A point at infinity represents a direction vector (rather than a point)



- 3 dimensional projective space
  - Each point $(x,\ y,\ z)$ in 3D is represented by the "punctured" line $(wx,\ wy,\ wz,\ w)$, $w \neq 0$
  - $(2,\ 3,\ 6,\ 2)$, $(4,\ 6,\ 12, 4)$, and $(1,\ \frac{3}{2},\ 3,\ 1)$ represent the same point in 3-dimensional projective space point
  - The "normalized" homogeneous points $(x,\ y,\ z,\ 1)$ is most used
  - A projective point $(x,\ y, z,\ 0)$ is considered a "point at infinity"
  - A point at infinity represents a direction vector (rather than a point)
- Homogeneous coordinates allow us to use one common device (matrix multiplication) in transforming points
- Translations of objects involve the homogeneous coordinate
- Homogeneous coordinates occur in perspective projections of 3D images to 2D displays

- Vectors in 2-dimensional Cartesian coordinate space
  $\vec{V} = \langle x,\, y \rangle$
- Vectors in 3-dimensional Cartesian coordinate space
  $\vec{V} = \langle x,\, y,\, z \rangle$
- Vectors have a length $\|\vec{V}\| = \sqrt{x^2 + y^2 + z^2}$
- A *unit vector* has length 1
- A (non-zero) vector can be "normalized to unit length" by dividing by its length ($\vec{V}/\|\vec{V}\|$ is a unit vector)
- Vectors are *invariant* under translation



2D Vector Space

- Vector addition:
  - The parallelogram rule:

    $\langle v_x, v_y, v_z\rangle + \langle u_x, u_y, u_z\rangle = \langle v_x+u_x, v_y+u_y, v_z+u_z\rangle$
  - Vector addition is *commutative* and *associative*
  - The is an *identity* vector $\vec{0} = \langle 0, 0, 0\rangle$, such that $\vec{V} + \vec{0} = \vec{V}$ for all vectors $\vec{V}$
  - Every vector $\vec{V}$ has an *additive inverse* vector $-\vec{V}$ such that $\vec{V} + (-\vec{V}) = \vec{0}$
- Scalar multiplication:
  - Scale the length of a vector

    $$c\langle x, y, z\rangle = \langle cx, cy, cz\rangle$$
  - For all vectors $\vec{V}$ and scalars α, β

    $$(\alpha\beta)\vec{V} = \alpha(\beta\vec{V})$$
  - For all vectors $\vec{V}$

    $$1\vec{V} = \vec{V}$$
  - For all vectors $\vec{V}$ and scalars α, β

    $$(\alpha + \beta)\vec{V} = \alpha\vec{V} + \beta\vec{V}$$
  - For all vectors $\vec{V}$ and $\vec{W}$, and scalars α

    $$\alpha(\vec{V} + \vec{W}) = \alpha\vec{V} + \alpha\vec{W}$$

- A *vector space* consists of a set of vector together with the operations of vector addition and scalar multiplication
- An *affine space* consists of a set of points, a derived vector space, and two operations
  - Given two points P and Q, we can form the *difference* $Q - P$ which is a vector
  - Given a point P and a vector $\mathbf{v}$ we can *add* the vector to the point to get a new point $P + \mathbf{v}$
  - The derived vector space is formed by taking all possible differences.
  - There is no distinguished origin in an affine space
- In affine spaces, with homogeneous coordinates, we will be able to rotate, scale, translate, and project objects using matrix operations (and normalization of the homogeneous coordinate)

- Let P and Q be points in an affine space
- $Q - P$ is the vector pointing from P to Q
- $t(Q - P)$ is a vector t times as long.
- The affine combination

$$P + t(Q - P), \quad -\infty < t < \infty$$

  describes the *line* through P and Q
- The affine combination

$$P + t(Q - P), \quad 0 \leq t \leq 1$$

  describes the *line segment* from P to Q
- The *parametric form* of a line segment is written as

$$P + t(Q - P) = (1 - t)P + tQ$$

- If P, Q, and R are three points in an affine space, that are not colinear, then the *plane* defined by P, Q, and R is the set of points

$$sP + tQ + uR, \quad s + t + u = 1$$

- Restricting s and t to the range $[0, 1]$ defines the triangle determined by the three points

- The *dot*, *inner*, or *scalar product* of two vectors $\vec{V} = \langle v_x,\, v_y,\, v_z \rangle$ and $\vec{W} = \langle w_x,\, w_y,\, w_z \rangle$ is the number (scalar)

$$\vec{V} \cdot \vec{W} = v_x w_x + v_y w_y + v_z w_z$$

- The *length* of a vector $\vec{V}$ is

$$\|\vec{V}\| = \sqrt{\vec{V} \cdot \vec{V}}$$

  (The Pythagorean Theorem)

- Any vector $\vec{V} \neq \vec{0}$ can be *normalized* by calculating

$$\vec{V}' = \frac{\vec{V}}{\|\vec{V}\|}$$

  $\vec{V}'$ is a *unit* vector

- The *angle* θ *between vectors* $\vec{V}$ and $\vec{W}$ is

$$\theta = \cos^{-1}\left(\frac{\vec{V} \cdot \vec{W}}{\|\vec{V}\|\|\vec{W}\|}\right)$$

- Two $\vec{V}$ and $\vec{W}$ vectors are *orthogonal* (perpendicular) if

$$\vec{V} \cdot \vec{W} = 0$$

- The *implicit* equation of a line in 2D,

  $$ax + by + c = 0,$$

  is given by the dot product equation

  $$\langle a,\, b,\, c\rangle \cdot \langle x,\, y,\, 1\rangle = 0$$

- In 2D, the *normal* vector $\langle a,\, b\rangle$ is orthogonal to the line
- The *implicit* equation of a plane in 3D,

  $$ax + by + cz + d = 0,$$

  is given by the dot product equation

  $$\langle a,\, b,\, c,\, d\rangle \cdot \langle x,\, y,\, z,\, 1\rangle = 0$$

- In 3D, the *normal* vector $\langle a,\, b,\, c\rangle$ is orthogonal to the plane
- If $\vec{N}$ is the "front-facing" (outward) normal to a surface at a point P and $\vec{E}$ is the vector from our eye to P, then we are seeing the "front" of the surface if

  $$\vec{N} \cdot \vec{E} < 0$$

  and the "back" of the surface if

  $$\vec{N} \cdot \vec{E} > 0$$

- Let $\vec{V} = \langle v_x,\, v_y,\, v_z \rangle$ and $\vec{W} = \langle w_x,\, w_y,\, w_z \rangle$ be two 3D vectors
- The *cross product* of $\vec{V}$ with $\vec{W}$ is a vector given by

$$\begin{aligned} \vec{V} \times \vec{W} &= \begin{vmatrix} i & j & k \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \\ &= \langle v_y w_z - w_y v_z,\, v_z w_x - w_z v_x,\, v_x w_y - w_x v_y \rangle \end{aligned}$$

- The cross product $\vec{V} \times \vec{W}$ is orthogonal to both $\vec{V}$ and $\vec{W}$

- A $m \times n$ *matrix* is a rectangular array

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix. Their product $C = AB$ is the $m \times n$ matrix where the element in the (i, j) position of C is the dot product of the ith row of A with the jth column of B
- Notice that $O(mnp)$ operations are required to compute the product of an $m \times n$ matrix with an $n \times p$ matrix
- Given an $m \times 4$ matrix P (a 3D polygon with m vertices) and a series $M_1, M_2, \ldots, M_n$ of 4×4 matrices
  - It costs $O(16m)$ operations to multiply P by each of $M_1, \ldots, M_n$ for a total cost of $O(16mn)$
  - It costs $O(64)$ operations to multiply M_i by M_{i+1} for $i = 1, \ldots, n-1$ forming one 4×4 composite transformation (costs $O(64(n-1))$) and $O(16m)$ to multiply P by the composite transform — for a total cost of $O(64(n-1) + 16m)$
  - The use of a single composite transform is almost always better!

- If

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

then

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

- Example:

$$\begin{bmatrix} 2 & 8 & 3 & 1 \\ -3 & 7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 & 0 \\ 3 & -4 & 1 & 0 \\ -1 & 6 & 2 & 4 \\ -1 & 6 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \\ & & \end{bmatrix}$$

- If $P = [x,\ y,\ z,\ 1]$ is a 3D projective point and A is a 4×4 matrix, the matrix product

$$\Big[\ a_{11}x + a_{21}y + a_{13}z + a_{14} \quad a_{12}x + a_{22}y + a_{32}z + a_{42}a_{13}x + a_{23}y + a_{33}$$

transforms P to a new point

- We will use matrix transformations to
  - *Translate* points along a straight line
  - *Scale* the distance of points from an origin
  - *Rotate* points around an axis
  - *Skew* points relative to an axis
  - *Project* points from 3D to 2D
  - There are many other uses for matrices in graphics

- A (closed) polygon is a sequence of points (vertices) $P_1, P_2, \ldots, P_n$ with edges drawn between adjacent vertices, i.e., there is an edge from P_1 to P_2, from P_2 to P_3, ..., from P_n to P_1
- Each point (vertex) is an *ordered pair* ($P = (x, y)$) in 2-space, or an *ordered triple* ($P = (x, y, z)$) in 3-space
- Often we append a homogeneous coordinate (with value) 1 to the points to work in projective space
- A polygon with n vertices in 3D projective space can be represented as an $n \times 4$ matrix

$$P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix}$$

- Special properties we'll usually require:
  - All vertices should lie in a plane (planar polygons)
  - No edge should intersect another edge
  - The polygon should be convex
  - The inside of the polygon is on our left as edges are traversed from P_1 to P_2 to $P_3 \cdots P_n$ to P_1

- Let $P_i = (x_i, y_i)$ and $P_{i+1} = (x_{i+1}, y_{i+1})$ be two adjacent vertices of a 2D polygon
- The edge vector from P_i to P_{i+1} is

  $$\vec{e}_{i,i+1} = P_{i+1} - P_i = \langle x_{i+1} - x_i, y_{i+1} - y_i \rangle$$

  and

  $$\vec{n}_{i,i+1} = \langle -(y_{i+1} - y_i), x_{i+1} - x_i \rangle$$

  is the *inward pointing edge normal*
- Let $P_i = (x_i, y_i, z_i)$ and $P_{i+1} = (x_{i+1}, y_{i+1}, z_{i+1})$ be two adjacent vertices of a 3D polygon lying in the plane $Ax + By + Cz + D = 0$
- The *inward pointing edge normal* is

  $$\vec{n}_{i,i+1} = \langle A, B, C \rangle \times \langle (x_{i+1} - x_i), (y_{i+1} - y_i), (z_{i+1} - z_i) \rangle$$

- Let P be a point on an edge of a convex polygon with inward edge normal $\vec{n}$ and let Q be any other point inside or on the boundary of the polygon. Then

$$\vec{n} \cdot (Q - P) \geq 0$$

- Let R be a point, let $\vec{n}$ be an inward pointing edge normal for a convex polygon, and P be a point on the edge of the polygon
  - If
  $$\vec{n} \cdot (P - R) > 0,$$
  then R is outside the polygon with respect to the given edge and the line from R to P *enters* the polygon
  - If
  $$\vec{n} \cdot (P - R) < 0,$$
  then R is inside the polygon with respect to the given edge and the line from R to P *exits* the polygon

- Let $P_1,\ P_2, P_3$ be any three consecutive vertices of a planar 3D polygon (listed in the counterclockwise direction so the inside in on the left)

- The cross product

$$\vec{N} = (P_2 - P_1) \times (P_3 - P_1)$$

  gives the *outward pointing surface normal* for the polygon

- Newell's technique for computing the *outward pointing surface normal* $\vec{N} = \langle n_1,\ n_2,\ n_3 \rangle$

$$N_x = \sum_{k=1}^{n} (y_k - y_{k+1})(z_k + z_{k+1})$$

$$N_y = \sum_{k=1}^{n} (z_k - z_{k+1})(x_k + x_{k+1})$$

$$N_z = \sum_{k=1}^{n} (x_k - x_{k+1})(y_k + y_{k+1})$$

  where $P_k = (x_k,\ y_k,\ z_k),\ k = 1,\ \ldots,\ n$ are the vertices of the polygon and $P_{n+1} = P_1$

- Newell's method gives a good approximation of the surface normal for non-planar polygons

1. Plot these points: $(2,\, 3)$, $(-4,\, 5)$, $(2,\, 3,\, 1)$, $(-4,\, 5,\, -2)$.
2. Transform the points above into homogeneous coordinates.
3. Transform the points above into homogeneous coordinates, with a normalized homogeneous coordinate.
4. Draw the vectors $\langle 2,\, 3\rangle$, $\langle -4,\, 5\rangle$, $\langle 2,\, 3,\, 1\rangle$, $\langle -4,\, 5,\, -2\rangle$.
5. What is the parametric form of the line segment from $(2,\, 3)$ to $(-4,\, 5)$?
6. What is the parametric form of the line segment from $(2,\, 3,\, 1)$ to $(-4,\, 5,\, -2)$?
7. What is the parametric form of the plane containing points $(2,\, 3,\, 1)$, $(-4,\, 5,\, -2)$, $(3,\, 4,\, -3)$?
8. What is the parametric form of the triangle with vertices $(2,\, 3,\, 1)$, $(-4,\, 5,\, -2)$, $(3,\, 4,\, -3)$?
9. What is the dot product of $\langle 2,\, 3,\, 1\rangle$ with $\langle -4,\, 5,\, -2\rangle$?
10. What is the length of each of the above vectors?
11. What is the inner product of two vectors $(x_0,\, y_0)$ and $(x_1,\, y_1)$?
12. What is the cosine of the angle between the above vectors?
13. What is the angle between the above vectors?
14. Find a vector orthogonal to $\langle 2,\, 3,\, 1\rangle$.
15. Given the plane $3x - 4y + 7z - 3 = 0$, what is its surface normal?
16. If the vector from our "eye" to the above plane is $\langle 2,\, 3,\, 1\rangle$ and the normal you found is the "front-facing" normal, are you looking at the front of back of the plane?
17. What is the cross product of $\langle 2,\, 3,\, 1\rangle$ with $\langle -4,\, 5,\, -2\rangle$?
18. What is the cross product of $\langle -4,\, 5,\, -2\rangle$ with $\langle 2,\, 3,\, 1\rangle$?

19. Compute the matrix product

$$\begin{bmatrix} 3 & 5 & 8 \\ 0 & 1 & 2 \\ 1 & 2 & -4 \\ -3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 & 2 \\ 3 & 0 & 4 & 1 \\ 0 & 1 & -4 & -3 \end{bmatrix}$$

20. Draw the (closed) polygon define by vertices
$(2,\ 4)$, $(4,\ 2)$, $(6,\ 4)$, $(8,\ 6)$, $(5,\ 7)$

21. Find the inward pointing normal to the edge from $(6, 4)$ to $(8,\ 6)$

22. Find the surface normal to the polygon.

23. What "polygon" do you get from the vertices
$(2,\ 4)$, $(5,\ 7)$, $(8,\ 6)$, $(6,\ 4)$, $(4,\ 2)$?

24. What is the surface normal to the polygon with vertices

$$(2,\ 4,\ -2),\ (4,\ 2,\ -3),\ (6,\ 4,\ 2),\ (8,\ 6,\ 0),\ (5,\ 7,\ 1)$$

(Note this is not a planar polygon, so I'm asking for an "approximate" normal.)