Quadson

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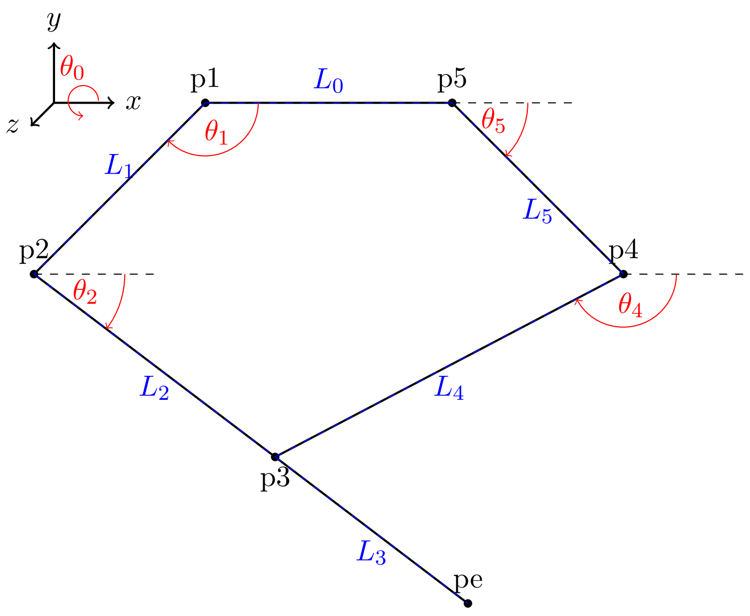
Spring 2025

Introduction

Quadson is a 3D printed quadruped robot designed as a platform for exploring advanced robotic motion and control. This document includes its forward kinematics, inverse kinematics, differential kinematics and expands to body kinematics and locomotion. Following these analysis, we will introduce the process of performing simultion in Pybullet and using deep reinforcement learning to improve the robot's control and performance.

Forward Kinematics

For each leg of the robot, we define the reference frame with its origin at point p_1. Two motors are positioned at p_1 and p_5 respectively, and the third motor connected to the entire leg assembly, controls the rotation around x axis. The remaining joints in the leg are all passive.



The following algorithm takes the angle of three motors as input and computes the position of the end effector. The process incorporates several safety check, prevent configurations that could potentially damage the mechanism.

Algorithm 1 Compute Leg End-Point

```
Require: Motor angles $\theta_0, \theta_1, \theta_5$
1: Compute initial 2D joint positions:
       $p_2 = (L_1 \cos\theta_1, -L_1 \sin\theta_1), p_5 = (L_0, 0),$
       $p_4 = p_5 + (L_5 \cos\theta_5, -L_5 \sin\theta_5)$
2: Compute distances:
       $L_{14} = \|p_4 - p_1\|, \quad L_{24} = \|p_4 - p_2\|$
3: Solve for $p_3$ using angles:
       $\theta_2 = \cos^{-1}\left(\frac{L_2^2 + L_{24}^2 - L_4^2}{2L_2L_{24}}\right) + \cos^{-1}\left(\frac{L_1^2 + L_{24}^2 - L_{14}^2}{2L_1L_{24}}\right) - (\pi - \theta_1)$
       $p_3 = p_2 + L_2(\cos\theta_2, -\sin\theta_2)$
4: Compute endpoint:
       $p_e = p_2 + (L_2 + L_3)(\cos\theta_2, -\sin\theta_2)$
5: Transform to 3D using rotation matrix:
       $R_{\theta_0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\theta_0) & -\sin(-\theta_0) \\ 0 & \sin(-\theta_0) & \cos(-\theta_0) \end{bmatrix}, p_e = R_{\theta_0} \times \begin{bmatrix} p_e[0] \\ p_e[1] \\ 0 \end{bmatrix}$
6: return $p_e$
```

Inverse Kinematics

In real world scenarios, controlling position of the end effector to reach to a certain position is often more pratical than setting the angles directly. For example, to move the end effector along a defined trajectory, achieving this by manually adjusting the three motor angles is nearly impossible due to the complexity of the joint movements.

Therefore we need the following algorithm to compute the angles of the motor corresponding to a given target position.

Algorithm 2 Compute Joint Angles from End-Point

```
Require: End-point position $(x, y, z)$
1: Calculate angle of motor 0:
       $$\theta_0 = -\tan^{-1}\left(\frac{z}{-y}\right)$$
2: Translate points from 3D to 2D using rotation matrix:
       $$R_{\theta_0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\theta_0) & -\sin(-\theta_0) \\ 0 & \sin(-\theta_0) & \cos(-\theta_0) \end{bmatrix}$$
       $$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R_{\theta_0} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
3: Calculate angle 1:
       $$L_{1e} = \|(x', y')\|, \quad \theta_{e15} = \tan^{-1}\left(\frac{-y'}{x'}\right)$$
       $$\theta_{e12} = \cos^{-1}\left(\frac{L_1^2 + L_{1e}^2 - (L_2 + L_3)^2}{2L_1L_{1e}}\right)$$
       $$\theta_1 = \theta_{e15} + \theta_{e12}$$
4: Compute positions of joints:
       $$p_1 = (0, 0), \quad p_2 = (L_1 \cos\theta_1, -L_1 \sin\theta_1)$$
       $$p_3 = \frac{(pe_2d \cdot L_2) + (p_2 \cdot L_3)}{L_2 + L_3}, \quad p_5 = (L_0, 0)$$
5: Calculate angle 5:
       $$\theta_{350} = \tan^{-1}\left(\frac{-p_3[1]}{L_0 - p_3[0]}\right)$$
       $$\theta_{354} = \cos^{-1}\left(\frac{L_{35}^2 + L_5^2 - L_4^2}{2L_{35}L_5}\right), \quad L_{35} = \|p_3 - p_5\|$$
       $$\theta_5 = \pi - (\theta_{350} + \theta_{354})$$
6: return $(\theta_0, \theta_1, \theta_5)$
```

Differential Kinematics

To control a quadruped robot's leg smoothly, we employ differential kinematics, which provides the relation between joint angular velocities and linear velocity of the end effector.

Given the motor angles $\boldsymbol{\theta} = [\theta_0, \theta_1, \theta_5]^T$ and their angular velocities $\dot{\boldsymbol{\theta}} = [\omega_0, \omega_1, \omega_5]^T$. The linear velocity of the end-effector $\mathbf{v} = [v_x, v_y, v_z]^T$ can be represented as following:

$$\mathbf{v} = \mathbf{J}(\boldsymbol{\theta}) \cdot \dot{\boldsymbol{\theta}}$$

Where $\mathbf{J}(\boldsymbol{\theta}) \in \mathbf{R}^{3\times3}$ is the Jacobian matrix. It is updated continuously as $\boldsymbol{\theta}$ changes during motion even if the given linear velocity remain the same. Note that angular velocity of the end-effector $(\omega_x, \omega_y, \omega_z)$ is ignored because the foot contact does not require rotational control in this context.

To find the inverse, the required motor angular velocities for a desired end-effector linear velocity:

$$\dot{\boldsymbol{\theta}} = \mathbf{J}^{+}(\boldsymbol{\theta}) \cdot \mathbf{v}$$

Where $\mathbf{J}^{+}$ is the Moore-Penrose pseudoinverse of $\mathbf{J}$.

Algorithm 3 Compute Joint Velocities from End-Effector Velocity

```
Require: End-effector velocity $\mathbf{v} = [v_x, v_y, v_z]^T$
1: Read current joint angles $\boldsymbol{\theta}$
2: Compute forward kinematics to get $\mathbf{p} = f(\boldsymbol{\theta})$
3: Compute numerical Jacobian $\mathbf{J} \in \mathbf{R}^{3\times3}$
4: Compute pseudoinverse $\mathbf{J}^{+}$
5: Compute $\dot{\boldsymbol{\theta}} = \mathbf{J}^{+} \cdot \mathbf{v}$
6: return $\dot{\boldsymbol{\theta}}$
```

Due to the difficulty in deriving $\mathbf{J}(\boldsymbol{\theta})$ analytically, we use the numerical finite difference method:

$$\mathbf{J}_{:,i} = \frac{f(\boldsymbol{\theta} + \delta \cdot \mathbf{e}_i) - f(\boldsymbol{\theta})}{\delta}$$

Where $f(\boldsymbol{\theta})$ is the forward kinematics function, δ is a small perturbation and $\mathbf{e}_i$ is a unit vector along the i-th motor axis.

Body Kinematics

To control the body pose of the robot when the legs are in contact with the ground, we take the positions of the end-effectors in the world frame and the orientation of the body as inputs. The goal is to transform the end-effector positions from the world frame to the shoulder frame, which is necessary for calculating the inverse kinematics of the legs.

The world frame, $\boldsymbol{W}$, is the global reference frame in which the robot operates. The body frame, $\boldsymbol{B}$, is defined as the frame attached to the robot's center, and the shoulder frame, $\boldsymbol{S}$, is defined as the local frame attached to each leg's joint 1.

Each coordinate transformation is represented using a 4x4 homogeneous transformation matrix, including a rotation matrix $\mathbf{R} \in \mathbf{R}^{3\times3}$ and a translation vector $\mathbf{p} \in \mathbf{R}^3$:

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

The steps to compute the foot positions in the shoulder frame are as follows:

Algorithm 4 Compute Foot Position in Shoulder Frame

```
Require: End effector positions $p_i^w$, body orientation $(roll, pitch, yaw)$
1: Compute $T_b$, transformation from body frame to world frame
2: for each leg $i$ do
3:     Get shoulder-to-body transform $T_{s_i}^b$ (predefined by geometry)
4:     Compute shoulder-to-world transform, $T_{s_i} = T_b \cdot T_{s_i}^b$
5:     Compute world-to-shoulder transform, $T_w^{s_i} = T_{s_i}^{-1}$
6:     Project world foot point into shoulder frame:
               $p_i^s = T_w^{s_i} p_i^w$
7: end for
8: return Foot positions $p_i^s$ in shoulder frames
```

Locomotion

Locomotion control is crucial for quadruped robots to enable them to move efficiently. The motion of the robot depends on the type of gait used, and different gaits have different characteristics such as step height, length, and timing.

The following first algorithm computes the phases of each leg based on time. Once the phase of each leg is determined, the next step is to calculate the end-effector positions for each leg. The second algorithm computes corresponding end-effector points. The position is calculated differently for the stance and swing phases of each gait cycle.

Algorithm 5 Locomotion Phase Calculation

```
Require: Gait type: gait_type ('walk', 'trot', 'pace', 'bound', or 'gallop')
1: Calculate the current phase of each leg based on the current time:
       $$\text{cycle\_progress} = \frac{\text{time\%cycle\_time}}{\text{cycle\_time}}$$
2: for each leg in the robot's leg configuration do
3:     Calculate phase based on cycle progress:
           $$\text{phase\_dict[leg\_name]} = (\text{cycle\_progress} + \text{phase\_offset})\%1.0$$
4: end for
5: return phase_dict
```

Algorithm 6 End-Effector Points Calculation

```
Require: Current time: time
1: Get current phase for each leg using get_current_phase(time)
2: for each leg in the robot's leg configuration do
3:     if phase of the leg is in the stance phase (i.e., less than duty_factor) then
4:         Interpolate between the stance start and end points:
               $$\text{p\_start} = (start\_x, base\_y),$$
               $$\text{p\_end} = (start\_x + \text{direction} \times \text{step\_length}, base\_y)$$
               $$p = \text{p\_start} + (\text{p\_end} - \text{p\_start}) \times \frac{\text{phase}}{\text{duty\_factor}}$$
5:         Set x, y coordinates based on interpolation
6:     else
7:         Interpolate using cubic Bezier curve for swing phase:
               $$p_0 = (start\_x + direction \times \text{step\_length}, base\_y)$$
               $$p_1 = (start\_x + direction \times \text{step\_length}/2, base\_y + \text{step\_height})$$
               $$p_2 = (start\_x, base\_y + \text{step\_height}/2), \quad p_3 = (start\_x, base\_y)$$
               $$(x, y) = \text{cubic\_bezier}(t, p_0, p_1, p_2, p_3)$$
8:     end if
9:     Set ee_points[leg_name] to the calculated coordinates
10: end for
11: return end-effector points: ee_points
```

The cubic Bezier curve is used for the swing phase to smoothly interpolate the leg's motion. The following formula is used to calculate the position on the cubic Bezier curve.

Algorithm 7 Cubic Bezier Curve

```
Require: Interpolation factor: t, Control points: p0, p1, p2, p3
1: Define the control points as a matrix:
       $\text{points} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$
2: Define the Bernstein coefficients for the cubic curve:
       $\text{coeffs} = \begin{bmatrix} (1-t)^3 & 3(1-t)^2 t & 3(1-t)t^2 & t^3 \end{bmatrix}$
3: Calculate the position on the curve:
       $(x, y) = \text{coeffs} \times \text{points}$
4: return position (x, y)
```

Simulation

Deep Reinforcement Learning

Conclusion