Van Allen Belt Particle Simulations - Documentation

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December 25, 2022

Contents

1 Modelling Earth's Magnetic Field

1.1 Dipole Field

Due to the complex nature of Earth's true magnetic field, we may model it to first approximation as a dipole field [1].

The components of this dipole field in spherical coordinates (in the physics convention) are given by [1]:

$$B_r = -2B_0 \left(\frac{R_E}{r}\right)^3 \cos(\theta) \tag{1.1}$$

$$B_\theta = -2B_0 \left(\frac{R_E}{r}\right)^3 \sin(\theta) \tag{1.2}$$

$$B_\phi = 0 \tag{1.3}$$

where R_E is the radius of Earth and B_0 is the mean value of the field on the equator at the Earth's surface. This has been experimentally determined to be $B_0 = 3.12 \times 10^{-5}$ T [1].

Note that $B_\phi = 0$. This is because the dipole field is symmetric about its axis.

Then, in unit vector notation:

$$\begin{aligned}
\mathbf{B} &= -2B_0 \left(\frac{R_E}{r}\right)^3 \cos(\theta)\hat{\mathbf{r}} - B_0 \left(\frac{R_E}{r}\right)^3 \sin(\theta)\hat{\boldsymbol{\theta}} + 0\hat{\boldsymbol{\phi}} && (1.4)\\
&= -2B_0 \left(\frac{R_E}{r}\right)^3 \cos(\theta)\hat{\mathbf{r}} - B_0 \left(\frac{R_E}{r}\right)^3 \sin(\theta)\hat{\boldsymbol{\theta}} && (1.5)
\end{aligned}$$

1.2 Cartesian Coordinates

Recall that:

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \sin(\theta)\cos(\phi) & \cos(\theta)\cos(\phi) & -\sin(\phi) \\ \sin(\theta)\sin(\phi) & \cos(\theta)\sin(\phi) & \cos(\phi) \\ \cos(\theta) & -\sin(\theta) & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix} \tag{1.6}$$

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix} = \begin{pmatrix} \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \\ \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} \tag{1.7}$$

And:

$$\begin{cases} x &= r\sin(\theta)\cos(\phi) \\ y &= r\sin(\theta)\sin(\phi) \\ z &= r\cos(\theta) \end{cases} \qquad \begin{cases} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \end{cases} \tag{1.8}$$

Manipulating:

$$
\begin{aligned}
\mathbf{B} &= -2B_0\left(\frac{R_E}{r}\right)^3\cos(\theta)\hat{\mathbf{r}} - B_0\left(\frac{R_E}{r}\right)^3\sin(\theta)\hat{\boldsymbol{\theta}} + 0\hat{\boldsymbol{\phi}} \\
&= -2B_0\left(\frac{R_E}{r}\right)^3\cos(\theta)\left(\sin(\theta)\cos(\phi)\hat{\mathbf{x}} + \sin(\theta)\sin(\phi)\hat{\mathbf{y}} + \cos(\theta)\hat{\mathbf{z}}\right) \\
&\quad - B_0\left(\frac{R_E}{r}\right)^3\sin(\theta)\left(\cos(\theta)\cos(\phi)\hat{\mathbf{x}} + \cos(\theta)\sin(\phi)\hat{\mathbf{y}} - \sin(\theta)\hat{\mathbf{z}}\right) \\
&= -2B_0\left(\frac{R_E}{r}\right)^3\cos(\theta)\sin(\theta)\cos(\phi)\hat{\mathbf{x}} - 2B_0\left(\frac{R_E}{r}\right)^3\cos(\theta)\sin(\theta)\sin(\phi)\hat{\mathbf{y}} - 2B_0\left(\frac{R_E}{r}\right)^3\cos^2(\theta)\hat{\mathbf{z}} \\
&\quad - B_0\left(\frac{R_E}{r}\right)^3\sin(\theta)\cos(\theta)\cos(\phi)\hat{\mathbf{x}} - B_0\left(\frac{R_E}{r}\right)^3\sin(\theta)\cos(\theta)\sin(\phi)\hat{\mathbf{y}} + B_0\left(\frac{R_E}{r}\right)^3\sin^2(\theta)\hat{\mathbf{z}} \\
&= -3B_0\left(\frac{R_E}{r}\right)^3\cos(\theta)\sin(\theta)\cos(\phi)\hat{\mathbf{x}} - 3B_0\left(\frac{R_E}{r}\right)^3\cos(\theta)\sin(\theta)\sin(\phi)\hat{\mathbf{y}} \\
&\quad + B_0\left(\frac{R_E}{r}\right)^3(\sin^2(\theta) - 2\cos^2(\theta))\hat{\mathbf{z}}
\end{aligned}
$$

To replace the angles, we apply some trigonometric identities:

$$
\begin{aligned}
\cos(\theta) &= \cos\left(\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)\right) \\
&= \frac{1}{\sqrt{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}} \\
&= \frac{1}{\sqrt{1+\frac{x^2+y^2}{z^2}}} \\
&= \frac{z}{\sqrt{x^2+y^2+z^2}} \\
&= \frac{z}{r}
\end{aligned}
\qquad
\begin{aligned}
\sin(\theta) &= \sin\left(\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)\right) \\
&= \frac{\frac{\sqrt{x^2+y^2}}{z}}{\sqrt{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}} \\
&= \frac{\sqrt{x^2+y^2}}{z\sqrt{1+\frac{x^2+y^2}{z^2}}} \\
&= \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \\
&= \frac{\sqrt{x^2+y^2}}{r}
\end{aligned}
$$

$$
\begin{aligned}
\cos(\phi) &= \cos\left(\tan^{-1}\left(\frac{y}{x}\right)\right) \\
&= \frac{1}{\sqrt{1+\left(\frac{y}{x}\right)^2}} \\
&= \frac{1}{\sqrt{1+\frac{y^2}{x^2}}} \\
&= \frac{x}{\sqrt{x^2+y^2}}
\end{aligned}
\qquad
\begin{aligned}
\sin(\phi) &= \sin\left(\tan^{-1}\left(\frac{y}{x}\right)\right) \\
&= \frac{\frac{y}{x}}{\sqrt{1+\left(\frac{y}{x}\right)^2}} \\
&= \frac{y}{x\sqrt{1+\frac{y^2}{x^2}}} \\
&= \frac{y}{\sqrt{x^2+y^2}}
\end{aligned}
$$

Then:

$$\mathbf{B} = -3B_0 \left(\frac{R_E}{r}\right)^3 \cos(\theta)\sin(\theta)\cos(\phi)\hat{\mathbf{x}} - 3B_0 \left(\frac{R_E}{r}\right)^3 \cos(\theta)\sin(\theta)\sin(\phi)\hat{\mathbf{y}} \tag{1.9}$$

$$+ B_0 \left(\frac{R_E}{r}\right)^3 (\sin^2(\theta) - 2\cos^2(\theta))\hat{\mathbf{z}} \tag{1.10}$$

$$= -3B_0 \left(\frac{R_E}{r}\right)^3 \frac{z}{r}\frac{\sqrt{x^2+y^2}}{r}\frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{x}} - 3B_0 \left(\frac{R_E}{r}\right)^3 \frac{z}{r}\frac{\sqrt{x^2+y^2}}{r}\frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{y}} \tag{1.11}$$

$$+ B_0 \left(\frac{R_E}{r}\right)^3 \left(\frac{x^2+y^2}{r^2} - 2\frac{z^2}{r^2}\right)\hat{\mathbf{z}} \tag{1.12}$$

$$= -3B_0 \left(\frac{R_E}{r}\right)^3 \left(\frac{xz}{r^2}\right)\hat{\mathbf{x}} - 3B_0 \left(\frac{R_E}{r}\right)^3 \left(\frac{yz}{r^2}\right)\hat{\mathbf{y}} + B_0 \left(\frac{R_E}{r}\right)^3 \left(\frac{x^2+y^2-2z^2}{r^2}\right)\hat{\mathbf{z}} \tag{1.13}$$

$$= -3B_0 \left(\frac{R_E}{r}\right)^3 \left(\frac{xz}{r^2}\right)\hat{\mathbf{x}} - 3B_0 \left(\frac{R_E}{r}\right)^3 \left(\frac{yz}{r^2}\right)\hat{\mathbf{y}} + B_0 \left(\frac{R_E}{r}\right)^3 \left(\frac{x^2+y^2+z^2-3z^2}{r^2}\right)\hat{\mathbf{z}} \tag{1.14}$$

$$= -3B_0 \left(\frac{R_E}{r}\right)^3 \left(\frac{xz}{r^2}\right)\hat{\mathbf{x}} - 3B_0 \left(\frac{R_E}{r}\right)^3 \left(\frac{yz}{r^2}\right)\hat{\mathbf{y}} + B_0 \left(\frac{R_E}{r}\right)^3 \left(\frac{r^2-3z^2}{r^2}\right)\hat{\mathbf{z}} \tag{1.15}$$

$$= -3B_0 \left(\frac{R_E}{r}\right)^3 \left(\frac{xz}{r^2}\right)\hat{\mathbf{x}} - 3B_0 \left(\frac{R_E}{r}\right)^3 \left(\frac{yz}{r^2}\right)\hat{\mathbf{y}} - 3B_0 \left(\frac{R_E}{r}\right)^3 \left(\frac{z^2}{r^2} - \frac{1}{3}\right)\hat{\mathbf{z}} \tag{1.16}$$

2 Equations of Motion

2.1 Lorentz Force

Recall that the Lorentz force acting on a charge q is given by:

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{2.1}$$

For simplicity, we assume that $\mathbf{E} = \mathbf{0}$. Although electric fields play an important role in the dynamics of plasmas, we are mainly interested in the dynamics of charged particles in the Van Allen radiation belts, where the magnetic contribution dominates [2].

Then the Lorentz force reduces to:

$$\mathbf{F} = q\left(\mathbf{v} \times \mathbf{B}\right) \tag{2.2}$$

2.2 Special Relativity

Because particles in the Van Allen belts frequently attain relativistic speeds, we must take special relativity into account.

Recall that the relativistic momentum of a particle with mass m and velocity $\mathbf{v}$ is given by:

$$\begin{aligned} \mathbf{p} &= \gamma m \mathbf{v} && (2.3) \\ &= \frac{m\mathbf{v}}{\sqrt{1-\frac{|\mathbf{v}|^2}{c^2}}} && (2.4) \end{aligned}$$

where γ is the Lorentz factor and c is the speed of light.

Then our force equation becomes:

$$\begin{aligned} \mathbf{F} &= q\left(\mathbf{v}\times\mathbf{B}\right) && (2.5) \\ \frac{d\mathbf{p}}{dt} &= q\left(\mathbf{v}\times\mathbf{B}\right) && (2.6) \\ \frac{d(\gamma m\mathbf{v})}{dt} &= q\left(\mathbf{v}\times\mathbf{B}\right) && (2.7) \end{aligned}$$

For simplicity, we assume that particle's speed $|\mathbf{v}|$ is constant. This is due to the cross product: the particle's acceleration vector will always be perpendicular to its velocity vector [2]. It follows then that the Lorentz factor γ is also constant.

Then:

$$\begin{aligned} \frac{d(\gamma m\mathbf{v})}{dt} &= q\left(\mathbf{v}\times\mathbf{B}\right) && (2.8) \\ \gamma m\frac{d\mathbf{v}}{dt} &= q\left(\mathbf{v}\times\mathbf{B}\right) && (2.9) \\ \gamma m\mathbf{a} &= q\left(\mathbf{v}\times\mathbf{B}\right) && (2.10) \\ \mathbf{a} &= \frac{q}{\gamma m}\left(\mathbf{v}\times\mathbf{B}\right) && (2.11) \end{aligned}$$

2.3 Gyration

As an aside, the gyration radius R can be derived by equating the magnetic force $\mathbf{F}_B$ to the centripetal force $\mathbf{F}_c$:

$$\begin{aligned} |\mathbf{F}_B| &= |\mathbf{F}_c| && (2.12) \\ qv_\perp B &= \frac{mv_\perp^2}{R} && (2.13) \\ qB &= \frac{mv_\perp}{R} && (2.14) \\ R &= \frac{mv_\perp}{qB} && (2.15) \end{aligned}$$

where $v_\perp$ is the component of the particle's velocity that is perpendicular to the field vector $\mathbf{B}$.

The gyration frequency f can be derived using the gyration radius:

$$
\begin{aligned}
f &= \frac{1}{T} && (2.16)\\
&= \frac{v_\perp}{2\pi R} && (2.17)\\
&= \frac{q v_\perp B}{2\pi m v_\perp} && (2.18)\\
&= \frac{qB}{2\pi m} && (2.19)
\end{aligned}
$$

Their relativistic counterparts are given by [2]:

$$
R = \frac{\gamma m v_\perp}{qB} \qquad\qquad f = \frac{qB}{2\pi\gamma m} \tag{2.20}
$$

3 Numerical Integration

We first break our equation of motion into components:

$$
\begin{aligned}
\ddot{x} &= \frac{q}{\gamma m}\left(\dot{y}B_z - \dot{z}B_y\right) && (3.1)\\
\ddot{y} &= \frac{q}{\gamma m}\left(\dot{z}B_x - \dot{x}B_z\right) && (3.2)\\
\ddot{z} &= \frac{q}{\gamma m}\left(\dot{x}B_y - \dot{y}B_x\right) && (3.3)
\end{aligned}
$$

With inspiration from García-Farieta and Hurtado [3], we then apply the Runge-Kutta fourth-order method to solve this system of equations. Its implementation can be viewed here.

References

[1] M. Walt. *Introduction to Geomagnetically Trapped Radiation*. Cambridge Atmospheric and Space Science Series. Cambridge University Press. ISBN: 9780521431439.

[2] M. Kaan Öztürk. "Trajectories of charged particles trapped in Earth's magnetic field". In: *American Journal of Physics* 80.5 (May 2012), pp. 420–428. DOI: 10.1119/1.3684537.

[3] Jorge García-Farieta and Alejandro Hurtado. "Simulation of charged particles in Earth's magnetosphere: an approach to the Van Allen belts". In: *Revista Mexicana de Física E* 65 (Jan. 2019), pp. 64–70. DOI: 10.31349/RevMexFisE.65.64.