

ON THE PRO-ÉTALE COHOMOLOGY OF QUOTIENT STACKS OF DRINFELD SPACES

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ABSTRACT. Let \mathcal{H}_K^{n-1} denote the $(n-1)$ -dimensional Drinfeld space over a p -adic field K . We give an explicit description of the ℓ -adic and p -adic pro-étale cohomology of quotient stacks $[\mathcal{H}_K^{n-1} / \mathrm{GL}_n(\mathcal{O}_K)]$ and $[\mathcal{H}_K^{n-1} / \mathrm{GL}_n(K)]$, which are moduli stacks of special formal \mathcal{O}_D -modules. The computation makes use of the isomorphism between the Lubin-Tate tower and the Drinfeld tower due to Faltings and Scholze–Weinstein, as well as the p -adic pro-étale cohomology of the Drinfeld spaces computed by Colmez–Dospinescu–Nizioł.

As an application, we also compute the continuous group cohomology of $\mathrm{GL}_n(\mathbb{Q}_p)$ over duals of generalized Steinberg representations over \mathbb{Q}_p .

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1. INTRODUCTION

Let K be a finite extension of \mathbb{Q}_p and let C be the completion of an algebraic closure of K . The cohomology of Drinfeld spaces was first studied by Schneider–Stuhler [SS91] for any cohomology theory satisfying certain axioms. Examples of such cohomology theories include de Rham cohomology and ℓ -adic ($\ell \neq p$) étale cohomology.

The $(n-1)$ -dimensional Drinfeld space \mathcal{H}_K^{n-1} admits an action of $\mathrm{GL}_n(K)$ and carries an interesting moduli interpretation. Specifically, if we consider the quotient stack $[\mathcal{H}_K^{n-1} / \mathrm{GL}_n(\mathcal{O}_K)]$ associated to \mathcal{H}_K^{n-1} , we get the moduli stack of isomorphism classes of special formal \mathcal{O}_D -modules (as defined in [Dri76]) of height n^2 ; similarly, the quotient stack $[\mathcal{H}_K^{n-1} / \mathrm{GL}_n(K)]$ gives rise to the moduli stack of height n^2 special formal \mathcal{O}_D -modules up to quasi-isogeny.

In this article, we compute the ℓ -adic and p -adic pro-étale cohomology of these two moduli stacks. For the ℓ -adic case, since the ℓ -adic pro-étale cohomology satisfies the axioms listed in [SS91], we have

$$(1.1) \quad H_{\mathrm{pro\acute{e}t}}^r(\mathcal{H}_C^{n-1}, \mathbb{Q}_\ell(r)) \cong \mathrm{Sp}_r(\mathbb{Q}_\ell)^*$$

for $0 \leq r \leq n-1$ where $\mathrm{Sp}_r(\mathbb{Q}_\ell)^*$ is the \mathbb{Q}_ℓ -dual of the generalized Steinberg representation $\mathrm{Sp}_r(\mathbb{Q}_\ell)$ of $\mathrm{GL}_n(K)$. By computing continuous group cohomology of $\mathrm{GL}_n(\mathcal{O}_K)$ and $\mathrm{GL}_n(K)$ over $\mathrm{Sp}_r(\mathbb{Q}_\ell)^*$, we get:

Theorem 1 (Theorem 4.4 & Theorem 5.2).

- (1) The cohomology group $H_{\mathrm{pro\acute{e}t}}^r([\mathcal{H}_K^{n-1} / \mathrm{GL}_n(\mathcal{O}_K)], \mathbb{Q}_\ell)$ is isomorphic to \mathbb{Q}_ℓ when $r = 0$ or 1 , and vanishes in all other degrees.
- (2) The cohomology group $H_{\mathrm{pro\acute{e}t}}^r([\mathcal{H}_K^{n-1} / \mathrm{GL}_n(K)], \mathbb{Q}_\ell)$ is isomorphic to \mathbb{Q}_ℓ when $r = 0$ or 2 , isomorphic to \mathbb{Q}_ℓ^2 when $r = 1$, and vanishes in all other degrees.

Our strategy to show Theorem 1 is fairly straightforward. First we descend (1.1) down to the field K . The proof then rests on the aforementioned results of group cohomology together with the use of the Hochschild-Serre spectral sequence for pro-étale cohomology.

As for the p -adic case, the computation is more involved. In [CDN20], Colmez–Dospinescu–Nizioł described the p -adic pro-étale cohomology of \mathcal{H}_C^{n-1} using a short exact sequence

$$0 \rightarrow \Omega^{r-1}(\mathcal{H}_C^{n-1}) / \ker d \rightarrow H_{\text{proét}}^r(\mathcal{H}_C^{n-1}, \mathbb{Q}_p(r)) \rightarrow \text{Sp}_r(\mathbb{Q}_p)^* \rightarrow 0.$$

It is then very tempting to mimic what we did in the ℓ -adic case. However, although the differential part $\Omega^{r-1}(\mathcal{H}_C^{n-1}) / \ker d$ wouldn't survive Galois descent, it is still difficult to compute the group cohomology $H_{\text{cts}}^*(\text{GL}_n(K), \text{Sp}_r(\mathbb{Q}_p)^*)$ and to control the spectral sequence converging to $H_{\text{proét}}^*([\mathcal{H}_K^{n-1} / \text{GL}_n(K)], \mathbb{Q}_p)$.

To salvage this, we use the isomorphism between the Drinfeld tower and the Lubin-Tate tower from [SW13]. Specifically, let \check{K} denote the completion of the maximal unramified extension of K , [SW13, Theorem E] allows us to trade the action of $\text{GL}_n(\mathcal{O}_K)$ on \mathcal{H}_K^{n-1} (resp. $\text{GL}_n(K)$ on \mathcal{H}_K^{n-1}) for the action of the Morava stabilizer group \mathbb{G}_n on the Lubin-Tate space $\text{LT}_{n,\check{K}}$ (resp. $\mathbb{G}_n^0 := \mathbb{G}_n \times \mathbb{Z}$ on $\mathbb{P}_{\check{K}}^{n-1}$), and gives us two isomorphisms of stacks:

$$[\mathcal{H}_K^{n-1} / \text{GL}_n(\mathcal{O}_K)] \cong [\text{LT}_{n,\check{K}} / \mathbb{G}_n] \quad \text{and} \quad [\mathcal{H}_K^{n-1} / \text{GL}_n(K)] \cong [\mathbb{P}_{\check{K}}^{n-1} / \mathbb{G}_n^0].$$

By transferring to the Lubin-Tate side, our problem is reduced to computing the continuous group cohomology of \mathbb{G}_n over the pro-étale cohomology of $\text{LT}_{n,\check{K}}$. Let $\Lambda_{\mathbb{Q}_p}(x_1, x_3, \dots, x_{2n-1})$ denote the exterior algebra over \mathbb{Q}_p generated by x_i in degree i and let $\Lambda_{\mathbb{Q}_p}(y)$ denote the exterior algebra generated by a single element y in degree 1. Our main result in the p -adic case is the following:

Theorem 2 (Theorem 4.12 & Theorem 5.4). *Suppose K/\mathbb{Q}_p is a finite extension of degree d .*

(1) *There is an isomorphism of graded \mathbb{Q}_p -vector spaces*

$$H_{\text{proét}}^*([\mathcal{H}_K^{n-1} / \text{GL}_n(\mathcal{O}_K)], \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_3, \dots, x_{2n-1})^{\otimes d} \otimes H_{\text{cts}}^*(\text{Gal}(\bar{K}/K), \mathbb{Q}_p).$$

(2) *There is an isomorphism of graded \mathbb{Q}_p -vector spaces*

$$H_{\text{proét}}^*([\mathcal{H}_K^{n-1} / \text{GL}_n(K)], \mathbb{Q}_p) \cong H_{\text{proét}}^*(\mathbb{P}_K^{n-1}, \mathbb{Q}_p) \otimes \Lambda_{\mathbb{Q}_p}(x_1, x_3, \dots, x_{2n-1})^{\otimes d} \otimes \Lambda_{\mathbb{Q}_p}(y),$$

where

$$H_{\text{proét}}^*(\mathbb{P}_K^{n-1}, \mathbb{Q}_p) \cong \begin{cases} \mathbb{Q}_p & \text{if } * = 0 \\ \mathbb{Q}_p^{d+1} & \text{if } * = 1 \\ \mathbb{Q}_p^d & \text{if } 3 \leq * \leq 2n-1 \text{ and } * \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Our computations conducted on the Lubin-Tate side also have some applications back to the Drinfeld side. Using Theorem 2, we show (Theorem 6.5) that there is an isomorphism $H_{\text{cts}}^i(\text{GL}_2(\mathbb{Q}_p), \text{Sp}_1(\mathbb{Q}_p)^*) \cong H_{\text{cts}}^{i-1}(\text{GL}_2(\mathbb{Q}_p), \mathbb{Q}_p)$ for every $i \geq 0$. Furthermore, Theorem 2 suggests that the cohomology groups of $\text{GL}_n(\mathbb{Q}_p)$ over the dual Steinberg representation $\text{Sp}_r(\mathbb{Q}_p)^*$ is a degree- r shift of that over \mathbb{Q}_p . Consider the exterior algebra $\Lambda_{\mathbb{Q}_p}(x, y, x_3, \dots, x_{2n-1})$ over \mathbb{Q}_p with degrees of generators given by $|x| = |y| = 1$ and $|x_i| = i$. We can then identify $H_{\text{cts}}^*(\text{GL}_n(\mathbb{Q}_p), \mathbb{Q}_p)$ with this exterior algebra (cf. Proposition 6.2). Using techniques from [Orl05] for computing Ext-groups between Steinberg representations and also the machinery of solid representations from [RJRC22] and [RJRC25], we show:

Theorem 3 (Theorem 7.1). *There is an isomorphism of graded \mathbb{Q}_p -vector spaces*

$$H_{\text{cts}}^*(\text{GL}_n(\mathbb{Q}_p), \text{Sp}_r(\mathbb{Q}_p)^*) \cong \Lambda_{\mathbb{Q}_p}(x, y, x_3, \dots, x_{2n-1})[-r]$$

for every $0 \leq r \leq n-1$.

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Notations. We use the following notations throughout the paper, unless otherwise stated.

We denote by K a finite extension of \mathbb{Q}_p , with ring of integers \mathcal{O}_K , uniformizer π , and residue field k . We let $W = W(\bar{k})$ be the ring of Witt vectors of \bar{k} , and let $\tilde{K} = W[\frac{1}{p}]$ be the completion of the maximal unramified extension of K . We let C be the completion of an algebraic closure \bar{K} of K , and the Galois group $\text{Gal}(\bar{K}/K)$ will be denoted by \mathcal{G}_K . We use ℓ to denote a prime different from p .

2. TWO PAIRS OF MODULI STACKS

Following [RZ96, Chapter 3] and [Far08, Chapitre II], we recall the definitions of Drinfeld tower and Lubin-Tate tower at infinite level, their related group actions, and also the moduli interpretation of associated quotient stacks. We will also state the isomorphism due to Scholze–Weinstein between the two towers and derive two pairs of isomorphic moduli stacks.

2.1. Moduli stacks arising from the Drinfeld tower. For $n \geq 2$, the $(n-1)$ -dimensional Drinfeld symmetric space over K is defined as

$$\mathcal{H}_K^{n-1} := \mathbb{P}_K^{n-1} \setminus \bigcup_{H \in \mathfrak{H}} H$$

where \mathfrak{H} is the set of K -rational hyperplanes inside \mathbb{P}_K^{n-1} . The space \mathcal{H}_K^{n-1} is a rigid analytic Stein space and has a natural action of $\text{GL}_n(K)$ on it. Let D be the unique (up to isomorphism) division algebra over K of invariant $1/n$, let \mathcal{O}_D be its ring of integers, and denote by ϖ the uniformizer of D . As defined in [Dri76], the level-0 Drinfeld moduli space $\mathcal{M}_0^{\text{Dr}}$ arises as the rigid fiber of certain deformation space of special formal \mathcal{O}_D -modules of dimension n and height n^2 . Moreover, Drinfeld also proved in [Dri76] that there is a $\text{GL}_n(K)$ -equivariant isomorphism $\mathcal{M}_0^{\text{Dr}} \cong \coprod_{\mathbb{Z}} \mathcal{H}_K^{n-1}$. The space $\mathcal{M}_0^{\text{Dr}}$ admits a tower of finite étale $\text{GL}_n(K)$ -coverings

$$\cdots \rightarrow \mathcal{M}_2^{\text{Dr}} \rightarrow \mathcal{M}_1^{\text{Dr}} \rightarrow \mathcal{M}_0^{\text{Dr}} \cong \coprod_{\mathbb{Z}} \mathcal{H}_K^{n-1},$$

which is called the *Drinfeld tower*.

Fix G_0 to be a special formal \mathcal{O}_D -module over k of dimension n and height n^2 .

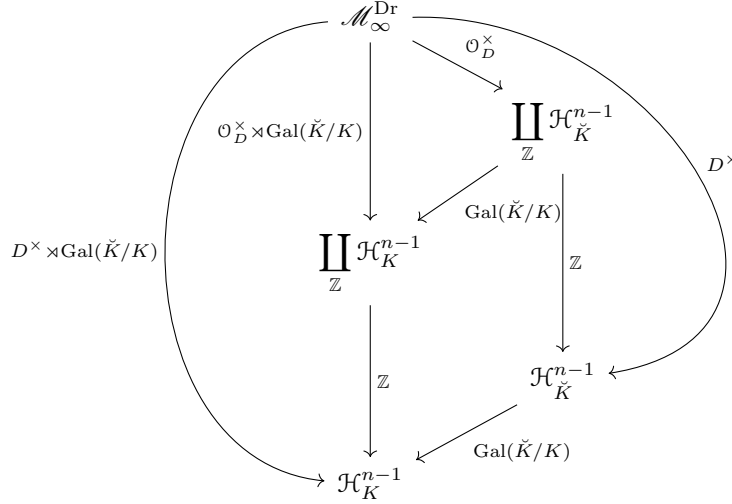
Definition 2.1. Let (R, R^+) be a complete affinoid (K, \mathcal{O}_K) -algebra. The *Drinfeld tower at infinite level* is a functor $\mathcal{M}_\infty^{\text{Dr}}$ on complete affinoid (K, \mathcal{O}_K) -algebras whose set of (R, R^+) -points is the set of quadruples (G, ι, ρ, η) up to isomorphism, where

- G is a special formal \mathcal{O}_D -module over R^+ ,
- $\iota : W \rightarrow R^+$ is a ring homomorphism,
- $\rho : G_0 \otimes_{W, \iota} R^+ / pR^+ \rightarrow G \otimes_{R^+} R^+ / pR^+$ is an \mathcal{O}_D -equivariant quasi-isogeny,
- $\eta : \mathcal{O}_D \xrightarrow{\sim} T_p(G) = \varprojlim G[p^n]$ is an isomorphism of \mathcal{O}_D -modules.

Two quadruples (G, ι, ρ, η) and $(G', \iota', \rho', \eta')$ are isomorphic if there is an isomorphism $f : G \xrightarrow{\sim} G'$ transferring one set of data to the other.

Remark 2.2. Here we have a slight caveat that a complete affinoid algebra (R, R^+) might not be sheafy, in which case its adic spectrum is not an adic space in the sense of Huber. However, as discussed in [SW13, Section 2.1], the category of complete affinoid algebras has the sheafy ones as a full subcategory, and the functor $(R, R^+) \mapsto \text{Spa}(R, R^+)$ is fully faithful when restricted to this subcategory. One could also restrict to pairs (R, R^+) where R is perfectoid, which guarantees the sheafiness and leads to the "diamond" version of the moduli space. See Remark 2.8 for more on this.

The space $\mathcal{M}_\infty^{\text{Dr}}$ bears an action of $\text{GL}_n(K) \times D^\times$. The group $\text{GL}_n(K)$ comes from the automorphism group $\text{Aut}^0(G_0)$ of G_0 in the isogeny category, and $g \in \text{GL}_n(K)$ acts on $\mathcal{M}_\infty^{\text{Dr}}$ by sending ρ to $\rho \circ g^{-1}$. An element $d \in \mathcal{O}_D^\times$ acts on $\mathcal{M}_\infty^{\text{Dr}}$ by sending η to $\eta \circ d^{-1}$. Let ϕ denote the quotient map $G \rightarrow G/G[\varpi]$. The action of $\varpi \in D^\times$ is defined by $\varpi \cdot (G, \iota, \rho, \eta) = (G/G[\varpi], \iota, \phi \circ \rho, \eta \circ \varpi)$. From the space $\mathcal{M}_\infty^{\text{Dr}}$, quotienting by \mathcal{O}_D^\times gives $\mathcal{M}_0^{\text{Dr}}$ and quotienting by D^\times places us at the level of \mathcal{H}_K^{n-1} . Let $\varphi \in \text{Gal}(\check{K}/K)$ be the lift of Frobenius, then φ acts on $\mathcal{M}_\infty^{\text{Dr}}$ by sending ι to $\iota \circ \varphi$. In the mean time, φ also induces an isomorphism $f_\varphi : G_0 \otimes_{W, \iota \circ \varphi} R^+/pR^+ \xrightarrow{\sim} G_0 \otimes_{W, \iota} R^+/pR^+$ and a quasi-isogeny $\rho_\varphi := \rho \circ f_\varphi : G_0 \otimes_{W, \iota \circ \varphi} R^+/pR^+ \rightarrow H \otimes R^+/pR^+$. The map $(G, \iota \circ \varphi, \rho, \eta) \mapsto (G, \iota, \rho_\varphi, \eta)$ then defines an isomorphism $\varphi^* \mathcal{M}_\infty^{\text{Dr}} \xrightarrow{\sim} \mathcal{M}_\infty^{\text{Dr}}$ and gives the functor $\mathcal{M}_\infty^{\text{Dr}}$ a Weil descent datum. To help visualize how these spaces in the Drinfeld tower relate to one another, we have the following diagram:



Building on the moduli interpretation of $\mathcal{M}_\infty^{\text{Dr}}$, we can similarly regard \mathcal{H}_K^{n-1} as a moduli space. Furthermore, since $\text{GL}_n(K)$ acts on \mathcal{H}_K^{n-1} , taking quotients by its subgroups naturally yields some interesting moduli stacks. Among these, we are particularly interested in the quotient stacks $[\mathcal{H}_K^{n-1}/\text{GL}_n(\mathcal{O}_K)]$ and $[\mathcal{H}_K^{n-1}/\text{GL}_n(K)]$. We will explain their moduli interpretation in the following.

We define a functor \mathcal{G}^+ from the category of affinoid adic spaces over $\text{Spa}(K, \mathcal{O}_K)$ to the category **Grpd** of groupoids by assigning to each affinoid adic space $\text{Spa}(R, R^+)$ the groupoid $\mathcal{G}^+(\text{Spa}(R, R^+))$ where:

- Objects in $\mathcal{G}^+(\text{Spa}(R, R^+))$ are special formal \mathcal{O}_D -modules over R^+ that are quasi-isogenous to the fixed module G_0 over R^+/pR^+ .
- Morphisms in $\mathcal{G}(\text{Spa}(R, R^+))$ are isomorphisms between such special formal \mathcal{O}_D -modules.

Similarly, we define another functor $\mathcal{G}^{0,+}$ from the category of affinoid adic spaces over $\text{Spa}(K, \mathcal{O}_K)$ to **Grpd** by assigning to each affinoid adic space $\text{Spa}(R, R^+)$ the groupoid $\mathcal{G}^{0,+}(\text{Spa}(R, R^+))$ where:

- Objects in $\mathcal{G}^{0,+}(\text{Spa}(R, R^+))$ are special formal \mathcal{O}_D -modules over R^+ which are quasi-isogenous to G_0 over R^+/pR^+ .
- Morphisms in $\mathcal{G}^{0,+}(\text{Spa}(R, R^+))$ are quasi-isogenies between such special formal \mathcal{O}_D -modules.

We let \mathcal{G} be the sheafification of \mathcal{G}^+ and let \mathcal{G}^0 be the sheafification of $\mathcal{G}^{0,+}$, both with respect to the pro-étale topology. So \mathcal{G} and \mathcal{G}^0 are functors from the category Adic_K of adic spaces over $\text{Spa}(K, \mathcal{O}_K)$ to the category **Grpd** which pro-étale locally on $\text{Spa}(R, R^+)$ are given by \mathcal{G}^+ and $\mathcal{G}^{0,+}$ respectively.

Proposition 2.3. *The quotient stacks $[\mathcal{H}_K^{n-1}/\text{GL}_n(\mathcal{O}_K)]$ and $[\mathcal{H}_K^{n-1}/\text{GL}_n(K)]$ represent the functors \mathcal{G} and \mathcal{G}^0 respectively.*

Proof. From the space $\mathcal{M}_\infty^{\text{Dr}}$, when we quotient out the action of D^\times , the resulting space \mathcal{H}_K^{n-1} only parametrizes triples (G, ι, ρ) . Using the Weil descent datum on $\mathcal{M}_\infty^{\text{Dr}}$, quotienting by $\text{Gal}(\check{K}/K)$ further forgets the homomorphism ι . Now two pairs (G, ρ) and (G', ρ') identify the same point in $\mathcal{M}_\infty^{\text{Dr}}$ if the quasi-isogeny

$\rho' \circ \rho^{-1} : G \bmod p \rightarrow G' \bmod p$ lifts to an isomorphism $G \rightarrow G'$ over R^+ . Thus in the stacky quotient $[\mathcal{H}_K^{n-1} / \mathrm{GL}_n(\mathcal{O}_K)]$, the data of ρ is forgotten and the morphisms between the objects are isomorphisms. When we instead quotient out by the full group $\mathrm{GL}_n(K)$, we are further allowing morphisms over R^+ which are only invertible upon inverting p . Thus morphisms in $[\mathcal{H}_K^{n-1} / \mathrm{GL}_n(K)]$ are quasi-isogenies. \square

2.2. Moduli stacks arising from the Lubin-Tate tower. The Lubin-Tate tower consists of spaces parametrizing formal groups. Similar to Section 2.1, we will first recall the construction of these spaces and then explain the moduli interpretation of the associated moduli stacks of our interest. For more details on this, see also [SW13, Section 6.4] and [BSSW24, Section 3].

Fix H_0 to be a 1-dimensional formal group over k of height n . The Lubin-Tate space arises as the deformation space of H_0 . Over a Noetherian local ring of residue characteristic p , [Tat67, Proposition 1] states that the functor $H \mapsto H[p^\infty]$ gives an equivalence between the category of p -divisible formal groups and the category of connected p -divisible groups over this ring. To simplify notation, we use the same symbol H for both a deformation of H_0 and its associated p -divisible group $H[p^\infty]$.

Definition 2.4. Let (R, R^+) be a complete affinoid (K, \mathcal{O}_K) -algebra. The *Lubin-Tate tower at infinite level* is a functor $\mathcal{M}_\infty^{\mathrm{LT}}$ on complete affinoid (K, \mathcal{O}_K) -algebras whose set of (R, R^+) -points is the set of quadruples (H, ι, ρ, η) up to isomorphism, where

- H is a 1-dimensional p -divisible formal group over R^+ ,
- $\iota : W \rightarrow R^+$ is a ring homomorphism,
- $\rho : H_0 \otimes_{W, \iota} R^+ / pR^+ \rightarrow H \otimes_{R^+} R^+ / pR^+$ is a quasi-isogeny,
- $\eta : \mathcal{O}_K^n \xrightarrow{\sim} T_p(H) = \varprojlim H[p^n]$ is an isomorphism of Galois modules.

Two quadruples (H, ι, ρ, η) and $(H', \iota', \rho', \eta')$ are isomorphic if there is an isomorphism $f : H \xrightarrow{\sim} H'$ over R^+ transferring one set of data to the other.

Both D^\times and $\mathrm{GL}_n(K)$ act naturally on the space $\mathcal{M}_\infty^{\mathrm{LT}}$. An element $d \in D^\times$ acts on the space by sending ρ to $\rho \circ d^{-1}$. The subgroup $\mathcal{O}_D^\times \subseteq D^\times$ arises as the group of automorphisms of the formal group H_0 , and the group D^\times is the group of automorphisms of H_0 in the isogeny category. An element $g \in \mathrm{GL}_n(\mathcal{O}_K)$ acts on the space via sending η to $\eta \circ g$. To extend the action to $\mathrm{GL}_n(K)$, let H be a fixed p -divisible group over R^+ and let S be the set of isomorphism classes of pairs (H', ϕ) where H' is another p -divisible group over R^+ and $\phi : H \rightarrow H'$ is a quasi-isogeny over R^+ . This set S is in bijection with the set of Galois stable lattices in $V_p(H)$, cf. [Far08, Lemme II.6.1]. Now for an element $g \in \mathrm{GL}_n(K)$, $\eta(g \cdot \mathcal{O}_K^n)$ is a Galois stable lattice in $V_p(H)$. Thus it gives a pair $(H_g, \phi_g) \in S$. Let $\phi_{g,*} : V_p(H) \xrightarrow{\sim} V_p(H_g)$ denote the induced isomorphism on the rational Tate modules. The action of $\mathrm{GL}_n(K)$ is defined via

$$(2.1) \quad g \cdot [(H, \iota, \rho, \eta)] = [(H_g, \iota, (\phi_g \bmod p) \circ \rho, \phi_{g,*} \circ \eta \circ g)], \quad g \in \mathrm{GL}_n(K).$$

Let $\mathrm{LT}_n = \mathrm{Spf} W[[u_1, \dots, u_{n-1}]]$ be the Lubin-Tate deformation space of H_0 as defined in [LT66] and let $\mathrm{LT}_{n, \check{K}}$ denote its generic fiber. We have the following diagram obtained from taking quotients on $\mathcal{M}_\infty^{\mathrm{LT}}$:

$$\begin{array}{ccc} & \mathcal{M}_\infty^{\mathrm{LT}} & \\ & \downarrow & \searrow \mathrm{GL}_n(\mathcal{O}_K) \\ & \mathrm{LT}_{n, \check{K}} & \searrow \mathbb{Z} \\ & \downarrow \text{Gross-Hopkins} & \\ & \mathbb{P}_{\check{K}}^{n-1} & \end{array} \quad \begin{array}{c} \mathrm{GL}_n(K) \\ \curvearrowright \end{array}$$

$\coprod_{\mathbb{Z}} \mathrm{LT}_{n, \check{K}}$

where the map $\mathrm{LT}_{n, \check{K}} \rightarrow \mathbb{P}_{\check{K}}^{n-1}$ is the Gross-Hopkins period map from [HG94].

As we noted earlier, the group \mathcal{O}_D^\times comes from $\text{Aut}_{\bar{k}}(H_0)$. If we treat H_0 as a functor on k -algebras and also take $\text{Gal}(\bar{k}/k)$ into consideration, we get the *Morava stabilizer group* $\mathbb{G}_n := \mathcal{O}_D^\times \rtimes \hat{\mathbb{Z}}$. We use $\mathbb{G}_n^0 := D^\times \rtimes \hat{\mathbb{Z}}$ to denote the analogue of \mathbb{G}_n in the isogeny category. These groups fit into the following diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_D^\times & \longrightarrow & \mathbb{G}_n & \longrightarrow & \hat{\mathbb{Z}} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & D^\times & \longrightarrow & \mathbb{G}_n^0 & \longrightarrow & \hat{\mathbb{Z}} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & \mathbb{Z} & \longrightarrow & \mathbb{Z} & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

Let $\varphi \in \text{Gal}(\bar{k}/k)$ be the Frobenius map, then the action of $\text{Gal}(\bar{k}/k)$ on $\mathcal{M}_\infty^{\text{LT}}$ is given by $\varphi \cdot (H, \iota, \rho, \eta) = (H, \iota \circ \varphi, \rho, \eta)$. Furthermore, φ induces an isomorphism $f_\varphi : H_0 \otimes_{W, \iota \circ \varphi} R^+ / pR^+ \xrightarrow{\sim} H_0 \otimes_{W, \iota} R^+ / pR^+$, which then gives a quasi-isogeny $\rho_\varphi := \rho \circ f_\varphi : H_0 \otimes_{W, \iota \circ \varphi} R^+ / pR^+ \rightarrow H \otimes R^+ / pR^+$. The map $(H, \iota \circ \varphi, \rho, \eta) \mapsto (H, \iota, \rho_\varphi, \eta)$ then gives an isomorphism $\varphi^* \mathcal{M}_\infty^{\text{LT}} \xrightarrow{\sim} \mathcal{M}_\infty^{\text{LT}}$ and equips the functor $\mathcal{M}_\infty^{\text{LT}}$ a Weil descent datum.

The moduli stacks we are interested in here are $[\text{LT}_{n, \check{K}} / \mathbb{G}_n]$ and $[\mathbb{P}_{\check{K}}^{n-1} / \mathbb{G}_n^0]$. To state the moduli interpretation of these two stacks, we first define two functors $\mathcal{H}^+, \mathcal{H}^{0,+}$ from the category of affinoid adic spaces over $\text{Spa}(K, \mathcal{O}_K)$ to \mathbf{Grpd} by assigning to $\text{Spa}(R, R^+)$ the following:

- Objects in both $\mathcal{H}^+(\text{Spa}(R, R^+))$ and $\mathcal{H}^{0,+}(\text{Spa}(R, R^+))$ are formal groups H over R^+ which are quasi-isogenous to H_0 over R^+ / pR^+ .
- Morphisms in $\mathcal{H}^+(\text{Spa}(R, R^+))$ are isomorphisms between the formal groups.
- Morphisms in $\mathcal{H}^{0,+}(\text{Spa}(R, R^+))$ are quasi-isogenies between the formal groups.

We let \mathcal{H} be the sheafification of \mathcal{H}^+ and let \mathcal{H}^0 be the sheafification of $\mathcal{H}^{0,+}$, both with respect to the pro-étale topology. Thus we have two functors

$$\mathcal{H}, \mathcal{H}^0 : \text{Adic}_K \rightarrow \mathbf{Grpd}$$

such that pro-étale locally they are defined as \mathcal{H}^+ and $\mathcal{H}^{0,+}$ above respectively.

Proposition 2.5. *The quotient stacks $[\text{LT}_{n, \check{K}} / \mathbb{G}_n]$ and $[\mathbb{P}_{\check{K}}^{n-1} / \mathbb{G}_n^0]$ represent the functors \mathcal{H} and \mathcal{H}^0 respectively.*

Proof. From the space $\mathcal{M}_\infty^{\text{LT}}$, quotienting out D^\times forgets the quasi-isogeny ρ . Quotienting by $\text{GL}_n(\mathcal{O}_K)$ further forgets the rigidification η . Using the Weil descent datum of $\mathcal{M}_\infty^{\text{LT}}$, quotienting out by $\hat{\mathbb{Z}}$ forgets the homomorphism ι . The moduli interpretation of $[\text{LT}_{n, \check{K}} / \mathbb{G}_n]$ then follows. As for the stack $[\mathbb{P}_{\check{K}}^{n-1} / \mathbb{G}_n^0]$, notice that the only difference from the earlier quotient is we are now quotienting $\mathcal{M}_\infty^{\text{LT}}$ by $\text{GL}_n(K)$. Let $g \in \text{GL}_n(K)$ and let (H, ι, ρ, η) be a point of $\mathcal{M}_\infty^{\text{LT}}$. Using [Far08, Lemme II.6.1] and the action of $\text{GL}_n(K)$ described in (2.1), as $\eta(g \cdot \mathcal{O}_K^n)$ runs through all Galois stable lattices in $V_p(H)$, H_g also runs through all formal groups over R^+ which are quasi-isogenous to H . This gives the moduli interpretation of $[\mathbb{P}_{\check{K}}^{n-1} / \mathbb{G}_n^0]$. \square

2.3. The isomorphism between the two towers. The Drinfeld tower and the Lubin-Tate tower are related through the following celebrated theorem of Scholze-Weinstein:

Theorem 2.6 ([SW13, Theorem E]). *The spaces $\mathcal{M}_\infty^{\text{Dr}}$ and $\mathcal{M}_\infty^{\text{LT}}$ are isomorphic as perfectoid spaces.*

Upon taking quotients of $\mathcal{M}_\infty^{\text{Dr}} \cong \mathcal{M}_\infty^{\text{LT}}$ by subgroups of $\text{GL}_n(K) \times D^\times$ and taking descent into consideration, we get:

Corollary 2.7. *There are isomorphisms of quotient stacks*

$$[\mathrm{LT}_{n,\check{K}}/\mathbb{G}_n] \cong [\mathcal{H}_K^{n-1}/\mathrm{GL}_n(\mathcal{O}_K)] \quad \text{and} \quad [\mathbb{P}_{\check{K}}^{n-1}/\mathbb{G}_n^0] \cong [\mathcal{H}_K^{n-1}/\mathrm{GL}_n(K)].$$

Based on their moduli interpretation, we will simply refer to the first pair of stacks $[\mathrm{LT}_{n,\check{K}}/\mathbb{G}_n]$ and $[\mathcal{H}_K^{n-1}/\mathrm{GL}_n(\mathcal{O}_K)]$ as the *isomorphism stacks*; similarly we shall call the two stacks in the second pair as *isogeny stacks*.

Remark 2.8. The spaces in the quotient stacks from Corollary 2.7 are the generic fibers of their formal models. One can also take the "diamond" generic fibers instead and reinterpret Corollary 2.7 as

$$[\mathrm{LT}_{n,\check{K}}^\diamond/\mathbb{G}_n] \cong [\mathcal{H}_K^{n-1,\diamond}/\mathrm{GL}_n(\mathcal{O}_K)] \quad \text{and} \quad [\mathbb{P}_{\check{K}}^{n-1,\diamond}/\mathbb{G}_n^0] \cong [\mathcal{H}_K^{n-1,\diamond}/\mathrm{GL}_n(K)].$$

Such stacks have analogous moduli interpretations as given earlier, except restricting the source from the category Adic_K to its subcategory of perfectoid spaces over $\mathrm{Spa}(K, \mathcal{O}_K)$, cf. [BSSW24, Section 3.7].

3. PRO-ÉTALE COHOMOLOGY OF SOME PERIOD DOMAINS

Our goal of this section is to review the pro-étale cohomology, both ℓ -adic and p -adic, of certain rigid spaces. After that, we explain how to equip such cohomology groups with a topology using the theory of condensed mathematics of Clausen–Scholze.

3.1. Results of Colmez–Dospinescu–Nizioł. The study of the cohomology of the Drinfeld spaces traces back to Schneider and Stuhler [SS91], where they give a general description of the cohomology groups for any cohomology theory satisfying certain axioms. Although the p -adic pro-étale cohomology theory does not satisfy the axiom of "homotopy invariance" in [SS91], Colmez–Dospinescu–Nizioł [CDN20] still manage to give a description of $H_{\mathrm{pro\acute{e}t}}^*(\mathcal{H}_C^{n-1}, \mathbb{Q}_p)$ in terms of differential forms and the generalized Steinberg representations, which is the key result we aim to highlight here. Along the way, we will also recall the pro-étale cohomology of some other spaces that already appeared in Section 2.

The pro-étale cohomology of projective spaces can be deduced directly from their étale cohomology, which is well-understood, see e.g. [Niz21].

Theorem 3.1. *Let ℓ be a prime (possibly equal to p) and let r be an integer. When r is even and $0 \leq r \leq 2n$, we have*

$$H_{\mathrm{pro\acute{e}t}}^r(\mathbb{P}_C^n, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell \left(-\frac{r}{2} \right)$$

For all other r , the cohomology group vanishes.

The space $\mathrm{LT}_{n,\check{K}}$ arises as the generic fiber of the Lubin–Tate space LT_n and is isomorphic to the $(n-1)$ -dimensional open unit ball $\mathring{\mathbb{B}}_{\check{K}}^{n-1}$. Let Ω^r denote the sheaf of differential r -forms and let $d : \Omega^r \rightarrow \Omega^{r+1}$ denote the differential map. The cohomology of open unit balls is given as follows.

Theorem 3.2. *Let $r \geq 0$ be an integer, we have*

- (1) $H_{\mathrm{pro\acute{e}t}}^0(\mathring{\mathbb{B}}_C^n, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$ and all higher cohomology groups vanish, for $\ell \neq p$.
- (2) $H_{\mathrm{pro\acute{e}t}}^0(\mathring{\mathbb{B}}_C^n, \mathbb{Q}_p) \cong \mathbb{Q}_p$ and $H_{\mathrm{pro\acute{e}t}}^r(\mathring{\mathbb{B}}_C^n, \mathbb{Q}_p(r)) \cong \Omega^{r-1}(\mathring{\mathbb{B}}_C^n)/\ker d$ for $r \geq 1$.

Proof. The ℓ -adic pro-étale cohomology of the open unit ball follows directly from the ℓ -adic pro-étale cohomology of the closed ball. For the p -adic case, see [CN20, Theorem 3]. \square

Before describing the pro-étale cohomology of Drinfeld spaces, we first recall the following class of $\mathrm{GL}_n(K)$ -representations; see also [CDN20, Section 5.2] for more details. Firstly, let $\Delta = \{1, \dots, n-1\}$, which can be identified with the set of simple roots of $\mathrm{GL}_n(K)$. Let W be its Weyl group and let $B \subseteq \mathrm{GL}_n(K)$ be the upper triangular Borel subgroup. For a subset $I \subseteq \Delta$, let $W_I \subseteq W$ be the subgroup generated by permutations $(i, i+1)$ for $i \in I$ and let $P_I = BW_I B$ be the associated parabolic subgroup. Then for any ring A and any subset $I \subseteq \Delta$, the *generalized Steinberg representation* (associated to I) is defined as

$$\mathrm{Sp}_I(A) = \frac{C^\infty(G/P_I, A)}{\sum_{i \in \Delta \setminus I} C^\infty(G/P_{I \cup \{i\}}, A)}$$

where C^∞ denotes the set of smooth functions. For $r \in \{0, \dots, n-1\}$, we simply write $\mathrm{Sp}_r(A)$ for the generalized Steinberg representation associated to the subset $\{1, \dots, n-1-r\} \subseteq \Delta$. Notice that when $r = 0$, its associated set is simply Δ and $\mathrm{Sp}_0(A)$ is the trivial representation over A . The ring A we consider will always be a topological ring, and this will give both $\mathrm{Sp}_I(A)$ and its dual $\mathrm{Sp}_I(A)^*$ a topology. As discussed in [CDN20, Section 5.2.2], when $A = \mathbb{Q}_p$, $\mathrm{Sp}_I(A)$ is an LF-space and $\mathrm{Sp}_I(A)^*$ is a Fréchet space.

Now for $\ell \neq p$, the following theorem can be directly deduced from [SS91] as the ℓ -adic pro-étale cohomology theory satisfies the axioms listed there.

Theorem 3.3 (Schneider-Stuhler). *Let $\ell \neq p$ be a prime. There is a $\mathrm{GL}_n(K) \times \mathcal{G}_K$ -equivariant isomorphism*

$$H_{\mathrm{pro\acute{e}t}}^r(\mathcal{H}_C^{n-1}, \mathbb{Q}_\ell(r)) \cong \mathrm{Sp}_r(\mathbb{Q}_\ell)^*$$

for every integer $r \geq 0$.

As for the p -adic case, we have the following result of Colmez–Dospinescu–Nizioł.

Theorem 3.4 ([CDN20, Theorem 5.13]). *There is a strictly exact sequence of $\mathrm{GL}_n(K) \times \mathcal{G}_K$ -Fréchet spaces*

$$0 \rightarrow \Omega^{r-1}(\mathcal{H}_C^{n-1}) / \ker d \rightarrow H_{\mathrm{pro\acute{e}t}}^r(\mathcal{H}_C^{n-1}, \mathbb{Q}_p(r)) \rightarrow \mathrm{Sp}_r(\mathbb{Q}_p)^* \rightarrow 0$$

for all integers $r \geq 0$.

3.2. Topology and condensed mathematics. After seeing the various cohomology groups from the previous section, one might ask: what is the topology on such cohomology groups, or more fundamentally, why should these cohomology groups possess a natural topology at all? The theory of condensed mathematics developed by Clausen and Scholze provides a natural way to endow cohomology groups with a topology. In this section, we briefly review some notions in condensed mathematics (following [CS19]) that we will need later and outline how a natural topology arises for cohomology groups under consideration.

Let $*_{\mathrm{pro\acute{e}t}}$ be the pro-étale site of a point, which is equivalent to the category of profinite sets. A condensed set/group/ring is a sheaf over $*_{\mathrm{pro\acute{e}t}}$ with values in **Set** / **Grp** / **Ring**. We denote the category of condensed sets by **Cond**. There is a functor from the category **Top** of topological spaces to the category **Cond** given by

$$\begin{aligned} (_) : \mathbf{Top} &\rightarrow \mathbf{Cond} \\ T &\mapsto (\underline{T} : S \mapsto \mathrm{Cont}(S, T)) \end{aligned}$$

where S is any profinite set and $\mathrm{Cont}(S, T)$ is the set of continuous maps from S to T . We let **Cond**(**Ab**) be the category of condensed abelian groups, which is an abelian category [CS19, Theorem 1.10] and has an internal Hom-functor denoted by $\underline{\mathrm{Hom}}$. We also let **Solid** \subseteq **Cond**(**Ab**) be the subcategory of solid abelian groups (see [CS19, Definition 5.1]), and tensor product of objects in **Solid** will be denoted by \otimes^\square .

Inside the category of profinite sets, we have the subcategory of extremally disconnected sets. These are the projective objects in the category $*_{\mathrm{pro\acute{e}t}}$, and we denote this subcategory by **EDis**. Given a condensed ring R , we denote by $\mathrm{Mod}_R^{\mathrm{cond}}$ the category of condensed R -modules. A pre-analytic ring \mathcal{A} consists of a condensed ring $\underline{\mathcal{A}}$ and a functor

$$\begin{aligned} \mathrm{EDis} &\rightarrow \mathrm{Mod}_{\underline{\mathcal{A}}}^{\mathrm{cond}} \\ S &\mapsto \mathcal{A}[S] \end{aligned}$$

sending finite disjoint unions to finite products, together with a natural transformation $S \rightarrow \mathcal{A}[S]$. Now given a pre-analytic ring \mathcal{A} , if for any complex $C : \dots \rightarrow C_i \rightarrow C_1 \rightarrow C_0 \rightarrow 0$ in $\mathbf{Ch}(\mathrm{Mod}_{\underline{\mathcal{A}}}^{\mathrm{cond}})$ with each C_i a direct sum of objects of the form $\mathcal{A}[T]$ for varying $T \in \mathrm{EDis}$, the map $R\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C) \rightarrow R\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C)$ is an isomorphism in the derived category $D(\mathbf{Cond}(\mathbf{Ab}))$, then we say \mathcal{A} is analytic.

Example 3.5. The primal example of a pre-analytic ring is \mathbb{Z}_\square , whose underlying condensed ring is given by $\underline{\mathbb{Z}}_\square = \mathbb{Z}$ and the associated functor is defined as

$$\begin{aligned} \mathrm{EDis} &\rightarrow \mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cond}} \\ S = \varprojlim_i S_i &\mapsto \mathbb{Z}_\square[S] := \varprojlim_i \mathbb{Z}[S_i]. \end{aligned}$$

In fact, \mathbb{Z}_\square is also analytic and we have $\mathbf{Solid} = \text{Mod}_{\mathbb{Z}_\square}^{\text{cond}}$ (see [CS19, Theorem 5.8]).

Example 3.6. For K/\mathbb{Q}_p a finite extension, we can give \mathcal{O}_K an analytic ring structure $\mathcal{O}_{K,\square}$ by setting $\mathcal{O}_{K,\square} = \mathcal{O}_K$ and defining $\mathcal{O}_{K,\square}[S] = \varprojlim_i \mathcal{O}_K[S_i]$ for $S = \varprojlim_i S_i \in \text{EDis}$. We can also give K an analytic ring structure K_\square by setting $K_\square = K$ and $K_\square[S] := \mathcal{O}_{K,\square}[S][\frac{1}{p}]$.

For the category $\text{Mod}_{\mathbb{Z}_\square}^{\text{cond}}$, we may also write it as $\text{Mod}_{\mathbb{Z}}^{\text{solid}}$ to emphasize solidity. When K is a finite extension of \mathbb{Q}_p , we let $\text{Mod}_K^{\text{solid}}$ denote the category of solid K -modules. Objects in $\text{Mod}_K^{\text{solid}}$ are solid abelian groups which are also K -vector spaces.

We recall the following classes of K -vector spaces from classical functional analysis.

Definitions 3.7. Let V be a topological K -vector space.

- (1) We say V is a K -Banach space if V admits a π -adically complete \mathcal{O}_K -module $V^0 \subseteq V$ such that $V^0 \otimes_{\mathcal{O}_K} K = V$ and $V^0/\pi^n V^0$ is discrete for all $n \in \mathbb{N}$.
- (2) We say V is a K -Fréchet space if V is locally convex, complete, and its topology can be given via a countable family of seminorms.

We call an object in $\text{Mod}_K^{\text{solid}}$ a *solid K -Banach/Fréchet space* if it arises as \underline{V} for some K -Banach/Fréchet space V .

Now for the cohomology groups appeared in Section 3.1, one way to endow them with a topology is by upgrading the constant sheaf \mathbb{Q}_p to a sheaf of condensed abelian groups \mathbb{Q}_p (with the p -adic topology). For an adic space X , this will also upgrade the pro-étale cohomology group $H_{\text{proét}}^i(X_C, \mathbb{Q}_p)$ into a condensed abelian group $H_{\text{cond}}^i(X_{C,\text{proét}}, \underline{\mathbb{Q}_p})$. If there exists a topological \mathbb{Q}_p -vector space V such that

$$H_{\text{cond}}^i(X_{C,\text{proét}}, \underline{\mathbb{Q}_p}) \cong \underline{V}$$

in $\mathbf{Cond}(\mathbf{Ab})$, the topology of V then offers us a way to topologize $H_{\text{proét}}^i(X_C, \mathbb{Q}_p)$.

When Colmez–Dospinescu–Nizioł first studied the p -adic (pro-)étale cohomology of Drinfeld spaces, the theory of condensed mathematics was not fully available yet. A condensed version of their calculations is contained in [Bos23], which describes the cohomology groups and their topology in a unified manner.

Example 3.8. We give an example of how one can endow a pro-étale cohomology group a natural topology through the case of $H_{\text{proét}}^r(\mathcal{H}_C^{n-1}, \mathbb{Q}_\ell(r))$ in Theorem 3.3. Following the condensed approach in [Bos23], once we upgrade the sheaf $\mathbb{Q}_\ell(r)$ into a sheaf of condensed abelian groups $\underline{\mathbb{Q}_\ell(r)}$, the isomorphism in Theorem 3.3 becomes

$$H_{\text{cond}}^r(\mathcal{H}_{C,\text{proét}}^{n-1}, \underline{\mathbb{Q}_\ell(r)}) \cong \underline{\text{Hom}_{\mathbb{Q}_\ell}(\text{Sp}_r(\mathbb{Q}_\ell), \mathbb{Q}_\ell)},$$

where \mathbb{Q}_ℓ is equipped with the ℓ -adic topology and the topology of $\text{Sp}_r(\mathbb{Q}_\ell)$ is induced by that of \mathbb{Q}_ℓ . One can then show that

$$\underline{\text{Hom}_{\mathbb{Q}_\ell}(\text{Sp}_r(\mathbb{Q}_\ell), \mathbb{Q}_\ell)} \cong \underline{\text{Hom}_{\text{cts}}(\text{Sp}_r(\mathbb{Q}_\ell), \mathbb{Q}_\ell)}$$

in the category $\text{Mod}_{\mathbb{Q}_\ell}^{\text{solid}}$, where $\text{Hom}_{\text{cts}}(\text{Sp}_r(\mathbb{Q}_\ell), \mathbb{Q}_\ell)$ is equipped with the weak topology. The topology of $H_{\text{proét}}^r(\mathcal{H}_C^{n-1}, \mathbb{Q}_\ell(r))$ is then given by that of $\text{Hom}_{\text{cts}}(\text{Sp}_r(\mathbb{Q}_\ell), \mathbb{Q}_\ell)$, which is a \mathbb{Q}_ℓ -Fréchet space.

Using the condensed formalism, we now elaborate on the Hochschild-Serre spectral sequence for pro-étale cohomology. Let G be a profinite group, X a rigid analytic space over K , and $Y \rightarrow X$ a pro-étale G -torsor. The pro-étale cohomology of X can be computed via the simplicial cover

$$\cdots \rightrightarrows Y \times_X Y \rightrightarrows Y$$

where the $(n+1)$ -fold product $(Y/X)^{n+1}$ is isomorphic to $Y \times G^n$. By [CGN23, Lemma 4.4] (see also [Bos23, Proposition 4.12]), we have a quasi-isomorphism in $D(\mathbf{Solid})$

$$R\Gamma_{\text{cond}}((Y \times G^n)_{\text{proét}}, \underline{\mathbb{Q}_p}) \simeq R\underline{\text{Hom}}(\mathbb{Z}_\square[G^n], R\Gamma_{\text{cond}}(Y_{\text{proét}}, \underline{\mathbb{Q}_p})).$$

This gives us a Hochschild-Serre spectral sequence

$$E_2^{i,j} = H_{\text{cond}}^i(G, H_{\text{cond}}^j(Y_{\text{proét}}, \underline{\mathbb{Q}_p})) \Rightarrow H_{\text{cond}}^{i+j}(X_{\text{proét}}, \underline{\mathbb{Q}_p})$$

or equivalently, if we remember that the cohomology groups have a topology,

$$E_2^{i,j} = H_{\text{cts}}^i(G, H_{\text{proét}}^j(Y, \mathbb{Q}_p)) \Rightarrow H_{\text{proét}}^{i+j}(X, \mathbb{Q}_p).$$

Such a spectral sequence allows us to compute the cohomology of a rigid analytic variety X over K using the \mathcal{G}_K -torsor $X_C \rightarrow X$ and also the cohomology of a quotient stack $[X/G]$ using the G -torsor $X \rightarrow [X/G]$.

In the rest of the paper, we will not emphasize the use of condensed formalism to prevent overloading the notation with too many "underlines". Instead, we will simply state what is the topology of the cohomology groups under consideration, and the reader should keep in mind that the topology comes from condensing the relevant objects.

4. COHOMOLOGY OF THE ISOMORPHISM STACKS

In this section, we compute the pro-étale cohomology of the isomorphism stack $[\mathcal{H}_K^{n-1} / \text{GL}_n(\mathcal{O}_K)]$.

4.1. ℓ -adic case. First recall the Galois cohomology of \mathcal{G}_K over \mathbb{Q}_ℓ , which can be summarized as follows.

Lemma 4.1. *For $j \in \mathbb{Z}$, let $\mathbb{Q}_\ell(j)$ be the j -th Tate twist of \mathbb{Q}_ℓ . We have*

$$\begin{aligned} H_{\text{cts}}^0(\mathcal{G}_K, \mathbb{Q}_\ell(j)) &\cong \begin{cases} \mathbb{Q}_\ell & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases} \\ H_{\text{cts}}^1(\mathcal{G}_K, \mathbb{Q}_\ell(j)) &\cong \begin{cases} \mathbb{Q}_\ell & \text{if } j = 0, 1 \\ 0 & \text{otherwise} \end{cases} \\ H_{\text{cts}}^2(\mathcal{G}_K, \mathbb{Q}_\ell(j)) &\cong \begin{cases} \mathbb{Q}_\ell & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and higher cohomology groups vanish for all j .

Proof. Since \mathcal{G}_K acts on $\mathbb{Q}_\ell(j)$ through the j -th power ℓ -adic cyclotomic character, we see that H^0 is nonzero only when $j = 0$. By local Tate duality, $H_{\text{cts}}^i(\mathcal{G}_K, \mathbb{Q}_\ell(j))$ and $H_{\text{cts}}^{2-i}(\mathcal{G}_K, \mathbb{Q}_\ell(1-j))$ are dual to each other, which gives the description of H^2 . Using the local Euler-Poincaré characteristic formula [NSW08, Corollary 7.3.8], we have $\sum_{i=0}^2 (-1)^i \dim_{\mathbb{Q}_\ell} H_{\text{cts}}^i(\mathcal{G}_K, \mathbb{Q}_\ell(j)) = 0$ for any fixed j , and this gives the description of H^1 . Finally, $H_{\text{cts}}^i(\mathcal{G}_K, \mathbb{Q}_\ell(j))$ vanishes for all $i \geq 3$ and all j as the cohomological degree of \mathcal{G}_K is 2. \square

Now we compute the ℓ -adic pro-étale cohomology of Drinfeld spaces over K through Galois descent.

Proposition 4.2. *There are $\text{GL}_n(K)$ -equivariant isomorphisms*

$$H_{\text{proét}}^r(\mathcal{H}_K^{n-1}, \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell & \text{if } r = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We use the Hochschild-Serre spectral sequence to carry out Galois descent from the geometric cohomology $H_{\text{proét}}^r(\mathcal{H}_C^{n-1}, \mathbb{Q}_\ell)$ to the arithmetic cohomology $H_{\text{proét}}^r(\mathcal{H}_K^{n-1}, \mathbb{Q}_\ell)$, where the E_2 -page is given by

$$E_2^{i,j} = H_{\text{cts}}^i(\mathcal{G}_K, H_{\text{proét}}^j(\mathcal{H}_C^{n-1}, \mathbb{Q}_\ell)) \Rightarrow H_{\text{proét}}^{i+j}(\mathcal{H}_K^{n-1}, \mathbb{Q}_\ell).$$

By Theorem 3.3, $H_{\text{proét}}^j(\mathcal{H}_C^{n-1}, \mathbb{Q}_\ell) \cong \text{Sp}_j(\mathbb{Q}_\ell)^*(-j)$ where \mathcal{G}_K acts through the Tate twist. So we have $E_2^{i,j} = H_{\text{cts}}^i(\mathcal{G}_K, \mathbb{Q}_\ell(-j)) \otimes \text{Sp}_j(\mathbb{Q}_\ell)^*$, which is isomorphic to $\text{Sp}_j(\mathbb{Q}_\ell)^*$ if $j = 0$ and $i = 0, 1$ and is 0 for all other i, j by Lemma 4.1. Thus the spectral sequence degenerates on the E_2 -page and the claim follows. \square

To compute the ℓ -adic pro-étale cohomology of the stack $[\mathcal{H}_K^{n-1} / \text{GL}_n(\mathcal{O}_K)]$, we first compute the continuous group cohomology of $\text{GL}_n(\mathcal{O}_K)$ over \mathbb{Q}_ℓ .

Lemma 4.3. *Let \mathbb{Q}_ℓ be the trivial 1-dimensional $\text{GL}_n(\mathcal{O}_K)$ -representation equipped with the ℓ -adic topology. Then*

$$H_{\text{cts}}^i(\text{GL}_n(\mathcal{O}_K), \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $P \subseteq \mathrm{GL}_n(\mathcal{O}_K)$ be a pro- p subgroup of finite index m . Since $\ell \neq p$, the ℓ -cohomological dimension of P is 0 [NSW08, Corollary 3.3.7]. Thus we have $H_{\mathrm{cts}}^i(P, \mathbb{Z}/\ell^r\mathbb{Z}) = 0$ for $i, r > 0$. This implies the system $\{H_{\mathrm{cts}}^i(P, \mathbb{Z}/\ell^r\mathbb{Z})\}_r$ is Mittag-Leffler for all $i \geq 0$. Thus we get $R^1 \varprojlim_r H_{\mathrm{cts}}^{i-1}(P, \mathbb{Z}/\ell^r\mathbb{Z}) = 0$ and $H_{\mathrm{cts}}^i(P, \mathbb{Z}_\ell) \cong \varprojlim_r H_{\mathrm{cts}}^i(P, \mathbb{Z}/\ell^r\mathbb{Z}) = 0$ for $i > 0$. Using the ℓ -cohomological dimension of P again, we also have $H_{\mathrm{cts}}^i(P, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ for $i > 0$. Together, such vanishing results imply $H_{\mathrm{cts}}^i(P, \mathbb{Q}_\ell) = 0$ for $i > 0$.

Now let $\mathrm{cor}_P^{\mathrm{GL}_n(\mathcal{O}_K)}$ and $\mathrm{res}_P^{\mathrm{GL}_n(\mathcal{O}_K)}$ denote the corestriction and restriction maps on group cohomology and also let $[m]$ denote the multiplication-by- m map. Using the vanishing of $H_{\mathrm{cts}}^i(P, \mathbb{Q}_\ell)$, we get

$$[m]|_{H_{\mathrm{cts}}^i(\mathrm{GL}_n(\mathcal{O}_K), \mathbb{Q}_\ell)} = \mathrm{cor}_P^{\mathrm{GL}_n(\mathcal{O}_K)} \circ \mathrm{res}_P^{\mathrm{GL}_n(\mathcal{O}_K)} = 0 \quad \text{for } i > 0.$$

As m is invertible in \mathbb{Q}_ℓ , we get $H_{\mathrm{cts}}^i(\mathrm{GL}_n(\mathcal{O}_K), \mathbb{Q}_\ell) = 0$ for $i > 0$. \square

Theorem 4.4. *There are isomorphisms*

$$H_{\mathrm{pro\acute{e}t}}^r([\mathcal{H}_K^{n-1}/\mathrm{GL}_n(\mathcal{O}_K)], \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell & \text{if } r = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We compute the pro-étale cohomology of the stack using the spectral sequence

$$E_2^{i,j} = H_{\mathrm{cts}}^i(\mathrm{GL}_n(\mathcal{O}_K), H_{\mathrm{pro\acute{e}t}}^j(\mathcal{H}_K^{n-1}, \mathbb{Q}_\ell)) \Rightarrow H_{\mathrm{pro\acute{e}t}}^{i+j}([\mathcal{H}_K^{n-1}/\mathrm{GL}_n(\mathcal{O}_K)], \mathbb{Q}_\ell).$$

By Proposition 4.2 and Lemma 4.3, we get $E_2^{i,j} \cong \mathbb{Q}_\ell$ when $j = 0, 1$ and $i = 0$ and vanishes everywhere else. This finishes the computation. \square

4.2. p -adic case. As one might have anticipated, the steps used in the ℓ -adic case do not translate to the p -adic case. Specifically, as we will see later in Proposition 4.11, while the differential part appeared in Theorem 3.4 does not survive the Galois descent from C to K , the Galois cohomology of \mathcal{G}_K over $\mathbb{Q}_p(j)$ is more intricate than over $\mathbb{Q}_\ell(j)$, making duals of nontrivial Steinberg representations appear in cohomology of Drinfeld spaces after descent. The challenge then comes from computing the group cohomology of $\mathrm{GL}_n(\mathcal{O}_K)$ over representations $\mathrm{Sp}_r(\mathbb{Q}_p)^*$.

To proceed with our computation of p -adic pro-étale cohomology of $[\mathcal{H}_K^{n-1}/\mathrm{GL}_n(\mathcal{O}_K)]$, we will pass to the Lubin-Tate side and compute the p -adic pro-étale cohomology of the stack $[\mathrm{LT}_{n,\tilde{K}}/\mathbb{G}_n]$ instead. As implied by Corollary 2.7, cohomology groups of these two quotient stacks are identical.

We again start by recalling Galois cohomology of \mathcal{G}_K over Tate twists of \mathbb{Q}_p .

Lemma 4.5. *Let d be the degree of K over \mathbb{Q}_p . Then for $j \in \mathbb{Z}$, we have*

$$\begin{aligned} H_{\mathrm{cts}}^0(\mathcal{G}_K, \mathbb{Q}_p(j)) &\cong \begin{cases} \mathbb{Q}_p & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases} \\ H_{\mathrm{cts}}^1(\mathcal{G}_K, \mathbb{Q}_p(j)) &\cong \begin{cases} \mathbb{Q}_p^{d+1} & \text{if } j = 0, 1 \\ \mathbb{Q}_p^d & \text{otherwise} \end{cases} \\ H_{\mathrm{cts}}^2(\mathcal{G}_K, \mathbb{Q}_p(j)) &\cong \begin{cases} \mathbb{Q}_p & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and higher cohomology groups vanish for all j .

Proof. Same as in Lemma 4.1, H^0 can be computed directly and local Tate duality gives the description of H^2 . For all fixed j , the local Euler-Poincaré characteristic formula [NSW08, Corollary 7.3.8] now implies $\sum_{i=0}^2 (-1)^i \dim_{\mathbb{Q}_p} H_{\mathrm{cts}}^i(\mathcal{G}_K, \mathbb{Q}_p(j)) = -[K : \mathbb{Q}_p]$, which gives the description of H^1 . \square

As for Galois cohomology over Tate twists of C , we have the following theorem of Tate.

Theorem 4.6 ([Tat67]). *Let K be a finite extension of \mathbb{Q}_p and let $C(j)$ be the 1-dimensional \mathcal{G}_K -representation over C where \mathcal{G}_K acts through the j -th power cyclotomic character. Then*

$$H_{\text{cts}}^i(\mathcal{G}_K, C(j)) \cong \begin{cases} K & \text{if } j = 0 \text{ and } i = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

On the Lubin-Tate side, since we are quotienting $\text{LT}_{n, \check{K}}$ by the Morava stabilizer group $\mathbb{G}_n \cong \mathcal{O}_D^\times \rtimes \hat{\mathbb{Z}}$, we evidently need to compute the continuous group cohomology of \mathcal{O}_D^\times over \mathbb{Q}_p . Let D_0 be the division algebra of invariant $1/n$ over \mathbb{Q}_p . The following result follows from Lazard's comparison theorem [Laz65, Théorème V.2.4.10] between continuous cohomology and Lie algebra cohomology for \mathbb{Q}_p -analytic groups.

Lemma 4.7 ([BSSW25, Proposition 3.8.1]). *Let G be either $\text{GL}_n(\mathbb{Z}_p)$ or $\mathcal{O}_{D_0}^\times$ with the trivial action on \mathbb{Q}_p . Then*

$$H_{\text{cts}}^*(G, \mathbb{Q}_p) \cong H_{\text{Lie}}^*(\mathfrak{gl}_n \mathbb{Q}_p, \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_3, \dots, x_{2n-1})$$

as graded \mathbb{Q}_p -algebras where $|x_i| = i$ in the exterior algebra on the right.

In Lemma 4.8 and Proposition 4.9 below, we generalize Lazard's result from \mathbb{Q}_p -analytic groups to K -analytic groups for finite extensions K/\mathbb{Q}_p . Our strategy is motivated by the proof of [HKN11, Lemma 4.1.1].

Lemma 4.8. *Let K be a finite extension of \mathbb{Q}_p of degree d and let GL_n be the general linear algebraic group defined over K , then*

$$(\text{Res}_{K/\mathbb{Q}_p} \text{GL}_n)_{\bar{K}} \cong (\text{GL}_{n, \bar{K}})^d$$

as group schemes over \bar{K} .

Proof. Since $[K : \mathbb{Q}_p] = d$, there are exactly d different embeddings of K into \bar{K} over \mathbb{Q}_p . Denote by $\Sigma = \{\sigma_i\}$ the set of embeddings of K into \bar{K} . Then we have $\bar{K} \otimes_{\mathbb{Q}_p} K \cong \prod_{\sigma_i \in \Sigma} \bar{K}$.

Now we proceed by comparing the functor of points associated with the two group schemes in question. Let X be a scheme over \bar{K} , then for each embedding $\sigma_i : K \hookrightarrow \bar{K}$, we have a corresponding map $f_i : \bar{K} \rightarrow \bar{K} \otimes_{\mathbb{Q}_p} K \cong \prod_{\sigma_i \in \Sigma} \bar{K}$ onto the copy of \bar{K} associated with σ_i . This induces a map

$$f_i : \text{Hom}_{\bar{K}}(X, \text{GL}_{n, \bar{K}}) \rightarrow \text{Hom}_{\bar{K}}(X \times_{\bar{K}} \text{Spec}(\bar{K} \otimes_{\mathbb{Q}_p} K), \text{GL}_{n, \bar{K}})$$

where

$$\begin{aligned} \text{Hom}_{\bar{K}}(X \times_{\bar{K}} \text{Spec}(\bar{K} \otimes_{\mathbb{Q}_p} K), \text{GL}_{n, \bar{K}}) &= \text{Hom}_K(X \times \text{Spec } K, \text{GL}_n) \\ &= \text{Hom}_{\mathbb{Q}_p}(X, \text{Res}_{K/\mathbb{Q}_p} \text{GL}_n) \\ &= \text{Hom}_{\bar{K}}(X, (\text{Res}_{K/\mathbb{Q}_p} \text{GL}_n)_{\bar{K}}). \end{aligned}$$

At the same time, the projection $s_i : \prod_{\sigma_i \in \Sigma} \bar{K} \rightarrow \bar{K}$ mapping the copy of \bar{K} corresponding to σ_i identically to \bar{K} gives a section of f_i . Thus we also get a map

$$s_i : \text{Hom}_{\bar{K}}(X, (\text{Res}_{K/\mathbb{Q}_p} \text{GL}_n)_{\bar{K}}) \rightarrow \text{Hom}_{\bar{K}}(X, \text{GL}_{n, \bar{K}})$$

such that $s_i \circ f_i = \text{id}$. This implies that $\text{GL}_{n, \bar{K}}$ is a direct summand of $(\text{Res}_{K/\mathbb{Q}_p} \text{GL}_n)_{\bar{K}}$. As this is true for all σ_i , we conclude that $(\text{Res}_{K/\mathbb{Q}_p} \text{GL}_n)_{\bar{K}} \cong (\text{GL}_{n, \bar{K}})^d$. \square

Proposition 4.9. *Suppose K/\mathbb{Q}_p is a finite extension of degree d and D is a division algebra over K of invariant $1/n$. Let G be either $\text{GL}_n(\mathcal{O}_K)$ or \mathcal{O}_D^\times with a trivial action on \mathbb{Q}_p . Then we have*

$$H_{\text{cts}}^*(G, \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_1, \dots, x_{2n-1})^{\otimes d}$$

as graded \mathbb{Q}_p -algebras where $|x_i| = i$ in the exterior algebra on the right.

Proof. We first consider the case of $G = \text{GL}_n(\mathcal{O}_K)$, which can be viewed as a \mathbb{Q}_p -analytic group via restriction of scalars. The Lie algebra of G is then given by $\text{Res}_{K/\mathbb{Q}_p} \mathfrak{gl}_n K$, which we denote by \mathfrak{g} . By Lemma 4.8, we have $\mathfrak{g} \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p \cong (\mathfrak{gl}_n \bar{\mathbb{Q}}_p)^d$ as Lie algebras over $\bar{\mathbb{Q}}_p$. This gives

$$(4.1) \quad H_{\text{Lie}}^*(\mathfrak{g} \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p, \bar{\mathbb{Q}}_p) \cong H_{\text{Lie}}^*((\mathfrak{gl}_n \bar{\mathbb{Q}}_p)^d, \bar{\mathbb{Q}}_p) \cong \Lambda_{\bar{\mathbb{Q}}_p}(x_1, x_3, \dots, x_{2n-1})^{\otimes d}$$

where the second isomorphism follows from Lemma 4.7 and the product formula for Lie algebra cohomology. In the meantime, base-change gives us an isomorphism

$$(4.2) \quad H_{\text{Lie}}^*(\mathfrak{g}, \mathbb{Q}_p) \otimes \overline{\mathbb{Q}_p} \cong H_{\text{Lie}}^*(\mathfrak{g} \otimes \overline{\mathbb{Q}_p}, \overline{\mathbb{Q}_p}).$$

Thus we have $H_{\text{Lie}}^*(\mathfrak{g}, \mathbb{Q}_p) \otimes \overline{\mathbb{Q}_p} \cong \Lambda_{\overline{\mathbb{Q}_p}}(x_1, x_3, \dots, x_{2n-1})^{\otimes d}$ and it suffices to show the latter description descends to \mathbb{Q}_p .

We claim that the base-change isomorphism (4.2) preserves primitive elements. Recall from [Kos50, § 10] that given a Lie algebra \mathfrak{a} over a field F , its Lie algebra cohomology carries a coproduct $\Delta : H_{\text{Lie}}^*(\mathfrak{a}, F) \rightarrow H_{\text{Lie}}^*(\mathfrak{a}, F) \otimes H_{\text{Lie}}^*(\mathfrak{a}, F)$ and the primitive elements in $H_{\text{Lie}}^*(\mathfrak{a}, F)$ are those such that $\Delta(x) = x \otimes 1 + 1 \otimes x$. The map Δ comes from the diagonal map $D : \mathfrak{a} \rightarrow \mathfrak{a} \times \mathfrak{a}$ of Lie algebras. As the diagonal map is compatible with base-change, we see the coproduct Δ commutes with base-change, hence the claim.

In the graded algebra $\Lambda_{\overline{\mathbb{Q}_p}}(x_1, x_3, \dots, x_{2n-1})^{\otimes d}$, the group of automorphisms which preserve primitive elements is given by $\prod_{i \in \{1, 3, \dots, 2n-1\}} \text{GL}_d(\overline{\mathbb{Q}_p})$. Using the claim above, to obtain the descent from $\overline{\mathbb{Q}_p}$ to \mathbb{Q}_p , it is enough to show $H^1(\mathcal{G}_{\mathbb{Q}_p}, \text{GL}_d(\overline{\mathbb{Q}_p}))$ is trivial. As $\mathcal{G}_{\mathbb{Q}_p}$ is profinite, any continuous 1-cocycle $\gamma : \mathcal{G}_{\mathbb{Q}_p} \rightarrow \text{GL}_d(\overline{\mathbb{Q}_p})$ has finite image. Thus the image of γ is contained in $\text{GL}_d(L)$ for some finite Galois extension L/\mathbb{Q}_p , and γ factors through the finite Galois group $\text{Gal}(L/\mathbb{Q}_p)$. Then using non-abelian Hilbert 90 ([Ser02, §III.1 Lemma 1]), we get

$$(4.3) \quad H^1(\mathcal{G}_{\mathbb{Q}_p}, \text{GL}_d(\overline{\mathbb{Q}_p})) \cong \varinjlim_{L/\mathbb{Q}_p \text{ finite Galois}} H^1(\text{Gal}(L/\mathbb{Q}_p), \text{GL}_d(L)) = 0.$$

This implies $H_{\text{Lie}}^*(\mathfrak{g}, \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_3, \dots, x_{2n-1})^{\otimes d}$, which proves the statement for $\text{GL}_n(\mathcal{O}_K)$.

When $G = \mathcal{O}_D^\times$, the Lie algebra of G and the Lie algebra of $\text{GL}_n(\mathcal{O}_K)$ are isomorphic over $\overline{\mathbb{Q}_p}$. Thus by (4.1) and (4.3), the continuous group cohomology of \mathcal{O}_D^\times over \mathbb{Q}_p is the same as that of $\text{GL}_n(\mathcal{O}_K)$. \square

We will need the following projection formula in our later computation.

Lemma 4.10. *Let G be a profinite group and $\mathcal{M}, \mathcal{N} \in \text{Mod}_{K[G]}^{\text{solid}}$. If the action of G on \mathcal{N} is trivial and \mathcal{N} is a solid K -Fréchet space, there is an isomorphism*

$$H^*(G, \mathcal{M} \otimes_K^\square \mathcal{N}) \cong H^*(G, \mathcal{M}) \otimes_K^\square \mathcal{N}.$$

Specifically, if M, N are K -Fréchet spaces and the action of G on N is trivial, then we have an isomorphism

$$H_{\text{cts}}^*(G, M \widehat{\otimes}_K N) \cong H_{\text{cts}}^*(G, M) \widehat{\otimes}_K N.$$

Proof. Consider \mathcal{M}, \mathcal{N} as objects in $D(\text{Mod}_{K[G]}^{\text{solid}})$ and let $R\Gamma(G, -)$ be the derived functor of $(-)^G : \text{Mod}_{K[G]}^{\text{solid}} \rightarrow \text{Mod}_K^{\text{solid}}$. Since the action of G on \mathcal{N} is trivial, we have an isomorphism

$$(4.4) \quad R\Gamma(G, \mathcal{M} \otimes_K^{L\square} \mathcal{N}) \cong R\Gamma(G, \mathcal{M}) \otimes_K^{L\square} \mathcal{N}$$

in the category $D(\text{Mod}_K^{\text{solid}})$ where $\otimes_K^{L\square}$ is the derived tensor product in $D(\text{Mod}_K^{\text{solid}})$. Because \mathcal{N} is a K -Fréchet space, it is flat for the tensor product \otimes_K^\square by [Bos23, Corollary A.65]. Therefore, we have $\mathcal{M} \otimes_K^{L\square} \mathcal{N} \cong \mathcal{M} \otimes_K^\square \mathcal{N}$ in $D(\text{Mod}_{K[G]}^{\text{solid}})$ and the cohomology of the left-hand side in (4.4) calculates $H^*(G, \mathcal{M} \otimes_K^\square \mathcal{N})$. The cohomology of the right-hand side in (4.4) admits a spectral sequence

$$E_2^{p,q} = H^p(H^q(R\Gamma(G, \mathcal{M})) \otimes_K^{L\square} \mathcal{N}) \Rightarrow H^{p+q}(R\Gamma(G, \mathcal{M}) \otimes_K^{L\square} \mathcal{N}).$$

Again as \mathcal{N} is flat, we have $H^q(R\Gamma(G, \mathcal{M})) \otimes_K^{L\square} \mathcal{N} \cong H^q(R\Gamma(G, \mathcal{M})) \otimes_K^\square \mathcal{N}$. Thus the cohomology of the right-hand side in (4.4) calculates $H^*(G, \mathcal{M}) \otimes_K^\square \mathcal{N}$ and the first claimed isomorphism of cohomology rings follows.

The second assertion follows directly from the first one together with the isomorphism between $V \widehat{\otimes}_K W$ and $V \otimes_K^\square W$ in the category $\text{Mod}_K^{\text{solid}}$ for K -Fréchet spaces V, W , cf. [Bos23, Proposition A.68]. \square

While our computation will be through the Lubin-Tate side, we nevertheless give a description of the p -adic pro-étale cohomology of Drinfeld spaces over K as an interlude.

Proposition 4.11. *Suppose K is a finite extension of \mathbb{Q}_p of degree d . There are $\mathrm{GL}_n(K)$ -equivariant isomorphisms*

$$H_{\mathrm{pro\acute{e}t}}^r(\mathcal{H}_K^{n-1}, \mathbb{Q}_p) \cong \begin{cases} \mathbb{Q}_p & \text{if } r = 0 \\ \mathbb{Q}_p^{d+1} & \text{if } r = 1 \\ \mathrm{Sp}_{r-1}(\mathbb{Q}_p)^* \otimes \mathbb{Q}_p^d & \text{if } r \geq 2. \end{cases}$$

Proof. By Theorem 3.4, we have a short exact sequence

$$(4.5) \quad 0 \rightarrow \Omega^{j-1}(\mathcal{H}_K^{n-1}) / \ker d \widehat{\otimes}_K C(-j) \rightarrow H_{\mathrm{pro\acute{e}t}}^j(\mathcal{H}_C^{n-1}, \mathbb{Q}_p) \rightarrow \mathrm{Sp}_j(\mathbb{Q}_p)^*(-j) \rightarrow 0$$

for $j \geq 0$. Notice that when $j = 0$, the differential part vanishes and $H_{\mathrm{pro\acute{e}t}}^0(\mathcal{H}_C^{n-1}, \mathbb{Q}_p)$ is simply \mathbb{Q}_p . Using Lemma 4.5, we have

$$H_{\mathrm{cts}}^i(\mathcal{G}_K, H_{\mathrm{pro\acute{e}t}}^0(\mathcal{H}_C^{n-1}, \mathbb{Q}_p)) \cong \begin{cases} \mathbb{Q}_p & \text{if } i = 0 \\ \mathbb{Q}_p^{d+1} & \text{if } i = 1 \\ 0 & \text{if } i \geq 2. \end{cases}$$

When $j \geq 1$, the Galois cohomology of (4.5) gives a long exact sequence

$$\begin{aligned} \cdots \rightarrow H_{\mathrm{cts}}^{i-1}(\mathcal{G}_K, \mathrm{Sp}_j(\mathbb{Q}_p)^*(-j)) &\rightarrow H_{\mathrm{cts}}^i(\mathcal{G}_K, \Omega^{j-1}(\mathcal{H}_K^{n-1}) / \ker d \widehat{\otimes}_K C(-j)) \\ &\rightarrow H_{\mathrm{cts}}^i(\mathcal{G}_K, H_{\mathrm{pro\acute{e}t}}^j(\mathcal{H}_C^{n-1}, \mathbb{Q}_p)) \rightarrow H_{\mathrm{cts}}^i(\mathcal{G}_K, \mathrm{Sp}_j(\mathbb{Q}_p)^*(-j)) \rightarrow \cdots \end{aligned}$$

Since the action of \mathcal{G}_K on $\Omega^{j-1}(\mathcal{H}_K^{n-1}) / \ker d \widehat{\otimes}_K C(-j)$ is through the Tate twist, we have

$$H_{\mathrm{cts}}^i(\mathcal{G}_K, \Omega^{j-1}(\mathcal{H}_K^{n-1}) / \ker d \widehat{\otimes}_K C(-j)) \cong H_{\mathrm{cts}}^i(\mathcal{G}_K, C(-j)) \widehat{\otimes}_K \Omega^{j-1}(\mathcal{H}_K^{n-1}) / \ker d$$

by Lemma 4.10, and Theorem 4.6 further implies that all such cohomology groups vanish. Since the Galois action on $\mathrm{Sp}_j(\mathbb{Q}_p)^*(-j)$ is also through the Tate twist, using Lemma 4.5, we get

$$H_{\mathrm{cts}}^i(\mathcal{G}_K, H_{\mathrm{pro\acute{e}t}}^j(\mathcal{H}_C^{n-1}, \mathbb{Q}_p)) \cong \begin{cases} \mathrm{Sp}_j(\mathbb{Q}_p)^* \otimes \mathbb{Q}_p^d & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

By applying these computations of Galois cohomology groups to the spectral sequence for Galois descent, we obtain the claimed isomorphisms. \square

Now we come back to the Lubin-Tate side and finish our computation of $H_{\mathrm{pro\acute{e}t}}^*([\mathrm{LT}_{n,\check{K}}/\mathbb{G}_n], \mathbb{Q}_p)$.

Theorem 4.12. *Let K be a finite extension of \mathbb{Q}_p of degree d . There is an isomorphism of graded \mathbb{Q}_p -algebras*

$$H_{\mathrm{pro\acute{e}t}}^*([\mathrm{LT}_{n,\check{K}}/\mathbb{G}_n], \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_3, \dots, x_{2n-1})^{\otimes d} \otimes H_{\mathrm{cts}}^*(\mathcal{G}_K, \mathbb{Q}_p)$$

where the degree of x_i is i in the exterior algebra on the right-hand side.

Proof. The generic fiber $\mathrm{LT}_{n,\check{K}}$ of the Lubin-Tate space is isomorphic to the rigid analytic $(n-1)$ -dimensional open ball $\mathring{\mathbb{B}}_{\check{K}}^{n-1}$. However, doing Galois descent on the pro-étale cohomology of $\mathring{\mathbb{B}}_C^{n-1}$ directly from C to \check{K} can be difficult. Instead, we will stitch together the quotient by $\mathrm{Gal}(\overline{K}/\check{K})$ and the quotient by the Morava stabilizer group \mathbb{G}_n . Over $\mathring{\mathbb{B}}_C^{n-1}$, we have actions of both \mathcal{G}_K and \mathbb{G}_n . Since both groups surject onto $\mathrm{Gal}(\check{K}/K) \cong \hat{\mathbb{Z}}$, we construct a group H as their equalizer so H fits into the diagram

$$\begin{array}{ccc} H & \longrightarrow & \mathcal{G}_K \\ \downarrow & & \downarrow f_2 \\ \mathbb{G}_n & \xrightarrow{f_1} & \mathrm{Gal}(\check{K}/K) \end{array}$$

where $\ker(f_1) = \mathcal{O}_D^\times$, $\ker(f_2) = \mathrm{Gal}(\overline{K}/\check{K})$, and we have a short exact sequence

$$(4.6) \quad 0 \rightarrow \mathcal{O}_D^\times \rightarrow H \rightarrow \mathcal{G}_K \rightarrow 0.$$

The construction of H allows us to study the actions of \mathcal{G}_K and \mathbb{G}_n together, and it identifies the pro-étale cohomology of $[\mathrm{LT}_{n,\check{K}}/\mathbb{G}_n]$ with that of $[\mathring{\mathbb{B}}_C^{n-1}/H]$.

We now compute the continuous group cohomology of H over $H_{\text{proét}}^j(\mathbb{B}_C^{n-1}, \mathbb{Q}_p)$ for $j \geq 0$ using (4.6) and the associated Hochschild-Serre spectral sequence (see [BSSW25, Section 3.3] for a justification of the Hochschild-Serre spectral sequence in the setting of continuous cohomology). When $j = 0$, we have $H_{\text{proét}}^0(\mathbb{B}_C^{n-1}, \mathbb{Q}_p) \cong \mathbb{Q}_p$ by Theorem 3.2. Then Lemma 4.5 and Proposition 4.9 imply that only the 0th column and the 1st column in the spectral sequence $E_2^{p,q} = H_{\text{cts}}^p(\mathcal{G}_K, H_{\text{cts}}^q(\mathcal{O}_D^\times, \mathbb{Q}_p)) \Rightarrow H_{\text{cts}}^{p+q}(H, \mathbb{Q}_p)$ are nonzero. Therefore, the spectral sequence degenerates and gives

$$H_{\text{cts}}^*(H, H_{\text{proét}}^0(\mathbb{B}_C^{n-1}, \mathbb{Q}_p)) \cong \Lambda_{\mathbb{Q}_p}(x_1, x_3, \dots, x_{2n-1})^{\otimes d} \otimes H_{\text{cts}}^*(\mathcal{G}_K, \mathbb{Q}_p).$$

When $j \geq 1$, Theorem 3.2 implies that $H_{\text{proét}}^j(\mathbb{B}_C^{n-1}, \mathbb{Q}_p) \cong \Omega^{j-1}(\mathbb{B}_K^{n-1}) / \ker d \widehat{\otimes}_K C(-j)$. Thus in the spectral sequence $E_2^{p,q} = H_{\text{cts}}^p(\mathcal{G}_K, H_{\text{cts}}^q(\mathcal{O}_D^\times, H_{\text{proét}}^j(\mathbb{B}_C^{n-1}, \mathbb{Q}_p))) \Rightarrow H_{\text{cts}}^{p+q}(H, H_{\text{proét}}^j(\mathbb{B}_C^{n-1}, \mathbb{Q}_p))$, the term $E_2^{p,q}$ is given by

$$\begin{aligned} E_2^{p,q} &= H_{\text{cts}}^p(\mathcal{G}_K, H_{\text{cts}}^q(\mathcal{O}_D^\times, \Omega^{j-1}(\mathbb{B}_K^{n-1}) / \ker d \widehat{\otimes}_K C(-j))) \\ &= H_{\text{cts}}^p(\mathcal{G}_K, H_{\text{cts}}^q(\mathcal{O}_D^\times, \Omega^{j-1}(\mathbb{B}_K^{n-1}) / \ker d) \widehat{\otimes}_K C(-j)) \\ &= H_{\text{cts}}^p(\mathcal{G}_K, C(-j)) \widehat{\otimes}_K H_{\text{cts}}^q(\mathcal{O}_D^\times, \Omega^{j-1}(\mathbb{B}_K^{n-1}) / \ker d). \end{aligned}$$

Using the projection formula from Lemma 4.10, the second equality follows from the fact that \mathcal{O}_D^\times only acts on the differential forms, and the third equality follows from the fact that \mathcal{G}_K acts through the Tate twist. By Theorem 4.6, we get $E_2^{p,q} = 0$ and $H_{\text{cts}}^*(H, H_{\text{proét}}^j(\mathbb{B}_C^{n-1}, \mathbb{Q}_p)) = 0$ when $j \geq 1$.

Finally, inserting these computations into the spectral sequence

$$E_2^{i,j} = H_{\text{cts}}^i(H, H_{\text{proét}}^j(\mathbb{B}_C^{n-1}, \mathbb{Q}_p)) \Rightarrow H_{\text{proét}}^{i+j}([\mathbb{B}_C^{n-1}/H], \mathbb{Q}_p),$$

we get the isomorphisms as desired. \square

Since $H_{\text{cts}}^*(\mathcal{G}_K, \mathbb{Q}_p)$ is already calculated in Lemma 4.5, Theorem 4.12 gives an explicit description of p -adic pro-étale cohomology of the stack $[\text{LT}_{n,\check{K}}/\mathbb{G}_n]$ in all degrees.

Remark 4.13. Following the same steps as in the proof of Theorem 4.12, one can also compute the ℓ -adic pro-étale cohomology of $[\mathcal{H}_K^{n-1}/\text{GL}_n(\mathcal{O}_K)]$ by passing to the Lubin-Tate side, recovering the description given in Theorem 4.4.

5. COHOMOLOGY OF THE ISOGENY STACKS

We now turn to the computation of pro-étale cohomology of the isogeny stack $[\mathcal{H}_K^{n-1}/\text{GL}_n(K)]$, where many of the auxiliary results from Section 4 are still helpful.

5.1. ℓ -adic case. We will compute $H_{\text{proét}}^*([\mathcal{H}_K^{n-1}/\text{GL}_n(K)], \mathbb{Q}_\ell)$ directly from the Drinfeld side, where the key step is to determine the continuous cohomology of $\text{GL}_n(K)$ over \mathbb{Q}_ℓ . The cohomology of $\text{GL}_n(K)$ over the trivial module over a ring R has been well-studied when R is equipped with the discrete topology. For example, [Dat06, Corollaire 2.1.7] (see also [Orl05, Corollary 2]) gives

$$(5.1) \quad H_{\text{cts}}^i(\text{GL}_n(K), R) = \begin{cases} R & \text{if } i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

if R is any characteristic 0 ring or $R = \mathbb{Z}/m\mathbb{Z}$ for suitably chosen m , where R is equipped with the discrete topology in both cases. However, as we pointed out in Section 3.2, the module \mathbb{Q}_ℓ here arises from the ℓ -adic pro-étale cohomology of Drinfeld spaces, so it is endowed with a natural ℓ -adic topology as justified by the condensed formalism. Therefore, the result (5.1) above does not apply directly to our setting; nevertheless, the following analogous statement still holds:

Lemma 5.1. *Let \mathbb{Q}_ℓ be the trivial 1-dimensional $\text{GL}_n(K)$ -representation equipped with the ℓ -adic topology. Then*

$$H_{\text{cts}}^i(\text{GL}_n(K), \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & \text{if } i = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We first compute the group cohomology of $\mathrm{SL}_n(K)$ over \mathbb{Q}_ℓ . Denote $\mathrm{SL}_n(K)$ by G . Following notations from [SS97, §II.1], we denote by X the Bruhat-Tits building of G , by X_i (resp. $X_{(i)}$) the set of i -dimensional facets (resp. oriented facets) in X , by R_i the finite set of G -orbit representatives in X_i . For a facet $F \in X_i$, let P_F^\dagger (resp. P_F) be the G -stabilizer (resp. pointwise G -stabilizer) of F and let $\varepsilon_F : P_F^\dagger \rightarrow \{\pm 1\}$ be the orientation character as defined in [SS97, §III.4], which is trivial when restricted to $P_F \subseteq P_F^\dagger$. By [SS97, Theorem II.3.1] and [SS97, proof of Proposition III.4.1], we have an exact sequence

$$0 \rightarrow \bigoplus_{F \in R_{n-1}} \mathrm{c}\text{-Ind}_{P_F^\dagger}^G \varepsilon_F \rightarrow \cdots \rightarrow \bigoplus_{F \in R_0} \mathrm{c}\text{-Ind}_{P_F^\dagger}^G \varepsilon_F \rightarrow \mathbb{Z} \rightarrow 0$$

of smooth representations of G ; see also [Fus22, eq. (4)]. Tensoring this with \mathbb{Q}_ℓ then gives an exact resolution of the representation \mathbb{Q}_ℓ :

$$(5.2) \quad 0 \rightarrow \bigoplus_{F \in R_{n-1}} \mathrm{c}\text{-Ind}_{P_F^\dagger}^G \varepsilon_F \rightarrow \cdots \rightarrow \bigoplus_{F \in R_0} \mathrm{c}\text{-Ind}_{P_F^\dagger}^G \varepsilon_F \rightarrow \mathbb{Q}_\ell \rightarrow 0.$$

Applying the continuous cochain functor $C^\bullet(G, -)$ to (5.2) then gives us a double complex. By Shapiro's lemma, the spectral sequence associated to such a double complex is given by

$$(5.3) \quad E_1^{i,j} = \bigoplus_{F \in R_i} H_{\mathrm{cts}}^j(P_F^\dagger, \varepsilon_F) \Rightarrow H_{\mathrm{cts}}^{i+j}(G, \mathbb{Q}_\ell).$$

By [SS97, §III.4], since our group G has anisotropic center, the stabilizer group P_F^\dagger is compact and $P_F \subseteq P_F^\dagger$ has finite index. Then [SS97, §II.1] further implies that for each facet F we can find an exact sequence $1 \rightarrow U_F \rightarrow P_F^\dagger \rightarrow I_F \rightarrow 1$ with $U_F \subseteq P_F \subseteq P_F^\dagger$ pro- p and I_F finite. Since $\varepsilon|_{P_F}$ is trivial, the same argument as in the proof of Lemma 4.3 implies that $H_{\mathrm{cts}}^j(P_F^\dagger, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$ when $j = 0$ and vanishes when $j > 0$. Thus the spectral sequence (5.3) collapses to the 0th row with $E_1^{i,0} = \bigoplus_{F \in R_i} \mathbb{Q}_\ell$. But notice that $(E_1^{\bullet,0}, d_1)$ is exactly the augmented cellular cochain complex of X with coefficient in \mathbb{Q}_ℓ . Since X is contractible, we have $E_2^{0,0} = \mathbb{Q}_\ell$ and $E_2^{i,0} = 0$ for $i > 0$. This implies $H_{\mathrm{cts}}^i(G, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$ when $i = 0$ and vanishes when $i > 0$.

Now consider the short exact sequence $1 \rightarrow \mathrm{SL}_n(K) \rightarrow \mathrm{GL}_n(K) \xrightarrow{\det} K^\times \rightarrow 1$. Using its associated spectral sequence and our computation of $H_{\mathrm{cts}}^*(\mathrm{SL}_n(K), \mathbb{Q}_\ell)$ above, we get $H_{\mathrm{cts}}^*(\mathrm{GL}_n(K), \mathbb{Q}_\ell) \cong H_{\mathrm{cts}}^*(K^\times, \mathbb{Q}_\ell)$. Write $K^\times \cong \pi^\mathbb{Z} \times \mathcal{O}_K^\times$, then Lemma 4.3 further implies $H_{\mathrm{cts}}^*(K^\times, \mathbb{Q}_\ell) \cong H_{\mathrm{cts}}^*(\mathbb{Z}, \mathbb{Q}_\ell)$, which is isomorphic to \mathbb{Q}_ℓ in degree 0 and 1 and vanishes in higher degrees. \square

Theorem 5.2. *We have isomorphisms*

$$H_{\mathrm{pro\acute{e}t}}^r([\mathcal{H}_K^{n-1}/\mathrm{GL}_n(K)], \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell & \text{if } r = 0, 2 \\ \mathbb{Q}_\ell^2 & \text{if } r = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Using the spectral sequence

$$E_2^{i,j} = H_{\mathrm{cts}}^i(\mathrm{GL}_n(K), H_{\mathrm{pro\acute{e}t}}^j(\mathcal{H}_K^{n-1}, \mathbb{Q}_\ell)) \Rightarrow H_{\mathrm{pro\acute{e}t}}^{i+j}([\mathcal{H}_K^{n-1}/\mathrm{GL}_n(K)], \mathbb{Q}_\ell),$$

the theorem follows directly from Proposition 4.2 and Lemma 5.1. \square

5.2. p -adic case. Same as in Section 4.2, carrying out computation directly from the Drinfeld side would lead us to the hurdle of determining $H_{\mathrm{cts}}^*(\mathrm{GL}_n(K), \mathrm{Sp}_r(\mathbb{Q}_p)^*)$ for r in between 0 and $n - 1$. The representations $\mathrm{Sp}_r(\mathbb{Q}_p)^*$ are Fréchet spaces over \mathbb{Q}_p , and we will see later in Section 6 that even computing such continuous cohomologies in the easiest case when $r = 0$ requires some effort.

Thus we again switch to the Lubin-Tate side and compute $H_{\mathrm{pro\acute{e}t}}^*([\mathcal{H}_K^{n-1}/\mathrm{GL}_n(K)], \mathbb{Q}_p)$ through computing $H_{\mathrm{pro\acute{e}t}}^*([\mathbb{P}_K^{n-1}/\mathbb{G}_n], \mathbb{Q}_p)$, as suggested by Corollary 2.7. Let H be the group constructed in the proof of Theorem 4.12. Then by construction of H , we have an isomorphism

$$(5.4) \quad [\mathbb{P}_K^{n-1}/\mathbb{G}_n] \cong [\mathbb{P}_C^{n-1}/H].$$

We first factorize the cohomology of $[\mathbb{P}_C^{n-1}/H]$.

Lemma 5.3. *The p -adic pro-étale cohomology of the stack $[\mathbb{P}_C^{n-1}/H]$ can be expressed as*

$$H_{\text{proét}}^*([\mathbb{P}_C^{n-1}/H], \mathbb{Q}_p) \cong H_{\text{proét}}^*([\mathbb{P}_C^{n-1}/\mathcal{G}_K], \mathbb{Q}_p) \otimes H_{\text{cts}}^*(\mathcal{O}_D^\times, \mathbb{Q}_p).$$

Proof. We start by describing the cohomology of $[\mathbb{P}_C^{n-1}/\mathcal{O}_D^\times]$. As the cohomology classes of \mathbb{P}_C^{n-1} are hyperplanes, \mathcal{O}_D^\times acts trivially on $H_{\text{proét}}^*(\mathbb{P}_C^{n-1}, \mathbb{Q}_p)$ thus the terms in the Hochschild-Serre spectral sequence converging to $H_{\text{proét}}^*([\mathbb{P}_C^{n-1}/\mathcal{O}_D^\times], \mathbb{Q}_p)$ are given by $E_2^{i,j} = H_{\text{cts}}^i(\mathcal{O}_D^\times, \mathbb{Q}_p) \otimes H_{\text{proét}}^j(\mathbb{P}_C^{n-1}, \mathbb{Q}_p)$. Let $\eta \in E_2^{0,2}$ be the hyperplane class. Since only the even rows in $E_2^{i,j}$ are nonzero, η is a permanent cycle. As the spectral sequence is multiplicative and its differentials are derivations, $d_r(\eta) = 0$ for all $r \geq 2$ implies $d_r(\eta^m) = 0$ for all $r \geq 2$ and all powers η^m . Since powers of η generate $H_{\text{proét}}^*(\mathbb{P}_C^{n-1}, \mathbb{Q}_p)$, the spectral sequence degenerates on the E_2 -page and gives $H_{\text{proét}}^*([\mathbb{P}_C^{n-1}/\mathcal{O}_D^\times], \mathbb{Q}_p) \cong H_{\text{proét}}^*(\mathbb{P}_C^{n-1}, \mathbb{Q}_p) \otimes H_{\text{cts}}^*(\mathcal{O}_D^\times, \mathbb{Q}_p)$.

Using such a tensor product decomposition, consider the spectral sequence corresponding to further quotienting $[\mathbb{P}_C^{n-1}/\mathcal{O}_D^\times]$ by \mathcal{G}_K

$$(5.5) \quad E_2^{i,j} = H_{\text{cts}}^i(\mathcal{G}_K, \bigoplus_{a+b=j} H_{\text{proét}}^a(\mathbb{P}_C^{n-1}, \mathbb{Q}_p) \otimes H_{\text{cts}}^b(\mathcal{O}_D^\times, \mathbb{Q}_p)) \Rightarrow H_{\text{proét}}^{i+j}([\mathbb{P}_C^{n-1}/H], \mathbb{Q}_p),$$

where \mathcal{G}_K acts on the factor $H_{\text{proét}}^a(\mathbb{P}_C^{n-1}, \mathbb{Q}_p)$ through Tate twist and acts on the factor $H_{\text{cts}}^b(\mathcal{O}_D^\times, \mathbb{Q}_p)$ through conjugation. By Theorem 3.1 and Lemma 4.5, we see that $E_2^{i,j} = 0$ for $i \geq 2$. Thus (5.5) degenerates on the E_2 -page and gives $H_{\text{proét}}^*([\mathbb{P}_C^{n-1}/H], \mathbb{Q}_p) \cong H_{\text{proét}}^*([\mathbb{P}_C^{n-1}/\mathcal{G}_K], \mathbb{Q}_p) \otimes H_{\text{cts}}^*(\mathcal{O}_D^\times, \mathbb{Q}_p)^{\mathcal{G}_K}$. By construction of the group H , the conjugation action of \mathcal{G}_K on \mathcal{O}_D^\times factors through the quotient $\hat{\mathbb{Z}}$, where $1 \in \hat{\mathbb{Z}}$ acts on \mathcal{O}_D^\times by $\text{Ad}(\varpi)$. In [BSSW25, Proposition 3.8.1], it is shown that the conjugation action of D^\times on $H_{\text{cts}}^*(\mathcal{O}_D^\times, \mathbb{Q}_p)$ is trivial if D is defined over \mathbb{Q}_p , and the argument there naturally extends to the case when D is defined over a finite extension K/\mathbb{Q}_p . Thus we have $H_{\text{cts}}^*(\mathcal{O}_D^\times, \mathbb{Q}_p)^{\mathcal{G}_K} = H_{\text{cts}}^*(\mathcal{O}_D^\times, \mathbb{Q}_p)$, which gives the result. \square

Theorem 5.4. *Suppose K is a finite extension of \mathbb{Q}_p of degree d . There is an isomorphism of graded \mathbb{Q}_p -vector spaces*

$$H_{\text{proét}}^*([\mathbb{P}_K^{n-1}/\mathbb{G}_n^0], \mathbb{Q}_p) \cong H_{\text{proét}}^*(\mathbb{P}_K^{n-1}, \mathbb{Q}_p) \otimes \Lambda_{\mathbb{Q}_p}(x_1, x_3, \dots, x_{2n-1})^{\otimes d} \otimes \Lambda_{\mathbb{Q}_p}(y),$$

where $|x_i| = i$, $|y| = 1$, and

$$H_{\text{proét}}^*(\mathbb{P}_K^{n-1}, \mathbb{Q}_p) = \begin{cases} \mathbb{Q}_p & \text{if } * = 0 \\ \mathbb{Q}_p^{d+1} & \text{if } * = 1 \\ \mathbb{Q}_p^d & \text{if } 3 \leq * \leq 2n-1 \text{ and } * \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall from Section 2 that $\mathbb{G}_n^0 \cong \mathbb{G}_n \times \mathbb{Z}$. Since cohomology classes of \mathbb{P}_K^{n-1} are hyperplanes, \mathbb{Z} acts trivially on $H_{\text{proét}}^*(\mathbb{P}_K^{n-1}, \mathbb{Q}_p)$. As the actions of \mathbb{Z} and \mathbb{G}_n on \mathbb{P}_K^{n-1} are also commutative, the spectral sequence for the quotient $\mathbb{P}_K^{n-1} \rightarrow [\mathbb{P}_K^{n-1}/\mathbb{G}_n]$ is \mathbb{Z} -equivariant. Thus \mathbb{Z} also acts trivially on $H_{\text{proét}}^*([\mathbb{P}_K^{n-1}/\mathbb{G}_n], \mathbb{Q}_p)$. By direct computation, the continuous group cohomology of \mathbb{Z} over the trivial module \mathbb{Q}_p is given by $H_{\text{cts}}^*(\mathbb{Z}, \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(y)$ with $|y| = 1$. This implies only the 0th and the 1st columns of the spectral sequence $E_2^{i,j} = H_{\text{cts}}^i(\mathbb{Z}, H_{\text{proét}}^j([\mathbb{P}_K^{n-1}/\mathbb{G}_n], \mathbb{Q}_p)) \Rightarrow H_{\text{proét}}^{i+j}([\mathbb{P}_K^{n-1}/\mathbb{G}_n^0], \mathbb{Q}_p)$ could be nonzero. Thus it degenerates on the E_2 -page and gives the factorization $H_{\text{proét}}^*([\mathbb{P}_K^{n-1}/\mathbb{G}_n^0], \mathbb{Q}_p) \cong H_{\text{proét}}^*([\mathbb{P}_K^{n-1}/\mathbb{G}_n], \mathbb{Q}_p) \otimes \Lambda_{\mathbb{Q}_p}(y)$. Using the isomorphism in (5.4) together with Lemma 5.3 and Proposition 4.9, we get

$$H_{\text{proét}}^*([\mathbb{P}_K^{n-1}/\mathbb{G}_n], \mathbb{Q}_p) \cong H_{\text{proét}}^*(\mathbb{P}_K^{n-1}, \mathbb{Q}_p) \otimes \Lambda_{\mathbb{Q}_p}(x_1, x_3, \dots, x_{2n-1})^{\otimes d}.$$

The computation of $H_{\text{proét}}^*(\mathbb{P}_K^{n-1}, \mathbb{Q}_p)$ is immediate from Theorem 3.1 and Lemma 4.5. \square

Remark 5.5. If we try to recover Theorem 5.2 via passing to the Lubin-Tate side and computing the cohomology groups $H_{\text{proét}}^*([\mathbb{P}_K^{n-1}/\mathbb{G}_n^0], \mathbb{Q}_\ell)$, the steps are much simpler than what we did here in the p -adic case since all of the spectral sequences are readily degenerating.

6. APPLICATION: GROUP COHOMOLOGY FOR $\mathrm{GL}_2(\mathbb{Q}_p)$

As we stated in Section 5.2, the problem of computing $H_{\mathrm{cts}}^*(\mathrm{GL}_n(K), \mathrm{Sp}_r(\mathbb{Q}_p)^*)$ appears to be challenging. As we will later see in Proposition 6.2, even when $r = 0$ (so that $\mathrm{Sp}_r(\mathbb{Q}_p)^* \cong \mathbb{Q}_p$) and $K = \mathbb{Q}_p$, the groups $H_{\mathrm{cts}}^*(\mathrm{GL}_n(K), \mathbb{Q}_p)$ already differ significantly from the discrete case in (5.1) and the ℓ -adic case in Lemma 5.1. But now since Theorem 5.4 gives us a description of the p -adic pro-étale cohomology of $[\mathcal{H}_K^{n-1}/\mathrm{GL}_n(K)]$, it is natural to ask whether it is possible to deduce some information about the continuous cohomology groups $H_{\mathrm{cts}}^*(\mathrm{GL}_n(K), \mathrm{Sp}_r(\mathbb{Q}_p)^*)$ for $0 \leq r \leq n-1$ in return using the spectral sequence

$$(6.1) \quad \begin{aligned} E_2^{i,j} &= H_{\mathrm{cts}}^i(\mathrm{GL}_n(K), H_{\mathrm{pro\acute{e}t}}^j(\mathcal{H}_{\mathbb{Q}_p}^{n-1}, \mathbb{Q}_p)) \\ &= H_{\mathrm{cts}}^i(\mathrm{GL}_n(K), \mathrm{Sp}_j(\mathbb{Q}_p)^*) \Rightarrow H_{\mathrm{pro\acute{e}t}}^{i+j}([\mathcal{H}_K^{n-1}/\mathrm{GL}_n(K)], \mathbb{Q}_p). \end{aligned}$$

For simplicity, we will focus on the case when $K = \mathbb{Q}_p$. We start with the case when $r = 0$, and $\mathrm{Sp}_0(\mathbb{Q}_p)^*$ is nothing but the trivial representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ over \mathbb{Q}_p . The following lemma is an immediate consequence of a theorem of Casselman-Wigner generalizing Lazard's comparison theorem between group cohomology and Lie algebra cohomology.

Lemma 6.1. *Let G be $\mathrm{SL}_n(\mathbb{Q}_p)$ or $\mathrm{PGL}_n(\mathbb{Q}_p)$ and let \mathbb{Q}_p be the trivial 1-dimensional G -representation equipped with the p -adic topology. Then we have*

$$H_{\mathrm{cts}}^*(G, \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_3, x_5, \dots, x_{2n-1})$$

as graded \mathbb{Q}_p -algebras where $|x_i| = i$ in the exterior algebra on the right-hand side.

Proof. First notice that G is the group of \mathbb{Q}_p -points of a connected semisimple group defined over \mathbb{Q}_p with $\mathrm{Lie} G = \mathfrak{sl}_n(\mathbb{Q}_p)$. Then by [CW74, Theorem 1], we have an isomorphism of cohomology rings $H_{\mathrm{cts}}^*(G, \mathbb{Q}_p) \cong H_{\mathrm{Lie}}^*(\mathfrak{sl}_n(\mathbb{Q}_p), \mathbb{Q}_p)$, where the latter one is further isomorphic to $H_{\mathrm{Lie}}^*(\mathfrak{sl}_n(\mathbb{Q}), \mathbb{Q}) \otimes \mathbb{Q}_p$. Now as $\mathrm{SU}(n)$ is a compact form of $\mathrm{SL}_n(\mathbb{R})$, the Lie algebra cohomology of $\mathfrak{sl}_n(\mathbb{Q})$ is isomorphic to the rational de Rham cohomology of $\mathrm{SU}(n)$. Thus we get $H_{\mathrm{Lie}}^*(\mathfrak{sl}_n(\mathbb{Q}), \mathbb{Q}) \cong H_{\mathrm{dR}}^*(\mathrm{SU}(n)) \cong \Lambda_{\mathbb{Q}}(x_3, x_5, \dots, x_{2n-1})$ where the degree of x_i is i . \square

Now we compute the continuous group cohomology ring $H_{\mathrm{cts}}^*(\mathrm{GL}_n(\mathbb{Q}_p), \mathbb{Q}_p)$.

Proposition 6.2. *Let \mathbb{Q}_p be the trivial 1-dimensional $\mathrm{GL}_n(\mathbb{Q}_p)$ -representation equipped with the p -adic topology. Then we have*

$$H_{\mathrm{cts}}^*(\mathrm{GL}_n(\mathbb{Q}_p), \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x, y, x_3, x_5, \dots, x_{2n-1})$$

as graded \mathbb{Q}_p -algebras where $|x| = |y| = 1$ and $|x_i| = i$ in the exterior algebra on the right-hand side.

Proof. We first compute the continuous cohomology of the center \mathbb{Q}_p^\times of $\mathrm{GL}_n(\mathbb{Q}_p)$. Consider the decomposition $\mathbb{Q}_p^\times \cong p^\mathbb{Z} \times \mathbb{Z}_p^\times$. By Lemma 4.7, we know $H_{\mathrm{cts}}^*(\mathbb{Z}_p^\times, \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x)$ with $|x| = 1$. For the other part, we have $H_{\mathrm{cts}}^0(p^\mathbb{Z}, \mathbb{Q}_p) = \mathbb{Q}_p$ and $H_{\mathrm{cts}}^1(p^\mathbb{Z}, \mathbb{Q}_p)$ is isomorphic to the group of continuous homomorphisms from \mathbb{Z} to \mathbb{Q}_p , which is again \mathbb{Q}_p . As all higher cohomology groups vanish, we have $H_{\mathrm{cts}}^*(p^\mathbb{Z}, \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(y)$ with $|y| = 1$. Thus by K nneth formula, we have $H_{\mathrm{cts}}^*(\mathbb{Q}_p^\times, \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x, y)$.

Next, since we have the short exact sequence

$$(6.2) \quad 1 \rightarrow \mathrm{SL}_n(\mathbb{Q}_p) \rightarrow \mathrm{GL}_n(\mathbb{Q}_p) \xrightarrow{\det} \mathbb{Q}_p^\times \rightarrow 1,$$

we want to access the cohomology ring of $\mathrm{GL}_n(\mathbb{Q}_p)$ using the Hochschild-Serre spectral sequence

$$(6.3) \quad E_2^{i,j} = H_{\mathrm{cts}}^i(\mathbb{Q}_p^\times, H_{\mathrm{cts}}^j(\mathrm{SL}_n(\mathbb{Q}_p), \mathbb{Q}_p)) \Rightarrow H_{\mathrm{cts}}^{i+j}(\mathrm{GL}_n(\mathbb{Q}_p), \mathbb{Q}_p).$$

By Lemma 6.1 and our computation of $H_{\mathrm{cts}}^*(\mathbb{Q}_p^\times, \mathbb{Q}_p)$ above, one immediately notices that $E_2^{i,j} = 0$ when $i \geq 3$. Since (6.2) splits, the differential maps $d_2^{0,j}$ on the E_2 -page are also 0 for any $j \geq 0$. Thus (6.3) degenerates on the E_2 -page and we get $H_{\mathrm{cts}}^*(\mathrm{GL}_n(\mathbb{Q}_p), \mathbb{Q}_p) \cong H_{\mathrm{cts}}^*(\mathrm{SL}_n(\mathbb{Q}_p), \mathbb{Q}_p) \otimes H_{\mathrm{cts}}^*(\mathbb{Q}_p^\times, \mathbb{Q}_p)$. \square

The following lemma gives a factorization of $H_{\mathrm{pro\acute{e}t}}^*([\mathcal{H}_{\mathbb{Q}_p}^{n-1}/\mathrm{GL}_n(\mathbb{Q}_p)], \mathbb{Q}_p)$ that we will need later.

Lemma 6.3. *The p -adic pro-étale cohomology of the stack $[\mathcal{H}_{\mathbb{Q}_p}^{n-1} / \mathrm{GL}_n(\mathbb{Q}_p)]$ can be expressed as*

$$H_{\mathrm{pro\acute{e}t}}^*([\mathcal{H}_{\mathbb{Q}_p}^{n-1} / \mathrm{GL}_n(\mathbb{Q}_p)], \mathbb{Q}_p) \cong H_{\mathrm{pro\acute{e}t}}^*([\mathcal{H}_{\mathbb{Q}_p}^{n-1} / \mathrm{PGL}_n(\mathbb{Q}_p)], \mathbb{Q}_p) \otimes H_{\mathrm{cts}}^*(\mathbb{Q}_p^\times, \mathbb{Q}_p).$$

Proof. Consider the short exact sequence $1 \rightarrow \mathbb{Q}_p^\times \rightarrow \mathrm{GL}_n(\mathbb{Q}_p) \rightarrow \mathrm{PGL}_n(\mathbb{Q}_p) \rightarrow 1$. From the definition of $\mathcal{H}_{\mathbb{Q}_p}^{n-1}$, we see that \mathbb{Q}_p^\times acts trivially on $\mathcal{H}_{\mathbb{Q}_p}^{n-1}$. Thus we have a commutative diagram of stacks:

$$\begin{array}{ccccc} B\mathbb{Q}_p^\times & \xhookrightarrow{i_1} & [\mathcal{H}_{\mathbb{Q}_p}^{n-1} / \mathrm{GL}_n(\mathbb{Q}_p)] & \xrightarrow{\pi_1} & [\mathcal{H}_{\mathbb{Q}_p}^{n-1} / \mathrm{PGL}_n(\mathbb{Q}_p)] \\ \parallel & & \downarrow f_1 & & \downarrow f_2 \\ B\mathbb{Q}_p^\times & \xhookrightarrow{i_2} & B\mathrm{GL}_n(\mathbb{Q}_p) & \xrightarrow{\pi_2} & B\mathrm{PGL}_n(\mathbb{Q}_p) \end{array}$$

where each row of this diagram gives a fibration of stacks. Specifically, the Leray-Serre spectral sequence corresponding to the fibration described by the first row gives

$$E_2^{i,j} = H_{\mathrm{pro\acute{e}t}}^i([\mathcal{H}_{\mathbb{Q}_p}^{n-1} / \mathrm{PGL}_n(\mathbb{Q}_p)], R^j i_{1,*} \underline{\mathbb{Q}_p}) \Rightarrow H_{\mathrm{pro\acute{e}t}}^{i+j}([\mathcal{H}_{\mathbb{Q}_p}^{n-1} / \mathrm{GL}_n(\mathbb{Q}_p)], \mathbb{Q}_p)$$

where $R^j i_{1,*} \underline{\mathbb{Q}_p}$ is the \mathbb{Q}_p -local system associated to the cohomology of the fiber $B\mathbb{Q}_p^\times$. Since the conjugation action of $\mathrm{PGL}_n(\mathbb{Q}_p)$ on \mathbb{Q}_p^\times is trivial, the monodromy action is trivial and the local system is in fact constant. Thus the spectral sequence above can be rewritten into

$$(6.4) \quad E_2^{i,j} = H_{\mathrm{pro\acute{e}t}}^i([\mathcal{H}_{\mathbb{Q}_p}^{n-1} / \mathrm{PGL}_n(\mathbb{Q}_p)], H_{\mathrm{cts}}^j(\mathbb{Q}_p^\times, \mathbb{Q}_p)) \Rightarrow H_{\mathrm{pro\acute{e}t}}^{i+j}([\mathcal{H}_{\mathbb{Q}_p}^{n-1} / \mathrm{GL}_n(\mathbb{Q}_p)], \mathbb{Q}_p)$$

and we are left to show this spectral sequence degenerates on the E_2 -page.

By Lemma 6.1 and Proposition 6.2 and dimension-counting, we see the Leray-Serre spectral sequence associated to the second row of the diagram above degenerates. So the edge homomorphism

$$i_2^* : H_{\mathrm{cts}}^j(\mathrm{GL}_n(\mathbb{Q}_p), \mathbb{Q}_p) \rightarrow H_{\mathrm{cts}}^j(\mathbb{Q}_p^\times, \mathbb{Q}_p)$$

is surjective for all $j \geq 0$. By commutativity of the diagram, we have $i_2^* = i_1^* \circ f_1^*$. Thus the edge homomorphism of the spectral sequence (6.4)

$$i_1^* : H_{\mathrm{pro\acute{e}t}}^j([\mathcal{H}_{\mathbb{Q}_p}^{n-1} / \mathrm{GL}_n(\mathbb{Q}_p)], \mathbb{Q}_p) \rightarrow E_\infty^{0,j} \subseteq E_2^{0,j} = H_{\mathrm{cts}}^j(\mathbb{Q}_p^\times, \mathbb{Q}_p)$$

is surjective for all $j \geq 0$ and (6.4) degenerates on the second page. \square

In the rest of this section, we restrict to the case when $n = 2$ and use Theorem 5.4 together with Proposition 6.2 to deduce a description of $H_{\mathrm{cts}}^*(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{Sp}_1(\mathbb{Q}_p)^*)$. Notice that when $n = 2$, the only $\mathrm{Sp}_r(\mathbb{Q}_p)^*$ appearing are for $r = 0$ and $r = 1$.

Lemma 6.4. *The Hochschild-Serre spectral sequence*

$$E_2^{i,j} = H_{\mathrm{cts}}^i(\mathrm{PGL}_2(\mathbb{Q}_p), H_{\mathrm{pro\acute{e}t}}^j(\mathcal{H}_{\mathbb{Q}_p}^1, \mathbb{Q}_p)) \Rightarrow H_{\mathrm{pro\acute{e}t}}^{i+j}([\mathcal{H}_{\mathbb{Q}_p}^1 / \mathrm{PGL}_2(\mathbb{Q}_p)], \mathbb{Q}_p)$$

degenerates on the E_2 -page.

Proof. By Proposition 4.11, only the first three rows of this spectral sequence are nonzero. By Lemma 6.1, we have $H_{\mathrm{cts}}^*(\mathrm{PGL}_2(\mathbb{Q}_p), \mathbb{Q}_p) \cong \Lambda_{\mathbb{Q}_p}(x_3)$, and we also write H^i for $H_{\mathrm{cts}}^i(\mathrm{PGL}_2(\mathbb{Q}_p), \mathrm{Sp}_1(\mathbb{Q}_p)^*)$. As many terms on the E_2 -page are zero, almost all $E_2^{i,j}$ are already stable with only two differentials we need to check more carefully. The first one is the map $d_3^{0,2} : H^0 \rightarrow E_3^{3,0} = \mathbb{Q}_p$. But by definition of the Steinberg representation $\mathrm{Sp}_1(\mathbb{Q}_p)$, there are no stable vectors in $\mathrm{Sp}_1(\mathbb{Q}_p)^*$ under the action of $\mathrm{PGL}_2(\mathbb{Q}_p)$, and thus $d_3^{0,2}$ is zero. The second map we need to check is $d_2^{1,2} : H^1 \rightarrow E_2^{3,1} = \mathbb{Q}_p^2$. By Proposition 4.11, we have $H_{\mathrm{pro\acute{e}t}}^1(\mathcal{H}_{\mathbb{Q}_p}^1, \mathbb{Q}_p) \cong \mathbb{Q}_p^2$ and the term $E_2^{3,1}$ here arises from $H_{\mathrm{cts}}^3(\mathrm{PGL}_2(\mathbb{Q}_p), \mathbb{Q}_p^2) \cong (H_{\mathrm{cts}}^3(\mathrm{PGL}_2(\mathbb{Q}_p), \mathbb{Q}_p))^2$. Now the stability of $E_2^{3,0}$ implies that every element of $H_{\mathrm{cts}}^3(\mathrm{PGL}_2(\mathbb{Q}_p), \mathbb{Q}_p)$ is a permanent cycle for the spectral sequence. So the elements in $E_2^{3,1}$ are also permanent cycles, and $d_2^{1,2}$ is zero. \square

Theorem 6.5. *For the dual Steinberg representation $\mathrm{Sp}_1(\mathbb{Q}_p)^*$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, we have*

$$H_{\mathrm{cts}}^*(\mathrm{PGL}_2(\mathbb{Q}_p), \mathrm{Sp}_1(\mathbb{Q}_p)^*) \cong H_{\mathrm{cts}}^*(\mathrm{PGL}_2(\mathbb{Q}_p), \mathbb{Q}_p)[-1]$$

and also

$$H_{\mathrm{cts}}^*(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{Sp}_1(\mathbb{Q}_p)^*) \cong H_{\mathrm{cts}}^*(\mathrm{GL}_2(\mathbb{Q}_p), \mathbb{Q}_p)[-1].$$

Proof. Following notations from Lemma 6.4, we write H^i for $H_{\mathrm{cts}}^i(\mathrm{PGL}_2(\mathbb{Q}_p), \mathrm{Sp}_1(\mathbb{Q}_p)^*)$ and also denote the dimension of H^i (as a \mathbb{Q}_p -vector space) by h^i . Then using Theorem 5.4 and Lemma 6.3, we get an explicit description of the dimension of the p -adic pro-étale cohomology group of $[\mathcal{H}_{\mathbb{Q}_p}^1 / \mathrm{PGL}_2(\mathbb{Q}_p)]$ in all degrees. The degeneration of the spectral sequence in Lemma 6.4 then allows us to determine the dimensions of H^i , yielding

$$h^0 = 0, \quad h^1 = 1, \quad h^2 = 0, \quad h^3 = 0, \quad h^4 = 1,$$

and $h^i = 0$ for $i \geq 5$. Comparing this with the description of the cohomology ring of $\mathrm{PGL}_2(\mathbb{Q}_p)$ over the trivial representation \mathbb{Q}_p given by Lemma 6.1 gives the claimed shifting.

For the case of $\mathrm{GL}_2(\mathbb{Q}_p)$, we first denote by \tilde{h}^i the dimension of the i -th continuous cohomology group of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\mathrm{Sp}_1(\mathbb{Q}_p)^*$. Then consider the spectral sequence

$$E_2^{i,j} = H_{\mathrm{cts}}^i(\mathrm{PGL}_2(\mathbb{Q}_p), H_{\mathrm{cts}}^j(\mathbb{Q}_p^\times, \mathrm{Sp}_1(\mathbb{Q}_p)^*)) \Rightarrow H_{\mathrm{cts}}^{i+j}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{Sp}_1(\mathbb{Q}_p)^*).$$

Since the center of $\mathrm{GL}_2(\mathbb{Q}_p)$ acts trivially on $\mathrm{Sp}_1(\mathbb{Q}_p)^*$ by definition, this spectral sequence can be written as

$$(6.5) \quad E_2^{i,j} = H_{\mathrm{cts}}^i(\mathrm{PGL}_2(\mathbb{Q}_p), H_{\mathrm{cts}}^j(\mathbb{Q}_p^\times, \mathbb{Q}_p) \otimes \mathrm{Sp}_1(\mathbb{Q}_p)^*) \Rightarrow H_{\mathrm{cts}}^{i+j}(\mathrm{GL}_2(\mathbb{Q}_p), \mathrm{Sp}_1(\mathbb{Q}_p)^*).$$

Proposition 6.2, together with our computation of the $\mathrm{PGL}_2(\mathbb{Q}_p)$ case above, gives all the terms $E_2^{i,j}$. Specifically, only the first three entries of the first and the fourth columns of this spectral sequence are nonzero. Therefore, all of the differential maps on the E_2 -page are zero except possibly $d_3^{1,2} : E_3^{1,2} \rightarrow E_3^{4,0}$. By direct computation, both $E_3^{1,2}$ and $E_3^{4,0}$ are one-dimensional. Thus if $d_3^{1,2}$ is nonzero, we would have $E_4^{4,0} = E_\infty^{4,0} = 0$, which further gives $\tilde{h}^4 = 0$. But this is impossible as we already have $h^4 = 1$. Therefore, the spectral sequence (6.5) degenerates on the E_2 -page which then gives

$$\tilde{h}^0 = 0, \quad \tilde{h}^1 = 1, \quad \tilde{h}^2 = 2, \quad \tilde{h}^3 = 1, \quad \tilde{h}^4 = 1, \quad \tilde{h}^5 = 2, \quad \tilde{h}^6 = 1,$$

and $\tilde{h}^i = 0$ for $i \geq 7$. Comparing this with Proposition 6.2 finishes the proof. \square

When $n > 2$, one cannot deduce descriptions of $H_{\mathrm{cts}}^*(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{Sp}_r(\mathbb{Q}_p)^*)$ in the same way as when $n = 2$: on one hand there would be too many r we need to deal with simultaneously; on the other hand, we also do not know whether the spectral sequence (6.1) degenerates on the E_2 -page.

Remark 6.6. Nevertheless, Theorem 6.5 still offers a plausible indication for the case of general n , suggesting that $H_{\mathrm{cts}}^*(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{Sp}_r(\mathbb{Q}_p)^*)$ might be given by a degree- r shift of the cohomology ring $H_{\mathrm{cts}}^*(\mathrm{GL}_n(\mathbb{Q}_p), \mathbb{Q}_p)$. Indeed, under the supposition that such a "shifting" holds, if we further assume (6.1) degenerates on the E_2 -page, then inserting these cohomology groups into (6.1) reproduces the cohomology groups of $[\mathcal{H}_{\mathbb{Q}_p}^{n-1} / \mathrm{GL}_n(\mathbb{Q}_p)]$ given in Theorem 5.4.

Remark 6.7. Another reason we find such a speculation of "shifting" reasonable is its similarity to the case of continuous cohomology of $\mathrm{GL}_n(\mathbb{Q}_p)$ with respect to duals of generalized Steinberg representations over \mathbb{C} . From [BW00, Proposition X.4.7] and taking into consideration the center of $\mathrm{GL}_n(\mathbb{Q}_p)$, one could deduce

$$H_{\mathrm{cts}}^i(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{Sp}_r(\mathbb{C})^*) \cong \begin{cases} \mathbb{C} & \text{if } i = r, r + 1 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the cohomology ring $H_{\mathrm{cts}}^*(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{Sp}_r(\mathbb{C})^*)$ is exactly a degree- r shift of $H_{\mathrm{cts}}^*(\mathrm{GL}_n(\mathbb{Q}_p), \mathbb{C})$.

In the following section, we will prove that such a speculation is indeed true, employing the language of condensed math from [CS19] and also the framework of solid representations recently developed by Rodrigues Jacinto–Rodríguez Camargo in [RJRC22] and [RJRC25].

7. GENERALIZATION TO $\mathrm{GL}_n(\mathbb{Q}_p)$

Our goal of this section is to generalize Theorem 6.5 to $\mathrm{GL}_n(\mathbb{Q}_p)$. Specifically, we will show:

Theorem 7.1. *For $0 \leq r \leq n-1$, we have an isomorphism of graded \mathbb{Q}_p -vector spaces*

$$H_{\mathrm{cts}}^*(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{Sp}_r(\mathbb{Q}_p)^*) \cong \Lambda_{\mathbb{Q}_p}(x, y, x_3, x_5, \dots, x_{2n-1})[-r],$$

where $|x| = |y| = 1$ and $|x_i| = i$ in the exterior algebra on the right-hand side.

For a discrete ring R , the cohomology groups $H_{\mathrm{cts}}^*(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{Sp}_r(R)^*)$ are well-understood if R has characteristic 0 or $R = \mathbb{Z}/m\mathbb{Z}$ for suitably chosen m by the work of [Orl05] and [Dat06]. For such a ring R , [Orl05, Corollary 2] and [Dat06, Corollaire 2.1.7] give a description of the extension groups

$$(7.1) \quad \mathrm{Ext}_G^*(\mathrm{Sp}_I(R), \mathrm{Sp}_J(R))$$

for any reductive p -adic Lie group G and any subsets I, J of simple roots of G . One can then compute $H_{\mathrm{cts}}^*(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{Sp}_r(R)^*)$ by realizing them as $\mathrm{Ext}_{\mathrm{GL}_n(\mathbb{Q}_p)}^*(\mathrm{Sp}_r(R), \mathrm{Sp}_0(R))$.

In an attempt to use such a strategy to compute the cohomology of $\mathrm{GL}_n(\mathbb{Q}_p)$ over $\mathrm{Sp}_r(\mathbb{Q}_p)^*$, one immediately encounters the following problem: the ring R and also the representations $\mathrm{Sp}_I(R)$ in [Orl05] and [Dat06] are equipped with the discrete topology, and the Ext-groups in (7.1) are computed in the abelian category of smooth G -representations with the discrete topology; on the other hand, the representations $\mathrm{Sp}_r(\mathbb{Q}_p)$ are equipped with the p -adic topology, but the category of smooth G -representations over topological abelian groups is not an abelian category. Thus it no longer makes sense to talk about Ext-groups in such a category.

To salvage this, we find the theory of solid representations developed in [RJRC22] and [RJRC25] perfect for our purpose as it provides us some abelian categories which contain representations like $\mathrm{Sp}_r(\mathbb{Q}_p)$. This allows us to study Ext-groups between generalized Steinberg representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ over \mathbb{Q}_p using strategies analogous to [Orl05] and [Dat06]. Specifically, the key ideas follow closely from those appeared in [Orl05].

For the rest of this section, let $G = \mathrm{GL}_n(\mathbb{Q}_p)$ and also let $K = \mathbb{Q}_p$. Following the notations¹ of [RJRC25], we let $\mathrm{Rep}_{K\Box}(G)$ be the category of continuous G -representations over solid K -vector spaces and let $\mathrm{Rep}_{K\Box}^\infty(G)$ be its subcategory of smooth G -representations. Both of these are abelian categories and thus have enough injectives (upon fixing the cardinal κ), see e.g. [Bos23, Remark 2.11]. The category $\mathrm{Rep}_{K\Box}(G)$ also has enough projectives. We denote by $\mathrm{Hom}_{K\Box[G]}$ the Hom-functor of $\mathrm{Rep}_{K\Box}(G)$ and we let $\mathrm{Ext}_{K\Box[G]}$ denote its derived functors. For $\mathcal{V} \in \mathrm{Rep}_{K\Box}(G)$, we denote its smooth vectors by \mathcal{V}^∞ and its smooth dual $(\mathcal{V}^*)^\infty$ by $\tilde{\mathcal{V}}$.

Lemma 7.2. *For any $I \subseteq \Delta$ and parabolic subgroup $P_I \subseteq G$, the representations $\underline{\mathrm{Ind}}_{P_I}^G \mathbb{1}, \underline{\mathrm{Sp}}_I(\mathbb{Q}_p), \underline{\mathrm{Sp}}_I(\mathbb{Q}_p)^*$ are all objects of $\mathrm{Rep}_{K\Box}(G)$.*

Proof. The representations $\underline{\mathrm{Ind}}_{P_I}^G \mathbb{1}, \underline{\mathrm{Sp}}_I(\mathbb{Q}_p), \underline{\mathrm{Sp}}_I(\mathbb{Q}_p)^*$ are all complete locally convex G -representations over \mathbb{Q}_p . The lemma then follows from [RJRC22, Proposition 3.7]. \square

Lemma 7.3. *The functor*

$$\begin{aligned} (-)^\infty : \mathrm{Rep}_{K\Box}(G) &\rightarrow \mathrm{Rep}_{K\Box}^\infty(G) \\ \mathcal{V} &\mapsto \mathcal{V}^\infty := \varinjlim_{\substack{H \leq G \\ \text{compact open}}} \mathcal{V}^H \end{aligned}$$

preserves injective resolutions.

Proof. For $\mathcal{V} \in \mathrm{Rep}_{K\Box}(G)$ and $H \leq G$ compact open, it's clear that the functor $\mathcal{V} \mapsto \mathcal{V}^H$ is exact. As $\mathrm{Rep}_{K\Box}(G)$ is abelian, filtered colimits of exact functors remain exact, which implies exactness of $(-)^\infty$. As both categories are abelian and have enough injectives, we deduce from [Stacks25, Tag 015Z] that $(-)^\infty$ preserves injective resolutions. \square

¹Unlike in [RJRC25], we will not use the derived language.

Proposition 7.4. For $\mathcal{U} \in \text{Rep}_{K_\square}^\infty(G)$ and $\mathcal{V} \in \text{Rep}_{K_\square}(G)$, we have $\text{Ext}_{K_\square[G]}^i(\mathcal{U}, \mathcal{V}) \cong \text{Ext}_{K_\square[G]}^i(\mathcal{U}, \mathcal{V}^\infty)$ for any $i \geq 0$. Specifically, if V is a complete locally convex representation of G over K , then $H_{\text{cts}}^i(G, V) \cong H_{\text{cts}}^i(G, V^\infty)$ for any $i \geq 0$.

Proof. Take an injective resolution $0 \rightarrow \mathcal{V} \rightarrow \mathcal{J}^\bullet$ of \mathcal{V} in $\text{Rep}_{K_\square}(G)$ and also consider the injective resolution $0 \rightarrow \mathcal{V}^\infty \rightarrow (\mathcal{J}^\bullet)^\infty$ of \mathcal{V}^∞ in $\text{Rep}_{K_\square}^\infty(G)$ given by Lemma 7.3. As \mathcal{U} is smooth, we have $f(\mathcal{U}) \subseteq \mathcal{W}^\infty$ for any continuous G -homomorphism $f : \mathcal{U} \rightarrow \mathcal{W}$. This implies $\text{Hom}_{K_\square[G]}(\mathcal{U}, \mathcal{J}^\bullet) \cong \text{Hom}_{K_\square[G]}(\mathcal{U}, (\mathcal{J}^\bullet)^\infty)$ and the first assertion follows. For V a complete locally convex G -representation, V^∞ is also complete and locally convex. The isomorphism $H_{\text{cts}}^i(G, V) \cong H_{\text{cts}}^i(G, V^\infty)$ then follows from [RJRC22, Lemma 5.2] and the first statement. \square

Lemma 7.5. Given $\mathcal{U}, \mathcal{V} \in \text{Rep}_{K_\square}^\infty(G)$, if \mathcal{V} is further admissible, then

$$\text{Ext}_{K_\square[G]}^i(\mathcal{U}, \mathcal{V}) \cong \text{Ext}_{K_\square[G]}^i(\widetilde{\mathcal{V}}, \widetilde{\mathcal{U}})$$

for any $i \geq 0$.

Proof. The classical case of such an isomorphism appeared in [Orl05, Lemma 6], and we follow the same strategy here. As $\text{Rep}_{K_\square}(G)$ is an abelian category with enough injectives and projectives, we can take a projective resolution $0 \leftarrow \mathcal{U} \leftarrow \mathcal{P}^\bullet$ in $\text{Rep}_{K_\square}(G)$ and its smooth dual gives an injective resolution of $\widetilde{\mathcal{U}}$. By [Vig96, Proposition I.4.13], we have $\text{Hom}_{K_\square[G]}(\mathcal{P}^i, \widetilde{\mathcal{V}}) \cong \text{Hom}_{K_\square[G]}(\mathcal{V}, \widetilde{\mathcal{P}}^i)$, which implies $\text{Ext}_{K_\square[G]}^i(\mathcal{U}, \widetilde{\mathcal{V}}) \cong \text{Ext}_{K_\square[G]}^i(\mathcal{V}, \widetilde{\mathcal{U}})$. As \mathcal{V} is admissible, we have $\widetilde{\mathcal{V}} \cong \mathcal{V}$ by [Cas95, Proposition 2.1.10] and the claim follows. \square

Lemma 7.6. For any parabolic subgroup $P_I \subseteq G$ associated to $I \subsetneq \Delta$, we have $\text{Ext}_{K_\square[G]}^*(\text{Ind}_{P_I}^G \mathbb{1}, \mathbb{1}) = 0$.

Proof. Let δ_{P_I} denote the modulus character of the parabolic subgroup P_I . Then by [Vig96, Chapitre I, 5.11] we have $\widetilde{\text{Ind}_{P_I}^G \mathbb{1}} = \text{Ind}_{P_I}^G \delta_{P_I}$. Thus we get

$$\begin{aligned} \text{Ext}_{K_\square[G]}^*(\text{Ind}_{P_I}^G \mathbb{1}, \mathbb{1}) &\cong \text{Ext}_{K_\square[G]}^*(\mathbb{1}, \widetilde{\text{Ind}_{P_I}^G \mathbb{1}}) \\ &\cong \text{Ext}_{K_\square[G]}^*(\mathbb{1}, \text{Ind}_{P_I}^G \delta_{P_I}) \\ &\cong \text{Ext}_{K_\square[P_I]}^*(\mathbb{1}, \delta_{P_I}) \end{aligned}$$

where we used Lemma 7.5 in the first isomorphism and the third isomorphism comes from Shapiro's lemma. Since I is a proper subset of Δ , the modulus character δ_{P_I} is not the trivial character and [BW00, Proposition XI.1.9] implies there does not exist any extension of $\mathbb{1}$ by δ_{P_I} . This gives $\text{Ext}_{K_\square[G]}^*(\mathbb{1}, \delta_{P_I}) = 0$, as desired. \square

Proof of Theorem 7.1. As the representations $\text{Sp}_r(\mathbb{Q}_p)^*$ are complete and locally convex, Proposition 7.4 and Lemma 7.5 together give $H_{\text{cts}}^*(\text{GL}_n(\mathbb{Q}_p), \text{Sp}_r(\mathbb{Q}_p)^*) \cong \text{Ext}_{K_\square[G]}^*(\text{Sp}_r(\mathbb{Q}_p), \mathbb{1})$. Since $\text{Ext}_{K_\square[G]}^*(\mathbb{1}, \mathbb{1})$ is just $H_{\text{cts}}^*(\text{GL}_n(\mathbb{Q}_p), \mathbb{Q}_p)$ which is isomorphic to $\Lambda_{\mathbb{Q}_p}(x, y, x_3, x_5, \dots, x_{2n-1})$ by Proposition 6.2, it suffices to show $\text{Ext}_{K_\square[G]}^*(\text{Sp}_r(\mathbb{Q}_p), \mathbb{1}) \cong \text{Ext}_{K_\square[G]}^*(\mathbb{1}, \mathbb{1})[-r]$.

Now recall that $\text{Sp}_r(\mathbb{Q}_p)$ is defined as the generalized Steinberg representation of $\text{GL}_n(\mathbb{Q}_p)$ associated to the subset of simple roots $\{1, \dots, n-1-r\} \subseteq \Delta$. Let us denote the set $\{1, \dots, n-1-r\}$ by I_r . Then [SS91, Proposition 13] (see also p.88 of loc. cit.) gives an acyclic resolution of $\text{Sp}_r(\mathbb{Z})$

$$0 \rightarrow \mathbb{Z} \rightarrow \bigoplus_{\substack{I_r \subseteq I \subseteq \Delta \\ |\Delta \setminus I|=1}} \text{Ind}_{P_I}^G \mathbb{1} \rightarrow \dots \rightarrow \bigoplus_{\substack{I_r \subseteq I \subseteq \Delta \\ |I \setminus I_r|=1}} \text{Ind}_{P_I}^G \mathbb{1} \rightarrow \text{Ind}_{P_{I_r}}^G \mathbb{1} \rightarrow \text{Sp}_r(\mathbb{Z}) \rightarrow 0$$

in the category of smooth $\text{GL}_n(\mathbb{Q}_p)$ -representations with discrete topology. Tensoring this exact sequence with \mathbb{Q}_p and condensing it, we get an acyclic resolution of $\text{Sp}_r(\mathbb{Q}_p)$

$$(7.2) \quad 0 \rightarrow \mathbb{Q}_p \rightarrow \bigoplus_{\substack{I_r \subseteq I \subseteq \Delta \\ |\Delta \setminus I|=1}} \text{Ind}_{P_I}^G \mathbb{1} \rightarrow \dots \rightarrow \bigoplus_{\substack{I_r \subseteq I \subseteq \Delta \\ |I \setminus I_r|=1}} \text{Ind}_{P_I}^G \mathbb{1} \rightarrow \text{Ind}_{P_{I_r}}^G \mathbb{1} \rightarrow \text{Sp}_r(\mathbb{Q}_p) \rightarrow 0$$

in the category $\text{Rep}_{K_\square}^\infty(G)$. Now apply the functor $\text{Hom}_{K_\square[G]}(-, \mathbb{1})$ to (7.2) and pick an injective resolution $0 \rightarrow \mathbb{1} \rightarrow \mathcal{J}^\bullet$ in the category $\text{Rep}_{K_\square}^\infty(G)$, we get a double complex of the form

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & \\
 \cdots \rightarrow & \text{Hom}_{K_\square[G]} \left(\bigoplus_{\substack{I_r \subseteq I \subseteq \Delta \\ |I \setminus I_r| = i}} \text{Ind}_{P_I}^G \mathbb{1}, \mathcal{J}^{j+1} \right) & \rightarrow & \text{Hom}_{K_\square[G]} \left(\bigoplus_{\substack{I_r \subseteq I \subseteq \Delta \\ |I \setminus I_r| = i+1}} \text{Ind}_{P_I}^G \mathbb{1}, \mathcal{J}^{j+1} \right) & \rightarrow \cdots \\
 & \uparrow & & \uparrow & \\
 \cdots \rightarrow & \text{Hom}_{K_\square[G]} \left(\bigoplus_{\substack{I_r \subseteq I \subseteq \Delta \\ |I \setminus I_r| = i}} \text{Ind}_{P_I}^G \mathbb{1}, \mathcal{J}^j \right) & \rightarrow & \text{Hom}_{K_\square[G]} \left(\bigoplus_{\substack{I_r \subseteq I \subseteq \Delta \\ |I \setminus I_r| = i+1}} \text{Ind}_{P_I}^G \mathbb{1}, \mathcal{J}^j \right) & \rightarrow \cdots \\
 & \uparrow & & \uparrow & \\
 & \vdots & & \vdots &
 \end{array}$$

The E_1 -page of the spectral sequence associated to such a double complex then gives

$$(7.3) \quad E_1^{i,j} = \text{Ext}_{K_\square[G]}^j \left(\bigoplus_{\substack{I_r \subseteq I \subseteq \Delta \\ |I \setminus I_r| = i}} \text{Ind}_{P_I}^G \mathbb{1}, \mathbb{1} \right) \Rightarrow \text{Ext}_{K_\square[G]}^{i+j} (\text{Sp}_r(\mathbb{Q}_p), \mathbb{1}).$$

By Lemma 7.6, the columns of (7.3) are all zero except the r -th column, which is $\text{Ext}_{K_\square[G]}^*(\mathbb{1}, \mathbb{1})$. Thus we get an isomorphism $\text{Ext}_{K_\square[G]}^*(\text{Sp}_r(\mathbb{Q}_p), \mathbb{1}) \cong \text{Ext}_{K_\square[G]}^*(\mathbb{1}, \mathbb{1})[-r]$, and this finishes the proof. \square

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