MTH 9821 Numerical Methods for Finance
Homework 1
Team 5, Member: Wenxin Shi, Mingxiang Jia, Leyao Zhang, Chenyu Zhao

(1)

At time t=0, construct a portfolio by

- Long $e^{-q\delta t}$ unit of the underlying asset
- Cash position: $-S(0)e^{-q\delta t}$

The present value $V(0)$ of this portfolio is zero, then according to the generalized law of one price, the future value of this portfolio should always be zero.

Thus, at time $t = \delta t$ (period 1), it follows that

$$V(\delta t)_{up} = \ S(0) \cdot u - S(0)e^{(r-q)\delta t} \geq 0$$
$$V(\delta t)_{down} = S(0) \cdot d - S(0)e^{(r-q)\delta t} \leq 0$$

Since $u > d$, if $V_{down} = 0$, then $V_{up} > 0$, thus at period 1, the future value of this portfolio will be non-negative with probability 1, and be positive with a positive probability, which is conflict with no-arbitrage assumption, thus $V_{down} < 0$.

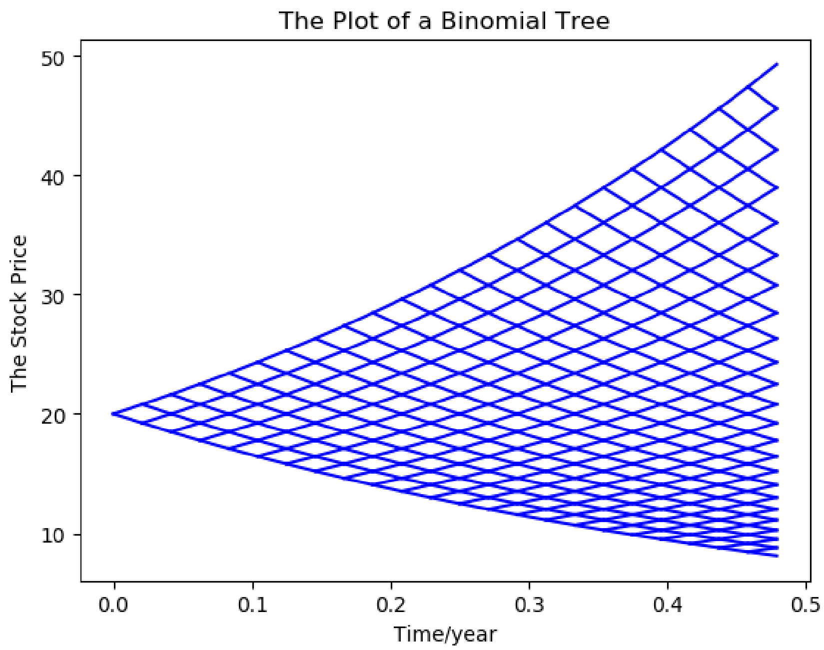
Similarly, it's easy to show that $V_{up} > 0$. In summary, we have

$$S(0) \cdot d < S(0)e^{(r-q)\delta t} < S(0) \cdot u$$

Simplify it, we get

$$d < e^{(r-q)\delta t} < u$$

(2)



(3)

View $p = \frac{e^{(r-q)\delta t} - d}{(u-d)}$ as a function of $\sqrt{\delta t}$, then to show the argument, we only need to compute $p(\sqrt{\delta t})$'s Taylor expansion to the first order, which means we only need to compute the value of $p(0)$, and $\frac{dp}{d\sqrt{\delta t}}$ $(\sqrt{\delta t} = 0)$.

Plug in the value of u and d, and use L'Hôpital's rule, we have

$$p(0)=\lim_{\sqrt{\delta t}\to 0}\frac{e^{(r-q)\delta t}-e^{-\sigma\sqrt{\delta t}}}{e^{\sigma\sqrt{\delta t}}-e^{-\sigma\sqrt{\delta t}}}=\lim_{\delta t\to 0}\frac{e^{(r-q)\delta t}\cdot 2(r-q)\sqrt{\delta t}+\sigma e^{-\sigma\sqrt{\delta t}}}{\sigma e^{\sigma\sqrt{\delta t}}+\sigma e^{-\sigma\sqrt{\delta t}}}=\frac{\sigma}{\sigma+\sigma}=\frac{1}{2}$$

For $\frac{\mathrm{dp}}{\mathrm{d}\sqrt{\delta t}}$ $(\sqrt{\delta t}=0)$,

$$\lim_{\sqrt{\delta t}\to 0}\frac{\mathrm{dp}}{\mathrm{d}\sqrt{\delta t}}=\lim_{\sqrt{\delta t}\to 0}\left[\frac{\left(e^{(r-q)\delta t}\cdot 2(r-q)\sqrt{\delta t}+\sigma e^{-\sigma\sqrt{\delta t}}\right)\left(e^{\sigma\sqrt{\delta t}}-e^{-\sigma\sqrt{\delta t}}\right)-\left(e^{(r-q)\delta t}-e^{-\sigma\sqrt{\delta t}}\right)\left(\sigma e^{\sigma\sqrt{\delta t}}+\sigma e^{-\sigma\sqrt{\delta t}}\right)}{\left(e^{\sigma\sqrt{\delta t}}-e^{-\sigma\sqrt{\delta t}}\right)^2}\right]$$

Abbreviate the above equation, we get

$$\lim_{\sqrt{\delta t}\to 0}\frac{\mathrm{dp}}{\mathrm{d}\sqrt{\delta t}}=\lim_{\sqrt{\delta t}\to 0}\frac{e^{(r-q)\delta t}\cdot 2(r-q)\sqrt{\delta t}+\sigma e^{-\sigma\sqrt{\delta t}}}{e^{\sigma\sqrt{\delta t}}-e^{-\sigma\sqrt{\delta t}}}-\lim_{\sqrt{\delta t}\to 0}\frac{\left(e^{(r-q)\delta t}-e^{-\sigma\sqrt{\delta t}}\right)\left(\sigma e^{\sigma\sqrt{\delta t}}+\sigma e^{-\sigma\sqrt{\delta t}}\right)}{\left(e^{\sigma\sqrt{\delta t}}-e^{-\sigma\sqrt{\delta t}}\right)^2}=\text{(i)}-\text{(ii)}$$

Apply L'Hôpital's rule to the first term (1), we have

$$\text{(i)}=\lim_{\sqrt{\delta t}\to 0}\frac{\left(e^{(r-q)\delta t}\cdot\left(2(r-q)\sqrt{\delta t}\right)^2+e^{(r-q)\delta t}\cdot 2(r-q)-\sigma^2 e^{-\sigma\sqrt{\delta t}}\right)}{\sigma e^{\sigma\sqrt{\delta t}}+\sigma e^{-\sigma\sqrt{\delta t}}}=\frac{2(r-q)-\sigma^2}{2\sigma}$$

Similarly, we can get

$$\text{(ii)}=\lim_{\sqrt{\delta t}\to 0}\frac{\left(e^{(r-q)\delta t}\cdot 2(r-q)\sqrt{\delta t}+\sigma e^{-\sigma\sqrt{\delta t}}\right)\left(\sigma e^{\sigma\sqrt{\delta t}}+\sigma e^{-\sigma\sqrt{\delta t}}\right)+\left(e^{(r-q)\delta t}-e^{-\sigma\sqrt{\delta t}}\right)\left(\sigma^2 e^{\sigma\sqrt{\delta t}}-\sigma^2 e^{-\sigma\sqrt{\delta t}}\right)}{2\left(e^{\sigma\sqrt{\delta t}}-e^{-\sigma\sqrt{\delta t}}\right)\left(\sigma e^{\sigma\sqrt{\delta t}}+\sigma e^{-\sigma\sqrt{\delta t}}\right)}=\frac{2(\mathrm{r}-\mathrm{q})-\sigma^2}{2\cdot 2\sigma}$$

Combine (i) and (ii), we have

$$\frac{\mathrm{dp}}{d\sqrt{\delta t}}\left(\sqrt{\delta t}=0\right)=\frac{1}{2}\left(\frac{r-q}{\sigma}-\frac{\sigma}{2}\right)$$

Then, using the formula for Taylor expansion, we have

$$\frac{\left(\mathrm{e}^{(\mathrm{r}-\mathrm{q})\delta\mathrm{t}}-d\right)}{u-d}=p(0)+\frac{dp}{d\sqrt{\delta t}}(0)\sqrt{\delta t}+O\left(\sqrt{\delta t}^2\right)=\frac{1}{2}+\frac{1}{2}\left(\frac{r-q}{\sigma}-\frac{\sigma}{2}\right)\sqrt{\delta t}+O(\delta t)$$

(4)

Assume the number of upward move over the time T is ξ, then

$$\mathrm{S_N}=\mathrm{S_0}\cdot\mathrm{u}^{\xi}\cdot\mathrm{d}^{\mathrm{N}-\xi}=S_0\cdot u^{2\xi-N}$$

So $log\left(\frac{\mathrm{S_N}}{\mathrm{S_0}}\right)=(2\xi-\mathrm{N})logu$, since $\mathrm{u}=\mathrm{e}^{\sigma\sqrt{\delta t}}$, we have $\log\left(\frac{\mathrm{S_N}}{\mathrm{S_0}}\right)=(2\xi-\mathrm{N})\sigma\sqrt{\frac{\mathrm{T}}{\mathrm{N}}}$, for calculation simplicity, let

$$\mathrm{X}=\frac{\log\left(\frac{\mathrm{S_N}}{\mathrm{S_0}}\right)}{\sigma\sqrt{\mathrm{T}}}=\frac{2\xi-\mathrm{N}}{\sqrt{\mathrm{N}}}$$

and we will calculate the moment generating function for X.

$$\mathrm{E}(\mathrm{e}^{\mathrm{tX}})=\sum_{\mathrm{n}=0}^{\mathrm{N}}\mathrm{e}^{\mathrm{t}\frac{2\mathrm{n}-\mathrm{N}}{\sqrt{\mathrm{N}}}}\binom{N}{n}p^n(1-p)^{N-n}=\sum_{n=0}^{N}\left(e^{\frac{2t}{\sqrt{N}}}\right)^n\cdot e^{-\sqrt{N}t}\binom{N}{2}\mathrm{p}^{\mathrm{n}}(1-\mathrm{p})^{\mathrm{N}-\mathrm{n}}$$

Combine $\left(e^{\frac{2t}{\sqrt{N}}}\right)^n$ with p^{n}, and use the formula of binomial expansion, we have

$$\mathrm{E}(e^{tX})=\left(\mathrm{p}\mathrm{e}^{\frac{2\mathrm{t}}{\sqrt{\mathrm{N}}}}+1-\mathrm{p}\right)^{\mathrm{N}}\mathrm{e}^{-\sqrt{\mathrm{N}}\mathrm{t}}=\left(\mathrm{p}\left(\mathrm{e}^{\frac{2\mathrm{t}}{\sqrt{\mathrm{N}}}}-1\right)+1\right)^N\cdot e^{-\sqrt{N}t}\qquad(1)$$

Next, I will plug in the value of p and use Taylor expansion to expand the above terms,

$$\left(\mathrm{p}\left(\mathrm{e}^{\frac{2\mathrm{t}}{\sqrt{\mathrm{N}}}}-1\right)+1\right)^{\mathrm{N}}=\left[\left[\frac{1}{2}+\frac{1}{2}\left(\frac{\mathrm{r}-\mathrm{q}}{\sigma}-\frac{\sigma}{2}\right)\sqrt{\frac{T}{N}}\right]\cdot\left[\frac{2t}{\sqrt{N}}+\frac{1}{2}\frac{(2t)^2}{N}+O\left(N^{-\frac{3}{2}}\right)\right]+1\right]^{N} \quad (2)$$

Expand the product of the above two terms and rearrange it according to the order of 1/N, we have

$$(2)=\left[1+\frac{\mathrm{t}}{\sqrt{\mathrm{N}}}+\left(\mathrm{t}^2+\left(\frac{\mathrm{r}-\mathrm{q}}{\sigma}-\frac{\sigma}{2}\right)\sqrt{\mathrm{T}}\mathrm{t}\right)\frac{1}{\mathrm{N}}+\mathrm{O}\left(\mathrm{N}^{-\frac{3}{2}}\right)\right]^{\mathrm{N}} \quad (3)$$

Then, include $e^{-\sqrt{N}t}$ into the power of N, and again use Taylor expansion to expand it, we have

$$\mathrm{e}^{-\sqrt{\mathrm{N}}\mathrm{t}}=\left(\mathrm{e}^{\frac{\mathrm{t}}{\sqrt{\mathrm{N}}}}\right)^{\mathrm{N}}=\left(1-\frac{\mathrm{t}}{\sqrt{\mathrm{N}}}+\frac{1}{2}\frac{\mathrm{t}^2}{\mathrm{N}}+\mathrm{O}\left(\mathrm{N}^{-\frac{3}{2}}\right)\right)^{\mathrm{N}} \quad (4)$$

According to (1), we have (1) = (3) * (4), plug in the expression we derived above, we get

$$(1)=\left(1-\frac{\mathrm{t}^2}{\mathrm{N}}+\frac{1}{2}\frac{\mathrm{t}^2}{\mathrm{N}}+\left(\mathrm{t}^2+\left(\frac{\mathrm{r}-\mathrm{q}}{\sigma}-\frac{1}{2}\right)\sqrt{\mathrm{T}}\mathrm{t}\right)\frac{1}{\mathrm{N}}+\mathrm{O}\left(\mathrm{N}^{-\frac{3}{2}}\right)\right)^{\mathrm{N}}=\left(1+\left(\frac{\mathrm{t}^2}{2}+\left(\frac{\mathrm{r}-\mathrm{q}}{\sigma}-\frac{\sigma}{2}\right)\sqrt{\mathrm{T}}\mathrm{t}\right)\frac{1}{\mathrm{N}}\right)^{\mathrm{N}} \quad (5)$$

Now, use the famous limit equation

$$\left(1+\frac{\mathrm{a}}{\mathrm{n}}\right)^{\mathrm{n}}=e^{a}$$

We have

$$\lim_{\mathrm{N}\to\infty}\mathrm{E}(\mathrm{e}^{\mathrm{tX}})=\lim_{\mathrm{N}\to\infty}\left(1+\frac{\left(\frac{t^2}{2}+\left(\frac{r-q}{\sigma}-\frac{\sigma}{2}\right)\sqrt{T}t\right)}{N}\right)^{N}=e^{\frac{t^2}{2}+\left(\frac{r-q}{\sigma}-\frac{\sigma}{2}\right)\sqrt{T}t}$$

Recall that $\log\frac{S_N}{S_0}=\sigma\sqrt{T}X$, we have

$$\lim_{N\to\infty}\mathrm{E}\mathrm{e}^{\mathrm{t}\cdot\log\left(\frac{\mathrm{S_N}}{\mathrm{S_0}}\right)}=\lim_{N\to\infty}Ee^{t\cdot\sigma\sqrt{T}X}=e^{\frac{(\sigma\sqrt{T}t)^2}{2}+\left(\frac{r-q}{\sigma}-\frac{\sigma}{2}\right)\sqrt{T}t\cdot\sigma\sqrt{T}}=e^{\frac{1}{2}\sigma^2Tt^2+\left(r-q-\frac{\sigma^2}{2}\right)Tt} \quad (6)$$

Note that, the RHS of (6) is the moment generating function of normal random variable $\mathrm{Z}\sim\mathrm{N}((\mathrm{r}-\mathrm{q}-\frac{\sigma^2}{2})T,\sigma^2\mathrm{T})$, then according the one-to-one relationship between moment generating function and distribution function, we have proven that

$$\lim_{\mathrm{N}\to\infty}\log\left(\frac{S_N}{S_0}\right)\sim N\left(\left(r-q-\frac{\sigma^2}{2}\right)T,\sigma^2T\right)=\left(r-q-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{\mathrm{T}}Z$$

where Z is the standard normal random variable.