MTH 9821 Numerical Methods for Finance

Fall 2018

Homework 10
Assigned: November 29; Due: December 6

This homework is to be done as a group. Each team will hand in one homework solution, using the blueprint solution provided by our teaching assistant.

Pricing Down–and–Out European Barrier Options using Finite Differences

The value of the down–and–out call with barrier B less than the strike K is

$$V(S,K) \;=\; C(S,K) \;-\; \left(\frac{B}{S}\right)^{2a} \; C\left(\frac{B^2}{S}, K\right), \tag{1}$$

where

$$a \;=\; \frac{r-q}{\sigma^2} - \frac{1}{2},$$

$C(S,K)$ is the value at time 0 of a plain vanilla call option with strike K and maturity T on the same underlying asset, and $C\left(\frac{B^2}{S}, K\right)$ is the Black–Scholes value at time 0 of a plain vanilla call with strike K and maturity T on an asset having spot price $\frac{B^2}{S}$ (and the same volatility as the underlying).

Value derived from the Closed Formula
Use formula (1) to find the value a seven months down–and–out call on an asset with spot price 42, paying 3% dividends continuously, and following a lognormal distribution with 25% volatility. The barrier is $B = 35$ and the strike for the call is $K = 40$. Assume that the continuously compounded risk–free interest rate is constant at 5%.

The following change of variables transforms x and τ into S and t, respectively, and maps $V(S,t)$, the value of the down–and–out call option, into $u(x,\tau)$, a solution to the heat equation:

$$V(S,t) \;=\; \exp(-ax - b\tau)u(x,\tau), \tag{2}$$

where

$$x \;=\; \ln\left(\frac{S}{K}\right); \quad \tau \;=\; \frac{(T-t)\sigma^2}{2},$$

and the constants a and b are given by

$$\begin{aligned} a &= \frac{r-q}{\sigma^2} - \frac{1}{2}; \\ b &= \left(\frac{r-q}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2q}{\sigma^2}. \end{aligned}$$

The function $u(x,\tau)$ satisfies the heat equation on the following bounded domain:

$$u_\tau(x,\tau) \;=\; u_{xx}(x,\tau) \;\forall\; (x,\tau) \in [x_{left}, x_{right}] \times [0, \tau_{final}],$$

where $x_{left} = \ln\left(\frac{B}{K}\right)$. The boundary conditions are

$$\begin{aligned} u(x,0) &= K\exp(ax)\;\max(\exp(x)-1, 0),\;\forall\, x_{left} \;\le\; x \;\le\; x_{right}; \\ u(x_{left},\tau) &= 0,\;\forall\; 0 \;\le\; \tau \;\le\; \tau_{final}; \\ u(x_{right},\tau) &= K\exp(ax_{right} + b\tau)\;\left(\exp\left(x_{right} - \frac{2q\tau}{\sigma^2}\right) - \exp\left(-\frac{2r\tau}{\sigma^2}\right)\right),\;\forall\; 0 \;\le\; \tau \;\le\; \tau_{final}. \end{aligned}$$

We will use a computational domain where $x_{compute} = \ln\left(\frac{S_0}{K}\right)$ is required to be a grid point in the finite difference discretization.

Computational Domain: $x_{compute}$ on the grid

In the (x, τ) space one of the nodal values will be

$$x_{compute} \;=\; \ln\left(\frac{S_0}{K}\right).$$

Also,

$$x_{left} \;=\; \ln\left(\frac{B}{K}\right).$$

The upper bound τ_{final} for τ is

$$\tau_{final} \;=\; \frac{T\sigma^2}{2}.$$

To choose x_{right}, start with M and α_{temp} given (α will be slightly smaller than α_{temp} in the end). Then,

$$\delta\tau \;=\; \frac{\tau_{final}}{M}$$

and

$$\delta x_{temp} \;=\; \sqrt{\frac{\delta\tau}{\alpha_{temp}}}.$$

Then,

$$N_{left} \;=\; \text{floor}\left(\frac{x_{compute} - x_{left}}{\delta x_{temp}}\right),$$

where floor(y) is the largest integer smaller than or equal to y.

Then,

$$\delta x \;=\; \frac{x_{compute} - x_{left}}{N_{left}}$$

and $\alpha < \alpha_{temp}$ is defined as

$$\alpha \;=\; \frac{\delta\tau}{(\delta x)^2}.$$

Choose the following temporary right end point:

$$\widetilde{x}_{right} \;=\; \ln\left(\frac{S_0}{K}\right) \;+\; \left(r - q - \frac{\sigma^2}{2}\right)T + 3\sigma\sqrt{T}.$$

Let

$$N_{right} \;=\; \text{ceil}\left(\frac{\widetilde{x}_{right} - x_{compute}}{\delta x}\right),$$

where ceil(y) is the smallest integer larger than or equal to y. Then,

$$N \;=\; N_{left} + N_{right}$$

and

$$x_{right} \;=\; x_{compute} + N_{right}\delta x.$$

Identify the computational domain, i.e., compute and record, for each $M \in \{4, 16, 64, 256\}$ and for $\alpha \in \{0.4, 4\}$, the following parameters describing the computational domain: α, x_{left}, x_{right}, N, δx, $\delta\tau$.

Finite difference solvers

Use Forward Euler with $\alpha_{temp} = 0.4$, Backward Euler with $\alpha_{temp} \in \{0.4, 4\}$, and Crank-Nicolson with $\alpha_{temp} \in \{0.4, 4\}$ to solve the diffusion equation for $u(x, \tau)$. For Backward Euler, use tridiagonal LU without pivoting to solve the linear system at each time step. For Crank-Nicolson, use SOR with relaxation parameter $\omega = 1.2$. The stopping criterion for SOR is that the norm of the difference between two consecutive approximations is less than $tol = 10^{-6}$.

Note: there is a total of 5 finite difference approximations, corresponding to:

(1) Forward Euler, for $\alpha_{temp} = 0.4$ (one);
(2) Backward Euler, using tridiagonal LU without pivoting, for $\alpha_{temp} \in \{0.4, 4\}$ (two);
(3) Crank-Nicolson using SOR with $\omega = 1.2$, for $\alpha_{temp} \in \{0.4, 4\}$ (two).

Choose $M = 4$ to begin with. Run each finite difference method for the initial value of M chosen above, and then quadruple the number of points on the τ-axis, i.e., choose $M \in \{4, 16, 64, 256\}$.

Pointwise Convergence and the Greeks:

Let U^M be the vector of length $N - 1$ which gives the finite difference solution after M time steps. Recall that $x_{compute} = x_{N_{left}}$. Then, $U^M(N_{left})$ is the finite difference approximation to $u(x_{compute}, \tau_{final})$.

Use the following change of variables to compute the approximate value of the option, $V_{approx}(S_0, 0)$ from $u(x_{compute}, \tau_{final})$:

(3) $$V_{approx}(S_0, 0) \;=\; \exp(-a x_{compute} - b\tau_{final})\; u(x_{compute}, \tau_{final}).$$

Let $V_{exact}(S_0, 0)$ be the value of the down–and–out call computed above using formula (1). The pointwise error is

(4) $$error\_pointwise \;=\; |V_{approx}(S_0, 0) - V_{exact}(S_0, 0)|.$$

Finite difference approximations for the Δ, Γ, and Θ of the option can be obtained as follows:

Let

$$\begin{aligned} S_{-1} &= K\exp(x_{N_{left}-1}) \;=\; K\exp(x_{compute} - \delta x); \\ S_0 &= K\exp(x_{N_{left}}) \;=\; K\exp(x_{compute}); \\ S_1 &= K\exp(x_{N_{left}+1}) \;=\; K\exp(x_{compute} + \delta x) \end{aligned}$$

be the values of S corresponding to the nodes

$$\begin{aligned} x_{N_{left}-1} &= x_{compute} - \delta x; \\ x_{N_{left}} &= x_{compute}; \\ x_{N_{left}+1} &= x_{compute} + \delta x, \end{aligned}$$

respectively, and let

$$\begin{aligned} V_{-1} &= \exp(-a x_{N_{left}-1} - b\tau_{final})\; u(x_{N_{left}-1}, \tau_{final}); \\ V_0 &= \exp(-a x_{N_{left}} - b\tau_{final})\; u(x_{N_{left}}, \tau_{final}); \\ V_1 &= \exp(-a x_{N_{left}=1} - b\tau_{final})\; u(x_{N_{left}+1}, \tau_{final}) \end{aligned}$$

be the corresponding finite difference approximate values of the option.

The central difference approximations for the Δ and Γ of the option are

(5) $$\Delta_{central} \;=\; \frac{V_1 - V_{-1}}{S_1 - S_{-1}};$$

(6) $$\Gamma_{central} \;=\; \frac{(S_0 - S_{-1})V_1 \;-\; (S_1 - S_{-1})V_0 \;+\; (S_1 - S_0)V_{-1}}{(S_0 - S_{-1})(S_1 - S_0)((S_1 - S_{-1})/2))}.$$

To compute an approximation for Θ, note that the next to last time step on the τ–axis, $\tau_{final} - \delta\tau$ corresponds to time

$$\delta t \;=\; \frac{2\delta\tau}{\sigma^2}.$$

Let

$$V_{approx}(S_0, \delta t) \;=\; \exp(-a x_{N_{left}} - b(\tau_{final} - \delta\tau))\; u(x_{N_{left}}, \tau_{final} - \delta\tau).$$

The forward finite difference approximation of $\Theta = -\frac{\partial V}{\partial t}$ is

(7) $$\Theta_{forward} \;=\; \frac{V_{approx}(S_0, 0) - V_{approx}(S_0, \delta t)}{\delta t}.$$

Finite Difference Solution

For each finite difference method, compute and record:

(1) $U^M(N_{left})$ as "u value"
(2) $V_{approx}(S_0, 0)$ given by (3) as "Option Value";
(3) $error\_pointwise$ given by (4) as "Pointwise Error";
(4) $\Delta_{central}$ given by (5);
(5) $\Gamma_{central}$ given by (6);
(6) $\Theta_{forward}$ given by (7).

To understand the numbers you provide, please include the following: for Forward Euler with $\alpha_{temp} = 0.4$ and for Backward Euler with $\alpha_{temp} = 0.4$, let $M = 4$. Run your codes and record the values of the finite difference approximations at each nodes, including at the boundary nodes. For $M = 4$ and $\alpha_{temp} = 0.4$ the corresponding value of N is $N = 5$.

Thus, for Forward Euler and Backward Euler for the domain including $x_{compute}$ you will have to fill out two tables with five rows (corresponding to time steps from 0 - boundary conditions, to 4) and 6 columns (including the boundary conditions at x_{left} and x_{right}).