Signals and Systems 1.2

---*Elementary Signals*

School of Information & Communication Engineering，BUPT

Reference:
1. Textbook: 1.6

Outline of Today's Lecture

- Elementary Signals: Several elementary signals feature prominently in the study of signals and systems. All of elementary signals serve as building blocks for construction of more complex signals.
  - Exponential Signals
  - Sinusoidal Signals
  - The Unit-Step Function
  - The Unit-Impulse Function

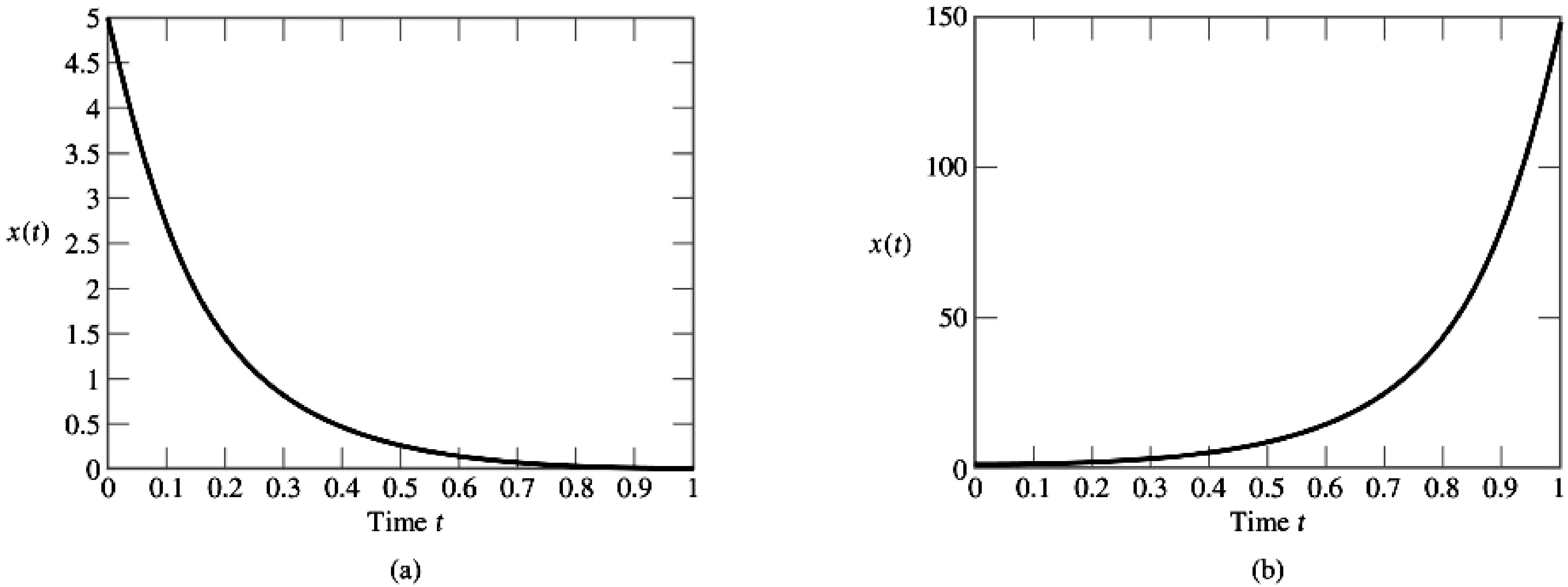
Elementary Signals---Exponential Signals

$$x(t) = Be^{at}$$

***B* and *a* are real parameters**

1. Decaying exponential, for which $a < 0$

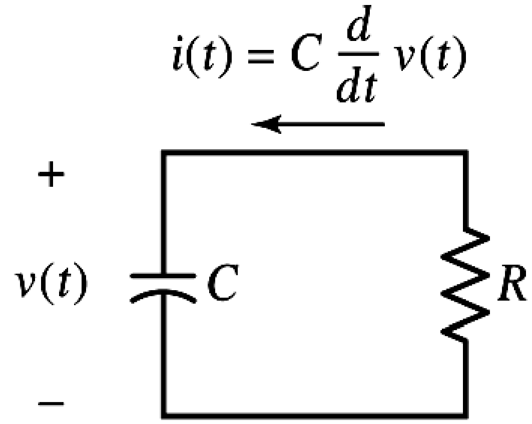
2. Growing exponential, for which $a > 0$



Elementary Signals---Exponential Signals

- For a physical example of an exponential signal, consider a so-call lossy capacitor. The capacitor has capacitance C, and the loss is represented by shunt resistance(分流电阻) R. The capacitor is charged by connecting a battery across it, and then the battery is removed at time t=0. Let V_0 denote the initial value of the voltage developed across the capacitor. For t≧0:

$$RC\frac{d}{dt}v(t)+v(t)=0$$



$$v(t)=V_0e^{-t/(RC)}$$

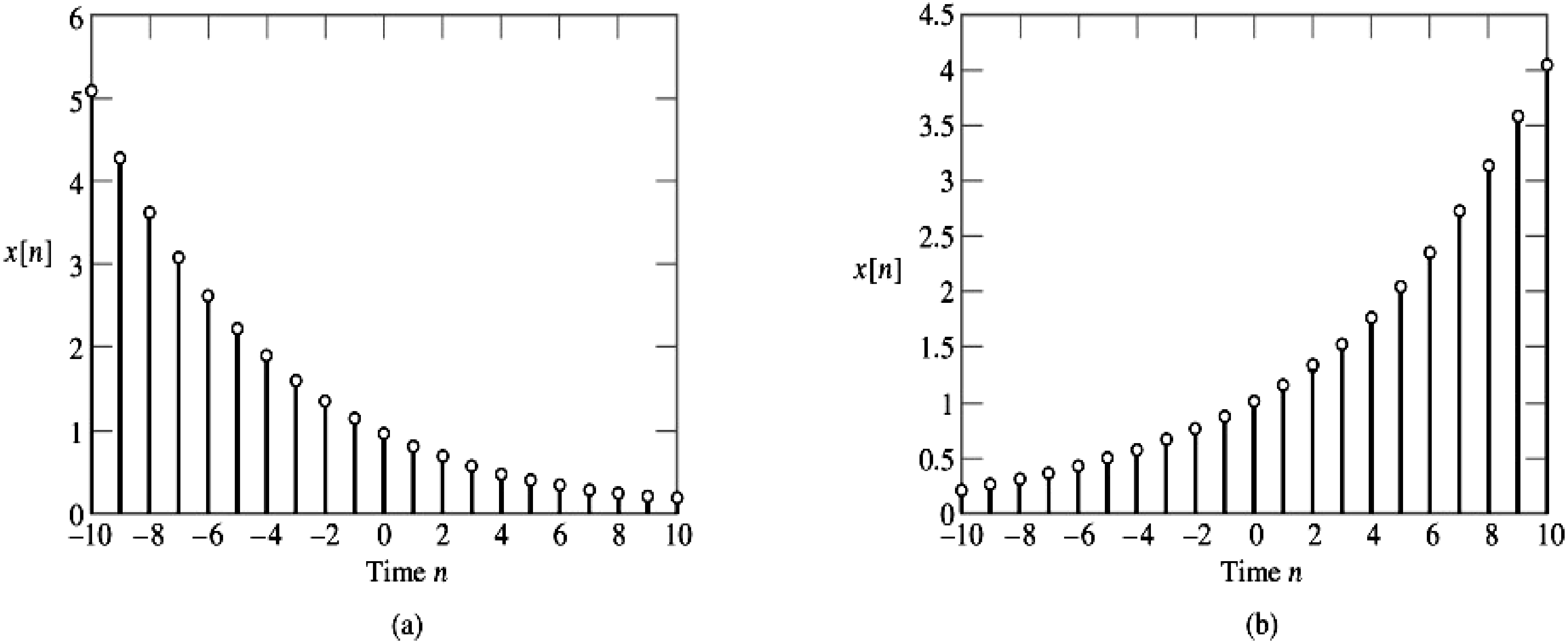
RC = Time constant

where $v(t)$ is the voltage measured across the capacitor at time t.

Elementary Signals---Exponential Signals

◆ Discrete-time case:

$$x[n] = Br^n \qquad r = e^{\alpha}$$



(a) (b)

Elementary Signals---Sinusoidal Signals

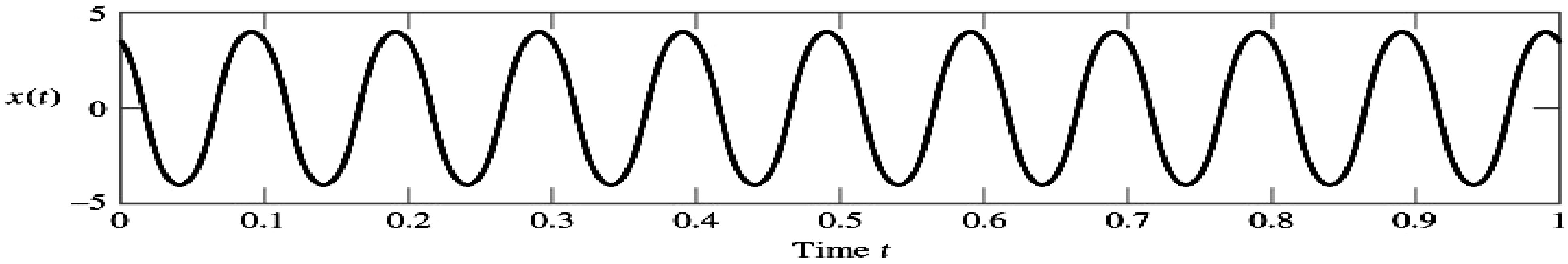
◆ **Continuous-time case:**

$$x(t) = A\cos(\omega t + \phi)$$

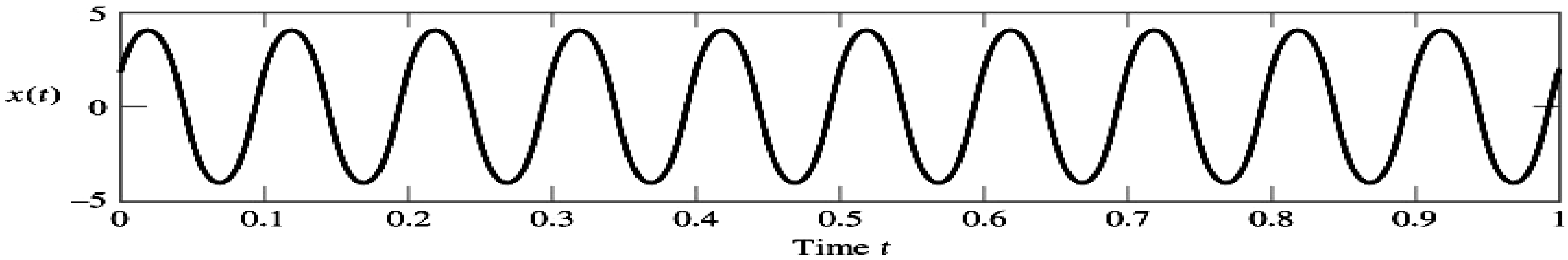
Periodicity $T = \dfrac{2\pi}{\omega}$

$$\begin{aligned} x(t+T) &= A\cos(\omega(t+T)+\phi) \\ &= A\cos(\omega t + \omega T + \phi) \\ &= A\cos(\omega t + 2\pi + \phi) \\ &= A\cos(\omega t + \phi) \\ &= x(t) \end{aligned}$$

A cos(ω *t* + Φ) with phase Φ = +π/6



(a)



(b)

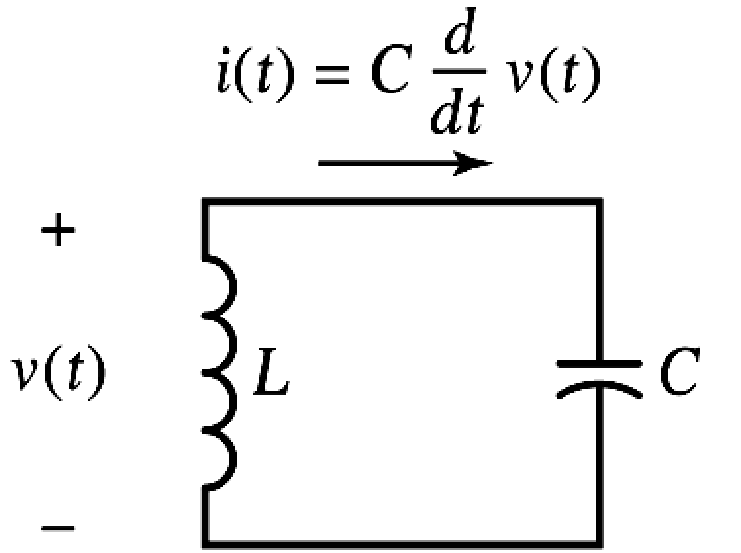
A sin (ω *t* + Φ) with phase Φ = +π/6

Elementary Signals---Sinusoidal Signals

Generation of a sinusoidal signal:

Parallel of an inductor and an capacitor, without loss.

$$LC\frac{d^2}{dt^2}v(t)+v(t)=0$$



$$v(t)=V_0\cos(\omega_0 t),\quad t\ge 0$$

$$\omega_0=\frac{1}{\sqrt{LC}}$$

Natural angular frequency of oscillation of the circuit

Elementary Signals---Sinusoidal Signals

◆ **Discrete-time case :**

Periodic condition

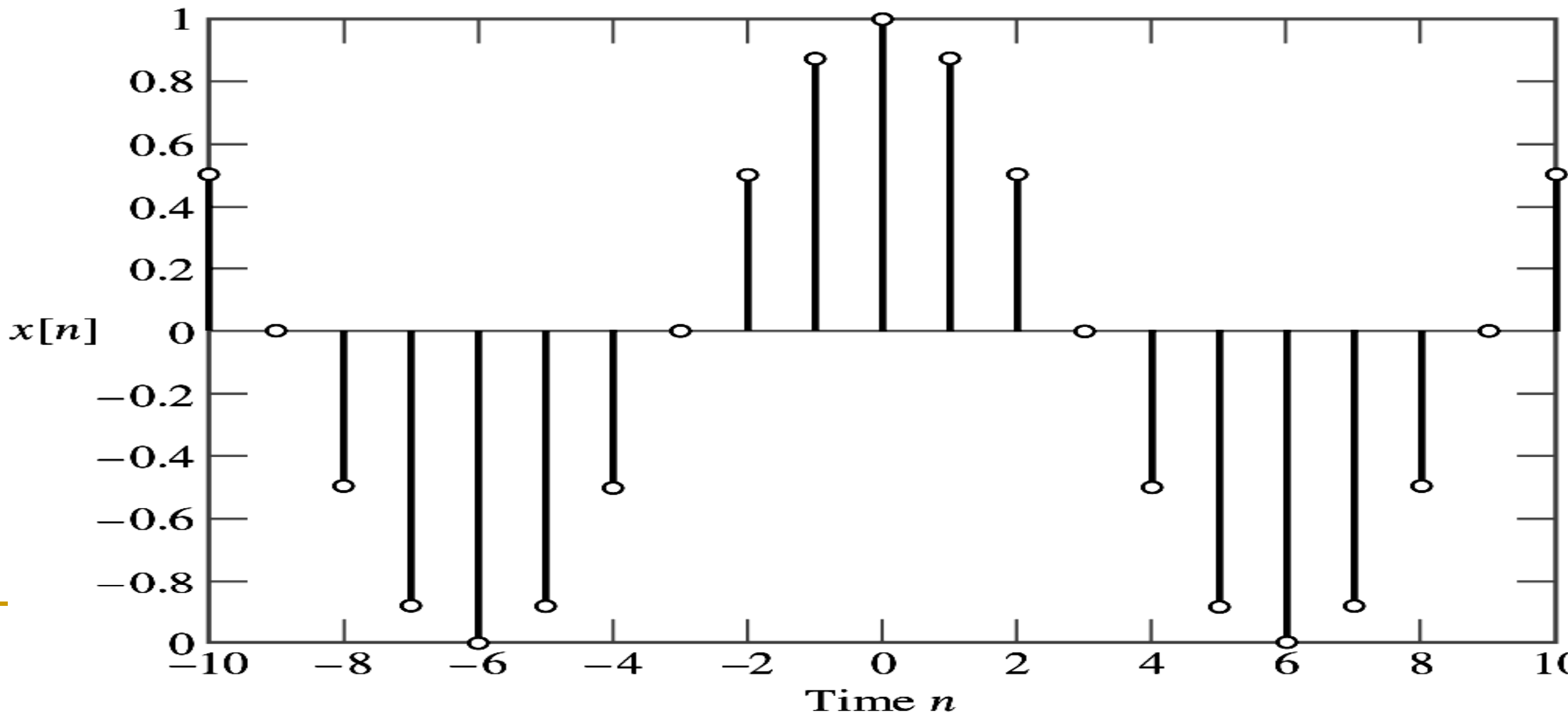
$$x[n] = A\cos(\Omega n + \phi)$$

$$x[n+N] = A\cos(\Omega n + \Omega N + \phi)$$

$$\Omega N = 2\pi m$$

$$\Omega = \frac{2\pi m}{N} \quad \text{radians/cycle,} \quad \text{integer} \quad m,\ N$$

Ex. A discrete-time sinusoidal signal: *A* = 1, ϕ = 0, and *N* = 12.



Elementary Signals---Sinusoidal Signals

Example 1.7 Discrete-Time Sinusoidal Signal

A pair of sinusoidal signals with a common angular frequency is defined by

$x_1[n] = \sin[5\pi n]$ **and** $x_2[n] = \sqrt{3}\cos[5\pi n]$

(a) Both $x_1[n]$ and $x_2[n]$ are periodic. Find their common fundamental period.

(b) Express the composite sinusoidal signal

$y[n] = x_1[n] + x_2[n]$

In the form $y[n] = A\cos(\Omega n + \phi)$, and evaluate the amplitude A and phase ϕ.

<Sol.>

(a) Angular frequency of both $x_1[n]$ and $x_2[n]$:

$\Omega = 5\pi$ radians/cycle ⇨ $N = \dfrac{2\pi m}{\Omega} = \dfrac{2\pi m}{5\pi} = \dfrac{2m}{5}$

⇨ **This can be only for m = 5, 10, 15, …, which results in N = 2, 4, 6, …**

Elementary Signals---Sinusoidal Signals

(b) Trigonometric identity:

$$A\cos(\Omega n+\phi)=A\cos(\Omega n)\cos(\phi)-A\sin(\Omega n)\sin(\phi)$$

Let Ω = 5π, then compare $x_1[n]$ + $x_2[n]$ with the above equation to obtain that

$$A\sin(\phi)=-1 \quad \text{and} \quad A\cos(\phi)=\sqrt{3}$$

$$\Rightarrow \tan(\phi)=\frac{\sin(\phi)}{\cos(\phi)}=\frac{\text{amplitude of } x_1[n]}{\text{amplitude of } x_2[n]}=\frac{-1}{\sqrt{3}} \Rightarrow \phi = -\pi/6$$

$$\Rightarrow A\sin(\phi)=-1$$

$$\Rightarrow A=\frac{-1}{\sin\left(-\pi/6\right)}=2$$

$$\boxed{y[n]=2\cos\left(5\pi n-\frac{\pi}{6}\right)}$$

Elementary Signals

-Relation Between Sinusoidal and Complex Exponential Signals

1. Euler's identity: $$e^{j\theta} = \cos\theta + j\sin\theta$$

Complex exponential signal: $$B = Ae^{j\phi}$$

$$x(t) = A\cos(\omega t + \phi)$$

$$\Rightarrow \quad A\cos(\omega t + \phi) = \mathrm{Re}\{Be^{j\omega t}\}$$

$$\begin{aligned} &Be^{j\omega t} \\ &= Ae^{j\phi}e^{j\omega t} \\ &= Ae^{j(\phi+\omega t)} \\ &= A\cos(\omega t + \phi) + jA\sin(\omega t + \phi) \end{aligned}$$

◇ **Continuous-time signal in terms of sine function:**

$$x(t) = A\sin(\omega t + \phi)$$

$$\Rightarrow \quad A\sin(\omega t + \phi) = \mathrm{Im}\{Be^{j\omega t}\}$$

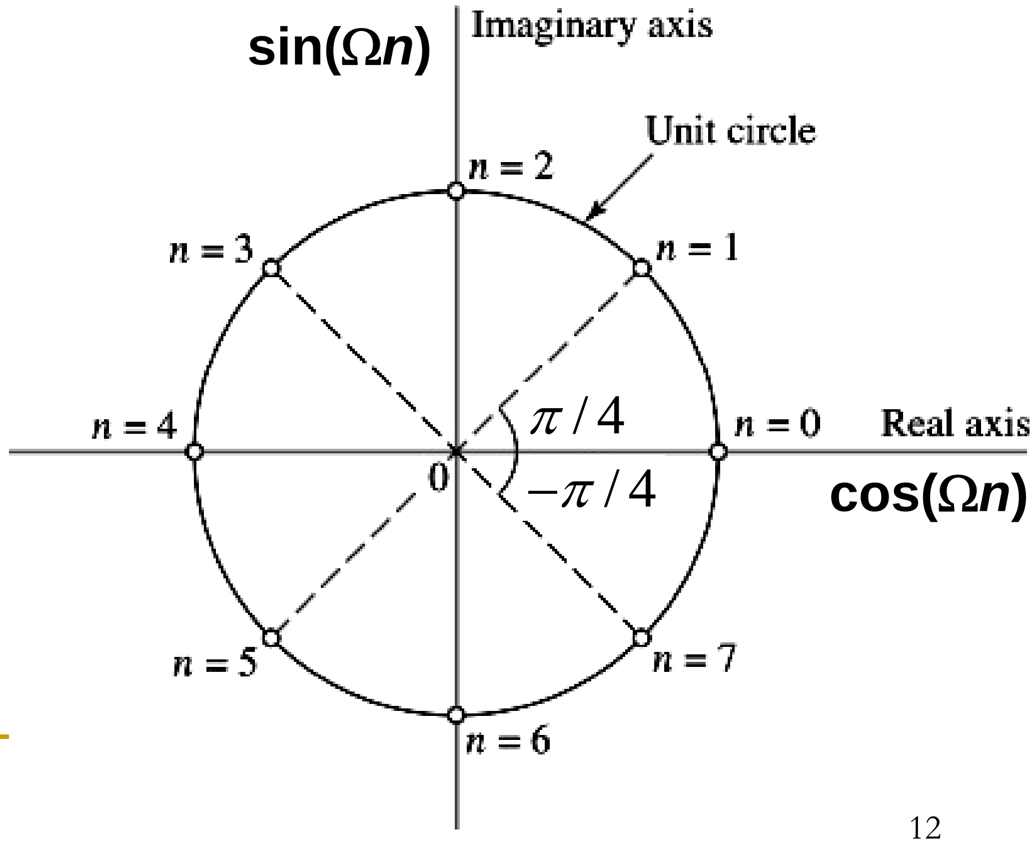
Elementary Signals

-Relation Between Sinusoidal and Complex Exponential Signals

2. Discrete-time case:

$$A\cos(\Omega n + \phi) = \mathrm{Re}\{Be^{j\Omega n}\} \qquad A\sin(\Omega n + \phi) = \mathrm{Im}\{Be^{j\Omega n}\}$$

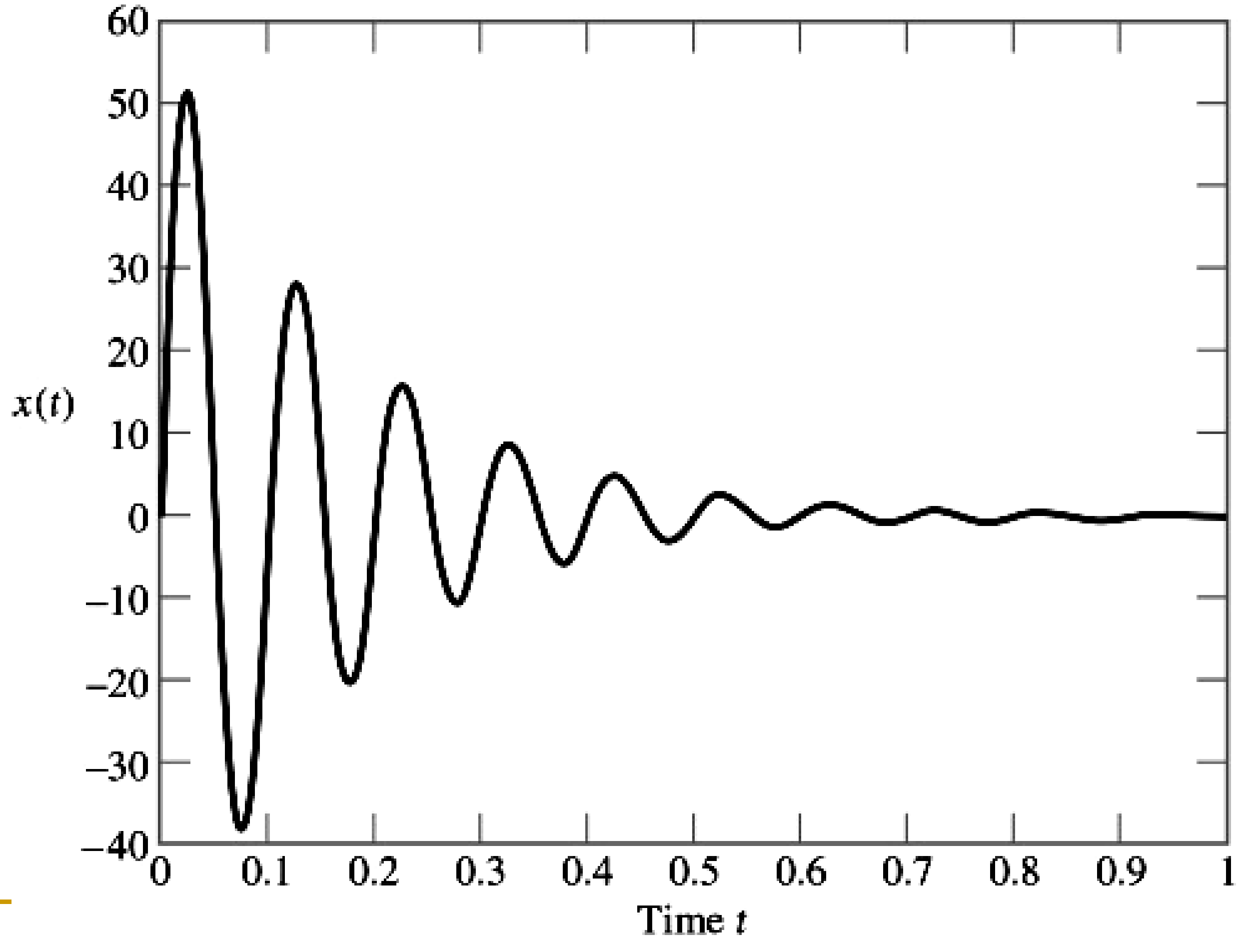
3. Two-dimensional representation of the complex exponential $e^{j\Omega n}$ for $\Omega = \pi/4$ and $n = 0, 1, 2, \ldots, 7$.



Elementary Signals -Exponential Damped Sinusoidal Signals

$$x(t) = Ae^{-\alpha t}\sin(\omega t + \phi), \quad \alpha > 0$$

Example for *A* = 60, α = 6, and ϕ = 0



Ex. Generation of an exponential damped sinusoidal signal

Let *Vo* denote the initial voltage developed across the capacitor at *t* =0.

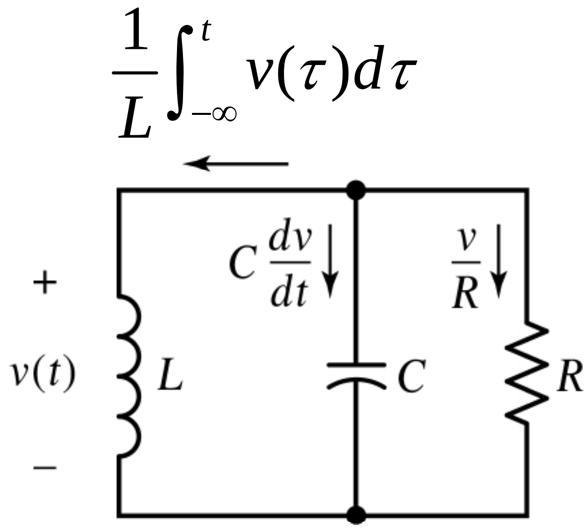
Circuit Eq.: $$C\frac{d}{dt}v(t)+\frac{1}{R}v(t)+\frac{1}{L}\int_{-\infty}^{t}v(\tau)d\tau=0$$

$$\Rightarrow \quad v(t)=V_0e^{-t/(2CR)}\cos(\omega_0 t) \quad t\geq 0$$

where *v(t)* is the voltage across the capacitor at time *t* >0.

$$\omega_0=\sqrt{\frac{1}{LC}-\frac{1}{4C^2R^2}}$$

$$R>\sqrt{L/(4C)}$$



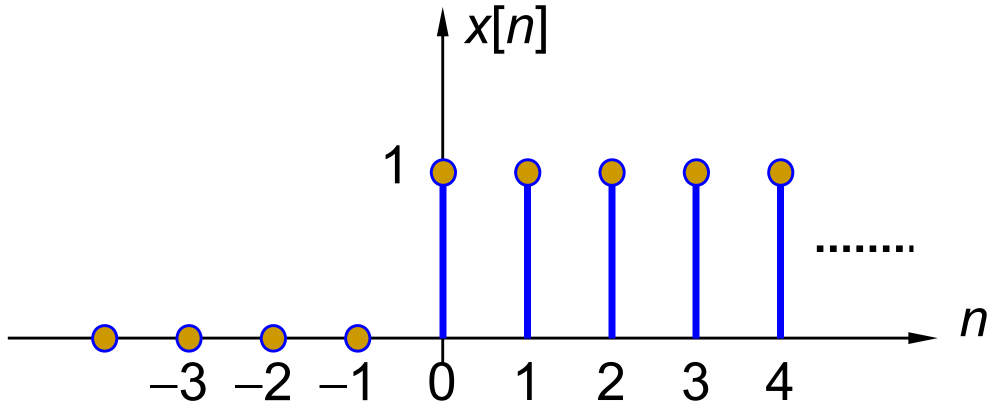
$$\boxed{x(t)=Ae^{-\alpha t}\sin(\omega t+\phi), \quad \alpha>0}$$

$$A=V_0, \quad \alpha=1/(2CR), \quad \omega=\omega_0, \quad \text{and} \quad \phi=\pi/2$$

Elementary Signals---Step Function

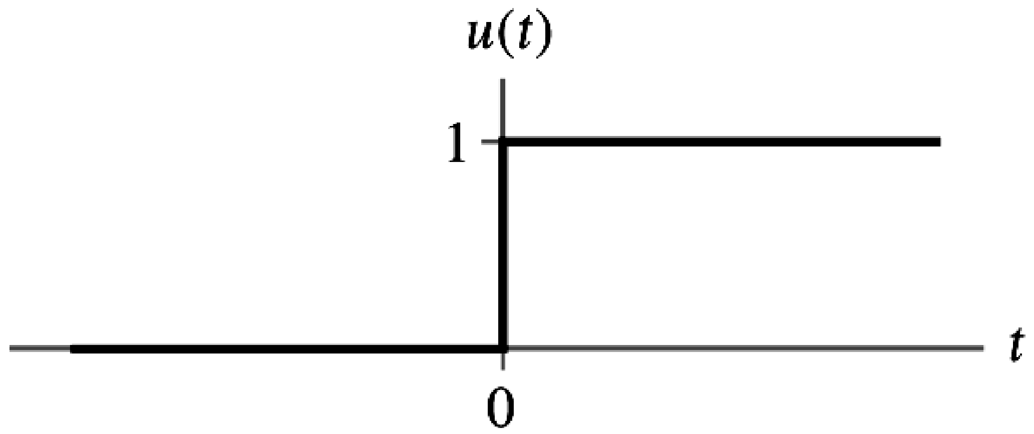
◆ **Discrete-time case:**

$$u[n]=\begin{cases}1, & n\geq 0\\ 0, & n<0\end{cases}$$



◆ **Continuous-time case:** unit-step function

$$u(t)=\begin{cases}1, & t>0\\ 0, & t<0\end{cases}$$



$u(0)\ (t=0)$ *is undifined*

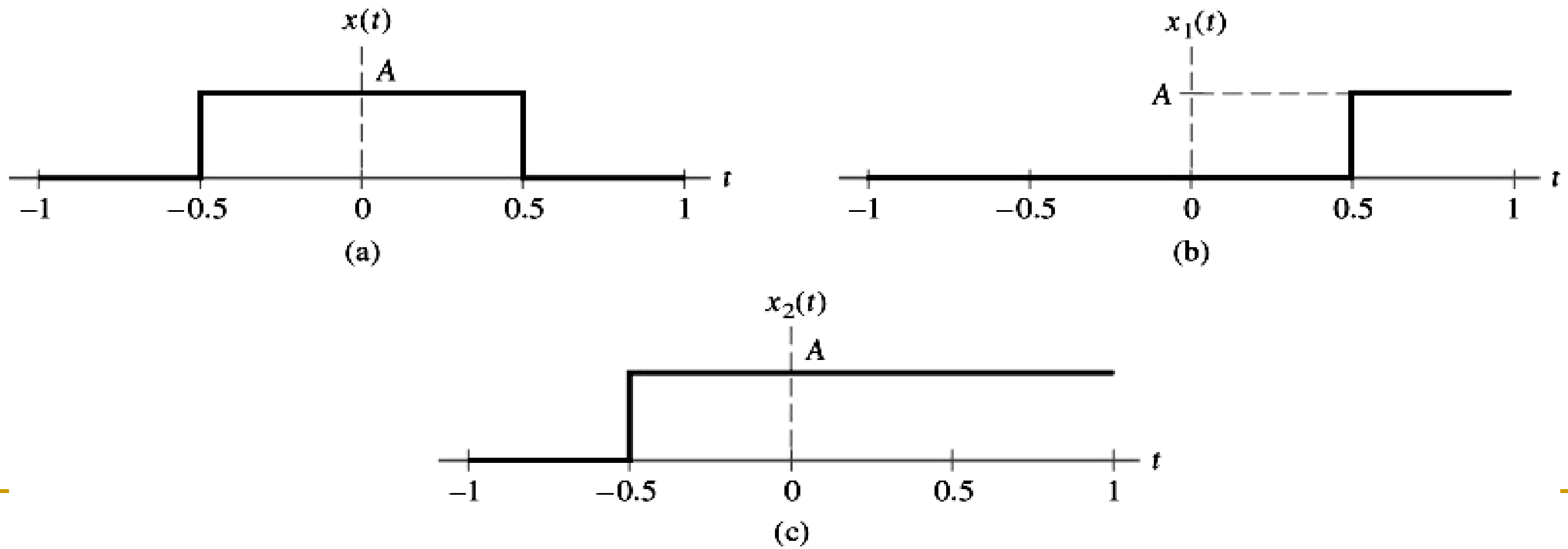
Elementary Signals---Step Function

Example 1.8 Rectangular Pulse

Consider the rectangular pulse *x*(*t*) . This pulse has an amplitude *A* and duration of 1 second. Express *x*(*t*) as a weighted sum of two step functions.

<Sol.>

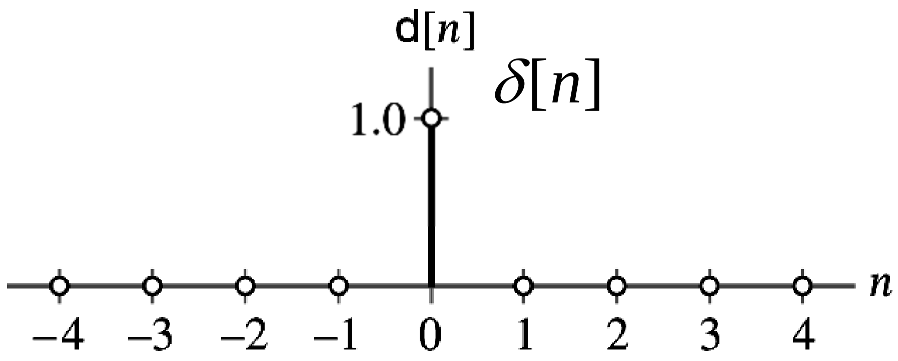
$$x(t) = \begin{cases} A, & 0 \le |t| < 0.5 \\ 0, & |t| > 0.5 \end{cases} \Rightarrow x(t) = Au\left(t + \frac{1}{2}\right) - Au\left(t - \frac{1}{2}\right)$$



Elementary Signals---Impulse Function

◆ Discrete-time case:

$$\delta[n]=\begin{cases}1, & n=0\\ 0, & n\neq 0\end{cases}$$



◆ Continuous-time case:

$$\delta(t)=0 \quad \text{for} \quad t\neq 0$$

Dirac delta function

$$\int_{-\infty}^{\infty}\delta(t)dt=1$$

Elementary Signals---Impulse Function

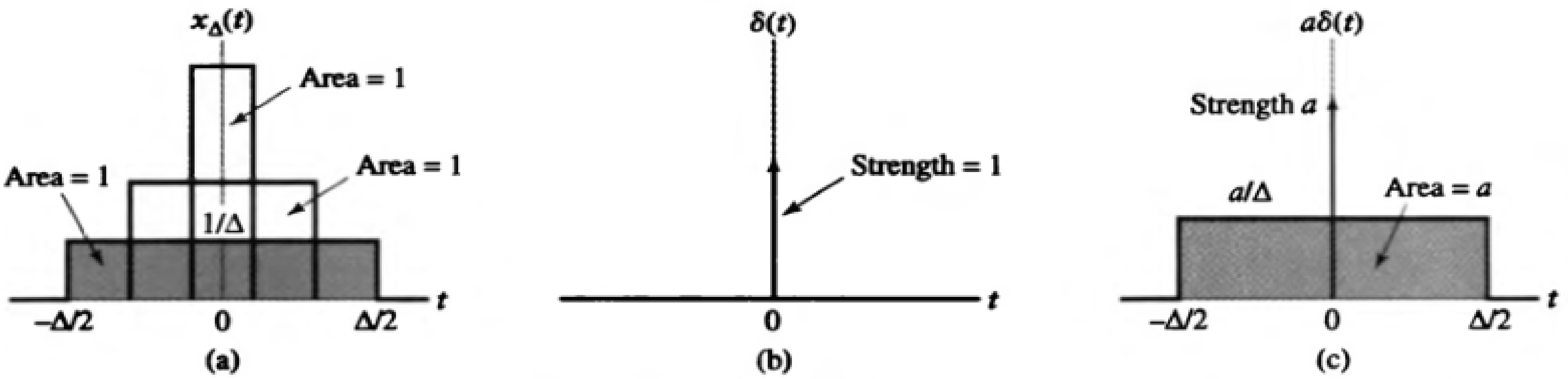


Figure 1.42 (p. 46)
(a) Evolution of a rectangular pulse of unit area into an impulse of unit strength (i.e., unit impulse). (b) Graphical symbol for unit impulse.
(c) Representation of an impulse of strength *a* that results from allowing the duration Δ of a rectangular pulse of area *a* to approach zero.

1. As the duration decreases, the rectangular pulse approximates the impulse more closely.

2. Mathematical relation between impulse and rectangular pulse function:

$$\delta(t) = \lim_{\Delta \to 0} x_\Delta(t)$$

1. **$x_\Delta(t)$: even function of t, Δ = duration.**
2. **$x_\Delta(t)$: Unit area.**

Elementary Signals---Impulse Function

The impulse and unit step function u(t) are related to each other

3. δ**(*t*) is the derivative of *u*(*t*):**

$$\delta(t) = \frac{d}{dt}u(t)$$

4. *u*(*t*) is the integral of δ**(*t*):**

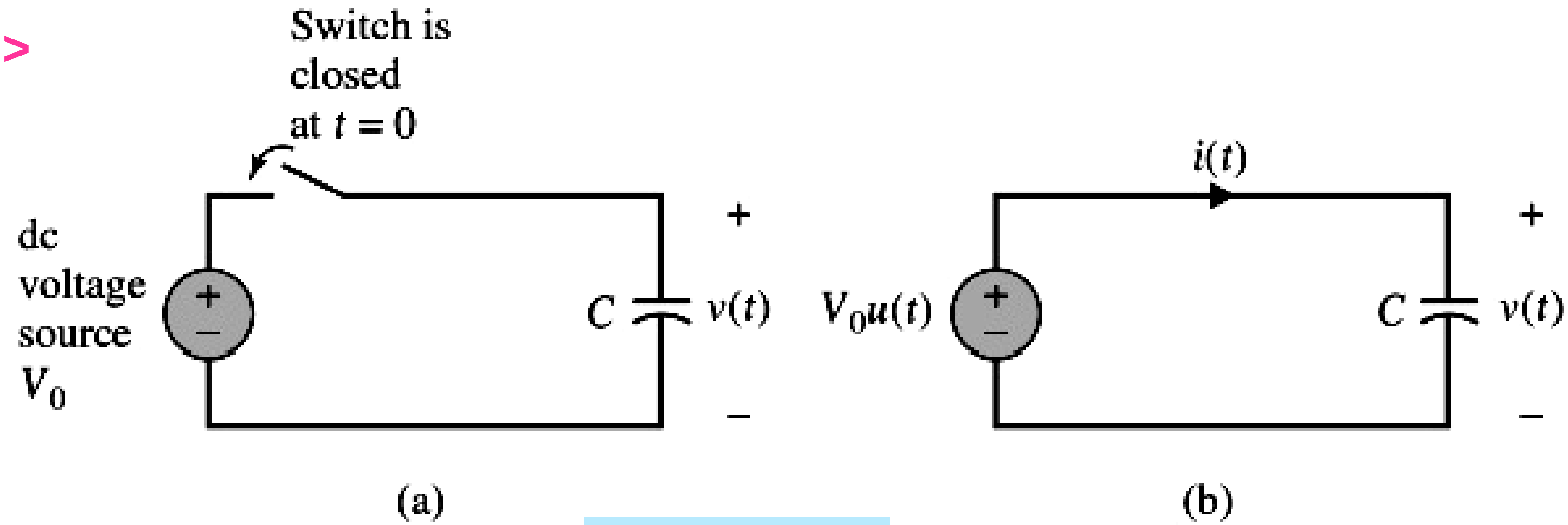
$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

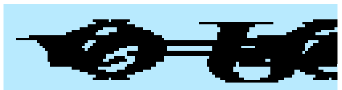
Elementary Signals---Impulse Function

Example 1.10 *RC* Circuit (Continued)

For the RC circuit shown in Fig. 1.43 (a), determine the current *i* (*t*) that flows through the capacitor for $t \geq 0$.

<Sol.>



1. Voltage across the capacitor: 

2. Current flowing through capacitor:

$$i(t) = C\frac{dv(t)}{dt} \Rightarrow i(t) = CV_0\frac{du(t)}{dt} = CV_0\delta(t)$$

Elementary Signals---Impulse Function

◆ **Properties of impulse function:**

1. Even function: $\delta(-t)=\delta(t)$

2. Sifting property:

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$$

3. Time-scaling property:

$$\delta(at)=\frac{1}{a}\delta(t), \quad a>0$$

$$\int_{-\infty}^{\infty}\delta\left(t-t_0\right)dt=1$$

$$\int_{-\infty}^{\infty} f\left(t\right)\delta\left(t\right)dt = f\left(0\right)$$

$$f\left(t\right)\delta\left(t\right)= f\left(0\right)\delta\left(t\right)$$

$$\int_{-\infty}^{\infty} f\left(t\right)\delta'\left(t\right)dt=-f'\left(0\right)$$

$$U\left(t\right)=\int_{-\infty}^{t}\delta\left(t\right)dt$$

Elementary Signals---Impulse Function

To represent the function $x_\Delta(t)$, we use the rectangular pulse shown in Fig. 1.44(a), which has duration Δ, amplitude $1/\Delta$, and therefore unit area. Correspondingly, the time-scaled function $x_\Delta(at)$ is shown in Fig. 1.44(b) for $a > 1$. The amplitude of $x_\Delta(at)$ is left unchanged by the time-scaling operation. Consequently, in order to restore the area under this pulse to unity, $x_\Delta(at)$ is scaled by the same factor a, as indicated in Fig. 1.44(c), in which the time function is thus denoted by $ax_\Delta(at)$. Using this new function in Eq. (1.67) yields

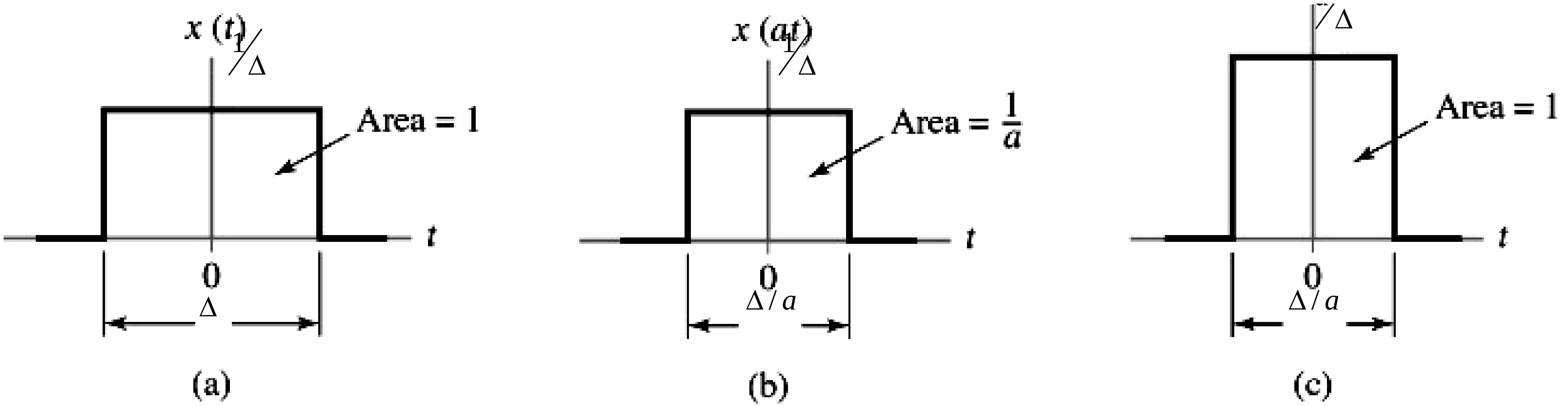


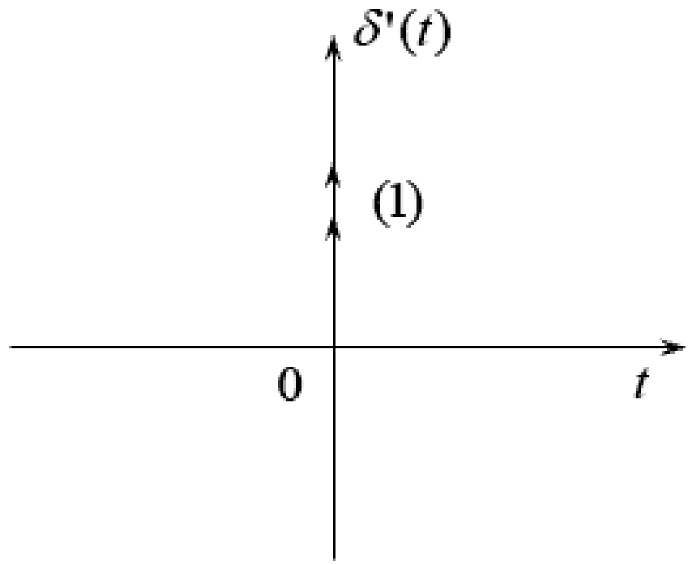
Figure 1.44 (p. 48)
Steps involved in proving the time-scaling property of the unit impulse. (a) Rectangular pulse $x\Delta(t)$ of amplitude $1/\Delta$ and duration Δ, symmetric about the origin. (b) Pulse $x\Delta(t)$ compressed by factor a. (c) Amplitude scaling of the compressed pulse, restoring it to unit area.

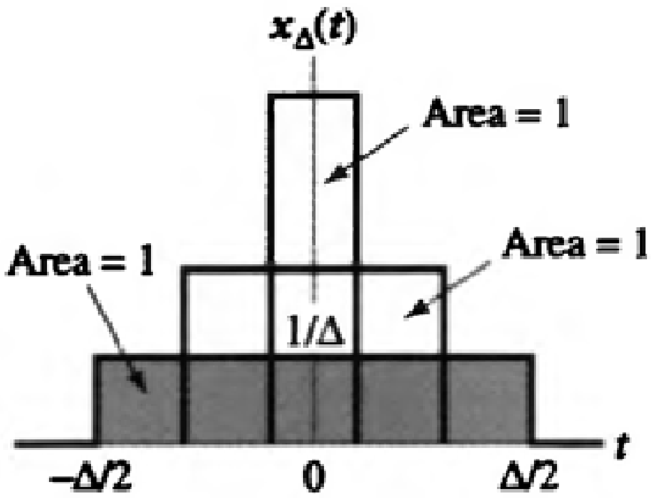
Elementary Signals---Impulse Function

Derivatives of The Impulse

Definitions:

$$\delta'(t) = \frac{\mathrm{d}\delta(t)}{\mathrm{d}t}$$





The first derivative of δ(t) as the limiting form of the first derivative of the same rectangular pulse.
The rectangular pulse is equal to the step function $(1/\triangle)[u(t+\triangle/2)-u(t-\triangle/2)]$

Elementary Signals---Impulse Function

Derivatives of The Impulse

1. Doublet:

$$\delta^{(1)}(t) = \lim_{\Delta \to 0} \frac{1}{\Delta}\left(\delta(t+\Delta/2) - \delta(t-\Delta/2)\right) \qquad \textbf{(1.70)}$$

2. Fundamental property of the doublet:

$$\int_{-\infty}^{\infty} \delta^{(1)}(t)dt = 0 \qquad \textbf{(1.71)}$$

$$\int_{-\infty}^{\infty} f(t)\delta^{(1)}(t-t_0)dt = \frac{d}{dt} f(t)\Big|_{t=t_0} \qquad \textbf{(1.72)}$$

3. Second derivative of impulse:

$$\frac{\partial^2}{\partial t^2}\delta(t) = \frac{d}{dt}\delta^{(1)}(t) = \lim_{\Delta \to 0} \frac{\delta^{(1)}(t+\Delta/2) - \delta^{(1)}(t-\Delta/2)}{\Delta} \qquad \textbf{(1.73)}$$

Problem 1.24

$$\int_{-\infty}^{\infty} f(t)\delta^{(2)}(t-t_0)dt = \frac{d^2}{dt^2} f(t)|_{t=t_0}$$

$$\int_{-\infty}^{\infty} f(t)\delta^{(n)}(t-t_0)dt = \frac{d^n}{dt^n} f(t)|_{t=t_0}$$

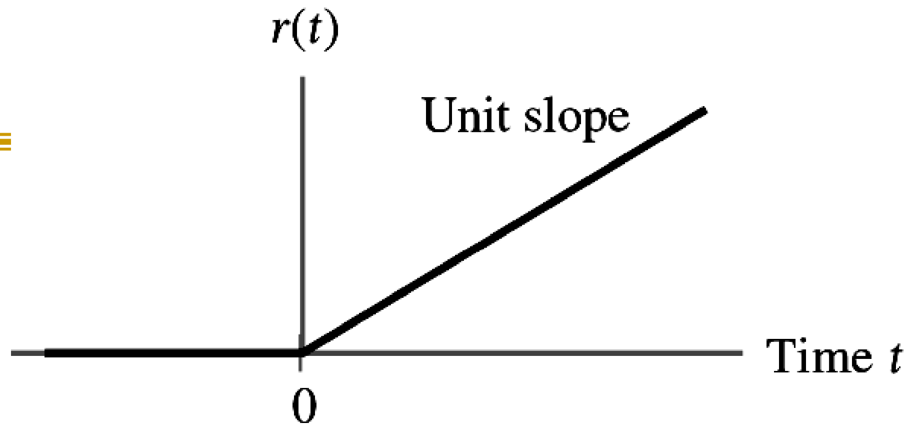
Other signals

★ 1 Ramp Function

1A. Continuous-time case

$$r(t)=\begin{cases} t, & t\geq 0 \\ 0, & t<0 \end{cases}$$

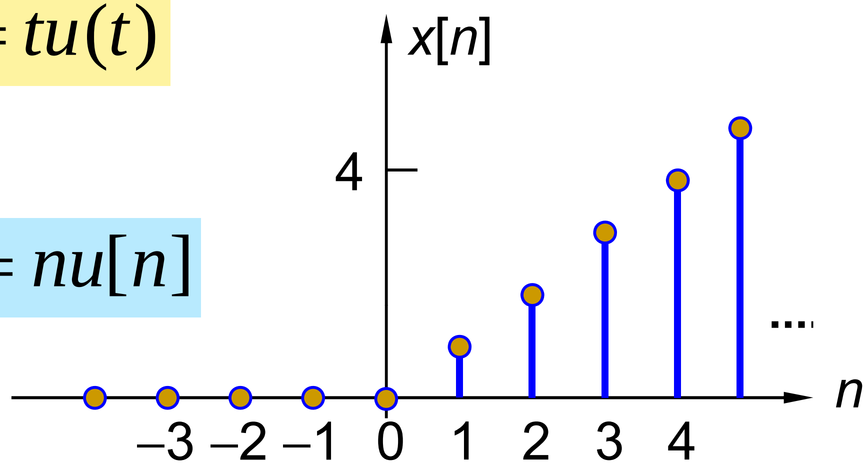
$$r(t)=tu(t)$$



1B. Discrete-time case

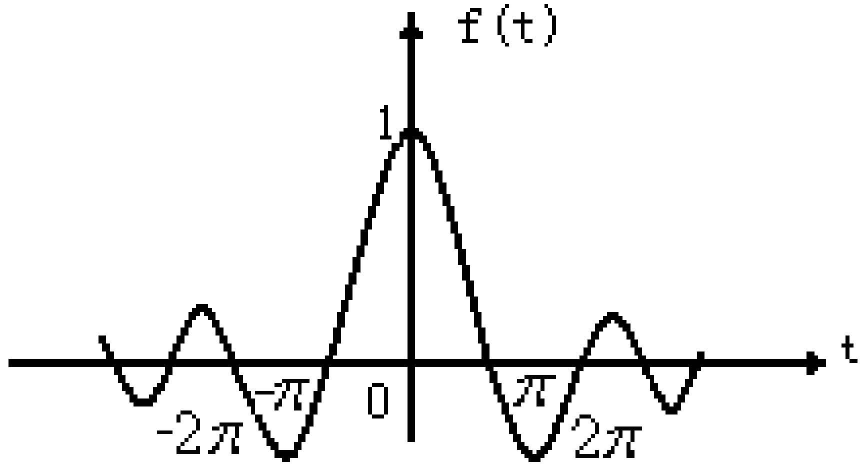
$$r[n]=\begin{cases} n, & n\geq 0 \\ 0, & n<0 \end{cases}$$

$$r[n]=nu[n]$$



★ 2 Sampling signals

$$f(t)=\frac{\sin t}{t}=Sa(t)$$



Summary and Exercises

- Summary
  - Exponential Signals
  - Sinusoidal Signals
  - The Unit-Step Function
  - The Unit-Impulse Function
- Exercises
  - P89: 1.54 (a, c, e)
  - P90: 1.56 (b, d, f, h, j), 1.57 (a, c, e, g, i) , 1.58, 1.60

Trigonometric identities

$$\sin A \pm \sin B = 2\sin\frac{A \pm B}{2}\cos\frac{A \mp B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

http://en.wikipedia.org/wiki/List_of_trigonometric_identities

Circuit Fundenmental

$$u_R(t) = Ri_R(t)$$

$$i_C(t) = C\frac{du_C(t)}{dt}$$

$$u_L(t) = L\frac{di_L(t)}{dt}$$