Signals and Systems 2

--- *Linear Time-invariant Systems*

School of Information & Communication Engineering，BUPT

Reference:

1. Textbook: Chapter 2

2. Schaum's outline of signals and systems, Hwei P. Hsu, McGraw-Hill, 1995. Chapter 2

Outline of Today's Lecture

- **L**inear **T**ime-**I**nvariant systems (**LTI**)
  - Introduction
  - Discrete time LTI systems: Convolution Sum
  - Continuous time LTI systems: Convolution Integral
  - Properties of LTI systems

Linear Time-invariant systems (LTI)

- A system satisfying both the **linearity** and the **time-invariance** property
- LTI systems are mathematically easy to analyze and characterize, and consequently, easy to design.
- Highly useful signal processing algorithms have been developed utilizing this class of systems over the last several decades.
- They possess superposition theorem: If we represent the input to an LTI system in terms of linear combination of a set of basic signals, we can then use superposition to compute the output of the system in terms of responses to these basic signals.

Representation of LTI systems

- Any LTI system, continuous-time or discrete-time, can be uniquely characterized by its
  - **Impulse response:** response of system to an impulse
  - **Frequency response**: response of system to a complex exponential $e^{j 2 \pi f}$ for all possible frequencies f.
  - **Transfer function**: Laplace transform of impulse response
- Given one of the three, we can find other two provided that they exist

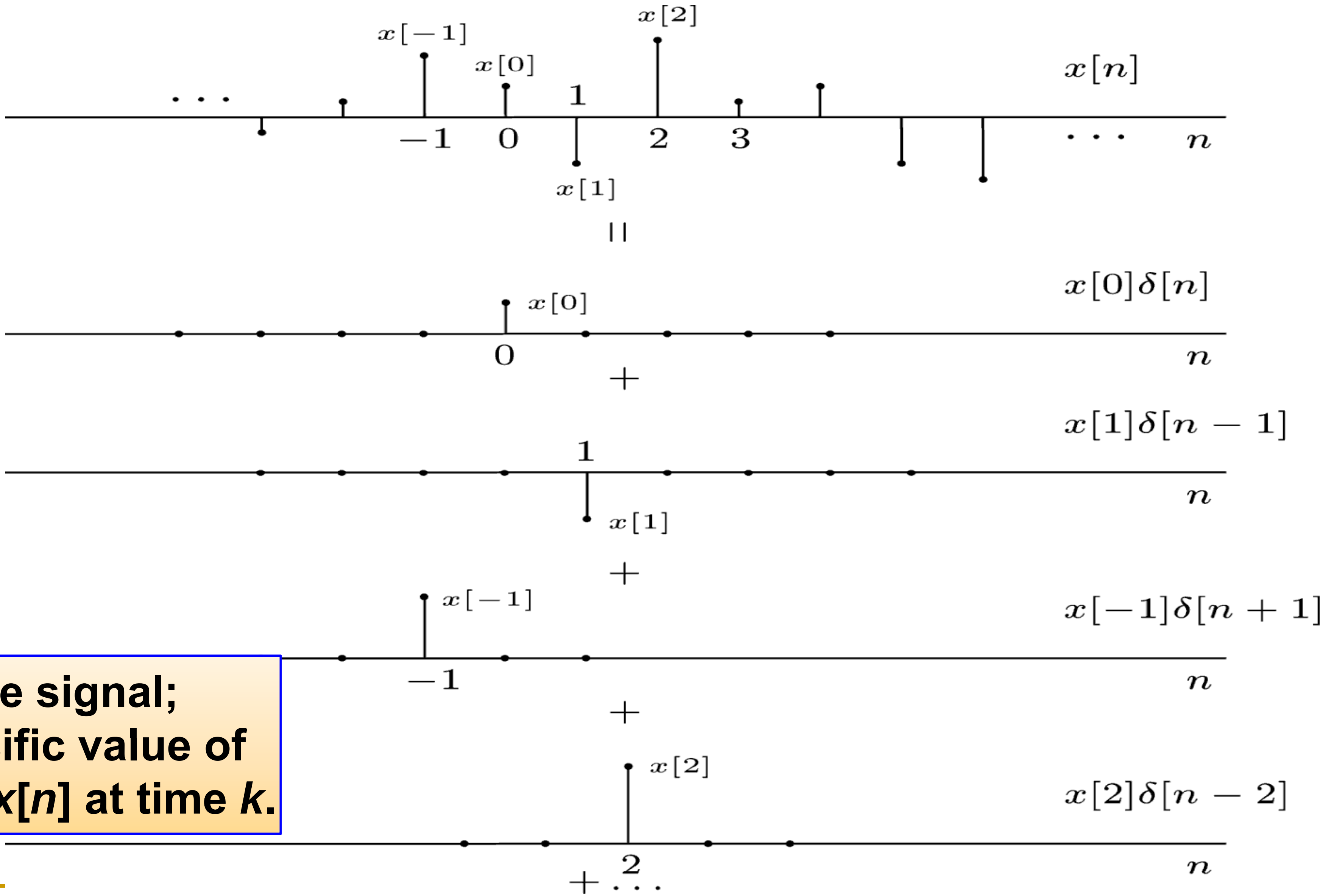
Significance of unit impulse

- Every signal whether large or small can be represented in terms of linear combination of delayed impulses(延迟冲激).
- Here two properties apply:
  - Linearity
  - Time Invariance

Basic building Blocks

- For DT or CT case; there are two natural choices for these two basic building blocks
  - For DT: Shifted unit samples
  - For CT: Shifted unit impulses.
- **DT: Discrete time**
- **CT: Continuous time**

Representing DT Signals with Sums of Unit Samples



***x*[*n*] = entire signal;**
***x*[*k*] = specific value of the signal *x*[*n*] at time *k*.**

$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

Representing DT Signals with Sums of Unit Samples

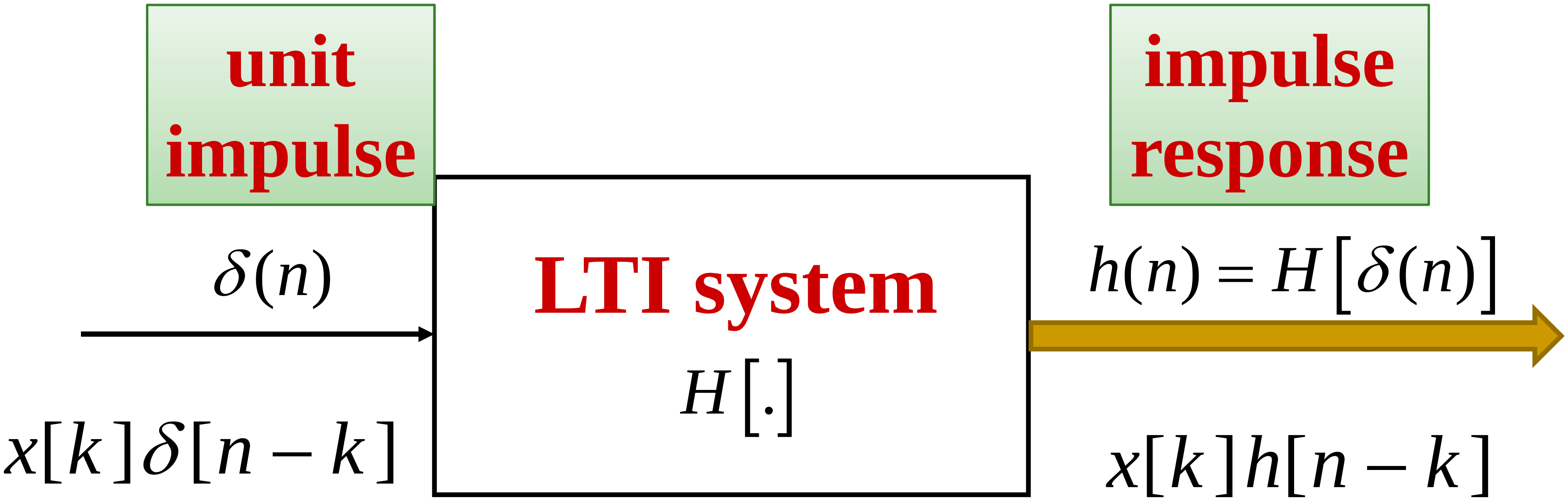
- An arbitrary signal (input sequence x[n]) can be expressed as a weighted superposition of time-shifted impulses.

$$\boxed{x[n]=\cdots+x[-2]\delta[n+2]+x[-1]\delta[n+1]+x[0]\delta[n]+x[1]\delta[n-1]+x[2]\delta[n-2]+\cdots}$$

$$x[n]=\sum_{k=-\infty}^{\infty}\underbrace{x[k]}_{\text{Coefficients}}\,\underbrace{\delta[n-k]}_{\text{Basic Signals}}$$

Impulse Response for LTI DT Systems

- The response of a discrete-time system to a unit sample sequence $\{\delta[n]\}$ is called the **unit sample response** or simply, the **impulse response**, and is denoted by $\{h[n]\}$



$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

Superposition Sum for LTI DT Systems

- Time-invariant $\quad h[n] \Longrightarrow h[n-k]$

$$
\begin{array}{c}
\text{input} \qquad \text{output} \\
\delta[n-2] \rightarrow h[n-2] \\
\delta[n-1] \rightarrow h[n-1] \\
\delta[n+1] \rightarrow h[n+1] \\
\delta[n+2] \rightarrow h[n+2]
\end{array}
$$

- Linear

$$x[n] = \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots$$

$$y[n] = \cdots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + \cdots$$

Time-Domain Characterization of LTI DT System

- Input-Output Relationship - A consequence of the linear, time-invariance property is that an LTI discrete-time system is completely characterized by its impulse response.
- A signal $x[n]$ can be as a weighted sum of time-shifted impulses. Then, the output response of LTI DT system is a weighted sum of the Impulse Response (response of the system to time-shifted impulses).

Example1: IR (Superposition Sum)

- Compute the output *y*[*n*] for the input *x*[*n*]

$$x[n] = 0.5\delta[n+2] + 1.5\delta[n-1] - \delta[n-2] + 0.75\delta[n-5]$$

input **output**

$$0.5\delta[n+2] \rightarrow 0.5h[n+2]$$

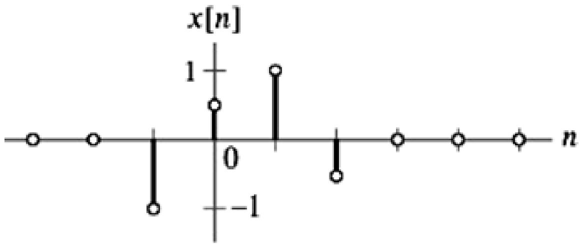
$$1.5\delta[n-1] \rightarrow 1.5h[n-1]$$

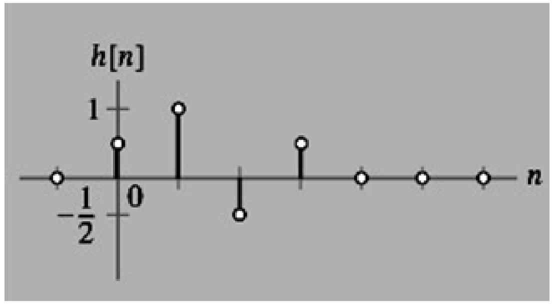
$$-\delta[n-2] \rightarrow -h[n-2]$$

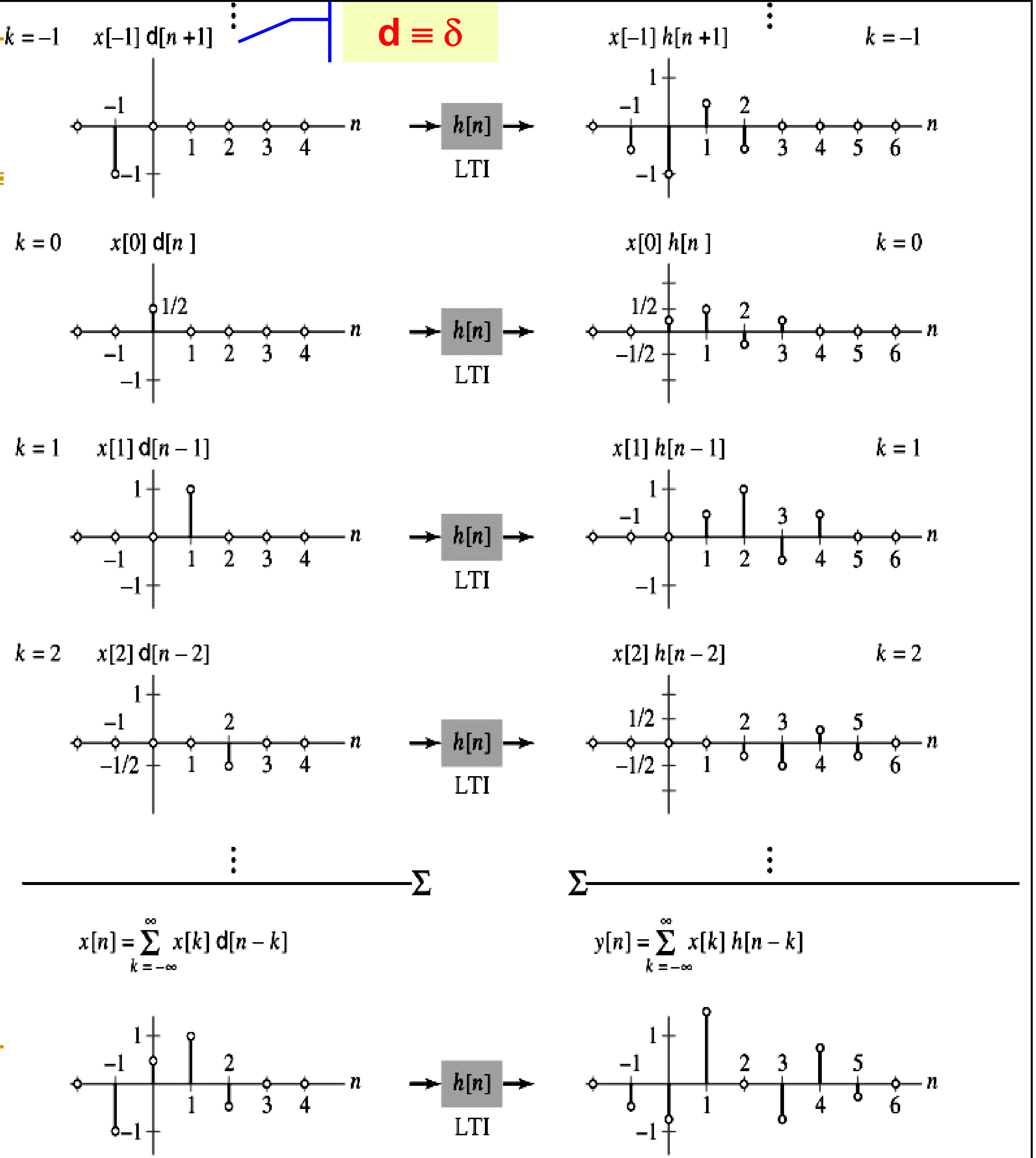
$$0.75\delta[n-5] \rightarrow 0.75h[n-5]$$

$$y[n] = 0.5h[n+2] + 1.5h[n-1] - h[n-2] + 0.75h[n-5]$$

Example2: IR


x[n]
1
0
−1
n


h[n]
1
0
−1/2
n


d ≡ δ
k = −1
x[−1] d[n + 1]
x[−1] h[n + 1]
k = 0
x[0] d[n]
x[0] h[n]
k = 1
x[1] d[n − 1]
x[1] h[n − 1]
k = 2
x[2] d[n − 2]
x[2] h[n − 2]
h[n]
LTI
Σ
$x[n] = \sum_{k=-\infty}^{\infty} x[k]\, d[n-k]$
$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

DT LTI Systems: THE CONVOLUTION SUM

DT LTI systems: Convolution Sum

$$x[n]=\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Input $x[n]$ → ***LTI system H*** → **Output $y[n]$**

$$y[n]=H\{x[n]\}=H\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}$$

Linearity ⇒

$$y[n]=\sum_{k=-\infty}^{\infty} H\{x[k]\delta[n-k]\}$$

Scaling ⇒

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]H\{\delta[n-k]\}$$

Time Invariant ⇒

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

***h*[*n*] = *H*{δ[*n*]} ≡ impulse response of the *LTI* system**

Convolution sum

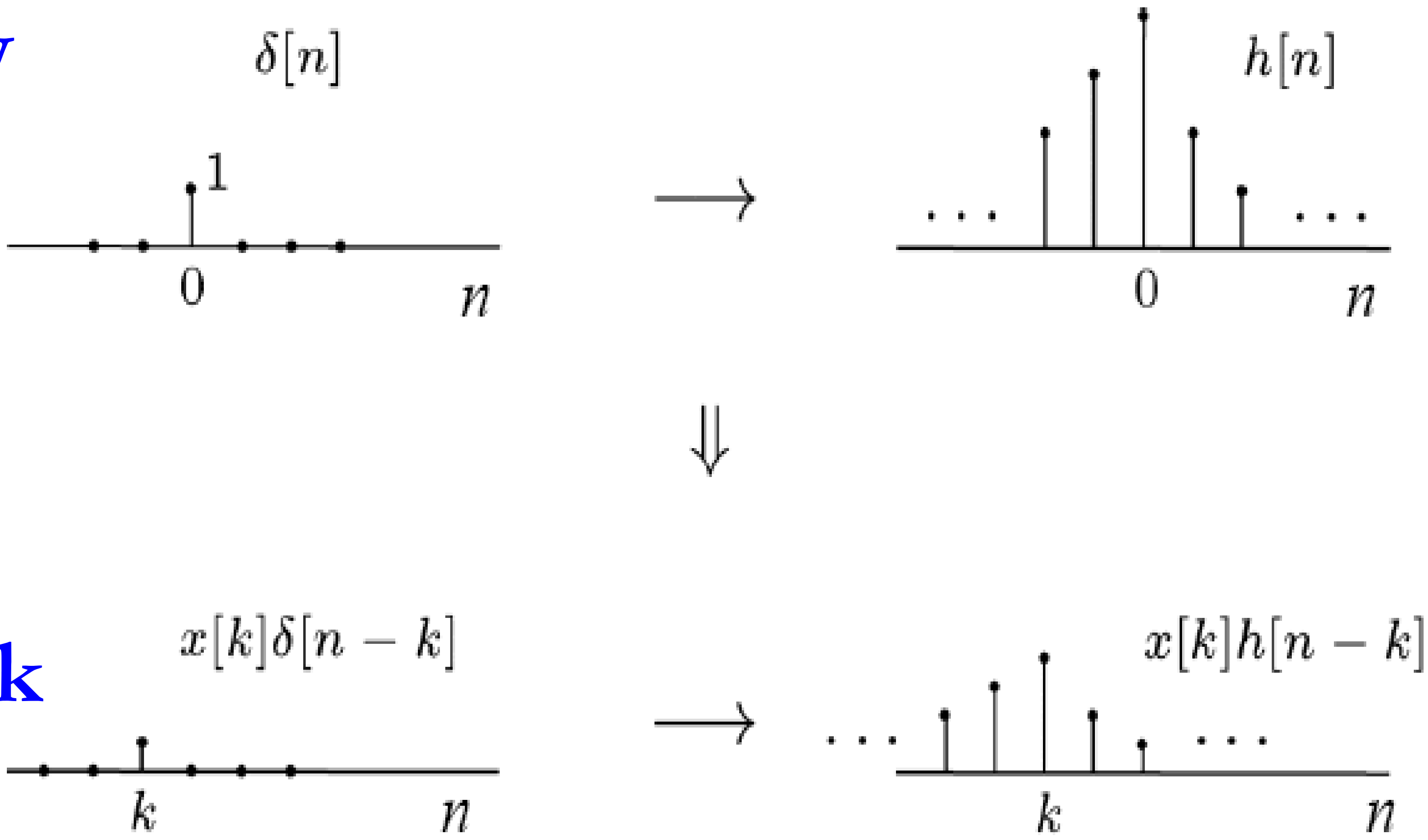
Mathematically

$$y[n]=x[n]*h[n]\triangleq\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

DT LTI systems: Convolution Sum 1

Graphically

For any k



Sum up all the responses for all k′s

Example 2.1 : Convolution Sum

Multipath Communication Channel: Direct Evaluation of the Convolution Sum. **Consider the discrete-time LTI system model representing a two-path propagation channel. If the strength of the indirect path is a = ½, then**

$$y[n] = x[n] + \frac{1}{2} x[n-1]$$

Letting $x[n] = \delta[n]$, the impulse response is h[n]. Then,
Determine the output of this system in response to input

$$x[n] = \begin{cases} 2, & n = 0 \\ 4, & n = 1 \\ -2, & n = 2 \\ 0, & \text{otherwise} \end{cases} \qquad h[n] = \begin{cases} 1, & n = 0 \\ \frac{1}{2}, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

Example 2.1 : Convolution Sum

<Sol.>

Input = 0 for $n < 0$ and $n > 2$

$$x[n]=\begin{cases}2, & n=0\\ 4, & n=1\\ -2, & n=2\\ 0, & \text{otherwise}\end{cases}$$

1. Input: $x[n]=2\delta[n]+4\delta[n-1]-2\delta[n-2]$

time-shifted impulse input

$\gamma\delta[n-k]$

time-shifted impulse response output

$\gamma h[n-k]$

2. Output:

$$y[n]=2h[n]+4h[n-1]-2h[n-2]$$

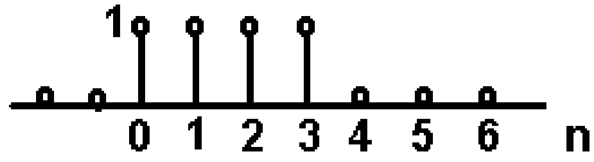
(convolution of x[n] and h[n])

$$h[n]=\begin{cases}1, & n=0\\ \frac{1}{2}, & n=1\\ 0, & \text{otherwise}\end{cases}$$

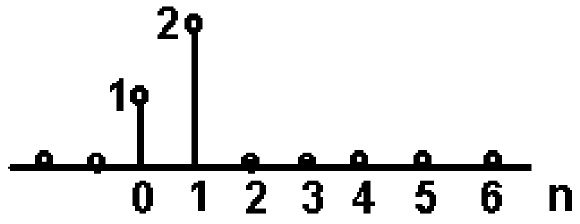
$$y[n]=\begin{cases}0, & n<0\\ 2, & n=0\\ 5, & n=1\\ 0, & n=2\\ -1, & n=3\\ 0, & n\geq 4\end{cases}$$

Example3: Convolution Sum

x[n]



h[n]



$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Superposition Sum

x(k)

n=0 h(-k)

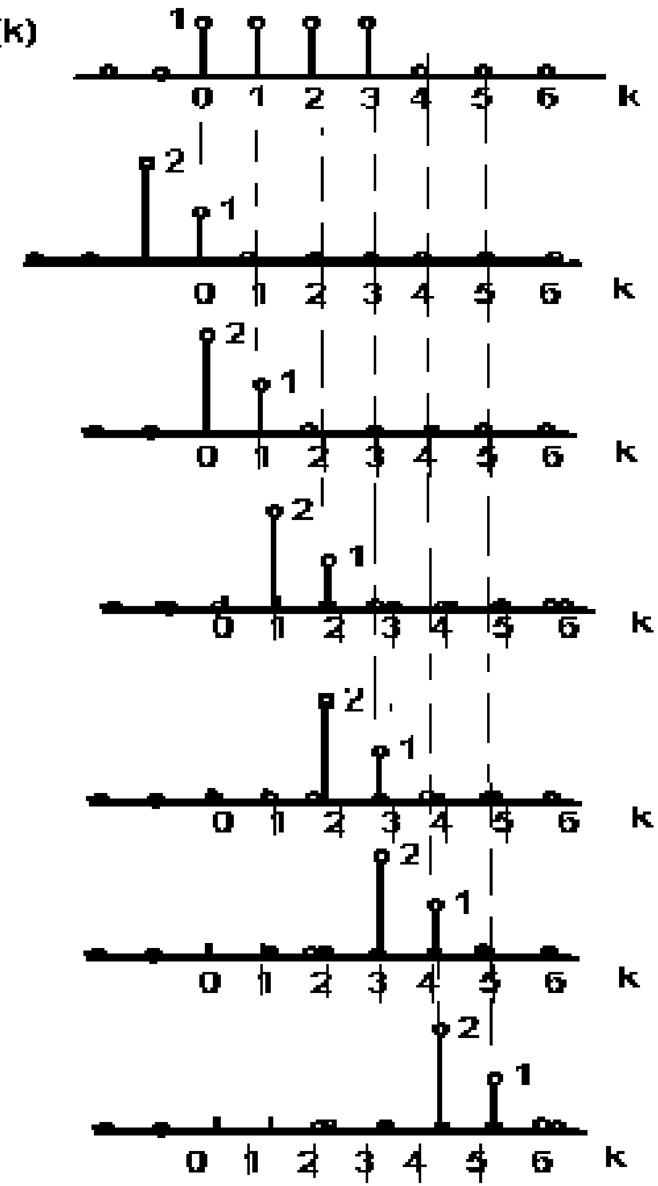
h(1-k)=h[-(k-1)] n=1 h(1-k)

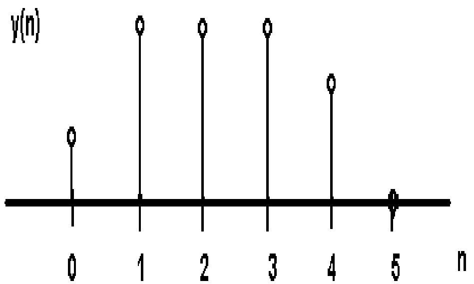
h(2-k)=h[-(k-2)] n=2 h(2-k)

n=3 h(3-k)

n=4 h(4-k)

n=5 h(5-k)





y(0)=1 y(0)=x(0)h(0)

y(1)=3 y(1)=x(0)h(1)

+x(1)h(0)

y(2)=3 y(2)=x(2)h(0)

+x(1)h(1)

y(3) =3

y(4)=2

y(5)=0

Convolution Sum **Evaluation Procedure**

1. Convolution sum:

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

2. Define the intermediate signal

$$\omega_n[k]=x[k]h[n-k]$$

***k* = independent variable, *n* is treated as a constant**

***h* [*n* – *k*] = *h* [– (*k* – *n*)] is a reflected (because of – *k*) and time-shifted (by – *n*) version of *h* [*k*].**

3. Calculate

$$y[n]=\sum_{k=-\infty}^{\infty} \omega_n[k]$$

The time shift *n* determines the time at which we evaluate the output of the system.

Example 2.2 : Convolution Sum

Convolution Sum Evaluation by using Intermediate Signal

Consider a system with impulse response h[n], determine the output of the system at time *n* = – 5, 5, and 10 when the input is *x* [*n*] = *u* [*n*].

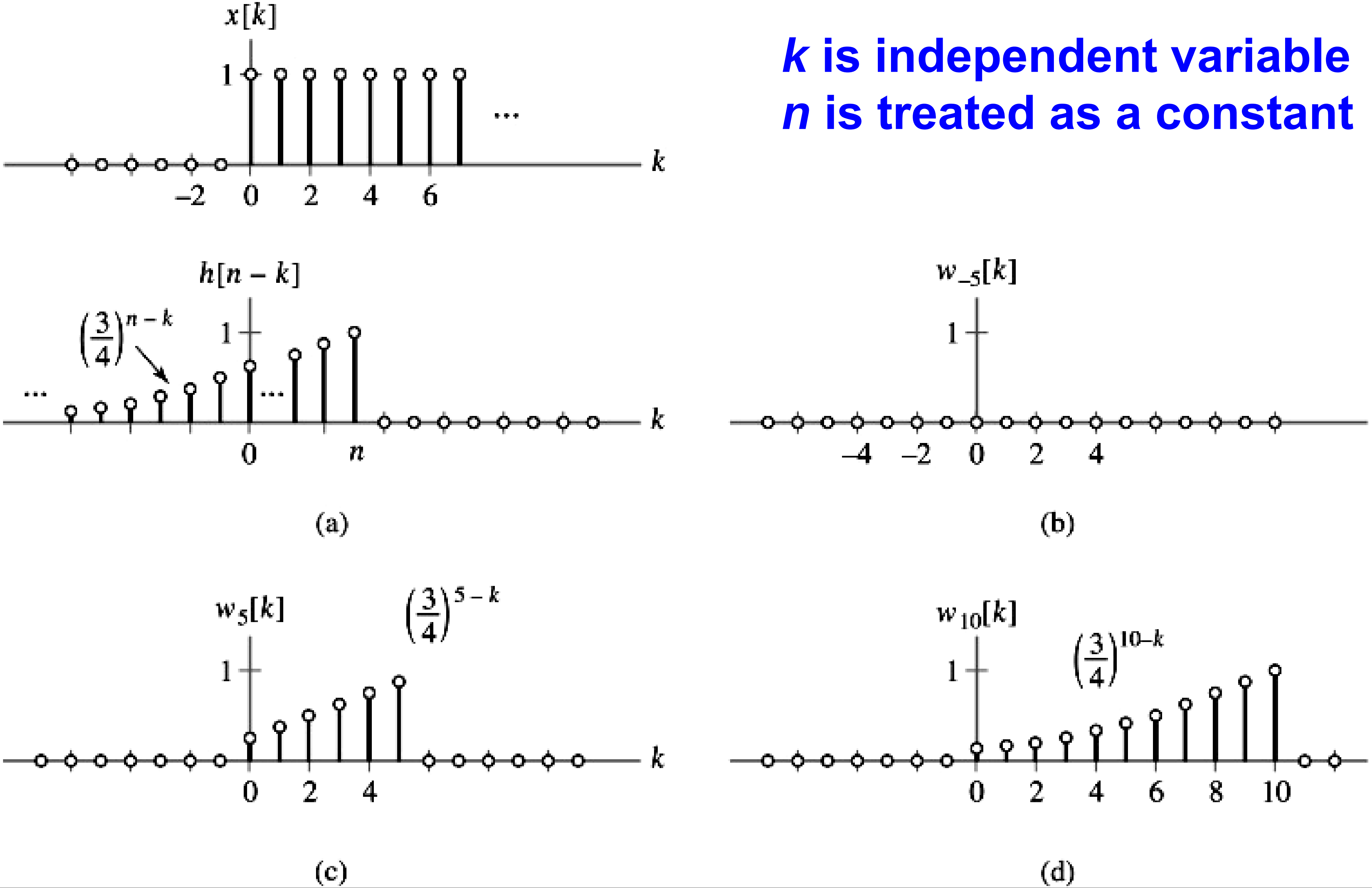
$$h[n] = \left(\frac{3}{4}\right)^n u[n]$$

1. $h[n-k] = (3/4)^{n-k} u[n-k]$

$$h[n-k] = \begin{cases} \left(\frac{3}{4}\right)^{n-k}, & k \le n \\ 0, & \text{otherwise} \end{cases}$$

2. Intermediate signal $w_n[k]$**:** $\omega_n[k] = x[k]h[n-k]$

Example 2.2 : Convolution Sum



***k* is independent variable**

***n* is treated as a constant**

Example 2.2 : Convolution Sum

2. $w_n[k]$:

For n = – 5: $w_{-5}[k]=0$

(x[k]=u[k]=0, k<=-5=n)

⇒ $y[-5]=0$

For n = 5:

$$w_5[k]=\begin{cases}\left(\frac{3}{4}\right)^{5-k}, & 0\le k\le 5\\ 0, & \text{otherwise}\end{cases}$$

For n = 10:

$$w_{10}[k]=\begin{cases}\left(\frac{3}{4}\right)^{10-k}, & 0\le k\le 10\\ 0, & \text{otherwise}\end{cases}$$

$$y[10]=\sum_{k=0}^{10}\left(\frac{3}{4}\right)^{10-k}=\left(\frac{3}{4}\right)^{10}\sum_{k=0}^{10}\left(\frac{3}{4}\right)^{k}=\left(\frac{3}{4}\right)^{10}\frac{1-\left(\frac{3}{4}\right)^{11}}{1-\left(\frac{3}{4}\right)}$$

$$=3.831$$

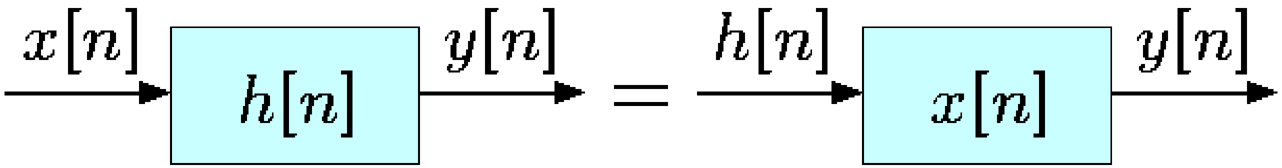
⇒ $$y[5]=\sum_{k=0}^{5}\left(\frac{3}{4}\right)^{5-k}=\left(\frac{3}{4}\right)^{5}\sum_{k=0}^{5}\left(\frac{3}{4}\right)^{k}=\left(\frac{3}{4}\right)^{5}\frac{1-\left(\frac{3}{4}\right)^{6}}{1-\left(\frac{3}{4}\right)}=3.288$$

Procedure 2.1: Reflect and Shift Convolution Sum Evaluation

- **1. Graph both $x[k]$ and $h[n-k]$ as a function of the independent variable k. To determine $h[n-k]$, first reflect $h[k]$ about $k = 0$ to obtain $h[-k]$. Then shift by $-n$.**
- **2. Begin with n large and negative. That is, shift $h[-k]$ to the far left on the time axis.**
- **3. Write the mathematical representation for the intermediate signal $w_n[k]$.**
- **4. Increase the shift n (i.e., move $h[n-k]$ toward the right) until the mathematical representation for $w_n[k]$ changes. The value of n at which the change occurs defines the end of the current interval and the beginning of a new interval.**
- **5. Let n be in the new interval. Repeat step 3 and 4 until all intervals of times shifts and the corresponding mathematical representations for $w_n[k]$ are identified. This usually implies increasing n to a very large positive number.**
- **6. For each interval of time shifts, sum all the values of the corresponding $w_n[k]$ to obtain $y[n]$ on that interval.**

Sequence Convolution Algebra: Commutative Property

$$x[n] * h[n] = h[n] * x[n]$$



Proof:

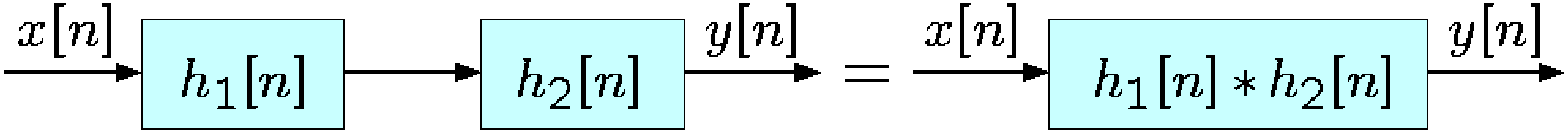
$$y[n] = x[n] * h[n] = \sum_m x[m]h[n-m].$$

Let $n - m = l$ then

$$y[n] = x[n] * h[n] = \sum_l x[n-l]h[l] = \sum_l h[l]x[n-l] = h[n] * x[n].$$

Sequence Convolution Algebra: Associative Property

$$y[n] = (x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$$



Proof:

$$y[n] = \sum_l \left(\sum_m x[m] h_1[l-m] \right) h_2[n-l] = \sum_m x[m] \left(\sum_l h_1[l-m] h_2[n-l] \right).$$

Let $k = l - m$ to obtain

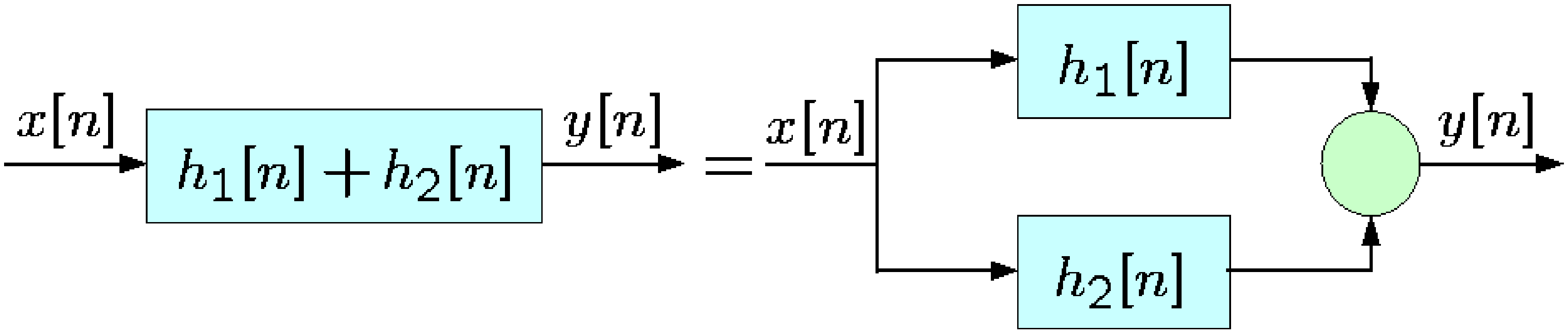
$$y[n] = \sum_m x[m] \left(\sum_k h_1[k] h_2[n-m-k] \right) = \sum_m x[m] h[n-m],$$

where

$$h[n] = h_1[n] * h_2[n].$$

Sequence Convolution Algebra: Distributive Property

$$y[n] = x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$



Proof:

$$\begin{aligned} y[n] &= \sum_m x[m](h_1[n-m] + h_2[n-m]) \\ &= \sum_m x[m]h_1[n-m] + \sum_m x[m]h_2[n-m] \end{aligned}$$

Goto http://mathworld.wolfram.com/Convolution.html
to get a dynamic appreciation of convolution.

Properties of DT-LTI Systems: Memoryless

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$y[n] = \cdots + h[-2]x[n+2] + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + \cdots$$

To be memoryless, $y[n]$ must depend only on $x[n]$ and therefore cannot depend on $x[n-k]$ for $k \neq 0$. Hence, A discrete-time LTI system is memoryless if and only if

$$y[n] = Kx[n]$$

The above equation implies

$$h[n] = K\delta[n]$$

$$\boxed{\therefore \quad if \;\; h[n_0] \neq 0 \quad for \quad n_0 \neq 0 \;\; \text{the DT-LTI has memory}}$$

Properties of DT-LTI Systems: Causality

The output of a causal DT LTI system is expressed as

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

$$\boxed{h[k] = 0 \quad \text{for} \quad k < 0}$$

The above equation shows that the only $x[n]$ needed to evaluate $y[n]$ are $k \le n$ those for which

So, $x[n]$ is causal and therefore, $x[n] = 0 \quad n < 0$

With $x[n]$ causal, $y[n]$ of a DT-LTI system is

$$y[n] = \sum_{k=0}^{\boxed{n}} h[k]x[n-k] = \sum_{\boxed{k=0}}^{n} x[k]h[n-k]$$

Properties of DT-LTI Systems: Stability

$$|y[n]| = |h[n] * x[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

$$|y[n]| \le \sum_{k=-\infty}^{\infty} |h[k]x[n-k]|$$

$$|a+b| \le |a| + |b|$$

$$|y[n]| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

$$|ab| = |a||b|$$

$$|x[n]| \le M_x \le \infty \qquad |x[n-k]| \le M_x$$

$$|y[n]| \le M_x \sum_{k=-\infty}^{\infty} |h[k]|$$

A DT-LTI system is BIBO-stable if its IR is *absolutely summable*; i.e.

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

CT LTI Systems: THE CONVOLUTION INTEGRAL

CT LTI system: Convolution Integral

1. A continuous-time signal can be expressed as a weighted superposition of time-shifted impulses.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

The sifting property of the impulse !

2. $h(t) = H\{\delta(t)\} \equiv$ impulse response of the *LTI* system *H*

$$y(t) = H\{x(t)\} = H\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\}$$

$$= \int_{-\infty}^{\infty} x(\tau)H\{\delta(t-\tau)\}d\tau$$

Linearity property

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$H\{\delta(t-\tau)\} = h(t-\tau)$$

$\delta(t)$

$x(t)$

LTI system

$H[.]$

$h(t) = H[\delta(t)]$

$y(t) = x(t) * h(t)$

Convolution integral

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Convolution Integral Evaluation Procedure

1. Convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

2. Define the intermediate signal

$$w_t(\tau) = x(\tau)h(t-\tau)$$

τ = independent variable
t = constant

***h* (*t* – τ) = *h* (– (τ – *t*)) is a reflected and shifted (by – *t*) version of *h*(τ).**

3. Output

$$y(t) = \int_{-\infty}^{\infty} w_t(\tau)d\tau$$

The time shift *t* determines the time at which we evaluate the output of the system.

Procedure : Reflect and Shift Convolution Integral Evaluation

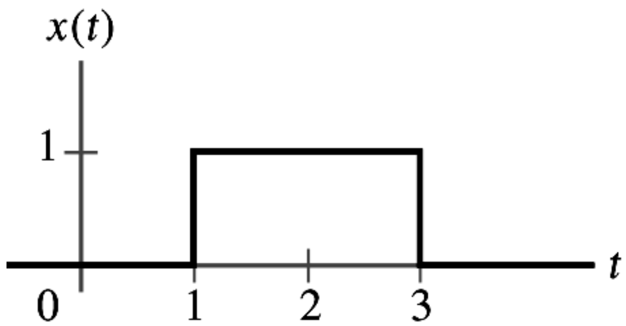
- 1. Graph both $x(\tau)$ and $h(t-\tau)$ as a function of the independent variable τ . To obtain $h(t-\tau)$, reflect $h(\tau)$ about $\tau = 0$ to obtain $h(-\tau)$ and then $h(-\tau)$ shift by $-t$.
- 2. Begin with the shift t large and negative. That is, shift $h(-\tau)$ to the far left on the time axis.
- 3. Write the mathematical representation for the intermediate signal $w_t(\tau)$.
- 4. Increase the shift t (i.e., move $h(t-\tau)$ toward the right) until the mathematical representation for $w_t(\tau)$ changes. The value of t at which the change occurs defines the end of the current set and the beginning of a new set.
- 5. Let t be in the new set. Repeat step 3 and 4 until all sets of shifts t and the corresponding mathematical representations for $w_t(\tau)$ are identified. This usually implies increasing t to a very large positive number.
- 6. For each sets of shifts t, integrate $w_t(\tau)$ from $\tau = -\infty$ to $\tau = \infty$ to obtain y(t).

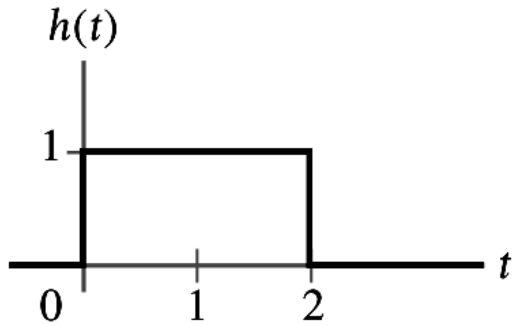
Example 2.6 *Reflect-and-shift Convolution Evaluation*

Evaluate the convolution integral $y(t) = x(t) * h(t)$.

$$x(t) = u(t-1) - u(t-3)$$

$$h(t) = u(t) - u(t-2)$$





<Sol.>

1. Graph of $x(\tau)$ and $h(t-\tau)$
2. Intervals of time shifts: Four

 1st interval: $t < 1$
 2nd interval: $1 \le t < 3$
 3rd interval: $3 \le t < 5$
 4th interval: $5 \le t$

3. 1st, 4th: $t < 1$ or $5 \le t$, $w_t(\tau) = 0$
4. 2nd: $1 \le t < 3$

$$w_t(\tau) = \begin{cases} 1, & 1 < \tau < t \\ 0, & \text{otherwise} \end{cases}$$

5. 3rd: $3 \le t < 5$

$$w_t(\tau) = \begin{cases} 1, & t-2 < \tau < 3 \\ 0, & \text{otherwise} \end{cases}$$

6. Convolution integral
 1) $t < 1$ and $t \ge 5$: $y(t) = 0$
 2) $1 \le t < 3$, $y(t) = t - 1$
 3) $3 \le t < 5$, $y(t) = 3 - (t - 2)$

$$y(t) = \begin{cases} 0, & t < 1 \\ t-1, & 1 \le t < 3 \\ 5-t, & 3 \le t < 5 \\ 0, & t \ge 5 \end{cases}$$

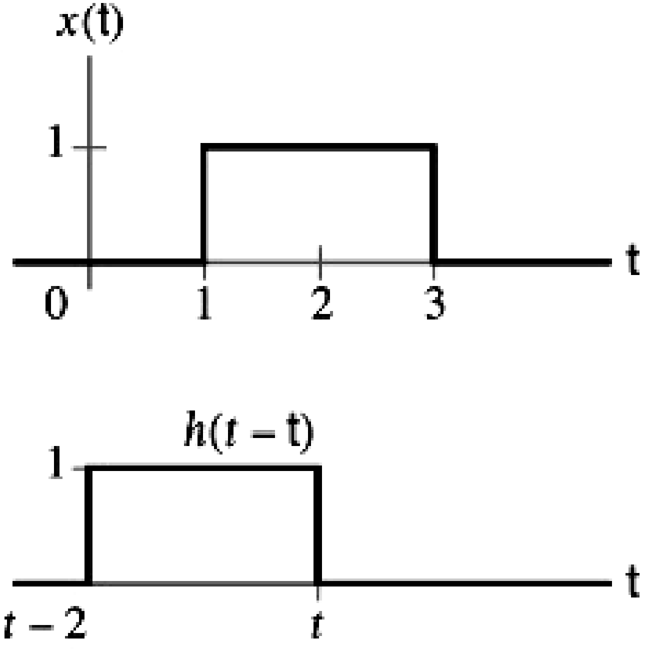
Example 2.6 *Reflect-and-shift Convolution Evaluation*

1'st interval: $t < 1$

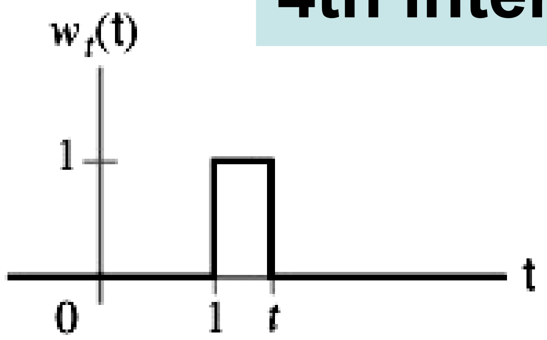
2'nd interval: $1 \le t < 3$

3'rd interval: $3 \le t < 5$

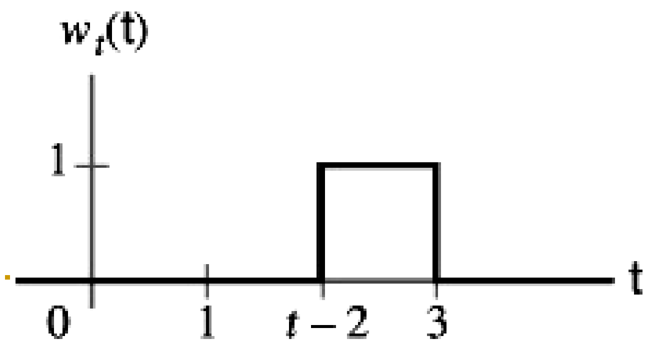
4th interval: $5 \le t$



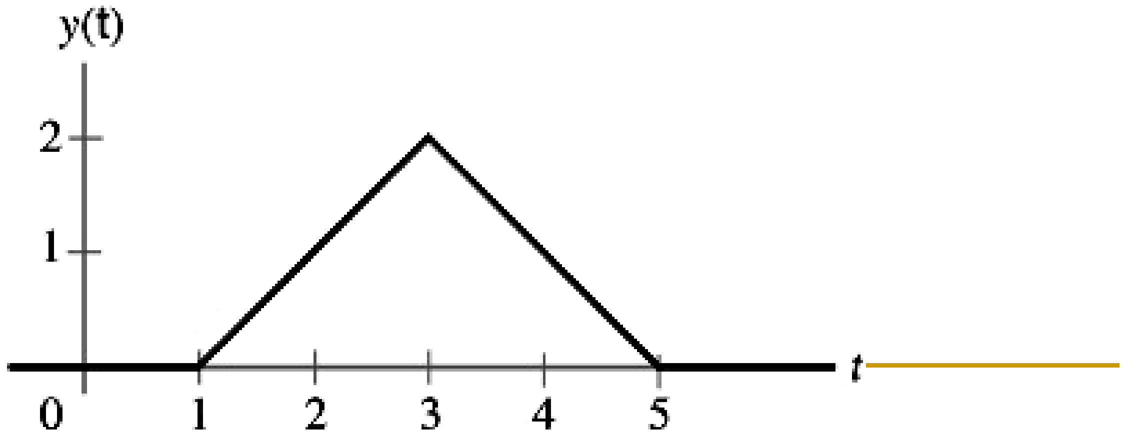
(a)



(b)



(c)



(d)

Convolution Algebra: CT LTI system

Commutation: $x(t)*h(t)=h(t)*x(t)$

Association: $\{x(t)*h_1(t)\}*h_2(t)=x(t)*\{h_1(t)*h_2(t)\}$

Distribution: $x(t)*\{h_1(t)+h_2(t)\}=x(t)*h_1(t)+x(t)*h_2(t)$

Goto
http://mathworld.wolfram.com/Convolution.html
to get a dynamic appreciation of convolution.

Step Response: CT LTI system

The response of a continuous-time LTI system to a step-excitation is given by
Response to a Step Excitation

$$s(t) = H\{u(t)\}$$

The step-response can be determined

$$s(t) = h(t) * u(t) \equiv \int_{-\infty}^{\infty} d\tau\, h(\tau) u(t-\tau) = \int_{-\infty}^{t} d\tau\, h(\tau)$$

i.e. the step-response can be got from integrating the impulse-response, but

$$s^{(1)}(t) = \frac{d}{dt}\{s(t)\} = h(t)$$

The impulse response is merely the differential of the step response

Properties of CT-LTI system: Memoryless

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau,$$

- By definition the response $y(t)$ of a system without memory is dependent only on the corresponding temporal excitation $x(t)$
.

- If also the system is LTI the excitation-response can only be formulated as $y(t) = Kx(t)$; $K =$ (gain) constant

$$\Rightarrow h(t) = K\delta(t)$$

A continuous-time LTI system is memoryless if and only if

so, if $h(\tau) \neq 0$ for $\tau \neq 0$, the continuous-time LTI system has memory.

Properties of CT-LTI system: Causality

A causal system does not respond to an input event until that event actually occurs. Therefore, for a causal continuous-time LTI system, we have h(t)=0 (t<0).

$$y(t) = \begin{cases} \int_0^{\infty} h(\tau)x(t-\tau)d\tau \\ \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau \end{cases} \quad for\ h(t)=0 \qquad t<0,$$

any signal $x(t)$ is called *causal* if

$$x(t) = 0 \qquad\qquad t < 0$$

when the input $x(t)$ is causal,

$$y(t) = \int_0^t h(\tau)x(t-\tau)\,d\tau = \int_0^t x(\tau)h(t-\tau)\,d\tau$$

Properties of CT-LTI system: Stability

Continuous-time LTI system is BIBO stable if its impulse response is absolutely integrable

$$\int_{-\infty}^{\infty} |h(\tau)| \, d\tau < \infty$$

◆ **Note: A system can be unstable even though the impulse response has a finite value.**

1. Ideal integrator:

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

If input x(τ)=δ(τ), then the output is *y*(*t*)=*h*(*t*)=*u*(*t*). *h*(*t*) is not absolutely integrable, Ideal integrator is not stable!

2. Ideal accumulator:

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

If input x[n]=δ[n], then the output is *y*[n]=*h*[*n*]=*u*[*n*]. *h*[*n*] is not absolutely summable, Ideal accumulator is not stable!

Example 2.12 *Properties of the First-Order Recursive System*

The first-order recursive system described by y[n] has the IR h[n], Is it causal, memoryless, and BIBO stable?

$$y[n] = \rho y[n-1] + x[n]$$

$$h[n] = \rho^n u[n]$$

<Sol.>

1. The system is causal, since $h[n] = 0$ for $n < 0$.

2. The system is not memoryless, since $h[n] \neq 0$ for $n > 0$.

3. Stability: Checking whether the impulse response is absolutely summable?

$$\sum_{k=-\infty}^{\infty} \left| h[k] \right| = \sum_{k=0}^{\infty} \left| \rho^k \right| = \sum_{k=0}^{\infty} \left| \rho \right|^k < \infty$$

if and only if $|\rho| < 1$

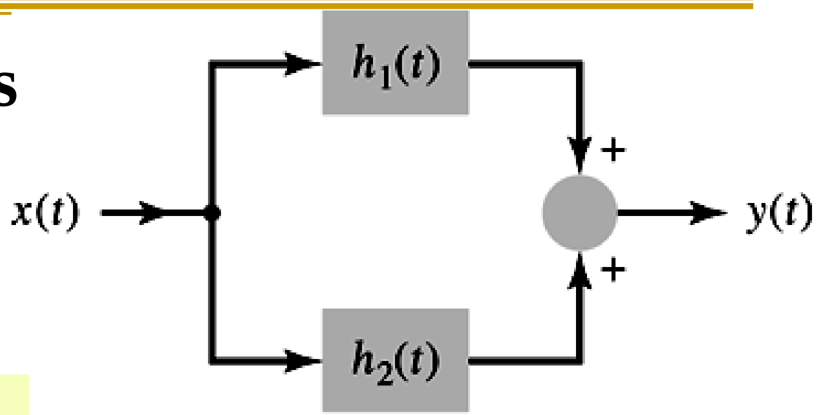
Summarizes the relation between LTI system properties and impulse response characteristics

Table 2.2 *Properties of the Impulse Response Representation for LTI Systems*

Property	Continuous-time system	Discrete-time system
Memoryless	$h(t) = c\delta(t)$	$h[n] = c\delta[n]$
Causal	$h(t) = 0 \quad \text{for} \quad t < 0$	$h[n] = 0 \quad \text{for} \quad n < 0$
Stability	$\int_{-\infty}^{\infty} \lvert h(t) \rvert dt < \infty$	$\sum_{n=-\infty}^{\infty} \lvert h[n] \rvert < \infty$
Invertibility	$h(t) * h^{inv} = \delta(t)$	$h[n] * h^{inv}[n] = \delta[n]$

2.6 Interconnection of LTI Systems: Parallel

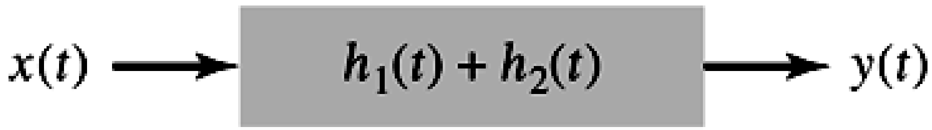
1. Parallel connection of Two LTI systems



$$
\begin{aligned}
y(t) &= y_1(t) + y_2(t) \\
&= x(t) * h_1(t) + x(t) * h_2(t)
\end{aligned}
$$

$$
y(t) = \int_{-\infty}^{\infty} x(\tau) h_1(t-\tau) d\tau + \int_{-\infty}^{\infty} x(\tau) h_2(t-\tau) d\tau
$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} x(\tau) \{ h_1(t-\tau) + h_2(t-\tau) \} d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = x(t) * h(t)
\end{aligned}
$$



$\boldsymbol{h(t) = h_1(t) + h_2(t)}$

2. Distributive property

$$
\mathrm{x(t) * h_1(t) + x(t) * h_2(t) = x(t) * \{h_1(t) + h_2(t)\}}
$$

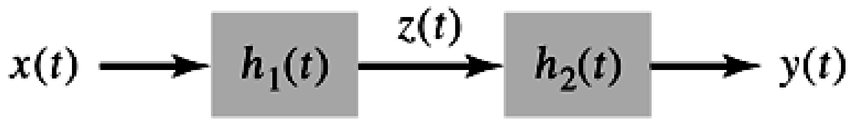
$$
\mathrm{x[n] * h_1[n] + x[n] * h_2[n] = x[n] * \{h_1[n] + h_2[n]\}}
$$

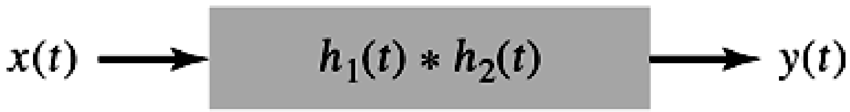
Interconnection of LTI Systems: Cascade

1. Cascade connection of Two LTI systems

$$y(t) = z(t) * h_2(t) = \int_{-\infty}^{\infty} z(\tau) h_2(t-\tau)\, d\tau$$

$$z(\tau) = x(\tau) * h_1(\tau) = \int_{-\infty}^{\infty} x(\nu) h_1(\tau-\nu)\, d\nu$$





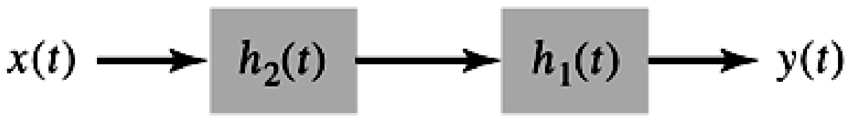
$$y(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(v) h_1(\tau - v) h_2(t-\tau)\, dv\, d\tau$$

$\eta = \tau - \nu$

$$y(t) = \int_{-\infty}^{\infty} x(\nu)\left[\int_{-\infty}^{\infty} h_1(\eta) h_2(t-\nu-\eta)\, d\eta\right] d\nu$$

$h(t) = h_1(t) * h_2(t)$

$$h(t-v) = \int_{-\infty}^{\infty} h_1(\eta) h_2(t-v-\eta)\, d\eta$$



$$y(t) = \int_{-\infty}^{\infty} x(\nu) h(t-\nu)\, d\nu = x(t) * h(t)$$

$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$$

Associative property

$$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$$

$$h(t) = h_1(t) * h_2(t) = \int_{-\infty}^{\infty} h_1(\tau) h_2(t-\tau)\, d\tau$$

$\nu = t - \tau$

$$= \int_{-\infty}^{\infty} h_1(t-\nu) h_2(\nu)\, d\nu = h_2(t) * h_1(t)$$

Commutative property

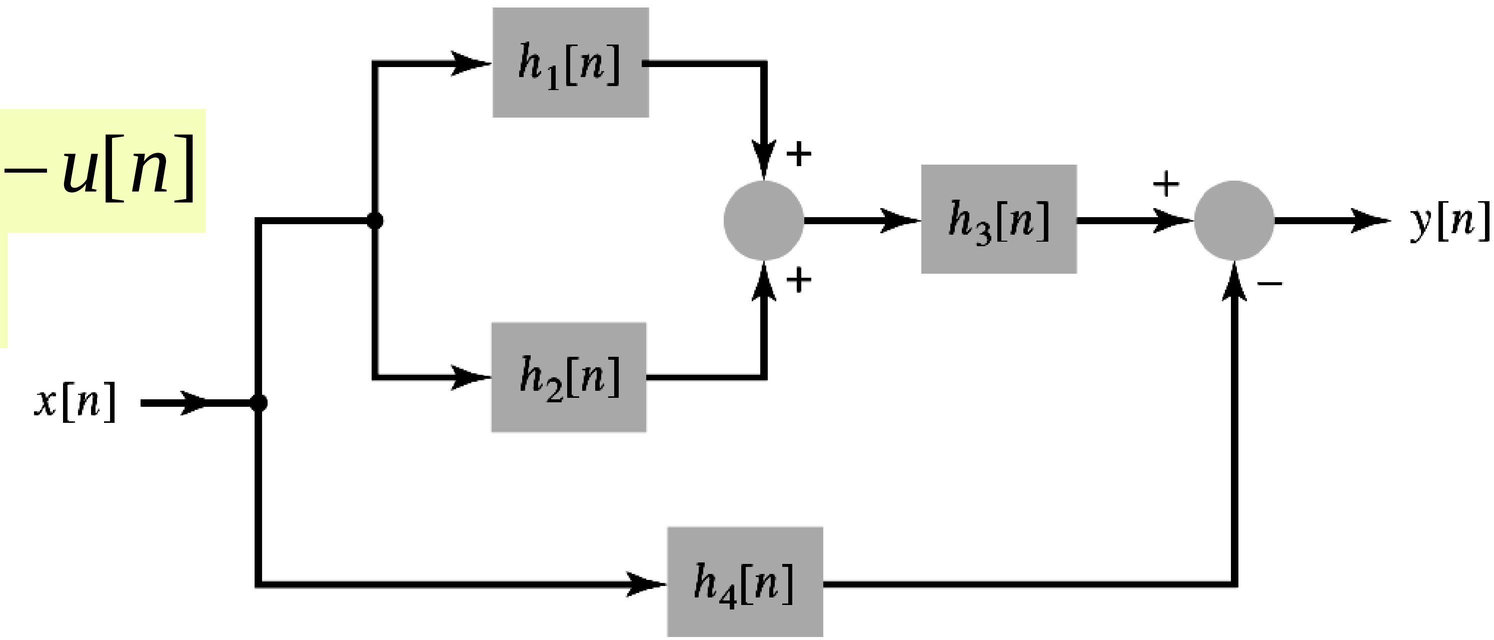
$$h_1[n] * h_2[n] = h_2[n] * h_1[n]$$

$$x(t) * \{h_1(t) * h_2(t)\} = x(t) * \{h_2(t) * h_1(t)\}$$

Example 2.11 *Equivalent System to Interconnected Systems*

Consider the interconnection of four *LTI* systems, Find the impulse response *h*[*n*] of the overall system.

$$h_1[n] = u[n]$$
$$h_2[n] = u[n+2] - u[n]$$
$$h_3[n] = \delta[n-2]$$
$$h_4[n] = \alpha^n u[n]$$



<Sol.>

1. Parallel combination of $h_1[n]$ and $h_2[n]$: $\quad h_{12}[n] = h_1[n] + h_2[n]$

2. $h_{12}[n]$ is in series with $h_3[n]$: $\quad h_{123}[n] = h_{12}[n] * h_2[n]$

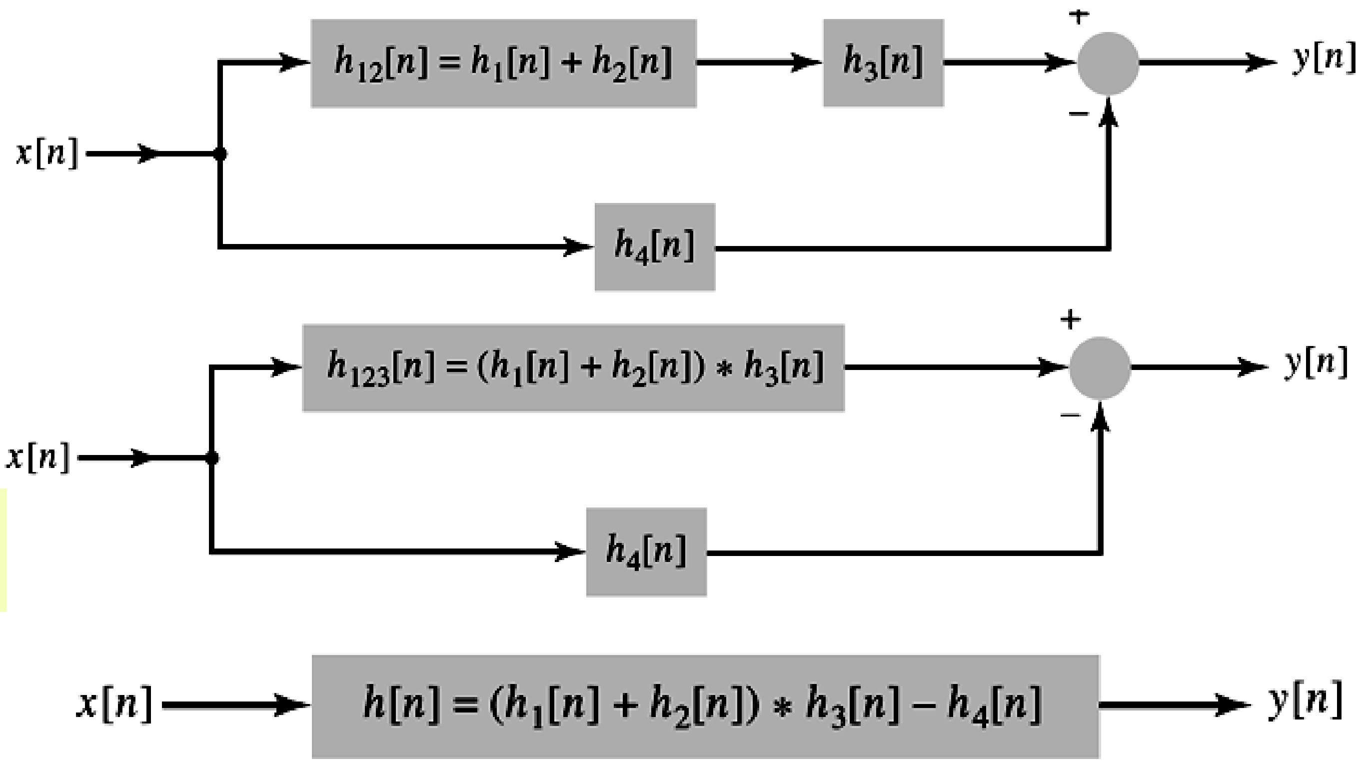
3. $h_{123}[n]$ is in parallel with $h_4[n]$: $\quad h[n] = h_{123}[n] - h_4[n]$

Example 2.11 *Equivalent System to Interconnected Systems*

$$h_{12}[n] = h_1[n] + h_2[n]$$

$$h_{123}[n] = h_{12}[n] \ * \ h_2[n]$$

$$h[n] = h_{123}[n] - h_4[n]$$



$$h_1[n] = u[n]$$

$$h_2[n] = u[n+2] - u[n]$$

$$h_3[n] = \delta[n-2]$$

$$h_4[n] = \alpha^n u[n]$$

$$h_{12}[n] = u[n] + u[n+2] - u[n] = u[n+2]$$

$$h_{123}[n] = u[n+2] * \delta[n-2] = u[n]$$

$$h[n] = (h_1[n] + h_2[n]) * h_3[n] - h_4[n] = \left\{1 - \alpha^n\right\} u[n]$$

Table 2.1 Interconnection Properties for LTI Systems

Table 2.1 Interconnection Properties for LTI Systems

Property	Continuous-time system	Discrete-time system
Distributive	$x(t)*h_1(t)+x(t)*h_2(t)=$ $x(t)*\{h_1(t)+h_2(t)\}$	$x[n]*h_1[n]+x[n]*h_2[n]=$ $x[n]*\{h_1[n]+h_2[n]\}$
Associative	$\{x(t)*h_1(t)\}*h_2(t)=x(t)*\{h_1(t)*h_2(t)\}$	$\{x[n]*h_1[n]\}*h_2[n]=x[n]*\{h_1[n]*h_2[n]\}$
Commutative	$h_1(t)*h_2(t)=h_2(t)*h_1(t)$	$h_1[n]*h_2[n]=h_2[n]*h_1[n]$

Equation Representations of LTI systems

Linear constant-coefficient difference and differential equations provide another representation for the input-output characteristics of DT and CT LTI systems, respectively.

1. Linear constant-coefficient differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

Input = $x(t)$, output = $y(t)$

2. Linear constant-coefficient difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Input = $x[n]$, output = $y[n]$

The *order* of the differential or difference equation is (N, M)

Solving Differential and Difference Equations

Complete solution: $y = y^{(h)} + y^{(p)}$

$y^{(h)}$ = homogeneous solution, set all terms involving the input to zero

$y^{(p)}$ = particular solution

2.10.1 The Homogeneous Solution

♣ **Continuous-time case:**

1. Homogeneous differential equation: $\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y^{(h)}(t) = 0$

2. Homogeneous solution:

$$y^{(h)}(t) = \sum_{i=0}^{N} c_i e^{r_i t}$$

Coefficients c_i is determined by I.C.

3. Characteristic eq.: $\sum_{k=0}^{N} a_k r^k = 0$

♣ **Discrete-time case:**

1. Homogeneous differential equation: $\sum_{k=0}^{N} a_k y^{(h)}[n-k] = 0$

2. Homogeneous solution:

$$y^{(h)}[n] = \sum_{i=1}^{N} c_i r_i^n$$

Coefficients c_i is determined by I.C.

3. Characteristic eq.: $\sum_{k=0}^{N} a_k r^{N-k} = 0$

♣ **If a root r_j is repeated p times in characteristic eqs., the corresponding solutions are**

Continuous-time case: $e^{r_j t}, \quad te^{r_j t}, \quad \ldots, \quad t^{p-1} e^{r_j t}$

Discrete-time case: $r_j^n, \quad nr_j^n, \quad \ldots, \quad n^{p-1} r_j^n$

2.10.2 The Particular Solution

A particular solution is usually obtained by assuming an output of the same general form as the input.

Table 2.3 ***Form of Particular Solutions Corresponding to Commonly Used Inputs***

Continuous Time		Discrete Time	
Input	Particular Solution	Input	Particular Solution
1	c	1	c
t	$c_1 t + c_2$	n	$c_1 n + c_2$
e^{-at}	ce^{-at}	α^n	$c\alpha^n$
$\cos(\omega t + \phi)$	$c_1 \cos(\omega t) + c_2 \sin(\omega t)$	$\cos(\Omega n + \phi)$	$c_1 \cos(\Omega n) + c_2 \sin(\Omega n)$

2.10.3 The Complete Solution

Complete solution: $y = y^{(h)} + y^{(p)}$

$y^{(h)}$ = homogeneous solution, $y^{(p)}$ = particular solution

The procedure for finding complete solution of differential or difference equations is summarized as follows:

Procedure 2.3: Solving a Differential or Difference equation

1. Find the form of the homogeneous solution $y^{(h)}$ from the roots of the characteristic equation.

2. Find a particular solution $y^{(p)}$ by assuming that it is of the same form as the input, yet is independent of all terms in the homogeneous solution.

3. Determine the coefficients in the homogeneous solution so that the complete solution $y = y^{(h)} + y^{(p)}$ satisfies the initial conditions.

★ Note that the initial translation is needed in some cases.

2.11 Characteristics of Systems Described by Differential and Difference Equations

Complete solution: $y = y^{(n)} + y^{(f)}$

$y^{(n)}$ = natural response,

$y^{(f)}$ = forced response

2.11.1 The Natural Response

The natural response is the system output for zero input and thus describes the manner in which the system dissipates any stored energy or memory of the past represented by non-zero initial conditions.

2.11.2 The Forced Response

The forced response is the system output due to the input signal assuming zero initial conditions.

The forced response is valid only for $t \geq 0$ or $n \geq 0$

2.11 Characteristics of Systems Described by Differential and Difference Equations

2.11.3 The Impulse Response

♣ **Relation between step response and impulse response**

1. Continuous-time case:

$$h(t) = \frac{d}{dt}s(t)$$

2. Discrete-time case:

$$h[n] = s[n] - s[n-1]$$

2.11.4 Linearity and Time Invariance

♣ **Forced response $\Rightarrow$ Linearity**

Input	Forced response
x_1	$y_1^{(f)}$
x_2	$y_2^{(f)}$
$\alpha x_1 + \beta x_2$	$\alpha y_1^{(f)} + \beta y_2^{(f)}$

♣ **Natural response $\Rightarrow$ Linearity**

Initial Cond.	Natural response
I_1	$y_1^{(n)}$
I_2	$y_2^{(n)}$
$\alpha I_1 + \beta I_2$	$\alpha y_1^{(n)} + \beta y_2^{(n)}$

♣ **The complete response of an LTI system is *not* time invariant.**

⇨ **Response due to initial condition will not shift with a time shift of the input.**

Summary and Exercises

- Summary
  - Convolution Sum
  - Convolution Integral
  - Properties of LTI systems
  - Equivalent systems
- Exercises
  - P91-93: 2.32, 2.34(a, c, d), 2.35, 2.38, 2.40(a, b, c, d), 2.47(c), 2.48