Signals and Systems 3.3

--- *Properties of Fourier Representation*

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Reference:
1. Textbook: Chapter 3.8 - 3.18

Properties of Fourier Representation

- Periodicity Properties
- Linearity Properties
- Symmetry Properties
- Convolution Property
- Differentiation and Integration Properties
- Time-shift Properties
- Frequency-shift Properties
- Multiplication Properties
- Scaling Properties
- Duality Properties
- Inverse Fourier representation
- Parseval relationship
- Time bandwidth products

Periodicity properties

1. Four Fourier representations: Table 3.2.

Table 3.2 The Four Fourier Representations

Time Domain	Periodic (t, n)	Non-periodic (t, n)	
Continuous (t)	**Fourier Series** $x(t)=\sum_{k=-\infty}^{\infty}X[k]e^{jk\omega_o t}$ $X[k]=\frac{1}{T}\int_0^T x(t)e^{-jk\omega_o t}dt$ $x(t)$ has period T $\omega_o=\frac{2\pi}{T}$	**Fourier Transform** $x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$ $X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$	Nonperiodic $(k,\ \omega)$
Discrete [n]	**Discrete-Time Fourier Series** $x[n]=\sum_{k=0}^{N-1}X[k]e^{jk\Omega_o n}$ $X[k]=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk\Omega_o n}$ $x[n]$ and $X[k]$ have period N $\Omega_o=\frac{2\pi}{N}$	**Discrete-Time Fourier Transform** $x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\Omega})e^{j\Omega n}d\Omega$ $X(e^{j\Omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}$ $X(e^{j\Omega})$ has period 2π	Periodic $(k,\ \Omega)$
	Discrete [k]	Continuous $(\omega,\ \Omega)$	Frequency Domain

Periodicity properties

2. Signals that are periodic in time have discrete frequency-domain representations, while nonperiodic time signals have continuous frequency- domain representations.

Table 3.3 Periodicity Properties of Fourier Representations

Time-Domain Property	Frequency-Domain Property
Continuous	Nonperiodic
Discrete	Periodic
Periodic	Discrete
Nonperiodic	Continuous

Linearity Properties

$$z(t)=ax(t)+by(t) \xleftrightarrow{FT} Z(j\omega)=aX(j\omega)+bY(j\omega)$$
$$z(t)=ax(t)+by(t) \xleftrightarrow{FS;\omega_o} Z[k]=aX[k]+bY[k]$$
$$z[n]=ax[n]+by[n] \xleftrightarrow{DTFT} Z(e^{j\Omega})=aX(e^{j\Omega})+bY(e^{j\Omega})$$
$$z[n]=ax[n]+by[n] \xleftrightarrow{DTFS;\Omega_o} Z[k]=aX[k]+bY[k]$$

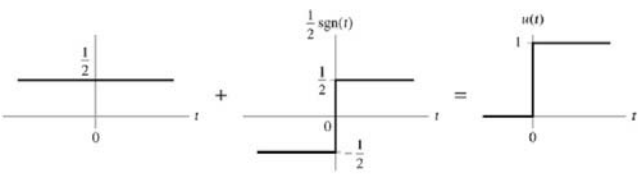
Both a and b are constant.

The linearity property is used to find Fourier representations of signals that are constructed as sums of signals whose representations are already known

Example 1 of Linearity in FT: : Representation of a Unit Step signal as a weighted sum of a constant and a signum signal

$$u(t)=\frac{1}{2}+\frac{1}{2}\mathrm{sgn}(t) \qquad F[u(t)]=\pi\delta(\omega)+\frac{1}{j\omega}$$

$$F[\frac{1}{2}]=2\pi\delta(\omega) \qquad F[\mathrm{sgn}(t)]=\frac{2}{j\omega}$$



Example 2 of Linearity in The FS: Representation of a periodic signal as a weighted sum of periodic square waves

Example 3.30

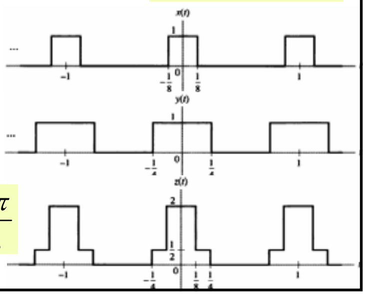
$$z(t)=\frac{3}{2}x(t)+\frac{1}{2}y(t)$$

$$x(t)\xleftrightarrow{FS;\omega_0}X[k]$$
$$y(t)\xleftrightarrow{FS;\omega_0}Y[k]$$
$$z(t)\xleftrightarrow{FS;\omega_0}Z[k]$$

$$X[k]=\frac{1}{k\pi}\sin\frac{k\pi}{4}$$

$$Y[k]=\frac{1}{k\pi}\sin\frac{k\pi}{2}$$

$$Z[k]=\frac{3}{2}X[k]+\frac{1}{2}Y[k]$$
$$=\frac{1}{2k\pi}\sin\frac{k\pi}{4}+\frac{3}{2k\pi}\sin\frac{k\pi}{2}$$



Symmetry Properties: Real and Imaginary Signals

$$x(t)\xleftrightarrow{FT}X(j\omega) \qquad x^*(t)\xleftrightarrow{FT}X^*(-j\omega)$$

$$F[x^*(t)]=\int_{-\infty}^{\infty}x^*(t)e^{-j\omega t}dt=\left[\int_{-\infty}^{\infty}x(t)e^{-j(-\omega)t}dt\right]^*=X^*(-j\omega)$$

$$X(j\omega)=|X(j\omega)|e^{j\phi(j\omega)}=\mathrm{Re}[X(j\omega)]+j\,\mathrm{Im}[X(j\omega)]$$

For continuous real valued signal *x(t)* = *x*(t)*

$$X(j\omega)=X^*(-j\omega) \qquad X^*(j\omega)=X(-j\omega)$$

$\mathrm{Re}\{X(j\omega)\}=\mathrm{Re}\{X(-j\omega)\}$ — Real of $X(j\omega)$ is oven

$\mathrm{Im}\{X(j\omega)\}=-\mathrm{Im}\{X(-j\omega)\}$ — Imagery of $X(j\omega)$ is odd

$|X(j\omega)|=|X(-j\omega)|$ — Magnitude spectrum of $X(j\omega)$ is oven

$\phi(j\omega)\triangleq\arg\{X(j\omega)\}=-\phi(-j\omega)$ — Phase spectrum of $X(j\omega)$ is odd

Symmetry Properties: Real and Imaginary Signals

Table 3.4 *Symmetry Properties for Fourier Representation of Real- and Imaginary-Valued Signals*

Representation	Real-Valued Time Signals	Imaginary-Valued Time Signals
FT	$X^*(j\omega)=X(-j\omega)$	$X^*(j\omega)=-X(-j\omega)$
FS	$X^*[k]=X[-k]$	$X^*[k]=-X[-k]$
DTFT	$X^*(e^{j\Omega})=X(e^{-j\Omega})$	$X^*(e^{j\Omega})=-X(e^{-j\Omega})$
DTFS	$X^*[k]=X[-k]$	$X^*[k]=-X[-k]$

Example: Symmetry Properties

1. *x*(*t*) is real valued and has even symmetry.

$x^*(t) = x(t)$

$x(-t) = x(t)$

$\Rightarrow$ $x^*(t) = x(-t)$

$$X^*(j\omega)=\int_{-\infty}^{\infty}x(-t)e^{-j\omega(-t)}dt$$ Change of variable $\tau=-t$

$$\Rightarrow X^*(j\omega)=-\int_{\infty}^{-\infty}x(\tau)e^{-j\omega\tau}d\tau=X(j\omega)$$

$X^*(j\omega)=X(j\omega)$ 1) If *x*(*t*) is real and even, then the imaginary part of $X(j\omega)$ = 0, $X(j\omega)$ is real.

2) If *x*(*t*) is real and odd, then $X^*(j\omega) = -X(j\omega)$ and $X(j\omega)$ is imaginary.

Convolution Properties: Nonperiodic CT Signals

- *Convolution of Nonperiodic Continuous-time Signals*

$$x(t)\xleftrightarrow{FT}X(j\omega) \qquad h(t)\xleftrightarrow{FT}H(j\omega)$$

$$y(t)=h(t)*x(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau$$
$$=\int_{-\infty}^{\infty}h(\tau)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}e^{-j\omega\tau}d\omega\right]d\tau$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}h(\tau)e^{-j\omega\tau}d\tau\right]X(j\omega)e^{j\omega t}d\omega$$
$$y(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}H(j\omega)X(j\omega)e^{j\omega t}d\omega$$

Convolution of *h(t)* and *x(t)* in the time domain corresponds to multiplication of their Fourier transforms, H(jω) and X(jω)in the frequency domain.

$$y(t)=h(t)*x(t)\xleftrightarrow{FT}Y(j\omega)=X(j\omega)H(j\omega)$$

Example1: Solving A Convolution Problem in The Frequency Domain

Example 3.31

$$x(t)=\frac{\sin(\pi t)}{\pi t} \rightarrow \boxed{h(t)=\frac{\sin(2\pi t)}{\pi t}} \rightarrow y(t)=h(t)*x(t)$$

$$x(t)\xleftrightarrow{FT}X(j\omega)=\begin{cases}1, & |\omega|<\pi\\ 0, & |\omega|>\pi\end{cases} \qquad h(t)\xleftrightarrow{FT}H(j\omega)=\begin{cases}1, & |\omega|<2\pi\\ 0, & |\omega|>2\pi\end{cases}$$

Time domain → Extremely Difficult
Frequency domain → Quite Simple

Convolution property

$$y(t)=h(t)*x(t)\xleftrightarrow{FT}Y(j\omega)=X(j\omega)H(j\omega)$$

$$Y(j\omega)=\begin{cases}1, & |\omega|<\pi\\ 0, & |\omega|>\pi\end{cases} \qquad y(t)=\frac{\sin(\pi t)}{\pi t}$$

Example2: Finding Inverse FT's by Means of The Convolution Property

Example 3.32 Use the convolution property to find $x(t)$

$$x(t) \xleftrightarrow{FT} X(j\omega) = \frac{4}{\omega^2}\sin^2(\omega)$$

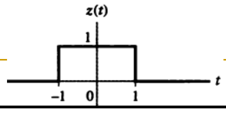
$X(j\omega) = Z(j\omega)\,Z(j\omega)$

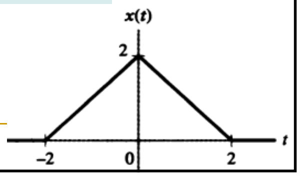
$$z(t) = \begin{cases} 1, & |t| < 1 \\ 0, & |t| > 1 \end{cases} \xleftrightarrow{FT} Z(j\omega)$$

$$Z(j\omega) = \frac{2}{\omega}\sin(\omega)$$

$$z(t) * z(t) \xleftrightarrow{FT} Z(j\omega)Z(j\omega)$$

$$x(t) = z(t) * z(t)$$





Convolution Properties: Nonperiodic DT Signals

♣ **Convolution of Nonperiodic Discrete-time Signals**

$$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$$

$$h[n] \xleftrightarrow{DTFT} H(e^{j\Omega})$$

$$y[n] = x[n] * h[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega})$$

Convolution of $h[n]$ and $x[n]$ in the time domain corresponds to multiplication of their Fourier transforms, $H(e^{j\Omega})$ and $X(e^{j\Omega})$ in the frequency domain.

Frequency response of the LTI system

- **The convolution property implies that the frequency response of a system may be expressed as the ratio of the FT or DTFT of the output to the input .**

For CT system:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

For DT system:

$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})}$$

Problem1: Find the system output (5~10min)

▶ **Problem 3.18** Use the convolution property to find the FT of the system output, either $Y(j\omega)$ or $Y(e^{j\Omega})$, for the following inputs and system impulse responses:

(a) $x(t) = 3e^{-t}u(t)$ and $h(t) = 2e^{-2t}u(t)$.

(b) $x[n] = \left(\frac{1}{3}\right)^{n+1} u[n]$ and $h[n] = \left(\frac{1}{6}\right)^n u[n]$

Sol (a):

$$x(t) = 3e^{-t}u(t) \Leftrightarrow X(j\omega) = \frac{3}{1+j\omega}$$

$$h(t) = 2e^{-2t}u(t) \Leftrightarrow H(j\omega) = \frac{2}{2+j\omega}$$

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{2}{(2+j\omega)}\frac{3}{(1+j\omega)}$$

Sol (b):

$$x[n] = \left(\tfrac{1}{3}\right)^{n+1} u[n] = \tfrac{1}{3}\left(\tfrac{1}{3}\right)^n u[n] \Leftrightarrow X(e^{j\Omega}) = \frac{1}{3\left(1-\frac{1}{3}e^{-j\Omega}\right)}$$

$$h[n] = \left(\tfrac{1}{6}\right)^n u[n] \Leftrightarrow H(e^{j\Omega}) = \frac{1}{1-\frac{1}{6}e^{-j\Omega}}$$

$$Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}) = \frac{1}{3\left(1-\frac{1}{3}e^{-j\Omega}\right)}\frac{1}{\left(1-\frac{1}{6}e^{-j\Omega}\right)}$$

Problem1: Find the system output (5~10min)

$$Y(j\omega) = \frac{2}{(2+j\omega)}\frac{3}{(1+j\omega)} = \frac{1}{6(1+j\omega)} - \frac{1}{6(2+j\omega)}$$

$$y(t) = \tfrac{1}{6}e^{-t}u(t) - \tfrac{1}{6}e^{-2t}u(t)$$

$$e^{-\alpha t}u(t) \xleftrightarrow{FT} \frac{1}{\alpha + j\omega} \qquad \alpha > 0$$

$$Y(e^{j\Omega}) = \frac{1}{3\left(1-\frac{1}{3}e^{-j\Omega}\right)}\frac{1}{\left(1-\frac{1}{6}e^{-j\Omega}\right)} = \frac{2}{3\left(1-\frac{1}{3}e^{-j\Omega}\right)} - \frac{1}{3\left(1-\frac{1}{6}e^{-j\Omega}\right)}$$

$$\alpha^n u[n] \xleftrightarrow{DTFT} \frac{1}{1-\alpha e^{-j\Omega}} \qquad |\alpha| < 1$$

$$y[n] = \tfrac{2}{3}\left(\tfrac{1}{3}\right)^n u[n] - \tfrac{1}{3}\left(\tfrac{1}{6}\right)^n u[n]$$

Problem2: Find the response of the system (5~10min)

The output of an LTI system in response to an input $x(t) = e^{-2t}u(t)$ is $y(t) = e^{-t}u(t)$. Find the frequency response and the impulse response of this system.

$$x(t) = e^{-2t}u(t) \Leftrightarrow X(j\omega) = \frac{1}{2+j\omega}$$

$$y(t) = e^{-t}u(t) \Leftrightarrow Y(j\omega) = \frac{1}{1+j\omega}$$

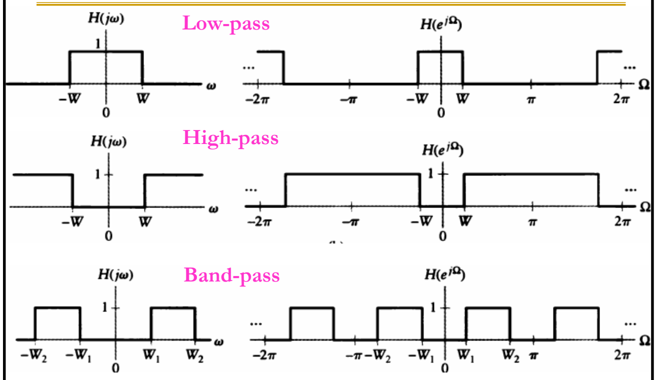
$$H(j\omega) = Y(j\omega)/X(j\omega) = \frac{2+j\omega}{1+j\omega} = 1 + \frac{1}{1+j\omega}$$

$$\Leftrightarrow h(t) = \delta(t) + e^{-t}u(t)$$

Filtering

1. The terms "*filtering*" implies that some frequency components of the input are eliminated while others are passed by the system unchanged.
2. Multiplication in frequency domain ↔ *Filtering*.
3. System Types of filtering:
   1) Low-pass filter
   2) High-pass filter
   3) Band-pass filter
4. Realistic filter:
   1) Gradual transition band
   2) Nonzero gain of stop band

Filtering



Filtering

5. Magnitude response of filter:

$20\log|H(j\omega)|$ $\quad$ $20\log|H(e^{j\Omega})|$ [dB]

6. The edge of the passband is usually defined by the frequencies for which the response is – 3 dB, corresponding to a magnitude response of ($1/\sqrt{2}$).

♣ Unity gain = 0 dB

7. Energy spectrum of filter: $|Y(j\omega)|^2 = |H(j\omega)|^2 |X(j\omega)|^2$

The – 3 dB point corresponds to frequencies at which the filter passes only half of the input power. – 3 dB point ⟹ Cutoff frequency

Convolution of Periodic Signals

♣ Basic Concept: → **Periodic *h*** →

1. Define the periodic convolution of two CT signals $x(t)$ and $z(t)$, each having period T, as

$$y(t) = x(t) \circledast z(t) = \int_0^T x(\tau) z(t-\tau)\, d\tau$$

where the symbol $\circledast$ denotes that integration is performed over a single period of the signals involved.

$$y(t) = x(t) \circledast z(t) \xleftrightarrow{FS;\frac{2\pi}{T}} Y[k] = TX[k]Z[k]$$

◆ Convolution in Time-Domain
↔ Multiplication in Frequency-Domain

Convolution in Time-Domain ↔ Multiplication in Frequency-Domain

Table 3.5 Convolution Properties

$x(t)*z(t)$	$\xleftrightarrow{FT}$	$X(j\omega)Z(j\omega)$
$x(t)\circledast z(t)$	$\xleftrightarrow{FS;\,\omega_0}$	$TX[k]Z[k]$
$x[n]*z[n]$	$\xleftrightarrow{FT}$	$X(e^{j\Omega})Z(e^{j\Omega})$
$x[n]\circledast z[n]$	$\xleftrightarrow{DTFS;\,\Omega_0}$	$NX[k]Z[k]$

Differentiation Properties: Time Domain

Differentiation in Time Domain

$$x(t) \xleftrightarrow{FT} X(j\omega) \qquad \frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega \qquad \frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) j\omega e^{j\omega t}\, d\omega$$

$$\frac{d^n}{dt^n}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)(j\omega)^n e^{j\omega t}\, d\omega \qquad \frac{d^n}{dt^n}x(t) \xleftrightarrow{FT} (j\omega)^n X(j\omega)$$

Differentiation of $x(t)$ in Time-Domain ↔ $(j\omega) \times X(j\omega)$ in Frequency-Domain

$$e^{-\alpha t}u(t) \xleftrightarrow{FT} \frac{1}{\alpha + j\omega} \qquad \frac{d}{dt}\left\{e^{-\alpha t}u(t)\right\} \xleftrightarrow{FT} \frac{j\omega}{\alpha + j\omega}$$

Differentiation Properties: Frequency Domain

Differentiation in Frequency Domain

$x(t) \xleftrightarrow{FT} X(j\omega)$

$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega} X(j\omega)$

$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$\frac{d}{d\omega} X(j\omega) = \int_{-\infty}^{\infty} -jtx(t) e^{-j\omega t} dt$

Differentiation of $X(j\omega)$ in Frequency-Domain ↔ $(-jt) \times x(t)$ in Time-Domain

Example

$\frac{d}{dt} x(t) \xleftrightarrow{FT} j\omega X(j\omega)$

$e^{-\alpha t} u(t) \xleftrightarrow{FT} \frac{1}{\alpha + j\omega}$

$\frac{d}{dt}\{e^{-\alpha t} u(t)\} \xleftrightarrow{FT} \frac{j\omega}{\alpha + j\omega}$

$$\frac{d}{dt}\{e^{-\alpha t} u(t)\} = -\alpha e^{-\alpha t} u(t) + e^{-\alpha t}\delta(t) = -\alpha e^{-\alpha t} u(t) + \delta(t)$$

$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega} X(j\omega)$

$te^{-\alpha t} u(t) \xleftrightarrow{FT} j\frac{d}{d\omega}\left\{\frac{1}{\alpha + j\omega}\right\} = \frac{1}{(\alpha + j\omega)^2}$

Integration Properties

$y(t) = \int_{-\infty}^{t} x(t) dt$

$\frac{d}{dt} y(t) = x(t)$

$x(t) \xleftrightarrow{FT} X(j\omega)$

$y(t) \xleftrightarrow{FT} Y(j\omega)$

$\Leftrightarrow j\omega Y(j\omega) = X(j\omega)$

$Y(j\omega) = \frac{X(j\omega)}{j\omega} \qquad \omega \neq 0$

c can be determined by the average value of $x(t)$

$\int_{-\infty}^{t} x(t) dt \xleftrightarrow{FT} \frac{X(j\omega)}{j\omega} + \pi X(j0)\delta(\omega)$

$X(j0) = \int_{-\infty}^{\infty} x(t) e^{-j0t} dt = \int_{-\infty}^{\infty} x(t) dt = 0$

$u(t) = \int_{-\infty}^{t} \delta(t) dt \xleftrightarrow{FT} \frac{1}{j\omega} + \pi\delta(\omega)$

Differentiation and Integration Properties

Table 3.6 summarizes the differentiation and integration properties of Fourier representations.

$x(t) \xleftrightarrow{FT} X(j\omega)$

$\frac{d}{dt} x(t)$	$\xleftrightarrow{FT}$	$j\omega X(j\omega)$
$\frac{d}{dt} x(t)$	$\xleftrightarrow{FS;\ \omega_0}$	$jk\omega_0 X[k]$
$-jtx(t)$	$\xleftrightarrow{FT}$	$\frac{d}{d\omega} X(j\omega)$
$-jnx[n]$	$\xleftrightarrow{DTFT}$	$\frac{d}{d\Omega} X(e^{j\Omega})$
$\int_{-\infty}^{t} x(\tau) d\tau$	$\xleftrightarrow{FT}$	$\frac{1}{j\omega} X(j\omega) + \pi X(j0)\delta(\omega)$

Time-Shift Properties

if $x(t) \xleftrightarrow{FT} X(j\omega)$

then $x(t - t_0) \xleftrightarrow{FT} X(j\omega) \cdot e^{-j\omega t_0}$

$$F[x(t - t_0)] = \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(t - t_0) e^{-j\omega(t - t_0)} dt$$

$$= X(j\omega) \cdot e^{-j\omega t_0} = |X(j\omega)| \cdot e^{j[\arg\{X(j\omega)\} - \omega t_0]}$$

A shift in time leaves the magnitude spectrum unchanged and introduces a phase shift that is a linear function of frequency.

Time-Shift Properties

- **The time-shifting properties of four Fourier representation are summarized in Table 3.7.**

$x(t - t_0)$	$\xleftrightarrow{FT}$	$e^{-j\omega t_0} X(j\omega)$
$x(t - t_0)$	$\xleftrightarrow{FT;\ \omega_0}$	$e^{-jk\omega_0 t_0} X(k)$
$x[n - n_0]$	$\xleftrightarrow{DTFT}$	$e^{-j\Omega n_0} X(e^{j\Omega})$
$x[n - n_0]$	$\xleftrightarrow{DTFS;\Omega_0}$	$e^{-jk\Omega_0 n_0} X[k]$

Frequency-Shift Properties

$$z(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}Z(j\omega)e^{j\omega t}d\omega \qquad x(t)\xleftrightarrow{FT}X(j\omega)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j(\omega-\gamma))e^{j\omega t}d\omega \qquad z(t)\xleftrightarrow{FT}Z(j\omega)$$

$Z(j\omega) = X(j(\omega - \gamma))$

$\eta = \omega - \gamma$

$$z(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\eta)e^{j(\eta+\gamma)t}d\eta \qquad e^{j\gamma t}x(t)\xleftrightarrow{FT}X(j(\omega-\gamma))$$

$$=e^{j\gamma t}\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\eta)e^{j\eta t}d\eta = e^{j\gamma t}x(t)$$

◆ **Frequency-shifting of $X(j\omega)$ by γ in Frequency-Domain [i.e. $X(j(\omega-\gamma))$] $\leftrightarrow$ ($e^{-\gamma t}$) $\times$ $x(t)$ in Time-Domain**

Frequency-Shift Properties

Table 3.8 Frequency-Shift Properties of Fourier Representations

$$e^{j\gamma t}x(t) \xleftrightarrow{FT} X(j(\omega-\gamma))$$

$$e^{jk_0\omega_0 t}x(t) \xleftrightarrow{FS;\,\omega_0} X[k-k_0]$$

$$e^{j\Gamma n}x[n] \xleftrightarrow{DTFT} X\left[e^{j(\Omega-\Gamma)}\right]$$

$$e^{jk_0\Omega_0 n}x[n] \xleftrightarrow{FS;\,\Omega_0} X[k-k_0]$$

Example

$$1\xleftrightarrow{FT}2\pi\delta(\omega) \qquad e^{j\gamma t}x(t)\xleftrightarrow{FT}X(j(\omega-\gamma))$$

$$e^{j\omega_0 t}\xleftrightarrow{FT}2\pi\delta(\omega-\omega_0)$$

$$\cos(\omega_0 t)=\frac{1}{2}\left(e^{j\omega_0 t}+e^{-j\omega_0 t}\right)\xleftrightarrow{FT}\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$$

$$\sin(\omega_0 t)=\frac{1}{2j}\left(e^{j\omega_0 t}-e^{-j\omega_0 t}\right)\xleftrightarrow{FT}-j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$$

Finding Inverse Fourier Transforms by Using Partial-Fraction Expansions

Frequency response ⟷ **A ratio of two polynomials in $j\omega$ or $e^{j\Omega}$**

$$X(j\omega)=\frac{b_M(j\omega)^M+\cdots+b_1(j\omega)+b_0}{(j\omega)^N+a_{N-1}(j\omega)^{N-1}+\cdots a_1(j\omega)+a_0}=\frac{B(j\omega)}{A(j\omega)}$$

$$=c_0+c_1(j\omega)+c_2(j\omega)^2+\cdots+c_{M-N}(j\omega)^{M-N}+\frac{B(j\omega)}{A(j\omega)}$$

$$\delta(t)\xleftrightarrow{FT}1$$

$$x(t)=c_0\delta(t)+c_1\delta'(t)+c_2\delta''(t)+\cdots+c_{M-N}\delta^{(M-N)}(t)+F^{-1}\left[\frac{B(j\omega)}{A(j\omega)}\right]$$

$$v^N+a_{N-1}v^{N-1}+\cdots+a_1v+a_0=0$$

Finding Inverse Fourier Transforms by Using Partial-Fraction Expansions

$$v^N+a_{N-1}v^{N-1}+\cdots+a_1v+a_0=0 \qquad d_k<0$$

$$X(j\omega)=\frac{\sum_{k=0}^{M}b_k(j\omega)^k}{\prod_{k=1}^{N}(j\omega-d_k)}=\sum_{k=1}^{N}\frac{C_k}{j\omega-d_k} \qquad e^{d_k t}u(t)\xleftrightarrow{FT}\frac{1}{j\omega-d_k}$$

$$=\frac{C_1}{j\omega-d_1}+\frac{C_2}{j\omega-d_2}+\cdots+\frac{C_N}{j\omega-d_N}$$

$$x(t)=\sum_{k=1}^{N}C_ke^{d_k t}u(t)\xleftrightarrow{FT}X(j\omega)=\sum_{k=1}^{N}\frac{C_k}{j\omega-d_k}$$

Example

$$X(j\omega)=\frac{j\omega+1}{(j\omega)^2+5j\omega+6}$$

$$=\frac{A}{2+j\omega}+\frac{B}{3+j\omega}$$

$$=\frac{A(3+j\omega)+B(2+j\omega)}{(2+j\omega)(3+j\omega)}$$

$$=\frac{3A+2B+j\omega(A+B)}{(2+j\omega)(3+j\omega)}$$

$$\begin{cases}3A+2B=1\\A+B=1\end{cases}\Rightarrow\begin{cases}A=-1\\B=2\end{cases}$$

$$=\frac{-1}{2+j\omega}+\frac{2}{3+j\omega}$$

$$e^{-\alpha t}u(t)\Leftrightarrow\frac{1}{\alpha+j\omega}$$

$$x(t)=-e^{-2t}u(t)+2e^{-3t}u(t)\Leftrightarrow X(j\omega)$$

Inverse Discrete-Time Fourier Transform

1. Suppose $X(e^{j\Omega})$ is given by a ratio of polynomial in $e^{j\Omega}$

$$X\left(e^{j\Omega}\right)=\frac{\beta_M e^{-j\Omega M}+\cdots+\beta_1 e^{-j\Omega}+\beta_0}{\alpha_N e^{-j\Omega N}+\alpha_{N-1}e^{-j\Omega(N-1)}+\cdots+\alpha_1 e^{-j\Omega}+1}$$

Normalized to 1

2. Factor the denominator polynomial as

$$\alpha_N e^{-j\Omega N}+\alpha_{N-1}e^{-j\Omega(N-1)}+\cdots+\alpha_1 e^{-j\Omega}+1=\prod_{k=1}^{N}\left(1-d_k e^{-j\Omega}\right)$$

$$v^N+\alpha_1 v^{N-1}+\alpha_2 v^{N-2}+\cdots+\alpha_{N-1}v+\alpha_N=0$$

3. Partial-fraction expansion: Assuming that $M < N$ and all the d_k are distinct, we may express $X(e^{j\Omega})$ as

$$\left(d_k\right)^n u[n] \xleftrightarrow{DTFT} \frac{1}{1-d_k e^{-j\Omega}},\ \left|d_k\right|<1$$

$$X\left(e^{j\Omega}\right)=\sum_{k=1}^{N}\frac{C_k}{1-d_k e^{-j\Omega}} \qquad x[n]=\sum_{k=1}^{N}C_k\left(d_k\right)^n u[n]$$

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Example 3.45 *Inverse by Partial-Fraction Expansion*

Find the inverse DTFT of

$$X\left(e^{j\Omega}\right)=\frac{-\frac{5}{6}e^{-j\Omega}+5}{1+\frac{1}{6}e^{-j\Omega}-\frac{1}{6}e^{-j2\Omega}}=\frac{C_1}{1+\frac{1}{2}e^{-j\Omega}}+\frac{C_2}{1-\frac{1}{3}e^{-j\Omega}}$$

1. Characteristic polynomial:

$$v^2+\frac{1}{6}v-\frac{1}{6}=(v+\frac{1}{2})(v-\frac{1}{3})=0$$

2. The roots of above polynomial: $d_1 = -1/2$ and $d_2 = 1/3$.
3. Partial-Fraction Expansion
4. Coefficients C_1 and C_2

$$x[n]=4(-1/2)^n u[n]+(1/3)^n u[n]$$

$$C_1=\left(1+\frac{1}{2}e^{-j\Omega}\right)\frac{-\frac{5}{6}e^{-j\Omega}+5}{1+\frac{1}{6}e^{-j\Omega}-\frac{1}{6}e^{-j2\Omega}}\Bigg|_{e^{-j\Omega}=-2}=\frac{-\frac{5}{6}e^{-j\Omega}+5}{1-\frac{1}{3}e^{-j\Omega}}\Bigg|_{e^{-j\Omega}=-2}=4$$

$$C_2=\left(1-\frac{1}{3}e^{-j\Omega}\right)\frac{-\frac{5}{6}e^{-j\Omega}+5}{1+\frac{1}{6}e^{-j\Omega}-\frac{1}{6}e^{-j2\Omega}}\Bigg|_{e^{-j\Omega}=3}=\frac{-\frac{5}{6}e^{-j\Omega}+5}{1+\frac{1}{2}e^{-j\Omega}}\Bigg|_{e^{-j\Omega}=3}=1$$

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Multiplication Property: Non-periodic continuous-time signals

Non-periodic signals: $x(t)$, $z(t)$, and $y(t) = x(t)z(t)$.

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(jv)e^{jvt}dv \qquad z(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}Z(j\eta)e^{j\eta t}d\eta$$

$$y(t)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}X(jv)Z(j\eta)e^{j(\eta+v)t}d\eta dv$$

Change variable: $\eta = \omega - v$

Inner Part: $Z(j\omega) * X(j\omega)$

$$y(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X(jv)Z(j(\omega-v))dv\right]e^{j\omega t}d\omega$$

Outer Part: FT of $y(t)$

$$y(t)=x(t)z(t)\xleftrightarrow{FT}Y(j\omega)=\frac{1}{2\pi}X(j\omega)*Z(j\omega)$$

◆ Multiplication of two signals in Time-Domain ↔ Convolution in Frequency-Domain × $(1/2\pi)$

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Multiplication Property: Non-periodic discrete-time signals

1. Non-periodic DT signals: $x[n]$, $z[n]$, and $y[n] = x[n]z[n]$.
2. DTFT of $y[n]$:

$$y[n]=x[n]z[n] \xleftrightarrow{DTFT} Y(e^{j\Omega})=\frac{1}{2\pi}X(e^{j\Omega})\circledast Z(e^{j\Omega})$$

where the symbol $\circledast$ denotes periodic convolution.

Here, $X(e^{j\Omega})$ and $X(e^{j\Omega})$ are 2π-periodic, so we evaluate the convolution over a 2π interval:

$$X(j\omega)\circledast Z(j\omega)=\int_{-\pi}^{\pi}X\left(e^{j\theta}\right)Z\left(e^{j(\Omega-\theta)}\right)d\theta$$

◆ Multiplication of two signals in Time-Domain ↔ Convolution in Frequency-Domain × $(1/2\pi)$

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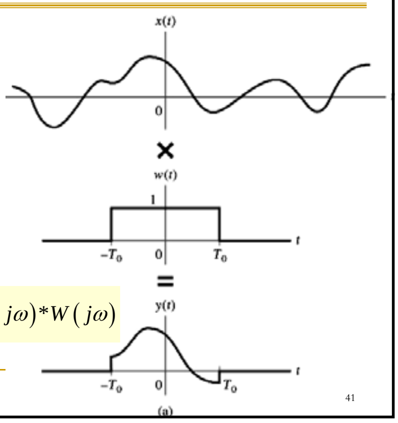
Multiplication Property : windowing

Truncate signal $x(t)$ by a window function $w(t)$ is represented by

$y(t) = x(t)w(t)$

$$W(j\omega)=\frac{2}{\omega}\sin(\omega T_0)$$

$$y(t)\xleftrightarrow{FT}Y(j\omega)=\frac{1}{2\pi}X(j\omega)*W(j\omega)$$



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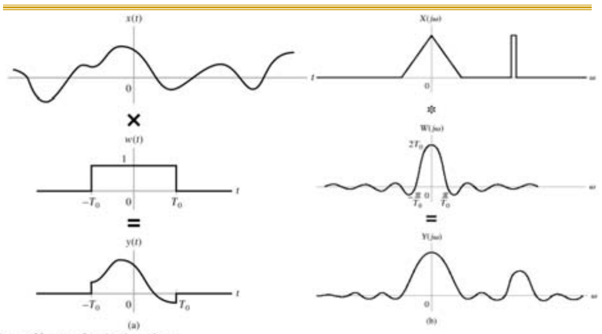
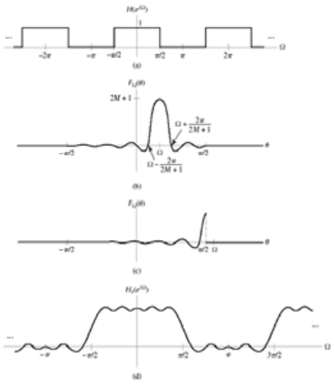
Multiplication Property



The effect of windowing.
(a) Truncating a signal in time by using a window function $w(t)$.

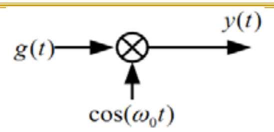
(b) Convolution of the signal and window FT's resulting from truncation in time.

$$x(n)z(n) \xleftrightarrow{DTFT} \frac{1}{2\pi}\left[X(e^{j\Omega}) \circledast Z(e^{j\Omega})\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Z(e^{j(\Omega-\theta)})d\theta$$



The effect of truncating the impulse response of a discrete-time system. (a) Frequency response of ideal system. (b) $F_{\Omega}(\theta)$ for Ω near zero. (c) $F_{\Omega}(\theta)$ for Ω slightly greater than $\pi/2$. (d) Frequency response of system with truncated impulse response.

Multiplication Property: Amplitude modulation



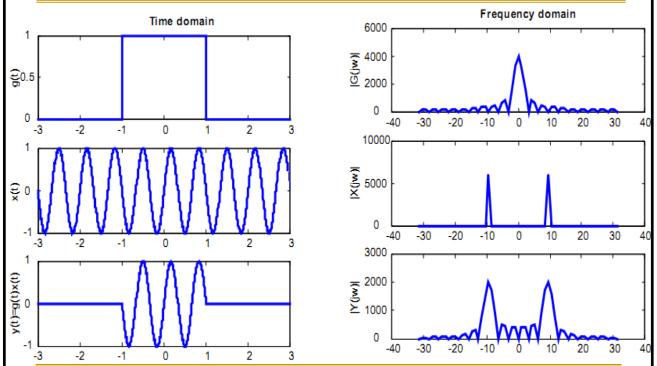
$$x(t) = \cos(\omega_0 t),\quad g(t) = \begin{cases} 1, & |t| \le 1 \\ 0, & |t| > 1 \end{cases},\quad \text{Find } Y(j\omega)$$

$$y(t) = g(t)\cos(\omega_0 t)$$

$$Y(j\omega) = \frac{1}{2\pi}\left[G(j\omega) * \pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]\right]$$

$$= \frac{1}{2}G[j(\omega+\omega_0)] + \frac{1}{2}G[j(\omega-\omega_0)]$$

Multiplication Property: Amplitude modulation



Multiplication Properties of Fourier Representations

Table 3.9 Multiplication Properties of Fourier Representations

$x(t)z(t)$	$\xleftrightarrow{FT}$	$\frac{1}{2\pi}X(j\omega) * Z(j\omega)$
$x(t)z(t)$	$\xleftrightarrow{FS;\ \omega_o}$	$X[k] * Z[k]$
$x[n]z[n]$	$\xleftrightarrow{DTFT}$	$\frac{1}{2\pi}X(e^{j\Omega}) \circledast Z(e^{j\Omega})$
$x[n]z[n]$	$\xleftrightarrow{DTFT;\ \Omega_o}$	$X[k] \circledast Z[k]$

Scaling Property

***z*(*t*) = *x*(*at*)**

$$Z(j\omega) = \int_{-\infty}^{\infty} z(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt$$

$$Z(j\omega) = \begin{cases} (1/a)\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a > 0 \\ (1/a)\int_{\infty}^{-\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a < 0 \end{cases}$$

Changing variable: $\tau = at$

$$Z(j\omega) = (1/|a|)\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau$$

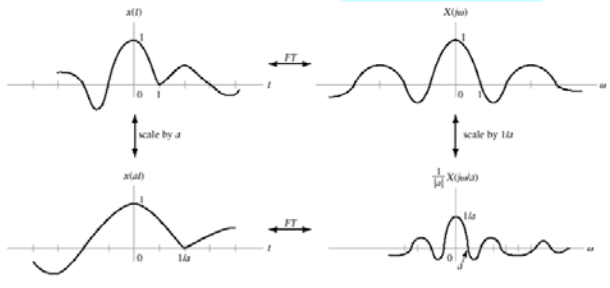
$$z(t) = x(at) \xleftrightarrow{FT} (1/|a|)X(j\omega/a).$$

◆ **Scaling in Time-Domain ↔ Inverse Scaling in Frequency-Domain**

➡ **Signal expansion or compression!**

Scaling Property

$x(t) \xleftrightarrow{FT} X(j\omega)$ $\quad$ $x(at) \xleftrightarrow{FT} \frac{1}{|a|}X(j\frac{\omega}{a})$



The FT scaling property. The figure assumes that 0 < *a* < 1.

Scaling Property: a Rectangular Pulse

$$z(t)=x(at)\xleftrightarrow{FT}(\frac{1}{|a|})X(\frac{j\omega}{a})$$

$$x(t)=\begin{cases}1, & |t|<1\\ 0, & |t|>1\end{cases}$$

$$\Leftrightarrow X(j\omega)=\frac{2}{\omega}\sin(\omega)$$

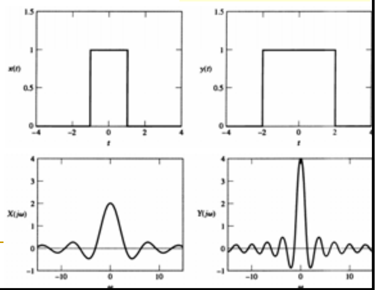
$$y(t)=\begin{cases}1, & |t|<2\\ 0, & |t|>2\end{cases}$$

$$y(t)=x(t/2)$$

$$Y(j\omega)=2X(j2\omega)$$

$$=2\left(\frac{2}{2\omega}\right)\sin(2\omega)$$

$$=\frac{2}{\omega}\sin(2\omega)$$



Parseval Relationships

♣ The energy or power in the time-domain representation of a signal is equal to the energy or power in the frequency-domain representation.

FT: $$W_x=\int_{-\infty}^{\infty}|x(t)|^2\,\mathrm{d}t=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\mathrm{j}\omega)|^2\,\mathrm{d}\omega$$

DTFT: $$W_x=\sum_{n=-\infty}^{\infty}|x(n)|^2=\frac{1}{2\pi}\int_{<2\pi>}|X(e^{\mathrm{j}\Omega})|^2\,\mathrm{d}\Omega$$

Parseval Relationships for CT nonperiodic signal

Energy in $x(t)$

$$|x(t)|^2=x(t)x^*(t)$$

$$x^*(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(j\omega)e^{-j\omega t}d\omega$$

$$W_x=\int_{-\infty}^{\infty}|x(t)|^2dt=\int_{-\infty}^{\infty}x(t)x^*(t)dt$$

$$W_x=\int_{-\infty}^{\infty}x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(j\omega)e^{-j\omega t}d\omega\right]dt$$

$$W_x=\frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(j\omega)\left\{\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt\right\}d\omega$$

$$W_x=\frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(j\omega)X(j\omega)d\omega$$

$$\int_{-\infty}^{\infty}|x(t)|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2\,d\omega$$

◆ Energy in Time-Domain Representation ↔ Energy in Frequency-Domain Representation × (1/2π)

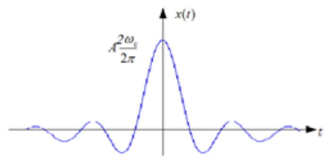
Parseval Relationships

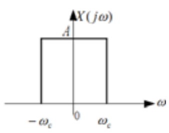
◆ The Parseval Relationships of all four Fourier representations are summarized in Table 3.10.

Representations	Parseval Relationships
FT	$\int_{-\infty}^{\infty}\|x(t)\|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\|X(j\omega)\|^2d\omega$
FS	$\frac{1}{T}\int_0^T\|x(t)\|^2\,dt=\sum_{k=-\infty}^{\infty}\|X[k]\|^2$
DTFT	$\sum_{n=-\infty}^{\infty}\|x[n]\|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\|X(e^{j\Omega})\|^2\,d\Omega$
DTFS	$\frac{1}{N}\sum_{n=0}^{N-1}\|x[n]\|^2=\sum_{k=0}^{N-1}\|X[k]\|^2$

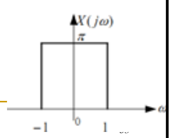
Example: Evaluate $\int_{-\infty}^{\infty}(\frac{\sin t}{t})^2\,\mathrm{d}t$

$$x(t)=\frac{\sin t}{t}=s_a(t)=\frac{A2\omega_c}{2\pi}s_a(\omega_c t)\qquad \begin{cases}A=\pi\\ \omega_c=1\end{cases}$$





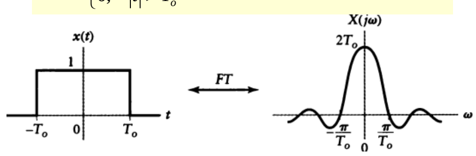
$$\int_{-\infty}^{\infty}(\frac{\sin t}{t})^2\,\mathrm{d}t=\frac{1}{2\pi}\int_{-1}^{1}\pi^2\mathrm{d}\omega=\pi$$



Time-Bandwidth Product

$$x(t)=\begin{cases}1, & |t|\le T_o\\ 0, & |t|>T_o\end{cases}\xleftrightarrow{FT}X(j\omega)=2\sin(\omega T_o)/\omega$$

$$2T_0(\pi/T_0)=2\pi$$



Compressing a signal in time leads to expansion in frequency and vice versa.

1. Effective duration of a signal $x(t)$

$$T_d=\left[\frac{\int_{-\infty}^{\infty}t^2|x(t)|^2\,dt}{\int_{-\infty}^{\infty}|x(t)|^2\,dt}\right]^{1/2}$$

2. Effective bandwidth of a signal $x(t)$

$$B_w=\left[\frac{\int_{-\infty}^{\infty}\omega^2|X(j\omega)|^2\,d\omega}{\int_{-\infty}^{\infty}|X(j\omega)|^2\,d\omega}\right]^{1/2}$$

The time-bandwidth product for any signal is lower bounded according to the relationship $T_dB_w\ge 1/2$

Time-Bandwidth Product

Example 3.51 ***Bounding the Bandwidth of a Rectangular Pulse***

$$x(t)=\begin{cases}1, & |t|\le T_o\\ 0, & |t|>T_o\end{cases}$$

Use the uncertainty principle to place a lower bound on the effective bandwidth of *x*(*t*).

<Sol.>

1. T_d of *x*(*t*):

$$T_d=\left[\frac{\int_{-T_o}^{T_o}t^2dt}{\int_{-T_o}^{T_o}dt}\right]^{1/2} \Rightarrow T_d=\left[\frac{\int_{-T_o}^{T_o}t^2dt}{\int_{-T_o}^{T_o}dt}\right]^{1/2}=[(1/(2T_o))(1/3)t^3|_{-T_o}^{T_o}]^{1/2}=T_d/\sqrt{3}$$

2. According to the uncertainty principle

$$B_w\ge 1/(2T_d) \Rightarrow B_w\ge\sqrt{3}/(2T_o)$$

Duality Property of FT

Difference in the factor 2π and the sign change in the complex sinusoid.

1. FT pair:

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega \qquad X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

2. General equation:

$$y(v)=\frac{1}{2\pi}\int_{-\infty}^{\infty}z(\eta)e^{jv\eta}dv$$

Choose $v=t$ and $\eta=\omega$

$$y(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}z(\omega)e^{j\omega t}d\omega \Rightarrow y(t)\xleftrightarrow{FT}z(\omega)$$

Let $v=-\omega$ and $\eta=t$ **Interchange the role of time and frequency**

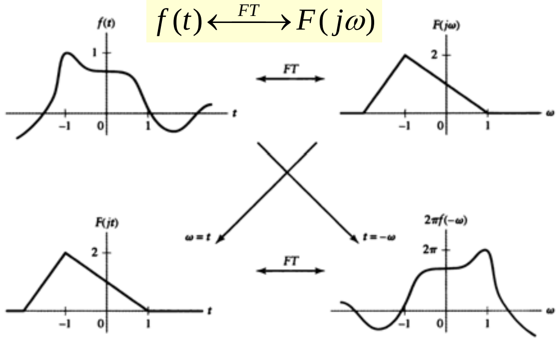
$$y(-\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}z(t)e^{-j\omega t}dt \Rightarrow z(t)\xleftrightarrow{FT}2\pi y(-\omega)$$

3. A new FT pair

$$y(t)\xleftrightarrow{FT}z(j\omega) \Rightarrow z(jt)\xleftrightarrow{FT}2\pi y(-\omega)$$

Duality Property of FT

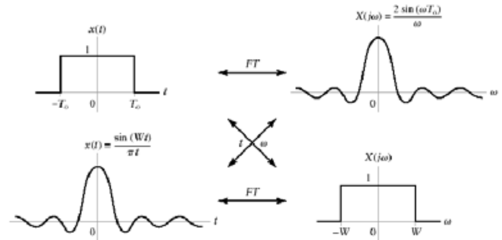
$$f(t)\xleftrightarrow{FT}F(j\omega)$$



$$F(jt)\xleftrightarrow{FT}2\pi f(-\omega)$$

Duality Property of Rectangular pulses and sinc functions

$$x(t)\xleftrightarrow{FT}X(j\omega) \qquad X(jt)\xleftrightarrow{FT}2\pi x(-\omega)$$



FT	$f(t)$	$\xleftrightarrow{FT}$	$F(j\omega)$	$F(jt)$	$\xleftrightarrow{FT}$	$2\pi f(-\omega)$
DTFS	$x[n]$	$\xleftrightarrow{DTFS;2\pi/N}$	$X[k]$	$X[n]$	$\xleftrightarrow{DTFS;2\pi/N}$	$(1/N)x[-k]$
FS-DTFT	$x[n]$	$\xleftrightarrow{DTFS}$	$X(e^{j\Omega})$	$X(e^{jt})$	$\xleftrightarrow{FS;1}$	$x[-k]$

Summary and Exercises

- Summary and Exercises
  - Complex Sinusoids and Frequency Response of LTI Systems
  - Fourier Representations for Four classes of Signals
  - Properties of Fourier Representations
- Exercises (**P322-333**)
  - **3.66(a-d), 3.67(a-e), 3.68(a), 3.69(b), 3.73(a, c), 3.74(a, c), 3.76(a, b)**

FT pairs

$$\delta[n]\xleftrightarrow{DTFT}1 \qquad 1\xleftrightarrow{FT}2\pi\delta(\omega) \qquad 1\xleftrightarrow{DTFT}2\pi\delta(\Omega)\ -\pi<\Omega\le\pi$$

$$\delta(t)\xleftrightarrow{FT}1 \qquad e^{j\omega_0 t}\xleftrightarrow{FT}2\pi\delta(\omega-\omega_0)$$

$$\alpha^n u[n]\xleftrightarrow{DTFT}\frac{1}{1-\alpha e^{-j\Omega}} \quad |\alpha|<1$$

$$\cos(\omega_0 t)\xleftrightarrow{FT}\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$$

$$\sin(\omega_0 t)\xleftrightarrow{FT}-j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$$

$$\alpha>0 \quad e^{-\alpha t}u(t)\xleftrightarrow{FT}\frac{1}{\alpha+j\omega}$$

$$u(t)\xleftrightarrow{FT}\frac{1}{j\omega}+\pi\delta(\omega)$$

$$sgn(t)\xleftrightarrow{FT}\frac{2}{j\omega}$$

$$te^{-\alpha t}u(t)\xleftrightarrow{FT}\frac{1}{(\alpha+j\omega)^2}$$

$$A\,rect\left(\frac{t}{\tau}\right)\xleftrightarrow{FT}A\tau\,\sin c\left(\frac{\omega\tau}{2}\right)$$

幅度A，宽度τ，关于y轴对称的矩形脉冲

任意周期信号 $f_0=1/T$

$$\sum_{n=-\infty}^{\infty}F_n e^{j2\pi nf_0 t}\xleftrightarrow{FT}\sum_{n=-\infty}^{\infty}F_n\delta(f-nf_0)$$

$$\delta_T(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)\xleftrightarrow{FT}f_0\sum_{n=-\infty}^{\infty}\delta(f-nf_0)=\sum_{m=-\infty}^{\infty}e^{-j2\pi fnT}$$