Signals and Systems 1.3

---*Systems Classification and Properties*

School of Information & Communication Engineering，BUPT

Reference:

1. Textbook: 1. 7,1.8

2. Schaum's outline of signals and systems, Hwei P. Hsu, McGraw-Hill, 1995. Section:1.5

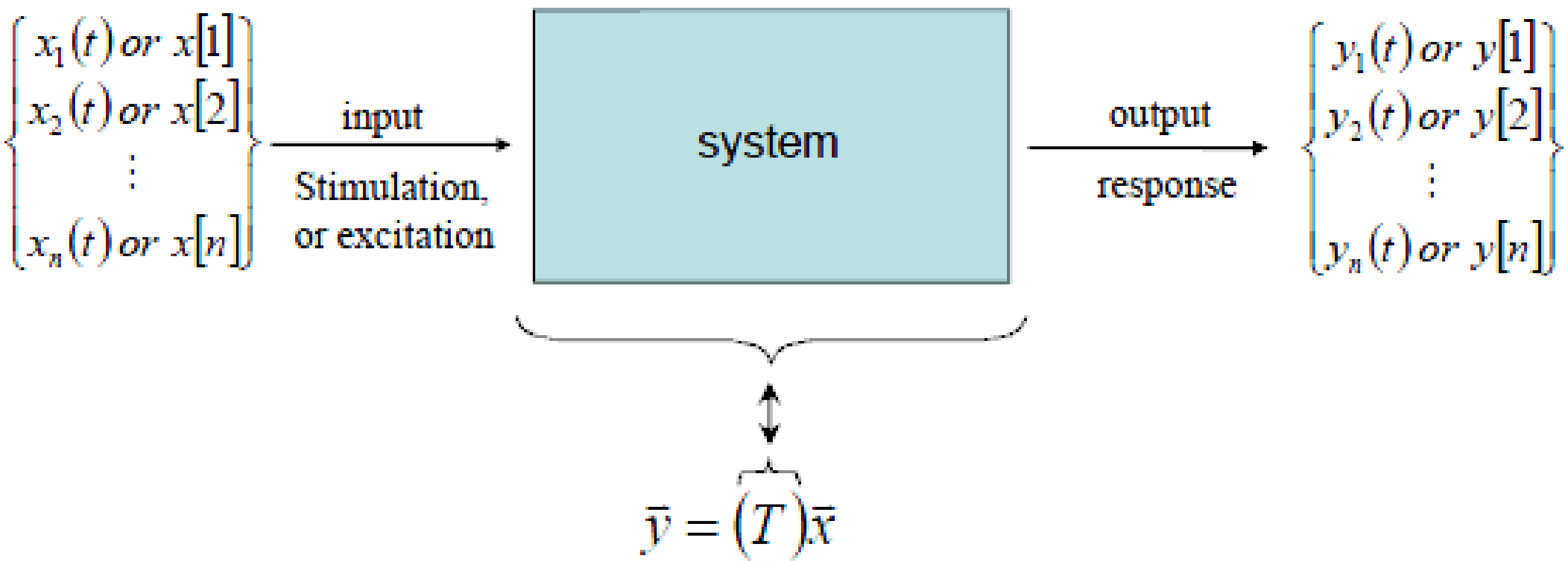
Outline of Today's Lecture

- Systems Classification and Properties
  - Continuous-time and Discrete-time Systems
  - Systems with and without memory
  - Causal and Non-causal Systems
  - Linear and Nonlinear Systems
  - Time-variant and Time-invariant Systems
  - Linear Time-invariant Systems
  - Stable Systems
  - Feedback Systems
  - Invertibility

System Representation

- A system is a mathematical model of a physical process that relates the input (or excitation) signal to the output (or response) signal.

Let x and y be the input and output signals, respectively, of a system. Then the system is viewed as a transformation (or mapping) of **x** into y.

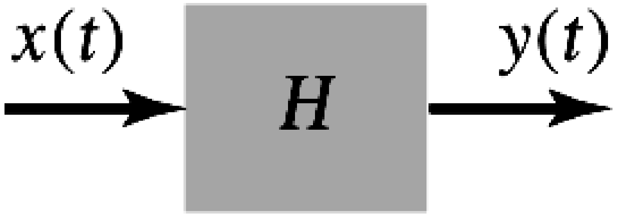


Continuous-time and Discrete-time Systems

If the input and output signals x and y are continuous/discrete-time signals, then the system is called a ***continuous/discrete-time system.***

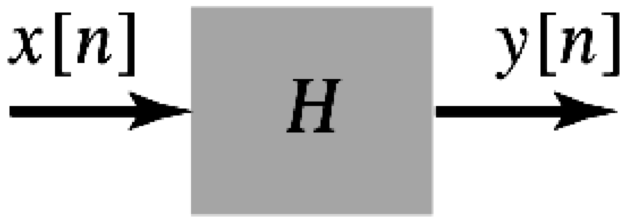
1. Continuous-time case

2. Discrete-time case



(a)

$$y(t) = H\{x(t)\}$$



(b)

$$y[n] = H\{x[n]\}$$

Example 1.12 Moving-average system

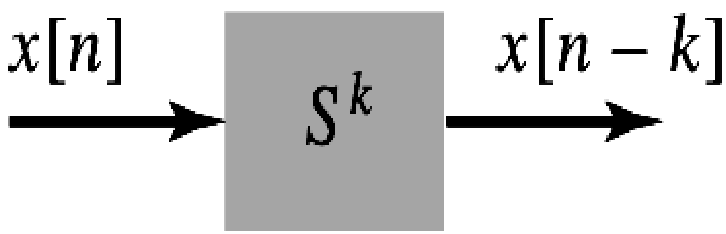
Consider a discrete-time system whose output signal $y[n]$ is the average of the three most recent values of the input signal $x[n]$, that is

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

Formulate the operator H for this system; hence, develop a block diagram representation for it.

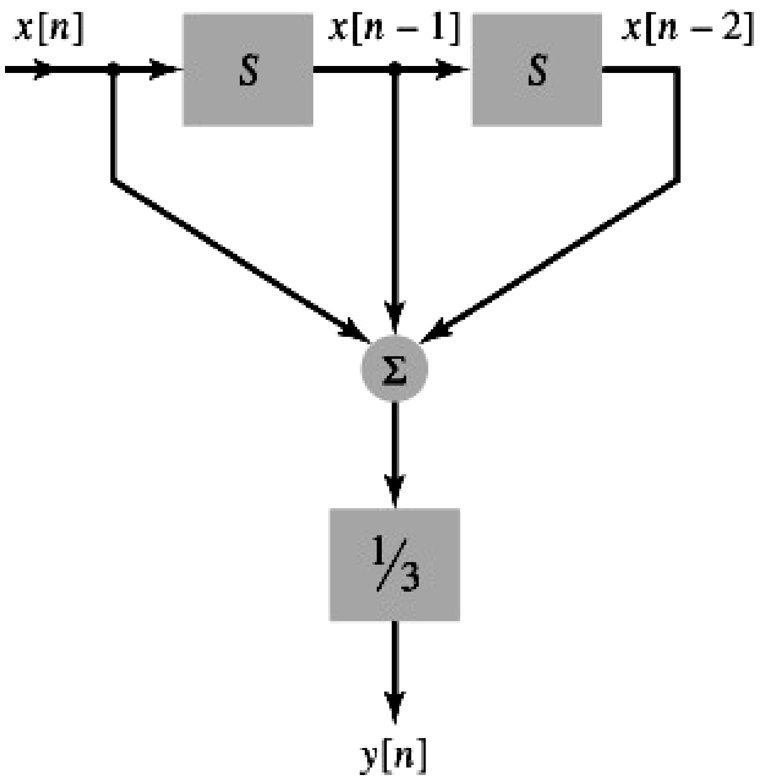
<Sol.> 1. Discrete-time-shift operator S^k.

Shifts the input $x[n]$ by k time units to produce an output equal to $x[n-k]$.

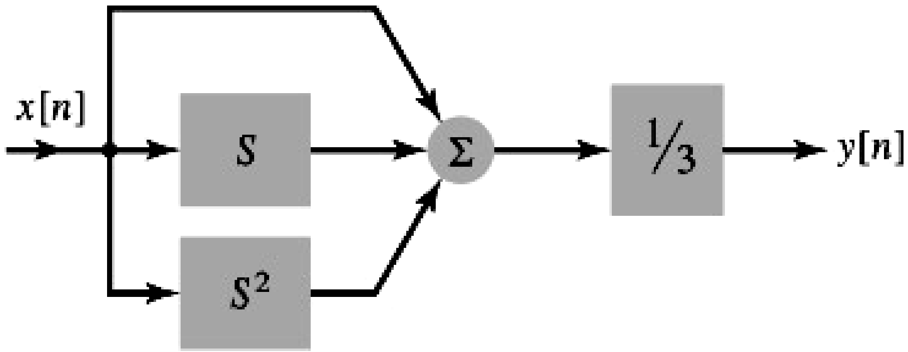


Example 1.12 Moving-average system

2. Overall operator *H* for the moving-average system



$$H = \frac{1}{3}(1 + S + S^2)$$



parallel form of implementation

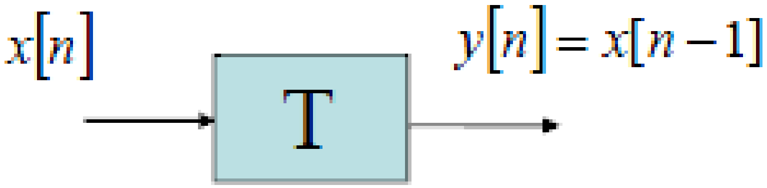
cascade form of implementation

▶ **Problem 1.25** Express the operator that describes the input–output relation

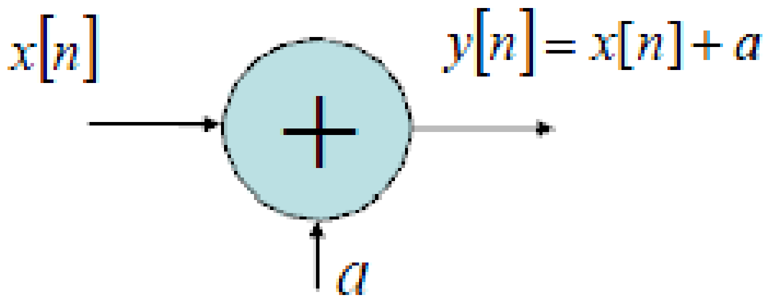
$$y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$$

in terms of the time-shift operator S. ***Answer:*** $H = \frac{1}{3}(S^{-1} + 1 + S^{1})$

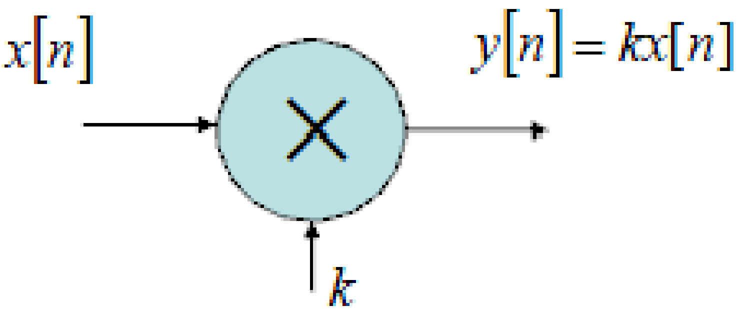
Representation of discrete-time operations



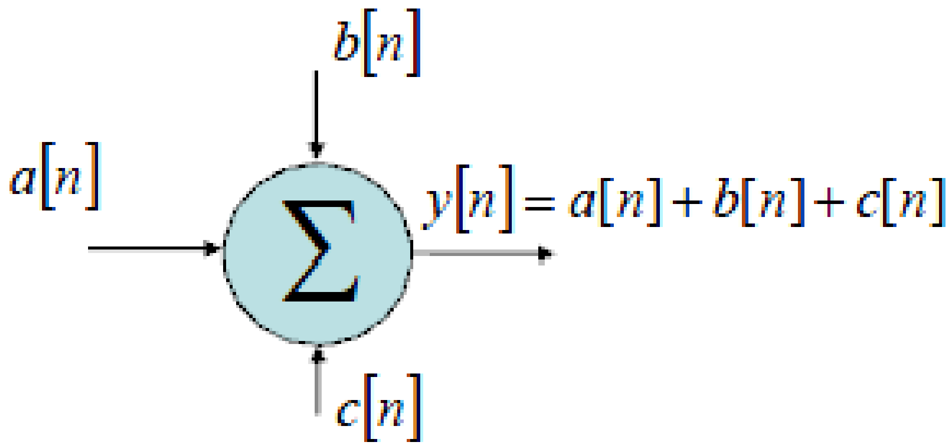
The delay block



The addition of a constant



The scaling of a constant



The summation of sequences

Systems with and without memory

A system is said to be memoryless if its output signal at any time depends only on the present values of the input signal (at that same time). Otherwise, the system is said to have memory. That is, its output signal depends on past or future values of the input signal.

Ex.: Resistor $$i(t) = \frac{1}{R} v(t)$$ ⟹ ***Memoryless !***

Ex.: Inductor $$i(t) = \frac{1}{L} \int_{-\infty}^{t} v(\tau) d\tau$$ ⟹ ***Memory !***

Ex.: Moving-average system

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$ ⟹ ***Memory !***

Ex.: A system described by the input-output relation

$$y[n] = x^2[n]$$ ⟹ ***Memoryless !***

Example: Memory

▶ **Problem 1.27** How far does the memory of the moving-average system described by the input–output relation

$$y[n] = \frac{1}{3}(x[n] + x[n-2] + x[n-4])$$

extend into the past?

Answer: Four time units.

▶ **Problem 1.28** The input–output relation of a semiconductor diode is represented by

$$i(t) = a_0 + a_1 v(t) + a_2 v^2(t) + a_3 v^3(t) + \cdots,$$

where $v(t)$ is the applied voltage, $i(t)$ is the current flowing through the diode, and a_0, $a_1, a_3, \ldots$ are constants. Does this diode have memory?

Answer: No.

▶ **Problem 1.29** The input–output relation of a capacitor is described by

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau.$$

What is the extent of the capacitor's memory?

Answer: The capacitor's memory extends from time t back to the infinite past.

Causal and Non-causal Systems

A system is said to be *causal* if its present value of the output signal depends only on the present or past values of the input signal.

A system is said to be *noncausal* if its output signal depends on one or more future values of the input signal.

Ex.: Moving-average system

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2]) \quad \Rightarrow \quad \textit{Causal !}$$

Ex.: Moving-average system

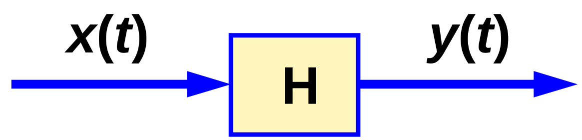
$$y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1]) \quad \Rightarrow \quad \textit{Noncausal !}$$

♣ **Causality is required for a systems to be capable of operating in *real time*.**

Linear and Nonlinear Systems

- If a system is linear, it has to satisfy the following two conditions.

1. The Addition Rule(叠加性): If given



$$x(t)=x_1(t) \Rightarrow y(t)=H\{x_1(t)\}=y_1(t)$$

$$x(t)=x_2(t) \Rightarrow y(t)=H\{x_2(t)\}=y_2(t)$$

$$x(t)=x_1(t)+x_2(t) \Rightarrow y(t)=H\{x_1(t)+x_2(t)\} = y_1(t)+y_2(t)$$

2. Homogeneity (or Scaling)(倍增性或者比例性):

$$x(t)=\alpha x_1(t) \Rightarrow y(t)=H\{\alpha x_1(t)\} = \alpha y_1(t)$$

※ Conditions 1. and 2. may be combined into the single condition

$$H\{\alpha_1 x_1(t)+\alpha_2 x_2(t)\} = \alpha_1 y_1(t)+\alpha_2 y_2(t)$$

Superposition property

- **Any system not satisfying these conditions is classified *nonlinear system.***

Linearity of continuous-time system

1. Operator *H* represent the continuous-tome system.

2. Input

$$x(t)=\sum_{i=1}^{N} a_i x_i(t)$$

$x_1(t)$, $x_2(t)$, …, $x_N(t)$ ≡ input signal;
a_1, a_2, …, a_N ≡ corresponding weighted factor

3. Output

$$y(t)=H\{x(t)\}=H\{\sum_{i=1}^{N} a_i x_i(t)\}$$

$$\Rightarrow\quad y(t)=\sum_{i=1}^{N} a_i y_i(t)$$

Superposition

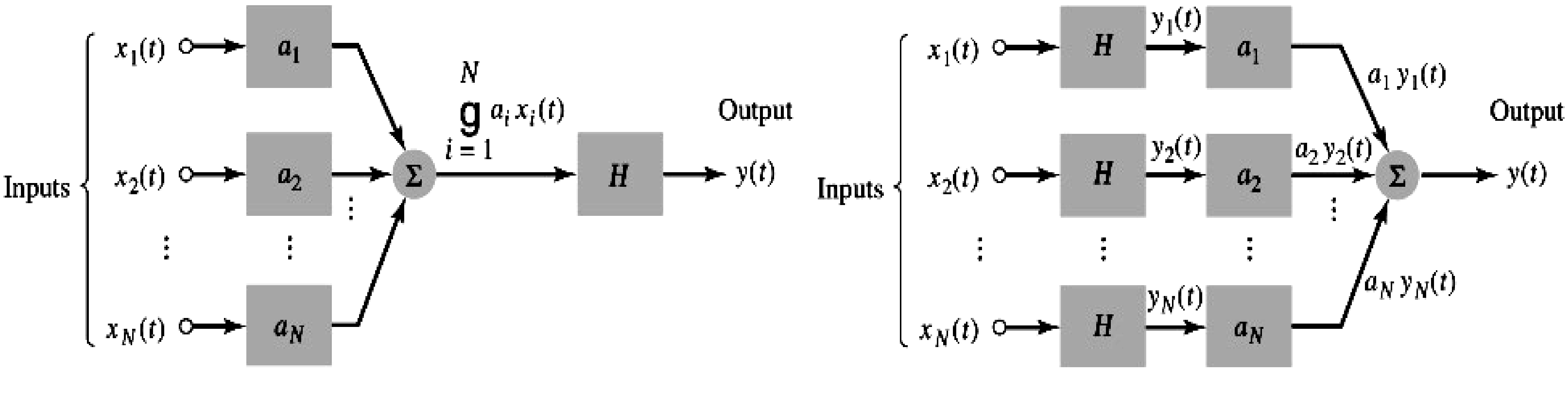
$$y_i(t)=H\{x_i(t)\},\quad i=1,\ 2,\ \ldots,\ N.$$

4. Commutation and Linearity

$$\begin{aligned} y(t) &= H\{\sum_{i=1}^{N} a_i x_i(t)\} \\ &= \sum_{i=1}^{N} a_i H\{x_i(t)\} \\ &= \sum_{i=1}^{N} a_i y_i(t) \end{aligned}$$

Linearity of continuous-time system

$$y(t) = H\{\sum_{i=1}^{N} a_i x_i(t)\} = \sum_{i=1}^{N} a_i H\{x_i(t)\} = \sum_{i=1}^{N} a_i y_i(t)$$



(a) (b)

Example 1.19 Linear Discrete-Time system

Consider a discrete-time system described by the input-output relation

$$y[n] = nx[n]$$

Show that this system is linear.

<p.f.>

1. Input:

$$x[n] = \sum_{i=1}^{N} a_i x_i[n]$$

2. Resulting output signal:

$$y[n] = n\sum_{i=1}^{N} a_i x_i[n] = \sum_{i=1}^{N} a_i n x_i[n] = \sum_{i=1}^{N} a_i y_i[n]$$

⇒ ***Linear system!***

$$y_i[n] = nx_i[n]$$

Example 1.20 Nonlinear Continuous-Time System

Consider a continuous-time system described by the input-output relation

$$y(t) = x(t)x(t-1)$$

Show that this system is nonlinear.

<p.f.>

1. Input:

$$x(t) = \sum_{i=1}^{N} a_i x_i(t)$$

2. Output:

$$y(t) = \sum_{i=1}^{N} a_i x_i(t) \sum_{j=1}^{N} a_j x_j(t-1) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j x_i(t) x_j(t-1)$$

Here we cannot write $y(t) = \sum_{i=1}^{N} a_i y_i(t)$ $\Rightarrow$ ***Nonlinear system!***

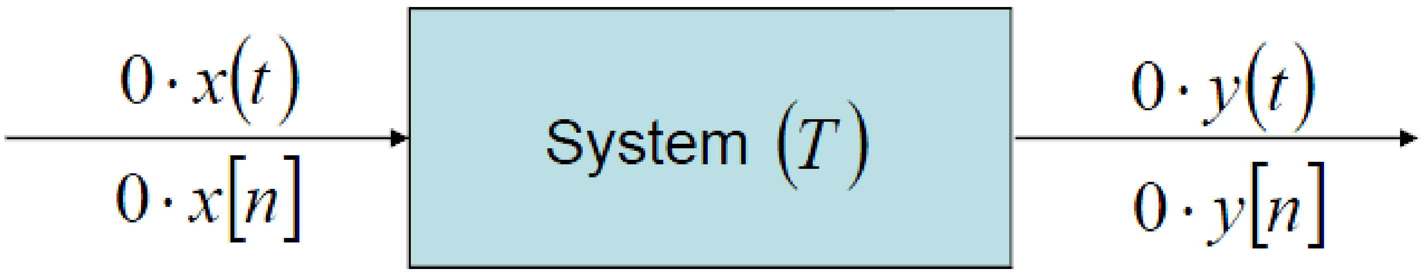
Linear and Nonlinear Systems

•Examples of linear-systems : $v(t) = Ri(t)$

Examples of non-linear systems: $y = x^2(t);$

NB:

• a consequence of the *scaling* property of linear-systems is zero in =zero out



Linear-system

Time-variant and Time-invariant Systems

A system is to be ***time-invariant*** if a time-shift (advance or delay) at the input causes an *identical* shift at the output. So for a continuous-time system, time-invariance exists if:

$$H\{x(t \pm \tau)\} = y(t \pm \tau); \quad \tau \in \mathbb{R}$$

For a discrete-time system, the system is time- or shift-invariant if

$$H\{x(n \pm k)\} = y(n \pm k); \quad k \in \mathbb{Z}$$

A system not satisfying equation above equations is **time-varying.** Time-invariance can be tested by *correlating* the shifted output with the output produced by a shifted input

Time-variant and Time-invariant Systems

1. Continuous-time system

$$y_1(t) = H\{x_1(t)\}$$

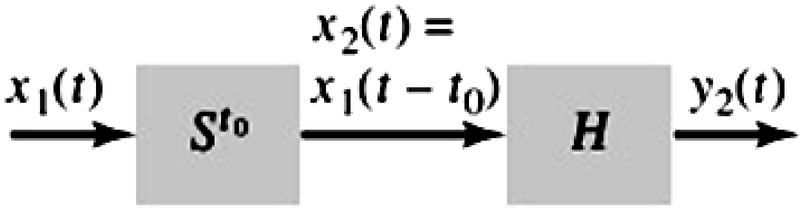
2. Input signal $x_1(t)$ is shifted in time by t_0 seconds

$$x_2(t) = x_1(t - t_0) = S^{t_0}\{x_1(t)\}$$

S^{t_0} = operator of a time shift equal to t_0

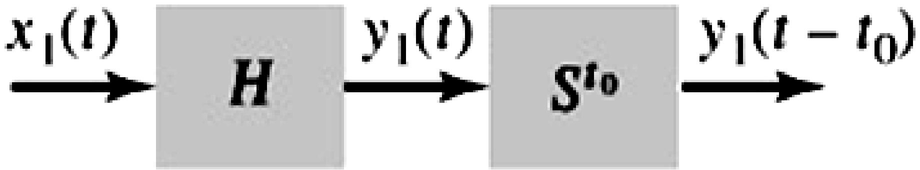
3. Output of system *H*:

$$\begin{aligned} y_2(t) &= H\{x_1(t-t_0)\} \\ &= H\{S^{t_0}\{x_1(t)\}\} \\ &= HS^{t_0}\{x_1(t)\} \end{aligned}$$



4. The output of system *H* is $y_1(t - t_0)$

$$\begin{aligned} y_1(t-t_0) &= S^{t_0}\{y_1(t)\} \\ &= S^{t_0}\{H\{x_1(t)\}\} \\ &= S^{t_0}H\{x_1(t)\} \end{aligned}$$



5. Condition for time-invariant system

$$HS^{t_0} = S^{t_0}H$$

These two situations are equivalent provided that *H* is time invariant.

Example 1.17 Inductor: Time-variance

The inductor shown in figure is described by the input-output relation:

$$y_1(t) = \frac{1}{L}\int_{-\infty}^{t} x_1(\tau)\,d\tau$$

where *L* is the inductance. Show that the inductor so described is time invariant.

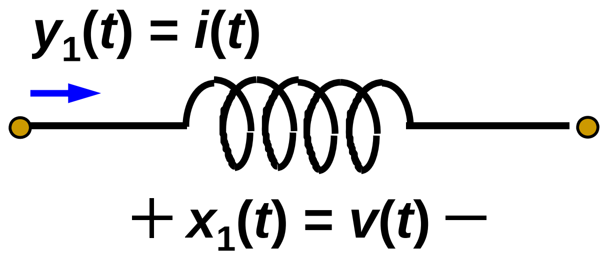
<Sol.>

1. Let $x_1(t)$ $\longrightarrow$ $x_1(t - t_0)$ $\longrightarrow$ Response $y_2(t)$ of the inductor to $x_1(t - t_0)$ is

$$y_2(t) = \frac{1}{L}\int_{-\infty}^{t} x_1(\tau - t_0)\,d\tau \qquad \text{(A)}$$

2. Let $y_1(t - t_0)$ = the original output of the inductor, shifted by t_0 seconds:

$$y_1(t - t_0) = \frac{1}{L}\int_{-\infty}^{t-t_0} x_1(\tau)\,d\tau \qquad \text{(B)}$$

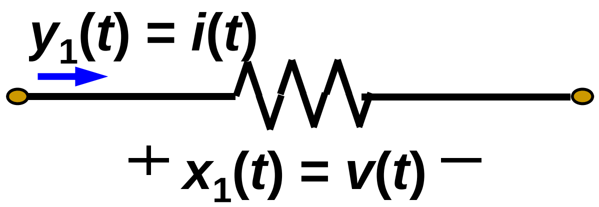


3. Changing variables: $\tau' = \tau - t_0$

(A) $\Longrightarrow$ $$y_2(t) = \frac{1}{L}\int_{-\infty}^{t-t_0} x_1(\tau')\,d\tau'$$ $\Longrightarrow$ **Inductor is time invariant.**

Example 1.18 Thermistor:Time-variance

Let **$R(t)$** denote the resistance of the thermistor, expressed as a function of time. the input-output relation of the device as $y_1(t)$.



Show that the thermistor so described is time variant.

$$y_1(t) = x_1(t)/R(t)$$

<Sol.>

1. Let response $y_2(t)$ of the thermistor to $x_1(t - t_0)$ is

$$y_2(t) = \frac{x_1(t - t_0)}{R(t)}$$

2. Let $y_1(t - t_0)$ = the original output of the thermistor due to $x_1(t)$, shifted by t_0 seconds:

$$y_1(t - t_0) = \frac{x_1(t - t_0)}{R(t - t_0)}$$

3. Since $R(t) \neq R(t - t_0)$ ⟹ $y_1(t - t_0) \neq y_2(t) \quad \text{for} \quad t_0 \neq 0$ ⟹ ***Time variant!***

Linear Time-Invariant Systems

If the system is linear and also time-invariant, then it is called a linear time-invariant (LTI) system.

Stability of a system

A system is ***bounded-input/bounded-output (BIBO) stable*** if for any bounded input ***x*** defined by

$$|x| \le k_1 \qquad \boxed{|x(t)| \le M_x < \infty \quad \text{for all } t.}$$

The corresponding output y is also bounded defined by

$$|y| \le k_2 \qquad \boxed{|y(t)| \le M_y < \infty \quad \text{for all } t}$$

where $\boldsymbol{k_1}$**,** and $\boldsymbol{k_2}$ **,** are finite real constants.
Note that there are many other definitions of stability.

Example 1.13 *Stable system*

Show that the moving-average system described in Example 1.12 is BIBO stable.

<p.f.>

1. Assume that:

$$|x[n]| \le M_x < \infty \qquad \text{for all} \quad n$$

2. Input-output relation:

$$y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$$

$$\begin{aligned} |y[n]| &= \frac{1}{3}\left|x[n] + x[n-1] + x[n-2]\right| \\ &\le \frac{1}{3}\left(|x[n]| + |x[n-1]| + |x[n-2]|\right) \\ &\le \frac{1}{3}\left(M_x + M_x + M_x\right) \\ &= M_x \end{aligned}$$

$\Rightarrow$ **The moving-average system is stable.**

Example 1.14 *Unstable system*

Consider a discrete-time system whose input-output relation is defined by

$$y[n] = r^n x[n]$$

where *r* > 1. Show that this system is unstable.

<p.f.>

1. Assume that:

$$|x[n]| \le M_x < \infty \qquad \text{for all } n$$

2. We find that

$$|y[n]| = |r^n x[n]| = |r^n| \, |x[n]|$$

With *r* > 1, the multiplying factor r^n diverges for increasing *n*.

⇨ **The system is unstable.**

Stability of a system

One famous example of an unstable system:

Figure 1.52a (p. 56)
Dramatic photographs showing the collapse of the Tacoma Narrows suspension bridge on November 7, 1940. (a) Photograph showing the twisting motion of the bridge's center span just before failure.
(b) A few minutes after the first piece of concrete fell, this second photograph shows a 600-ft section of the bridge breaking out of the suspension span and turning upside down as it crashed in Puget Sound, Washington. Note the car in the top right-hand corner of the photograph.

(Courtesy of the Smithsonian Institution.)

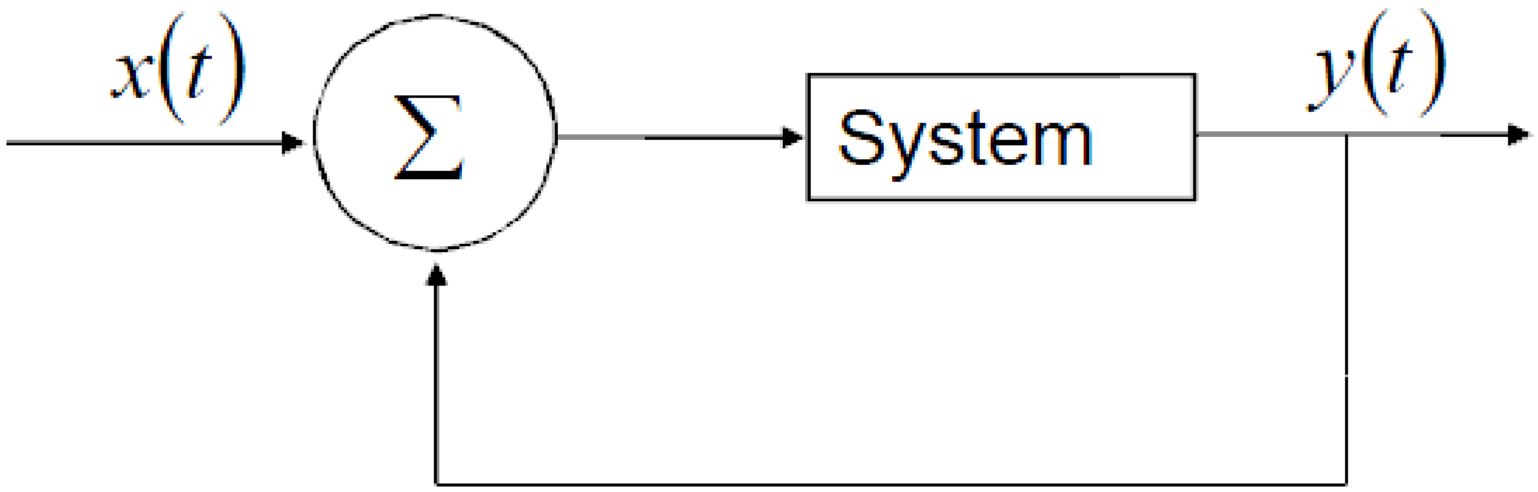
(a)

(b)

Feedback Systems:

A special class of systems of great importance consists of systems having ***feedback.***

In a ***feedback system,*** the output signal is fed back and added to the input to the system

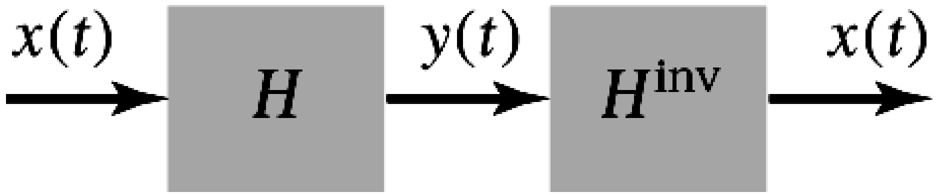


System feedback

Invertibility Systems

A system is said to be *invertible* if the input of the system can be recovered from the output.

1. Continuous-time system



$x(t)$ = input; $y(t)$ = output

H = first system operator;
H^{inv} = second system operator

2. Output of the second system

$$H^{inv}\{y(t)\} = H^{inv}\{H\{x(t)\}\} = H^{inv}H\{x(t)\}$$

H^{inv} = inverse operator

3. Condition for invertible system

$$H^{inv}H = I$$

I = identity operator

Example: Invertibility

Example 1.15 Inverse of System

Consider the time-shift system described by the input-output relation, where the operator S^{t_0} represents a time shift of t_0 seconds. Find the inverse of this system.

$$y(t) = x(t - t_0) = S^{t_0}\{x(t)\}$$

<Sol.>

1. Inverse operator S^{-t_0}

$$S^{-t_0}\{y(t)\} = S^{-t_0}\{S^{t_0}\{x(t)\}\} = S^{-t_0}S^{t_0}\{x(t)\}$$

2. Invertibility condition

$$S^{-t_0}S^{t_0} = I \quad \Rightarrow \quad S^{-t_0} \equiv$$ **Time shift of $-t_0$**

Example 1.16 Non-Invertible System

Show that a square-law system described by the input-output relation is not invertible.

$$y(t) = x^2(t)$$

<p.f.> **Since the distinct inputs $x(t)$ and $-x(t)$ produce the same output $y(t)$. Accordingly, the square-law system is not invertible.**

Summary and Exercises

- Summary
  - Classification : Continuous-time and Discrete-time Systems
  - Properties: memory, Causality, Linearity, Time-variance, Stablity, Invertibility
- Exercises
  - P183-184: 1.63, 1.65, 1.66, 1.67, 1.75 (a), 1.77 (a, b)