Signals and Systems 3.3

--- *Properties of Fourier Representation*

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Reference:
1. Textbook: Chapter 3.8 - 3.18

Properties of Fourier Representation

- Periodicity Properties
- Linearity Properties
- Symmetry Properties
- Convolution Property
- Differentiation and Integration Properties
- Time-shift Properties
- Frequency-shift Properties
- Multiplication Properties
- Scaling Properties
- Duality Properties
- Inverse Fourier representation
- Parseval relationship
- Time bandwidth products

Periodicity properties

1. **Four Fourier representations: Table 3.2.**

Table 3.2 The Four Fourier Representations

Time Domain	**Periodic** (t, n)	**Non-periodic** (t, n)	
Continuous (t)	**Fourier Series** $$x(t)=\sum_{k=-\infty}^{\infty}X[k]e^{jk\omega_o t}$$ $$X[k]=\frac{1}{T}\int_0^T x(t)e^{-jk\omega_o t}\,dt$$ $x(t)$ has period T $$\omega_o=\frac{2\pi}{T}$$	**Fourier Transform** $$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}\,d\omega$$ $$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}\,dt$$	Nonperiodic $(k,\ \omega)$
Discrete [n]	**Discrete-Time Fourier Series** $$x[n]=\sum_{k=0}^{N-1}X[k]e^{jk\Omega_o n}$$ $$X[k]=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk\Omega_o n}$$ $x[n]$ and $X[k]$ have period N $$\Omega_o=\frac{2\pi}{N}$$	**Discrete-Time Fourier Transform** $$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\Omega})e^{j\Omega n}\,d\Omega$$ $$X(e^{j\Omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}$$ $X(e^{j\Omega})$ has period 2π	Periodic $(k,\ \Omega)$
	Discrete [k]	Continuous $(\omega,\ \Omega)$	Frequency Domain

Periodicity properties

2. Signals that are periodic in time have discrete frequency-domain representations, while nonperiodic time signals have continuous frequency- domain representations.

Table 3.3 Periodicity Properties of Fourier Representations

Time-Domain Property	Frequency-Domain Property
Continuous	*Nonperiodic*
Discrete	*Periodic*
Periodic	*Discrete*
Nonperiodic	*Continuous*

Linearity Properties

$$z(t) = ax(t) + by(t) \xleftrightarrow{FT} Z(j\omega) = aX(j\omega) + bY(j\omega)$$

$$z(t) = ax(t) + by(t) \xleftrightarrow{FS;\omega_o} Z[k] = aX[k] + bY[k]$$

$$z[n] = ax[n] + by[n] \xleftrightarrow{DTFT} Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega})$$

$$z[n] = ax[n] + by[n] \xleftrightarrow{DTFS;\Omega_o} Z[k] = aX[k] + bY[k]$$

Both a and b are constant.

The linearity property is used to find Fourier representations of signals that are constructed as sums of signals whose representations are already known

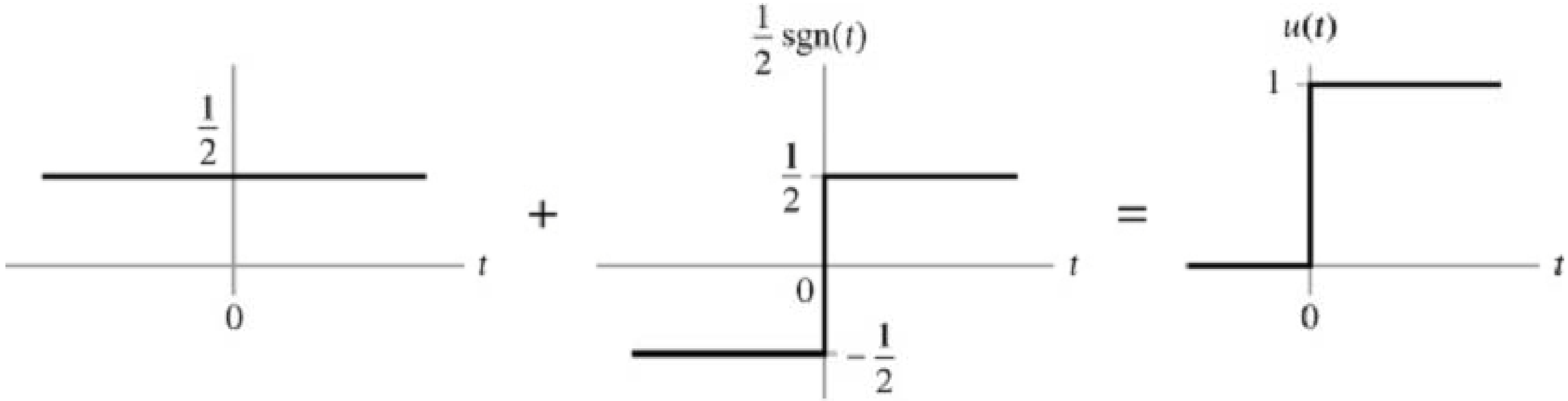
Example 1 of Linearity in FT: : Representation of a Unit Step signal as a weighted sum of a constant and a signum signal

$$u(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)$$

$$F[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$F[\frac{1}{2}] = 2\pi\delta(\omega)$$

$$F[\mathrm{sgn}(t)] = \frac{2}{j\omega}$$



Example 2 of Linearity in The FS: Representation of a periodic signal as a weighted sum of periodic square waves

Example 3.30

$$z(t)=\frac{3}{2}x(t)+\frac{1}{2}y(t)$$

$$x(t)\xleftrightarrow{FS;\omega_0}X[k]$$

$$y(t)\xleftrightarrow{FS;\omega_0}Y[k]$$

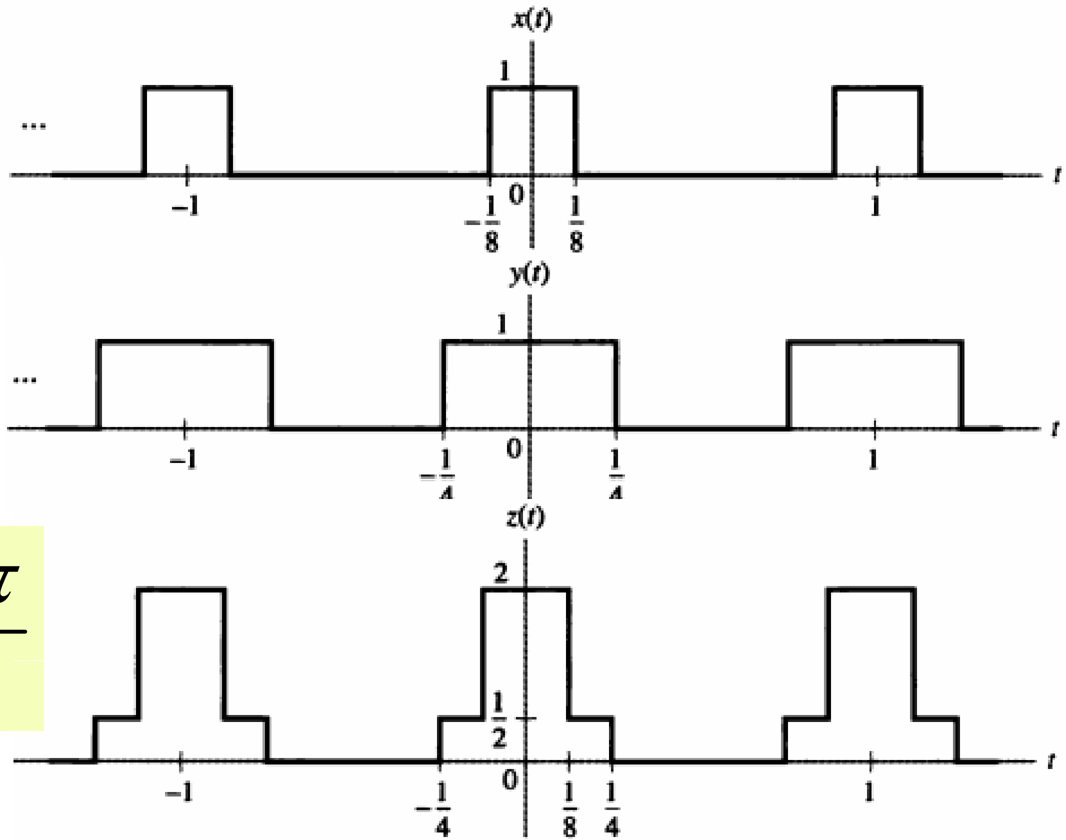
$$z(t)\xleftrightarrow{FS;\omega_0}Z[k]$$

$$X[k]=\frac{1}{k\pi}\sin\frac{k\pi}{4}$$

$$Y[k]=\frac{1}{k\pi}\sin\frac{k\pi}{2}$$

$$Z[k]=\frac{3}{2}X[k]+\frac{1}{2}Y[k]$$

$$=\frac{1}{2k\pi}\sin\frac{k\pi}{4}+\frac{3}{2k\pi}\sin\frac{k\pi}{2}$$



Symmetry Properties: Real and Imaginary Signals

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

$$x^*(t) \xleftrightarrow{FT} X^*(-j\omega)$$

$$F[x^*(t)] = \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt = \left[\int_{-\infty}^{\infty} x(t) e^{-j(-\omega)t} dt \right]^* = X^*(-j\omega)$$

$$X(j\omega) = |X(j\omega)| e^{j\phi(j\omega)}$$

$$= \mathrm{Re}[X(j\omega)] + j\,\mathrm{Im}[X(j\omega)]$$

For continuous real valued signal *x*(*t*) = *x(*t*)**

$$X(j\omega) = X^*(-j\omega)$$

$$X^*(j\omega) = X(-j\omega)$$

$$\mathrm{Re}\{X(j\omega)\} = \mathrm{Re}\{X(-j\omega)\}$$

Real of $X(j\omega)$ is oven

$$\mathrm{Im}\{X(j\omega)\} = -\mathrm{Im}\{X(-j\omega)\}$$

Imagery of $X(j\omega)$ is odd

$$|X(j\omega)| = |X(-j\omega)|$$

Magnitude spectrum of $X(j\omega)$ is oven

$$\phi(j\omega) \triangleq \arg\{X(j\omega)\} = -\phi(-j\omega)$$

Phase spectrum of $X(j\omega)$ is odd

Symmetry Properties: Real and Imaginary Signals

Table 3.4 ***Symmetry Properties for Fourier Representation of Real- and Imaginary-Valued Signals***

Representation	Real-Valued Time Signals	Imaginary-Valued Time Signals
FT	$X^*(j\omega) = X(-j\omega)$	$X^*(j\omega) = -X(-j\omega)$
FS	$X^*[k] = X[-k]$	$X^*[k] = -X[-k]$
DTFT	$X^*(e^{j\Omega}) = X(e^{-j\Omega})$	$X^*(e^{j\Omega}) = -X(e^{-j\Omega})$
DTFS	$X^*[k] = X[-k]$	$X^*[k] = -X[-k]$

Example: Symmetry Properties

1. $x(t)$ is real valued and has even symmetry.

$$x^*(t) = x(t)$$

$$x(-t) = x(t)$$

$$\Rightarrow \quad x^*(t) = x(-t)$$

$$X^*(j\omega) = \int_{-\infty}^{\infty} x(-t) e^{-j\omega(-t)} dt$$

Change of variable $\tau = -t$

$$\Rightarrow \quad X^*(j\omega) = -\int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau = X(j\omega)$$

$$X^*(j\omega) = X(j\omega)$$

1) If $x(t)$ is real and even, then the imaginary part of $X(j\omega) = 0$, $X(j\omega)$ is real.

2) If $x(t)$ is real and odd, then $X^*(j\omega) = -X(j\omega)$ and $X(j\omega)$ is imaginary.

Convolution Properties: Nonperiodic CT Signals

Convolution of Nonperiodic Continuous-time Signals

$$x(t) \xleftrightarrow{FT} X(j\omega) \qquad h(t) \xleftrightarrow{FT} H(j\omega)$$

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}e^{-j\omega\tau}d\omega\right]d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau\right]X(j\omega)e^{j\omega t}d\omega$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)X(j\omega)e^{j\omega t}d\omega$$

Convolution of *h(t)* and *x(t)* in the time domain corresponds to multiplication of their Fourier transforms, H(jω) and X(jω)in the frequency domain.

$$y(t) = h(t) * x(t) \xleftrightarrow{FT} Y(j\omega) = X(j\omega)H(j\omega)$$

Example1: Solving A Convolution Problem in The Frequency Domain

Example 3.31

$$x(t)=\frac{\sin(\pi t)}{\pi t} \longrightarrow \boxed{h(t)=\frac{\sin(2\pi t)}{\pi t}} \xrightarrow{y(t)} = h(t)*x(t)$$

$$x(t)\xleftrightarrow{FT}X(j\omega)=\begin{cases}1, & |\omega|<\pi \\ 0, & |\omega|>\pi\end{cases}$$

$$h(t)\xleftrightarrow{FT}H(j\omega)=\begin{cases}1, & |\omega|<2\pi \\ 0, & |\omega|>2\pi\end{cases}$$

Time domain $\rightarrow$ Extremely Difficult
Frequency domain $\rightarrow$ Quite Simple

Convolution property

$$y(t)=h(t)*x(t)\xleftrightarrow{FT}Y(j\omega)=X(j\omega)H(j\omega)$$

$$Y(j\omega)=\begin{cases}1, & |\omega|<\pi \\ 0, & |\omega|>\pi\end{cases}$$

$$y(t)=\frac{\sin(\pi t)}{\pi t}$$

Example2: Finding Inverse FT's by Means of The Convolution Property

Example 3.32 **Use the convolution property to find $x(t)$**

$$x(t) \xleftrightarrow{FT} X(j\omega) = \frac{4}{\omega^2}\sin^2(\omega)$$

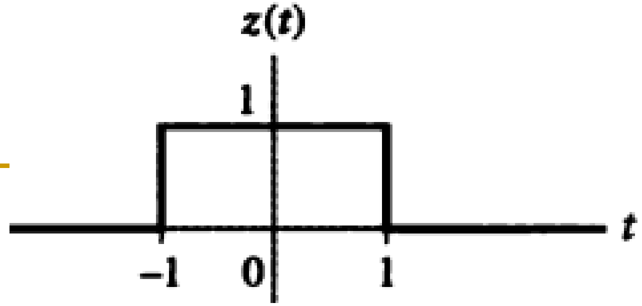
$$X(j\omega) = Z(j\omega)\, Z(j\omega)$$

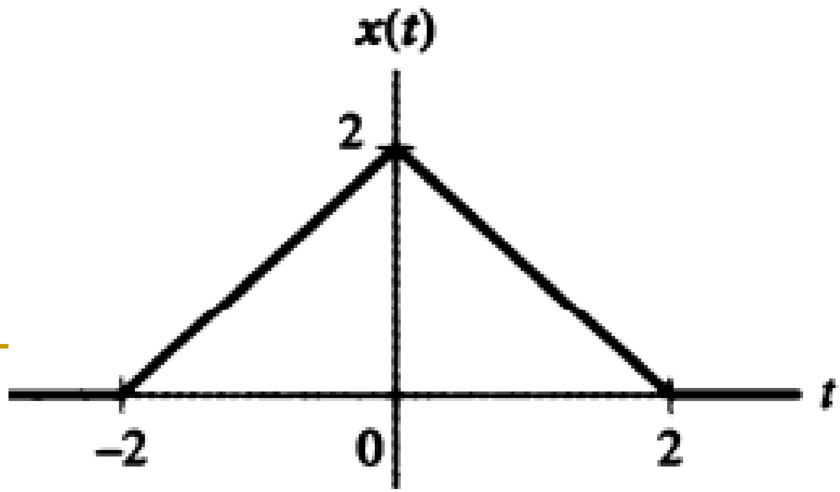
$$z(t) = \begin{cases} 1, & |t| < 1 \\ 0, & |t| > 1 \end{cases} \xleftrightarrow{FT} Z(j\omega)$$

$$Z(j\omega) = \frac{2}{\omega}\sin(\omega)$$

$$z(t) * z(t) \xleftrightarrow{FT} Z(j\omega)Z(j\omega)$$

$$x(t) = z(t) * z(t)$$





Convolution Properties: Nonperiodic DT Signals

♣ ***Convolution of Nonperiodic Discrete-time Signals***

$$x[n] \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right)$$

$$h[n] \xleftrightarrow{DTFT} H\left(e^{j\Omega}\right)$$

$$y[n] = x[n] * h[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega})$$

Convolution of *h[n]* and *x[n]* in the time domain corresponds to multiplication of their Fourier transforms, $H(e^{j\Omega})$ and $X(e^{j\Omega})$ in the frequency domain.

Frequency response of the LTI system

- **The convolution property implies that the frequency response of a system may be expressed as the ratio of the FT or DTFT of the output to the input .**

For CT system:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

For DT system:

$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})}$$

Problem1: Find the system output (5~10min)

▶ **Problem 3.18** Use the convolution property to find the FT of the system output, either $Y(j\omega)$ or $Y(e^{j\Omega})$, for the following inputs and system impulse responses:

(a) $x(t) = 3e^{-t}u(t)$ and $h(t) = 2e^{-2t}u(t)$.

(b) $x[n] = \left(\frac{1}{3}\right)^{n+1} u[n]$ and $h[n] = \left(\frac{1}{6}\right)^{n} u[n]$

Sol (a):

$$x(t) = 3e^{-t}u(t) \Leftrightarrow X(j\omega) = \frac{3}{1+j\omega}$$

$$h(t) = 2e^{-2t}u(t) \Leftrightarrow H(j\omega) = \frac{2}{2+j\omega}$$

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{2}{(2+j\omega)}\frac{3}{(1+j\omega)}$$

Sol (b):

$$x[n] = \left(\tfrac{1}{3}\right)^{n+1} u[n] = \tfrac{1}{3}\left(\tfrac{1}{3}\right)^{n} u[n] \Leftrightarrow X(e^{j\Omega}) = \frac{1}{3\left(1-\frac{1}{3}e^{-j\Omega}\right)}$$

$$h[n] = \left(\tfrac{1}{6}\right)^{n} u[n] \Leftrightarrow H(e^{j\Omega}) = \frac{1}{1-\frac{1}{6}e^{-j\Omega}}$$

$$Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}) = \frac{1}{3\left(1-\frac{1}{3}e^{-j\Omega}\right)}\frac{1}{\left(1-\frac{1}{6}e^{-j\Omega}\right)}$$

Problem1: Find the system output (5~10min)

$$Y(j\omega)=\frac{2}{(2+j\omega)}\frac{3}{(1+j\omega)}=\frac{1}{6(1+j\omega)}-\frac{1}{6(2+j\omega)}$$

$$y(t)=\tfrac{1}{6}e^{-t}u(t)-\tfrac{1}{6}e^{-2t}u(t)$$

$$\alpha>0$$

$$e^{-\alpha t}u(t)\xleftrightarrow{FT}\frac{1}{\alpha+j\omega}$$

$$Y(e^{j\Omega})=\frac{1}{3\left(1-\tfrac{1}{3}e^{-j\Omega}\right)}\frac{1}{\left(1-\tfrac{1}{6}e^{-j\Omega}\right)}$$

$$=\frac{2}{3\left(1-\tfrac{1}{3}e^{-j\Omega}\right)}-\frac{1}{3\left(1-\tfrac{1}{6}e^{-j\Omega}\right)}$$

$$\alpha^n u[n]\xleftrightarrow{DTFT}\frac{1}{1-\alpha e^{-j\Omega}}$$

$$|\alpha|<1$$

$$y[n]=\tfrac{2}{3}\left(\tfrac{1}{3}\right)^n u[n]-\tfrac{1}{3}\left(\tfrac{1}{6}\right)^n u[n]$$

Problem2: Find the response of the system (5~10min)

The output of an LTI system in response to an input $x(t) = e^{-2t}u(t)$ is $y(t) = e^{-t}u(t)$. Find the frequency response and the impulse response of this system.

$$x(t) = e^{-2t}u(t) \Leftrightarrow X(j\omega) = \frac{1}{2 + j\omega}$$

$$y(t) = e^{-t}u(t) \Leftrightarrow Y(j\omega) = \frac{1}{1 + j\omega}$$

$$H(j\omega) = Y(j\omega) / X(j\omega)$$

$$= \frac{2 + j\omega}{1 + j\omega} = 1 + \frac{1}{1 + j\omega}$$

$$\Leftrightarrow h(t) = \delta(t) + e^{-t}u(t)$$

Filtering

1. The terms "*filtering*" implies that some frequency components of the input are eliminated while others are passed by the system unchanged.

2. Multiplication in frequency domain $\leftrightarrow$ *Filtering*.

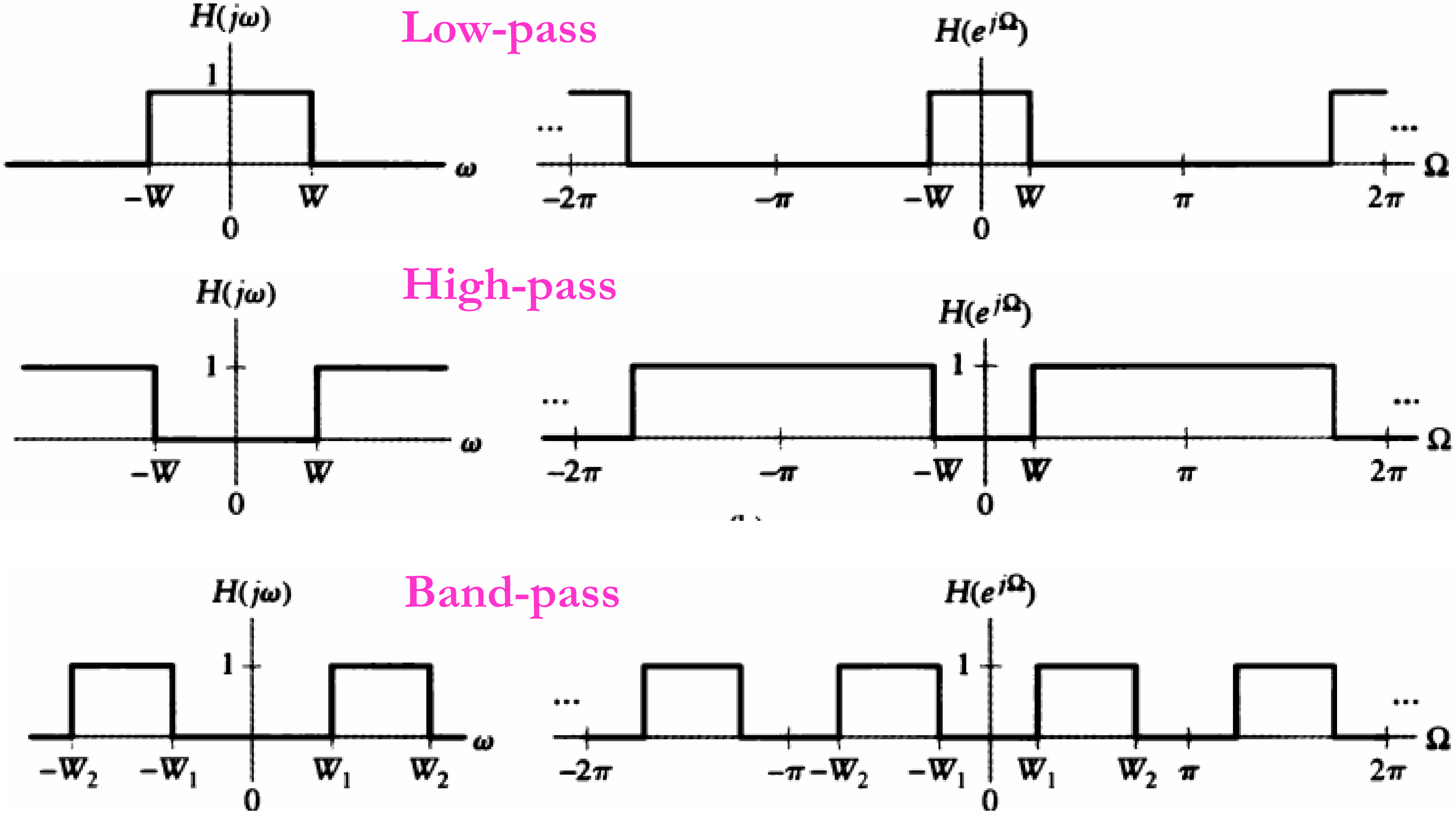
3. System Types of filtering:

   1) Low-pass filter
   2) High-pass filter
   3) Band-pass filter

4. Realistic filter:

   1) Gradual transition band
   2) Nonzero gain of stop band

Filtering


Low-pass
H(jω)
1
−W
W
0
ω
H(e^{jΩ})
...
−2π
−π
−W
W
0
π
2π
Ω
High-pass
H(jω)
1
−W
W
0
ω
H(e^{jΩ})
1
...
−2π
−π
−W
W
0
π
2π
Ω
Band-pass
H(jω)
1
−W₂
−W₁
W₁
W₂
0
ω
H(e^{jΩ})
1
...
−2π
−π −W₂
−W₁
W₁
W₂
π
0
2π
Ω

Filtering

5. Magnitude response of filter:

$$20\log\left|H(j\omega)\right| \qquad\qquad 20\log\left|H\left(e^{j\Omega}\right)\right| \qquad \text{[dB]}$$

6. The edge of the passband is usually defined by the frequencies for which the response is – 3 dB, corresponding to a magnitude response of (1/ $\sqrt{2}$).

♣ **Unity gain = 0 dB**

7. Energy spectrum of filter: $$\left|Y(j\omega)\right|^2 = \left|H(j\omega)\right|^2 \left|X(j\omega)\right|^2$$

➡ **The – 3 dB point corresponds to frequencies at which the filter passes only half of the input power.**

– 3 dB point ➡ **Cutoff frequency**

Convolution of Periodic Signals

♣ **Basic Concept:**

$$\longrightarrow \boxed{\textit{Periodic } h} \longrightarrow$$

1. Define the periodic convolution of two CT signals *x*(*t*) and *z*(*t*), each having period *T*, as

$$y(t) = x(t) \circledast z(t) = \int_0^T x(\tau)z(t-\tau)d\tau$$

where the symbol ⊛ denotes that integration is performed over a single period of the signals involved.

$$y(t) = x(t) \circledast z(t) \quad \xleftrightarrow{FS;\frac{2\pi}{T}} \quad Y[k] = TX[k]Z[k]$$

◆ **Convolution in Time-Domain**

↔ **Multiplication in Frequency-Domain**

Convolution in Time-Domain ↔ Multiplication in Frequency-Domain

Table 3.5 Convolution Properties

$$x(t) * z(t) \xleftrightarrow{FT} X(j\omega)Z(j\omega)$$

$$x(t) \circledast z(t) \xleftrightarrow{FS;\omega_0} TX[k]Z[k]$$

$$x[n] * z[n] \xleftrightarrow{FT} X(e^{j\Omega})Z(e^{j\Omega})$$

$$x[n] \circledast z[n] \xleftrightarrow{DTFS;\Omega_0} NX[k]Z[k]$$

Differentiation Properties: Time Domain

Differentiation in Time Domain

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)j\omega e^{j\omega t}\,d\omega$$

$$\frac{d^n}{dt^n}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)(j\omega)^n e^{j\omega t}\,d\omega$$

$$\frac{d^n}{dt^n}x(t) \xleftrightarrow{FT} (j\omega)^n X(j\omega)$$

Differentiation of $x(t)$ in Time-Domain $\leftrightarrow$ $(j\omega) \times X(j\omega)$ in Frequency-Domain

$$e^{-\alpha t}u(t) \xleftrightarrow{FT} \frac{1}{\alpha + j\omega}$$

$$\frac{d}{dt}\left\{e^{-\alpha t}u(t)\right\} \xleftrightarrow{FT} \frac{j\omega}{\alpha + j\omega}$$

Differentiation Properties: Frequency Domain

Differentiation in Frequency Domain

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

$$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega} X(j\omega)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\frac{d}{d\omega} X(j\omega) = \int_{-\infty}^{\infty} -jtx(t) e^{-j\omega t} dt$$

Differentiation of $X(j\omega)$ in Frequency-Domain $\leftrightarrow$ $(-jt) \times x(t)$ in Time-Domain

Example

$$e^{-\alpha t}u(t) \xleftrightarrow{FT} \frac{1}{\alpha + j\omega}$$

$$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)$$

$$\frac{d}{dt}\left\{e^{-\alpha t}u(t)\right\} \xleftrightarrow{FT} \frac{j\omega}{\alpha + j\omega}$$

$$\frac{d}{dt}\left\{e^{-\alpha t}u(t)\right\}$$

$$= -\alpha e^{-\alpha t}u(t) + e^{-\alpha t}\delta(t)$$

$$= -\alpha e^{-\alpha t}u(t) + \delta(t)$$

$$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega)$$

$$te^{-\alpha t}u(t) \xleftrightarrow{FT} j\frac{d}{d\omega}\left\{\frac{1}{\alpha + j\omega}\right\} = \frac{1}{(\alpha + j\omega)^2}$$

Integration Properties

$$y(t)=\int_{-\infty}^{t}x(t)dt$$

$$\frac{d}{dt}y(t)=x(t)$$

$$x(t)\xleftrightarrow{FT}X(j\omega)$$

$$y(t)\xleftrightarrow{FT}Y(j\omega)$$

$$\Leftrightarrow\ j\omega Y(j\omega)=X(j\omega)$$

$$Y(j\omega)=\frac{X(j\omega)}{j\omega}\qquad \omega\neq 0$$

c can be determined by the average value of *x*(*t*)

$$\int_{-\infty}^{t}x(t)dt\xleftrightarrow{FT}\frac{X(j\omega)}{j\omega}+\pi X(j0)\delta(\omega)$$

$$X(j0)=\int_{-\infty}^{\infty}x(t)e^{-j0t}\mathrm{dt}=\int_{-\infty}^{\infty}x(t)\mathrm{dt}=0$$

$$u(t)=\int_{-\infty}^{t}\delta(t)dt\xleftrightarrow{FT}\frac{1}{j\omega}+\pi\delta(\omega)$$

Differentiation and Integration Properties

Table 3.6 summarizes the differentiation and integration properties of Fourier representations.

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)$$

$$\frac{d}{dt}x(t) \xleftrightarrow{FS;\ \omega_0} jk\omega_0 X[k]$$

$$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega)$$

$$-jnx[n] \xleftrightarrow{DTFT} \frac{d}{d\Omega}X\left(e^{j\Omega}\right)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega)+\pi X(j0)\delta(\omega)$$

Time-Shift Properties

if $x(t) \xleftrightarrow{FT} X(\mathrm{j}\omega)$

then $x(t-t_0) \xleftrightarrow{FT} X(\mathrm{j}\omega)\cdot \mathrm{e}^{-\mathrm{j}\omega t_0}$

$$F[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \mathrm{e}^{-\mathrm{j}\omega t_0}\int_{-\infty}^{\infty} x(t-t_0)\mathrm{e}^{-\mathrm{j}\omega(t-t_0)}\mathrm{dt}$$

$$= X(\mathrm{j}\omega)\cdot \mathrm{e}^{-\mathrm{j}\omega t_0} = \left|X(\mathrm{j}\omega)\right|\cdot \mathrm{e}^{j[\arg\{X(\mathrm{j}\omega)\}-\omega t_0]}$$

A shift in time leaves the magnitude spectrum unchanged and introduces a phase shift that is a linear function of frequency.

Time-Shift Properties

◆ The time-shifting properties of four Fourier representation are summarized in Table 3.7.

$$x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(j\omega)$$

$$x(t-t_0) \xleftrightarrow{FT;\ \ \omega_0} e^{-jk\omega_0 t_0} X(k)$$

$$x[n-n_0] \xleftrightarrow{DTFT} e^{-j\Omega n_0} X(e^{j\Omega})$$

$$x[n-n_0] \xleftrightarrow{DTFS;\Omega_0} e^{-jk\Omega_0 n_0} X[k]$$

Frequency-Shift Properties

$$z(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(j\omega)e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X\left(j(\omega-\gamma)\right)e^{j\omega t}\,d\omega$$

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

$$z(t) \xleftrightarrow{FT} Z(j\omega)$$

$$Z(j\omega) = X(j(\omega - \gamma))$$

$$\eta = \omega - \gamma$$

$$z(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\eta)e^{j(\eta+\gamma)t}\,d\eta$$

$$e^{j\gamma t}x(t) \xleftrightarrow{FT} X\left(j(\omega-\gamma)\right)$$

$$= e^{j\gamma t}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\eta)e^{j\eta t}\,d\eta \qquad = e^{j\gamma t}x(t)$$

◆ **Frequency-shifting of $X(j\omega)$ by γ in Frequency-Domain [i.e. $X(j(\omega-\gamma))$] $\leftrightarrow$ $(e^{-\gamma t}) \times x(t)$ in Time-Domain**

Frequency-Shift Properties

Table 3.8 Frequency-Shift Properties of Fourier Representations

$$e^{j\gamma t}x(t) \xleftrightarrow{FT} X(j(\omega-\gamma))$$

$$e^{jk_0\omega_0 t}x(t) \xleftrightarrow{FS;\,\omega_0} x[k-k_0]$$

$$e^{j\Gamma n}x[n] \xleftrightarrow{DTFT} X\left[e^{j(\Omega-\Gamma)}\right]$$

$$e^{jk_0\Omega_0 n}x[n] \xleftrightarrow{FS;\,\Omega_0} X[k-k_0]$$

Example

$$1 \xleftrightarrow{FT} 2\pi\delta(\omega)$$

$$e^{j\gamma t}x(t) \xleftrightarrow{FT} X(j(\omega-\gamma))$$

$$e^{j\omega_0 t} \xleftrightarrow{FT} 2\pi\delta(\omega-\omega_0)$$

$$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) \xleftrightarrow{FT} \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$

$$\sin(\omega_0 t) = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) \xleftrightarrow{FT} -j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

Finding Inverse Fourier Transforms by Using Partial-Fraction Expansions

Frequency response ⟷ **A ratio of two polynomials in $j\omega$ or $e^{j\Omega}$**

$$X(j\omega)=\frac{b_M(j\omega)^M+\cdots+b_1(j\omega)+b_0}{(j\omega)^N+a_{N-1}(j\omega)^{N-1}+\cdots a_1(j\omega)+a_0}=\frac{B(j\omega)}{A(j\omega)}$$

$$=c_0+c_1(j\omega)+c_2(j\omega)^2+\cdots+c_{M-N}(j\omega)^{M-N}+\frac{B(j\omega)}{A(j\omega)}$$

$$\delta(t)\xleftrightarrow{FT}1$$

$$x(t)=c_0\delta(t)+c_1\delta'(t)+c_2\delta''(t)+\cdots+c_{M-N}\delta^{(M-N)}(t)+F^{-1}\left[\frac{B(j\omega)}{A(j\omega)}\right]$$

$$v^N+a_{N-1}v^{N-1}+\cdots+a_1v+a_0=0$$

Finding Inverse Fourier Transforms by Using Partial-Fraction Expansions

$$v^N + a_{N-1}v^{N-1} + \cdots + a_1 v + a_0 = 0$$

$$d_k < 0$$

$$X(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\prod_{k=1}^{N}(j\omega - d_k)} = \sum_{k=1}^{N} \frac{C_k}{j\omega - d_k}$$

$$e^{d_k t}u(t) \xleftrightarrow{FT} \frac{1}{j\omega - d_k}$$

$$= \frac{C_1}{j\omega - d_1} + \frac{C_2}{j\omega - d_2} + \cdots + \frac{C_N}{j\omega - d_N}$$

$$x(t) = \sum_{k=1}^{N} C_k e^{d_k t}u(t) \xleftrightarrow{FT} X(j\omega) = \sum_{k=1}^{N} \frac{C_k}{j\omega - d_k}$$

Example

$$X(j\omega)=\frac{j\omega+1}{(j\omega)^2+5j\omega+6}$$

$$=\frac{A}{2+j\omega}+\frac{B}{3+j\omega}$$

$$=\frac{-1}{2+j\omega}+\frac{2}{3+j\omega}$$

$$=\frac{A(3+j\omega)+B(2+j\omega)}{(2+j\omega)(3+j\omega)}$$

$$=\frac{3A+2B+j\omega(A+B)}{(2+j\omega)(3+j\omega)}$$

$$\begin{cases}3A+2B=1\\ A+B=1\end{cases}\Rightarrow\begin{cases}A=-1\\ B=2\end{cases}$$

$$e^{-\alpha t}u(t)\Leftrightarrow\frac{1}{\alpha+j\omega}$$

$$x(t)=-e^{-2t}u(t)+2e^{-3t}u(t)\Leftrightarrow X(j\omega)$$

Inverse Discrete-Time Fourier Transform

1. Suppose $X(e^{j\Omega})$ is given by a ratio of polynomial in $e^{j\Omega}$

$$X\left(e^{j\Omega}\right)=\frac{\beta_M e^{-j\Omega M}+\cdots+\beta_1 e^{-j\Omega}+\beta_0}{\alpha_N e^{-j\Omega N}+\alpha_{N-1}e^{-j\Omega(N-1)}+\cdots+\alpha_1 e^{-j\Omega}+1}$$

Normalized to 1

2. Factor the denominator polynomial as

$$\alpha_N e^{-j\Omega N}+\alpha_{N-1}e^{-j\Omega(N-1)}+\cdots+\alpha_1 e^{-j\Omega}+1=\prod_{k=1}^{N}\left(1-d_k e^{-j\Omega}\right)$$

$$v^N+\alpha_1 v^{N-1}+\alpha_2 v^{N-2}+\cdots+\alpha_{N-1}v+\alpha_N=0$$

3. Partial-fraction expansion:Assuming that $M < N$ and all the d_k are distinct, we may express $X(e^{j\Omega})$as

$$\left(d_k\right)^n u[n]\xleftrightarrow{DTFT}\frac{1}{1-d_k e^{-j\Omega}},\ \left|d_k\right|<1$$

$$X\left(e^{j\Omega}\right)=\sum_{k=1}^{N}\frac{C_k}{1-d_k e^{-j\Omega}}$$

$$x[n]=\sum_{k=1}^{N}C_k\left(d_k\right)^n u[n]$$

Example 3.45 *Inverse by Partial-Fraction Expansion*

Find the inverse DTFT of

$$X\left(e^{j\Omega}\right)=\frac{-\frac{5}{6}e^{-j\Omega}+5}{1+\frac{1}{6}e^{-j\Omega}-\frac{1}{6}e^{-j2\Omega}}$$

1. Characteristic polynomial:

$$v^2+\frac{1}{6}v-\frac{1}{6}=(v+\frac{1}{2})(v-\frac{1}{3})=0$$

2. The roots of above polynomial:

$d_1 = -1/2$ and $d_2 = 1/3$.

3. Partial-Fraction Expansion

$$=\frac{C_1}{1+\frac{1}{2}e^{-j\Omega}}+\frac{C_2}{1-\frac{1}{3}e^{-j\Omega}}$$

4. Coefficients C_1 and C_2

$$x[n]=4(-1/2)^n u[n]+(1/3)^n u[n]$$

$$C_1=\left(1+\frac{1}{2}e^{-j\Omega}\right)\frac{-\frac{5}{6}e^{-j\Omega}+5}{1+\frac{1}{6}e^{-j\Omega}-\frac{1}{6}e^{-j2\Omega}}\Bigg|_{e^{-j\Omega}=-2}=\frac{-\frac{5}{6}e^{-j\Omega}+5}{1-\frac{1}{3}e^{-j\Omega}}\Bigg|_{e^{-j\Omega}=-2}=4$$

$$C_2=\left(1-\frac{1}{3}e^{-j\Omega}\right)\frac{-\frac{5}{6}e^{-j\Omega}+5}{1+\frac{1}{6}e^{-j\Omega}-\frac{1}{6}e^{-j2\Omega}}\Bigg|_{e^{-j\Omega}=3}=\frac{-\frac{5}{6}e^{-j\Omega}+5}{1+\frac{1}{2}e^{-j\Omega}}\Bigg|_{e^{-j\Omega}=3}=1$$

Multiplication Property: Non-periodic continuous-time signals

Non-periodic signals: $x(t)$, $z(t)$, and $y(t) = x(t)z(t)$.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jv)e^{jvt}\,dv \qquad z(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(j\eta)e^{j\eta t}\,d\eta$$

$$\Rightarrow\; y(t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(jv)Z(j\eta)e^{j(\eta+v)t}\,d\eta\,dv$$

Change variable: $\eta = \omega - v$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(jv)Z(j(\omega-v))\,dv\right]e^{j\omega t}\,d\omega$$

Inner Part: $Z(j\omega) * X(j\omega)$

Outer Part: FT of $y(t)$

$$\boxed{y(t) = x(t)z(t) \xleftrightarrow{FT} Y(j\omega) = \frac{1}{2\pi}X(j\omega) * Z(j\omega)}$$

◆ **Multiplication of two signals in Time-Domain $\leftrightarrow$ Convolution in Frequency-Domain $\times$ (1/2π)**

Multiplication Property：Non-periodic discrete-time signals

1. Non-periodic DT signals: $x[n]$, $z[n]$, and $y[n] = x[n]z[n]$.
2. DTFT of $y[n]$:

$$y[n] = x[n]z[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = \frac{1}{2\pi} X(e^{j\Omega}) \circledast Z(e^{j\Omega})$$

where the symbol $\circledast$ denotes periodic convolution.

Here, $X(e^{j\Omega})$ and $X(e^{j\Omega})$ are 2π-periodic, so we evaluate the convolution over a 2π interval:

$$X(j\omega) \circledast Z(j\omega) = \int_{-\pi}^{\pi} X\left(e^{j\theta}\right) Z\left(e^{j(\Omega-\theta)}\right) d\theta$$

◆ Multiplication of two signals in Time-Domain $\leftrightarrow$ Convolution in Frequency-Domain $\times$ $(1/2\pi)$

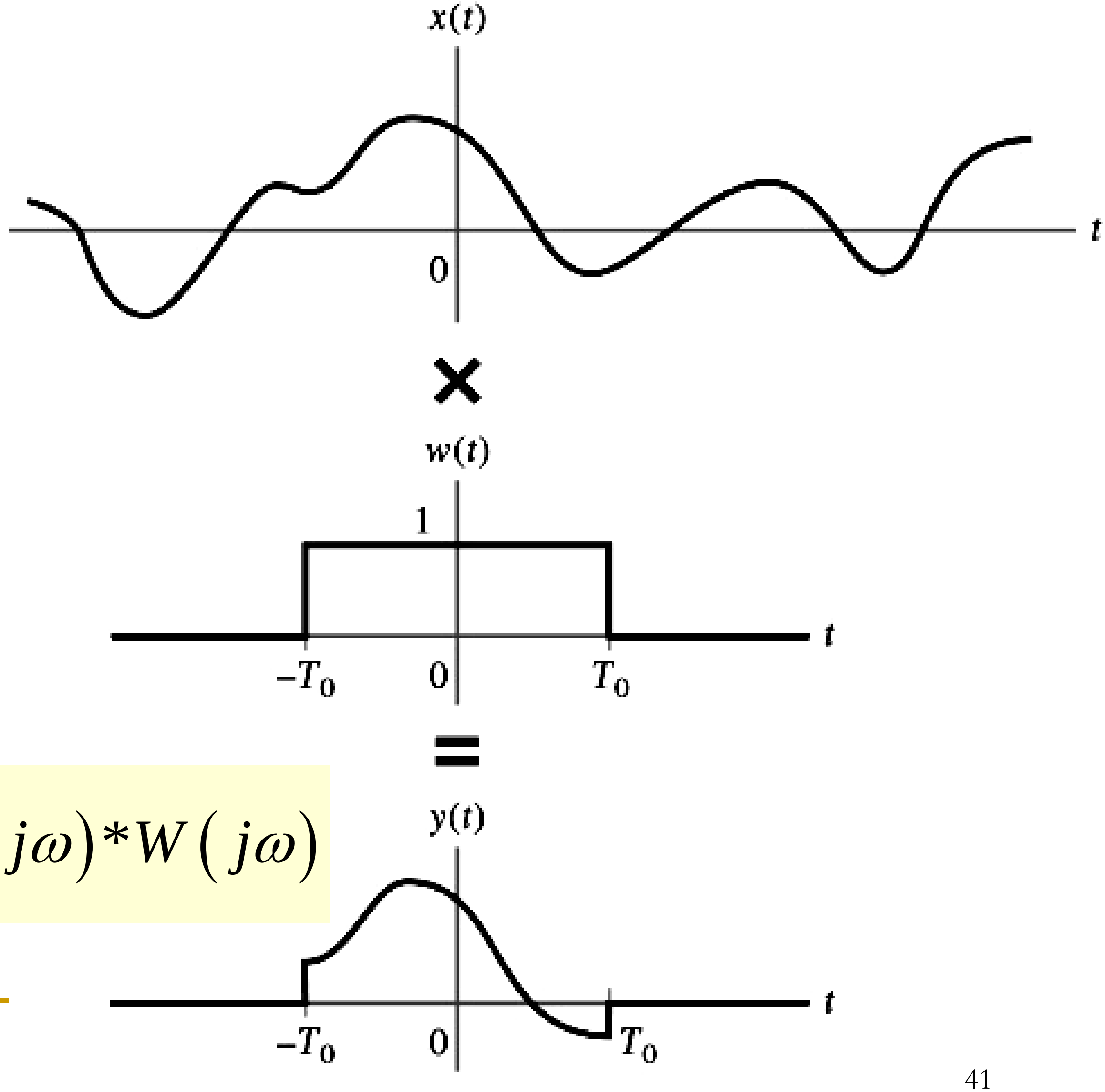
Multiplication Property : windowing

Truncate signal *x*(*t*) by a window function *w*(*t*) is represented by

$$y(t) = x(t)w(t)$$

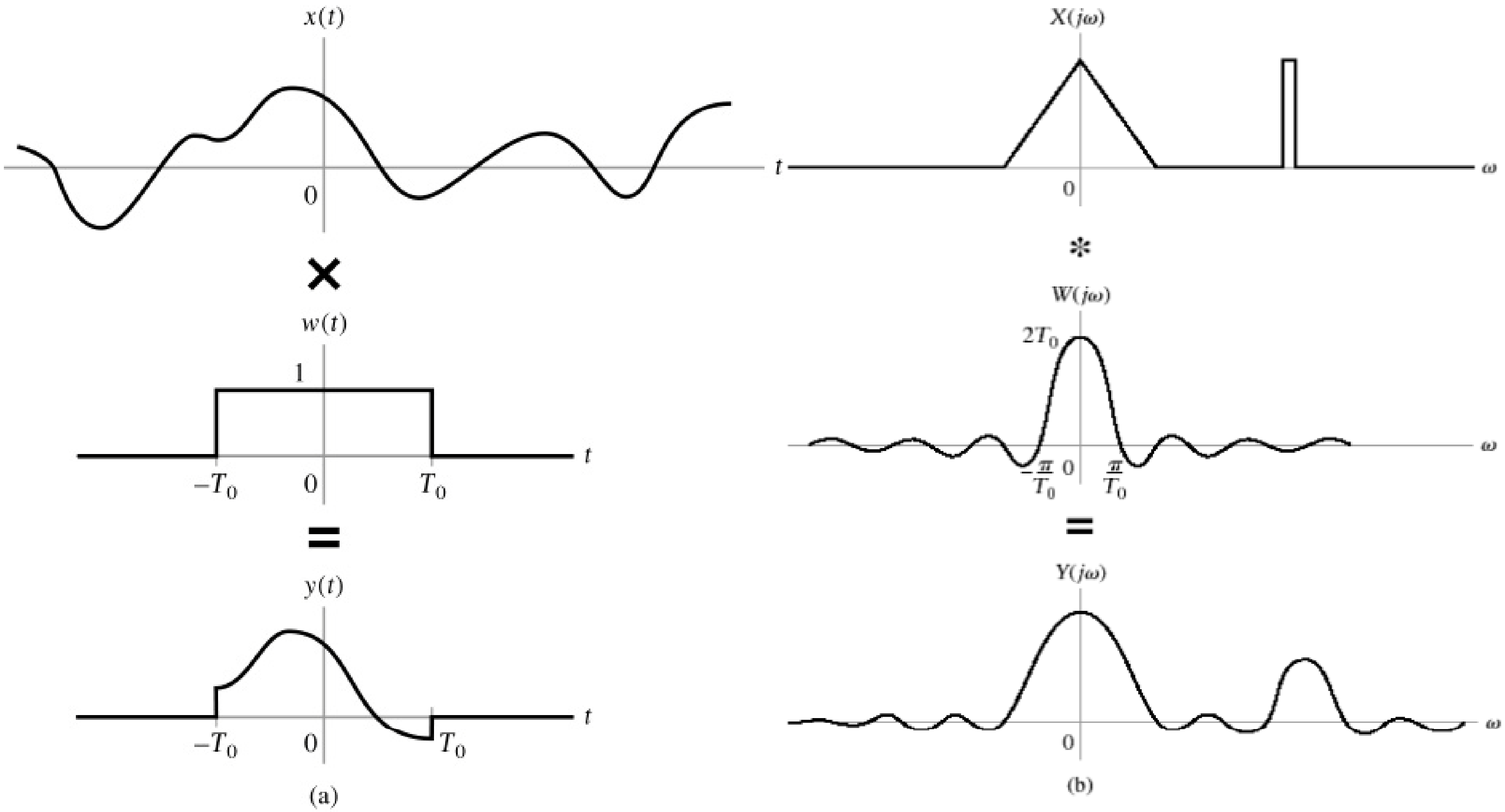
$$W(j\omega) = \frac{2}{\omega}\sin(\omega T_0)$$

$$y(t) \xleftrightarrow{FT} Y(j\omega) = \frac{1}{2\pi} X(j\omega) * W(j\omega)$$



(a)

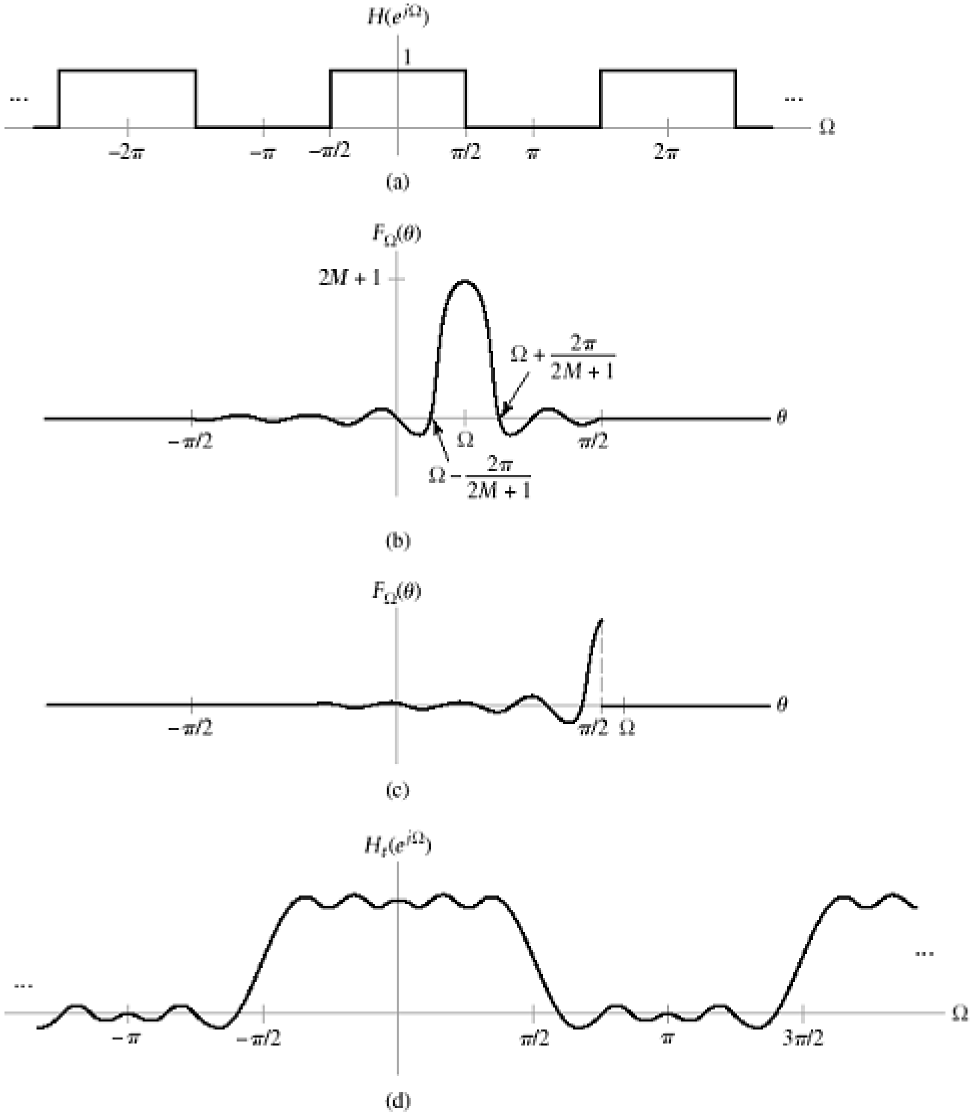
Multiplication Property



The effect of windowing.
(a) Truncating a signal in time by using a window function $w(t)$.

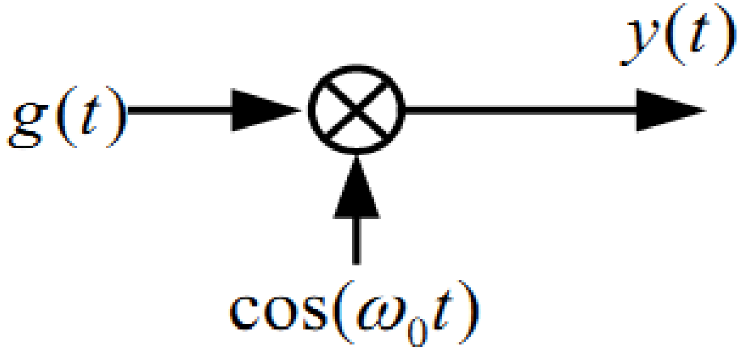
(b) Convolution of the signal and window FT's resulting from truncation in time.

$$x(n)z(n) \xleftrightarrow{DTFT} \frac{1}{2\pi}\left[X(e^{j\Omega}) \otimes Z(e^{j\Omega})\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Z(e^{j(\Omega-\theta)})d\theta$$



The effect of truncating the impulse response of a discrete-time system.
(a) Frequency response of ideal system. (b) $F_\Omega(\theta)$ for Ω near zero. (c) $F_\Omega(\theta)$ for Ω slightly greater than $\pi/2$.
(d) Frequency response of system with truncated impulse response.

Multiplication Property: Amplitude modulation



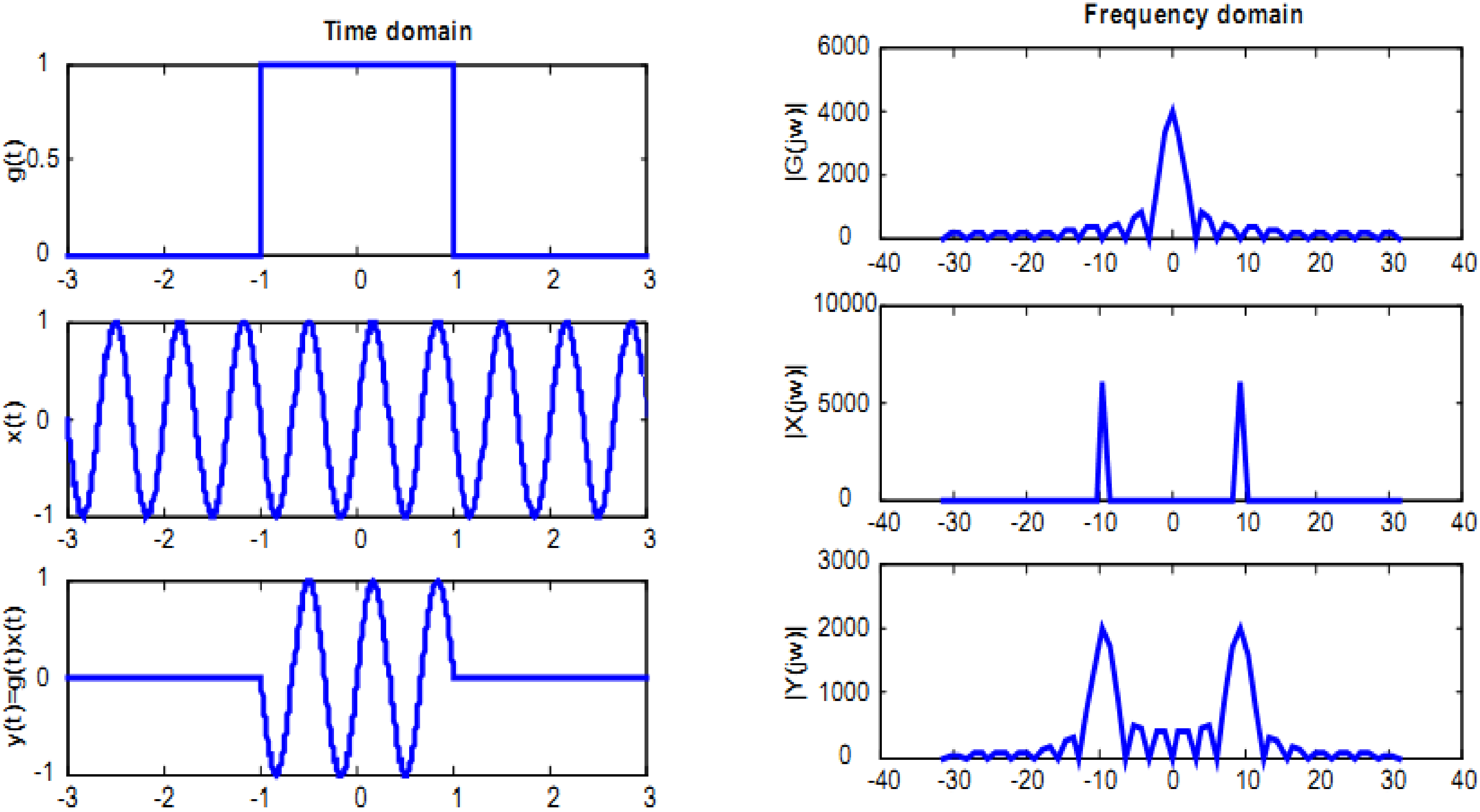
$$x(t) = \cos(\omega_0 t), \quad g(t) = \begin{cases} 1, & |t| \le 1 \\ 0, & |t| > 1 \end{cases} \quad \text{, Find } Y(j\omega)$$

$$y(t) = g(t)\cos(\omega_0 t)$$

$$Y(j\omega) = \frac{1}{2\pi}\left[G(j\omega) * \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]\right]$$

$$= \frac{1}{2}G\left[j(\omega+\omega_0)\right] + \frac{1}{2}G\left[j(\omega-\omega_0)\right]$$

Multiplication Property: Amplitude modulation



Multiplication Properties of Fourier Representations

Table 3.9 Multiplication Properties of Fourier Representations

$$x(t)z(t) \xleftrightarrow{FT} \frac{1}{2\pi} X(j\omega) * Z(j\omega)$$

$$x(t)z(t) \xleftrightarrow{FS;\quad \omega_o} X[k] * Z[k]$$

$$x[n]z[n] \xleftrightarrow{DTFT} \frac{1}{2\pi} X(e^{j\Omega}) \circledast Z(e^{j\Omega})$$

$$x[n]z[n] \xleftrightarrow{DTFT;\, \omega_o} X[k] \circledast Z[k]$$

Scaling Property

***z*(*t*) = *x*(*at*)**

$$Z(j\omega)=\int_{-\infty}^{\infty} z(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt$$

$$Z(j\omega)=\begin{cases}(1/a)\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a>0\\(1/a)\int_{\infty}^{-\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a<0\end{cases}$$

Changing variable: $\tau = at$

$$Z(j\omega)=(1/|a|)\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau$$

$$z(t)=x(at) \quad \xleftrightarrow{FT} \quad (1/|a|)X(j\omega/a).$$

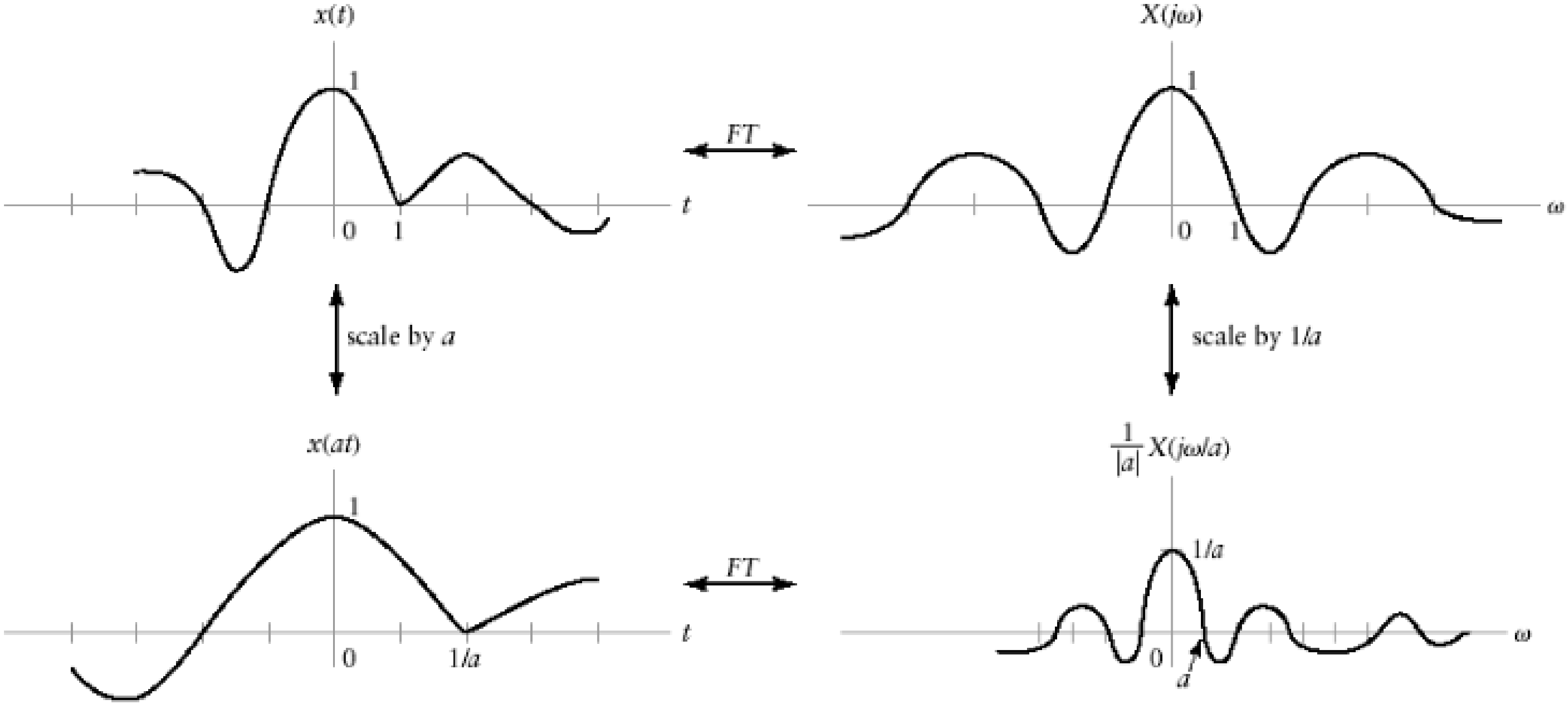
◆ **Scaling in Time-Domain $\leftrightarrow$ Inverse Scaling in Frequency-Domain**

Signal expansion or compression!

Scaling Property

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

$$x(at) \xleftrightarrow{FT} \frac{1}{|a|} X(j\frac{\omega}{a})$$



The FT scaling property. The figure assumes that $0 < a < 1$.

Scaling Property: a Rectangular Pulse

$$z(t) = x(at) \xleftrightarrow{FT} \left(\frac{1}{|a|}\right) X\left(\frac{j\omega}{a}\right)$$

$$x(t) = \begin{cases} 1, & |t| < 1 \\ 0, & |t| > 1 \end{cases}$$

$$\Leftrightarrow X(j\omega) = \frac{2}{\omega}\sin(\omega)$$

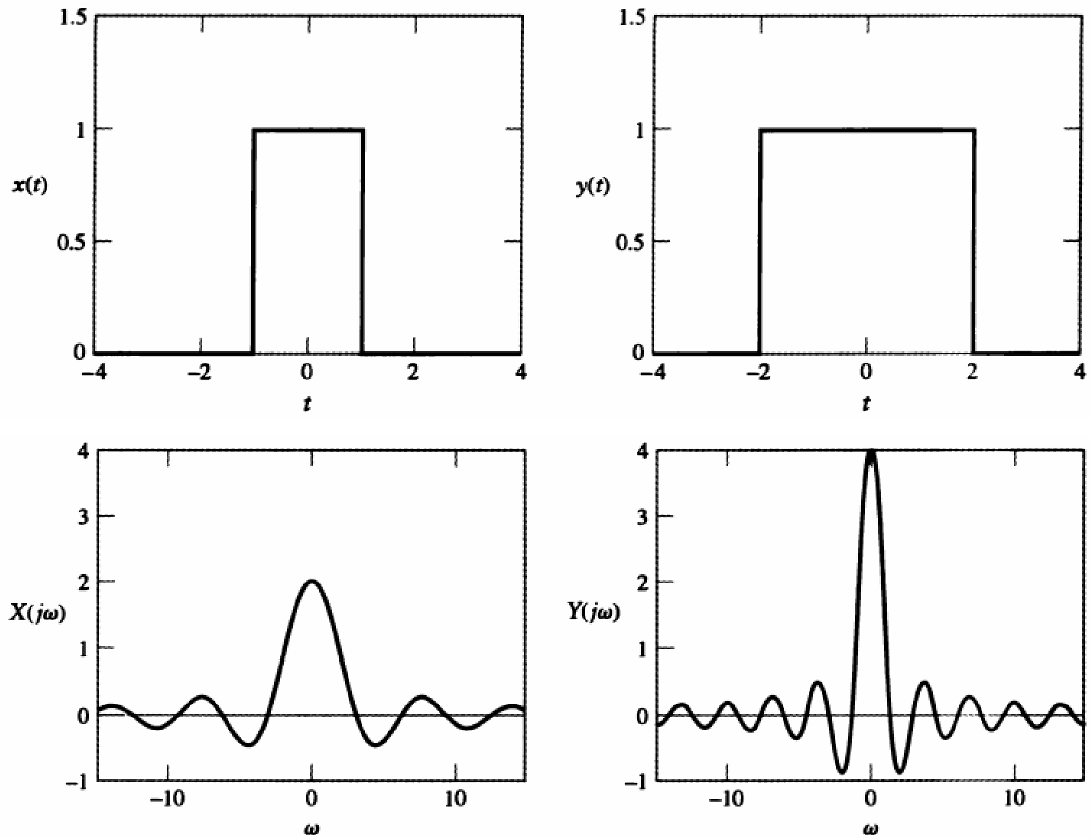
$$y(t) = \begin{cases} 1, & |t| < 2 \\ 0, & |t| > 2 \end{cases}$$

$$y(t) = x(t/2)$$

$$Y(j\omega) = 2X(j2\omega)$$

$$= 2\left(\frac{2}{2\omega}\right)\sin(2\omega)$$

$$= \frac{2}{\omega}\sin(2\omega)$$



Parseval Relationships

♣ **The energy or power in the time-domain representation of a signal is equal to the energy or power in the frequency-domain representation.**

FT:

$$W_x = \int_{-\infty}^{\infty} \left|x(t)\right|^2 \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\mathrm{j}\omega)|^2 \,\mathrm{d}\omega$$

DTFT:

$$W_x = \sum_{n=-\infty}^{\infty} \left|x(n)\right|^2 = \frac{1}{2\pi}\int_{<2\pi>} |X(e^{\mathrm{j}\Omega})|^2 \,\mathrm{d}\Omega$$

Parseval Relationships for CT nonperiodic signal

Energy in $x(t)$

$$W_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)x^*(t)dt$$

$$|x(t)|^2 = x(t)x^*(t)$$

$$W_x = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega\right]dt$$

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}d\omega$$

$$W_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\left\{\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right\}d\omega$$

$$W_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)X(j\omega)d\omega$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

◆ **Energy in Time-Domain Representation ↔ Energy in Frequency-Domain Representation × (1/2π)**

Parseval Relationships

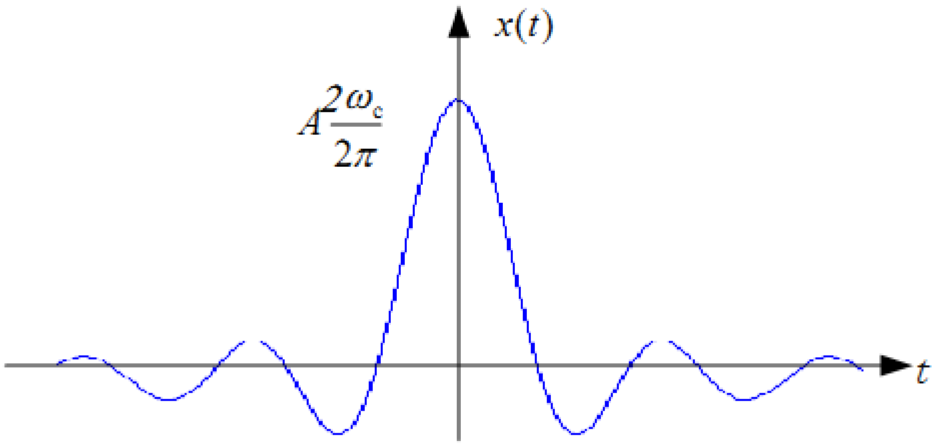
- The Parseval Relationships of all four Fourier representations are summarized in Table 3.10.

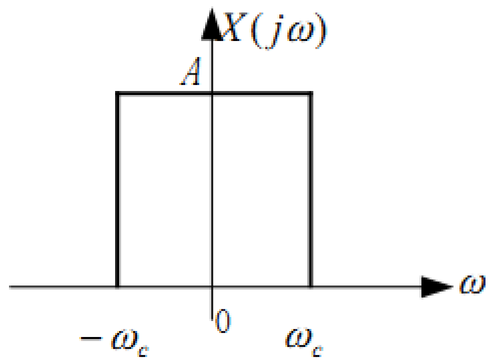
Representations	Parseval Relationships
FT	$\int_{-\infty}^{\infty} \lvert x(t) \rvert^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(j\omega) \rvert^2 d\omega$
FS	$\frac{1}{T} \int_0^T \lvert x(t) \rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert X[k] \rvert^2$
DTFT	$\sum_{n=-\infty}^{\infty} \lvert x[n] \rvert^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \lvert X(e^{j\Omega}) \rvert^2 d\Omega$
DTFS	$\frac{1}{N} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2 = \sum_{k=0}^{N-1} \lvert X[k] \rvert^2$

Example:

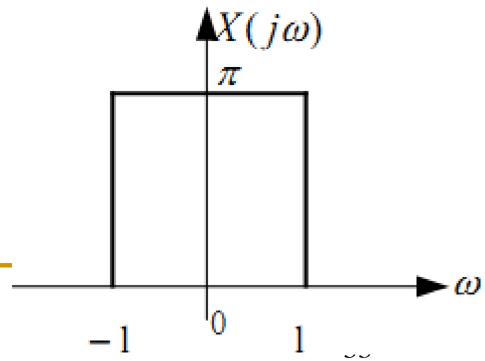
Evaluate $\int_{-\infty}^{\infty} (\frac{\sin t}{t})^2 \mathrm{d}t$

$$x(t) = \frac{\sin t}{t} = s_a(t) = \frac{A2\omega_c}{2\pi} s_a(\omega_c t) \qquad \begin{cases} A = \pi \\ \omega_c = 1 \end{cases}$$





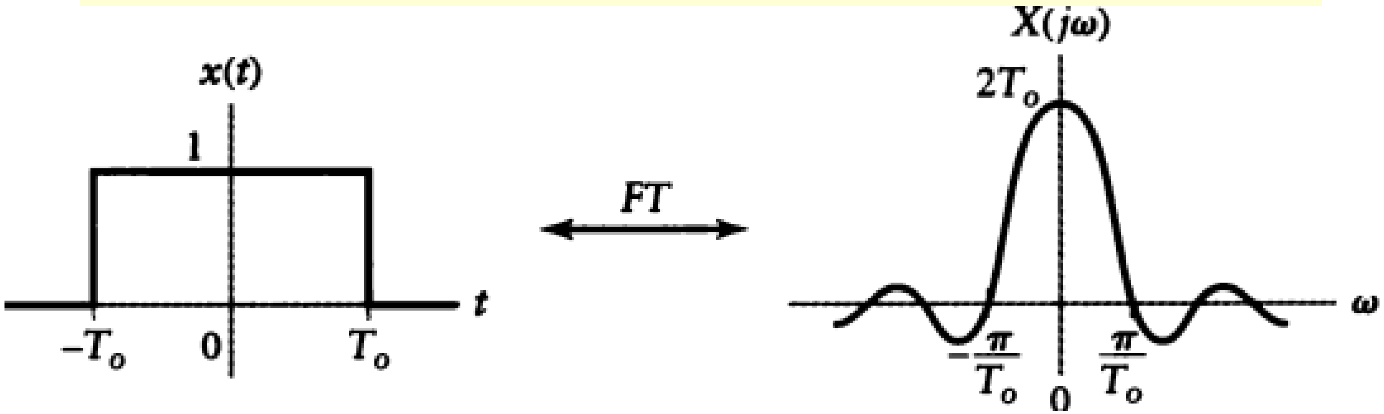
$$\int_{-\infty}^{\infty} (\frac{\sin t}{t})^2 \mathrm{d}t = \frac{1}{2\pi} \int_{-1}^{1} \pi^2 \mathrm{d}\omega = \pi$$



Time-Bandwidth Product

$$x(t) = \begin{cases} 1, & |t| \le T_o \\ 0, & |t| > T_o \end{cases} \quad \xleftrightarrow{FT} \quad X(j\omega) = 2\sin(\omega T_o)/\omega$$

$$2T_0\left(\pi / T_0\right) = 2\pi$$



Compressing a signal in time leads to expansion in frequency and vice versa.

1. Effective duration of a signal *x*(*t*)

$$T_d = \left[\frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 \, dt}{\int_{-\infty}^{\infty} |x(t)|^2 \, dt} \right]^{1/2}$$

2. Effective bandwidth of a signal *x*(*t*)

$$B_w = \left[\frac{\int_{-\infty}^{\infty} \omega^2 |X(j\omega)|^2 \, d\omega}{\int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega} \right]^{1/2}$$

The time-bandwidth product for any signal is lower bounded according to the relationship

$$T_d B_w \ge 1/2$$

Time-Bandwidth Product

Example 3.51 ***Bounding the Bandwidth of a Rectangular Pulse***

$$x(t) = \begin{cases} 1, & |t| \le T_o \\ 0, & |t| > T_o \end{cases}$$

- **Use the uncertainty principle to place a lower bound on the effective bandwidth of $x(t)$.**

<Sol.>

1. T_d of $x(t)$:

$$T_d = \left[\frac{\int_{-T_o}^{T_o} t^2 dt}{\int_{-T_o}^{T_o} dt} \right]^{1/2} \Rightarrow T_d = \left[\frac{\int_{-T_o}^{T_o} t^2 dt}{\int_{-T_o}^{T_o} dt} \right]^{1/2} = [(1/(2T_o))(1/3)t^3 \Big|_{-T_o}^{T_o}]^{1/2} = T_d / \sqrt{3}$$

2. According to the uncertainty principle

$$B_w \ge 1/(2T_d) \Rightarrow B_w \ge \sqrt{3}/(2T_o)$$

Duality Property of FT

Difference in the factor 2π and the sign change in the complex sinusoid.

1. FT pair:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

2. General equation:

$$y(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} z(\eta)e^{jv\eta}dv$$

Choose $\nu = t$ and $\eta = \omega$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} z(\omega)e^{j\omega t}d\omega \Rightarrow y(t) \xleftrightarrow{FT} z(\omega)$$

Let $\nu = -\omega$ and $\eta = t$ **Interchange the role of time and frequency**

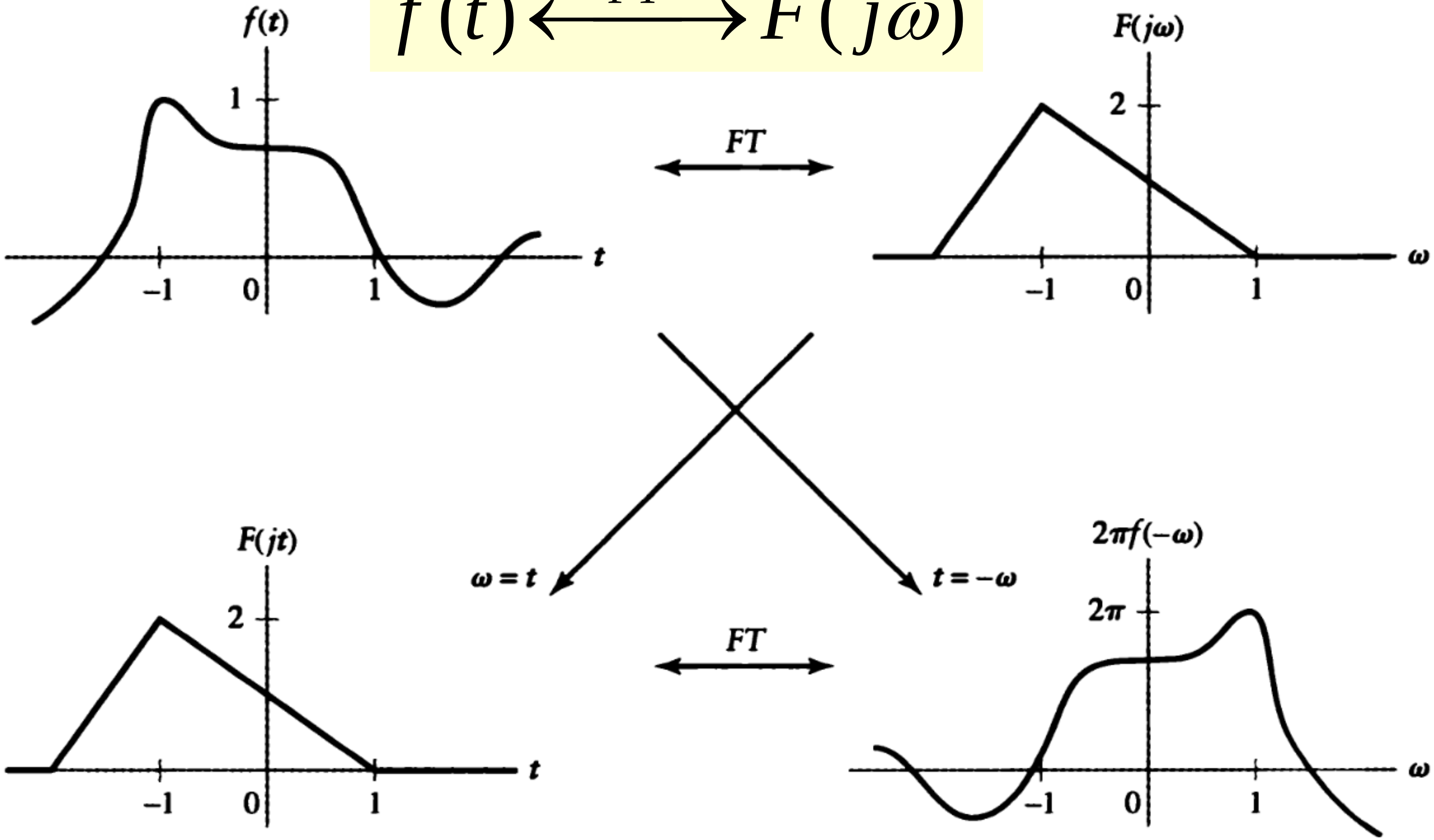
$$y(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} z(t)e^{-j\omega t}dt \Rightarrow z(t) \xleftrightarrow{FT} 2\pi y(-\omega)$$

3. A new FT pair

$$y(t) \xleftrightarrow{FT} z(j\omega) \Rightarrow z(jt) \xleftrightarrow{FT} 2\pi y(-\omega)$$

Duality Property of FT

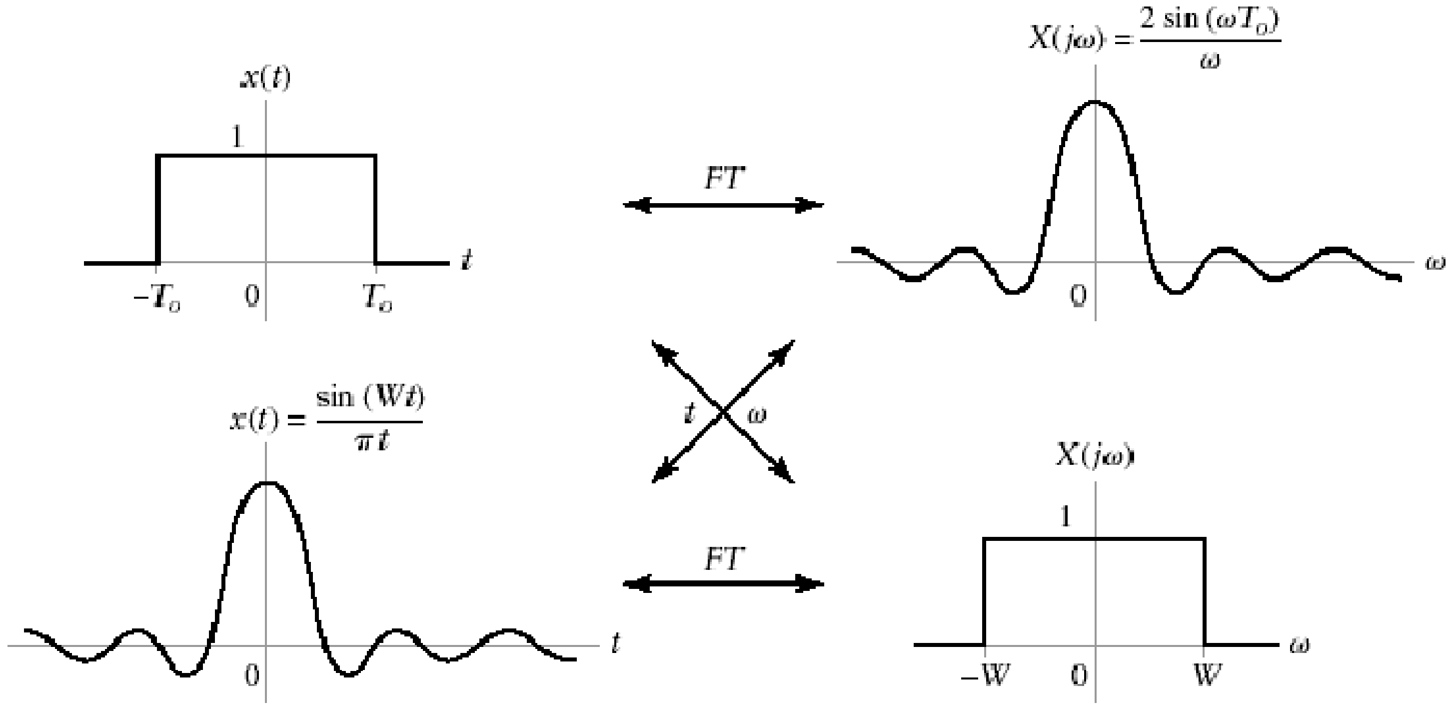
$$f(t) \xleftrightarrow{FT} F(j\omega)$$



$$F(jt) \xleftrightarrow{FT} 2\pi f(-\omega)$$

Duality Property of Rectangular pulses and sinc functions

$$x(t) \xleftrightarrow{FT} X(j\omega) \qquad X(jt) \xleftrightarrow{FT} 2\pi x(-\omega)$$



FT	$f(t)$	$\xleftrightarrow{FT}$	$F(j\omega)$	$F(jt)$	$\xleftrightarrow{FT}$	$2\pi f(-\omega)$
DTFS	$x[n]$	$\xleftrightarrow{DTFS;\,2\pi/N}$	$X[k]$	$X[n]$	$\xleftrightarrow{DTFS;\,2\pi/N}$	$(1/N)x[-k]$
FS-DTFT	$x[n]$	$\xleftrightarrow{DTFS}$	$X(e^{j\Omega})$	$X(e^{jt})$	$\xleftrightarrow{FS;1}$	$x[-k]$

Summary and Exercises

- Summary and Exercises
  - Complex Sinusoids and Frequency Response of LTI Systems
  - Fourier Representations for Four classes of Signals
  - Properties of Fourier Representations
- Exercises (**P322-333**)
  - **3.66(a-d), 3.67(a-e), 3.68(a), 3.69(b), 3.73(a, c), 3.74(a, c), 3.76(a, b)**

FT pairs

$\delta[n] \xleftrightarrow{DTFT} 1$

$\delta(t) \xleftrightarrow{FT} 1$

$1 \xleftrightarrow{FT} 2\pi\delta(\omega)$

$1 \xleftrightarrow{DTFT} 2\pi\delta(\Omega) \quad -\pi < \Omega \le \pi$

$e^{j\omega_0 t} \xleftrightarrow{FT} 2\pi\delta(\omega-\omega_0)$

$\cos(\omega_0 t) \xleftrightarrow{FT} \pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$

$\sin(\omega_0 t) \xleftrightarrow{FT} -j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$

$\alpha^n u[n] \xleftrightarrow{DTFT} \dfrac{1}{1-\alpha e^{-j\Omega}}$

$|\alpha|<1$

$\alpha > 0$

$e^{-\alpha t}u(t) \xleftrightarrow{FT} \dfrac{1}{\alpha + j\omega}$

$te^{-\alpha t}u(t) \xleftrightarrow{FT} \dfrac{1}{(\alpha + j\omega)^2}$

$u(t) \xleftrightarrow{FT} \dfrac{1}{j\omega} + \pi\delta(\omega)$

$\mathrm{sgn}(t) \xleftrightarrow{FT} \dfrac{2}{j\omega}$

$A\,rect\left(\dfrac{t}{\tau}\right) \xleftrightarrow{FT} A\tau \sin c\left(\dfrac{\omega\tau}{2}\right)$

幅度A，宽度τ，关于y轴对称的矩形脉冲

任意周期信号 $f_0 = 1/T$

$$\sum_{n=-\infty}^{\infty} F_n e^{j2\pi n f_0 t} \xleftrightarrow{FT} \sum_{n=-\infty}^{\infty} F_n \delta(f - nf_0)$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \xleftrightarrow{FT} f_0 \sum_{n=-\infty}^{\infty} \delta(f-nf_0) = \sum_{m=-\infty}^{\infty} e^{-j2\pi fmT}$$