Signals and Systems 3.4

--- *Frequency Representations of LTI systems*

School of Information & Communication Engineering，BUPT

Reference:
1. Textbook: Chapter 3

Frequency Representations of LTI systems

- Frequency Response of LTI systems
- Filtering
- Representations and Solutions of LTI systems in frequency domain
- Conditions of Distortionless Transmission

Output of LTI System in frequency domain

$$e^{j\omega t} \longrightarrow \boxed{\textbf{CT}} \longrightarrow H(j\omega)e^{j\omega t}$$

$$x(t) = \sum_{k=1}^{M} a_k e^{j\omega_k t} \qquad y(t) = \sum_{k=1}^{M} a_k H(j\omega) e^{j\omega_k t}$$

$$e^{j\Omega n} \longrightarrow \boxed{\textbf{DT}} \longrightarrow H\left(e^{j\Omega}\right)e^{j\Omega n}$$

$$x[n] = \sum_{k=1}^{M} a_k e^{j\Omega_k n} \qquad y[n] = \sum_{k=1}^{M} a_k H\left(e^{j\Omega}\right) e^{j\Omega_k n}$$

Convolution operation $\Rightarrow$ Multiplication

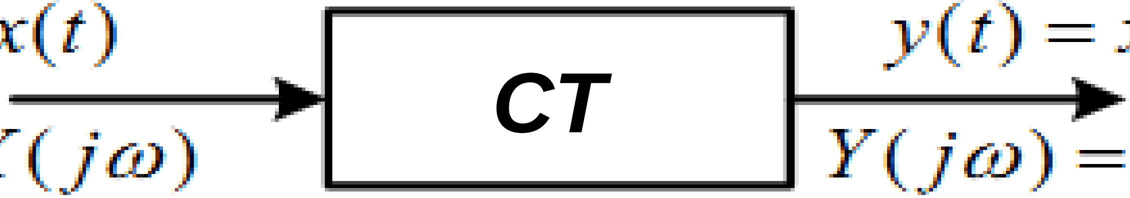
$$\underset{X(j\omega)}{x(t)} \longrightarrow \boxed{\textbf{CT}} \longrightarrow \underset{Y(j\omega) = X(j\omega)H(j\omega)}{y(t) = x(t) * h(t)}$$

$$\underset{X(e^{j\Omega})}{x[n]} \longrightarrow \boxed{\textbf{DT}} \longrightarrow \underset{Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega})}{y[n] = x[n] * h[n]}$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt \qquad H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$$

Transfer function of LTI system

$x(t)$, $X(j\omega)$



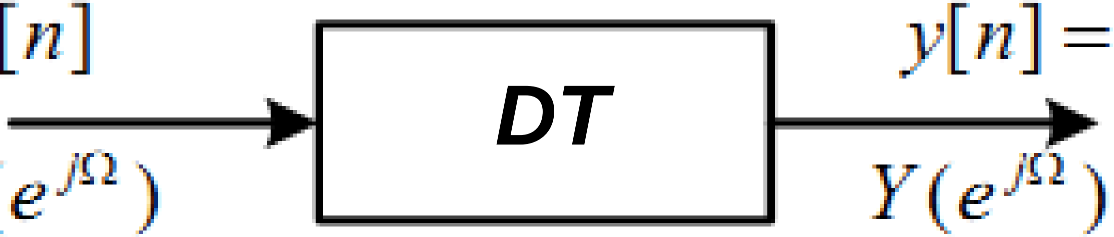
$y(t) = x(t) * h(t)$

$Y(j\omega) = X(j\omega)H(j\omega)$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$x[n]$, $X(e^{j\Omega})$



$y[n] = x[n] * h[n]$

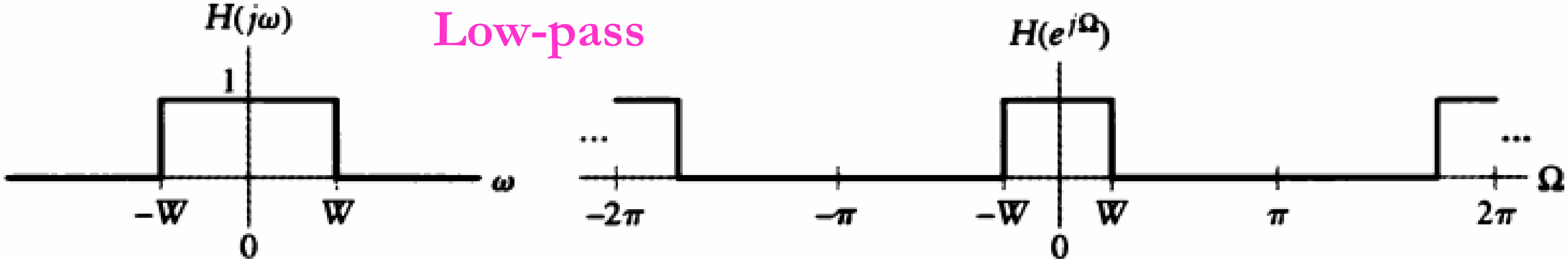
$Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega})$

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$$

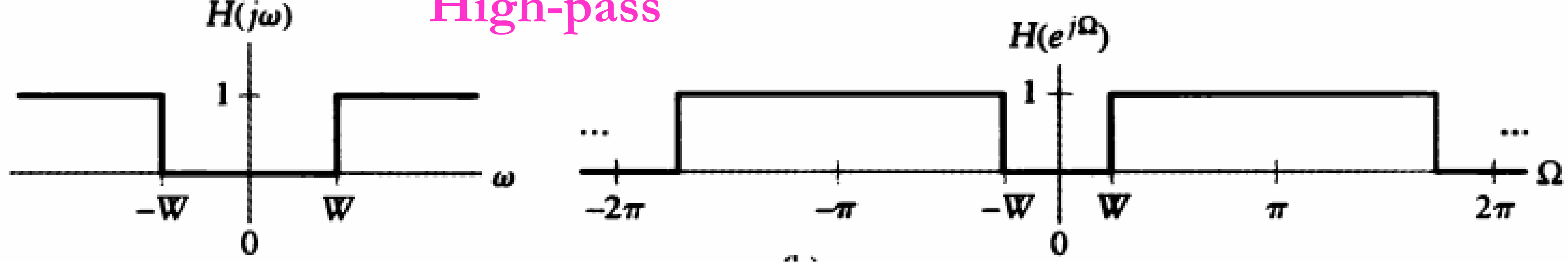
$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})}$$

Filtering: some frequency are eliminated while others are passed

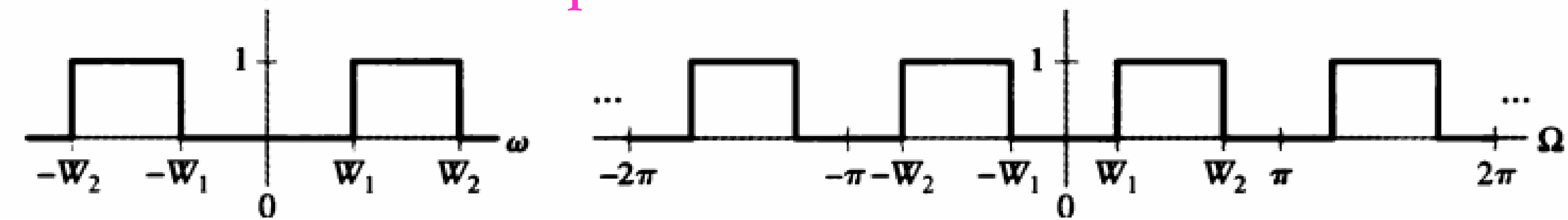
Low-pass



High-pass



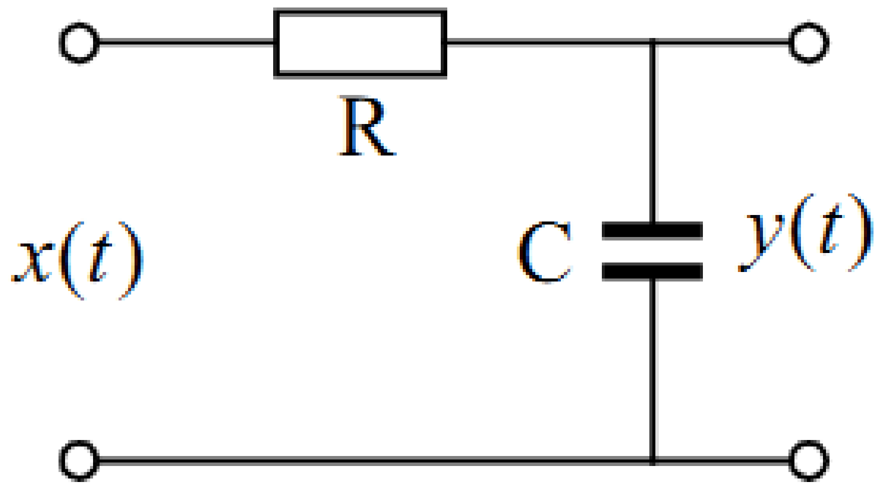
Band-pass



Example: RC circuit (Filtering)

$$h(t) = \frac{1}{\mathrm{RC}} \mathrm{e}^{-\frac{t}{\mathrm{RC}}} u(t)$$

$$H(\mathrm{j}\omega) = \frac{\frac{1}{\mathrm{RC}}}{\mathrm{j}\omega + \frac{1}{\mathrm{RC}}}$$



Applications

- Find the output of the system, given its input and impulse response.

$$x(t) = 3e^{-t}u(t) \Leftrightarrow X(j\omega) = \frac{3}{1 + j\omega}$$

$$h(t) = 2e^{-2t}u(t) \Leftrightarrow H(j\omega) = \frac{2}{2 + j\omega}$$

$$Y(j\omega) = X(j\omega)H(j\omega)$$

$$= \frac{2}{(2 + j\omega)}\frac{3}{(1 + j\omega)}$$

$$y(t) = \tfrac{1}{6}e^{-t}u(t) - \tfrac{1}{6}e^{-2t}u(t)$$

- Identifying a system, given its input and output.

$$x(t) = e^{-2t}u(t) \Leftrightarrow X(j\omega) = \frac{1}{2 + j\omega}$$

$$y(t) = e^{-t}u(t) \Leftrightarrow Y(j\omega) = \frac{1}{1 + j\omega}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$= 1 + \frac{1}{(1 + j\omega)}$$

$$h(t) = \delta(t) + e^{-t}u(t)$$

Representations of LTI Systems in Frequency Domain

$$x(t) \longrightarrow \boxed{\mathbf{CT}} \longrightarrow y(t)$$

$$\sum_{k=0}^{\mathrm{N}} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{\mathrm{M}} b_k \frac{d^k x(t)}{dt^k}$$

$$x[n] \longrightarrow \boxed{\mathbf{DT}} \longrightarrow y[n]$$

$$\sum_{k=0}^{\mathrm{N}} a_i y[n-k] = \sum_{k=0}^{\mathrm{M}} b_j x[n-k]$$

Example1: solutions of the CT LTI system

$$y''(t)+3y'(t)+2y(t)=x'(t)+4x(t) \qquad x(t)=e^{-3t}u(t)$$

Find the impulse response h(t) and output y(t).

$$(j\omega)^2Y(j\omega)+3j\omega Y(j\omega)+2Y(j\omega)=j\omega X(j\omega)+4X(j\omega)$$

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{4+j\omega}{(j\omega)^2+3(j\omega)+2}=\frac{-2}{j\omega+2}+\frac{3}{j\omega+1}$$

$$h(t)=-2e^{-2t}u(t)+3e^{-t}u(t)$$

$$Y(j\omega)=X(j\omega)H(j\omega)=\frac{1}{j\omega+3}\bullet\frac{j\omega+4}{(j\omega+2)(j\omega+1)}$$

$$=\frac{1/2}{3+j\omega}-\frac{2}{2+j\omega}+\frac{3/2}{1+j\omega}$$

$$y(t)=\left[\tfrac{1}{2}e^{-3t}-2e^{-2t}+\tfrac{3}{2}e^{-t}\right]u(t)$$

Example2: solutions of the DT LTI system

$$y[n] = x[n] + x[n\text{-}1] + x[n\text{-}2]$$

Find the impulse response h(t) and H(e$^{j\Omega}$).

$$(1 + e^{j\Omega} + e^{j2\Omega})X(e^{j\Omega}) = Y(e^{j\Omega})$$

$$H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = 1 + e^{j\Omega} + e^{j2\Omega}$$

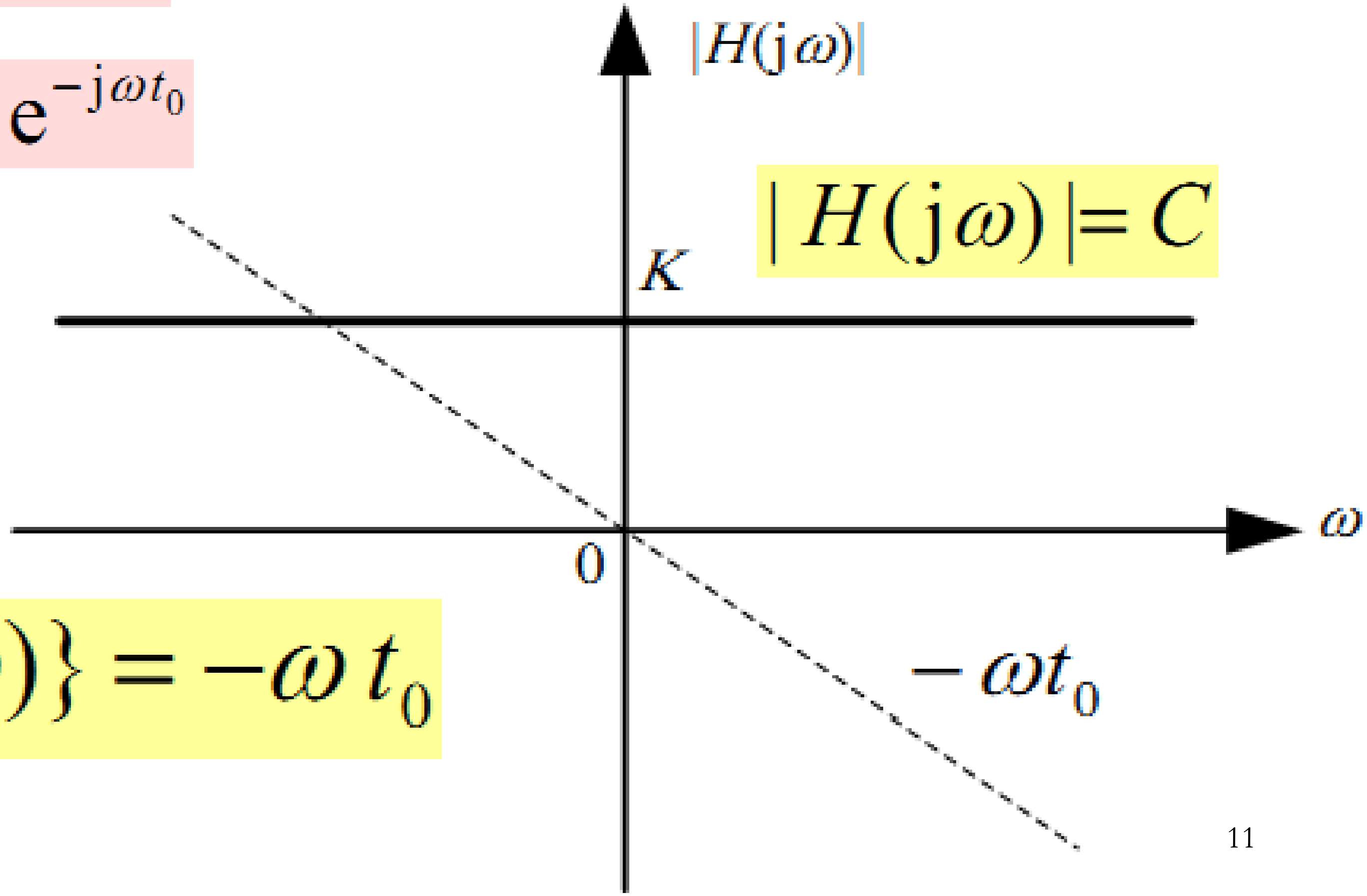
$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$$

Conditions of Distorionless Transmission

$$y(t) = Cx(t - t_0)$$

$$h(t) = C \cdot \delta(t - t_0)$$

$$H(j\omega) = C \cdot e^{-j\omega t_0}$$



$$\arg\{H(j\omega)\} = -\omega t_0$$