# Signals and Systems 7.1

*--- Representing Signals by Using Discrete-Time Complex Exponentials: The z-Transform*

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Reference:
1. Textbook: Chapter 7.1~7.7

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## *Outline*

- z-Transform
  - Introduction
  - The z-Transform
  - Properties of the z-Transform
  - Inversion of the z-Transform
  - The Transfer Function
  - Causality and Stability

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## Introduction

- The z-Transform is a more general discrete-time signal and system representation based on complex exponential signals.
  - To study a much broader class of discrete-time LTI systems and signals, e.g. the impulse response for unstable LTI systems.
- Main usage
  - study of system characteristics and the derivation of computational structures for implementing discrete-time system on computers.
  - to solve difference equations subject to initial conditions.

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## *From DTFT to z-Transform*

$$F\{x[n]r^{-n}\} = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n](re^{j\Omega})^{-n}$$

$$\xrightarrow{z=re^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \triangleq X(z)$$

- The z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad \text{or} \quad X(z) = Z\{x[n]\}$$

- The inverse z-Transform

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz \quad \text{or} \quad x[n] = Z^{-1}\{X(z)\}$$

Integration around a circle of radius $|z|=r$ in a counter-clockwise direction represents $x[n]$ as a weighted superposition of complex exponentials $z^n$.
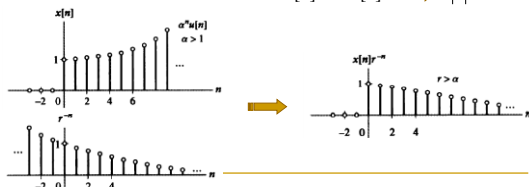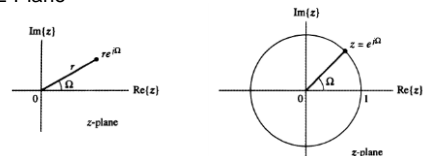
$$x[n] \xleftrightarrow{z} X(z)$$

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## *Convergence*

- necessary condition for convergence: absolutely summability of $x[n]z^{-n}$.

$$\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty \implies \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

- **Region of convergence(ROC):** the range of $r$ which the z-Transform converges.

Ex. $x[n]=\alpha^n u[n] \implies |z| = r > a.$



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## *Relations between the z-Transform and DTFT*

- The z-Plane



- If $x[n]$ is absolutely summable, the DTFT is obtained from the z-transform by setting $r = 1$, i.e.

$$X(e^{j\Omega}) = X(z)\big|_{z=e^{j\Omega}}$$

- If ROC does not include the unit circle, z-Transform exists while DTFT is nonexistent.

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## Poles and Zeros

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\tilde{b} \prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}, \quad \tilde{b} = \frac{b_0}{a_0}.$$

- Zeros of $X(z)$: the roots of the numerator polynomial $c_k$. **"o"**

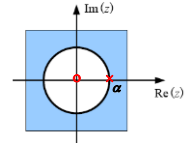- Poles of $X(z)$: the roots of the denominator polynomial $d_k$. **" $\times$ "**

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## z-Transform of Signals

**Example 7.2** Find the z-Transform of $x_1[n] = \alpha^n u[n]$ and $x_2[n] = -\alpha^n u[-n-1]$.

**<Sol.>** $X_1(z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$

$$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1 - \alpha z^{-1}}$$
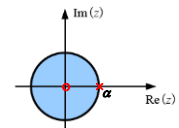
**ROC:** $|z| > \alpha$



$$X_2(z) = \sum_{n=-\infty}^{\infty} -\alpha^n u[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} (\alpha z^{-1})^{-n} = 1 - \sum_{n=0}^{\infty} (\alpha^{-1} z)^n$$

$$= 1 - \frac{1}{1 - \alpha^{-1} z} = \frac{1}{1 - \alpha z^{-1}}$$

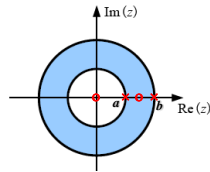**ROC:** $|z| < |\alpha|$

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## z-Transform of Signals

**Example 7.4** Find the z-Transform of $x[n] = a^n u[n] - b^n u[-n-1]$.

**<Sol.>** $X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} -b^n u[-n-1] z^{-n}$

$$= \frac{1}{1 - az^{-1}} + \frac{1}{1 - bz^{-1}}$$

$$= \frac{2 \left[ 1 - \frac{a+b}{2} z^{-1} \right]}{(1 - az^{-1})(1 - bz^{-1})}$$

**ROC:** $|a| < |z| < |b|$



The z-Transform only exists when $|b| > |a|$.

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## z-Transform for Elementary Signals

$$\delta[n] \overset{z}{\longleftrightarrow} 1, \quad \text{All } z$$

$$u[n] \overset{z}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

$$nu[n] \overset{z}{\longleftrightarrow} \frac{z^{-1}}{(1 - z^{-1})^2}, \quad |z| > |a|$$

$$\alpha^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1 - \alpha z^{-1}}, \quad |z| > |a|$$

$$n\alpha^n u[n] \overset{z}{\longleftrightarrow} \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}, \quad |z| > |a|$$

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## Properties of z-Transform

$x[n] \overset{z}{\longleftrightarrow} X(z)$ with ROC $R_x$ $\qquad y[n] \overset{z}{\longleftrightarrow} Y(z)$ with ROC $R_y$

- **Linearity**

$ax[n] + by[n] \overset{z}{\longleftrightarrow} aX(z) + bY(z)$ with ROC at least $R_x \cap R_y$.

- **Time shift**

$x[n - n_0] \overset{z}{\longleftrightarrow} z^{-n_0} X(z)$ with ROC $R_x$, except possibly $z = 0$ or $|z| = \infty$.

**Ex.** Find the z-Transform of $x[n] = u[n] - u[n-5]$.

$$X(z) = \frac{1}{1 - z^{-1}} + \frac{z^{-5}}{1 - z^{-1}} = \frac{1 - z^{-5}}{1 - z^{-1}}, \quad |z| > 0$$

**Ex.** If $X(z) = \frac{1}{z - a}$, $|z| > a$, determine $x[n]$.

$$X(z) = z^{-1} \frac{1}{1 - az^{-1}} \quad \Longrightarrow \quad x[n] = a^{n-1} u[n-1]$$

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## Properties of z-Transform

- **Convolution**

$x[n] * y[n] \overset{z}{\longleftrightarrow} X(z) Y(z)$ with ROC at least $R_x \cap R_y$.

**Ex.** $Z\left\{ \sum_{k=0}^{n} x[k] \right\} = Z\{x[n] * u[n]\} = \frac{X(z)}{1 - z^{-1}}$

- **Multiplication by an exponential sequence**

$$a^n x[n] \overset{z}{\longleftrightarrow} X\left(\frac{z}{a}\right) \quad \text{with ROC } |a| R_x.$$

- **Differential in the z-Domain**

$$nx[n] \overset{z}{\longleftrightarrow} -z \frac{d}{dz} X(z) \quad \text{with ROC } R_x.$$

**Ex.** $u[n] \overset{z}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, \quad |z| > 1$

$$\Longrightarrow nu[n] \overset{z}{\longleftrightarrow} -z \frac{d}{dz}\left(\frac{1}{1 - z^{-1}}\right) = \frac{1}{(1 - z^{-1})^2}$$

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## Inversion of z-Transform

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

- Direct inversion by residue computation
- Power series expansion
- Inversion by partial-fraction expansion (部分分式法)

$$X(z) = \frac{b_M z^{-M} + \cdots + b_1 z^{-1} + b_0}{a_N z^{-N} + \cdots a_1 z^{-1} + a_0} \quad (M \geq N)$$

$$= c_0 + c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)} + \frac{D(z)}{A(z)} \implies$$

$$x[n] = c_0\delta[n] + c_1\delta[n-1] + \cdots + c_{M-N}\delta[n-(M-N)] + Z^{-1}\left[\frac{D(z)}{A(z)}\right]$$

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## Inversion by Partial-fraction Expansions

$$\frac{D(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_P z^{-P}}{a_0 \prod_{k=1}^{N}(1-d_k z^{-1})} = \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}} \quad (P < N)$$

where $A_k = (1-d_k z^{-1})\dfrac{D(z)}{A(z)}\bigg|_{z=d_k}$

$$A_k(d_k)^n u[n] \xleftrightarrow{z} \frac{A_k}{1-d_k z^{-1}} \quad \text{with ROC } |z| > d_k.$$

$$\text{or} \quad -A_k(d_k)^n u[-n-1] \xleftrightarrow{z} \frac{A_k}{1-d_k z^{-1}} \quad \text{with ROC } |z| < d_k.$$

$$A_k(d_k)^n nu[n] \xleftrightarrow{z} \frac{A_k}{(1-d_k z^{-1})^2} \quad \text{with ROC } |z| > d_k.$$

$$\text{or} \quad -A_k(d_k)^n(n+1)u[-n-1] \xleftrightarrow{z} \frac{A_k}{(1-d_k z^{-1})^2} \quad \text{with ROC } |z| < d_k.$$

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## Inversion by Partial-fraction Expansions

**Ex.** Find the inverse z-Transform of $X(z) = \dfrac{1}{(1-2z^{-1})(1-3z^{-1})}$.

**<Sol.>** $X(z) = \dfrac{A_1}{1-2z^{-1}} + \dfrac{A_2}{1-3z^{-1}} = \dfrac{-2}{1-2z^{-1}} + \dfrac{3}{1-3z^{-1}}$

$$A_1 = (1-2z^{-1})X(z)\big|_{z=2} = -2$$

$$A_2 = (1-3z^{-1})X(z)\big|_{z=3} = 3$$

- $|z| > 3$: $x[n] = (-2\cdot 2^n + 3\cdot 3^n)u[n] = (-2^{n+1} + 3^{n+1})u[n]$
- $2 < |z| < 3$: $x[n] = -2^{n+1}u[n] - 3^{n+1}u[-n-1]$
- $|z| < 2$: $x[n] = 2^{n+1}u[-n-1] - 3^{n+1}u[-n-1]$

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## Inversion by Partial-fraction Expansions

**Ex.** Find the inverse z-Transform of $X(z) = \dfrac{1}{(1-2z^{-1})^2(1-4z^{-1})}, |z| > 4$.

**<Sol.>** $X(z) = \dfrac{A_1}{1-2z^{-1}} + \dfrac{A_2}{(1-2z^{-1})^2} + \dfrac{A_3}{1-4z^{-1}} = \dfrac{-2}{1-2z^{-1}} + \dfrac{3}{1-3z^{-1}}$

$$A_3 = (1-4z^{-1})X(z)\big|_{z=4} = 4$$

$$A_2 = \frac{1}{0!}(1-2z^{-1})^2 X(z)\big|_{z=2} = -1$$

$$A_1 = \frac{1}{1!}\cdot\frac{d}{dz}\left[(1-2z^{-1})^2 X(z)\right]\big|_{z=2} = -2$$

$$\implies x[n] = \{-2\cdot 2^n - (n+1)2^n + 4\cdot 4^n\}u[n]$$

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## The Transfer Function (系统函数)

- Transfer function: for an LTI system with impulse response $h[n]$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

$$y[n] = h[n] * x[n]$$

$$Y(z) = H(z)X(z) \implies H(z) = \frac{Y(z)}{X(z)}$$

$x[n] \rightarrow \boxed{\begin{array}{c}h[n]\\H(z)\end{array}} \rightarrow y[n]$

- Furthermore, for an input $x[n] = z^n$ to the LTI system

$$y[n] = h[n]*x[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n\sum_{k=-\infty}^{\infty} h[k]z^{-k} = H(z)z^n$$

- Eigenfunction of the system: $z^n$
- Eigenvalue: $H(z)$

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## Transfer Function and Differential Equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

If Initial conditions equal zero, and $x[n] = z^n$

$$z^n\sum_{k=0}^{N} a_k z^{-k}H(z) = z^n\sum_{k=0}^{M} b_k z^{-k}$$

$$\implies H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Transfer function $\iff$ Difference equation system description

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## Transfer Function and Differential Equation

**Ex.** Find the transfer function of the LTI system described by the difference equation

$$y[n]-0.7y[n-1]+0.1y[n-2]=x[n-1], \quad n \ge 0$$

**<Sol.>**

$$\left(1-0.7z^{-1}+0.1z^{-2}\right)Y(z)=z^{-1}X(z) \implies H(z)=\frac{Y(z)}{X(z)}=\frac{z^{-1}}{1-0.7z^{-1}+0.1z^{-2}}$$

**Ex.** Find the impulse response of the LTI system described by the following difference equation

$$y[n]-5y[n-1]+6y[n-2]=x[n-1], \quad n \ge 0$$

**<Sol.>** $\left(1+5z^{-1}+6z^{-2}\right)Y(z)=z^{-1}X(z)$
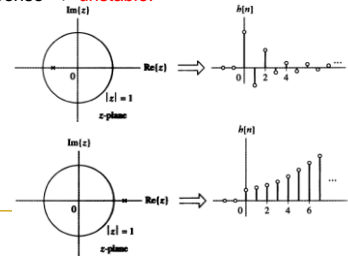
$$\implies H(z)=\frac{Y(z)}{X(z)}=\frac{z^{-1}}{1+5z^{-1}+6z^{-2}}=\frac{1}{1+2z^{-1}}-\frac{1}{1+3z^{-1}}$$

$$\implies h[n]=\left\{(-2)^{n}-(-3)^{n}\right\}u[n]$$

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## Causality and Stability (因果性与稳定性)

- For a causal system: $h[n]=0$ for $n<0$.
  - A pole inside the unit circle in the z-plane ($|d_k| < 1$) corresponds to an exponentially decaying impulse response.
  - A pole outside the unit circle in the z-plane ($|d_k| > 1$) corresponds to an exponentially increasing impulse response --> unstable.



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## Causality and Stability (因果性与稳定性)

- For a stable system: $\sum_{k=-\infty}^{\infty}|h[k]|<\infty.$
  - A pole inside the unit circle in the z-plane *corresponds to a right-sided impulse response.*
  - A pole outside the unit circle in the z-plane corresponds to a left-sided impulse response → noncausal.
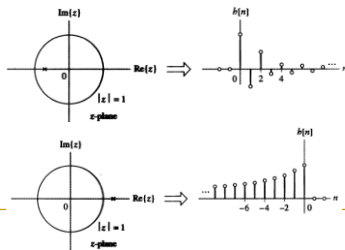


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## Causality and Stability (因果性与稳定性)

- A system that is both stable and causal must have a transfer function with all of its poles inside the unit circle.
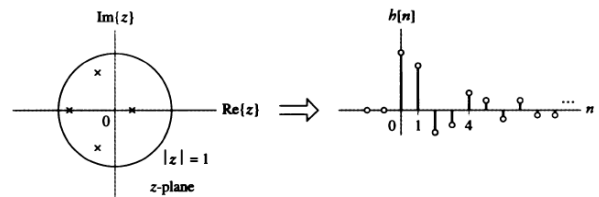


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## Causality and Stability (因果性与稳定性)

**Ex.** A causal system has the transfer function

**Find the impulse response. Is the system stable?**

$$H(z)=\frac{2+z^{-1}}{1-0.7z^{-1}+0.1z^{-2}}$$

**<Sol.>** The system has two poles: $z=0.2, \quad z=0.5.$

**The system is stable.**

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## Summary

- z-Transform
  - Introduction
  - The z-Transform
  - Properties of the z-Transform
  - Inversion of the z-Transform
  - The Transfer Function
  - Causality and Stability

- **Homework: 7.17, 7.22, 7.31, 7.34**

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