Signals and Systems 3.1

---*Fourier series*

School of Information & Communication Engineering，BUPT

Reference:

1. Textbook: Chapter 3.5

2. Schaum's outline of signals and systems, Hwei P. Hsu, McGraw-Hill, 1995. Section: 5.1-5.3

Clue of this chapter

- In **chapter 2**, by representing **signals** as linear combinations of **shifted impulses**, we **analyzed LTI systems** through the **convolution sum (integral).**
- An alternative representation for signals and LTI systems: represent **signals** as linear combinations of a set of basic signals---**complex exponentials**. The resulting representations are known as the **continuous-time and discrete-time Fourier series and transform.**
  - which convert time-domain signals into frequency-domain (or ***spectral)*** representations

Outline of Today's Lecture

- Fourier series and transform
  - Fourier series
  - Dirichlet conditions
  - Gibbs phenomenon

Fourier series

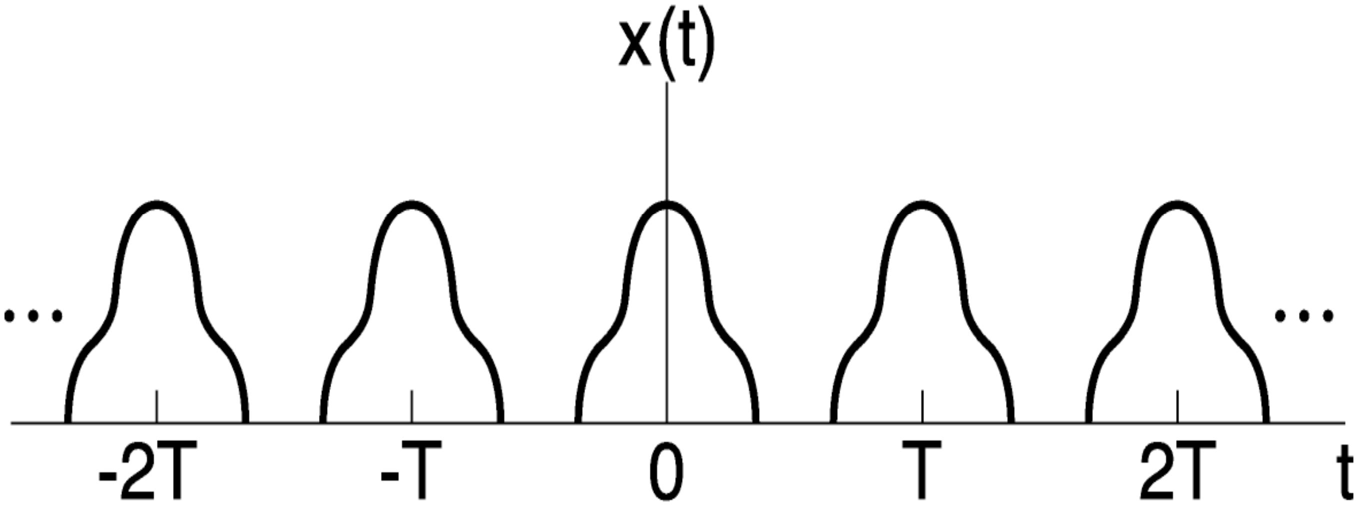
- Periodic signals can be expressed as a sum of sinusoids. In this case, the frequency spectrum can be generated by computation of the ***Fourier series***.
- The Fourier series is named after the French physicist Jean Baptiste Fourier (1768-1830), who was the first one to propose that **periodic** waveforms could be represented by a sum of sinusoids (or complex exponentials).
- An example showing how the Fourier series work http://www.falstad.com/fourier/

Period signal

$$x(t) = x(t+T) \quad \text{for all } t$$

a positive nonzero value of ***T***

- smallest such T is the *fundamental period*
- $\omega_o = \dfrac{2\pi}{T}$ is the *fundamental angular frequency*



$$x(t) = \cos(\omega_0 t + \phi)$$

$$x(t) = e^{j\omega_0 t}$$

Trigonometric Fourier Series 1

A periodic signal, x(t), whose period is T, can be represented by the appropriate sum of sine and cosine components:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n.\cos(n.\omega.t) + \sum_{n=1}^{\infty} b_n.\sin(n.\omega.t) \qquad (1)$$

a_0 is the ***mean value***, or ***zero frequency*** term.

Integrating both sides of eqn (1), between = -T/2 and T/2 :

$$\int_{-T/2}^{T/2} x(t)\,dt = \int_{-T/2}^{T/2} a_0 + \int_{-T/2}^{T/2} \left[\sum_{n=1}^{\infty} a_n.\cos(n.\omega.t) + \sum_{n=1}^{\infty} b_n.\sin(n.\omega.t) \right] dt$$

Trigonometric Fourier Series 2

$$\int_{-T/2}^{T/2} x(t)\,dt = \int_{-T/2}^{T/2} a_0 + \int_{-T/2}^{T/2} \cancel{\left[\sum_{n=1}^{\infty} a_n.\cos(n.\omega.t) + \sum_{n=1}^{\infty} b_n.\sin(n.\omega.t)\right] dt}$$

$$\int_{-T/2}^{T/2} x(t)\,dt = \int_{-T/2}^{T/2} a_0\,dt = a_0.T$$

$$a_0 = 1/T \int_{-T/2}^{T/2} x(t)\,dt$$

Trigonometric Fourier Series 3

To find a formula for a_n it is necessary to multiply both sides of eqn(1) by cos(m.ω.t) and then integrate over the same limits:

$$\int_{-T/2}^{T/2} x(t)\cos(m.\omega.t)\ dt = \int_{-T/2}^{T/2} a_{0.}\cos(m.\omega.t)\ +$$

$$+\int_{-T/2}^{T/2}\Big[\underbrace{\sum_{n=1}^{\infty}\cos(m.\omega.t).a_n.\cos(n.\omega.t)}_{\text{the "cos.cos" terms}} + \underbrace{\sum_{n=1}^{\infty}\cos(m.\omega.t)\,b_n.\sin(n.\omega.t)}_{\text{the "cos.sin" terms}}\Big]dt$$

•Using the appropriate trigonometric identities we can see that the cos.sin terms all produce
cos(A).sin(B) = ½ (sin(A+B) +/- sin(A-B))
odd waveforms which all disappear under integration.
•The cos.cos terms produce:
cos(A).cos(B) = ½ (cos(A+B) +/- cos(A-B))
which will not necessarily disappear under integration:

Trigonometric Fourier Series 4

$$\int_{-T/2}^{T/2} \sum_{n=1}^{\infty} \quad \cos(m.\omega.t)\ .\ a_n.\cos(n.\omega.t)$$

$$a_n\ \tfrac{1}{2}\ (\ \cos((m+n).\omega.t)\ + \cos((m-n).\omega.t)\ )$$

HOWEVER, we are integrating over $-T/2 \rightarrow +T/2$ and this represents an integer number of cycles of the sinusoid, whatever the value of 'm' and 'n'. BUT when m=n, we have a non-zero term after integration:

$$\int_{-T/2}^{T/2} x(t).\cos(m.\omega.t)\ \ dt\ = \int_{-T/2}^{T/2} \cancel{a_0.\cos(m.\omega.t)}\ + \int_{-T/2}^{T/2} a_n.\ \ \tfrac{1}{2}\ \cos((0).\omega.t)\ )$$

$$+ \int_{-T/2}^{T/2} [\ \sum_{n=1}^{\infty} \cancel{\cos(m.\omega.t).a_n.\cos(n.\omega.t) + \sum_{n=1}^{\infty} \cos(m.\omega.t)\ b_n.\sin(n.\omega.t)}\ ]dt$$

$$\int_{-T/2}^{T/2} x(t)\ \cos(m.\omega.t)\ \ dt\ -\ (a_n./2)\ |\ t\ |_{-T/2}^{T/2}\ \ -\ a_n\ .\ \mathbf{T}\ /2$$

Trigonometric Fourier Series 5

BUT m=n, so:

$$\int_{-T/2}^{T/2} x(t)\cos(n.\omega.t)\ dt = a_n./2 \mid t \mid_{-T/2}^{T/2} = a_n . T/2$$

$$a_n = 2/T \int_{-T/2}^{T/2} x(t).\cos(n.\omega.t)\ dt$$

And by similar reasoning:

$$b_n = 2/T \int_{-T/2}^{T/2} x(t).\sin(n.\omega.t)\ dt$$

Trigonometric Fourier Series 6

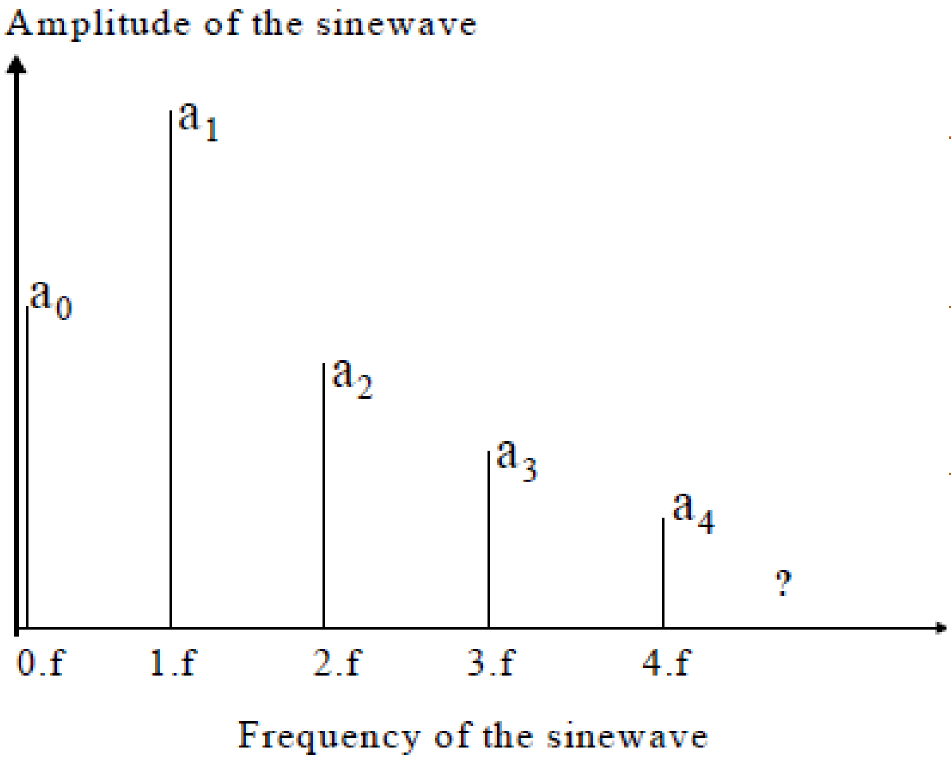
The trigonometric Fourier series given by equation (1) can also be written in the cosine-in-phase from

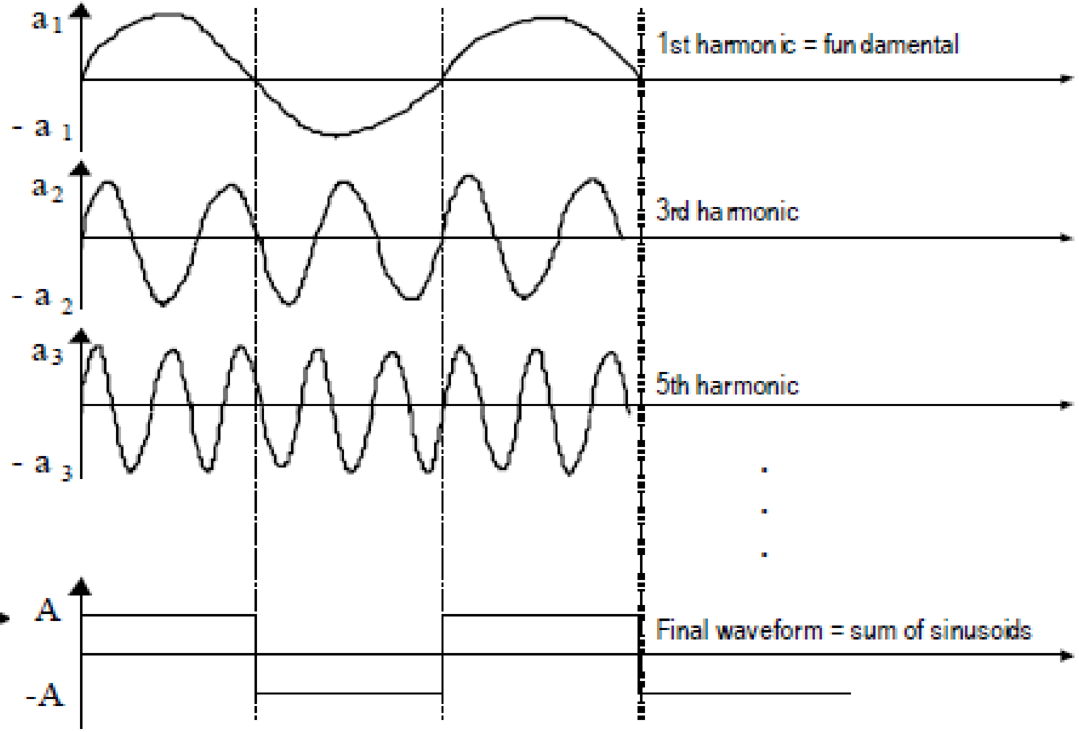
$$x(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n) \qquad -\infty < t < \infty$$

$$A_n = \sqrt{a_n^2 + b_n^2} \quad , n = 1,2,\ldots.$$

$$\theta_n = \begin{cases} \tan^{-1}(-\dfrac{b_n}{a_n}) , & n = 1,2,\ldots, when \;\; a_n \geq 0 \\ \pi + \tan^{-1}(-\dfrac{b_n}{a_n}) , & n = 1,2,\ldots, when \;\; a_n < 0 \end{cases}$$

Fourier series





This diagram represents the frequency domain

This diagram represents the time domain

Convergence (收敛) of Fourier Series:

- Fourier believed that any periodic signal could be expressed as a sum of sinusoids. However, this turned out not to be the case, although virtually all periodic signals arising in engineering do have a Fourier series representation.
- In particular, a periodic signal x(t) has a Fourier series if it satisfies the following *Dirichlet* （*狄里赫利*）*conditions*:

1. x(t) is absolutely integrable over any period; that is

$$\int_a^{a+T} |x(t)| dt < \infty \qquad for\ any\ a$$

2. x(t) has only a finite number of maxima and minima over any period.

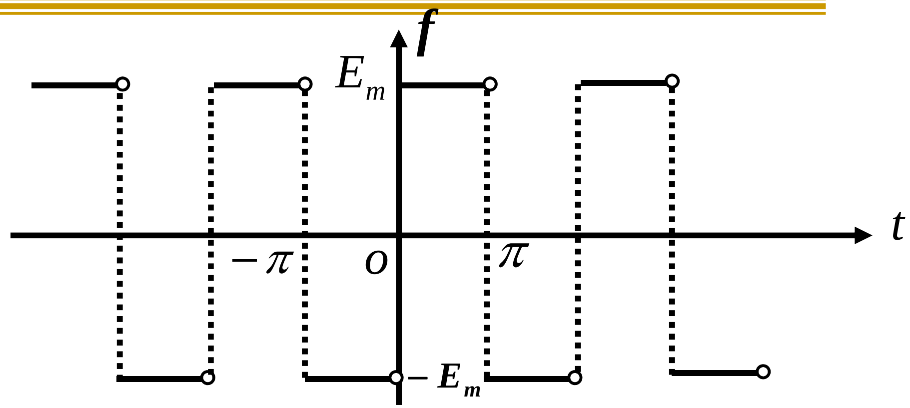
3. x(t) has only a finite number of discontinuities over any period.

Note that the Dirichlet conditions are sufficient but not necessary conditions for the Fourier series representation

Application of the Fourier series 1

- A rectangle impulse with period 2π

$$f(t)=\begin{cases}-E_m, -\pi\le t<0\\ E_m, 0\le t<\pi\end{cases}$$



Determine it is Fourier series representation

it satisfies the Dirichlet conditions.

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-E_m)\cos ntdt+\frac{1}{\pi}\int_{0}^{\pi}E_m\cos ntdt \quad =0 \quad (n=0,1,2,\cdots)$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ntdt$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-E_m)\sin ntdt+\frac{1}{\pi}\int_{0}^{\pi}E_m\sin ntdt$$

Application of the Fourier series 2

$$= \frac{2E_m}{n\pi}(1-\cos n\pi) = \frac{2E_m}{n\pi}[1-(-1)^n]$$

$$= \begin{cases} \dfrac{4E_m}{(2k-1)\pi}, & n = 2k-1,\ k = 1,2,\cdots \\ 0, & n = 2k,\ k = 1,2,\cdots \end{cases}$$

So, the Fourier series representation

$$f(t) = \sum_{n=1}^{\infty} \frac{4E_m}{(2n-1)\pi} \sin(2n-1)t$$

$$(-\infty < t < +\infty;\ t \neq 0, \pm\pi, \pm 2\pi, \cdots)$$

Application of the Fourier series 3

If set $E_m = 1$, Period= 2π,

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{2n-1}$$

$$(-\infty < t < +\infty;\ t \neq 0, \pm\pi, \pm 2\pi, \cdots)$$

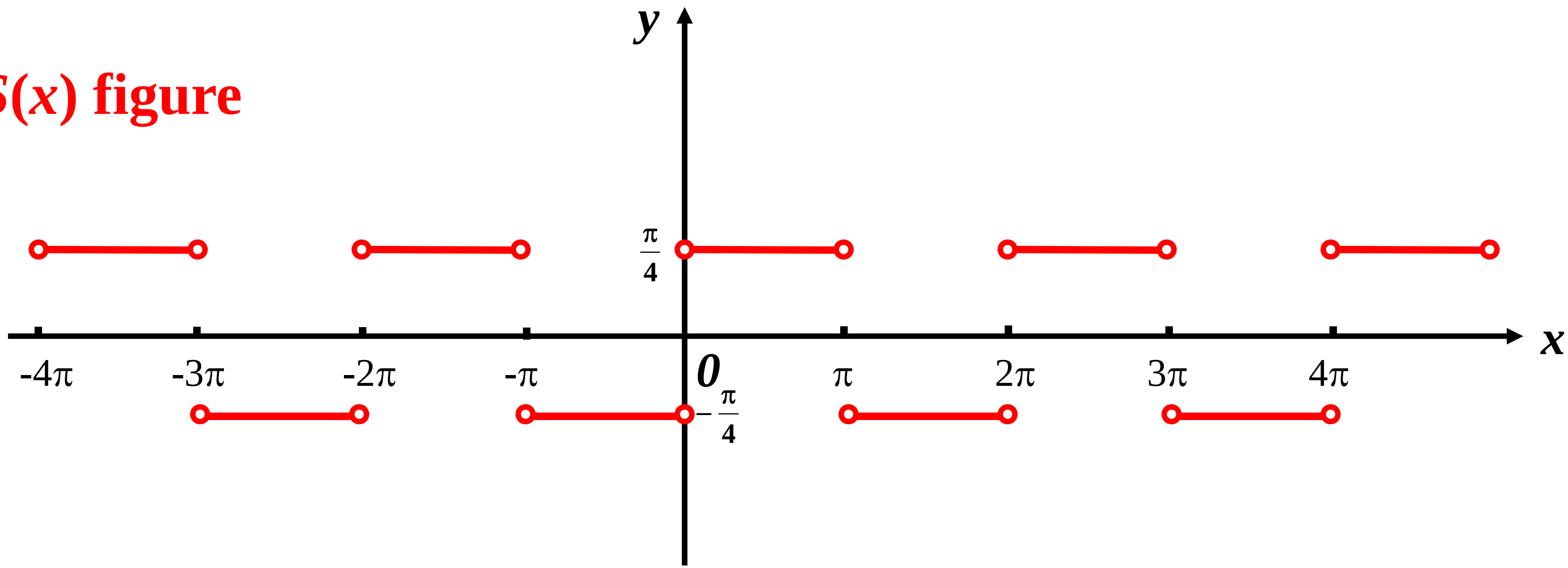
so

$$\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1} = \begin{cases} \dfrac{\pi}{4}, x \in (0,\pi) \\ -\dfrac{\pi}{4}, x \in (-\pi,0) \end{cases} = S(x)$$

Application of the Fourier series 4

$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} = \begin{cases} \dfrac{\pi}{4}, x \in (0, \pi) \\ -\dfrac{\pi}{4}, x \in (-\pi, 0) \end{cases} = S(x)$$

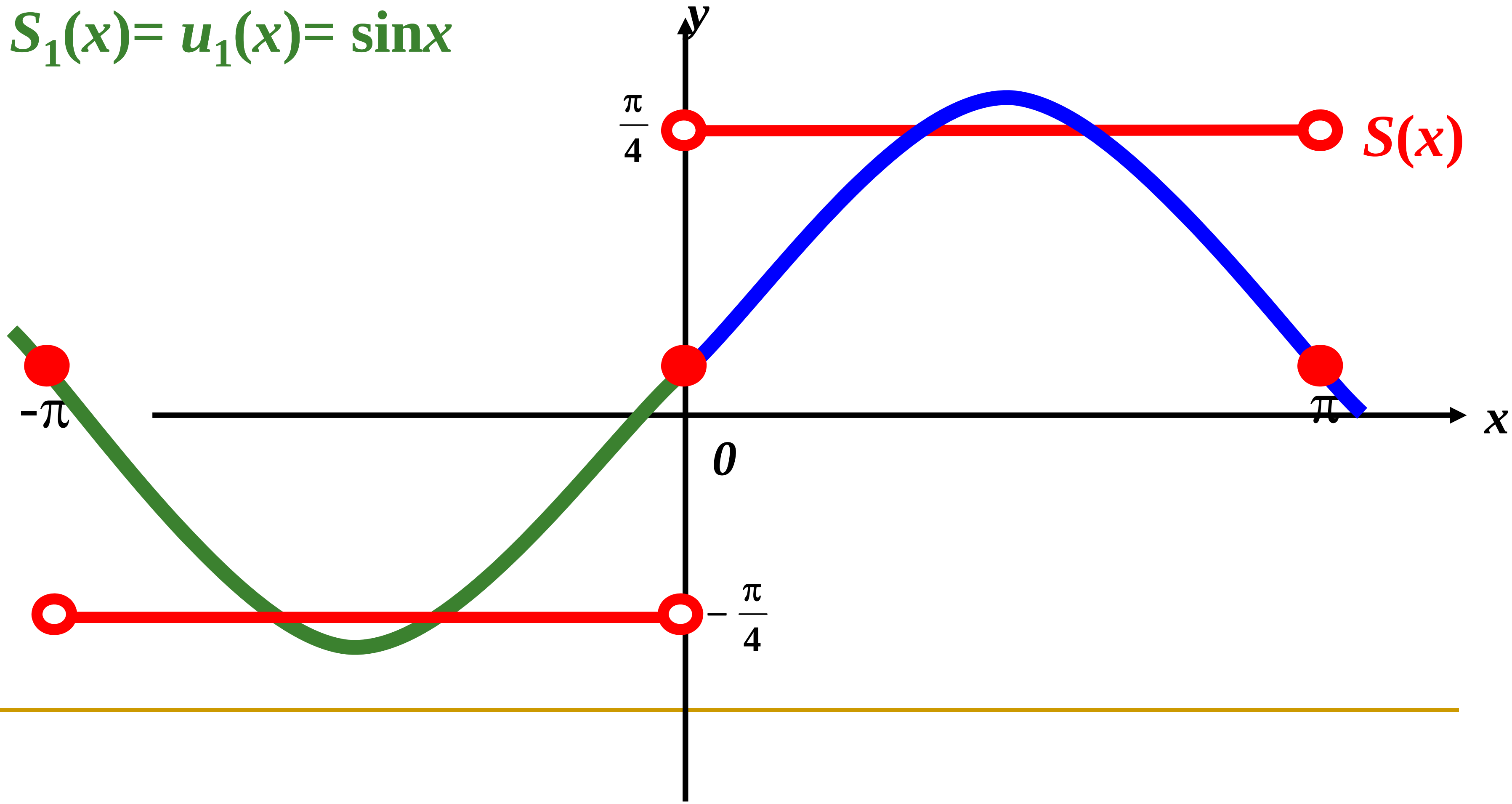
***S*(*x*) figure**



Application of the Fourier series 5

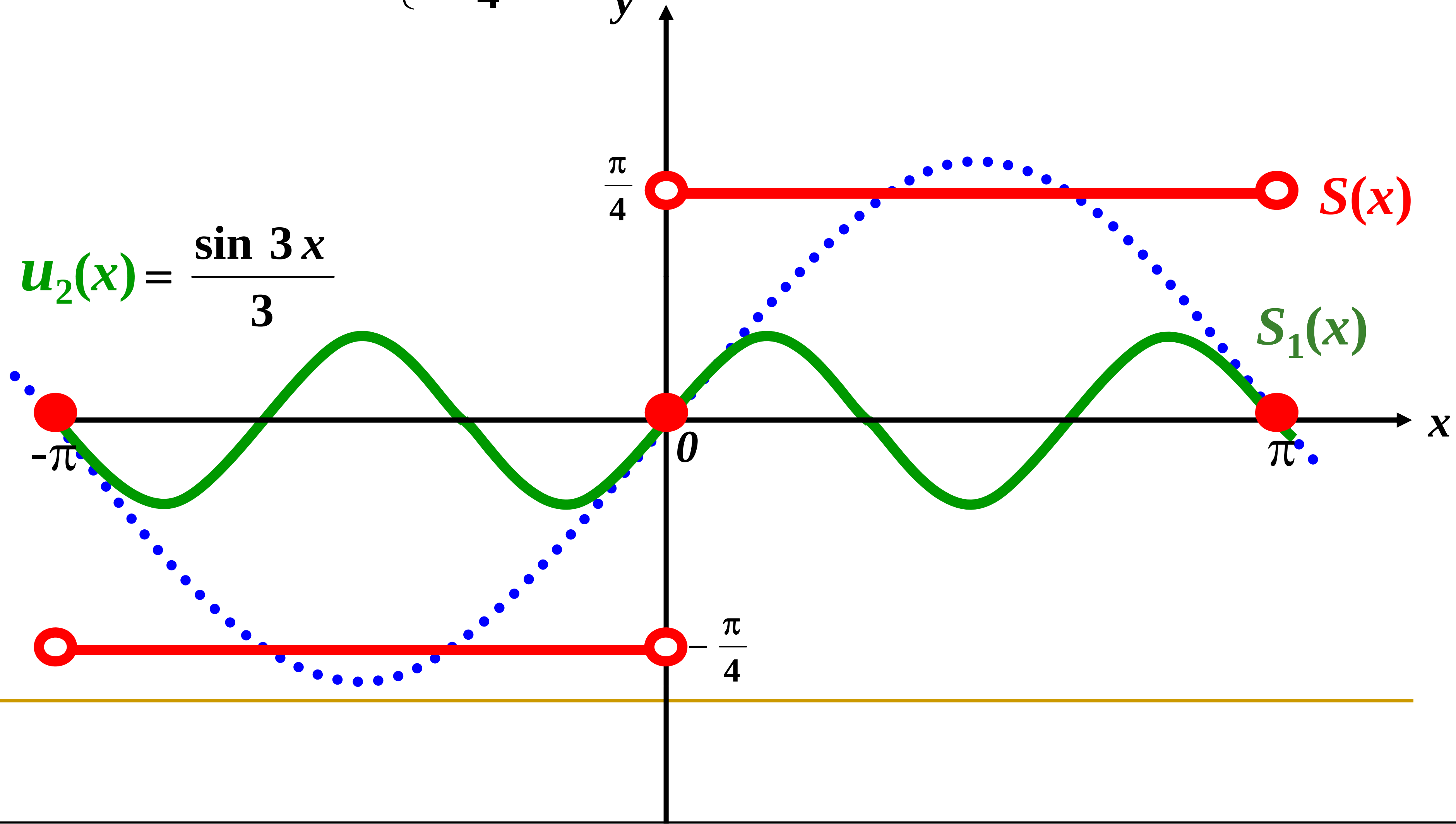
$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} = \begin{cases} \frac{\pi}{4}, x \in (0,\pi) \\ -\frac{\pi}{4}, x \in (-\pi,0) \end{cases} = S(x)$$

$S_1(x) = u_1(x) = \sin x$



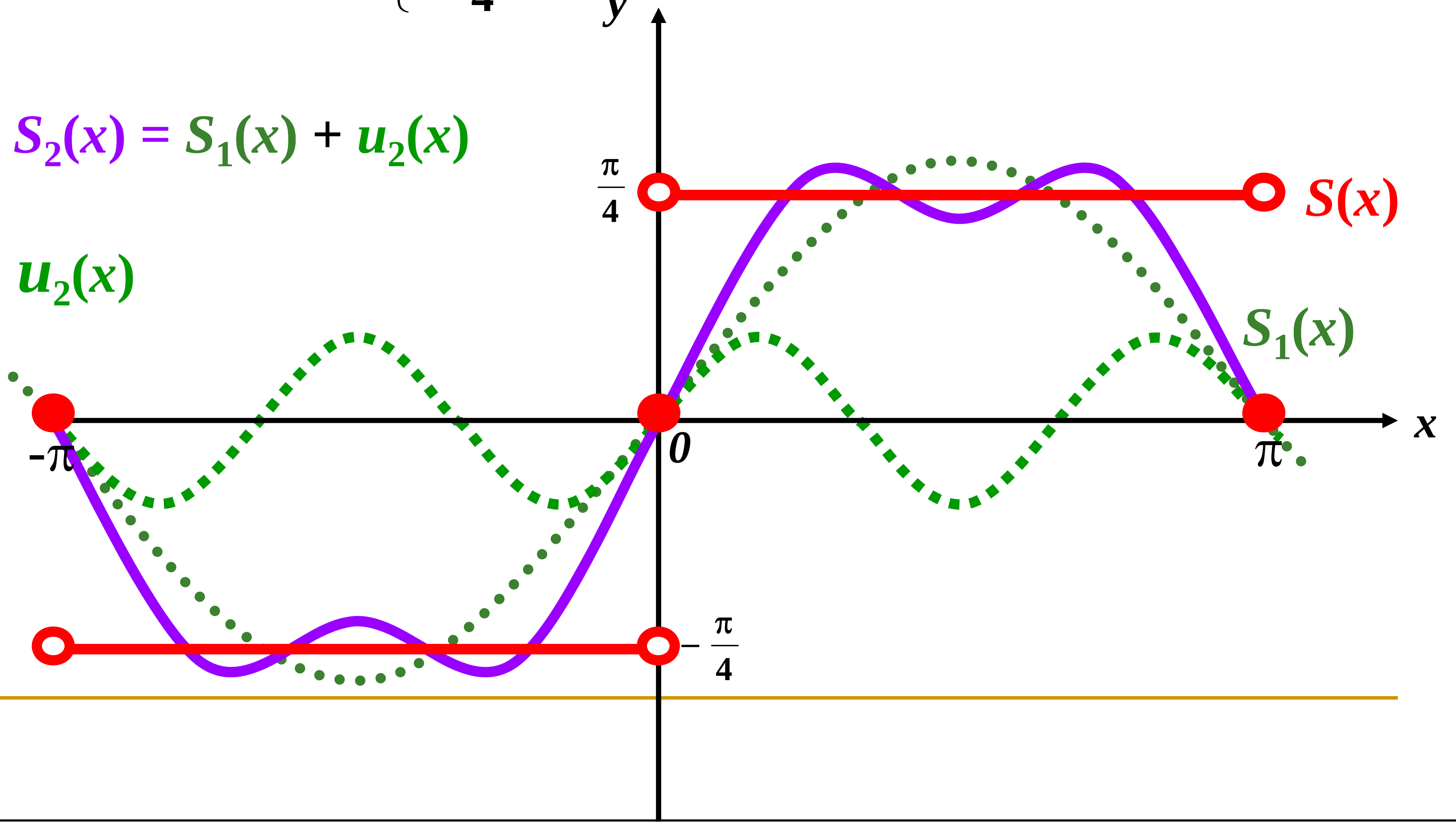
Application of the Fourier series 6

$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} = \begin{cases} \dfrac{\pi}{4}, & x \in (0,\pi) \\ -\dfrac{\pi}{4}, & x \in (-\pi,0) \end{cases} = S(x)$$



Application of the Fourier series 7

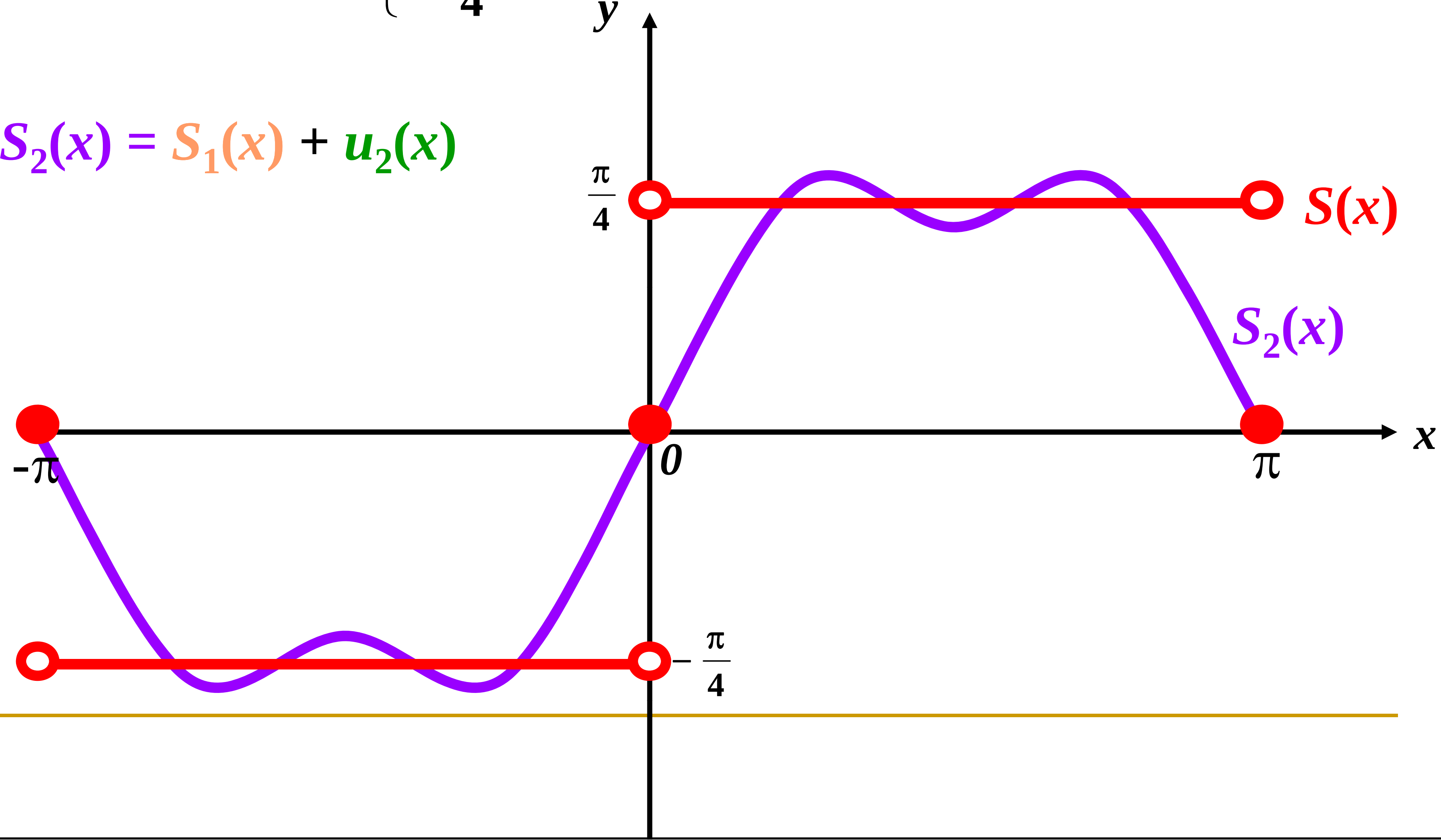
$$\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}=\begin{cases}\dfrac{\pi}{4}, & x\in(0,\pi)\\ -\dfrac{\pi}{4}, & x\in(-\pi,0)\end{cases}=S(x)$$



Application of the Fourier series 8

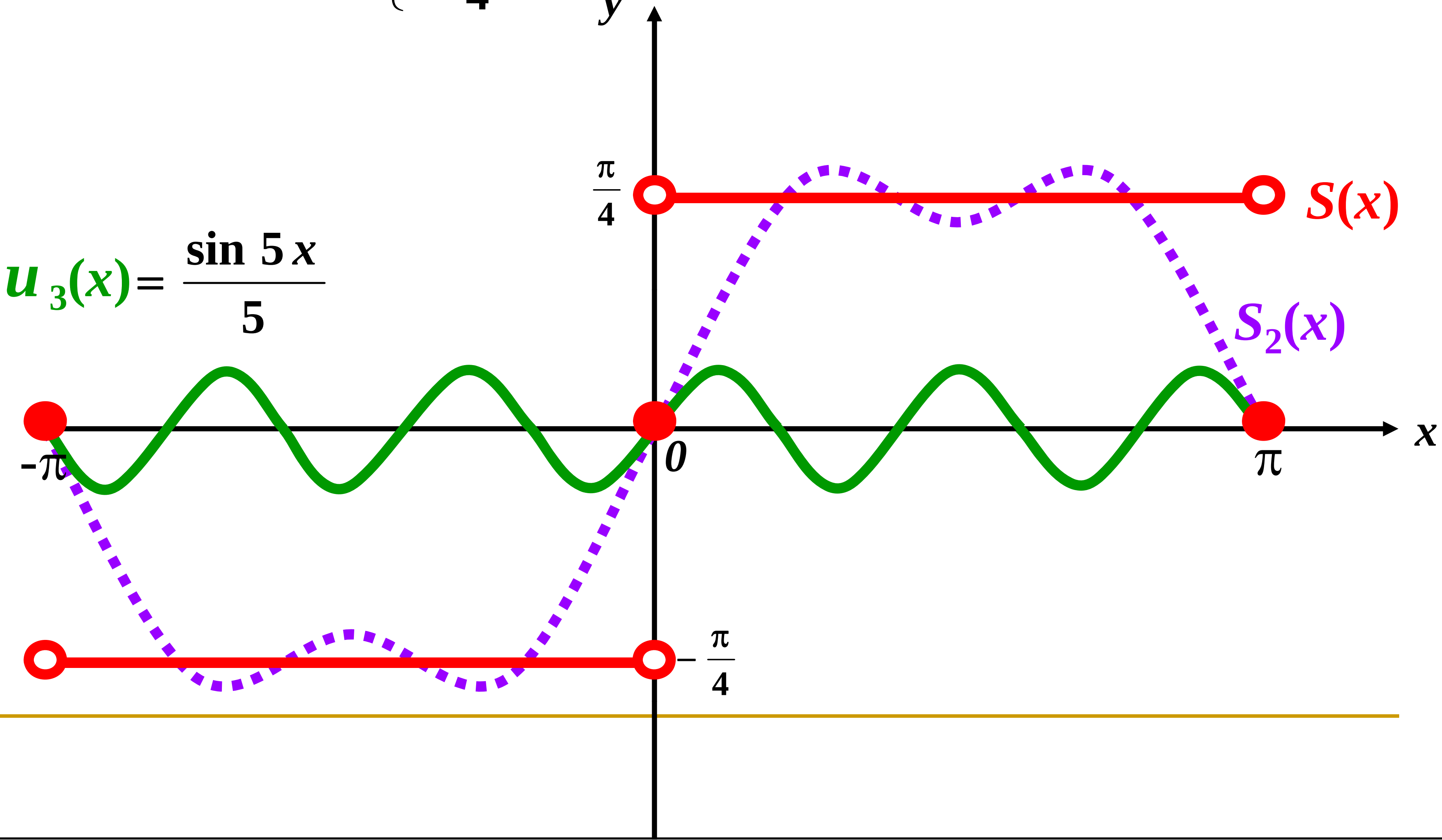
$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} = \begin{cases} \frac{\pi}{4}, x \in (0,\pi) \\ -\frac{\pi}{4}, x \in (-\pi,0) \end{cases} = S(x)$$

$S_2(x) = S_1(x) + u_2(x)$



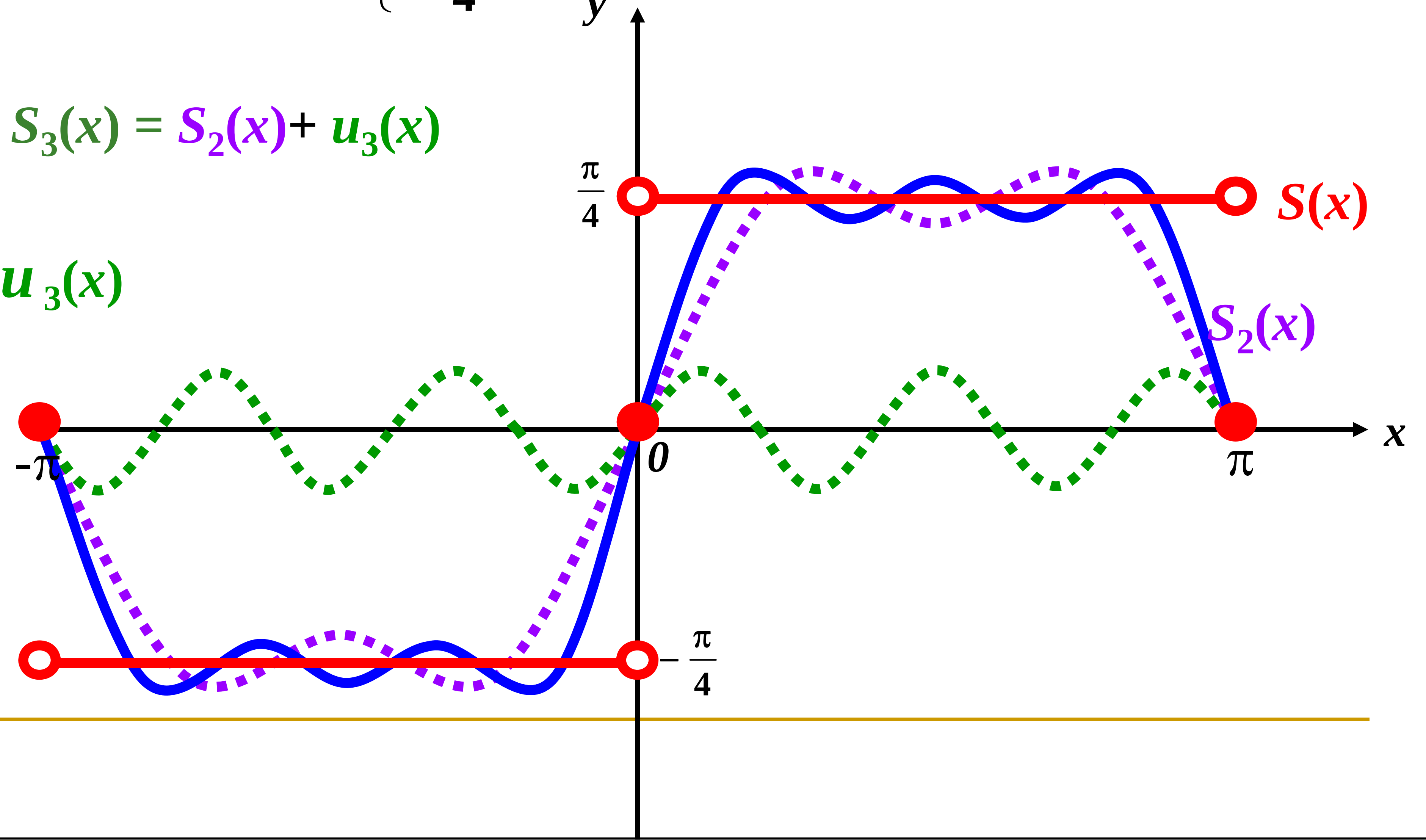
Application of the Fourier series 9

$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} = \begin{cases} \dfrac{\pi}{4}, & x \in (0,\pi) \\ -\dfrac{\pi}{4}, & x \in (-\pi,0) \end{cases} = S(x)$$



Application of the Fourier series 10

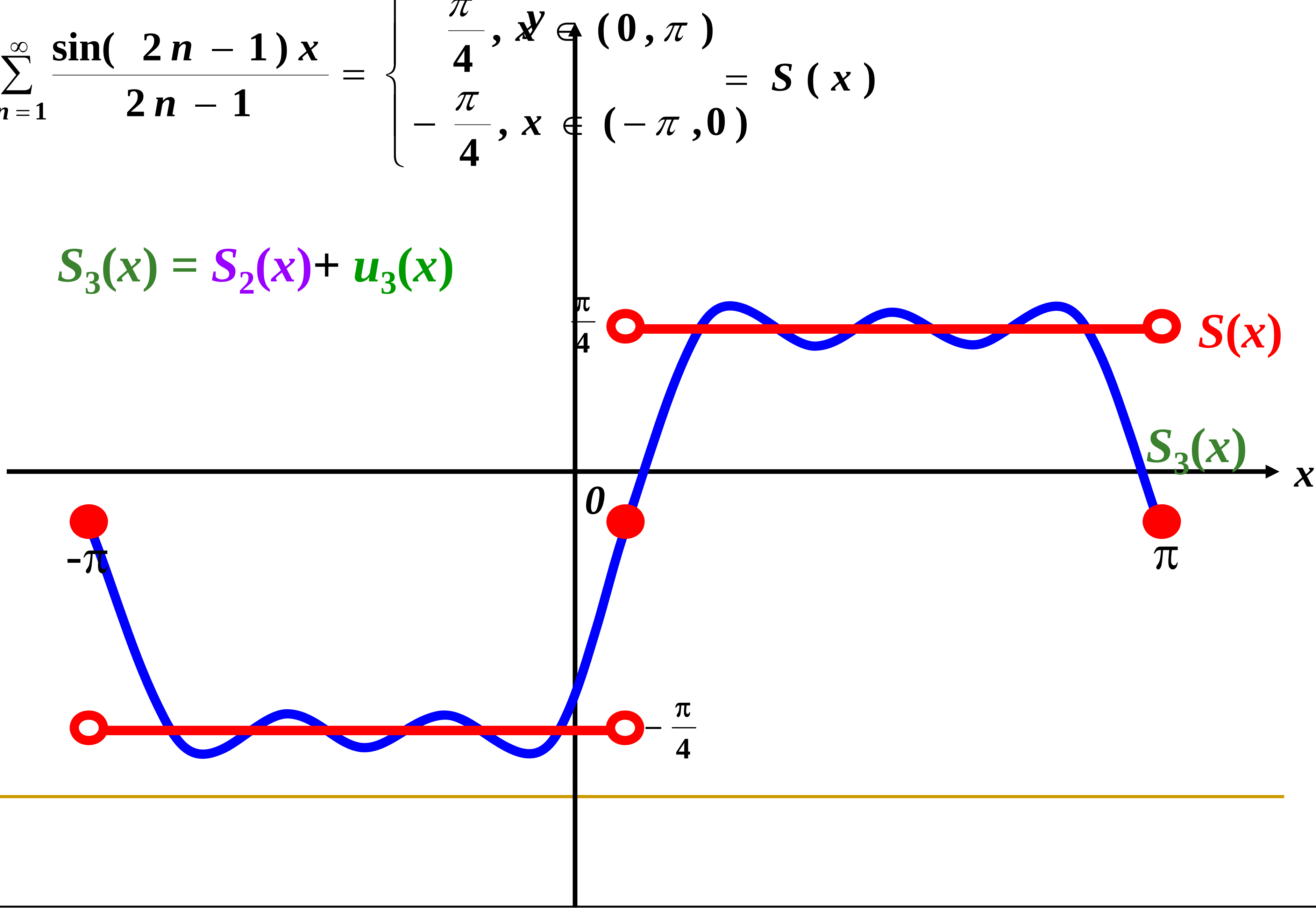
$$\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}=\begin{cases}\dfrac{\pi}{4}, & x\in(0,\pi)\\ -\dfrac{\pi}{4}, & x\in(-\pi,0)\end{cases}=S(x)$$



Application of the Fourier series 11

$$\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}=\begin{cases}\dfrac{\pi}{4}, & x\in(0,\pi)\\ -\dfrac{\pi}{4}, & x\in(-\pi,0)\end{cases}=S(x)$$

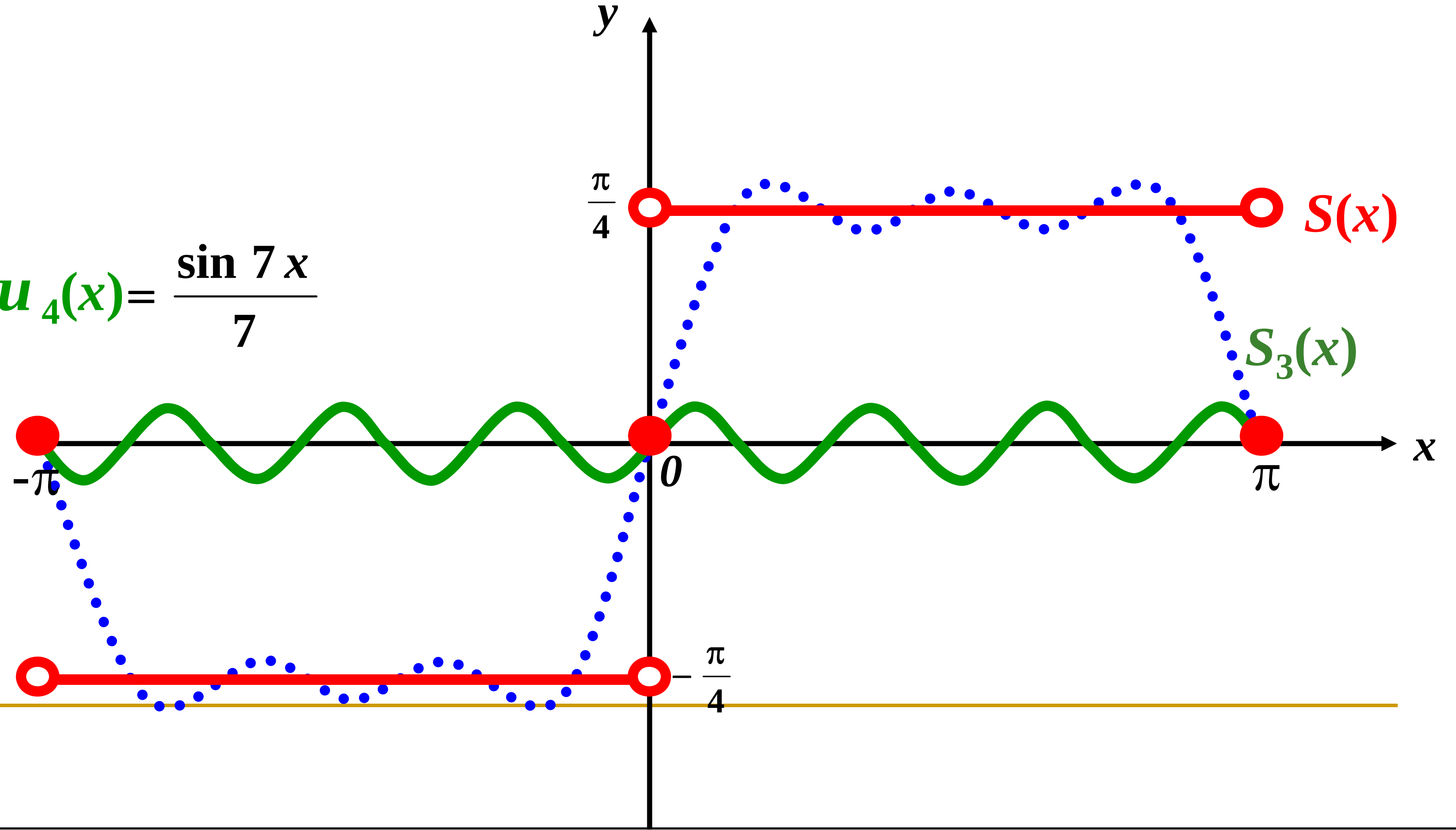
$S_3(x) = S_2(x) + u_3(x)$



Application of the Fourier series 12

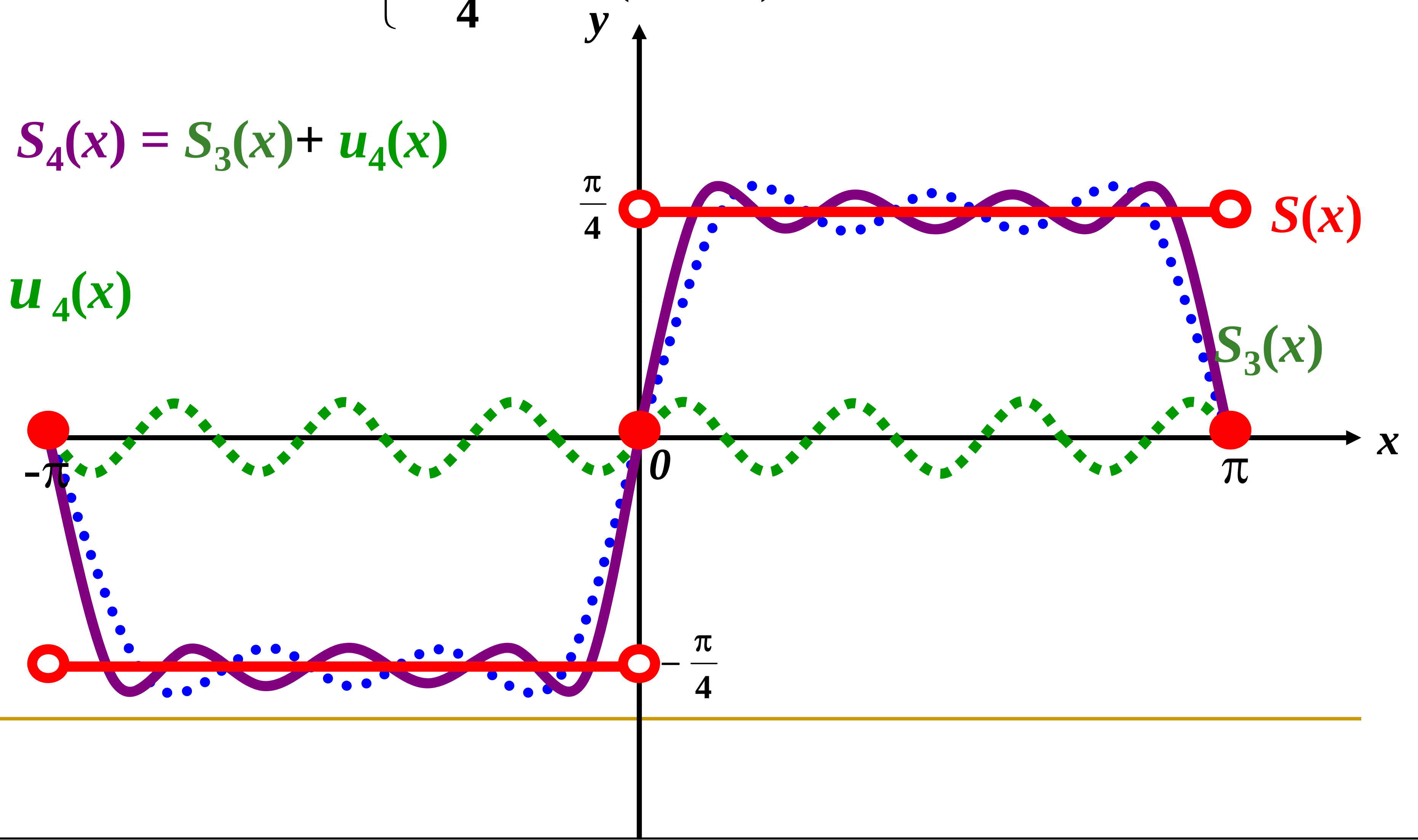
$$\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}=\begin{cases}\dfrac{\pi}{4}, & x\in(0,\pi)\\ -\dfrac{\pi}{4}, & x\in(-\pi,0)\end{cases}=S(x)$$

$$u_4(x)=\frac{\sin 7x}{7}$$



Application of the Fourier series 13

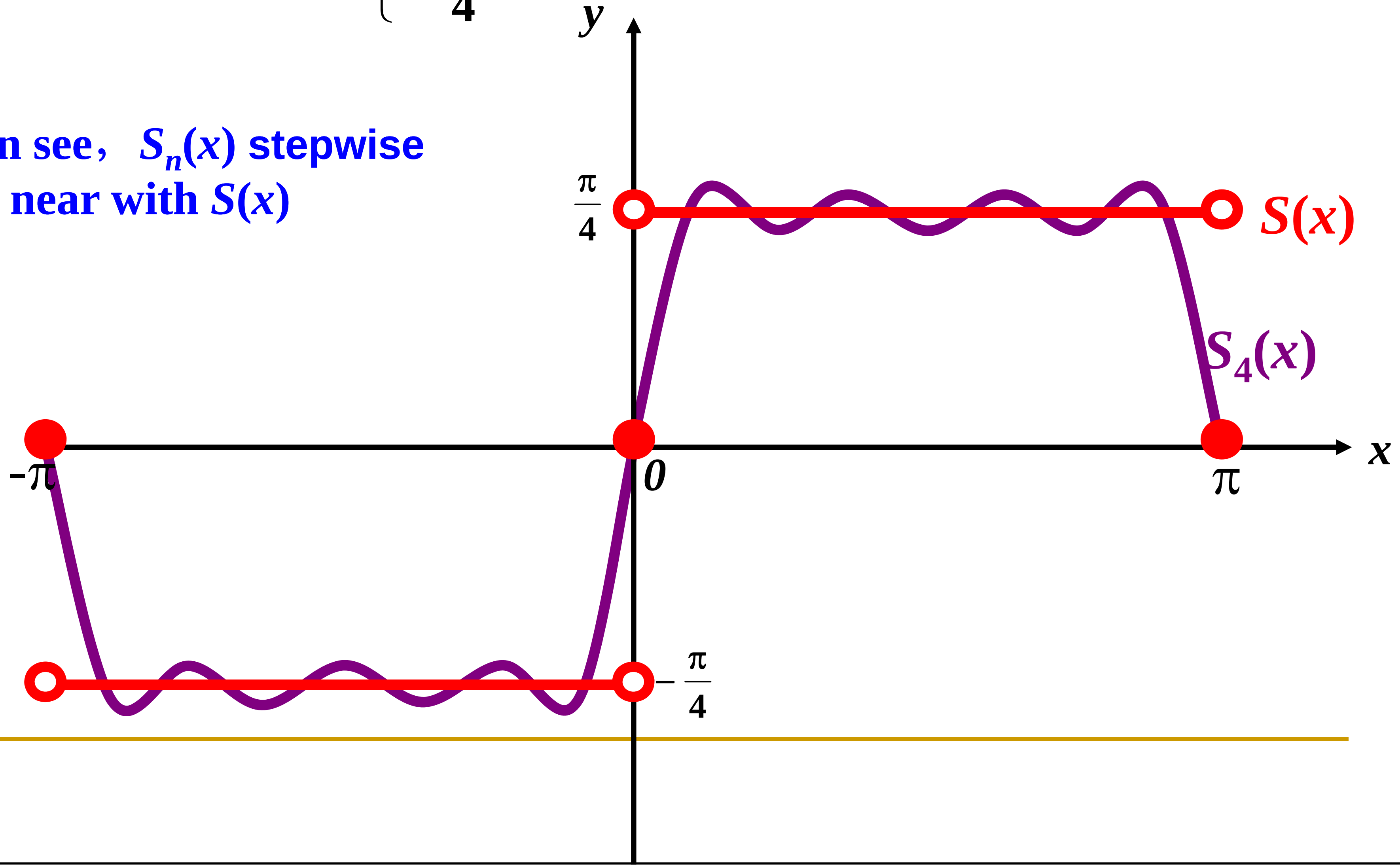
$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} = \begin{cases} \dfrac{\pi}{4}, x \in (0,\pi) \\ -\dfrac{\pi}{4}, x \in (-\pi,0) \end{cases} = S(x)$$



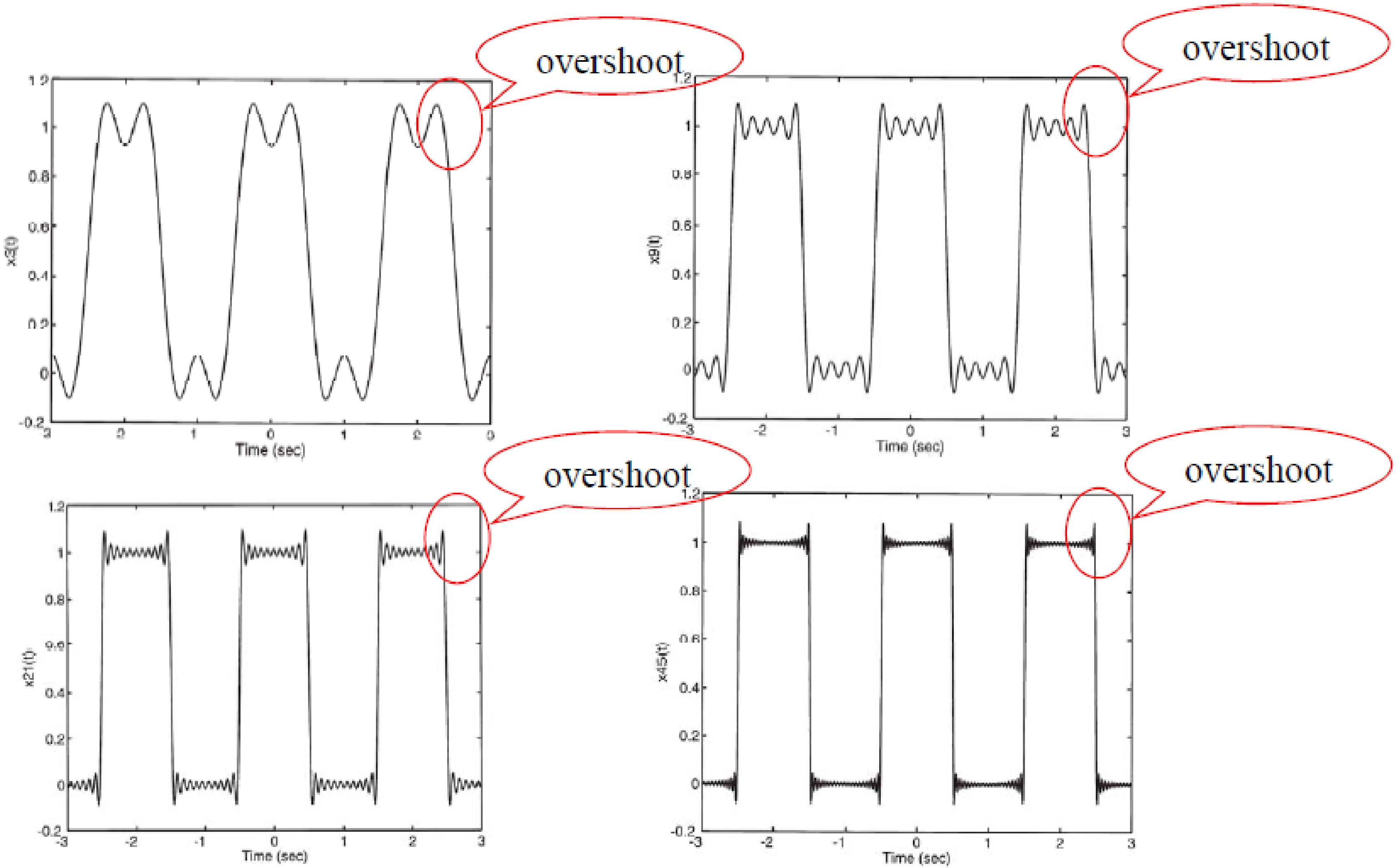
Application of the Fourier series 14

$$\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} = \begin{cases} \frac{\pi}{4}, x \in (0,\pi) \\ -\frac{\pi}{4}, x \in (-\pi,0) \end{cases} = S(x)$$

We can see， $S_n(x)$ stepwise near with $S(x)$



Gibbs Phenomenon 1



Anything in common of the four diagrams?

Gibbs Phenomenon 2

- The overshoot at the corners is till present even in the limit as N approaches to infinity.
- This characteristic was first discovered by Josiah Willard Gibbs (1893-1903), and thus overshoot is referred to as the *Gibbs phenomenon*.

- Now let x(t) be an arbitrary periodic signal. As a consequence of the Gibbs phenomenon, the Fourier series representation of x(t) is not actually equal to the true value of x(t) at any points where x(t) is discontinuous.
- If x(t) is discontinuous at $t=t_1$, the Fourier series representation is off by approximately 9% at t_1^- and t_1^+

The exponential form of the Fourier series

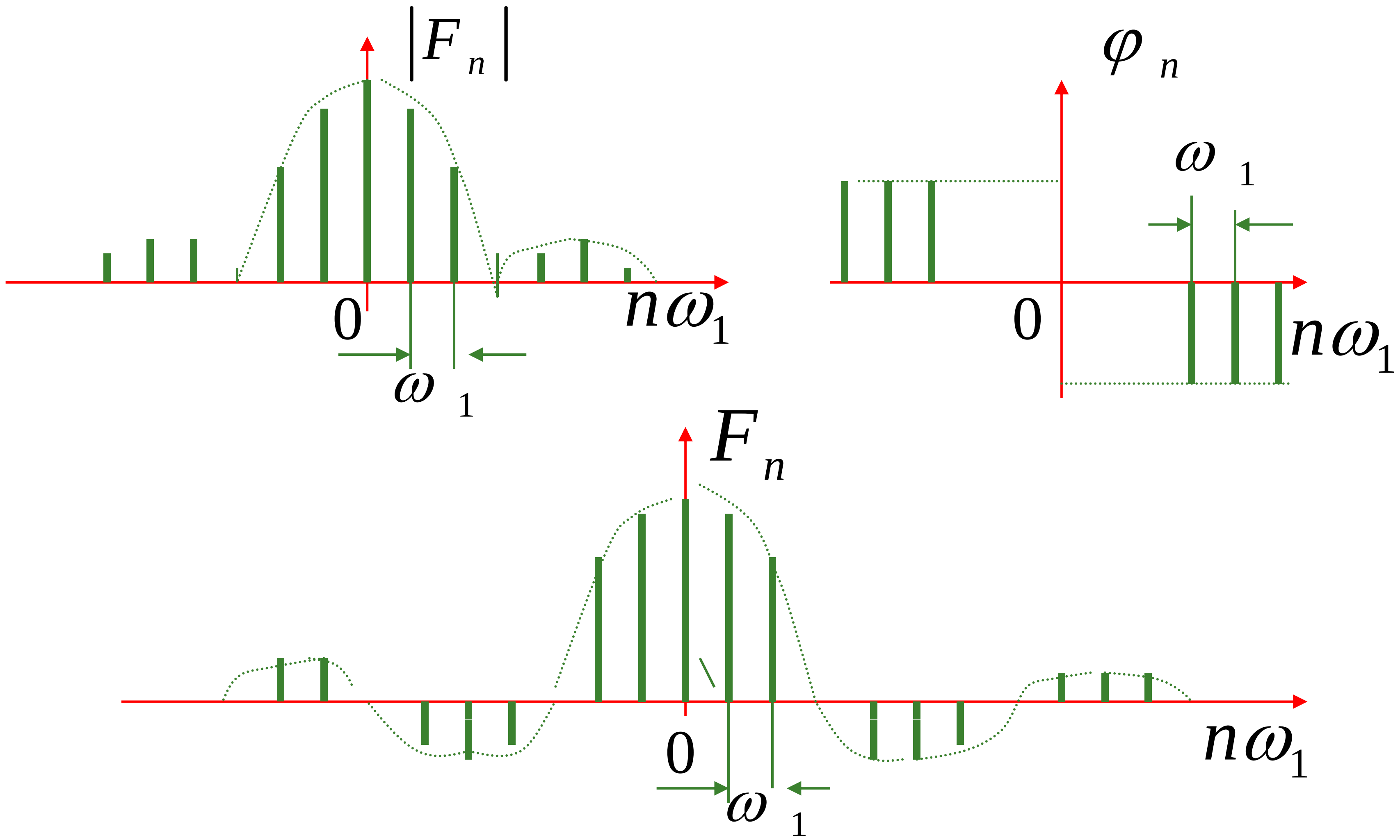
$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_1 t + b_n \sin n\omega_1 t)$$

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} F(n\omega_1) e^{jn\omega_1 t}}$$

$$F(0) = a_0 \qquad F(n\omega_1) = \frac{1}{2}(a_n - jb_n) \qquad F(-n\omega_1) = \frac{1}{2}(a_n + jb_n)$$

$$F(n\omega_1) = F_n \qquad \boxed{F_n = \frac{1}{T_1}\int_{t_0}^{t_0+T_1} f(t) e^{-jn\omega_1 t} dt}$$

The exponential form of the Fourier series



The exponential form of the Fourier series

- Let's recall the original form of Fourier series:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n.\cos(n.\omega.t) + \sum_{n=1}^{\infty} b_n.\sin(n.\omega.t)$$

- In order to reduce the amount of 'writing out' the Fourier series, an exponential form can be expressed as:

$$a_n.\cos(n.\omega.t) = (a_n/2) . [e^{jn\omega t} + e^{-jn\omega t}]$$

$$b_n.\sin(n.\omega.t) = (b_n/2j) . [e^{jn\omega t} - e^{-jn\omega t}]$$

$$a_n.\cos(n.\omega.t) + b_n.\sin(n.\omega.t) = (a_n/2) [e^{jn\omega t} + e^{-jn\omega t}] + (b_n/2j) [e^{jn\omega t} - e^{-jn\omega t}]$$

$$= X_n . e^{jn\omega t} + X_{-n} . e^{-jn\omega t}$$

where: $X_n = \frac{1}{2} (a_n - j\, b_n) \quad n \neq 0$

$X_{-n} = \frac{1}{2} (a_n + j\, b_n) \quad n \neq 0$

So the original Fourier Series can be written out as: $x(t) = \sum_{n=-\infty}^{\infty} X_n.\, e^{jn\omega t}$

Where we have defined: $X_0 = a_0$

The Relationships of coefficients

$$F_n = |F_n| e^{j\varphi_n} = \tfrac{1}{2}(a_n - jb_n) \qquad F_0 = c_0 = d_0 = a_0$$

$$F_{-n} = |F_{-n}| e^{-j\varphi_n} = \tfrac{1}{2}(a_n + jb_n)$$

$$|F_n| = |F_{-n}| = \tfrac{1}{2}c_n = \tfrac{1}{2}d_n = \tfrac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$$|F_n| + |F_{-n}| = c_n$$

$$F_n + F_{-n} = a_n \qquad j(F_n - F_{-n}) = b_n$$

$$c_n^2 = d_n^2 = a_n^2 + b_n^2 = 4F_nF_{-n}$$

Summary of the Fourier series

- Three forms
  - Original (sine and cosine components)

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

  - Cosine-with-phase form

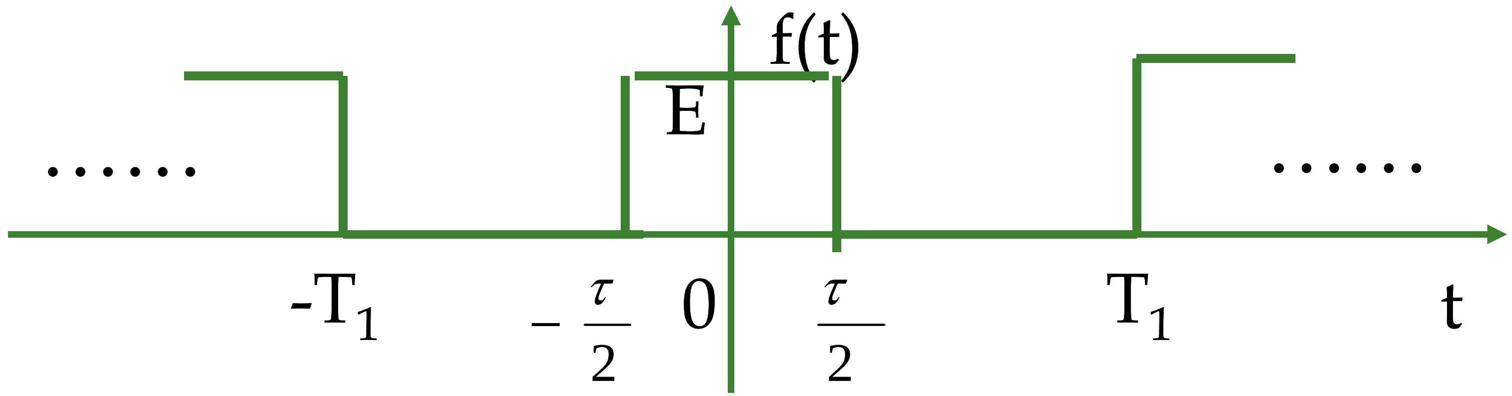
$$x(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t + \theta_k) \qquad -\infty < t < \infty$$

  - Exponential form

$$x(t) = \sum_{n=-\infty}^{\infty} X_n . e^{jn\omega t} \qquad X_n = \frac{1}{T}\int_T x(t) e^{-jn\omega t} dt$$

- Dirichlet conditions
- Gibbs phenomenon

Example: Fourier series of Rectangle pulse



$$f(t) = \begin{cases} E & (|t| \leq \frac{\tau}{2}) \\ 0 & (|t| > \frac{\tau}{2}) \end{cases}$$

Example: Fourier series of Rectangle pulse

$$f(t)=\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_1 t}$$

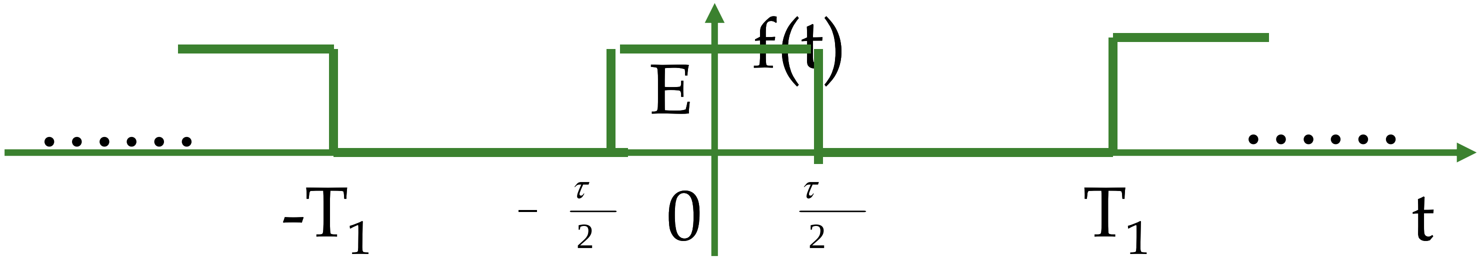
$$F_n=\frac{1}{T_1}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} E e^{-jn\omega_1 t}\,dt$$

$$=\frac{E}{T_1(-jn\omega_1)}\left(e^{-jn\omega_1\tau/2}-e^{jn\omega_1\tau/2}\right)$$

$$=\frac{E\tau}{T_1}\frac{\sin\left(\frac{n\omega_1\tau}{2}\right)}{\left(\frac{n\omega_1\tau}{2}\right)}$$

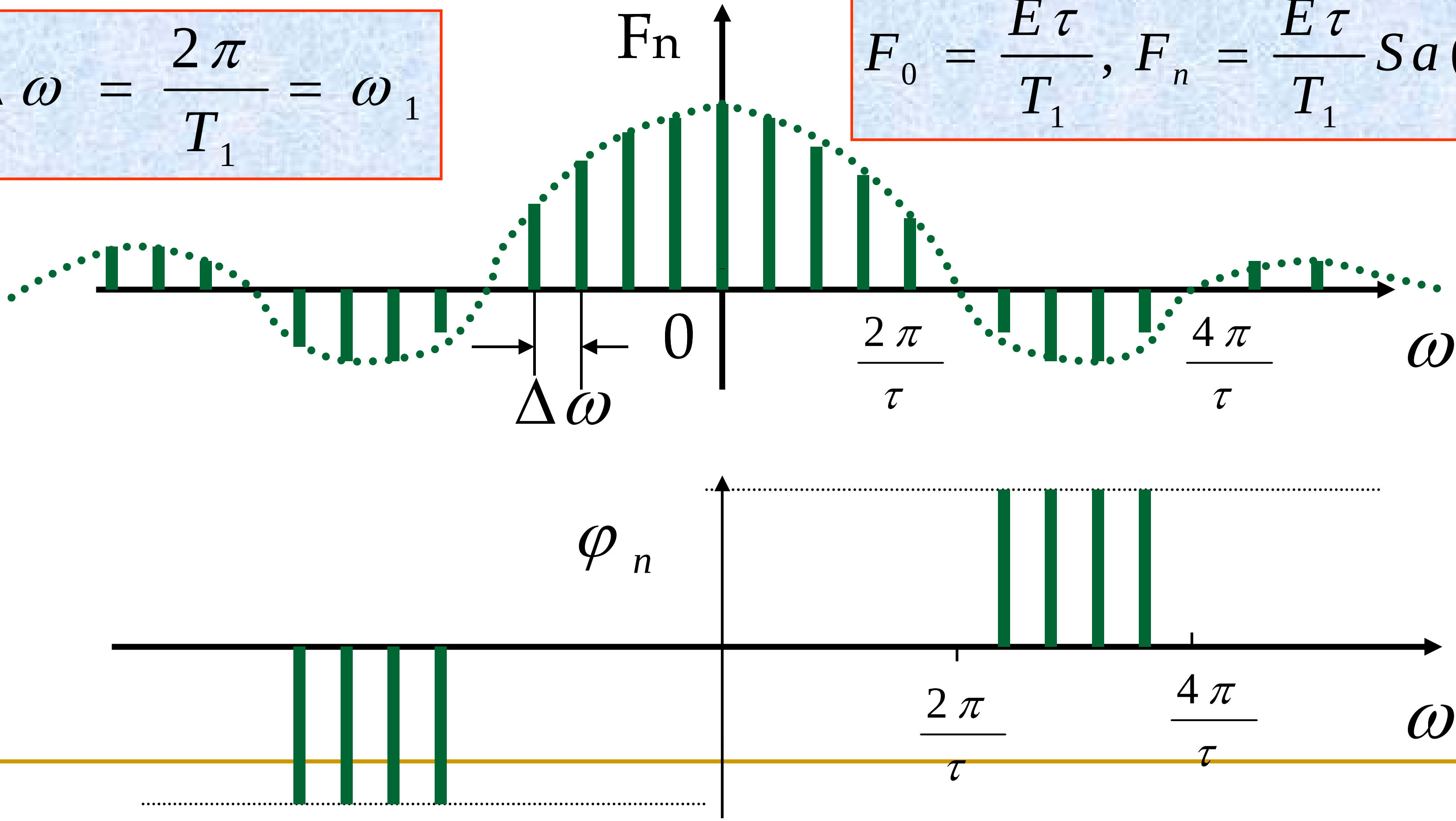
$$Sa\left(\frac{n\pi\tau}{T_1}\right)$$

Example: Fourier series of Rectangle pulse

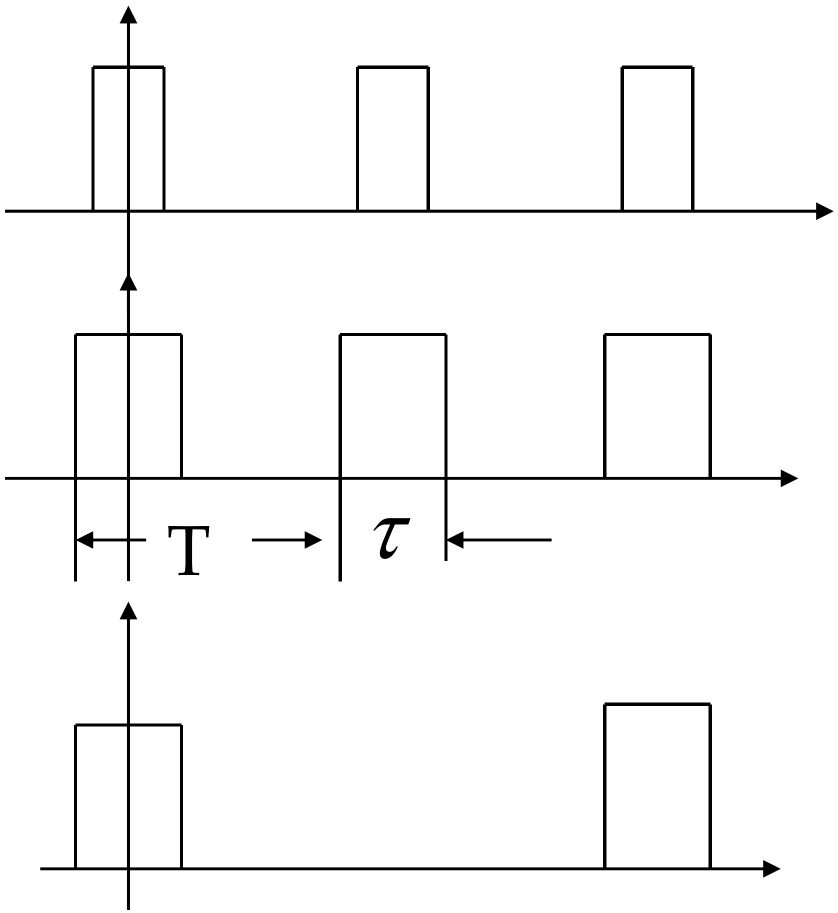


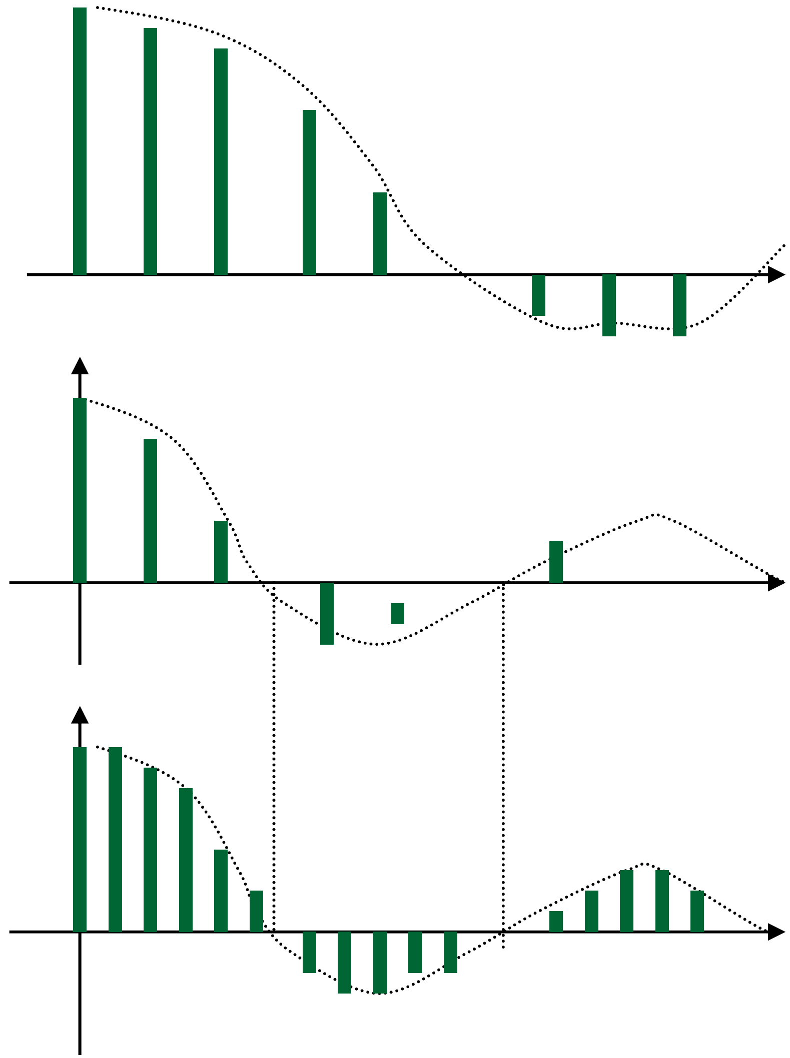
$$\Delta\omega = \frac{2\pi}{T_1} = \omega_1$$

$$F_0 = \frac{E\tau}{T_1},\ F_n = \frac{E\tau}{T_1} Sa\left(\frac{n\pi\tau}{T_1}\right)$$

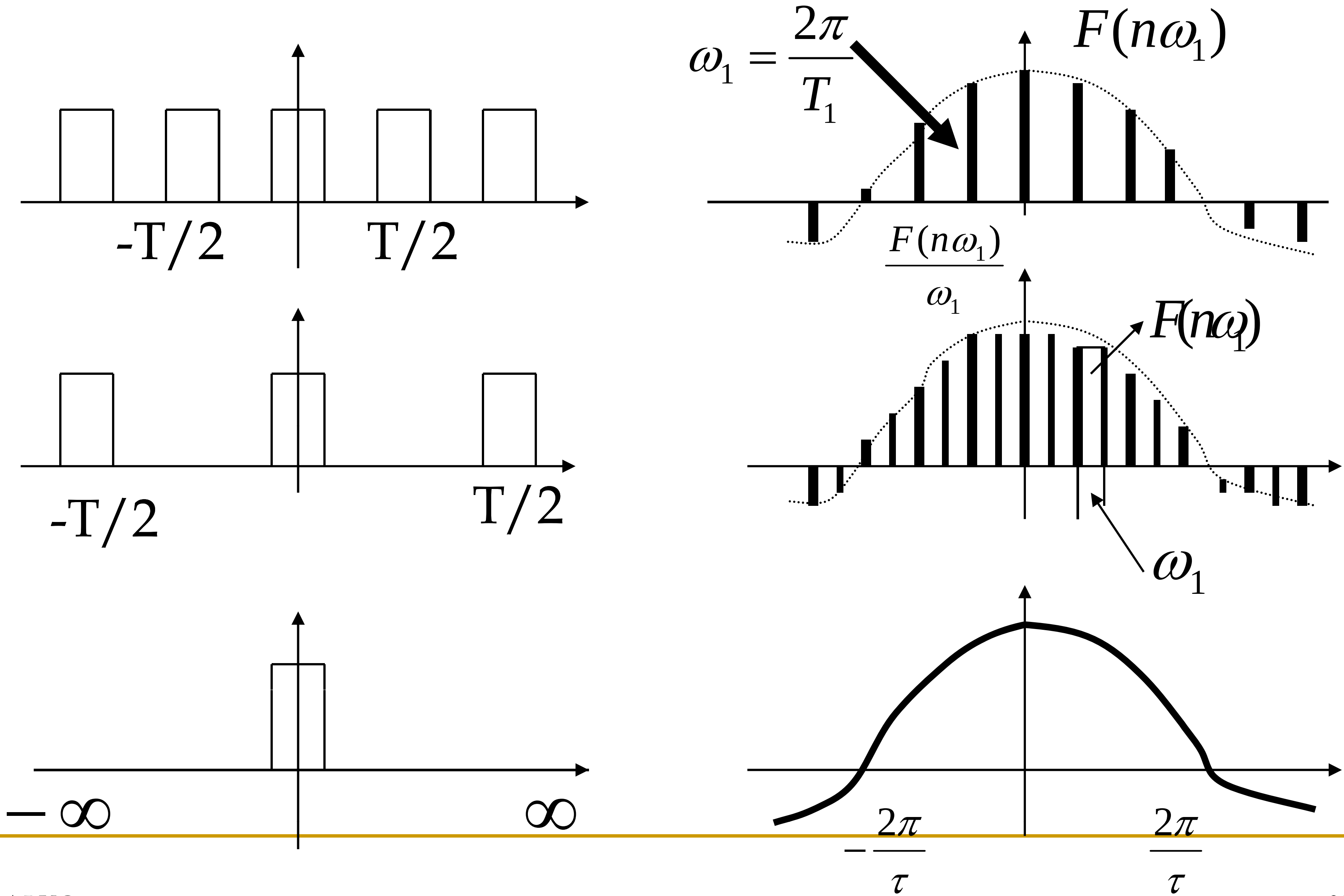


Example: Fourier series of Rectangle pulse





Example: Fourier series of Rectangle pulse



Fourier Series → Fourier Transform

$$T_1 \rightarrow \infty \qquad \omega_1 = \frac{2\pi}{T_1} \rightarrow 0 \rightarrow d\omega \qquad n\omega_1 \rightarrow \omega$$

$$\tilde{f}(t) = \sum_{n=-\omega}^{\infty} F(n\omega_1) e^{jn\omega_1 t} \qquad F(n\omega_1) = \frac{1}{T_1} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} \tilde{f}(t) e^{-jn\omega_1 t} dt$$

$$F(n\omega_1) \frac{2\pi}{\omega_1} = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-jn\omega_1 t} dt$$

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt}$$

Inverse Fourier Transform

$$\tilde{f}(t) = \sum_{n=-\omega}^{\infty} F(n\omega_1).e^{jn\omega_1 t} \quad F(n\omega_1) \to F(\omega) \quad \sum_{n=-\omega}^{\infty} \to \int_{-\infty}^{\infty}$$

$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} \frac{F(n\omega_1)}{\omega_1}.e^{jn\omega_1 t}.\omega_1 = \frac{T_1}{2\pi} \sum_{\Delta n\omega_1=-\infty}^{\infty} F(n\omega_1).e^{jn\omega_1 t}.\Delta(n\omega_1)$$

$$T_1 \to \infty \quad \omega_1 \to 0 \quad n\omega_1 \to \omega \quad \Delta(n\omega_1) \to d\omega$$

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega).e^{j\omega t} d\omega}$$