

Signals and Systems 6.1

--- The Laplace Transform

School of Information & Communication Engineering, BUPT

Reference:
1. Textbook: Chapter 6.1~6.6

Outline

■ The Laplace Transform

- Introduction
- Definition
- The Unilateral Laplace Transform
- Property of The Unilateral Laplace Transform
- Inversion of The Unilateral Laplace Transform
- Solving Differential Equations with Initial Conditions
- The Transfer Function
- Causality and Stability

2

Introduction

- The Laplace Transform is a more **general continuous-time signal and system representation** based on complex exponential signals.
 - There are some functions of interest, such as the ramp function which do not have a Fourier transform.
 - We wish to determine a system's response from a specific time, and also include any initial conditions in the system's response.
- Main usage: **transient** and **stability** analysis of causal LTI system.
 - Unilateral (one sided) Laplace Transform: solving differential equations with initial conditions.
 - Bilateral (two sided) Laplace Transform: analysis on the system characteristics such as stability, causality, and frequency response.

2011-12-6

3

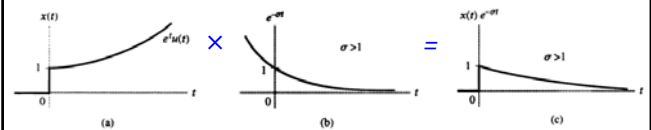
From Fourier Transform to Laplace Transform

- Eg. The Fourier transform of $x(t)=e^{at}u(t)$, $a > 0$ non-exists.

$$F[x(t)e^{-\sigma t}] = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt = \int_0^{\infty} e^{at}e^{-(\sigma+j\omega)t}dt$$

$$s = \sigma + j\omega \Rightarrow \int_0^{\infty} e^{-(s-a)t}dt = \frac{-1}{s-a}e^{-(s-a)t}\bigg|_0^{\infty} = \frac{1}{s-a} \text{ if } \sigma > a.$$

For $a=1$:



2011-12-6

4

From Fourier Transform to Laplace Transform

To be generalized

$$F[x(t)e^{-\sigma t}] = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$$

$$s = \sigma + j\omega \Rightarrow \int_{-\infty}^{\infty} x(t)e^{-st}dt \triangleq X(s)$$

- The Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \text{ or } X(s) = L[x(t)]$$

- The inverse Laplace transform

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{-st}ds \text{ or } x(t) = L^{-1}[X(s)]$$

$$x(t) \xleftrightarrow{L} X(s)$$

2011-12-6

5

Complex Exponentials

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{-st}ds$$

represents $x(t)$ as a weighted superposition of complex exponentials e^{st} .

$$e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t} \cos \omega t + je^{\sigma t} \sin \omega t$$

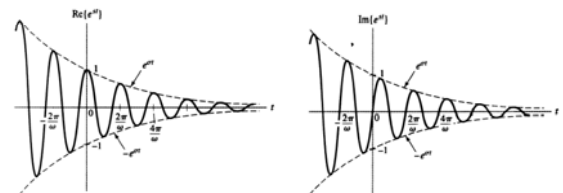


Figure 6.1 Real and imaginary parts of the complex exponential e^{st} , $s = \sigma + j\omega$.

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6

Convergence

- necessary condition for convergence: absolutely integrability of $x(t)e^{-\sigma t}$.

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt < \infty \implies \lim_{t \rightarrow \infty} x(t)e^{-\sigma t} = 0$$

- Region of convergence (ROC):** the region of σ which the Laplace transform converges.

Ex. 6.1 $x(t) = e^{at}u(t)$

$$\implies X(s) = \frac{1}{s-a},$$

$$\text{Re}(s) = \sigma > a.$$

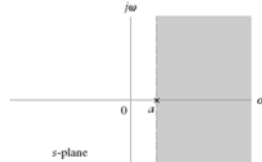


Figure 6.4 ROC for $x(t) = e^{at}u(t)$

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7

Region of convergence

Example 6.2 An anticausal signal is zero for $t > 0$. Determine the Laplace transform and ROC for the anticausal signal $y(t) = -e^{at}u(-t)$

<Sol.>

$$Y(s) = \int_{-\infty}^{\infty} -e^{at}u(-t)e^{-st} dt$$

$$= -\int_{-\infty}^0 e^{-(s-a)t} dt$$

$$= \frac{1}{s-a} e^{-(s-a)t} \Big|_{-\infty}^0$$

$$= \frac{1}{s-a},$$

$$\text{Re}(s) = \sigma < a.$$

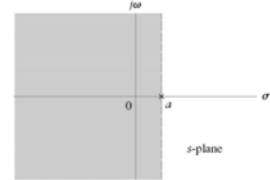


Figure 6.5 ROC for $y(t) = -e^{at}u(-t)$.

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8

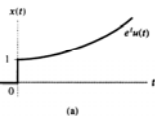
Region of convergence

- Laplace transforms of left- and right-sided exponentials have the **same form**; with left- and right-sided ROCs, respectively.

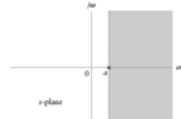
Time function

Laplace transform

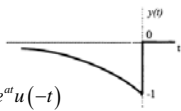
$$x(t) = e^{at}u(t)$$



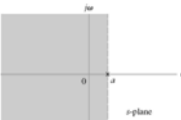
$$\frac{1}{s-a}$$



$$y(t) = -e^{at}u(-t)$$



$$\frac{1}{s-a}$$



2011-12-6

9

Relations between Laplace and Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$$

- If ROC includes the imaginary axis ($\sigma=0$), both Laplace transform and Fourier transform for $x(t)$ exist.

$$X(j\omega) = X(s) \Big|_{\sigma=0}$$

- If ROC does not include the imaginary axis, Laplace transform exists while Fourier transform is nonexistent.

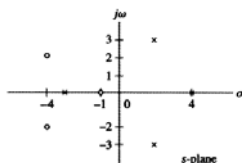
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10

Poles and Zeros

$$X(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \dots + b_0}{s^N + a_{N-1} s^{N-1} + \dots + a_1 s + a_0} = \frac{b_M \prod_{k=1}^M (s - c_k)}{\prod_{k=1}^N (s - d_k)}$$

- Zeros of $X(s)$:** the roots of the numerator polynomial c_k . "o"
- Poles of $X(s)$:** the roots of the denominator polynomial d_k . "x"



Zeros:

$$s = -1, \quad s = -4 + 2j$$

Poles:

$$s = -3, \quad s = 2 + 3j, \quad s = 4$$

11

Unilateral Laplace Transform

$$X(s) = \int_0^{\infty} x(t)e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{-st} ds$$

$$x(t) \xleftrightarrow{L} X(s)$$

- Lower limit of 0^- implies to include discontinuities and impulses that occur at $t = 0$ in the integral.
- The unilateral and bilateral Laplace transforms are equivalent for signals that are zero for $t < 0$.

Ex.

$$e^{at}u(t) \xleftrightarrow{L} \frac{1}{s-a} \quad \text{and} \quad e^{at}u(t) \xleftrightarrow{L} \frac{1}{s-a} \quad \text{with ROC } \text{Re}\{s\} > a.$$

12

Laplace Transform for Elementary Signals

$$\begin{aligned}
 e^{\lambda t} u(t) &\xleftrightarrow{L} \frac{1}{s-\lambda} \quad \text{Re}\{s\} > \lambda \\
 e^{-\lambda t} u(t) &\xleftrightarrow{L} \frac{1}{s+\lambda} \quad \text{Re}(s) > -\lambda \\
 e^{-j\omega_0 t} u(t) &\xleftrightarrow{L} \frac{1}{s+j\omega_0} \quad \text{Re}(s) > 0 \\
 e^{j\omega_0 t} u(t) &\xleftrightarrow{L} \frac{1}{s-j\omega_0} \quad \text{Re}(s) > 0 \\
 \cos \omega_0 t u(t) &= \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} u(t) \xleftrightarrow{L} \frac{1}{2} \left(\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0} \right) = \frac{s}{s^2 + \omega_0^2} \quad \text{Re}(s) > 0 \\
 \sin \omega_0 t u(t) &= \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} u(t) \xleftrightarrow{L} \frac{1}{2j} \left(\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0} \right) = \frac{\omega_0}{s^2 + \omega_0^2} \quad \text{Re}(s) > 0
 \end{aligned}$$

13

Laplace Transform for Elementary Signals

$$\begin{aligned}
 \delta(t) &\xleftrightarrow{L} 1 \quad \text{Re}(s) > -\infty \\
 \delta^{(n)}(t) &\xleftrightarrow{L} s^n \quad \text{Re}(s) > -\infty \\
 u(t) &\xleftrightarrow{L} \frac{1}{s} \quad \text{Re}(s) > 0 \\
 tu(t) &\xleftrightarrow{L} \frac{1}{s^2} \quad \text{Re}(s) > 0 \\
 t^n u(t) &\xleftrightarrow{L} \frac{n!}{s^{n+1}} \quad \text{Re}(s) > 0 \\
 te^{-\lambda t} u(t) &\xleftrightarrow{L} \frac{1}{(s+\lambda)^2} \quad \text{Re}(s) > -\lambda
 \end{aligned}$$

14

Laplace Transform for Elementary Signals

$$\begin{aligned}
 e^{-\sigma_0 t} \cos \omega_0 t u(t) &\xleftrightarrow{L} \frac{s + \sigma_0}{(s + \sigma_0)^2 + \omega_0^2} \quad \text{Re}(s) > -\sigma_0 \\
 e^{-\sigma_0 t} \sin \omega_0 t u(t) &\xleftrightarrow{L} \frac{\omega_0}{(s + \sigma_0)^2 + \omega_0^2} \quad \text{Re}(s) > -\sigma_0 \\
 t \cos \omega_0 t u(t) &\xleftrightarrow{L} \frac{s^2 - \omega_0^2}{(s^2 + \omega_0^2)^2} \quad \text{Re}(s) > 0 \\
 t \sin \omega_0 t u(t) &\xleftrightarrow{L} \frac{2\omega_0 s}{(s^2 + \omega_0^2)^2} \quad \text{Re}(s) > 0
 \end{aligned}$$

15

Properties of Unilateral Laplace Transform

$$x(t) \xleftrightarrow{L_y} X(s) \quad y(t) \xleftrightarrow{L_y} Y(s)$$

Linearity

$$ax(t) + by(t) \xleftrightarrow{L_y} aX(s) + bY(s)$$

Scaling

$$x(at) \xleftrightarrow{L_y} \frac{1}{a} X\left(\frac{s}{a}\right) \quad \text{for } a > 0.$$

Time shift

$$x(t-\tau) \xleftrightarrow{L_y} e^{-s\tau} X(s)$$

for all τ such that $x(t-\tau)u(t) = x(t-\tau)u(t-\tau)$.

16

Properties of Unilateral Laplace Transform

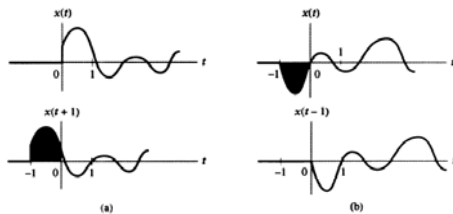


Figure 6.6 Time shifts for which unilateral Laplace transform time-shift property does not apply.

Ex. $u(t) \xleftrightarrow{L_y} 1/s$

$$u(t+3) \xleftrightarrow{L_y} \int_0^\infty u(t+3)e^{-st} dt = \int_0^\infty u(t)e^{-st} dt = 1/s$$

$$u(t-3) \xleftrightarrow{L_y} \int_0^\infty u(t-3)e^{-st} dt = \int_{-3}^\infty e^{-st} dt = e^{-3s}/s$$

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17

Properties of Unilateral Laplace Transform

s-Domain Shift

$$e^{s_0 t} x(t) \xleftrightarrow{L_y} X(s-s_0)$$

Ex. $\cos \omega_0 t u(t) \xleftrightarrow{L_y} \frac{s}{s^2 + \omega_0^2}$

$$\Rightarrow e^{-\lambda t} \cos \omega_0 t u(t) \xleftrightarrow{L_y} \frac{s + \lambda}{(s + \lambda)^2 + \omega_0^2}$$

Convolution

$$x(t) * y(t) \xleftrightarrow{L_y} X(s)Y(s)$$

only when $x(t)=0$ and $y(t)=0$ for $t < 0$.

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18

Properties of Unilateral Laplace Transform

Differentiation in the s-Domain

$$-tx(t) \xleftrightarrow{L_t} \frac{d}{ds} X(s)$$

Ex. $u(t) \xleftrightarrow{L_t} \frac{1}{s}$

$$tu(t) \xleftrightarrow{L_t} -\frac{d}{ds} \left(\frac{1}{s} \right) = \frac{1}{s^2}$$

$$t^2 u(t) \xleftrightarrow{L_t} -\frac{d}{ds} \left(\frac{1}{s^2} \right) = \frac{2}{s^3}$$

$$t^n u(t) \xleftrightarrow{L_t} \frac{n!}{s^{n+1}} \quad t^n e^{-\lambda t} u(t) \xleftrightarrow{L_t} \frac{n!}{(s+\lambda)^{n+1}}$$

19

Properties of Unilateral Laplace Transform

Example 6.3. Find the unilateral Laplace transform of

$$x(t) = (-e^{3t} u(t)) * (tu(t)).$$

<Sol.>

$$u(t) \xleftrightarrow{L_t} \frac{1}{s}$$

$$\Rightarrow x_1(t) = -e^{3t} u(t) \xleftrightarrow{L_t} X_1(s) = -\frac{1}{s-3}$$

$$x_2(t) = tu(t) \xleftrightarrow{L_t} X_2(s) = \frac{1}{s^2}$$

$$x(t) = x_1(t) * x_2(t) \xleftrightarrow{L_t} X(s) = -\frac{1}{s-3} \cdot \frac{1}{s^2} = -\frac{1}{s^2(s-3)}$$

2011-12-6

20

Properties of Unilateral Laplace Transform

Differentiation in the time domain

$$\frac{d}{dt} x(t) \xleftrightarrow{L_t} sX(s) - x(0^-)$$

<p.f.> $L \left[\frac{d}{dt} x(t) \right] = \int_0^\infty \left(\frac{d}{dt} x(t) \right) e^{-st} dt$

$$= x(t) e^{-st} \Big|_0^\infty - \int_0^\infty x(t) (-se^{-st}) dt$$

$$= -x(0^-) + s \int_0^\infty x(t) e^{-st} dt = sX(s) - x(0^-)$$

$$\frac{d^2 x(t)}{dt^2} \xleftrightarrow{L_t} s^2 X(s) - sx(0^-) - x'(0^-)$$

$$\frac{d^n x(t)}{dt^n} \xleftrightarrow{L_t} s^n X(s) - s^{n-1} x(0^-) - s^{n-2} x'(0^-) - \dots - x^{(n-1)}(0^-)$$

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21

Properties of Unilateral Laplace Transform

Integration property

$$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{L_t} \frac{X(s)}{s} + \frac{x^{(-1)}(0^-)}{s} \quad \text{where } x^{(-1)}(0^-) = \int_{-\infty}^0 x(\tau) d\tau.$$

Ex. $tu(t) = \int_{-\infty}^t u(\tau) d\tau \xleftrightarrow{L_t} \frac{L[u(t)]}{s} + \frac{1}{s} \int_{-\infty}^0 u(\tau) d\tau = \frac{1}{s^2}$

Initial-value theorem

$$\lim_{s \rightarrow \infty} sX(s) = x(0^+)$$

Final-value theorem

$$\lim_{s \rightarrow 0} sX(s) = x(\infty)$$

22

Inversion of Unilateral Laplace Transform

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{-st} ds$$

Direct inversion by contour integration

Inversion by partial-fraction expansion (部分分式法)

$$X(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \dots + b_0}{s^N + a_{N-1} s^{N-1} + \dots + a_1 s + a_0} \quad (M \geq N)$$

$$= c_0 + c_1 s + c_2 s^2 + \dots + c_{M-N} s^{M-N} + \frac{D(s)}{A(s)} \Rightarrow$$

$$x(t) = c_0 \delta(t) + c_1 \delta'(t) + c_2 \delta''(t) + \dots + c_{M-N} \delta^{(M-N)}(t) + L^{-1} \left[\frac{D(s)}{A(s)} \right]$$

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23

Inversion of Unilateral Laplace Transform

$$\frac{D(s)}{A(s)} = \frac{b_P s^P + b_{P-1} s^{P-1} + \dots + b_0}{\prod_{k=1}^N (s - d_k)} \quad (P < N)$$

Case1: all the poles are distinct: $s = d_1, d_2, \dots, d_N$

$$\frac{D(s)}{A(s)} = \frac{A_1}{s - d_1} + \frac{A_2}{s - d_2} + \dots + \frac{A_N}{s - d_N} \quad \text{where } A_k = (s - d_k) \frac{D(s)}{A(s)} \Big|_{s=d_k}$$

$$A_k e^{d_k t} u(t) \xleftrightarrow{L_t} \frac{A_k}{s - d_k}$$

$$\Rightarrow L^{-1} \left[\frac{D(s)}{A(s)} \right] = (A_1 e^{d_1 t} + A_2 e^{d_2 t} + \dots + A_N e^{d_N t}) u(t)$$

2011-12-6

24

Inversion of Unilateral Laplace Transform

Ex. Find the inverse Laplace transform of $X(s) = \frac{s+2}{s^3+4s^2+3s}$.

<Sol.> $X(s) = \frac{s+2}{s^3+4s^2+3s} = \frac{s+2}{s(s+1)(s+3)} = \frac{A_1}{s} + \frac{A_2}{s+1} + \frac{A_3}{s+3}$

$$A_1 = (s-0)X(s)\Big|_{s=0} = \frac{s+2}{(s+1)(s+3)}\Big|_{s=0} = \frac{2}{3}$$

$$A_2 = (s+1)X(s)\Big|_{s=-1} = \frac{s+2}{s(s+3)}\Big|_{s=-1} = -\frac{1}{2}$$

$$A_3 = (s+3)X(s)\Big|_{s=-3} = \frac{s+2}{s(s+1)}\Big|_{s=-3} = -\frac{1}{6}$$

$$\Rightarrow x(t) = (2/3 - e^{-t}/2 - e^{-3t}/6)u(t)$$

2011-12-6

25

Inversion of Unilateral Laplace Transform

□ Case 2: a pole is repeated N times $s = d_1 = d_2 = \dots = d_N = d$

$$\frac{D(s)}{A(s)} = \frac{D(s)}{(s-d)^N} = \frac{A_1}{s-d} + \frac{A_2}{(s-d)^2} + \dots + \frac{A_N}{(s-d)^N}$$

where $A_k = \frac{1}{(N-k)!} \cdot \frac{d^{N-k}}{ds^{N-k}} \left[(s-d)^N \frac{D(s)}{A(s)} \right] \Big|_{s=d}$

$$\frac{A_k t^{k-1}}{(n-1)!} e^{dt} u(t) \xleftarrow{L^{-1}} \frac{A_k}{(s-d)^n}$$

$$\Rightarrow L^{-1} \left[\frac{D(s)}{A(s)} \right] = \left(A_1 + A_2 t + \dots + \frac{A_N t^{N-1}}{N-1} \right) e^{dt} u(t)$$

26

Inversion of Unilateral Laplace Transform

Ex. Find the inverse Laplace transform of $X(s) = \frac{s-2}{s(s+1)^3}$.

<Sol.> $X(s) = \frac{s-2}{s(s+1)^3} = \frac{A_1}{s} + \frac{A_2}{s+1} + \frac{A_3}{(s+1)^2} + \frac{A_4}{(s+1)^3}$

$$A_1 = sX(s)\Big|_{s=0} = \frac{s-2}{(s+1)^3}\Big|_{s=0} = -2$$

$$A_4 = \frac{1}{0!} (s+1)^3 X(s)\Big|_{s=-1} = \frac{s-2}{s}\Big|_{s=-1} = 3$$

$$A_3 = \frac{1}{1!} \cdot \frac{d}{ds} (s+1)^3 X(s)\Big|_{s=-1} = \left(\frac{s-2}{s} \right)' \Big|_{s=-1} = 2$$

$$A_2 = \frac{1}{2!} \cdot \frac{d^2}{ds^2} (s+1)^3 X(s)\Big|_{s=-1} = \frac{1}{2} \left(\frac{s-2}{s} \right)'' \Big|_{s=-1} = 2$$

$$\Rightarrow x(t) = (-2 + 2e^{-t} + 2te^{-t} - 3t^2 e^{-t}/2)u(t)$$

2011-12-6

27

Inversion of Unilateral Laplace Transform

Ex 6.8 Inverting an improper rational Laplace transform.
Find the inverse Laplace transform of

$$X(s) = \frac{2s^3 - 9s^2 + 4s + 10}{s^2 - 3s - 4}$$

<Sol.> $X(s) = \frac{2s^3 - 9s^2 + 4s + 10}{s^2 - 3s - 4} = \frac{2s-3}{s^2-3s-4} + \frac{s^2-3s-4}{s^2-3s-4} \cdot \frac{2s-3}{s^2-3s-4}$

$$= 2s-3 + \frac{3s-2}{s^2-3s-4}$$

$$= 2s-3 + \frac{1}{s+1} + \frac{2}{s-4}$$

$$\Rightarrow x(t) = 2\delta^{(1)}(t) - 3\delta(t) + e^{-t}u(t) + 2e^{4t}u(t)$$

28

Inversion of Unilateral Laplace Transform

□ Case 3: a pair of complex-conjugate poles $s = \sigma \pm j\omega$

$$\frac{D(s)}{A(s)} = \frac{A_1}{s-(\sigma+j\omega)} + \frac{A_2}{s-(\sigma-j\omega)}$$

In order for this sum to represent a real-valued signal, A_1 and A_2 must be complex conjugates of each other.

$$\frac{D(s)}{A(s)} = \frac{B_1 s + B_2}{(s-\sigma-j\omega)(s-\sigma+j\omega)} = \frac{C_1(s-\sigma)}{(s-\sigma)^2 + \omega^2} + \frac{C_2 \omega}{(s-\sigma)^2 + \omega^2}$$

where $C_1 = B_1$, $C_2 = \frac{B_1 \sigma + B_2}{\omega}$.

$$\Rightarrow L^{-1} \left[\frac{D(s)}{A(s)} \right] = C_1 e^{\sigma t} \cos(\omega t) u(t) + C_2 e^{\sigma t} \sin(\omega t) u(t)$$

29

Inversion of Unilateral Laplace Transform

Ex. Find the inverse Laplace transform of $X(s) = \frac{4s^2+6}{s^3+s^2-2}$.

<Sol.> $X(s) = \frac{4s^2+6}{(s^3-1)+(s^2-1)} = \frac{A}{s-1} + \frac{B_1 s + B_2}{(s+1)^2 + 1}$

$$A = (s-1)X(s)\Big|_{s=1} = \frac{4s^2+6}{(s+1)^2+1}\Big|_{s=1} = 2$$

$$4s^2+6 = 2((s+1)^2+1) + (B_1 s + B_2)(s+1) \Rightarrow B_1 = 2, B_2 = -2$$

$$\left\{ \begin{array}{l} C_1 = 2, \\ C_2 = \frac{B_1 \sigma + B_2}{\omega} = -4 \end{array} \right. \Rightarrow X(s) = \frac{2}{s-1} + 2 \frac{s+1}{(s+1)^2+1} - 4 \frac{1}{(s+1)^2+1}$$

$$\Rightarrow x(t) = (2e^t + 2e^{-t} \cos t - 4e^{-t} \sin t)u(t)$$

30

Inversion of Unilateral Laplace Transform

Ex. Find the inverse Laplace transform of $X(s) = \frac{1}{3s^2(s^2+4)}$.

<Sol.> $X(s) = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{B_1s+B_2}{s^2+4}$

$$A_2 = s^2 X(s) \Big|_{s=0} = \frac{1}{3(s^2+4)} \Big|_{s=0} = \frac{1}{12}; \quad A_1 = \frac{1}{1!} \cdot \frac{d}{ds} s^2 X(s) \Big|_{s=0} = \left(\frac{1}{3(s^2+4)} \right) \Big|_{s=0} = 0$$

$$1/3 = (s^2+4)/12 + (B_1s+B_2)s^2 \Rightarrow B_1=0, \quad B_2 = -\frac{1}{12}$$

$$C_1 = 2, \quad C_2 = \frac{B_1\sigma + B_2}{\omega} = -4 \Rightarrow X(s) = \frac{1}{12} \cdot \frac{1}{s^2} - \frac{1}{24} \cdot \frac{2}{s^2+4}$$

$$\Rightarrow x(t) = \frac{1}{12} \left(t - \frac{1}{2} \sin 2t \right) u(t)$$

31

Inversion of Unilateral Laplace Transform

Ex. Find the inverse Laplace transform of $X(s) = \frac{1-e^{-2s}}{s(s^2+4)}$.

<Sol.> $X(s) = \frac{1}{s(s^2+4)} + \frac{-e^{-2s}}{s(s^2+4)}$

$$X_1(s) = \frac{1}{s(s^2+4)} = \frac{A_1}{s} + \frac{B_1s+B_2}{s^2+4}, \quad A_1 = sX(s) \Big|_{s=0} = \frac{1}{(s^2+4)} \Big|_{s=0} = \frac{1}{4};$$

$$1 = (s^2+4)/4 + (B_1s+B_2)s \Rightarrow B_1 = -1/4, \quad B_2 = 0$$

$$C_1 = -1/4, \quad C_2 = B_1\sigma + B_2/\omega = 0$$

$$X_1(s) = \frac{1}{4} \cdot \frac{1}{s^2} - \frac{1}{4} \cdot \frac{s}{s^2+4} \Rightarrow x_1(t) = \frac{1}{4} (1 - \cos 2t) u(t)$$

$$x(t) = x_1(t) - x_1(t-2) = \frac{1}{4} (1 - \cos 2t) u(t) - \frac{1}{4} [1 - \cos 2(t-2)] u(t-2)$$

32

Solving Differential Equations with Initial Conditions

$$\frac{d^2y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_2 y(t) = b_0 \frac{d^2x(t)}{dt^2} + b_1 \frac{dx(t)}{dt} + b_2 x(t)$$

Determine $y(t)$ with specified $x(t)$ and initial conditions $y(0^-)$, $y'(0^-)$.

$$[s^2Y(s) - sy(0^-) - y'(0^-)] + a_1[sY(s) - y(0^-)] + a_2Y(s) = b_0s^2X(s) + b_1sX(s) + b_2X(s)$$

$$Y(s) = \frac{b_0s^2 + b_1s + b_2}{s^2 + a_1s + a_2} X(s) + \frac{sy(0^-) + y'(0^-) + a_1y(0^-)}{s^2 + a_1s + a_2}$$

$$= Y^{(f)}(s) + Y^{(n)}(s) \Rightarrow y(t) = y^{(f)}(t) + y^{(n)}(t)$$

- Forced response $Y^{(f)}(s)$: response to the input
- Natural response $Y^{(n)}(s)$: response to the initial conditions

33

Solving Differential Equations with Initial Conditions

Ex. Use the unilateral Laplace transform to determine the output of a system

$$y''(t) + 5y'(t) + 6y(t) = 2x'(t) + 8x(t)$$

in response to input $x(t) = e^{-t}u(t)$, and initial conditions $y(0^-) = 3$, $y'(0^-) = 2$.

<Sol.> $[s^2Y(s) - sy(0^-) - y'(0^-)] + 5[sY(s) - y(0^-)] + 6Y(s) = 2sX(s) + 8X(s)$

$$Y(s) = \frac{2s+8}{s^2+5s+6} X(s) + \frac{sy(0^-) + y'(0^-) + 5y(0^-)}{s^2+5s+6}$$

$$Y^{(f)}(s) = \frac{2s+8}{s^2+5s+6} \cdot \frac{1}{s+1} = \frac{3}{s+1} - \frac{4}{s+2} + \frac{1}{s+3}$$

$$\Rightarrow y^{(f)}(t) = (3e^{-t} - 4e^{-2t} + e^{-3t})u(t)$$

$$Y^{(n)}(s) = \frac{3s+17}{s^2+5s+6} = \frac{11}{s+2} - \frac{8}{s+3} \Rightarrow y^{(n)}(t) = (11e^{-2t} - 8e^{-3t})u(t)$$

$$\Rightarrow y(t) = y^{(f)}(t) + y^{(n)}(t) = (3e^{-t} + 7e^{-2t} - 7e^{-3t})u(t)$$

34

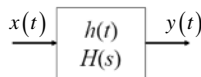
The Transfer Function (系统函数)

- Transfer function: for an LTI system with impulse response $h(t)$

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

$$y(t) = h(t) * x(t)$$

$$Y(s) = H(s) X(s) \Rightarrow H(s) = \frac{Y(s)}{X(s)}$$



- Furthermore, for an input $x(t) = e^{st}$ to the LTI system

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = e^{st} H(s)$$

- Eigenfunction of the system: e^{st}
- Eigenvalue: $H(s)$

35

Transfer Function and Differential Equation

$$\sum_{k=0}^N a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^M b_k \frac{d^k}{dt^k} x(t)$$

If Initial conditions equal zero, and $x(t) = e^{st}$

$$\left(\sum_{k=0}^N a_k \frac{d^k}{dt^k} \{e^{st}\} \right) H(s) = \sum_{k=0}^M b_k \frac{d^k}{dt^k} \{e^{st}\}$$

$$\frac{d^k}{dt^k} \{e^{st}\} = s^k e^{st} \Rightarrow H(s) = \frac{\sum_{k=0}^M b_k s^k}{\sum_{k=0}^N a_k s^k}$$

Transfer function \longleftrightarrow Differential equation system description

36

Transfer Function and Differential Equation

Ex. Find the transfer function of the LTI system described by the differential equation

$$y''(t) + 7y'(t) + 10y(t) = 2x'(t) + x(t)$$

<Sol.>

$$(s^2 + 7s + 10)Y(s) = (2s + 1)X(s) \Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{2s + 1}{s^2 + 7s + 10}$$

Ex. Find a differential-equation description of the systems described by the following transfer function

$$(a) H(s) = \frac{s^2 - 2}{s^3 - 3s + 1} \Rightarrow y'''(t) - 3y'(t) + y(t) = x''(t) - 2x(t)$$

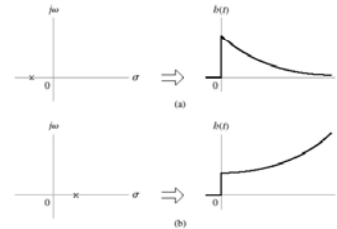
$$(b) H(s) = \frac{2(s+1)(s-1)}{s(s+2)(s+1)} = \frac{2s^2 - 2}{s^3 + 3s^2 + 2s} \Rightarrow y'''(t) + 3y''(t) + 2y'(t) = 2x''(t) - 2x'(t)$$

37

Causality and Stability (因果性与稳定性)

■ For a causal system: $h(t) = 0$ for $t < 0$.

- A pole in the left half of the s-plane corresponds to an exponentially decaying impulse response.
- A pole in the right half of the s-plane corresponds to an exponentially increasing impulse response \rightarrow **unstable**.

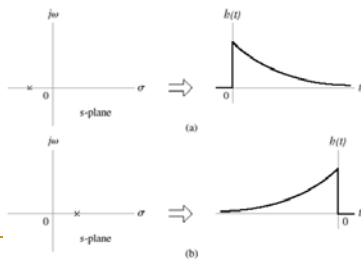


38

Causality and Stability (因果性与稳定性)

■ For a stable system: $\int_{-\infty}^{\infty} |h(\tau)| d\tau = S < \infty$.

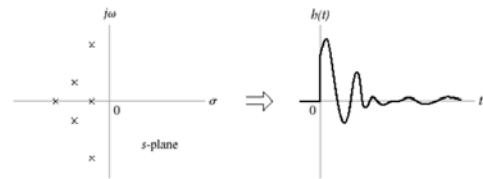
- A pole in the left half of the s-plane corresponds to a right-sided impulse response.
- A pole in the right half of the s-plane corresponds to a left-sided impulse response \rightarrow **noncausal**.



39

Causality and Stability (因果性与稳定性)

■ A system that is both stable and causal must have a transfer function with **all of its poles in the left half of the s-plane**.



40

Causality and Stability (因果性与稳定性)

Ex. A system has the transfer function $H(s) = \frac{2}{s+3} + \frac{1}{s-2}$

Find the impulse response: a) the system is stable; b) the system is causal. Can this system be both stable and causal?

<Sol.> The system has two poles: $s = -3$, $s = 2$.

It cannot be both stable and causal!

a) the system is stable

$$y(t) = 2e^{-3t}u(t) - e^{2t}u(-t)$$

b) the system is causal

$$y(t) = 2e^{-3t}u(t) + e^{-2t}u(t)$$

41

Inverse Systems

$$h^{inv}(t) * h(t) = \delta(t)$$

$$\Rightarrow H^{inv}(s)H(s) = 1$$

$$\text{or } H^{inv}(s) = \frac{1}{H(s)} = \frac{\prod_{k=1}^N (s - d_k)}{b_M \prod_{k=1}^M (s - c_k)}$$

- The zeros of the inverse system are the poles of $H(s)$, and the poles of the inverse system are the zeros of $H(s)$.
- **Minimum phase system:** have a transfer function with all of its **poles and zeros** in the left half of the s-plane.
 - Unique relationship between the magnitude and phase response.

42

Inverse Systems

Ex. Consider an LTI system described by

a) differential equation: $y'(t) + 3y(t) = x''(t) + x'(t) - 2x(t)$

b) Impulse response: $h(t) = \delta(t) + e^{-3t}u(t) + 2e^{-t}u(t)$

Find the transfer function of the inverse system. Does a stable and causal inverse system exist?

<Sol.>

a) $H(s) = \frac{s^2 + s - 2}{s + 3} \Rightarrow H^{inv}(s) = \frac{s + 3}{s^2 + s - 2} = \frac{s + 3}{(s - 1)(s + 2)}$

The system has two poles: $s = 1, s = -2$. **cannot be both stable and causal!**

b) $H(s) = 1 + \frac{1}{s + 3} + \frac{2}{s + 1} = \frac{s^2 + 7s + 10}{(s + 3)(s + 1)}$
 $\Rightarrow H^{inv}(s) = \frac{(s + 3)(s + 1)}{s^2 + 7s + 10} = \frac{s^2 + 4s + 3}{(s + 2)(s + 5)}$

The system has two poles: $s = -2, s = -5$. **both stable and causal!**

43

Summary

■ The Laplace Transform

- Introduction
- Definition
- The Unilateral Laplace Transform
- Property of The Unilateral Laplace Transform
- Inversion of The Unilateral Laplace Transform
- Solving Differential Equations with Initial Conditions
- The Transfer Function
- Causality and Stability

■ Reference in textbook: 6.1~6.6

■ Homework: 6.29, 6.36, 6.37(a,c,f,h), 6.38(a,c)

44