Signals and Systems 4.1

--- *Applications of Fourier Representations to Mixed Signal Classes*

School of Information & Communication Engineering，BUPT

Reference:
1. Textbook: Chapter 4

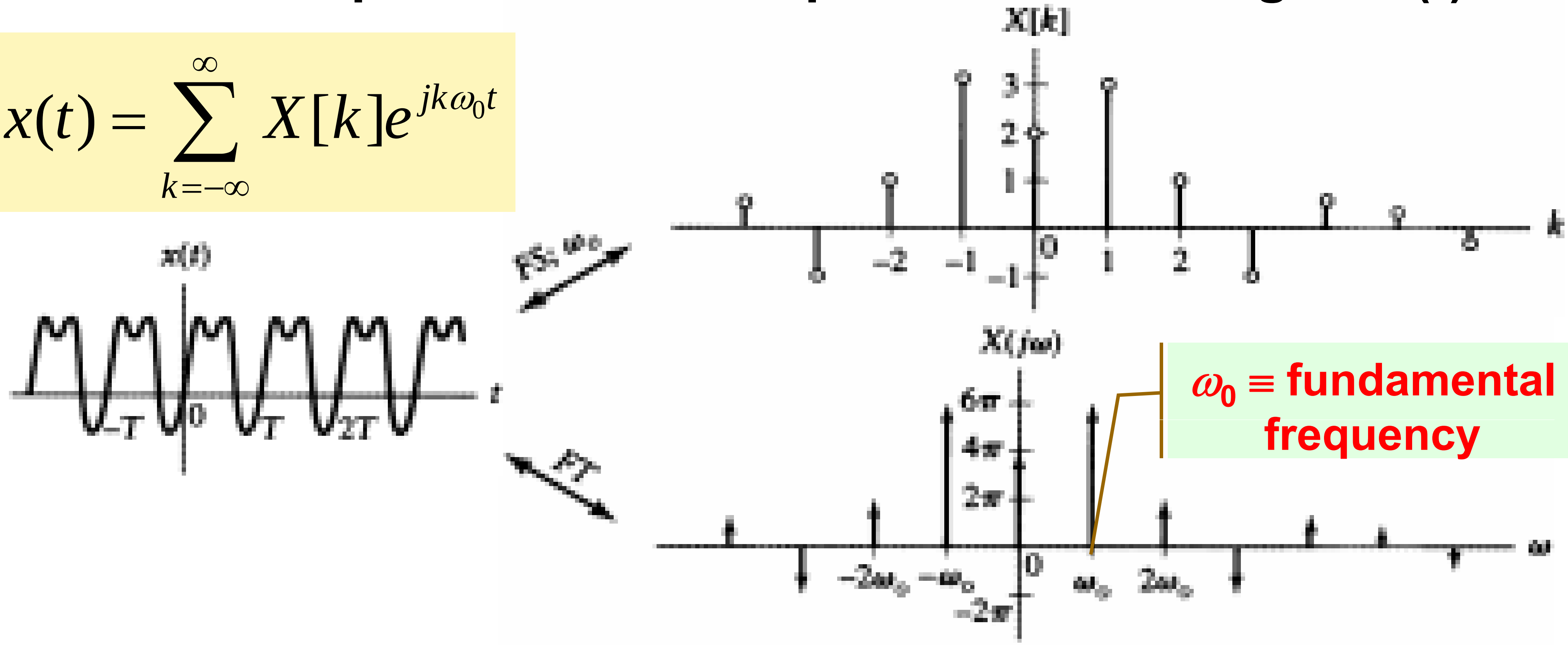
Applications of Fourier Representations to Mixed Signal Classes

- Two cases of mixed signals to be studied
  - Periodic and nonperiodic signals
  - Continuous- and discrete-time signals
- Fourier Transform of Periodic Signals
- Convolution and modulation with mixtures of periodic and Non-periodic signals
- Fourier Transform Representations of Discrete-time signals
- Sampling and Reconstruction

FT of CT Periodic Signals

FS and FT representation of a periodic CT signal $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$$



$\omega_0 \equiv$ fundamental frequency

$$1 \xleftrightarrow{FT} 2\pi\delta(\omega)$$

Frequency shift

$$e^{jk\omega_0 t} \xleftrightarrow{FT} 2\pi\delta(\omega - k\omega_0)$$

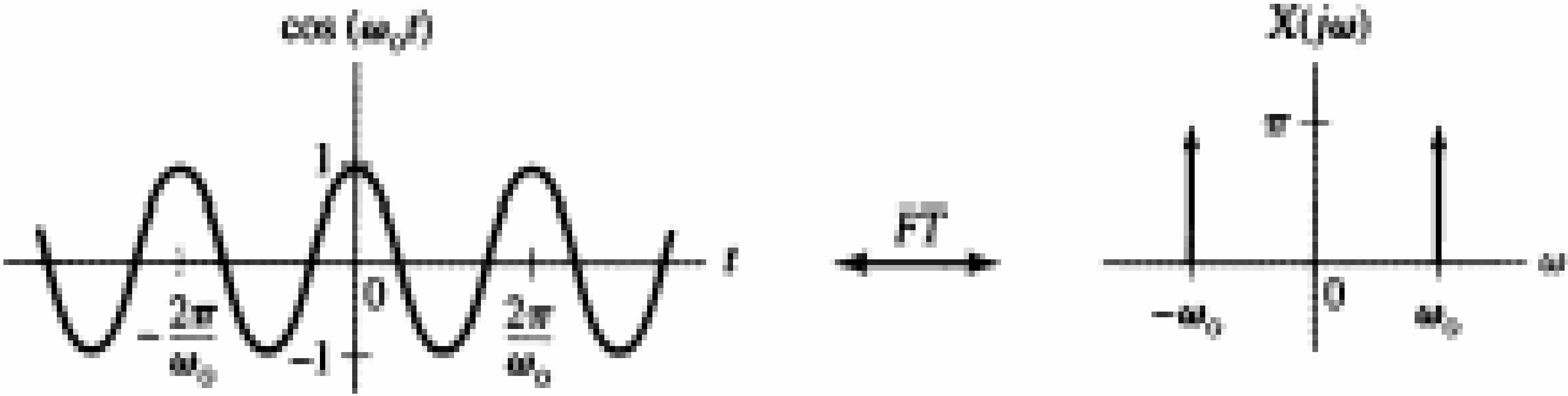
$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} \xleftrightarrow{FT} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$$

Example 4.1 *FT of a Cosine*

Find the FT representation of $x(t) = \cos(\omega_o t)$

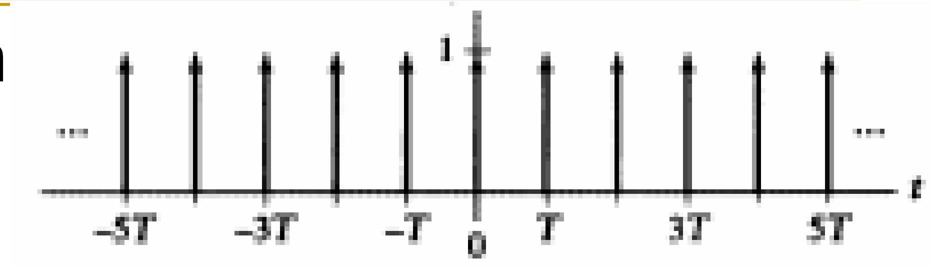
$$\cos(\omega_0 t) \xleftrightarrow{FS;\omega_0} X[k] = \begin{cases} \frac{1}{2}, & k = \pm 1 \\ 0, & k \neq 1 \end{cases}$$

$$\cos(\omega_0 t) \xleftrightarrow{FT} X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



Example 4.2 *FT of a Unit Impulse Train*

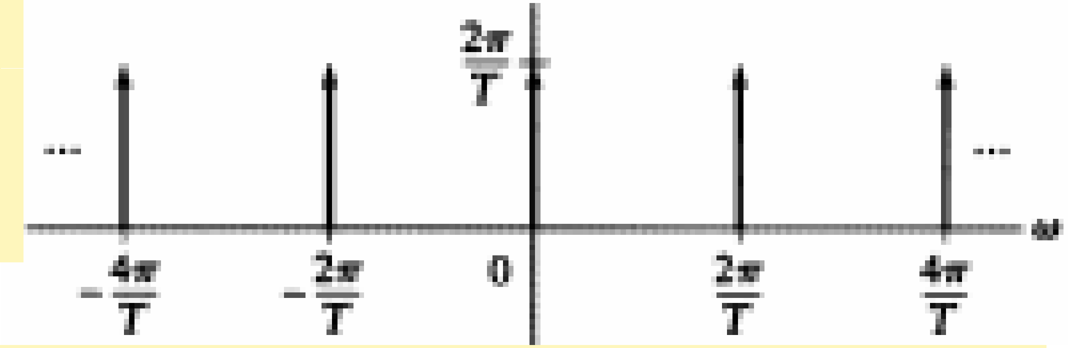
Find the FT of the impulse train



$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Impulse spacing is inversed each other; the strength of impulses differ by a factor of 2π/*T*

$$X[k] = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt$$
$$= 1/T$$



$$\delta_T(t) \xleftrightarrow{FT} 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{T} \delta(\omega - k\omega_0) = \omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

ω_0 **= 2π/*T***

Example 1

Suppose a periodic signal x(t) with the period *T* = 2.

$$X[k] = \begin{cases} (\frac{1}{3})^k & |k| \le 5 \\ 0 & otherwise \end{cases}$$

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$$

$$\omega_0 = \frac{2\pi}{T} = \pi$$

$$= 2\pi \sum_{k=-5}^{5} (\frac{1}{3})^k \delta(\omega - k\pi)$$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} \xleftrightarrow{FT} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$$

DTFT of DT Periodic Signals

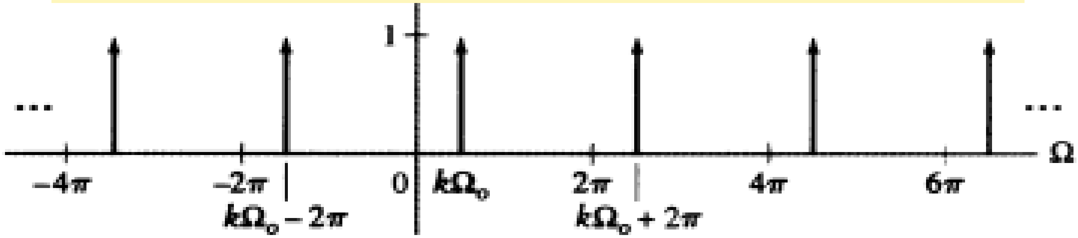
DTFS and DTFT representation of an *N*-periodic DT signal *x*[*n*]

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n}$$

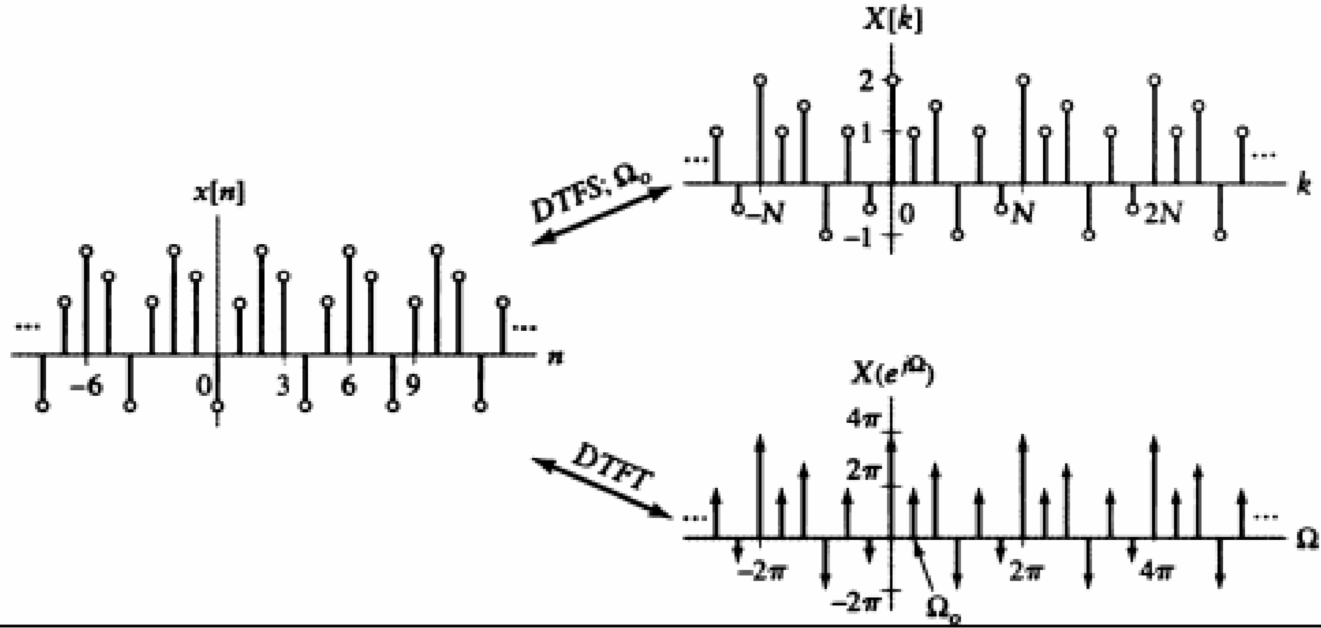
$$x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_0 n}$$

DTFS

$$e^{jk\Omega_0 n} \xleftrightarrow{DTFT} 2\pi\sum_{m=-\infty}^{\infty}\delta\left(\Omega - k\Omega_0 - m2\pi\right).$$

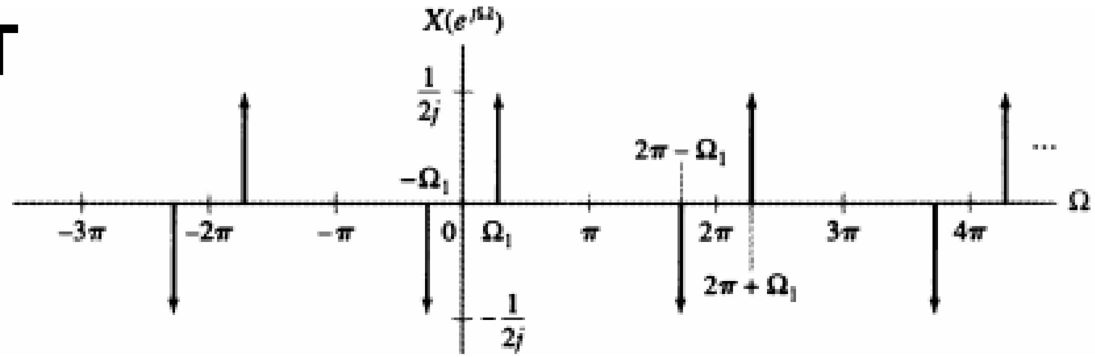


$$\xleftrightarrow{DTFT} X\left(e^{j\Omega}\right) = 2\pi\sum_{k=-\infty}^{\infty} X[k]\sum_{m=-\infty}^{\infty}\delta\left(\Omega - k\Omega_0 - m2\pi\right) = 2\pi\sum_{k=-\infty}^{\infty} X[k]\delta\left(\Omega - k\Omega_0\right)$$



Example 4.3 *DTFT of a Periodic DT Signal*

Determine the inverse DTFT of the frequency-domain representation depicted in Fig, where $\Omega_1 = \pi / N$.



$$X\left(e^{j\Omega}\right) = \frac{1}{2j}\delta\left(\Omega - \Omega_1\right) - \frac{1}{2j}\delta\left(\Omega + \Omega_1\right), \quad -\pi < \Omega \le \pi$$

$$X[k] = \begin{cases} 1/(4\pi j), & k = 1 \\ -1/(4\pi j), & k = -1 \\ 0, & \text{otherwise on } -1 \le k \le N-2 \end{cases}$$

$$x[n] = \frac{1}{2\pi}\left[\frac{1}{2j}\left(e^{j\Omega_1 n} - e^{-j\Omega_1 n}\right)\right] = \frac{1}{2\pi}\sin\left(\Omega_1 n\right)$$

Convolution with Mixtures of Periodic and Nonperiodic Signals

$y(t) = x(t) * h(t)$

Periodic input $x(t)$ → **Stable filter** Nonperiodic IR $h(t)$ →

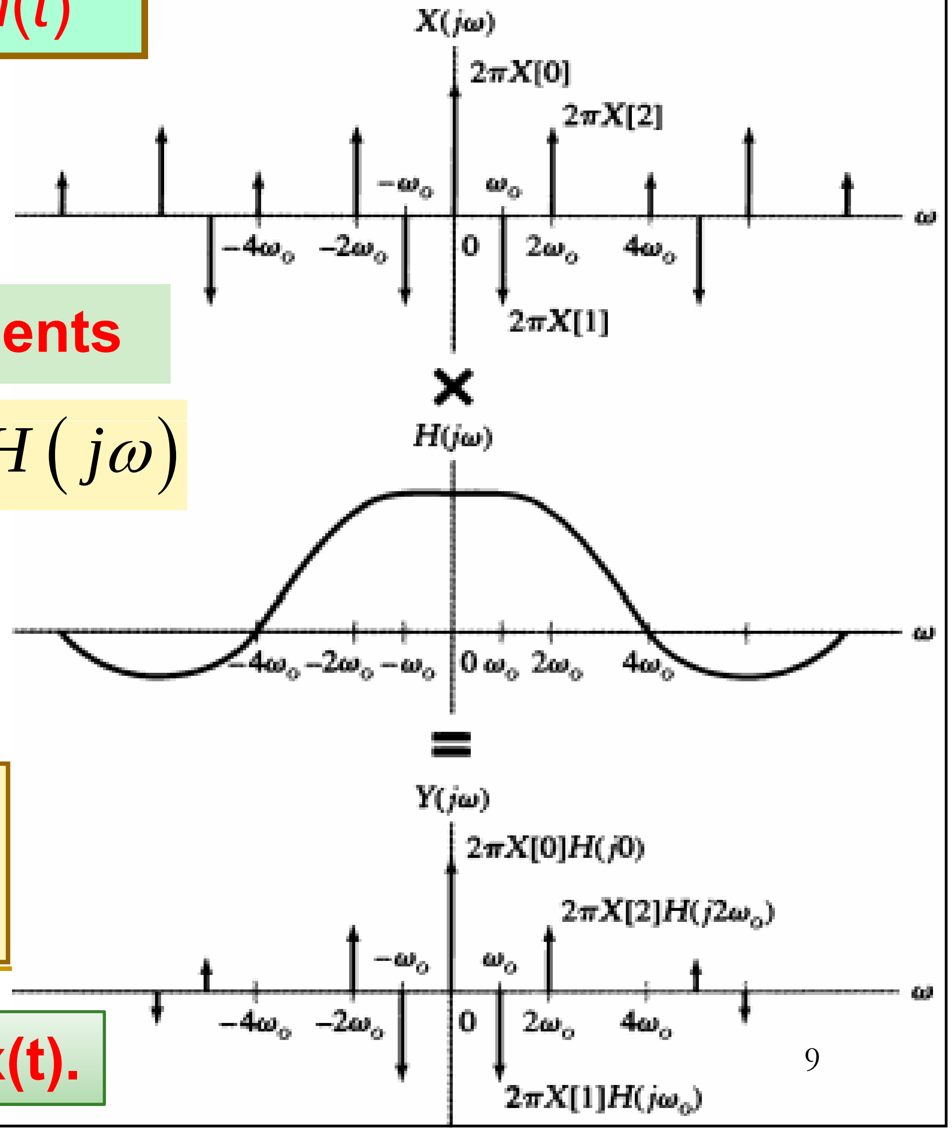
$$x(t) \xleftrightarrow{FT} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)$$

FS coefficients

$$y(t) = x(t) * h(t) \xleftrightarrow{FT} Y(j\omega) = X(j\omega)H(j\omega)$$

$$Y(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_0)H(j\omega)$$

$$\boxed{= 2\pi \sum_{k=-\infty}^{\infty} H(jk\omega_0)X[k]\delta(\omega - k\omega_0)}$$



Y(t) is periodic with the same period as x(t).

Multiplication with Mixtures of Periodic and Nonperiodic Signals

$$y(t)=g(t)x(t)\xleftrightarrow{FT}Y(j\omega)=\frac{1}{2\pi}G(j\omega)*X(j\omega)$$

$$x(t)=\sum_{k=-\infty}^{\infty}X[k]e^{jk\omega_0 t}$$

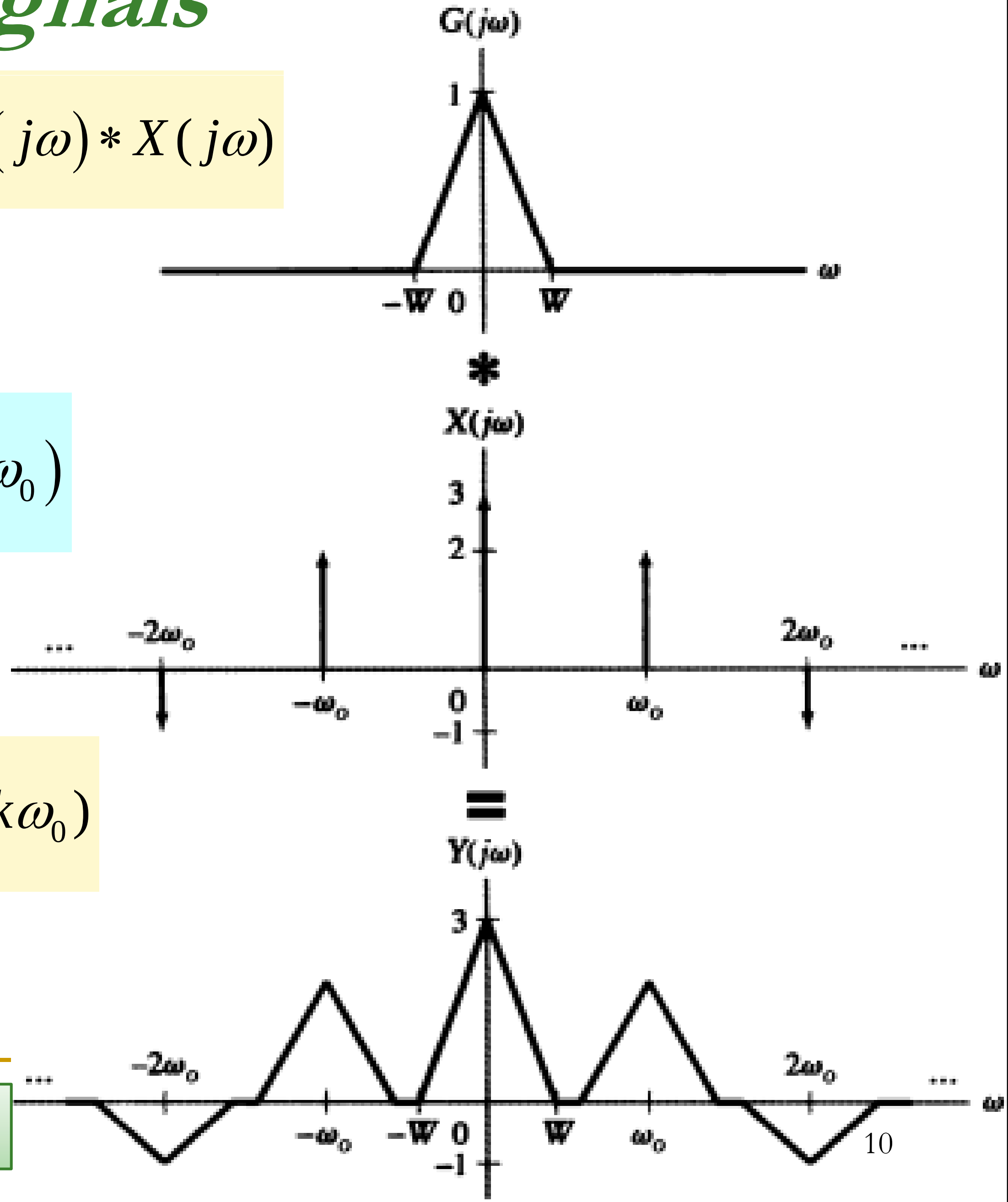
$$\xleftrightarrow{FT}X(j\omega)=2\pi\sum_{k=-\infty}^{\infty}X[k]\delta(\omega-k\omega_0)$$

$$y(t)=g(t)x(t)$$

$$\xleftrightarrow{FT}Y(j\omega)=G(j\omega)*\sum_{k=-\infty}^{\infty}X[k]\delta(\omega-k\omega_0)$$

$$Y(j\omega)=\sum_{k=-\infty}^{\infty}X[k]G(j(\omega-k\omega_0))$$

***y*(*t*) becomes nonperiodic signal !**



FT of Discrete-Time Signals

$CT: \; x(t) = e^{j\omega t}$

$g[n] \triangleq x(t)|_{t=nT_s} = x(nT_s) = e^{j\omega nT_s}$

$DT: \; g[n] = e^{j\Omega n}$ ⇒ $e^{j\Omega n} = e^{j\omega T_s n}$

$\Omega = \omega T_s$

Relationship between CT and DT frequency: $\Omega = \omega T_s$

$DT: x[n] \xleftrightarrow{DTFT} X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

$|_{\Omega=\omega T_s} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega T_s n}$

$x_\delta(t)$ **is a CT signal corresponds to** $x[n]$

$$X_\delta(j\omega) \triangleq \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega T_s n}$$

$CT: x_\delta(t) \xleftrightarrow{FT} X_\delta(j\omega)$

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$$

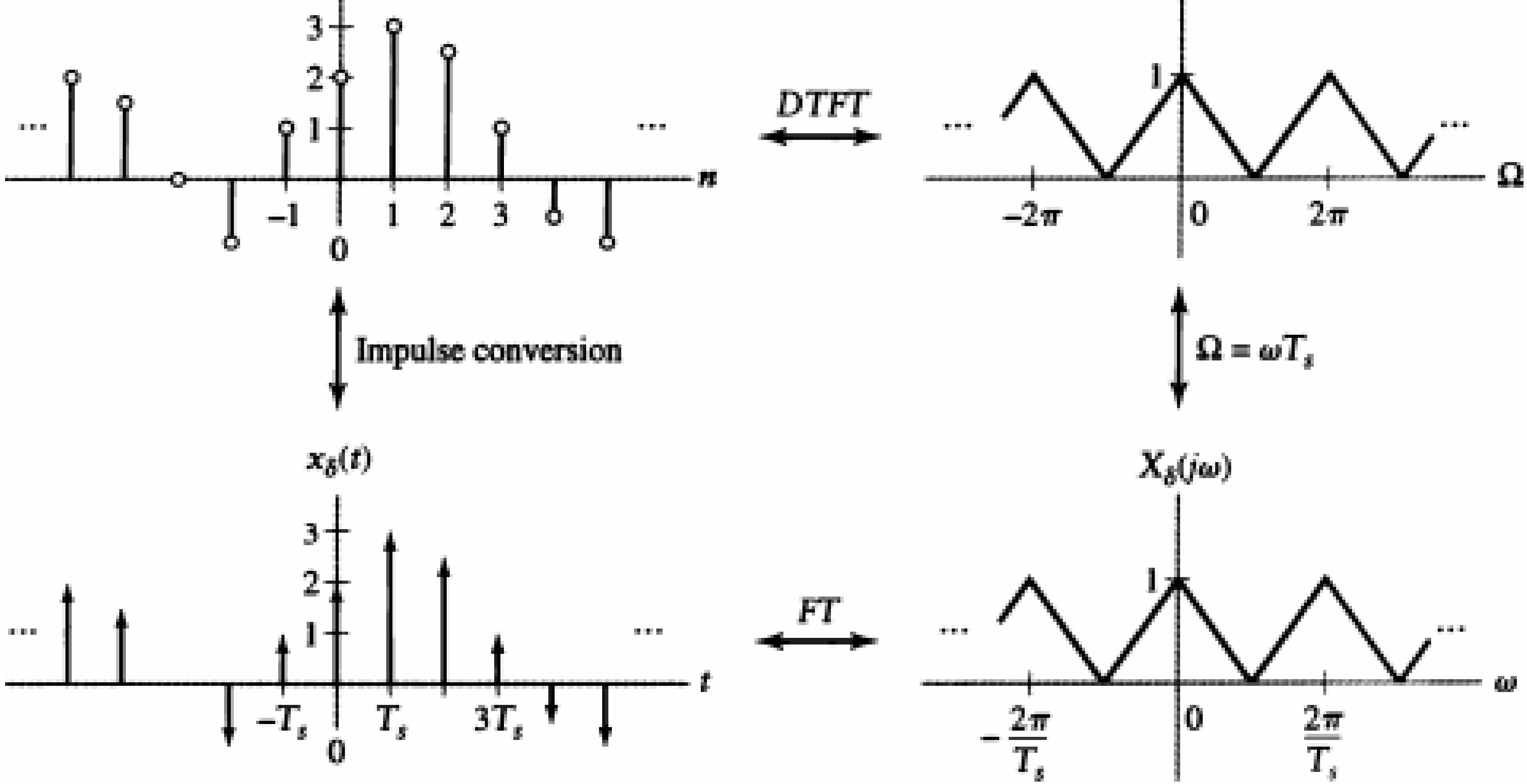
$\delta(t) \xleftrightarrow{FT} 1$

$\delta(t - nT_s) \xleftrightarrow{FT} 1 \cdot e^{-j\omega T_s n}$

$$CT: x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) \xleftrightarrow{FT} X_\delta(j\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega T_s n}$$

Relationship between FT and DTFT representations of a DT signal

$$DT : x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$$



$$CT : x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) \xleftrightarrow{FT} X_\delta(j\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega T_s n}$$

Example 4.8 *FT from the DTFT*

$$DT: x[n] \xleftrightarrow{DTFT} X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \Big|_{\Omega=\omega T_s} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega T_s n}$$

$$CT: x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) \xleftrightarrow{FT} X_\delta(j\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega T_s n}$$

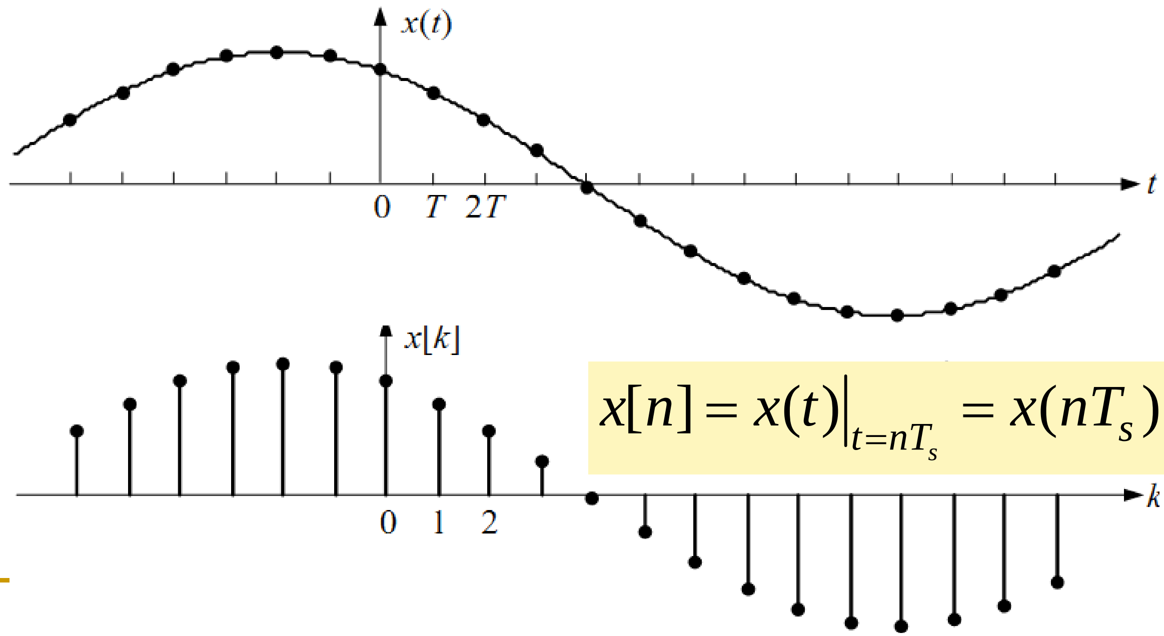
$$x[n] = a^n u[n] \xleftrightarrow{DTFT} X(e^{j\Omega}) = \frac{1}{1 - ae^{-j\Omega}}$$

$$|a| < 1$$

$$x_\delta(t) = \sum_{n=0}^{\infty} a^n \delta(t - nT_s) \xleftrightarrow{FT} X_\delta(j\omega) = \frac{1}{1 - ae^{-j\omega T_s}}$$

Sampling of CT signals

- **What is sampling?**
  - **Sampling: taking snap shots of CT signal x(t) every Ts seconds. Ts is the sampling period. After the signal sampling, we can get the samples x[n].**
- **Why to sample?**
- **How to sample?**



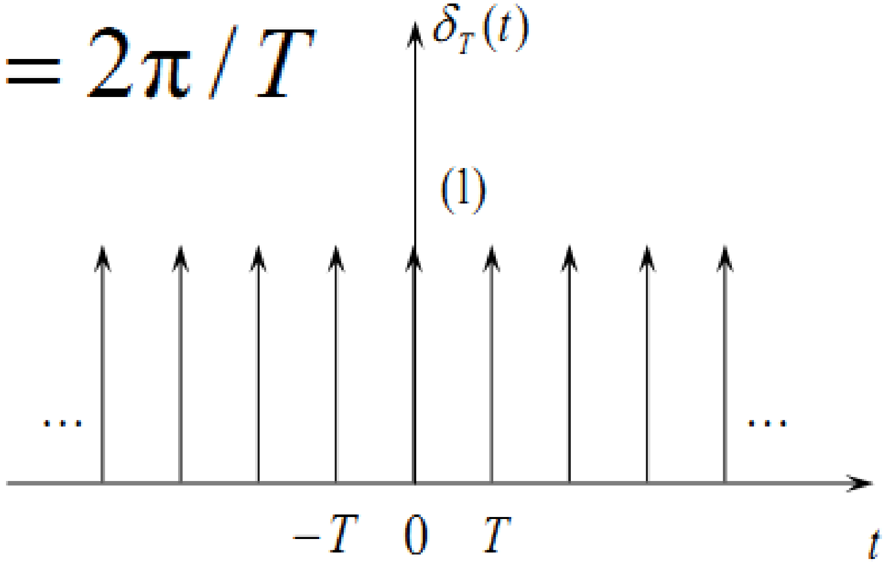
$$x[n] = x(t)\big|_{t=nT_s} = x(nT_s)$$

How to Sampling of CT signals

$$\begin{tikzcd} x(t) \arrow[r, shift left, "t=kT"] \arrow[d, leftrightarrow] & x[k] \arrow[l, shift left, "?"] \arrow[d, leftrightarrow] \\ X(j\omega) & X(e^{j\Omega}) \arrow[l] \end{tikzcd}$$

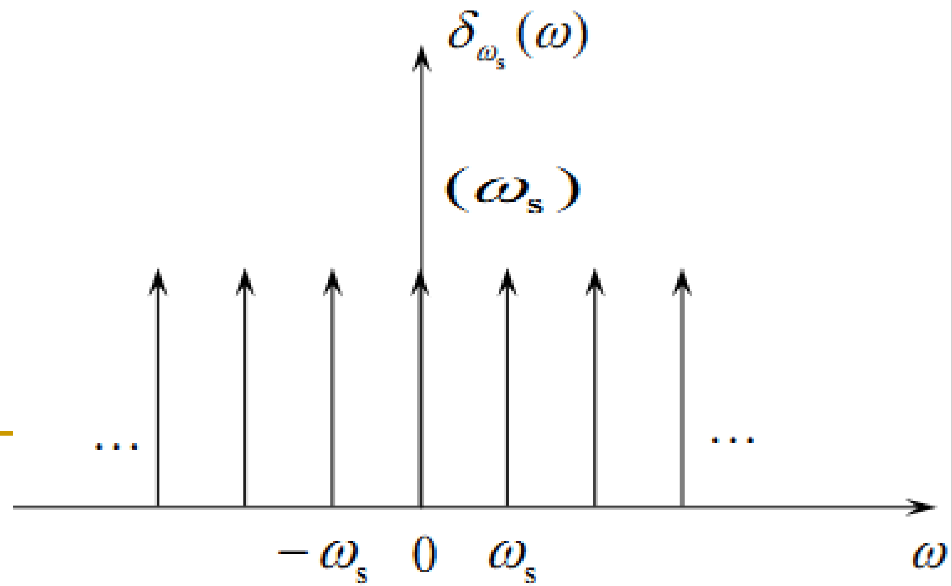
$$\delta_T(t) \xleftrightarrow{FT} \delta_{\omega_s}(\omega)$$

$$\omega_s = 2\pi / T$$



$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$\delta_{\omega_s}(\omega) = \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$



Theoretical derivation of sampling theorem

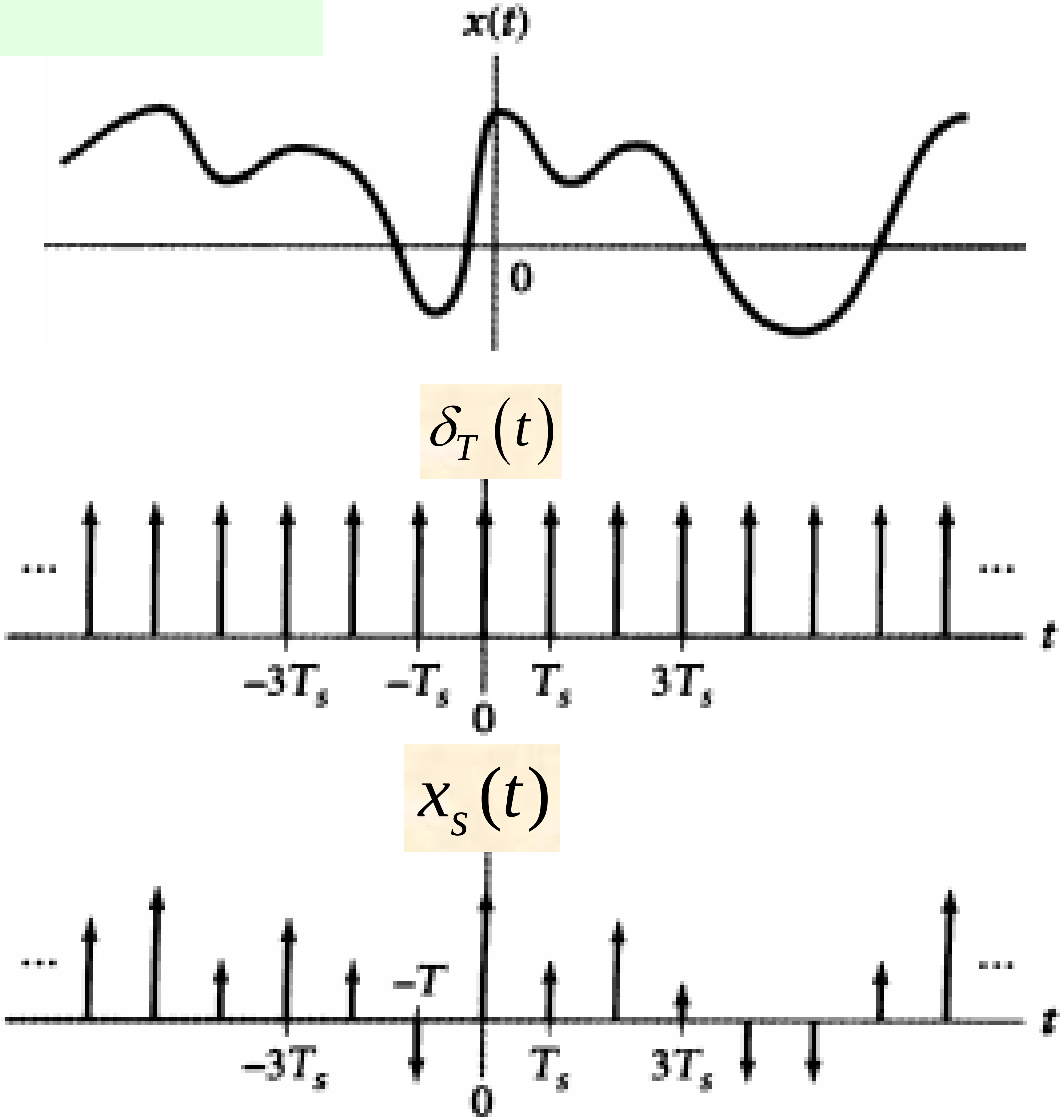
Time Domain

$$x_s(t) = x(t)\delta_T(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

$$\delta_T(t) \xleftrightarrow{FT} \delta_{\omega_s}(\omega)$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$\delta_{\omega_s}(\omega) = \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$



Theoretical derivation of sampling theorem

$$x_s(t) = x(t)\delta_T(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$X_s(j\omega) = F\left[x(t)\cdot\sum_{k=-\infty}^{\infty}\delta(t-kT)\right] = F\left[\sum_{k=-\infty}^{\infty}x(kT)\delta(t-kT)\right]$$

Frequency Domain

$$= \sum_{k=-\infty}^{\infty} x(kT)e^{-j\omega kT} = \sum_{k=-\infty}^{\infty} x(kT)e^{-j\Omega k} = X(e^{j\Omega})$$

$$X_s(j\omega) = \frac{1}{2\pi}X(j\omega) * \delta_{\omega_s}(\omega) = \frac{1}{2\pi}X(j\omega) * \omega_s\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s)$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} X[j(\omega-n\omega_s)]$$

$$\delta_{\omega_s}(\omega) = \omega_s\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_s)$$

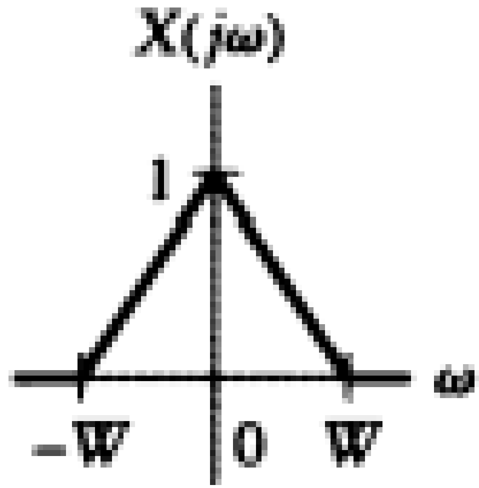
$$X(e^{j\Omega}) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} X[j(\omega-n\omega_s)] \qquad (\Omega=\omega T)$$

$$\delta_T(t) \xleftrightarrow{FT} \delta_{\omega_s}(\omega)$$

Sampling theorem - Spectrum

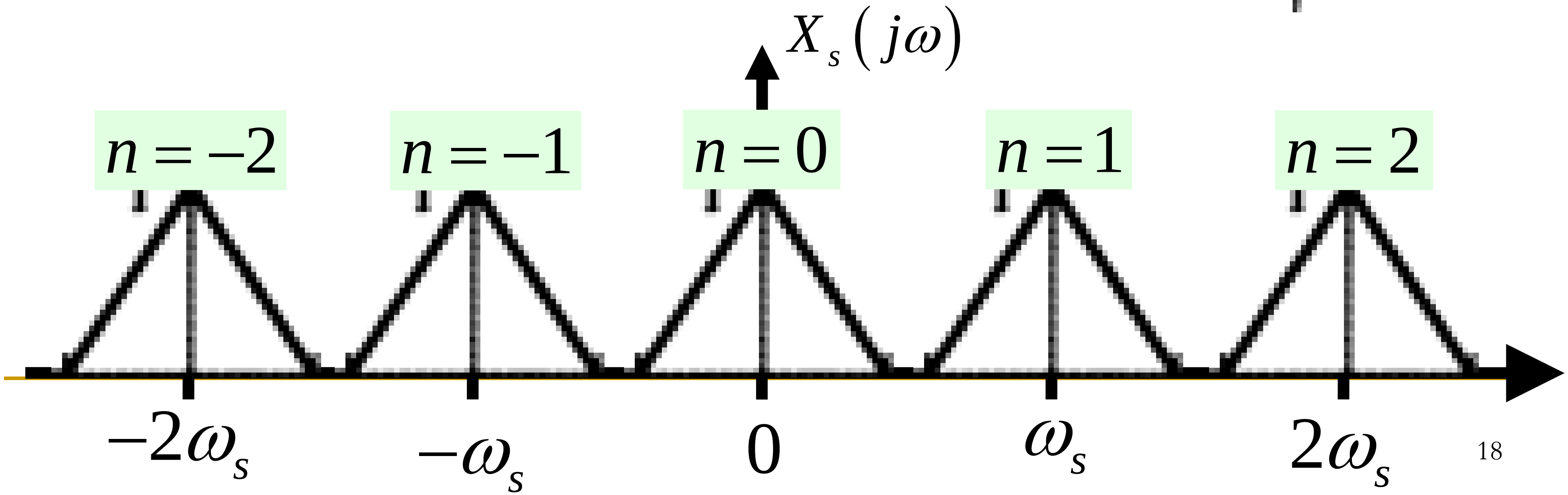
$$x_s(t) = x(t)\delta_T(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

$$\omega_s = 2\pi / T$$



$$X_s(j\omega) = \frac{1}{2\pi} X(j\omega) * \delta_{\omega_s}(\omega)$$

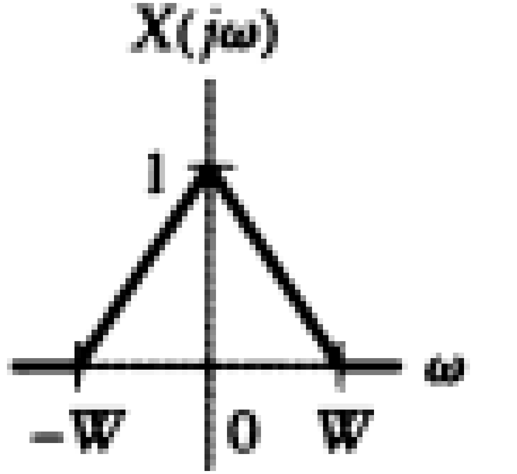
$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} X[j(\omega - n\omega_s)]$$



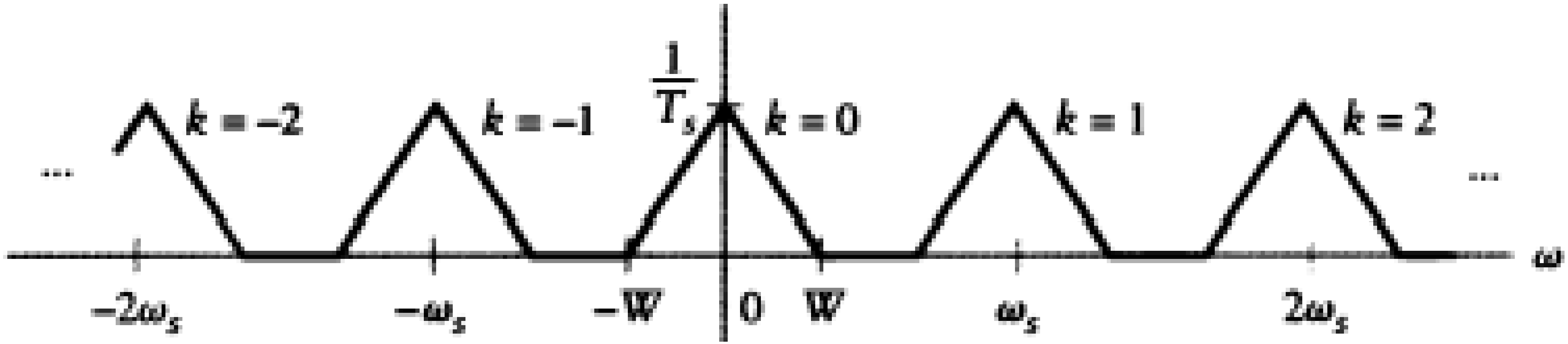
How to sample signals? sampling period1

$$X_s(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X[j(\omega - k\omega_s)]$$

$$\omega_s > 2W$$

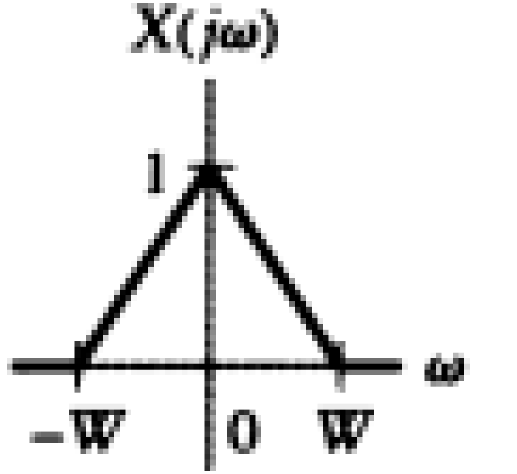


$$X_s(j\omega)$$



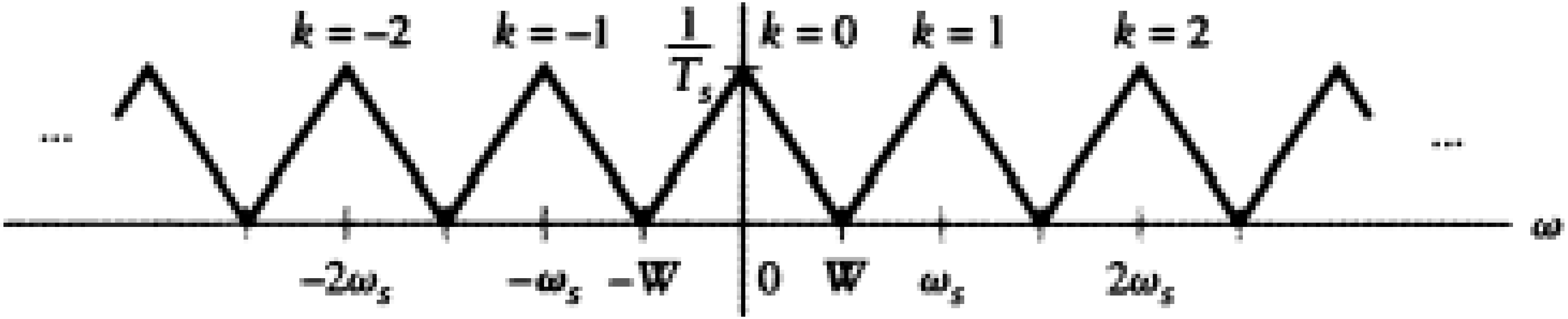
How to sample signals? sampling period2

$$X_s(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X[j(\omega - k\omega_s)]$$



$$\omega_s = 2W$$

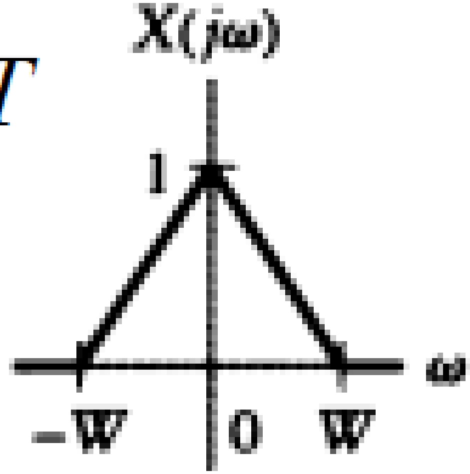
$X_s(j\omega)$



How to sample signals? sampling period3

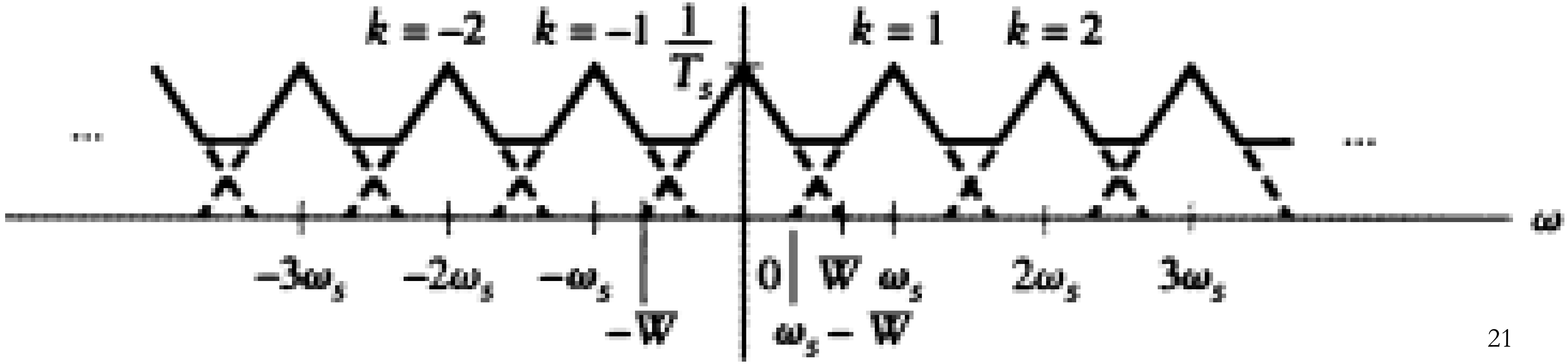
$$X_s(j\omega)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X[j(\omega-k\omega_s)]$$

$$\omega_s = 2\pi / T$$



When ω_s < 2*W*, the shifted replicas of *X*($j\omega$) moves closer together, finally overlapping one another. Overlap in the shifted replicas of the original spectrum is termed *aliasing*.

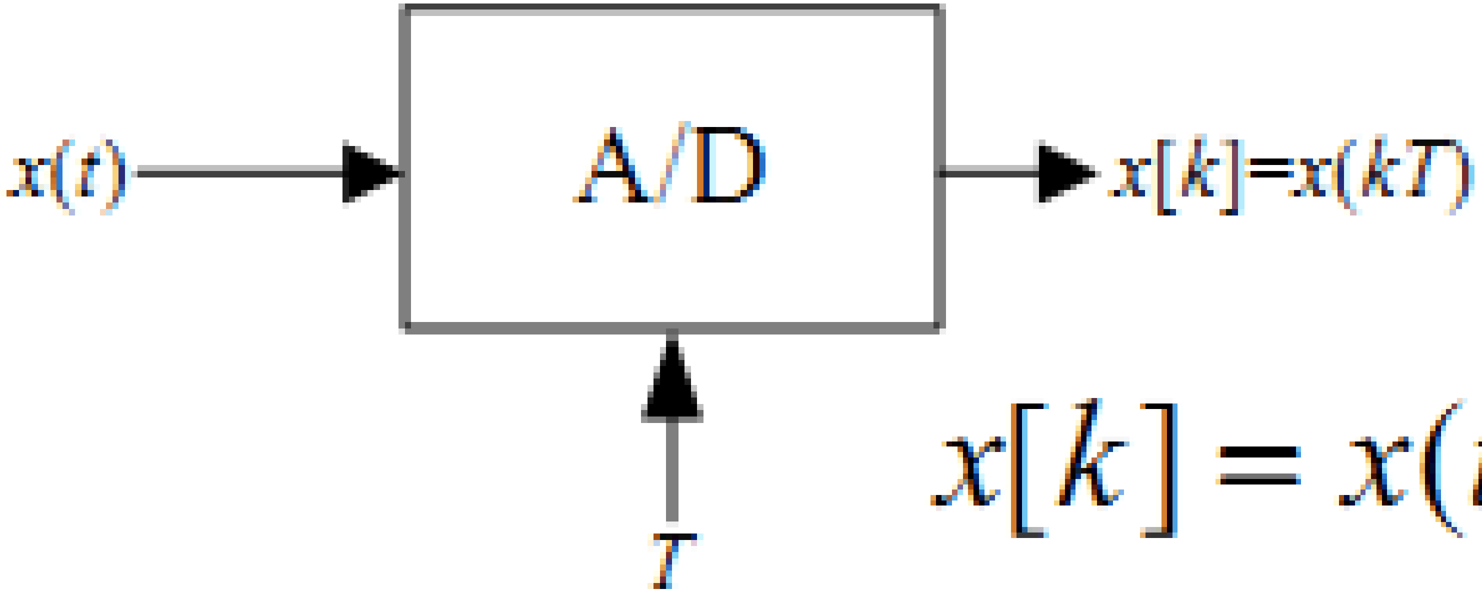
$$\omega_s < 2W$$



Sampling theorem

- Let x(t) be a band-limited signal. If the highest frequency of x(t) is W , then x(t) can be uniquely determined by its samples under the following condition:
  - Sampling Period T： $T \le \pi/W = 1/(2f_m)$
    or： $\omega_s \ge 2W$ (or $f_s \ge 2f_m$)
- The minimum sampling frequency is $f_s = 2f_m$, which is called Nyquist Rate.

Implement of sampling

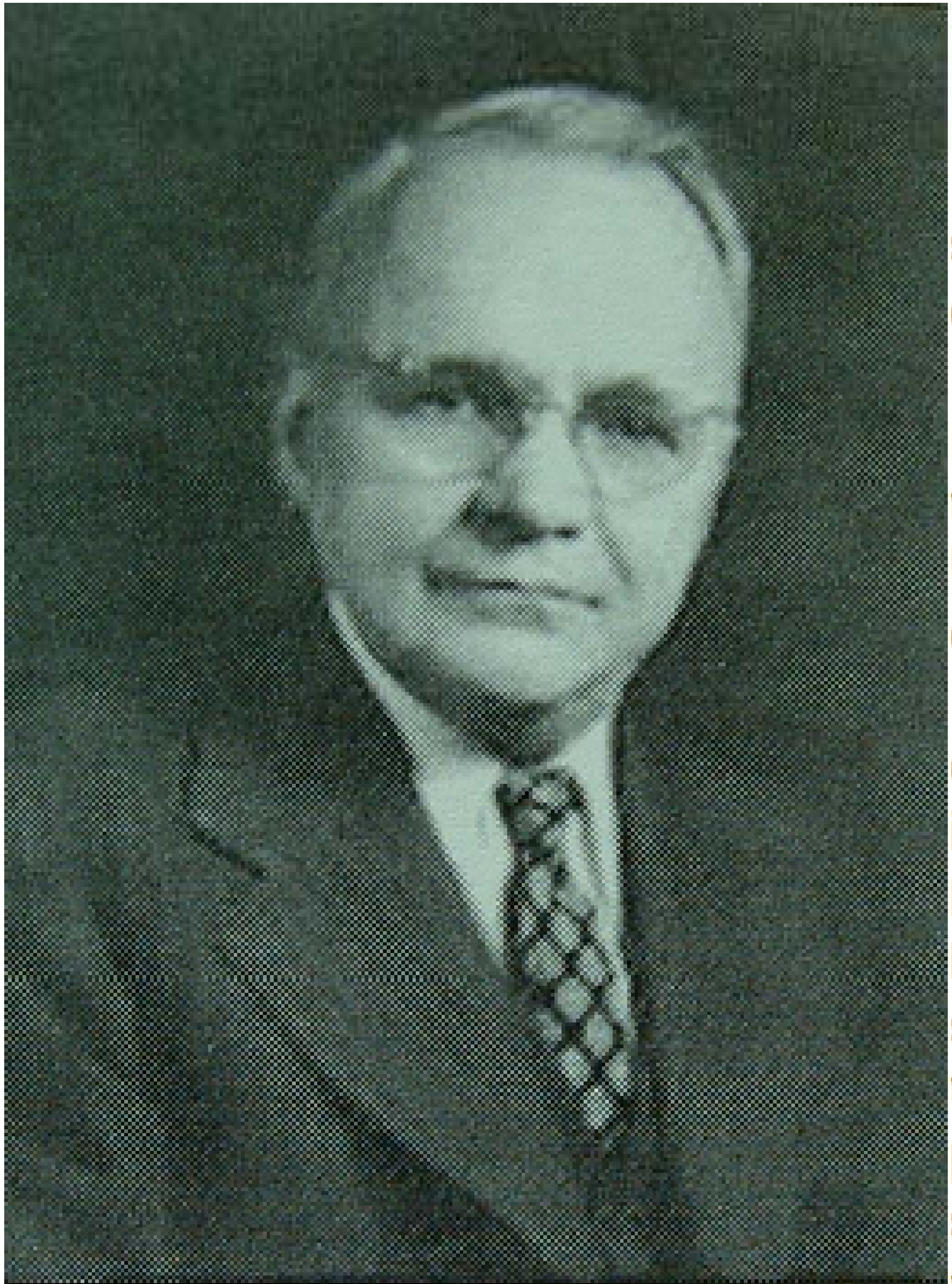


$$x[k] = x(t)\big|_{t=kT}$$

Sampling period is $T \le \pi/W = 1/(2f_m)$

Sampling frequency is $f_s = 1/T$

$$\omega_s = 2\pi / T$$



Harry Nyquist
1889-1976

BIO

Harry Nyquist, American physical scientist, was born in Sweden, 1889, died in Texas 1976.

He made significant contribution to Information Theory. In 1907, he migrated to America, and studied at Betact University. In 1917, he received the Ph.D degree at physics from Yale. In 1917~1934, he worked at AT&T company, later worked at Bell Lab.
in 1927, Nyquist presented the famous Nyquist sampling theory: if a band-limited CT signal is sampled with sampling frequency up to the threshold, the time function can be recovered perfectly from the samples. The sampling frequency is no less than 2 times bandwidth of the CT signal.

Example 2

- Find the maximum sampling interval for the signal (in Example1) that will prevent aliasing.

$$X(j\omega) = 2\pi \sum_{k=-5}^{5} \left(\frac{1}{3}\right)^k \delta(\omega - k\pi)$$

$$T = 2$$

$$\omega_0 = \frac{2\pi}{T} = \pi$$

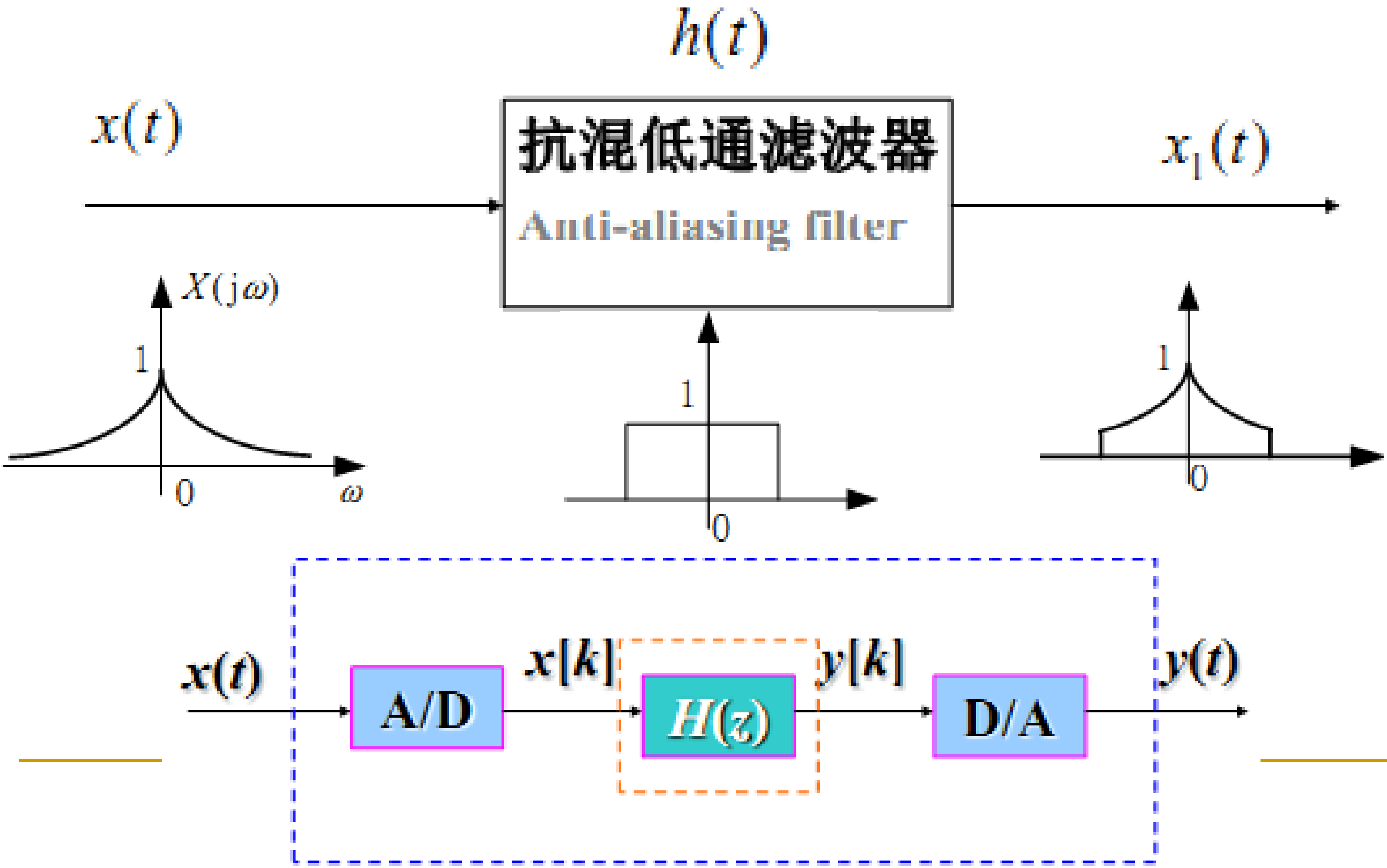
$$W = 5\omega_0 = 5\pi$$

$$\frac{2\pi}{T_s} \geq 2W \Rightarrow T_s \leq \frac{1}{5} \Rightarrow \max T_s = \frac{1}{5}$$

$$\omega_s = \frac{2\pi}{T_s} \leq 10\pi$$

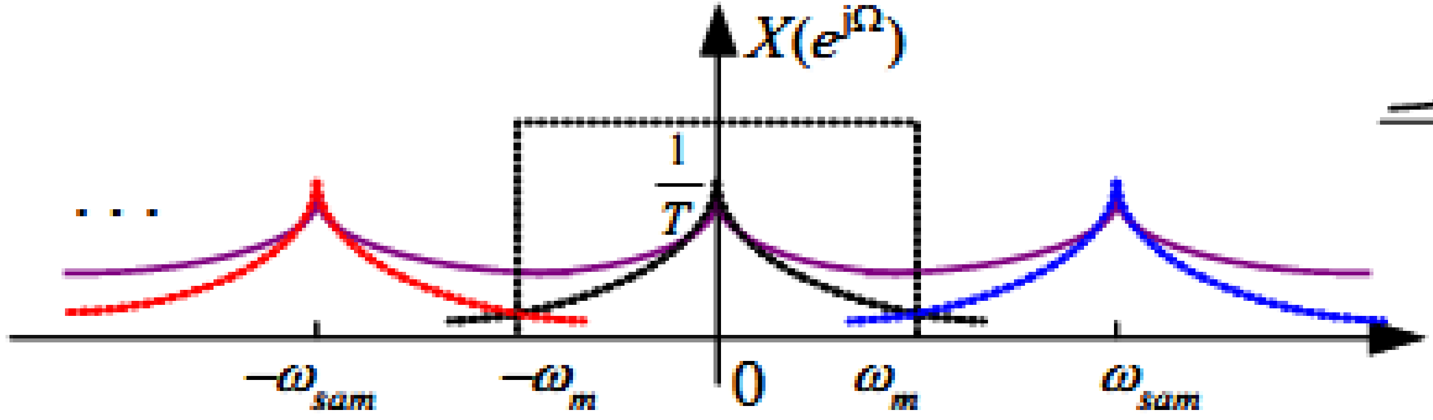
Practical Applications of Sampling Theorem 1

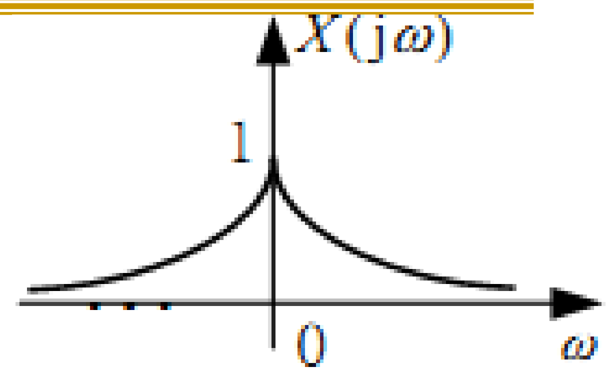
Many practical signal is not a band-limited signal.



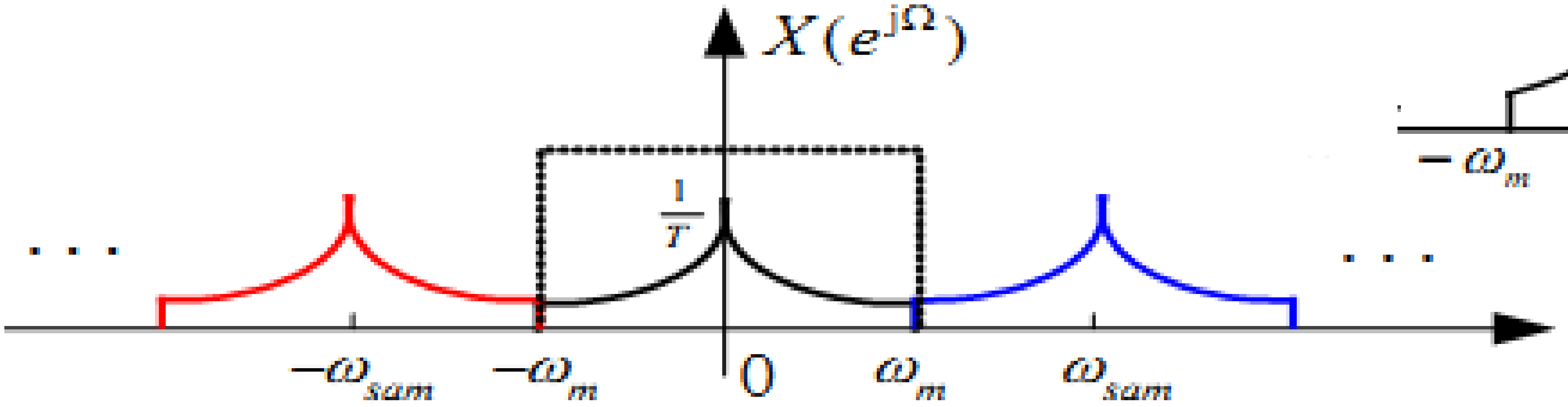
Practical Applications of Sampling Theorem 2

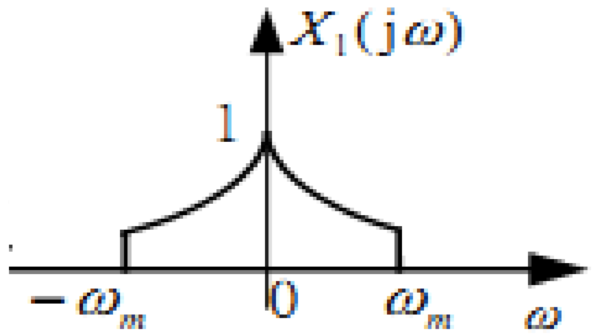
Aliasing





Truncation





Conclusions

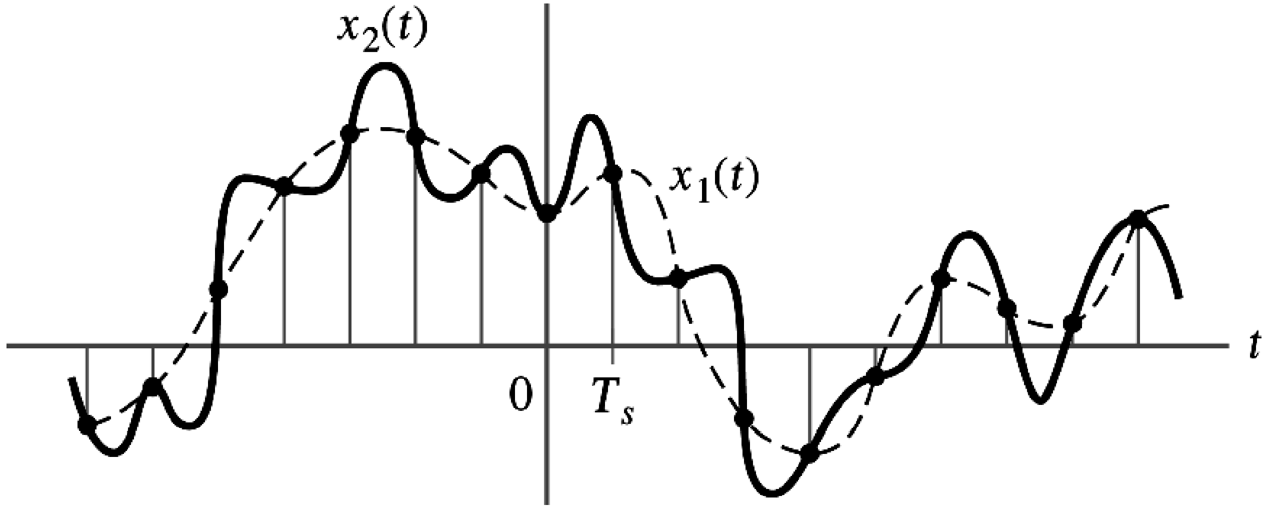
- (1) Sampling in time domain will result in periodicity in frequency domain. The spectra of DT signal x[k] is the periodicity of CT signal x(t)'s spectra.
- (2) Nyquist sampling theory: if a CT signal is band-limited, and it is sampled with sampling frequency up to the threshold, the time function can be recovered perfectly from the samples. the sampling frequency is no less than 2 times bandwidth of the CT signal.
- (3) Engineering applications of Sampling: if the practical signal is not a band-limited signal, x(t) can be recovered by passing the CT signal through an anti-aliasing filter with cutoff frequency at f = fm .

Discussion

- **(1) Based on Nyquist Sampling Theorem, when we sample a CT signal, the sampling frequency is no less than 2 times bandwidth of the CT signal. But in practical engineering, the sampling frequency is no less than 3~5 times bandwidth of the CT signal. Why?**
- **(2) If the highest frequency of the CT signal is unknown, how do we choose the signal sampling period T?**

Reconstruction of CT Signals from Samples

The samples of a signal do not always uniquely determine the corresponding CT signal.



♣ **Reconstruction of signal: There must be a unique correspondence between the FTs of the continuous-time signal and its sampled signal.**

The sampling process must not introduce aliasing!

♣ Aliasing distorts the spectrum of the original signal and destroys the one-to-one relationship between the FT's of the CT signal and the sampled signal.

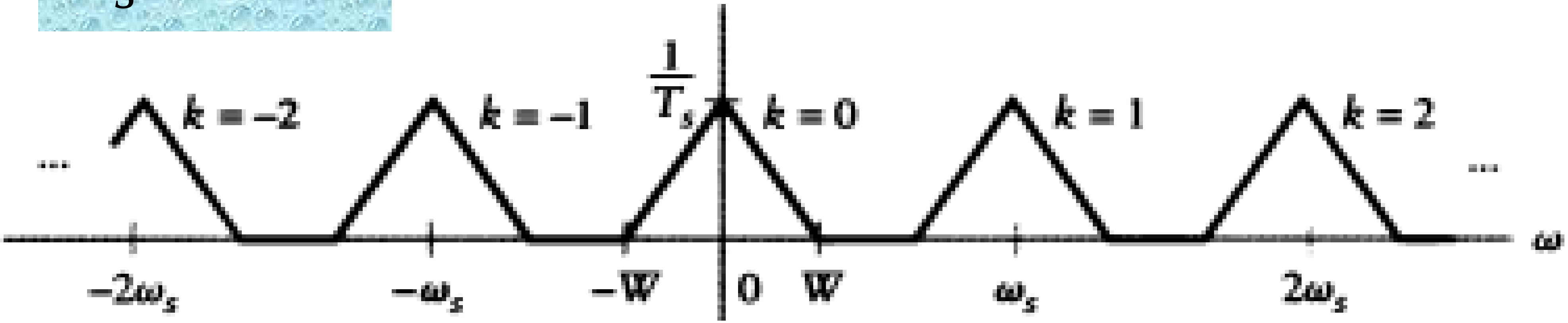
Reconstruction of CT Signals from Samples

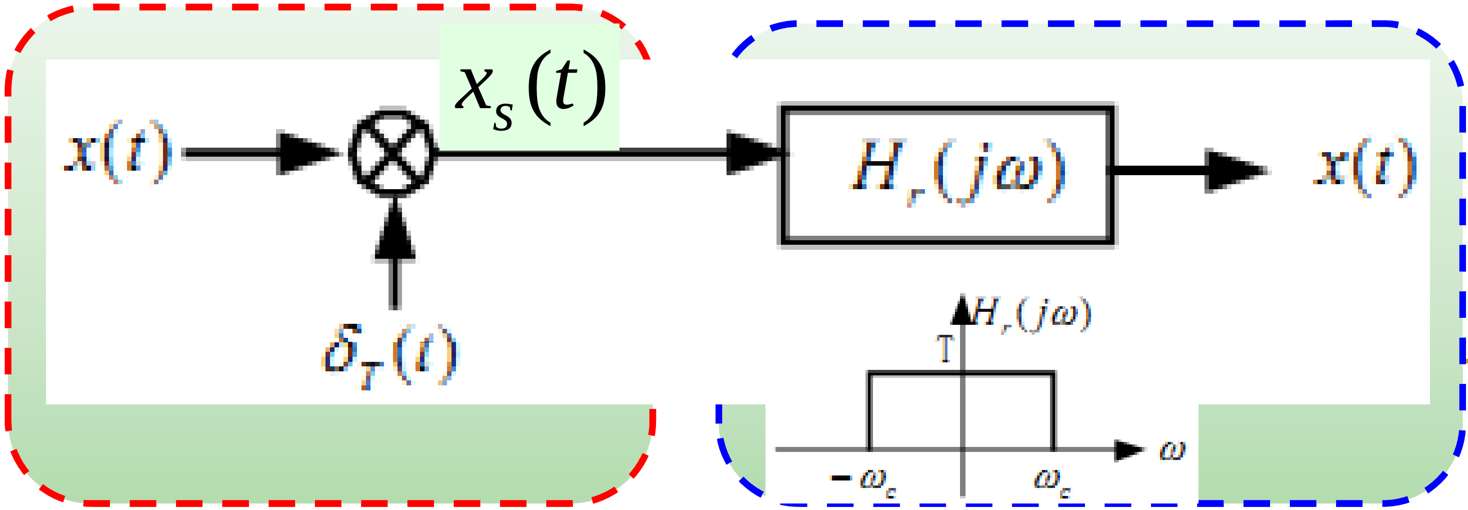
$$X_s(j\omega)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X[j(\omega-k\omega_s)]$$

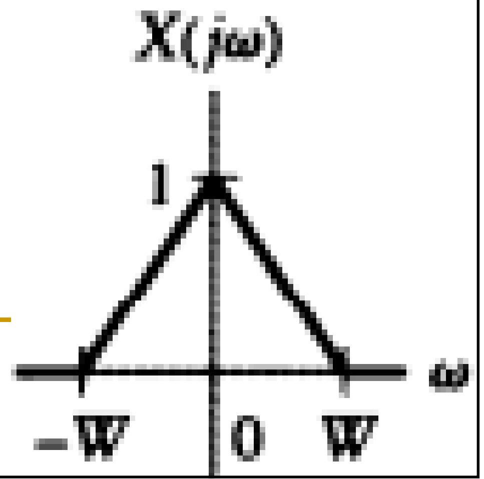
$$x_s(t)=x(t)\delta_T(t)$$
$$=\sum_{n=-\infty}^{\infty}x(nT)\delta(t-nT)$$

$$\omega_s \ge 2W$$

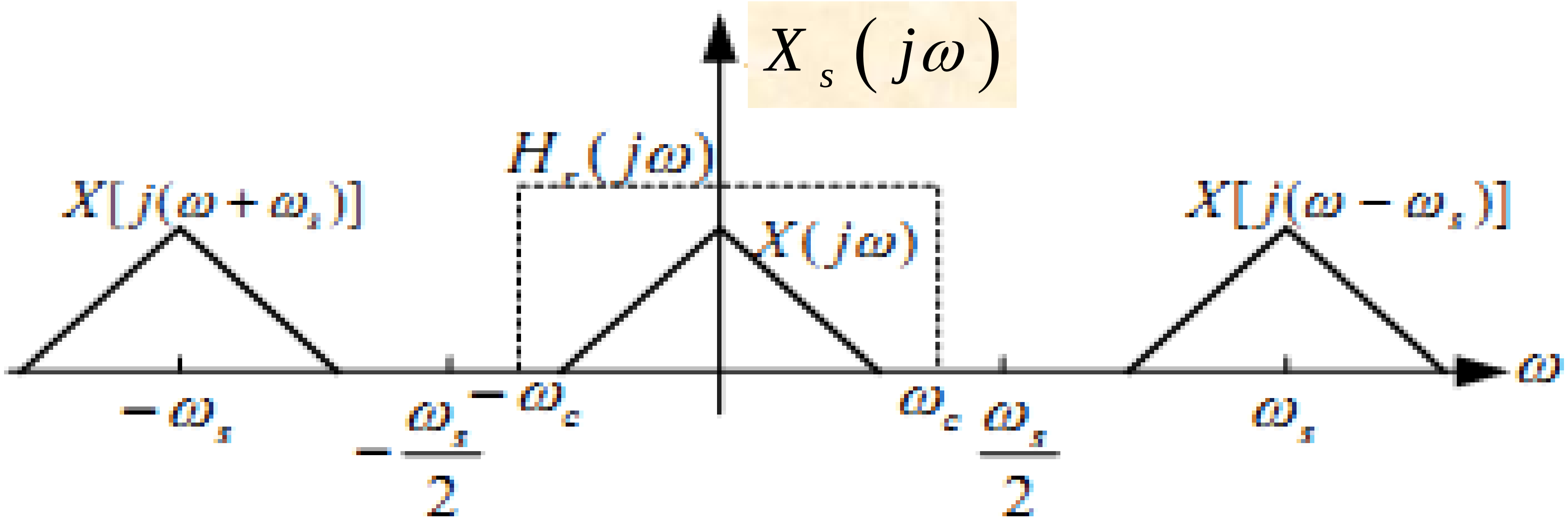
$X_s(j\omega)$







Reconstruction of CT Signals from Samples



$$X(j\omega) = H_r(j\omega)X_s(j\omega)$$

$$H_r(\omega) = \begin{cases} T & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

$$h_r(t) = F^{-1}[H_r(\omega)] = \mathrm{Sa}\left(\frac{\omega_s t}{2}\right)$$

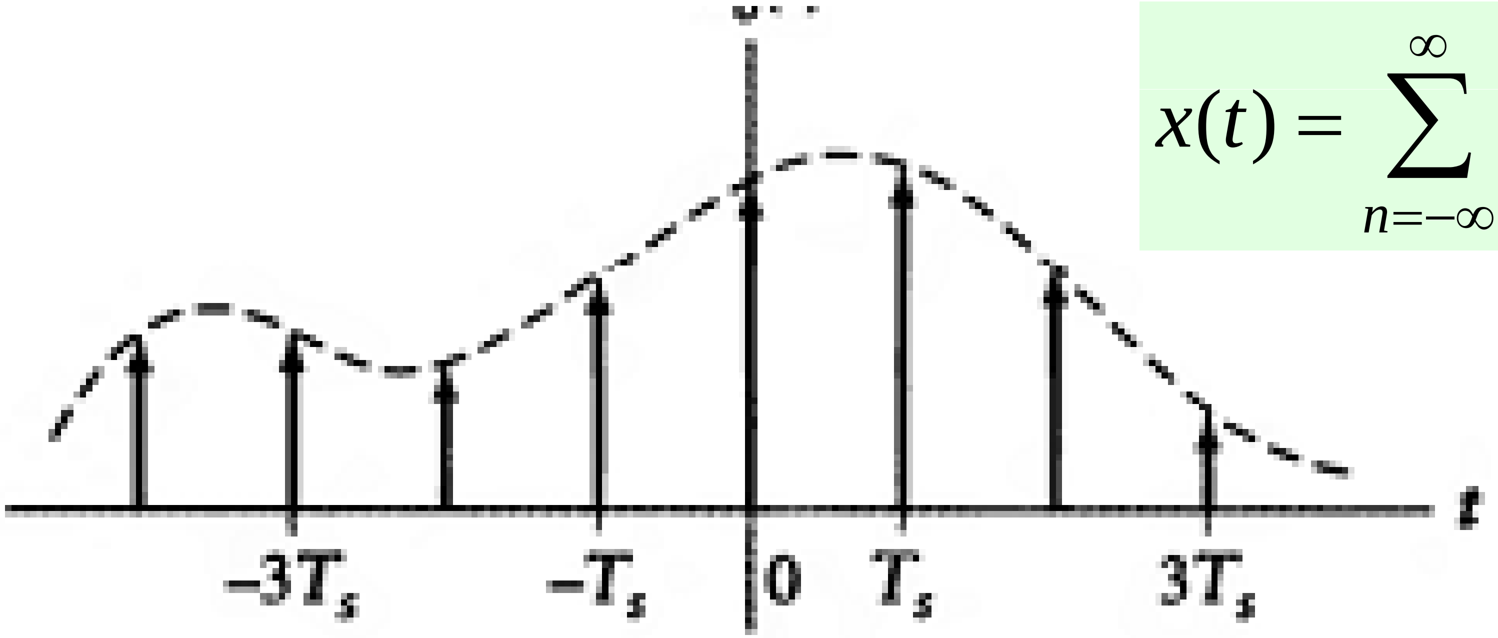
$$x(t) = x_s(t) * h_r(t)$$

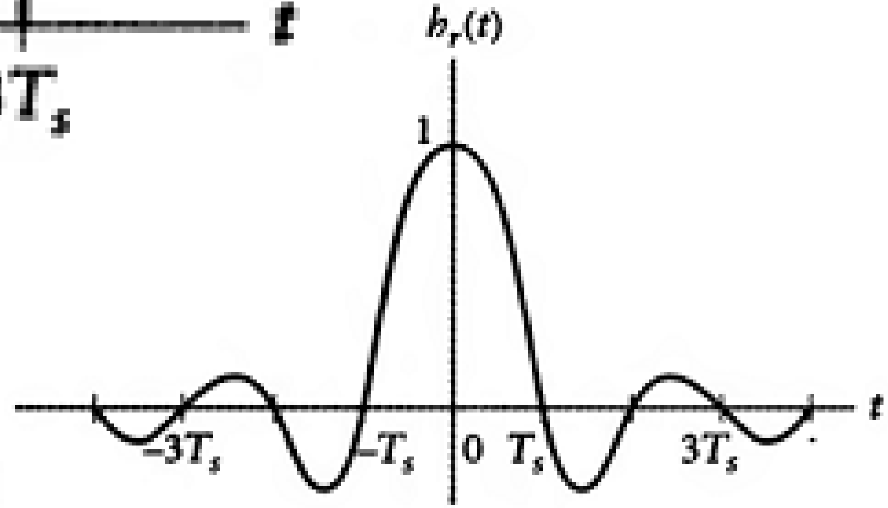
$$= \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) * h_r(t)$$

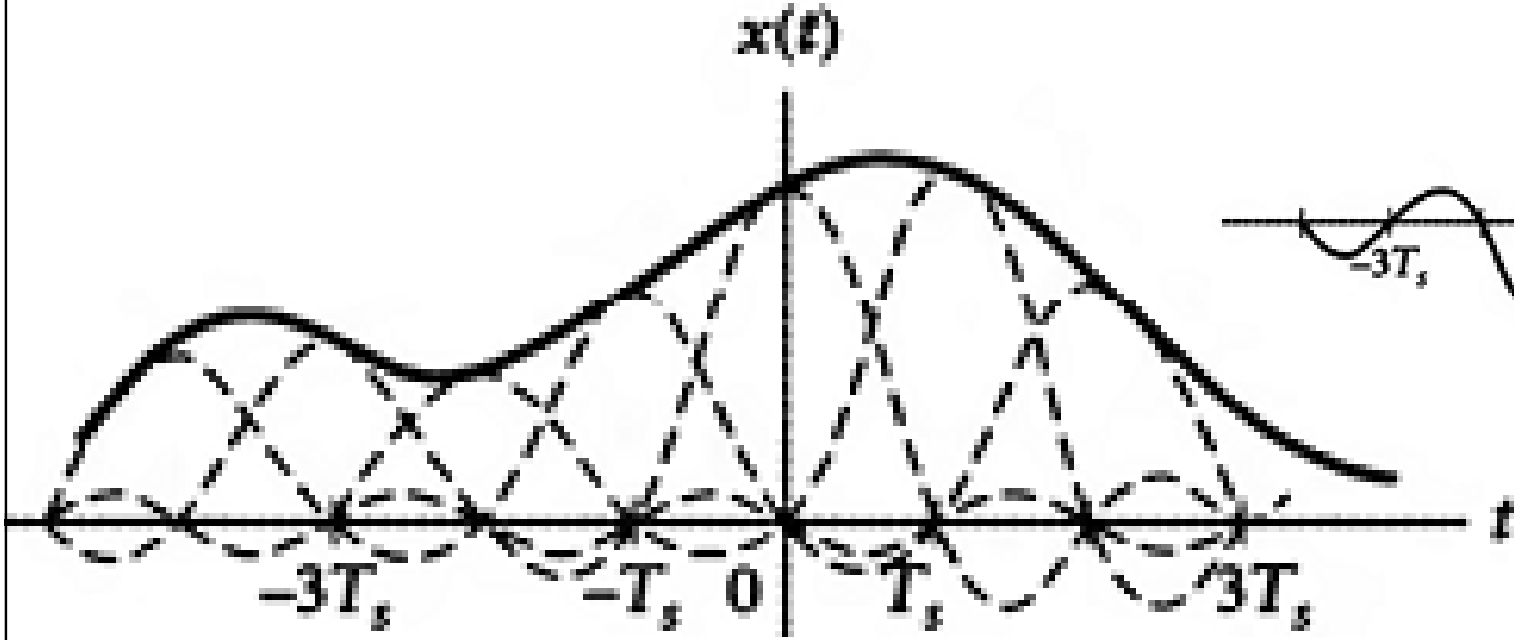
$$= \sum_{n=-\infty}^{\infty} x(nT)h_r(t-nT)$$

Reconstruction of CT Signals from Samples

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)h_r\left(t - nT\right)$$







Summary and Exercises

- Summary
  - Fourier Transform of Periodic Signals
  - Convolution and modulation with mixtures of periodic and Non-periodic signals
  - Fourier Transform Representations of Discrete-time signals
  - Sampling and Reconstruction
- Exercises (**P413-418**)
  - **4.16(a), 4.17(a), 4.18(a,b), 4.20(a), 4.22(a), 4.24(a), 4.29(a), 4.30**