Signals and Systems 3.2

--- *Fourier transform*

School of Information & Communication Engineering，BUPT

Reference:
1. Textbook: Chapter 3

Clue of this chapter

- In **chapter 2**, by representing **signals** as linear combinations of **shifted impulses**, we **analyzed LTI systems** through the **convolution sum (integral).**
- An alternative representation for signals and LTI systems: represent **signals** as linear combinations of a set of basic signals---**complex exponentials**. The resulting representations are known as the **continuous-time and discrete-time Fourier series and transform.**
  - which convert time-domain signals into frequency-domain (or ***spectral)*** representations

Outline of Today's Lecture

- Fourier transform
  - Complex Sinusoids and Frequency Response of LTI Systems
  - Fourier Representations for Four classes of Signals
    - Discrete-time periodic signals - DTFS
    - Discrete-time nonperiodic signals - DTFT
    - Continuous-time periodic signals - FS
    - Continuous-time nonperiodic signals - FT
  - Properties of Fourier Representations

Summary of the Fourier series

- Three forms
  - Original (sine and cosine components)

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

  - Cosine-with-phase form

$$x(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t + \theta_k) \qquad -\infty < t < \infty$$

  - Exponential form

$$x(t) = \sum_{n=-\infty}^{\infty} X_n . e^{jn\omega t}$$

- Dirichlet conditions
- Gibbs phenomenon

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_1 t}}$$

The response of an LTI system to a *complex exponentials* input lead to a characterization of system behavior that is termed the *frequency response* of the LTI system

♣ **Frequency response ≡ The response of an *LTI* system to a complex exponentials input.**

- **Frequency response of** a **Discrete-time *LTI* system**

1. Impulse response of discrete-time *LTI* system = $h[n]$, input = $x[n] = e^{j\Omega n}$

2. Output

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]e^{j\Omega(n-k)}$$

Frequency response of Discrete-time *LTI* system

$$y[n]=e^{j\Omega n}\sum_{k=-\infty}^{\infty}h[k]e^{-j\Omega k}=H\left(e^{j\Omega}\right)e^{j\Omega n}$$

Complex scaling factor

3. Frequency response:

$$H(e^{j\Omega})=\sum_{k=-\infty}^{\infty}h[k]e^{-j\Omega k}$$

A function of frequency Ω

$$e^{j\Omega n}\longrightarrow \boxed{h[n]}\longrightarrow H(e^{j\Omega})e^{j\Omega n}$$

The output of a complex *exponentials* input to an LTI system is a complex *exponentials* of the same frequency as the input, multiplied by the frequency response of the system.

Frequency response of Continuous-time *LTI* system

1. Impulse response of continuous-time *LTI* system = *h*(*t*), input = *x*(*t*) = $e^{j\omega t}$

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)} d\tau = e^{j\omega t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

$$= H(j\omega) e^{j\omega t}$$

2. Frequency response

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

$$H(j\omega) = |H(j\omega)| e^{j\arg\{H(j\omega)\}}$$

$$|H(j\omega)| = \text{Magnitude response}$$

$$\arg\{H(j\omega)\} = \text{Phase response}$$

$$y(t) = |H(j\omega)| e^{j\left(\omega t + \arg\{H(j\omega)\}\right)}$$

The systems modifies the amplitude of the input by |H(jw)| and the phase by arg{H(jw)}.

Summary: Frequency Response of LTI systems

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega)e^{j\omega t}$$

$$y(t) = \mathrm{e}^{\mathrm{j}\omega t} * h(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{j}\omega(t-\tau)} h(\tau)\mathrm{d}\tau$$

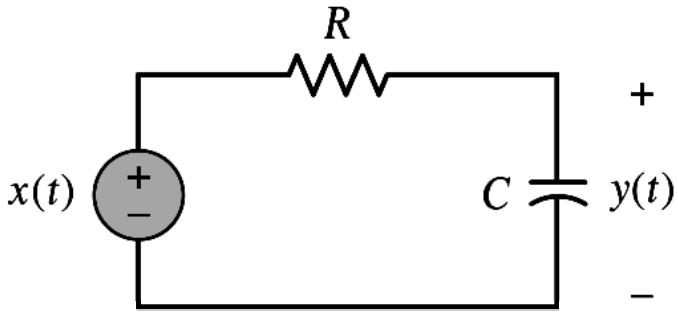
$$= \mathrm{e}^{\mathrm{j}\omega t} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}\omega\tau} h(\tau)\mathrm{d}\tau \;= \mathrm{e}^{\mathrm{j}\omega t} H(\mathrm{j}\omega)$$

$$H(\mathrm{j}\omega) = \int_{-\infty}^{\infty} h(t)e^{-\mathrm{j}\omega t}\mathrm{d}t \;= \left|H(j\omega)\right| e^{j\arg\{H(j\omega)\}}$$

$$y(t) = \mathrm{e}^{\mathrm{j}\omega t} H(\mathrm{j}\omega) \;= \left|H(j\omega)\right| e^{j(\omega t + \arg\{H(j\omega)\})}$$

Example 3.1 *RC Circuit: Frequency response*

$$h(t)=\frac{1}{RC}e^{-t/RC}u(t)$$



<Sol.> **Frequency response:**

$$H(j\omega)=\frac{1}{RC}\int_{-\infty}^{\infty}e^{-\frac{\tau}{RC}}u(\tau)e^{-j\omega\tau}d\tau=\frac{1}{RC}\int_{0}^{\infty}e^{-\left(j\omega+\frac{1}{RC}\right)\tau}d\tau$$

$$=\frac{1}{RC}\frac{-1}{\left(j\omega+\frac{1}{RC}\right)}e^{-\left(j\omega+\frac{1}{RC}\right)\tau}\Bigg|_{0}^{\infty}$$

$$=\frac{1}{RC}\frac{-1}{\left(j\omega+\frac{1}{RC}\right)}(0-1)=\frac{\frac{1}{RC}}{j\omega+\frac{1}{RC}}$$

Complex Exponentials and Frequency Response of LTI Systems

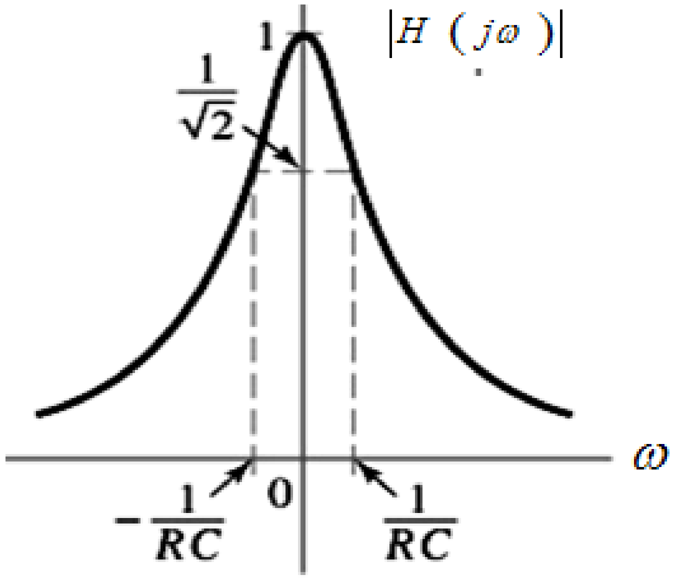
$$H(j\omega)=\frac{\frac{1}{RC}}{j\omega+\frac{1}{RC}}$$

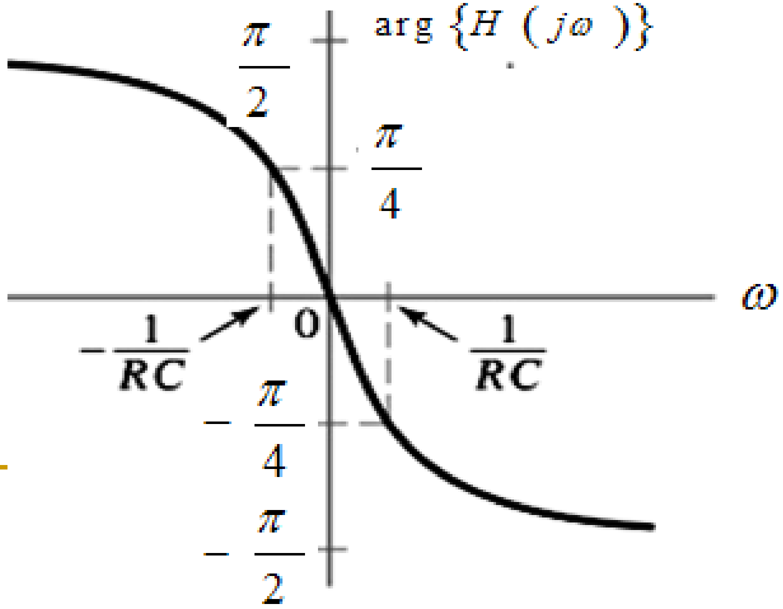
Magnitude response:

$$|H(j\omega)|=\frac{\frac{1}{RC}}{\sqrt{\omega^2+\left(\frac{1}{RC}\right)^2}}$$

Phase response:

$$\arg\{H(j\omega)\}=-\arctan(\omega RC)$$





Eigenvalue and eigenfunction of *LTI* system

If $\boldsymbol{e}_k$ is an eigenvector of a matrix **A** with eigenvalue λ_k, then

$$\mathbf{A}\mathbf{e}_k = \lambda_k \mathbf{e}_k$$

Arbitrary input = weighted superpositions of eigenfunctions

Eigenrepresentation

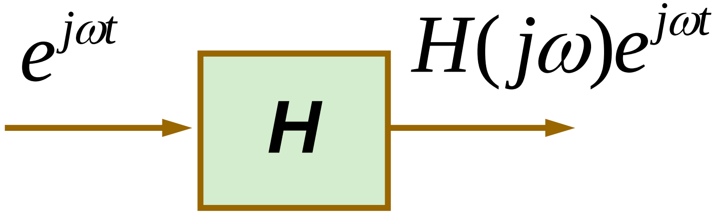
$$\psi(t) \;\; (\psi[n]) \longrightarrow \boxed{H} \longrightarrow \lambda\psi(t) \;\; (\underbrace{\lambda}_{\text{eigenvalue}} \underbrace{\psi[n]}_{\text{eigenfunction}})$$

The action of the system on an eigenfunction input
is multiplication by the corresponding eigenvalue.

Eigenvalue and eigenfunction of *LTI* system

- **Continuous-time case:**

$$H\{\psi(t)\} = \lambda\psi(t)$$



Eigenfunction: $\psi(t) = e^{j\omega t}$

Eigenvalue: $\lambda = H(j\omega)$

Arbitrary input = weighted superpositions of eigenfunctions

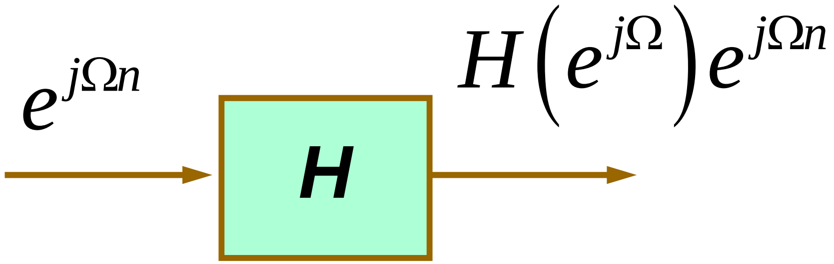
⇒ **Convolution operation $\Rightarrow$ Multiplication**

$$x(t) = \sum_{k=1}^{M} a_k e^{j\omega_k t} \quad \Rightarrow \quad y(t) = \sum_{k=1}^{M} a_k H(j\omega) e^{j\omega_k t}$$

Eigenvalue and eigenfunction of *LTI* system

■ **Discrete-time case:**

$$H(\psi[n]) = \lambda \psi[n]$$



Eigenfunction: $\psi[n] = e^{j\Omega n}$

Eigenvalue: $\lambda = H(e^{j\Omega})$

$$x[n] = \sum_{k=1}^{M} a_k e^{j\Omega_k n}$$

$$y[n] = \sum_{k=1}^{M} a_k H\left(e^{j\Omega}\right) e^{j\Omega_k n}$$

Eigenvalue and eigenfunction of *LTI* system

By representing arbitrary signals as weighted superpositions of eigenfunctions, **we transform the operation of convolution to multiplication.**

Convolution operation $\Rightarrow$ Multiplication

$$x(t)=\sum_{k=1}^{M} a_k e^{j\omega_k t} \quad \Longrightarrow \quad y(t)=h(t)*x(t)=\sum_{k=1}^{M} a_k H(j\omega)e^{j\omega_k t}$$

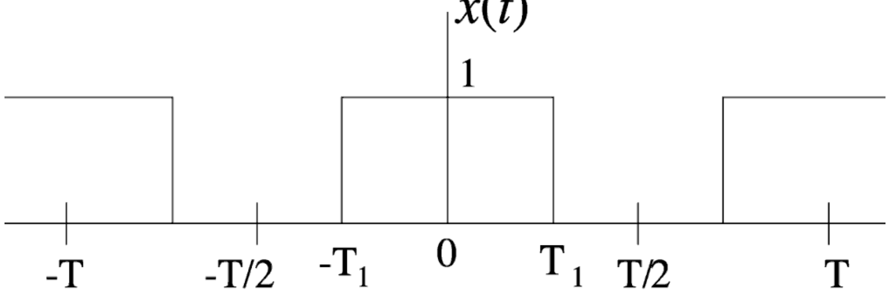
$$x[n]=\sum_{k=1}^{M} a_k e^{j\Omega_k n} \quad \Longrightarrow \quad y[n]=h(n)*x(n)=\sum_{k=1}^{M} a_k H\left(e^{j\Omega}\right)e^{j\Omega_k n}$$

$$H(j\omega)=\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau \qquad H(e^{j\Omega})=\sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k}$$

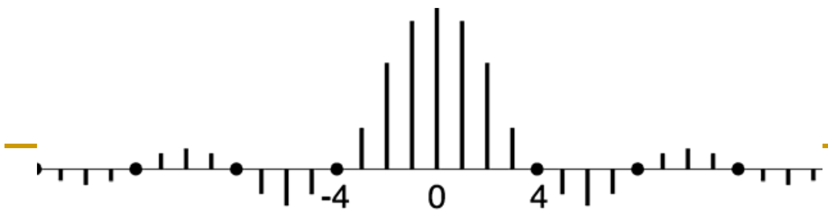
Fourier Representations for Four classes of Signals

Table 3.1 Relationship between Time Properties of a Signal and the Approximate Fourier Representation

Time Property	Periodic	Nonperiodic
Continuous (*t*)	Fourier Series (FS)	Fourier Transform (FT)
Discrete [*n*]	Discrete-Time Fourier Series (DTFS)	Discrete-Time Fourier Transform (DTFT)



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$



$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

Periodic Signals: Fourier Series Representations

1. $x[n]$ is a discrete-time signal with fundamental period N. Its DTFS is

$$\hat{x}[n] = \sum_{k} A[k] e^{jk\Omega_0 n}$$

$\Omega_o = 2\pi/N \equiv$ Fundamental frequency of $x[n]$

2. $x(t)$ is a continuous-time signal with fundamental period T. Its FS is

$$\hat{x}(t) = \sum_{k} A[k] e^{jk\omega_0 t}$$

$\omega_o = 2\pi/T \equiv$ Fundamental frequency of $x(t)$

♣ "^" denotes approximate value. $A[k]$ = the weight applied to the k^{th} harmonic.

♣ $e^{jk\omega_0 t}$ is the k th harmonic of $x(t)$.

♣ Mean-square error (MSE) between the signal and its series representation

$$MSE = \frac{1}{N} \sum_{n=0}^{N-1} \left| x[n] - \hat{x}[n] \right|^2 dt \qquad MSE = \frac{1}{T} \int_0^T \left| x(t) - \hat{x}(t) \right|^2 dt$$

Nonperiodic Signals: Fourier-Transform Representations

1. *x*(*t*) is a continuous-time signal. Its FT is

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

***X*(*j*ω)/(2π) = the weight applied to a sinusoid of frequency ω in the FT representation.**

2. *x*[*n*] is a discrete-time signal. Its DTFT is

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\Omega})e^{j\Omega n}d\omega$$

$X(e^{j\Omega})/(2\pi)$ = the weight applied to the sinusoid $e^{j\Omega n}$ in the DTFT representation.

▶ **Problem 3.1** Identify the appropriate Fourier representation for each of the following signals:

(a) $x[n] = (1/2)^n u[n]$

(b) $x(t) = 1 - \cos(2\pi t) + \sin(3\pi t)$

(c) $x(t) = e^{-t}\cos(2\pi t)u(t)$

(d) $x[n] = \sum_{m=-\infty}^{\infty}\delta[n - 20m] - 2\delta[n - 2 - 20m]$

Answers:

(a) DTFT

(b) FS

(c) FT

(d) DTFS

① CT Periodic Signals: The Fourier Series

1. FS pair of *T*-periodic signal *x*(*t*):

$$x(t) \xleftrightarrow{FS;\omega_0} X[k]$$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$$

$-\infty \le k \le \infty$

fundamental period *T* and fundamental frequency $\omega_o = 2\pi/T$

$$X[k] = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} dt$$

Frequency domain representation of *x*(*t*)

2. If *x*(*t*) is square integrable

$$\frac{1}{T}\int_0^T |x(t)|^2 dt < \infty \Rightarrow \frac{1}{T}\int_0^T \left|x(t) - \hat{x}(t)\right|^2 dt = 0$$

$\Rightarrow x(t) = \hat{x}(t)$ **at all values of *t*; it simply implies that there is zero power in their difference.**

Example 1: Determine the FS coefficients for CT Periodic Signals Using Defination

1. The period of $x(t)$ is $T = 2$, so $\omega_o = 2\pi/2 = \pi$.

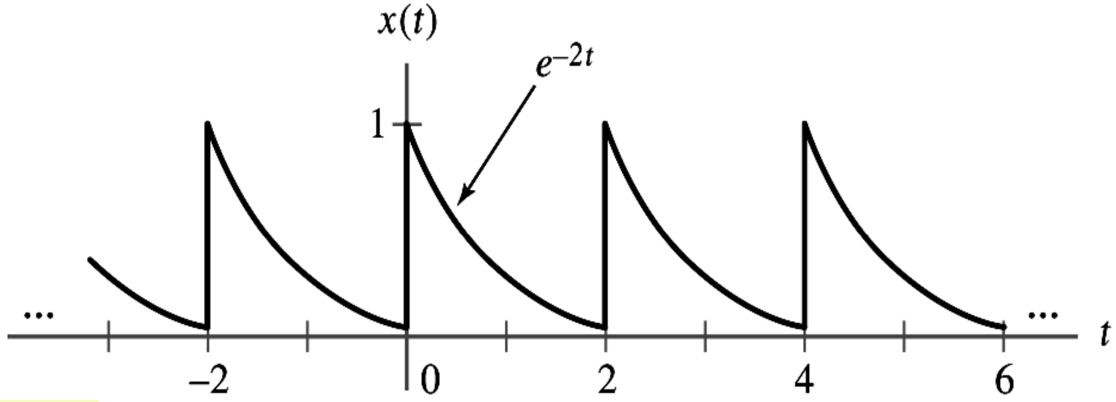
2. One period of $x(t)$: $x(t) = e^{-2t}$, $0 \le t \le 2$.

$$X[k] = \frac{1}{2}\int_0^2 e^{-2t} e^{-jk\pi t}\, dt$$

$$= \frac{1}{2}\int_0^2 e^{-(2+jk\pi)t}\, dt$$

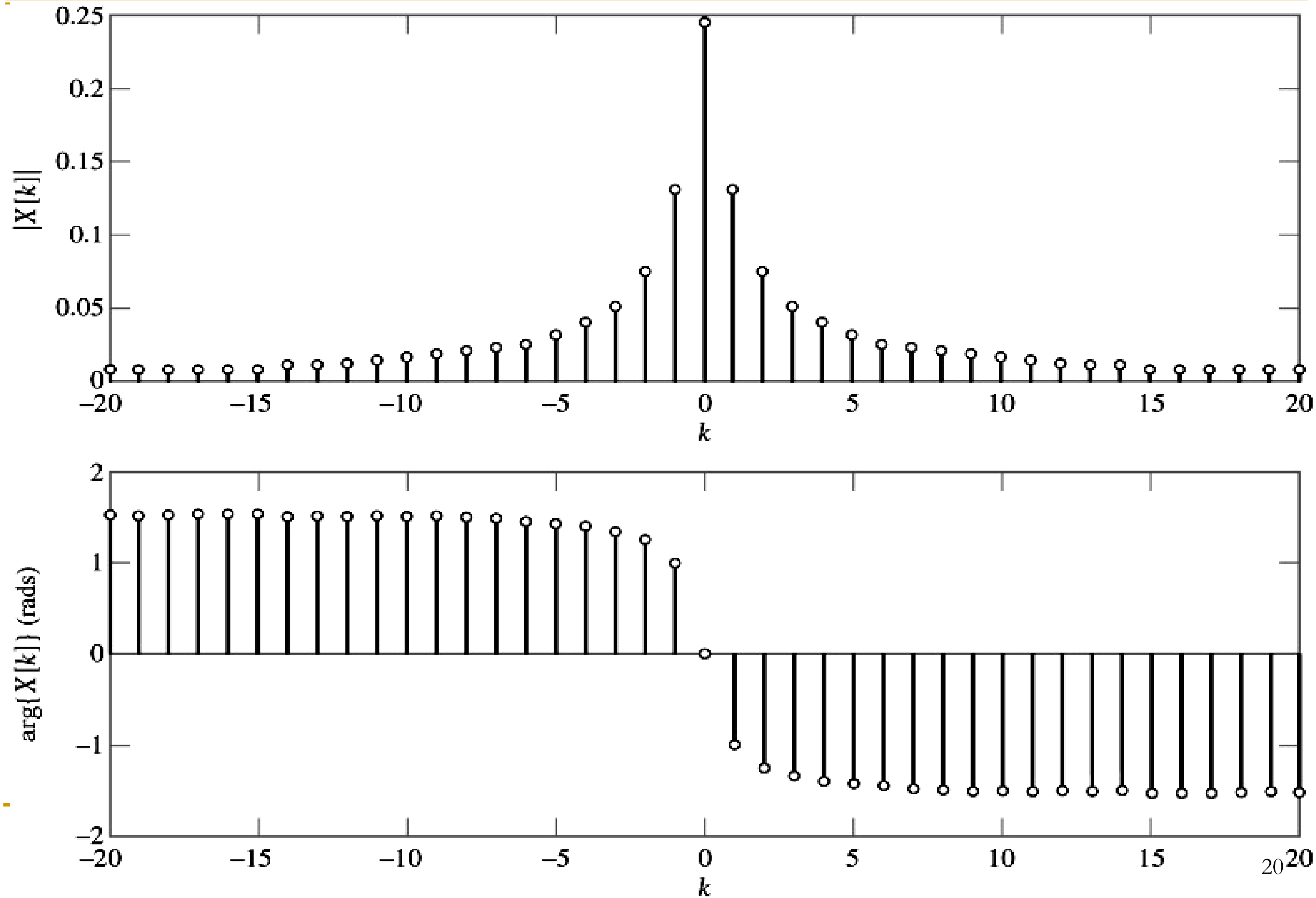
$$= \frac{-1}{2(2+jk\pi)} e^{-(2+jk\pi)t}\Big|_0^2$$

$$= \frac{1}{4+jk2\pi}\left(1-e^{-4}e^{-jk2\pi}\right) = \frac{1-e^{-4}}{4+jk2\pi}$$



$e^{-jk2\pi} = 1$

Example 1: FS for CT Periodic Signals



Example 2(3.10): *FS Coefficients for An Impulse Train*

$$x(t)=\sum_{l=-\infty}^{\infty}\delta(t-4l)$$

<Sol.>

1. Fundamental period of $x(t)$ is $T = 4$, each period contains an impulse. frequency $\omega_o = 2\pi/T$

2. By integrating over a period that is symmetric about the origin $-2 < t \le 2$, to obtain $X[k]$

$$X[k]=\frac{1}{4}\int_{-2}^{2}\delta(t)e^{-jk(\pi/2)t}dt=\frac{1}{4}$$

3. The magnitude spectrum is constant and the phase spectrum is zero.

Example 3: Determine the FS coefficients of CT Periodic Signals using the method of inspection

$$x(t) = 3\cos(\pi t/2 + \pi/4)$$

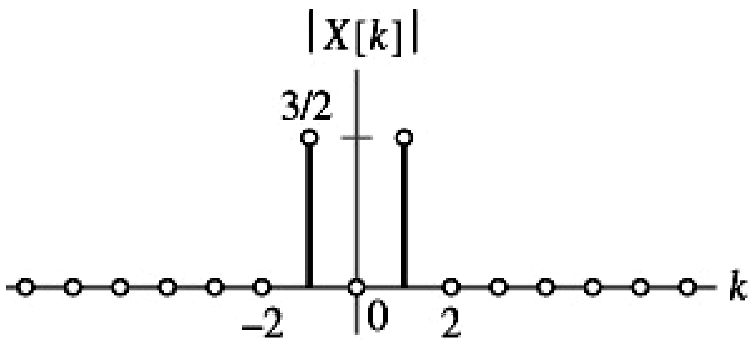
$T = 4, \quad \omega_o = 2\pi/4 = \pi/2$

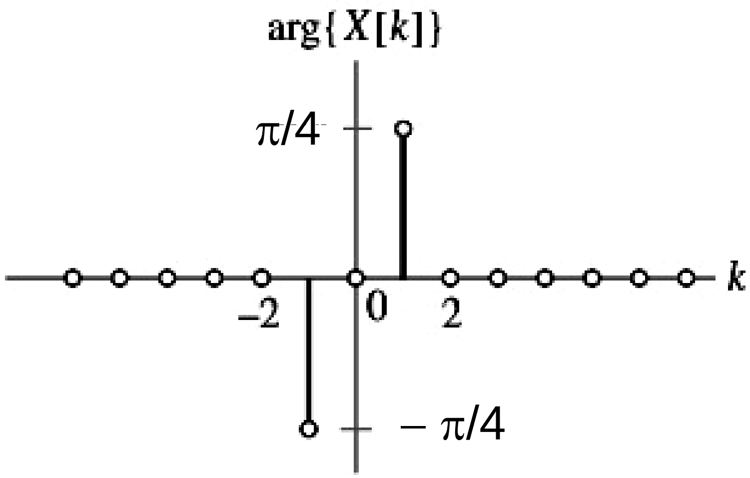
$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\pi t/2}$$

$$x(t) = \tfrac{3}{2}\left(e^{j(\pi t/2+\pi/4)} + e^{-j(\pi t/2+\pi/4)}\right)$$

$$= \tfrac{3}{2} e^{j\pi/4} e^{j\pi t/2} + \tfrac{3}{2} e^{-j\pi/4} e^{-j\pi t/2}$$

$$X[k] = \begin{cases} \dfrac{3}{2} e^{-j\pi/4}, & k = -1 \\ \dfrac{3}{2} e^{j\pi/4}, & k = 1 \\ 0, & \text{otherwise} \end{cases}$$





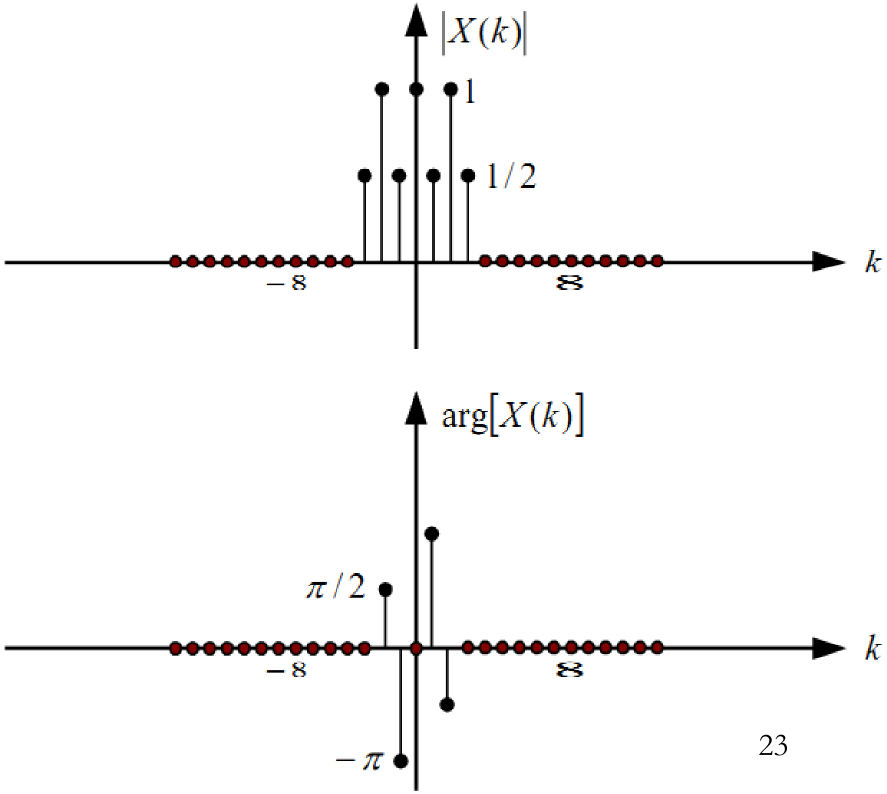
Example 4: FS for CT Periodic Signals

Determine the DTFS coefficients of $x(t) = 1 - \cos(\pi t) + 2\sin(2\pi t) + \cos(3\pi t)$ using the method of inspection.

$$x(t) = 1 - \cos(\pi t) + 2\sin(2\pi t) + \cos(3\pi t)$$

$$= 1 - \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} + 2\frac{e^{j2\omega_0 t} - e^{-j2\omega_0 t}}{2j} + \frac{e^{j3\omega_0 t} + e^{-j3\omega_0 t}}{2}$$

$$X(k\omega_0) = \begin{cases} 1, & k = 0 \\ -\frac{1}{2}, & k = \pm 1 \\ \mp j, & k = \pm 2 \\ \frac{1}{2}, & k = \pm 3 \\ 0, & others \end{cases}$$



Example 5(3.12) *Inverse FS*

Find the time-domain signal *x*(*t*) corresponding to the FS coefficients

$$X[k] = (1/2)^{|k|} e^{jk\pi/20}$$

***T* = 2**

<Sol.>

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$$

$$X[k] = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$= \sum_{k=0}^{\infty} (1/2)^k e^{jk\pi/20} e^{jk\pi t} + \sum_{k=-1}^{-\infty} (1/2)^{-k} e^{jk\pi/20} e^{jk\pi t}$$

$$= \sum_{k=0}^{\infty} (1/2)^k e^{jk\pi/20} e^{jk\pi t} + \sum_{l=1}^{\infty} (1/2)^l e^{-jl\pi/20} e^{-jl\pi t}$$

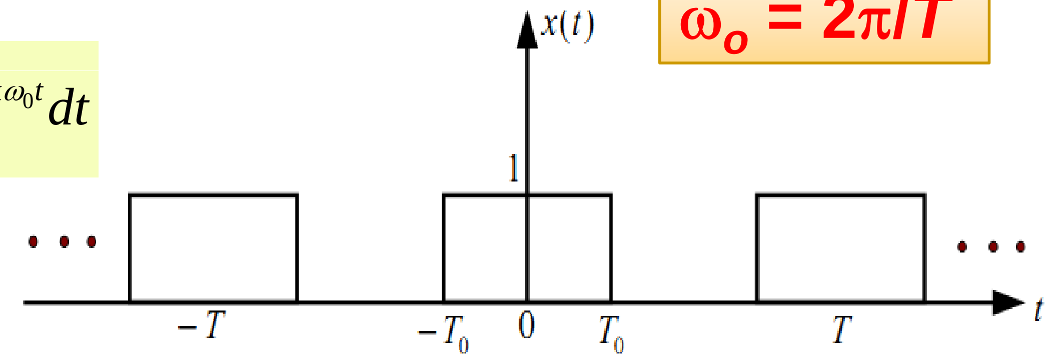
$$= \frac{1}{1-(1/2)e^{j(\pi t+\pi/20)}} + \frac{1}{1-(1/2)e^{-j(\pi t+\pi/20)}} - 1$$

$$= \frac{3}{5-4\cos(\pi t+\pi/20)}$$

Example 6: FS for CT Periodic Signals

Periodic square wave

$\omega_o = 2\pi/T$



$$X[k] = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_0}^{T_0} e^{-jk\omega_0 t}dt$$

$$= \begin{cases} X[0] = \dfrac{1}{T}\displaystyle\int_{-T_0}^{T_0} dt = \dfrac{2T_0}{T}, & k = 0 \\ X[k] = \dfrac{-1}{Tjk\omega_0} e^{-jk\omega_0 t}\Big|_{-T_0}^{T_0}, & k \neq 0 \end{cases}$$

$$= \frac{2}{Tk\omega_0}\left(\frac{e^{jk\omega_0 T_0} - e^{-jk\omega_0 T_0}}{2j}\right), \quad k \neq 0$$

$$= \frac{2\sin(k\omega_0 T_0)}{Tk\omega_0}, \quad k \neq 0$$

Sinc function

$$\operatorname{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$$

$$X[k] = \frac{2\sin(k\omega_0 T_0)}{Tk\omega_0}$$

$$= \frac{2\sin(k2\pi T_0/T)}{k2\pi}$$

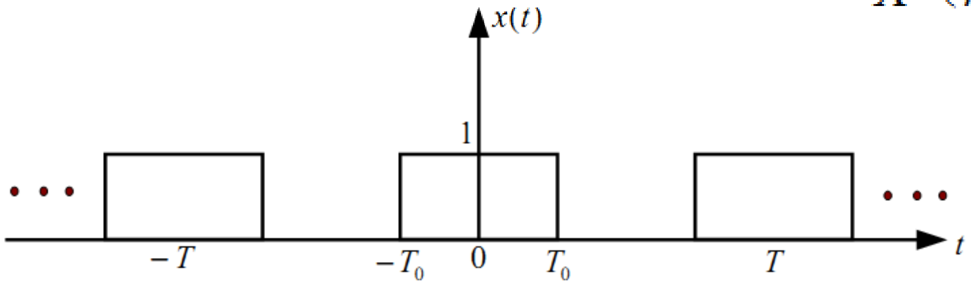
$$= 2T_0/T \operatorname{sinc}(2kT_0/T)$$

L'Hôpital's rule

$$\lim_{k\to 0}\frac{2\sin(k\omega_0 T_0)}{Tk\omega_0} = \frac{2T_0}{T}$$

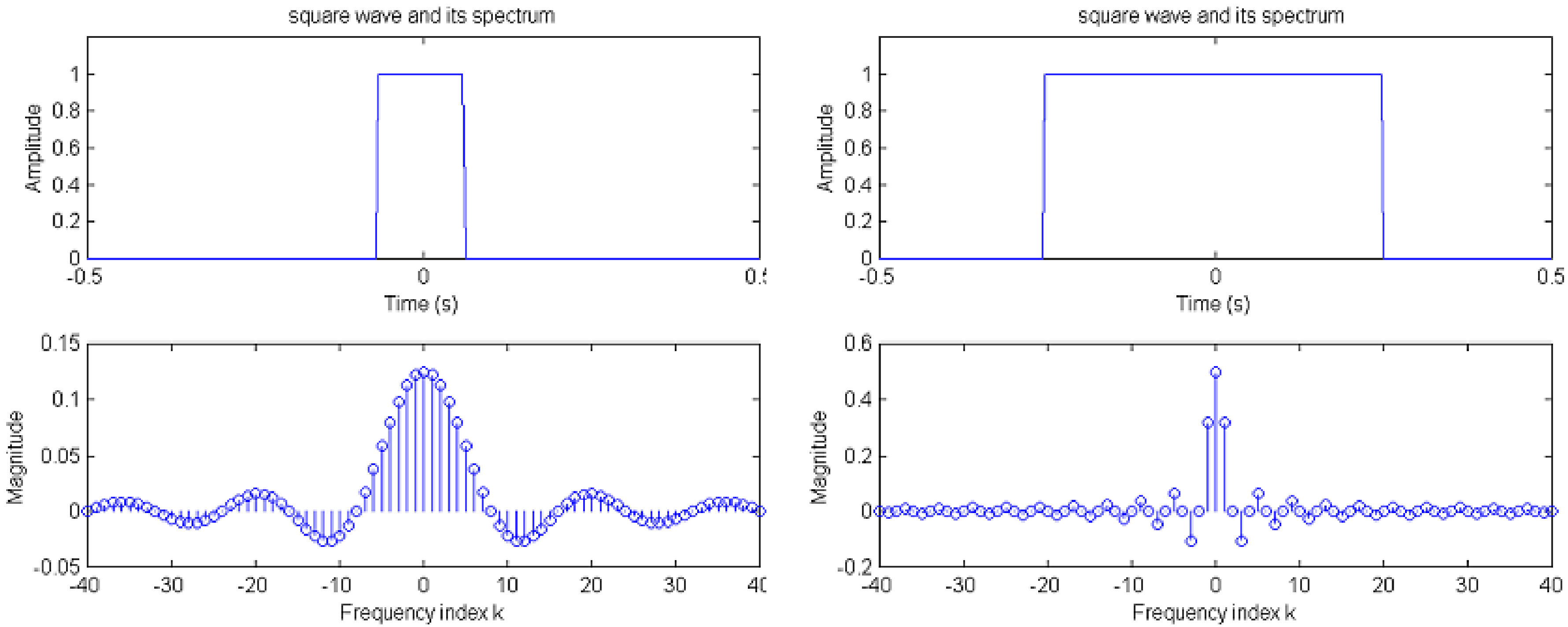
Example 6: FS for CT Periodic Signals

Periodic square wave

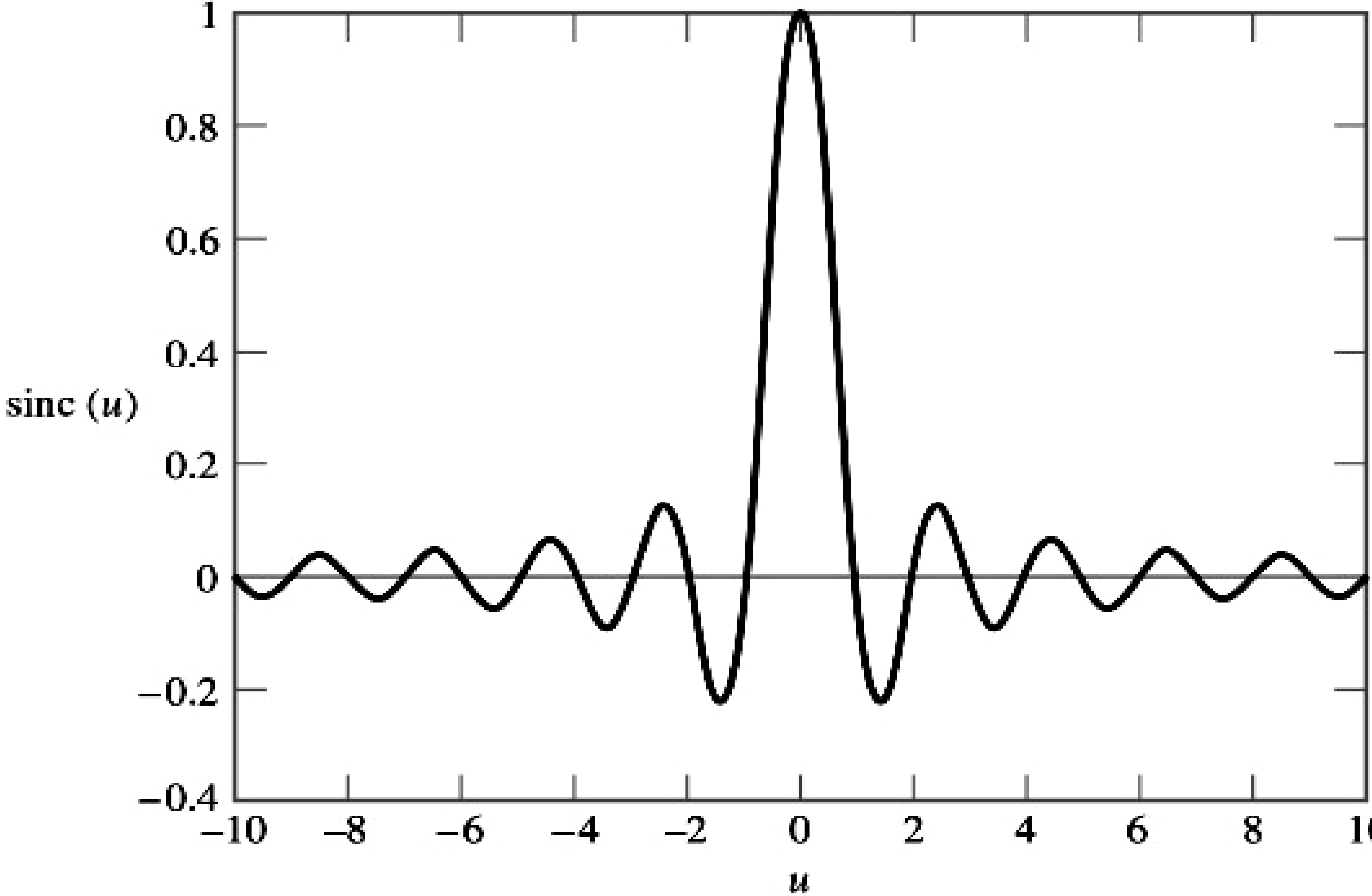


$$X(k)=\frac{1}{T}\int_{-T/2}^{T/2} e^{-jk\omega_0 t}=\frac{2T_0}{T}\frac{\sin(k2\pi T_0/T)}{k2\pi T_0/T}$$

$$=\begin{cases}\dfrac{2T_0}{T}, & k=0,\pm 2m\pi,\cdots \\ \dfrac{2T_0}{T}\sin\left(k2\pi\dfrac{T_0}{T}\right), & k\neq 0,\pm 2m\pi,\cdots\end{cases}$$



Sinc function sinc(u) = sin(πu)/(πu)



② DT Periodic Signals: DT Fourier Series (DTFS)

1. DTFS pair of *N*-periodic signal *x*[*n*]: $x[n] \xleftrightarrow{DTFS;\ \Omega_0} X[k]$

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n}$$

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_0 n}$$

Fundamental period *N*;
Fundamental frequency $\Omega_o = 2\pi/N$

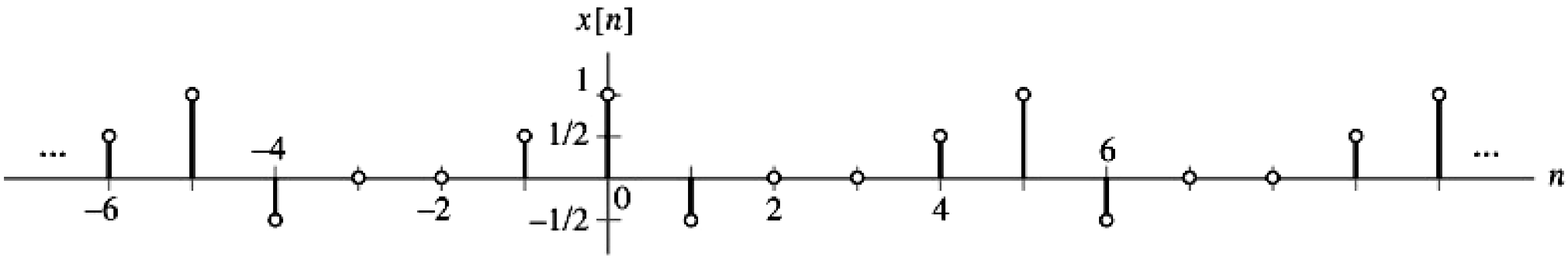
$$0 \le n, k \le N-1$$

Fourier coefficients; Frequency domain representation

2. The complex sinusoids exp($jk\Omega_o n$) are *N*-periodics in the frequency index *k*. There are only *N* distinct complex sinusoids of the form exp($jk\Omega_0 n$) should be used.

$$e^{j(N+k)\Omega_0 n} = e^{jN\Omega_0 n} e^{jk\Omega_0 n} = e^{j2\pi n} e^{jk\Omega_0 n} = e^{jk\Omega_0 n}$$

Example 1: Determine the DTFS coefficients of DT Periodic Signals using defination



<Sol.> 1. Period: $N = 5$ ⟹ $\Omega_o = 2\pi/5$

2. Odd symmetry ⟹ $n = -2$ to $n = 2$

$$X[k] = \frac{1}{5}\sum_{n=-2}^{2} x[n]e^{-jk2\pi n/5}$$

If $x[n]$ is symmetric in odd or even, we can choose k as: $k = -(N-1)/2$ to $(N-1)/2$

$$= \frac{1}{5}\left\{x[-2]e^{jk4\pi/5} + x[-1]e^{jk2\pi/5} + x[0]e^{j0} + x[1]e^{-jk2\pi/5} + x[2]e^{-jk4\pi/5}\right\}$$

$$= \frac{1}{5}\left\{\frac{1}{2}e^{jk2\pi/5} + 1 - \frac{1}{2}e^{-jk2\pi/5}\right\} = \frac{1}{5}\{1 + j\sin(k2\pi/5)\}$$

Example 1: DTFS

$$X[k]=\frac{1}{5}\{1+\frac{1}{2}e^{jk2\pi/5}-\frac{1}{2}e^{-jk2\pi/5}\}$$
$$=\frac{1}{5}\{1+j\sin(k2\pi/5)\}$$

3. One period of *X*[*k*]

***n* = – 2 to n= 2**

4. Calculate *X*[*k*] using

***n* = 0 to *n* = 4**

$$X[-2]=\frac{1}{5}-j\frac{\sin(4\pi/5)}{5}=0.232e^{-j0.531}$$

$$X[-1]=\frac{1}{5}-j\frac{\sin(2\pi/5)}{5}=0.276e^{-j0.760}$$

$$X[0]=\frac{1}{5}=0.2e^{j0}$$

$$X[1]=\frac{1}{5}+j\frac{\sin(2\pi/5)}{5}=0.276e^{j0.760}$$

$$X[2]=\frac{1}{5}+j\frac{\sin(4\pi/5)}{5}=0.232e^{j0.531}$$

$$X[k]=\frac{1}{5}\left\{x[0]e^{j0}+x[1]e^{-jk2\pi/5}+x[2]e^{-jk4\pi/5}+x[3]e^{-jk6\pi/5}+x[4]e^{-jk8\pi/5}\right\}$$

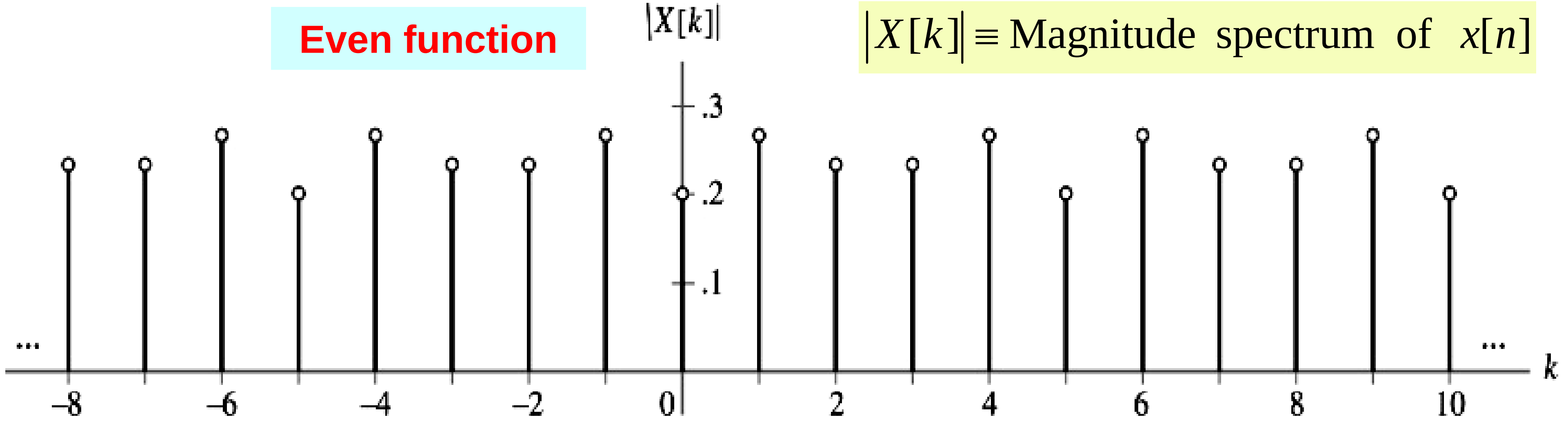
$$=\frac{1}{5}\left\{1-\frac{1}{2}e^{-jk2\pi/5}+\frac{1}{2}e^{-jk8\pi/5}\right\}$$

$$e^{-jk8\pi/5}=e^{-jk2\pi}e^{jk2\pi/5}=e^{jk2\pi/5}$$

Example 1: DTFS

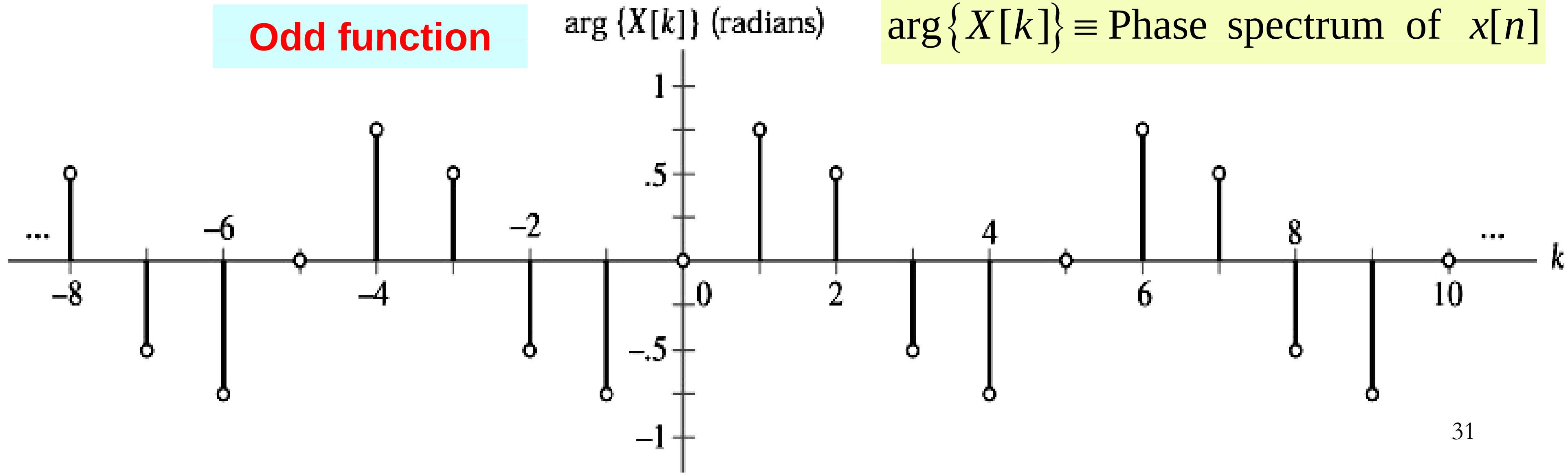
Even function

$|X[k]| \equiv \text{Magnitude spectrum of } x[n]$



Odd function

$\arg\{X[k]\} \equiv \text{Phase spectrum of } x[n]$



Example 2: Determine the DTFS coefficients of DT Periodic Signals using the method of inspection

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n}$$

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\Omega_0 n}$$

$$x[n] = \cos(\frac{\pi}{8} n + \frac{\pi}{3})$$

$\Omega_o = \pi / 8, N=16$

$$x[n] = \tfrac{1}{2} e^{j(\frac{\pi}{8}n + \frac{\pi}{3})} + \tfrac{1}{2} e^{-j(\frac{\pi}{8}n + \frac{\pi}{3})}$$

$$= \tfrac{1}{2} e^{j\frac{\pi}{3}} e^{j\frac{\pi}{8}n} + \tfrac{1}{2} e^{-j\frac{\pi}{3}} e^{-j\frac{\pi}{8}n}$$

$$X[k] = \begin{cases} \tfrac{1}{2} e^{j\frac{\pi}{3}}, & k = 1 \\ \tfrac{1}{2} e^{-j\frac{\pi}{3}}, & k = -1 \\ 0, & k \neq \pm 1 \end{cases}$$

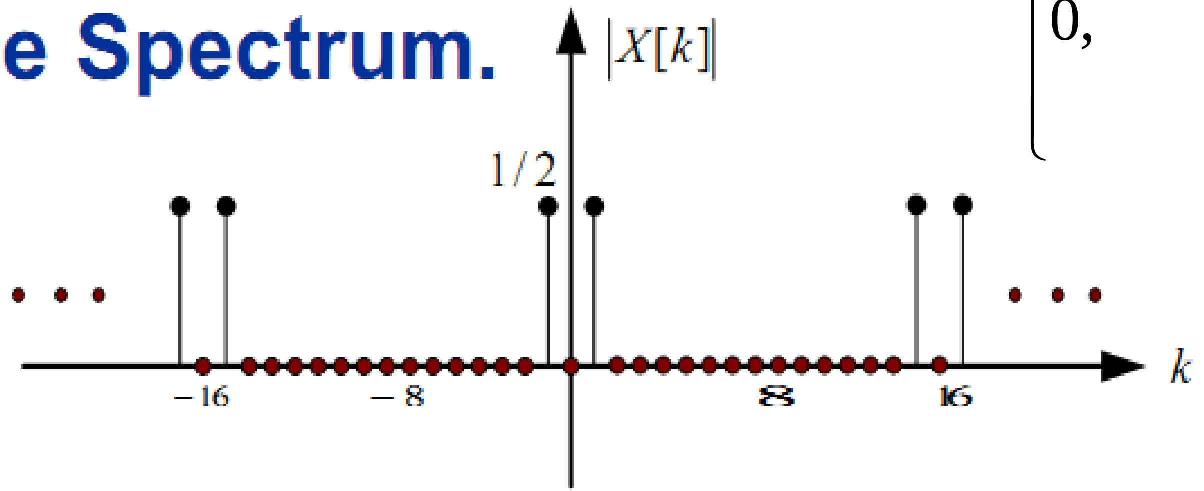
$$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\frac{\pi}{8}n}$$

$$X[k] = |X[k]| e^{j\arg\{X[k]\}}$$

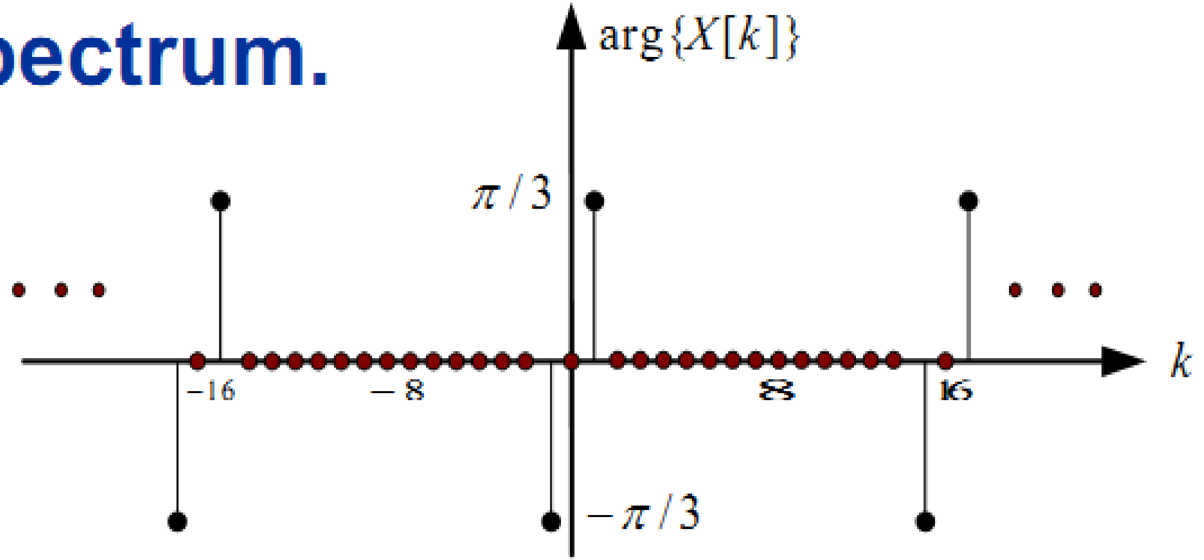
Example 2: DTFS

$$X[k]=\begin{cases}\frac{1}{2}e^{j\frac{\pi}{3}}, & k=1\\ \frac{1}{2}e^{-j\frac{\pi}{3}}, & k=-1\\ 0, & k\neq\pm1\end{cases}$$

Magnitude Spectrum.

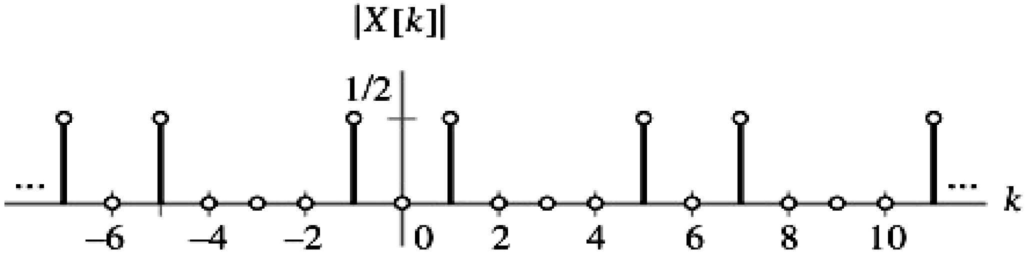


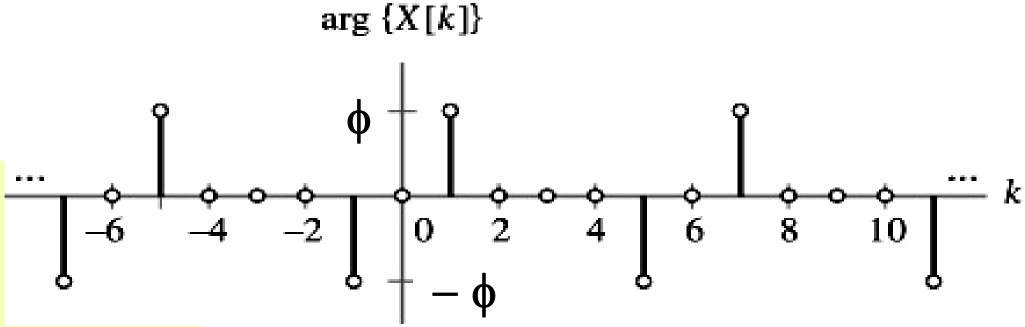
Phase Spectrum.



Example 3: $x[n] = \cos(n\pi/3 + \phi)$

$N = 6,\ \Omega_o = 2\pi/6 = \pi/3$





$$x[n] = \frac{1}{2}\{e^{j(\frac{\pi}{3}n+\phi)} + e^{-j(\frac{\pi}{3}n+\phi)}\}$$

$$= \frac{1}{2}e^{-j\phi}e^{-j\frac{\pi}{3}n} + \frac{1}{2}e^{j\phi}e^{j\frac{\pi}{3}n}$$

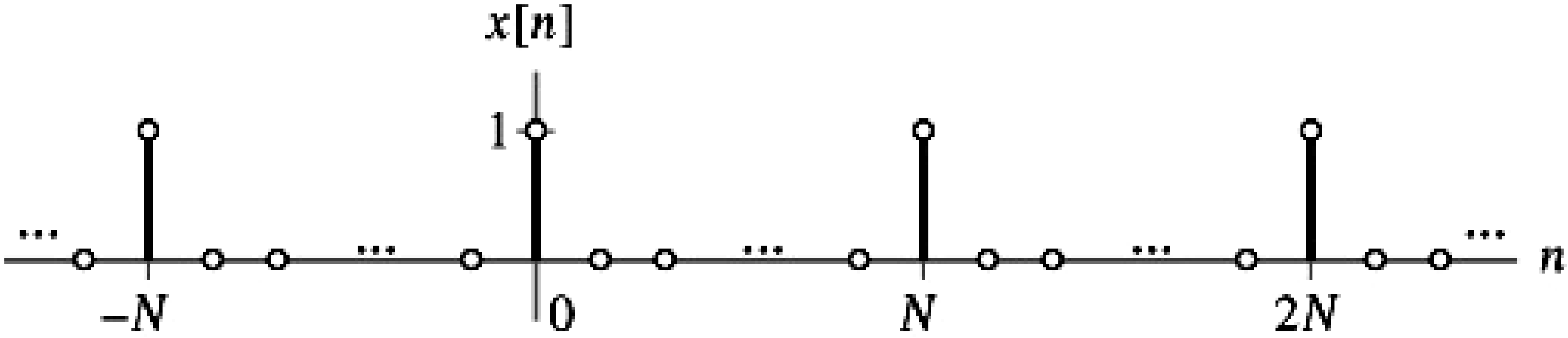
$$x[n] = \sum_{k=-2}^{3} X[k]e^{jk\pi n/3}$$

$$= X[-2]e^{-j2\pi n/3} + X[-1]e^{-j\pi n/3} + X[0]$$

$$+ X[1]e^{j\pi n/3} + X[2]e^{j2\pi n/3} + X[3]e^{j\pi n}$$

$$x[n] \xleftrightarrow{DTFS;\ \frac{\pi}{3}} X[k] = \begin{cases} e^{-j\phi}/2, & k=-1 \\ e^{j\phi}/2, & k=1 \\ 0, & \text{otherwise on } -2 \le k \le 3 \end{cases}$$

Example 4: DTFS Representation of An Impulse Train



$$x[n] = \sum_{l=-\infty}^{\infty} \delta[n - lN] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_0 n} = \frac{1}{N}\sum_{k=0}^{N-1} e^{j2\pi kn/N}$$

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\Omega_0 n} = \frac{1}{N}\sum_{n=0}^{N-1} \delta[n] e^{-jkn2\pi/N} = \frac{1}{N}$$

$$\boxed{\sum_{l=-\infty}^{\infty} \delta[n - lN] = \frac{1}{N}\sum_{k=0}^{N-1} e^{j2\pi kn/N}}$$

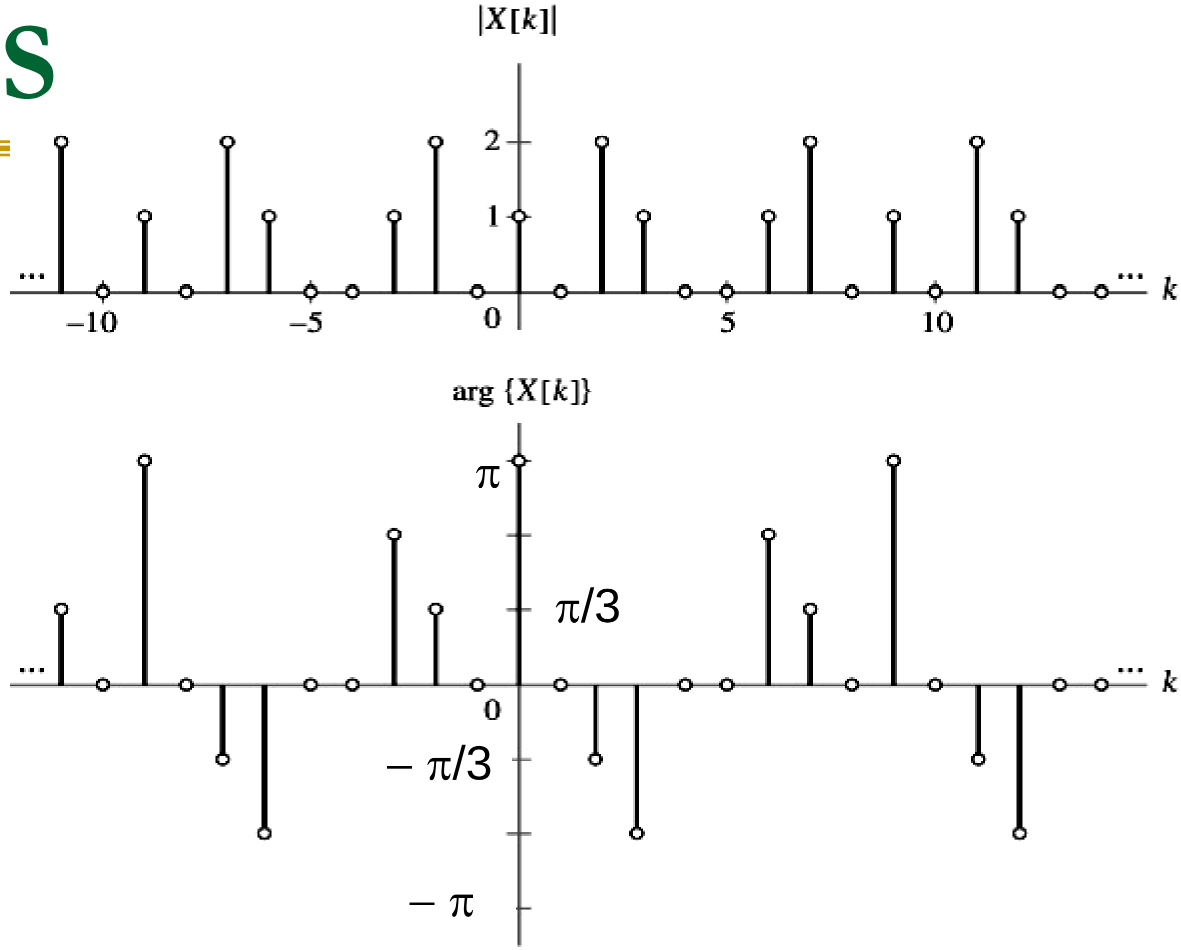
Inverse DTFS

$$x[n]=\sum_{k=0}^{N-1}X[k]e^{jk\Omega_0 n}$$

$$X[k]=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk\Omega_0 n}$$

N=9; Ω_o = 2π/9

$$x[n]=\sum_{k=-4}^{4}X[k]e^{jk2\pi n/9}$$

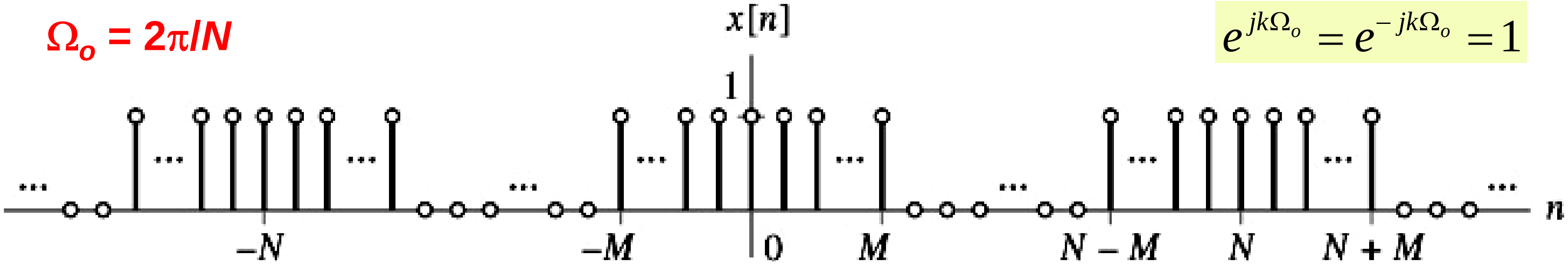


$$=e^{j2\pi/3}e^{-j6\pi n/9}+2e^{j\pi/3}e^{-j4\pi n/9}-1+2e^{-j\pi/3}e^{j4\pi n/9}+e^{-j2\pi/3}e^{j6\pi n/9}$$

$$=2\cos\left(6\pi n/9-2\pi/3\right)+4\cos\left(4\pi n/9-\pi/3\right)-1$$

Example 5: DTFS Representations of A Square Wave

$\Omega_o = 2\pi/N$



$$e^{jk\Omega_o} = e^{-jk\Omega_o} = 1$$

$$x[n] = \begin{cases} 1, & -M \le n \le M \\ 0, & M < n < N - M \end{cases}$$

$$X[k] = \frac{1}{N}\sum_{m=0}^{2M} 1 = \frac{2M+1}{N}, \quad k = 0, \ \pm N, \ \pm 2N, \ \ldots$$

$$X[k] = \frac{1}{N}\sum_{n=-M}^{N-M-1} x[n] e^{-jk\Omega_0 n} = \frac{1}{N}\sum_{n=-M}^{M} e^{-jk\Omega_0 n}$$

$$\xrightarrow{m=n+M} \frac{1}{N}\sum_{m=0}^{2M} e^{-jk\Omega_0 (m-M)} = \frac{1}{N} e^{jk\Omega_0 M} \sum_{m=0}^{2M} e^{-jk\Omega_0 m}$$

$$X[k] = \frac{e^{jk\Omega_0 M}}{N}\left(\frac{1-e^{-jk\Omega_0(2M+1)}}{1-e^{-jk\Omega_0}}\right), \ k \ne 0, \pm N, \pm 2N \ldots$$

$$= \frac{1}{N}\left(\frac{e^{jk\Omega_0(2M+1)/2}}{e^{jk\Omega_0/2}}\right)\left(\frac{1-e^{-jk\Omega_0(2M+1)}}{1-e^{-jk\Omega_0}}\right)$$

$$= \frac{1}{N}\left(\frac{e^{jk\Omega_0(2M+1)/2} - e^{-jk\Omega_0(2M+1)/2}}{e^{jk\Omega_0/2} - e^{-jk\Omega_0/2}}\right)$$

$$= \frac{1}{N}\frac{\sin\left(k\Omega_0\left(2M+1\right)/2\right)}{\sin\left(k\Omega_0/2\right)}$$

Example 5: DTFS Representations of A Square Wave

Substituting $\Omega_o = 2\pi/N$, yields

$$X[k] = \begin{cases} \dfrac{1}{N}\dfrac{\sin\left(k\pi(2M+1)/N\right)}{\sin\left(k\pi/N\right)}, & k \neq 0, \ \pm N, \ \pm 2N, \ \ldots \\ (2M+1)/N, & k = 0, \ \pm N, \ \pm 2N, \ \ldots \end{cases}$$

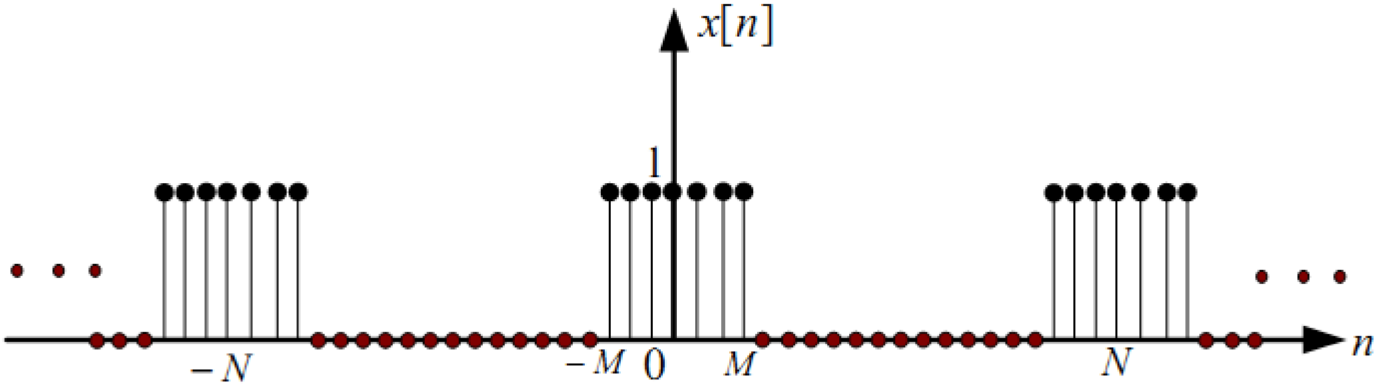
$$X[k] = \frac{1}{N}\frac{\sin\left(k\pi(2M+1)/N\right)}{\sin\left(k\pi/N\right)}$$

The value of $X[k]$ for $k = 0, \pm N, \ldots$, is obtained from the limit as $k \to 0$.

$$\lim_{k \to 0, \pm N, \pm 2N, \ldots} \frac{1}{N}\frac{\sin\left(k\pi(2M+1)/N\right)}{\sin\left(k\pi/N\right)} = \frac{2M+1}{N}$$

L'Hôpital's Rule

Example 5: DTFS Representations of A Square Wave



$$X[k]=\frac{1}{N}\sum_{n=-M}^{M}e^{-jk\Omega_0 n}\quad=\frac{1}{N}\frac{\sin\left(\frac{2M+1}{2}k\Omega_0\right)}{\sin\left(\frac{k\Omega_0}{2}\right)}$$

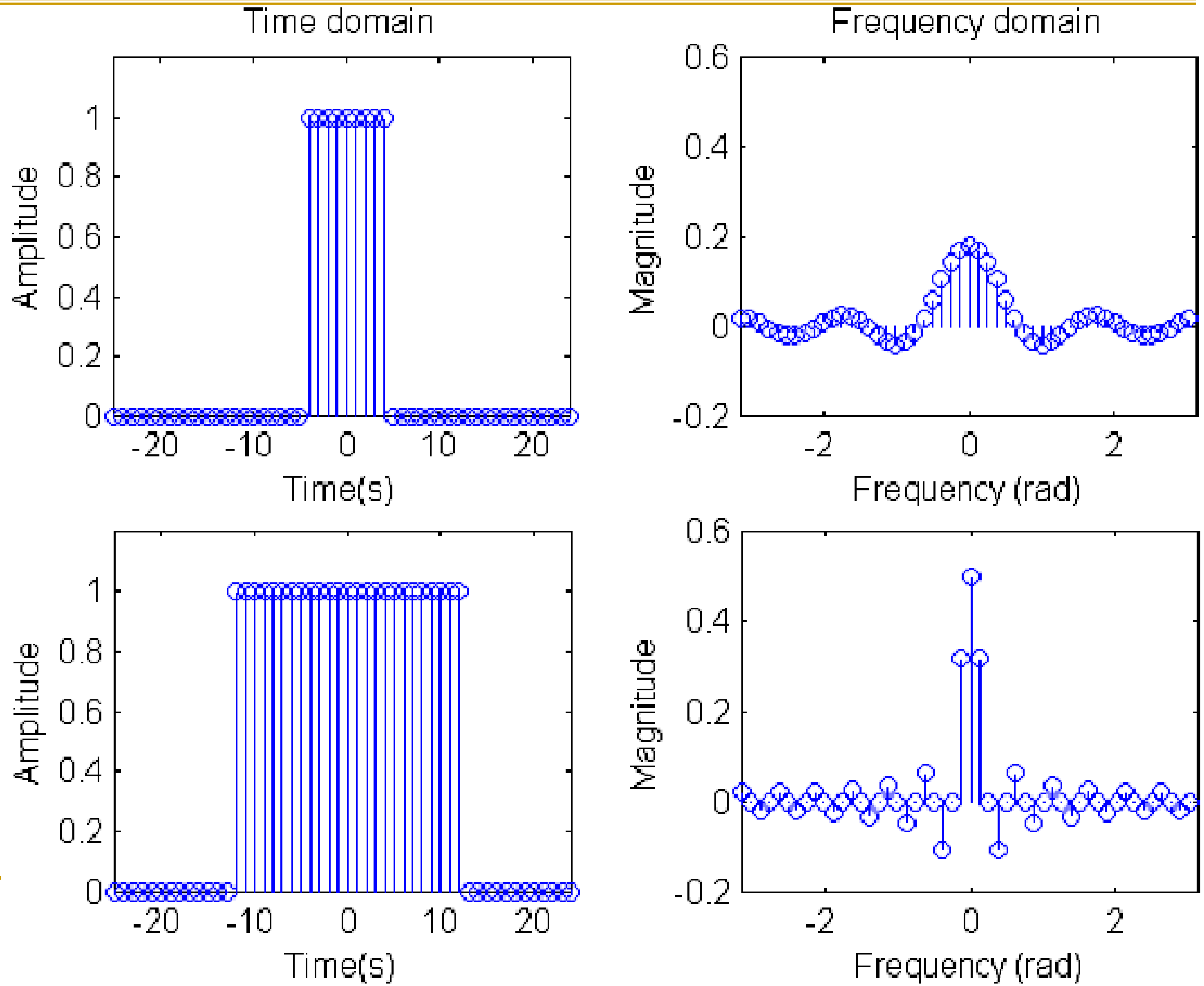
$$=\begin{cases}\dfrac{2M+1}{N}, & k=0,\pm N,\pm 2N,\cdots\\[2ex] \dfrac{1}{N}\dfrac{\sin\left(\frac{2M+1}{2}k\Omega_0\right)}{\sin\left(\frac{k\Omega_0}{2}\right)}, & k\neq 0,\pm N,\pm 2N,\cdots\end{cases}$$

Example 5: DTFS for DT Periodic Signals

N = 5

M = 4

M = 12

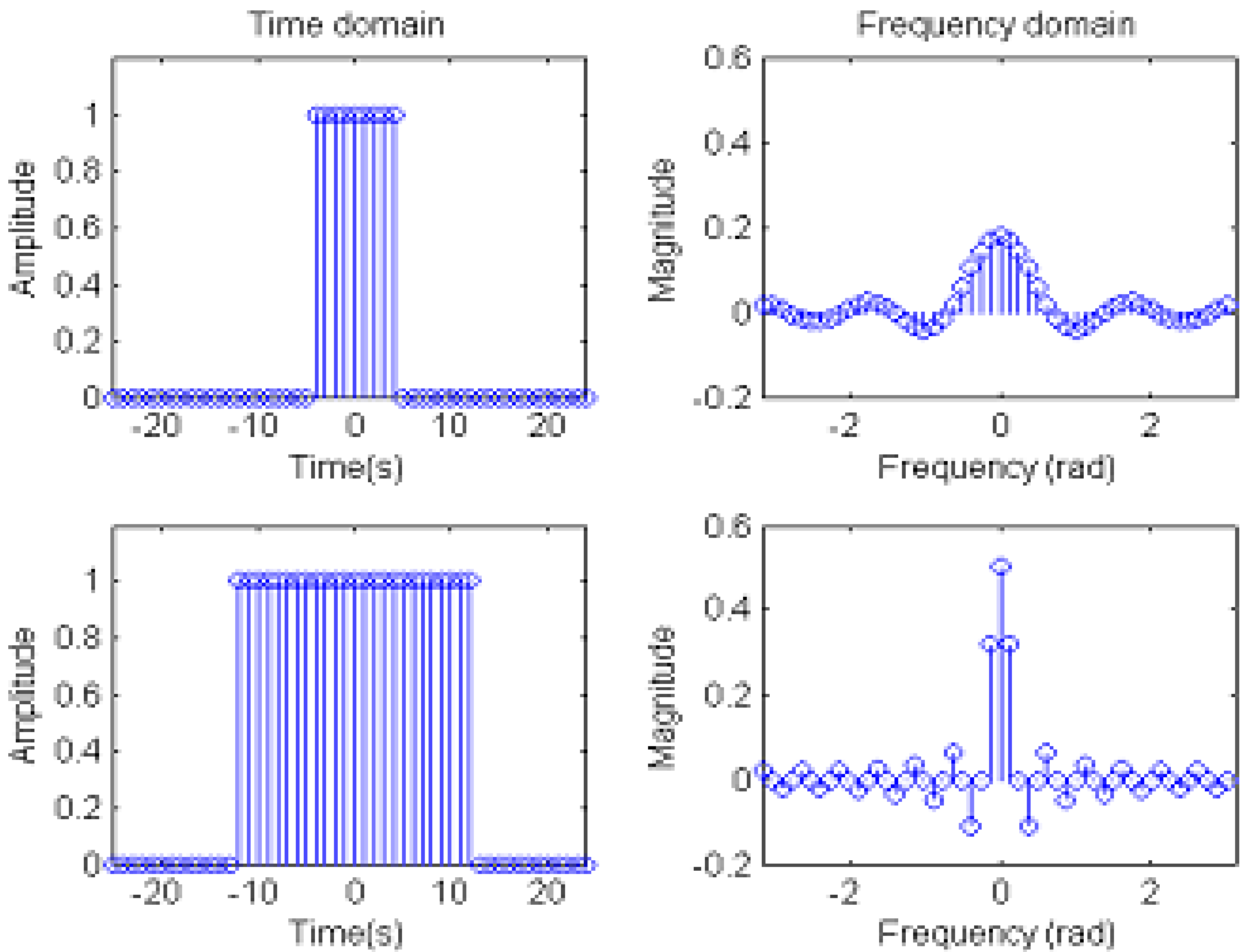


Example 5: DTFS for DT Periodic Signals

$N = 50$

$M = 4$

$M = 12$



③DT Nonperiodic Signals: The DT Fourier Transform (DTFT)

The DTFT is used to represent a discrete-time non-periodic signal as a superposition of complex sinusoids.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}\,d\Omega$$

$$x[n] \xleftrightarrow{DTFT} X\left(e^{j\Omega}\right)$$

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

Frequency-domain representation *x*[*n*]

Condition for convergence of DTFT:

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right| < \infty$$

Converges uniformly to a continuous function of Ω.

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 < \infty$$

Converges in a mean-square error sense, but does not converge pointwise(逐点).

Example 3.17 **Find the DTFT of the sequence $x[n] = \alpha^n u[n]$.**

<Sol.>

$$X\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\Omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega n}$$

diverges for $|\alpha| \geq 1$

2. For $|\alpha| < 1$

$$X(e^{j\Omega}) = \sum_{n=0}^{\infty} \left(\alpha e^{-j\Omega}\right)^n = \frac{1}{1-\alpha e^{-j\Omega}}, \quad |\alpha| < 1$$

3. If α is real valued

$$X\left(e^{j\Omega}\right) = \frac{1}{1-\alpha\cos\Omega + j\alpha\sin\Omega}$$

Euler's Formula

$$\left|X\left(e^{j\Omega}\right)\right| = \frac{1}{\left(\left(1-\alpha\cos\Omega\right)^2 + \alpha^2\sin^2\Omega\right)^{1/2}} = \frac{1}{\left(\alpha^2 + 1 - 2\alpha\cos\Omega\right)^{1/2}}$$

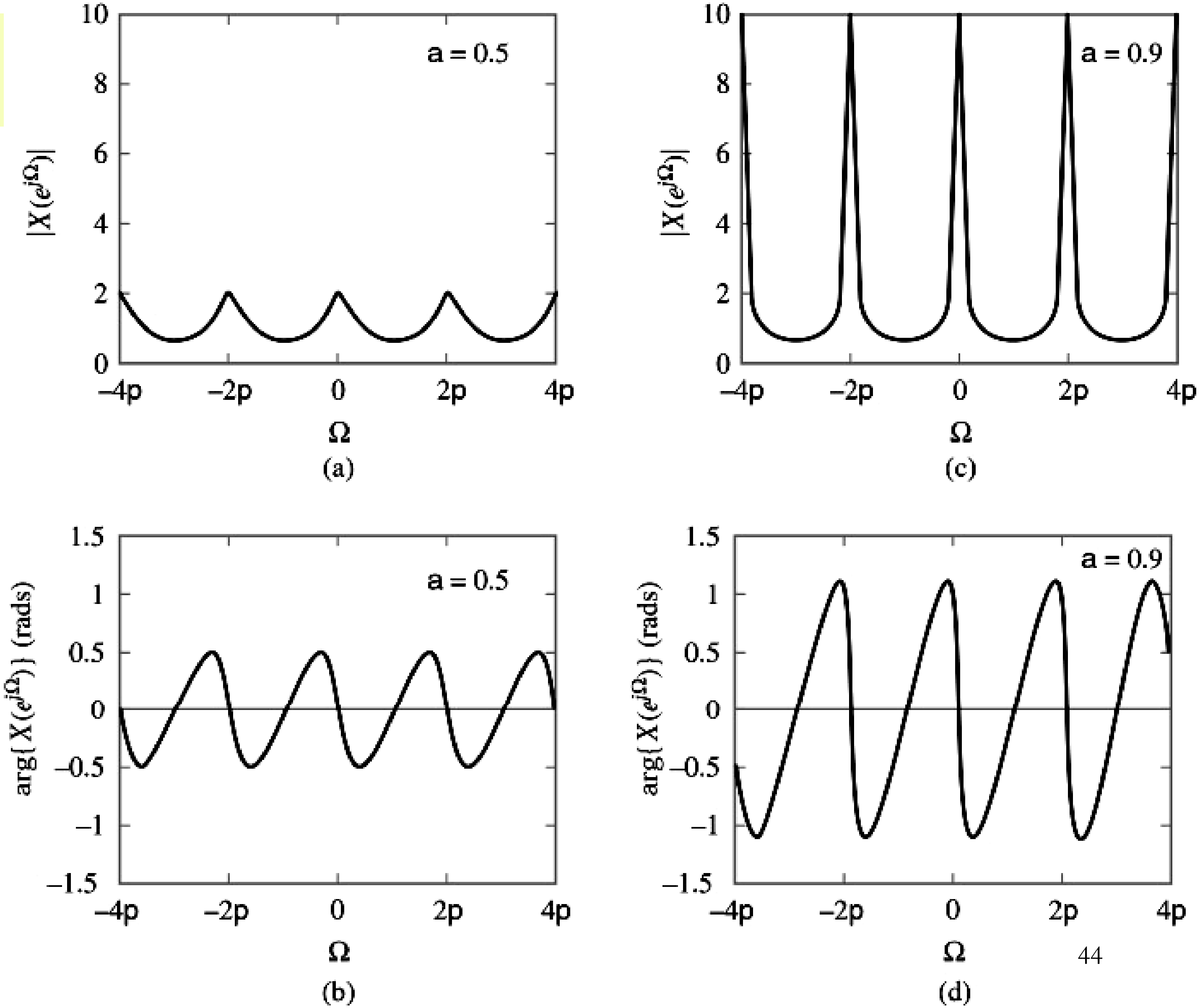
Even function

$$\arg\left\{X\left(e^{j\Omega}\right)\right\} = -\arctan\left(\frac{\alpha\sin\Omega}{1-\alpha\cos\Omega}\right)$$

Odd function

DT Nonperiodic Signals: DTFT of An Exponential Sequence

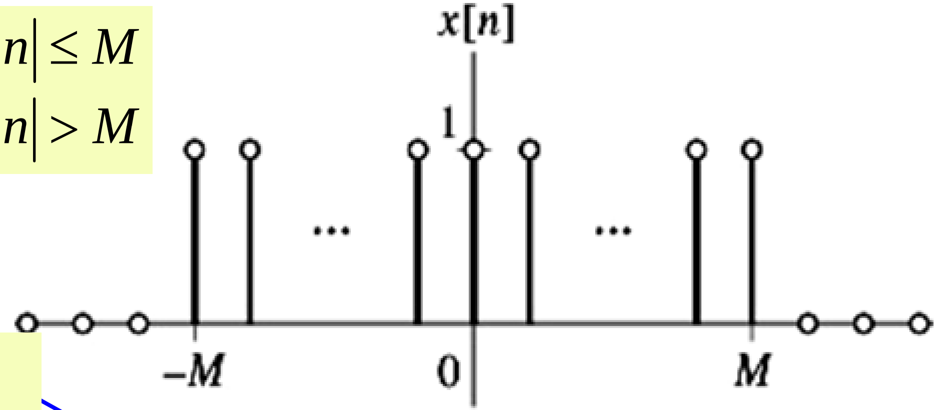
$$x[n] = \alpha^n u[n]$$



DT Nonperiodic Signals: DTFT of A Rectangular Pulse

$$X\left(e^{j\Omega}\right)=\sum_{n=-M}^{M} 1e^{-j\Omega n}.$$

$$x[n]=\begin{cases}1, & |n|\le M\\ 0, & |n|>M\end{cases}$$



$$=\sum_{m=0}^{2M} e^{-j\Omega(m-M)}=e^{j\Omega M}\sum_{m=0}^{2M} e^{-j\Omega m}$$

Change of variable
m = n + M

$$=\begin{cases}2M+1, & \Omega=0,\pm 2\pi,\pm 4\pi,\dots\\ e^{j\Omega M}\dfrac{1-e^{-j\Omega 2(M+1)}}{1-e^{-j\Omega}}, & \Omega\neq 0,\ \pm 2\pi,\pm 4\pi,\dots\end{cases}$$

$$=e^{j\Omega M}\frac{e^{-j\Omega(2M+1)/2}\left(e^{j\Omega(2M+1)/2}-e^{-j\Omega(2M+1)/2}\right)}{e^{-j\Omega/2}\left(e^{j\Omega/2}-e^{-j\Omega/2}\right)}$$

$$=\frac{\left(e^{j\Omega(2M+1)/2}-e^{-j\Omega(2M+1)/2}\right)}{e^{j\Omega/2}-e^{-j\Omega/2}}$$

L'Hôptital's Rule

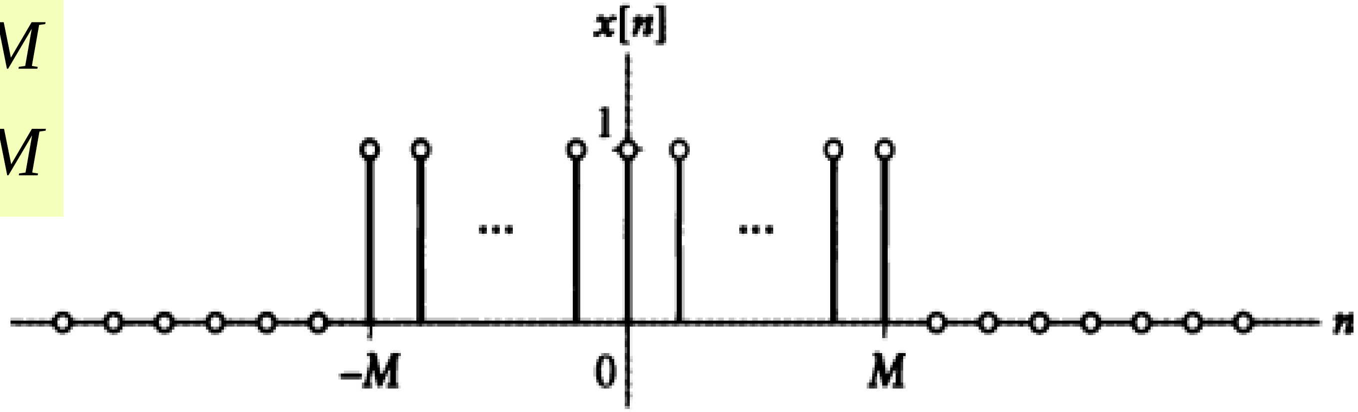
$$\lim_{\Omega\to 0,\pm 2\pi,\pm 4\pi,\dots,}\frac{\sin\left(\Omega\left(2M+1\right)/2\right)}{\sin\left(\Omega/2\right)}=2M+1$$

$$X(e^{j\Omega})=\frac{\sin\left(\Omega\left(2M+1\right)/2\right)}{\sin\left(\Omega/2\right)}$$

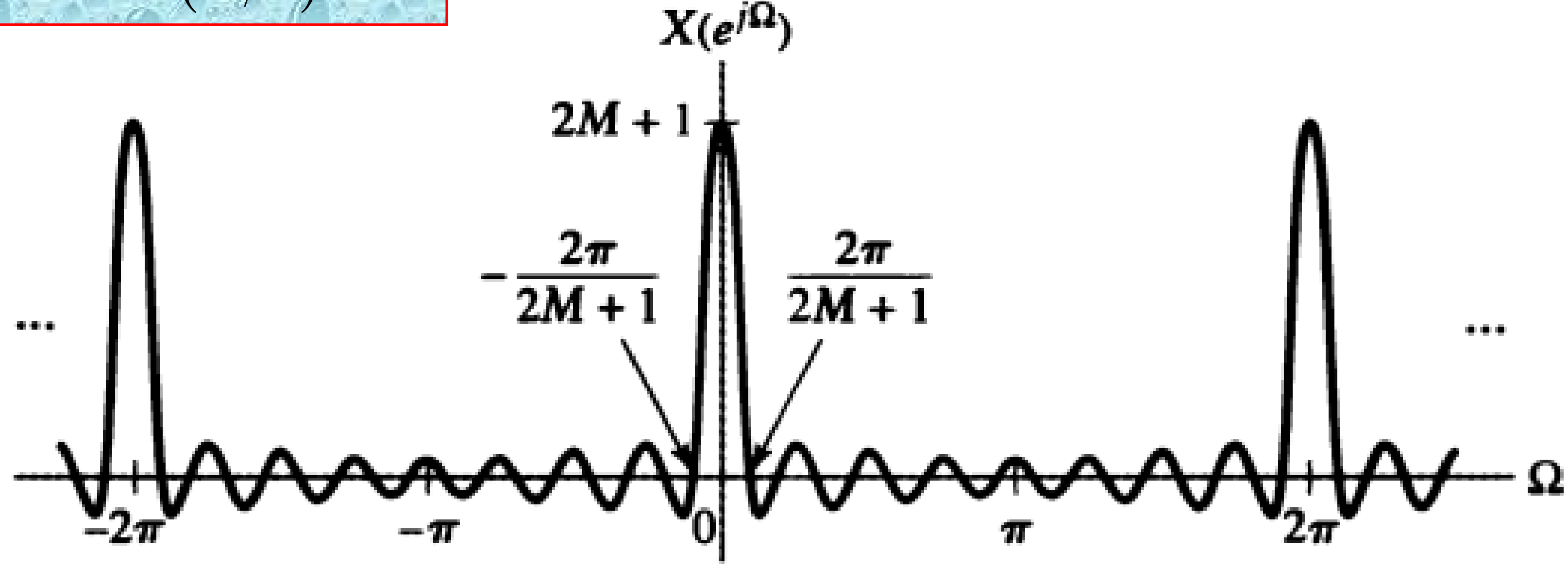
With understanding that $X(e^{j\Omega})$ for $\Omega\neq\pm 2m\pi$ is obtained as limit.

DT Nonperiodic Signals: DTFT of A Rectangular Pulse

$$x[n] = \begin{cases} 1, & |n| \le M \\ 0, & |n| > M \end{cases}$$



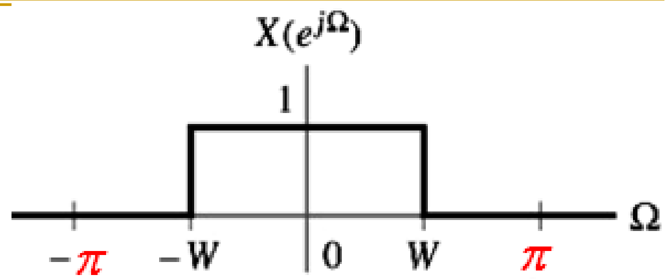
$$X(e^{j\Omega}) = \frac{\sin\left(\Omega(2M+1)/2\right)}{\sin\left(\Omega/2\right)}$$



DT Nonperiodic Signals: Inverse DTFT of A Rectangular Pulse

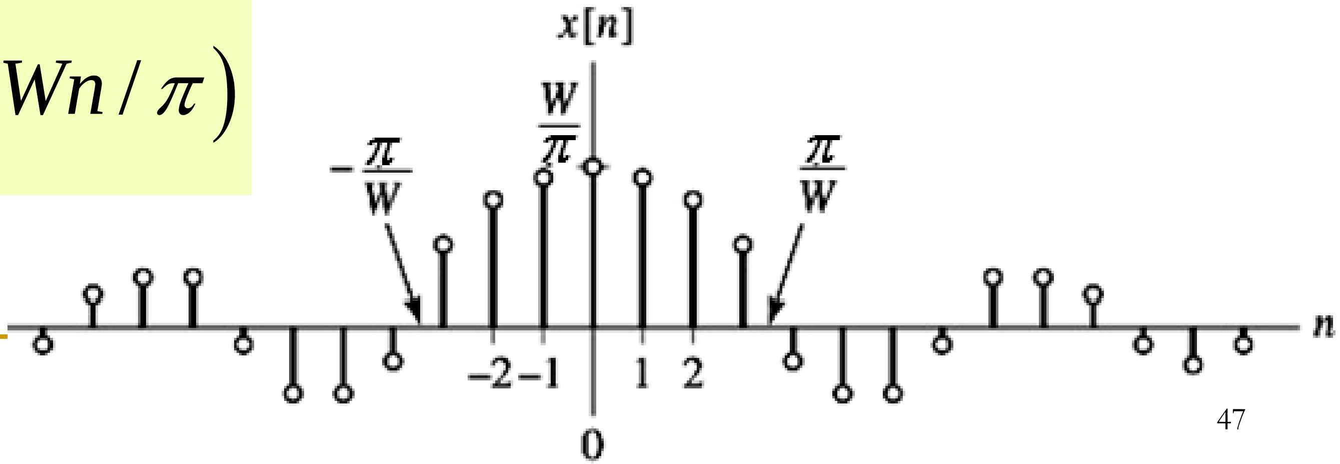
***X*(e$^{j\Omega}$) is specified only for $-\pi < \Omega \le \pi$.**

$$X\left(e^{j\Omega}\right)=\begin{cases}1, & |\Omega| \le W \\ 0, & W<|\Omega|<\pi\end{cases}$$



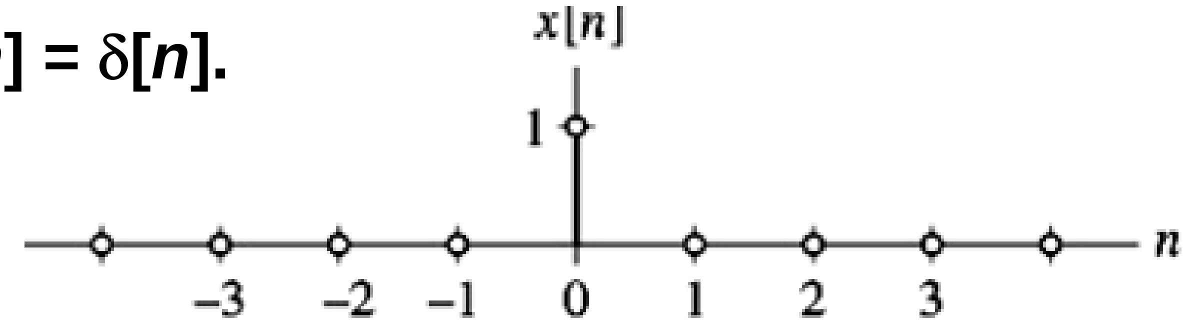
$$x[n]=\frac{1}{2\pi}\int_{-W}^{W} e^{j\Omega n}\,d\Omega=\begin{cases} n=0 & x[n]=\dfrac{1}{\pi n}\sin(Wn) \quad \lim\limits_{n\to 0}\dfrac{1}{n\pi}\sin(Wn)=\dfrac{W}{\pi} \\ n\neq 0 & =\dfrac{1}{2\pi n j}e^{j\Omega n}\Big|_{-W}^{W} \quad =\dfrac{1}{\pi n}\sin(Wn),\quad n\neq 0. \end{cases}$$

$$x[n]=\frac{W}{\pi}\operatorname{sinc}(Wn/\pi)$$



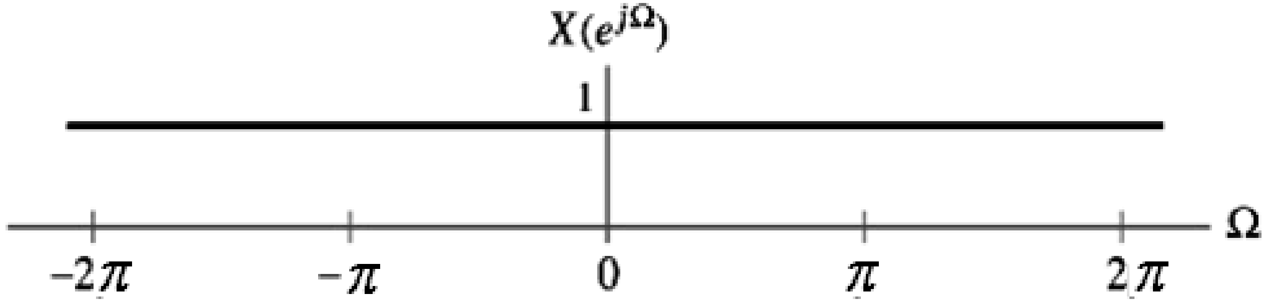
DT Nonperiodic Signals: DTFT of The Unit Impulse

Find the DTFT of *x*[*n*] = δ[*n*].



$$X\left(e^{j\Omega}\right)=\sum_{n=-\infty}^{\infty}\delta[n]e^{-j\Omega n}=1$$

$$\delta[n] \xleftrightarrow{DTFT} 1$$



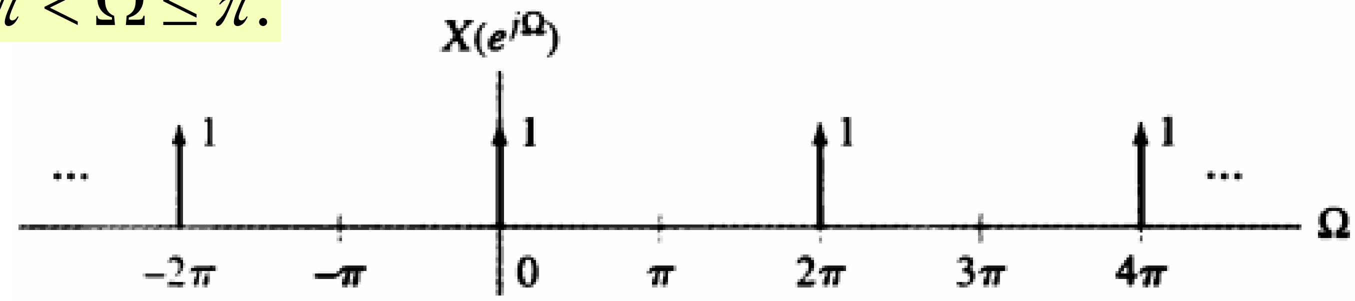
DT Nonperiodic Signals: Inverse DTFT of A Unit Impulse Spectrum

Find the inverse DTFT of $X(e^{j\Omega}) = \delta(\Omega), -\pi < \Omega \le \pi$.

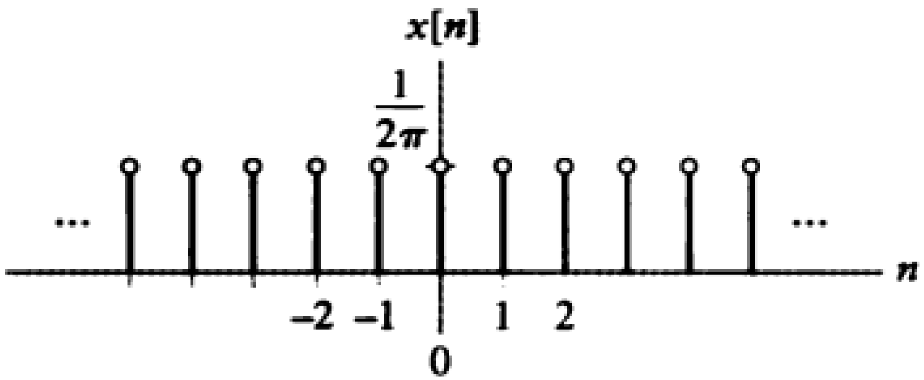
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\Omega) e^{j\Omega n} d\Omega. \quad = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\Omega) d\Omega = \frac{1}{2\pi}$$

Sifting property of impulse function

$$\frac{1}{2\pi} \xleftrightarrow{DTFT} \delta(\Omega) \quad -\pi < \Omega \le \pi.$$



$$X\left(e^{j\Omega}\right) = \sum_{k=-\infty}^{\infty} \delta(\Omega - k2\pi).$$



④CT Nonperiodic Signals: The Fourier Transform (FT)

1. FT is used to represent a continuous-time non-periodic signal as a superposition of complex sinusoids.

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

$$x(t)\xleftrightarrow{FT}X(j\omega)$$

Frequency-domain representation of the signal *x*(*t*)

2. Convergence

$$\hat{x}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$$

$$\int_{-\infty}^{\infty}\left|x(t)\right|^2dt<\infty$$

$$\int_{-\infty}^{\infty}\left|x(t)\right|dt<\infty$$

$$\Rightarrow \int_{-\infty}^{\infty}\left|x(t)-\hat{x}(t)\right|^2dt\to 0$$

$$\Rightarrow x(t)=\hat{x}(t)$$

CT Nonperiodic Signals: FT of A Real Decaying Exponential

Find the FT of $x(t) = e^{-at}u(t)$

1. For $a \le 0$, since $x(t)$ is not absolutely integrable,

$$\int_0^{\infty} e^{-at}dt = \infty, \quad a \le 0$$

The FT of $x(t)$ does not converge

2. For $a > 0$, the FT of $x(t)$ is

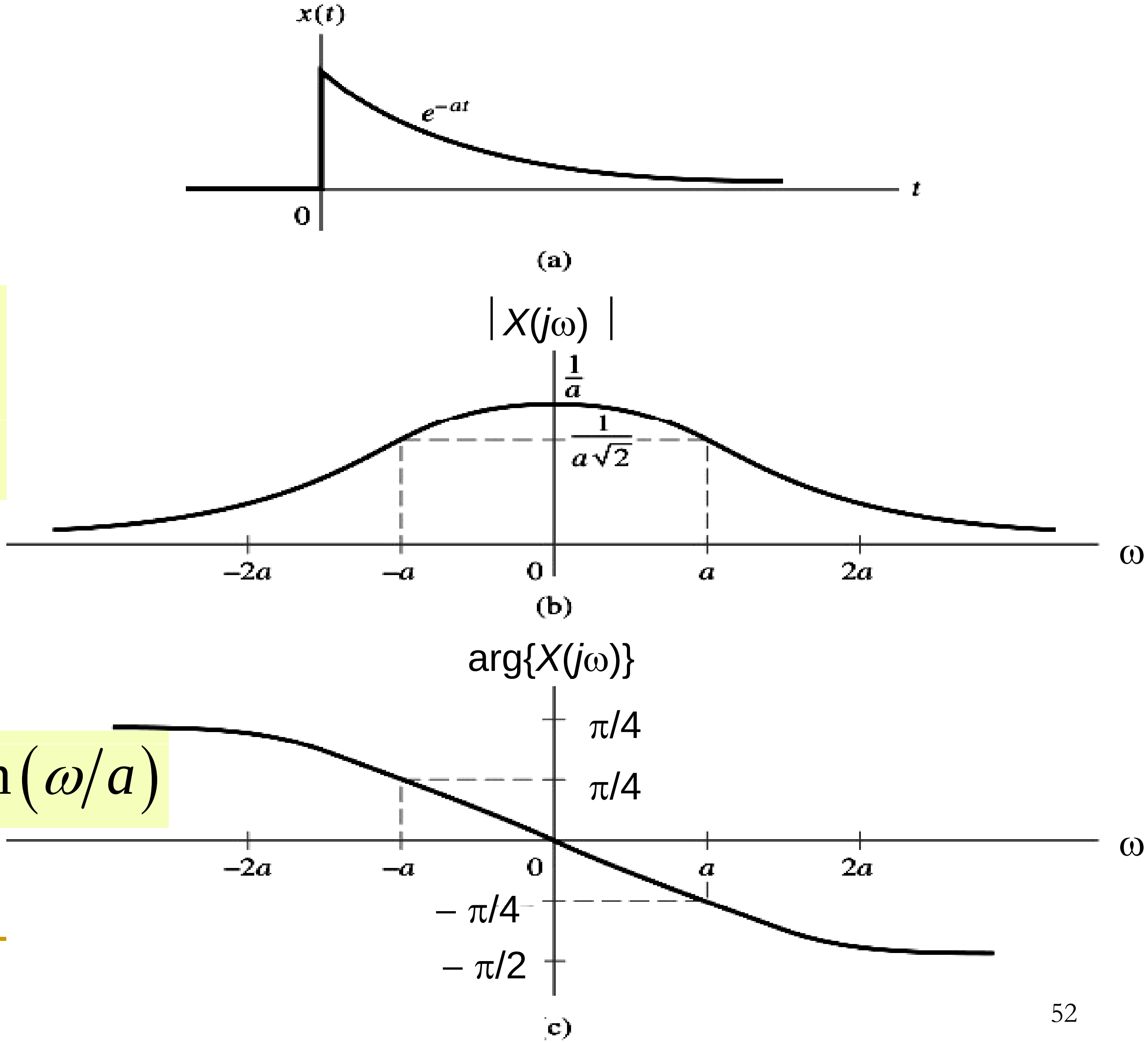
$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_0^{\infty} e^{-(a+j\omega)t}dt$$

$$= -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}$$

$$\left|X(j\omega)\right| = \frac{1}{\left(a^2+\omega^2\right)^{\frac{1}{2}}}$$

$$\arg\left\{X(j\omega)\right\} = -\arctan(\omega/a)$$

CT Nonperiodic Signals: FT of A Real Decaying Exponential



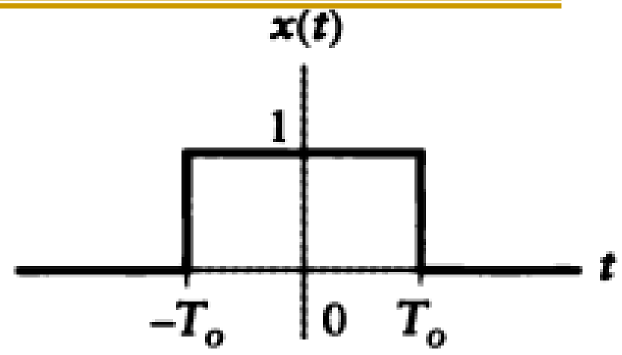
$$\left|X(j\omega)\right| = \frac{1}{\left(a^2+\omega^2\right)^{\frac{1}{2}}}$$

$$\arg\left\{X(j\omega)\right\} = -\arctan\left(\omega/a\right)$$

CT Nonperiodic Signals: FT of A Rectangular Pulse

Example 3.25

$$x(t)=\begin{cases}1, & -T_0<t<T_0\\ 0, & |t|>T_0\end{cases}$$



$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{-T_0}^{T_0}e^{-j\omega t}dt$$

$$=\begin{cases}2T_0 & \omega=0\\ -\dfrac{1}{j\omega}e^{-j\omega t}\Big|_{-T_0}^{T_0}=\dfrac{2}{\omega}\sin(\omega T_0) & \omega\neq 0\end{cases}$$

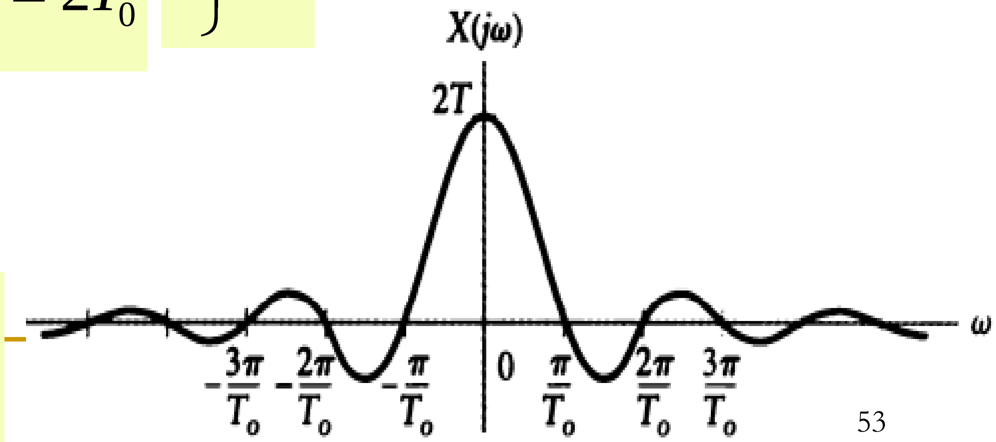
$$\lim_{\omega\to 0}\frac{2}{\omega}\sin(\omega T_0)=2T_0$$

$$\Rightarrow\quad X(j\omega)=\frac{2}{\omega}\sin(\omega T_0)$$

$$=2T_0\,\mathrm{sinc}(\omega T_0/\pi)$$

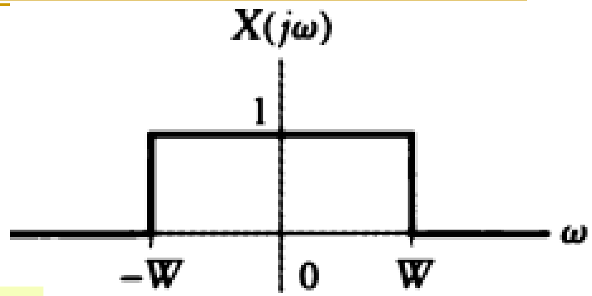
$$|X(j\omega)|=2\left|\frac{\sin(\omega T_0)}{\omega}\right|$$

$$\arg\{X(j\omega)\}=\begin{cases}0, & \sin(\omega T_0)/\omega>0\\ \pi, & \sin(\omega T_0)/\omega<0\end{cases}$$



Example 3.26

$$X(j\omega)=\begin{cases}1, & -W<\omega<W\\ 0, & |\omega|>W\end{cases}$$



$$x(t)=\frac{1}{2\pi}\int_{-W}^{W}e^{j\omega t}\,d\omega$$

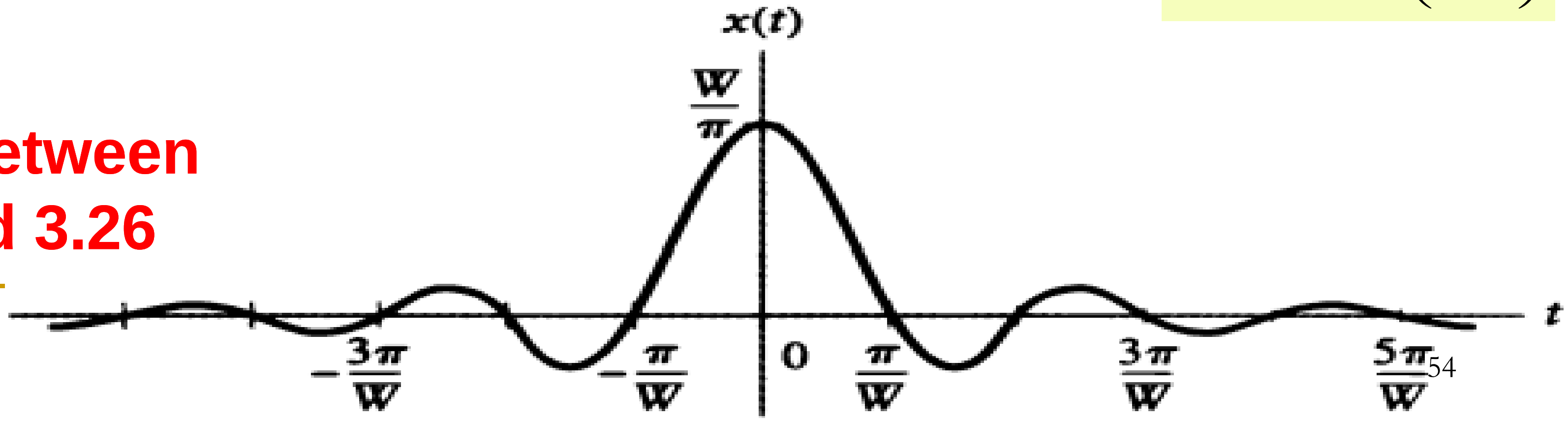
$$\lim_{t\to 0}\frac{1}{\pi t}\sin(Wt)=W/\pi$$

$$=\begin{cases}W/\pi & t=0\\ -\dfrac{1}{j\pi t}e^{j\omega t}\Big|_{-W}^{W}=\dfrac{1}{\pi t}\sin(Wt) & t\neq 0\end{cases}$$

$$\Rightarrow\quad x(t)=\frac{1}{\pi t}\sin(Wt)$$

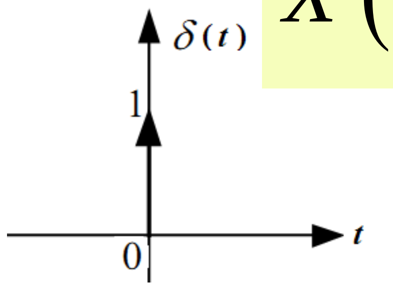
$$=\frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)$$

Duality between 3.25 and 3.26



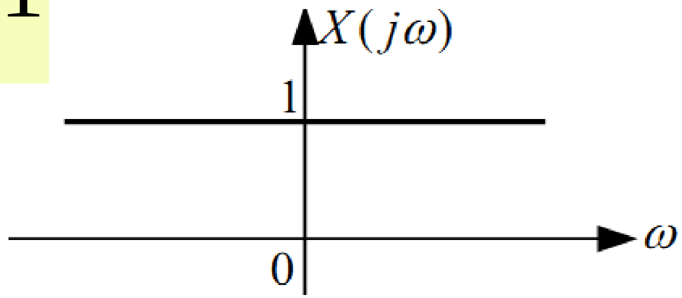
CT Nonperiodic Signals: FT

Example 3.27 **Find the FT of *The Unit Impulse* $x(t) = \delta(t)$.**



$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

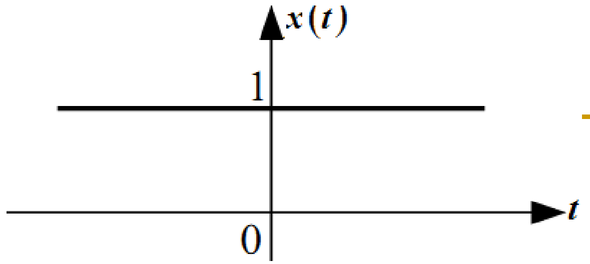
$$\delta(t) \xleftrightarrow{FT} 1$$



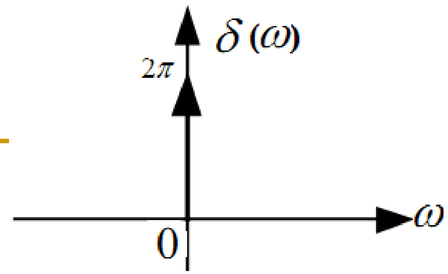
Example 3.28 **Find the inverse FT of *An Impulse Spectrum* $X(j\omega) = 2\pi\delta(\omega)$.**

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega) e^{j\omega t} d\omega = 1$$

Duality between 3.27 and 3.28



$$1 \xleftrightarrow{FT} 2\pi\delta(\omega)$$



CT Nonperiodic Signals: FT of the Signum function

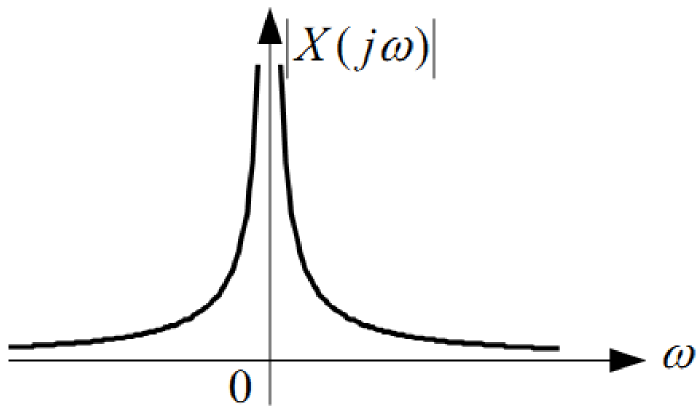
$$F[\mathrm{sgn}(t)] = \lim_{\sigma\to 0}\left\{F[\mathrm{sgn}(t)e^{-\sigma|t|}]\right\}$$

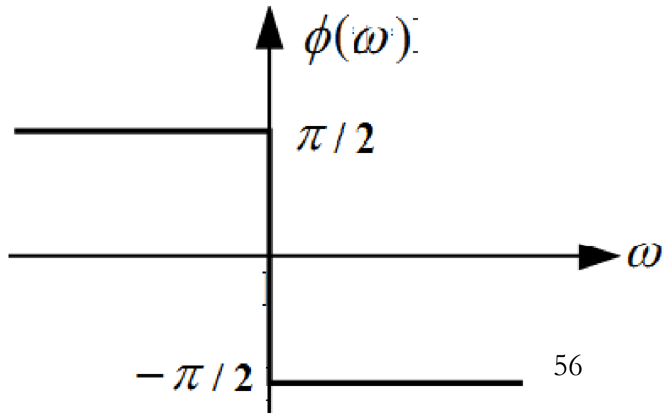
$$\mathrm{sgn}(t) = \begin{cases} -1 & t<0 \\ 0 & t=0 \\ 1 & t>0 \end{cases}$$

$$= \int_{-\infty}^{0} (-1)e^{\sigma t}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-\sigma t}e^{-j\omega t}dt$$

$$= -\left.\frac{e^{(\sigma-j\omega)t}}{\sigma-j\omega}\right|_{t=-\infty}^{0} - \left.\frac{e^{-(\sigma+j\omega)t}}{\sigma+j\omega}\right|_{t=0}^{\infty}$$

$$= \frac{-1}{\sigma-j\omega} + \frac{1}{\sigma+j\omega} = \frac{2}{j\omega}$$

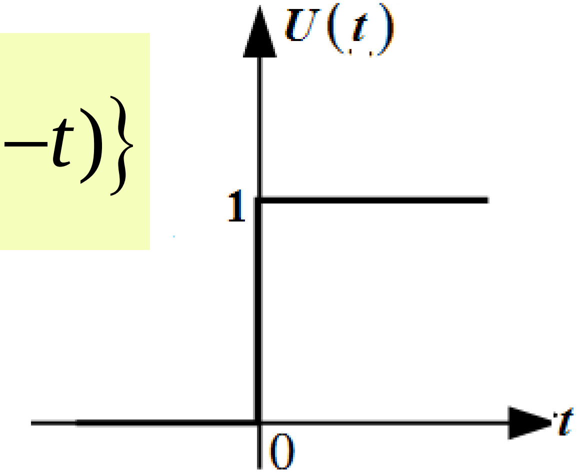




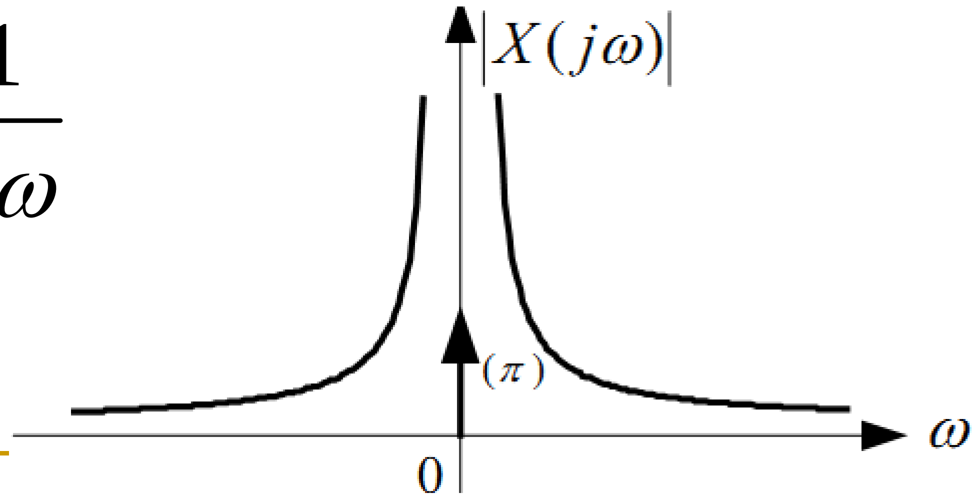
CT Nonperiodic Signals: FT of the Unit Step

$$u(t) = \frac{1}{2}\{u(t)+u(-t)\} + \frac{1}{2}\{u(t)-u(-t)\}$$

$$= \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)$$



$$F[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}$$



Properties of Fourier Representation

1. Four Fourier representations: Table 3.2.

Table 3.2 The Four Fourier Representations

Time Domain	**Periodic** (t, n)	**Non-periodic** (t, n)	
Continuous (t)	**Fourier Series** $$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_o t}$$ $$X[k] = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_o t}\, dt$$ $x(t)$ has period T $$\omega_o = \frac{2\pi}{T}$$	**Fourier Transform** $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$ $$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$	*Nonperiodic* (k, ω)
Discrete $[n]$	**Discrete-Time Fourier Series** $$x[n] = \sum_{k=0}^{N-1} X[k] e^{jk\Omega_o n}$$ $$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\Omega_o n}$$ $x[n]$ and $X[k]$ have period N $$\Omega_o = \frac{2\pi}{N}$$	**Discrete-Time Fourier Transform** $$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n}\, d\Omega$$ $$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$ $X(e^{j\Omega})$ has period 2π	*Periodic* (k, Ω)
	Discrete $[k]$	*Continuous* (ω, Ω)	*Frequency Domain*

Properties of Fourier Representation

2. Signals that are periodic in time have discrete frequency-domain representations, while nonperiodic time signals have continuous frequency- domain representations.

Table 3.3 Periodicity Properties of Fourier Representations

Time-Domain Property	Frequency-Domain Property
Continuous	*Nonperiodic*
Discrete	*Periodic*
Periodic	*Discrete*
Nonperiodic	*Continuous*

Summary and Exercises

- Summary and Exercises
  - Complex Sinusoids and Frequency Response of LTI Systems
  - Fourier Representations for Four classes of Signals
  - Properties of Fourier Representations
- Exercises (**P322-333**)
  - **3.48(a, c), 3.49(a, c), 3.50(a, b), 3.51(a, b), 3.52(a, d), 3.53(a,c), 3.54(a, d), 3.55(a, b)**