MATH100
Applied Linear Algebra

Zed Chance

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Contents

1 Vectors

Definition 1 (Vectors)**.** Vectors are directed line segments, they have both magnitude and direction. They exist in a "space," such as the plane $\mathbb{R}^2$, ordinary space $\mathbb{R}^3$, or an n-dimensional space $\mathbb{R}^n$. Jan 23

- In $\mathbb{R}^3$, the vector v can be represented by its components as $v = [v_1, v_2, v_3]$.
- v can also be represented as a line segment with an arrowhead pointing in the direction of v.

Properties Vectors can be combined to form new vectors. Whether we are combining our vectors algebraically (manipulating their components) or geometrically (manipulating their graphs), the following **properties** apply: Let u, v, and w be vectors, and c and d be real numbers, then

$u + v = v + u$	commutative
$(u + v) + w = v + (u + w)$	associative
$c(du) = (cd)u$	associative
$u + 0 = u$	additive identity
$u + (-u) = 0$	additive inverse
$c(u + v) = cu + cv$	distributive
$(c + d)u = cu + du$	distributive
$1u = u$	multiplicative identity

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Representing vectors Row vector:

$$\bar{V} = [2, 3]$$

Column vector:

$$\bar{V} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Note 1. Vectors u and v are equivalent if they have the same length and direction.

1.1 Vector properties

Let $\bar{u} = [1, 2]$ and $\bar{v} = [3, 1]$.

$$\begin{aligned} \bar{u} + \bar{v} &= [1, 2] + [3, 1] \\ &= [1 + 3, 2 + 1] \\ &= [4, 3] \end{aligned}$$

Geometrically, this is the "tip to tail" method. Any two vectors define a parallelogram.

Let $\bar{u} = [1, 2]$, and think about $\bar{u} + \bar{u}$.

$$\begin{aligned}\bar{u} + \bar{u} &= [1, 2] + [1, 2] \\ &= [2, 4] \\ 2\bar{u} &= 2[1, 2] \\ &= [2, 4]\end{aligned}$$

Also think about multiplying $\bar{u}$ by -1:

$$\begin{aligned}(-1)\bar{u} &= (-1)[1, 2] \\ &= [-1, -2]\end{aligned}$$

This points the vector in the opposite direction, which is considered "antiparallel". So if the scalar in the multiplication is a negative number, it will point the vector in the other direction (as well as being scaled).

Definition 2 (Scalar multiplication)**.** For constant c and $\bar{V} = [v_1, v_2, v_3]$, then

$$c\bar{V} = [cv_1, cv_2, cv_3]$$

Definition 3 (Vector subtraction)**.**

$$\bar{u} - \bar{v} = \bar{u} + (-\bar{v})$$

So with our existing vectors:

$$\begin{aligned}\bar{u} - \bar{v} &= \bar{u} + (-\bar{v}) \\ &= [1, 2] + [-3, -1] \\ &= [-2, 1]\end{aligned}$$

The sum and difference is the diagonals of the parallelogram created by adding the vectors.

Note 2. Vector addition is commutative, but vector subtraction is not (it is anticommutative).

$$\begin{aligned}\bar{v} - \bar{u} &= [3, 1] + [-1, -2] \\ &= [2, -1]\end{aligned}$$

Note 3. All of these properties hold true in all dimensions: $\mathbb{R}^n$.

Concerning the **additive identity**: In $\mathbb{R}^3$ the "zero vector" is $\bar{0} = [0, 0, 0]$.

How to represent the length of a vector:

$$\begin{aligned}\bar{u} &= [1, 2] \\ &= \sqrt{1^2 + 2^2} \\ &= \sqrt{5} \\ ||\bar{u}|| &= \sqrt{5}\end{aligned}$$

We use the double bars to represent the length of a vector.

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1.2 Linear combinations and coordinates

Definition 4. $\bar{v}$ is a linear combination of a set of vectors, $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$, if $\bar{v} = c_1\bar{v}_1, \ldots, c_k, \bar{v}_k$ for scalars c_i.

Example 1. See *Handout 1.*

Definition 5 (Standard Basis Vectors and Standard Coordinates)**.** In $\mathbb{R}^2$: $\bar{e}_1 = [1, 0]$, $\bar{e}_2 = [0, 1]$, these are the standard basis vectors.

Then $\bar{v} = [v_1, v_2]$,

and the standard coordinates of $\bar{v}$ are v_1, v_2.

1.3 Dot Product

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Definition 6 (Dot Product)**.** If $\bar{u} = [u_1, u_2, \ldots, u_n]$, $\bar{v} = [v_1, v_2, \ldots, v_n]$, then the **dot product** of $\bar{u}$ with $\bar{v}$ is

$$\bar{u} \cdot \bar{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Example 2.

$$\begin{aligned}[2, -1, 7] \cdot [3, 5, -2] &= (2)(3) + (-1)(5) + (7)(-2) \\ &= 6 - 5 - 14 \\ &= -13\end{aligned}$$

Properties of dot products (scalar products) Let $\bar{u}, \bar{v}, \bar{w}$ be vectors, and c be a scalar, then

$\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$	commutative
$\bar{u} \cdot (\bar{v} + \bar{w}) = (\bar{u} \cdot \bar{v}) + (\bar{u} \cdot \bar{w})$	distributive
$(c\bar{u}) \cdot \bar{v} = c(\bar{u} \cdot \bar{v})$	
$\bar{0} \cdot \bar{v} = 0$	
$\bar{v} \cdot \bar{v} = v_1^2 + v_2^2 + \cdots + v_n^2$	

Length In $\mathbb{R}^2$: $||\bar{v}|| = \sqrt{v_1^2 + v_2^2}$

In general: $||\bar{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

Example 3. If $\bar{v} = [2, -1, 7]$, then the length is

$$\begin{aligned} ||\bar{v}|| &= \sqrt{2^2 + (-1)^2 + 7^2} \\ &= \sqrt{4 + 1 + 49} \\ &= 3\sqrt{6} \end{aligned}$$

Note 4.

$$||\bar{v}|| = \sqrt{\bar{v} \cdot \bar{v}}$$

Definition 7. A vector of length 1 is called **unit vector**. For any vector $\bar{v} \neq \bar{0}$: $\frac{\bar{v}}{||\bar{v}||}$ is a unit vector in the same direction as $\bar{v}$.

Note 5.

$$\bar{v}\left(\frac{||\bar{v}||}{||\bar{v}||}\right) = ||\bar{v}||\left(\frac{\bar{v}}{||\bar{v}||}\right)$$

Example 4. In $\mathbb{R}^2 : \bar{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, these are unit vectors.

Important inequalities

- Triangle inequality

  The triangle created by the parallelogram of a vector addition, the length of any one side cannot be greater than the sum of the other two sides.

$$||\bar{u} + \bar{v}|| \leq ||\bar{u}|| + ||\bar{v}||$$

- Cauchy-Schwarz inequality

$$|\bar{u} \cdot \bar{v}| \leq ||\bar{u}||\,||\bar{v}||$$

  Proof by the Law of Cosines:

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos\theta \\
||\bar{u}-\bar{v}||^2 &= ||\bar{u}||^2 + ||\bar{v}||^2 - 2||\bar{u}||\,||\bar{v}||\cos\theta \\
(\bar{u}-\bar{v})\cdot(\bar{u}-\bar{v}) &= \\
||\bar{u}||^2 - 2(\bar{u}\cdot\bar{v}) + ||\bar{v}||^2 &= \\
||\bar{u}||^2 - 2(\bar{u}\cdot\bar{v}) + ||\bar{v}||^2 &= ||\bar{u}||^2 + ||\bar{v}||^2 - 2||\bar{u}||\,||\bar{v}||\cos\theta \\
\bar{u}\cdot\bar{v} &= ||\bar{u}||\,||\bar{v}||\cos\theta \\
|\bar{u}\cdot\bar{v}| &= ||\bar{u}||\,||\bar{v}|||\cos\theta| \\
|\bar{u}\cdot\bar{v}| &\le ||\bar{u}||\,||\bar{v}||.
\end{aligned}
$$

Angle θ between vectors $\bar{u}$ and $\bar{v}$ Excluding the zero vector:

Let $0 \le \theta \le \pi$,

$$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{||\bar{u}||\,||\bar{v}||}$$

So, $\theta = \cos^{-1}\left(\frac{\bar{u}\cdot\bar{v}}{||\bar{u}||\,||\bar{v}||}\right)$

> **Note 6.** If $\bar{u}, \bar{v} \neq 0$, then $\theta = \frac{\pi}{2}$, if and only if $\bar{u}\cdot\bar{v} = 0$.
>
> $$\bar{u} \perp \bar{v}, \text{ iff } \bar{u}\cdot\bar{v} = 0$$

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1.4 Distance between vectors

Definition 8. The distance between two vectors is the distance between their tips.

If $\bar{u} = [u_1, u_2]$ and $\bar{v} = [v_1, v_2]$, then

$$
\begin{aligned}
d(\bar{u},\bar{v}) &= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2} \\
&= ||\bar{u} - \bar{v}|| \\
&= d(\bar{v},\bar{u}) \\
&= ||\bar{v} - \bar{u}||
\end{aligned}
$$

Example 5. In $\mathbb{R}^3$:

For $\bar{u} = [2, -1, 7]$ and $\bar{v} = [3, 5, -2]$:

Find the distance:

$$
\begin{aligned}
d(\bar{u}, \bar{v}) &= ||\bar{u} - \bar{v}|| \\
&= ||[(2-3), (-1-5), (7+2)]|| \\
&= ||[-1, -6, 9]|| \\
&= \sqrt{(-1)^2 + (-6)^2 + 9^2} \\
&= \sqrt{1 + 36 + 81} \\
&= \sqrt{118}
\end{aligned}
$$

1.5 Projections

Definition 9. Let $proj_{\bar{u}}\bar{v}$ be the vector projection of $\bar{v}$ onto $\bar{u}$, then the signed length of $proj_{\bar{u}}\bar{v}$ is given by

$$
\begin{aligned}
||\bar{v}|| \cos\theta &= ||\bar{v}|| \frac{\bar{u} \cdot \bar{v}}{||\bar{v}||||\bar{u}||} \\
&= \frac{\bar{v} \cdot \bar{u}}{||\bar{u}||}
\end{aligned}
$$

So,

$$
\begin{aligned}
proj_{\bar{u}}\bar{v} &= \left(\frac{\bar{v} \cdot \bar{u}}{||\bar{u}||}\right) \frac{\bar{u}}{||\bar{u}||} \\
&= \left(\frac{\bar{v} \cdot \bar{u}}{\bar{u} \cdot \bar{u}}\right) \bar{u}
\end{aligned}
$$

Note 7. Recall, $\bar{u} \cdot \bar{u} = ||\bar{u}||^2$

Note 8. Remember, $\frac{\bar{u}}{||\bar{u}||}$ is the unit vector.

$$
\frac{\bar{v} \cdot \bar{u}}{||\bar{u}||} = \bar{v} \frac{\bar{u}}{||\bar{u}||}
$$

Example 6. For $\bar{u} = [2, 1, -2]$ and $\bar{v} = [3, 0, 8]$, find the projection of $\bar{v}$ onto $\bar{u}$:

$$
\begin{aligned}
proj_{\bar{u}}\bar{v} &= \frac{[3,0,8]\cdot[2,1,-2]}{[2,1,-2]\cdot[2,1,-2]}[2,1,-2] \\
&= \frac{6+0-16}{4+1+4}[2,1,-2] \\
&= \frac{-10}{9}[2,1,-2]
\end{aligned}
$$

Since the coefficient is negative, the angle between the two vectors is more than 90 degrees.

1.6 Lines and planes

See *Handout 2*

Comparing vector and parametric forms:

$$
\begin{aligned}
\bar{x} &= \bar{p} + t\bar{d} \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + t\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\
x &= p_1 + td_1 \\
y &= p_2 + td_2
\end{aligned}
$$

The solution to this is the line l.

Comparing the normal and general forms:

Let $\bar{n} = \begin{bmatrix} a \\ b \end{bmatrix}$:

$$
\begin{aligned}
\bar{n}\cdot\bar{x} &= \bar{n}\cdot\bar{p} \\
\begin{bmatrix} a \\ b \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} a \\ b \end{bmatrix}\cdot\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \\
ax + by &= ap_1 + bp_2 \\
ax + by &= c
\end{aligned}
$$

Remember, $ap_1 + bp_2$ are constants, so we can call them c.

Example 7. See *Handout 1.*

Find an equation for that line that passes through the point (-3, 2) and is parallel to the vector [2, 1].

1. Vector form

$$\bar{x} = \bar{p} + t\bar{d}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2. Parametric form

$$\begin{aligned} x &= -3 + 2t \\ y &= 2 + t \end{aligned}$$

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Example 8. *Cont from previous example*

3. General form

$$\begin{aligned} t = \frac{x+3}{2} &= \frac{y-2}{1} \\ x + 3 &= 2y - 4 \\ x - 2y &= -7 \end{aligned}$$

4. Normal form

$$\bar{n} \cdot \bar{x} = \bar{n} \cdot \bar{p}$$
$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Making sense of the normal form *See handout 2's graph*

1. Note that $\bar{x} - \bar{p}$ is parallel to the line l.
2. Also, $\bar{n} \perp (\bar{x} - \bar{p})$ by definition of $\bar{n}$.
3. Then, $\bar{n} \cdot (\bar{x} - \bar{p}) = 0$ by a property of dot products.

   We can use the distributive property:

$$\begin{aligned} \bar{n} \cdot \bar{x} - \bar{n} \cdot \bar{p} &= 0 \\ \bar{n} \cdot \bar{x} &= \bar{n} \cdot \bar{p} \end{aligned}$$

1.7 Lines in $\mathbb{R}^3$

See handout 2

Vector form

$$\bar{x} = \bar{p} + t\bar{d}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Parametric form

$$x = p_1 + td_1$$
$$y = p_2 + td_2$$
$$z = p_3 + td_3$$

Example 9. *See handout 2*

Find vector and parametric forms for the equation for the line containing the points (2, 4, -3) and (3, -1, 1).

$$\bar{p_1} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$$
$$\bar{p_2} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$
$$\bar{d} = \bar{p_2} - \bar{p_1}$$
$$= \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$$
$$\bar{d} = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

Pick one of the points for our point vector $\bar{p}$.

1. Vector form

$$\bar{x} = \bar{p} + t\bar{d}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

2. Parametric form

$$\begin{aligned} x &= 2 + t \\ y &= 4 - 5t \\ z &= -3 + 4t \end{aligned}$$

1.8 Planes in $\mathbb{R}^3$

See handout 2

Normal form Let $\bar{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\bar{n} \cdot \bar{x} = \bar{n} \cdot \bar{p}$$
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

General form

$$\begin{aligned} ax + by + cz &= ap_1 + bp_2 + cp_3 \\ ax + by + cz &= d \end{aligned}$$

We can combine the constants on the right into one single constant, d.

Example 10. *See handout 2*

Find normal and general forms for the equation of the plane orthogonal to the vector [2,3,4] that passes through the point (2,4,-1).

Let $\bar{n} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, and $\bar{p} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$. We can start off by putting this in the normal form.

1. Normal form

$$\bar{n} \cdot \bar{x} = \bar{n} \cdot \bar{p}$$
$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$$

2. General form

$$\begin{aligned} 2x + 3y + 4z &= (2)(2) + (3)(4) + (4)(-1) \\ 2x + 3y + 4z &= 12 \end{aligned}$$

Example 11. *See handout 2*

Find a vector form for the plane in the previous example.

Let $\bar{u}, \bar{v}$ be in the plane. Then, $\bar{u} \perp \bar{n}$, $\bar{v} \perp \bar{n}$, so

$$\bar{u} \cdot \bar{n} = 0 \qquad \bar{v} \cdot \bar{n} = 0$$

And, $\bar{u}$ is not parallel to $\bar{v}$.

Let $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}$, where $u_3 = 0$. Then,

$$\begin{aligned} \bar{n} \cdot \bar{u} &= 0 \\ \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} &= 0 \\ 2u_1 + 3u_2 &= 0 \end{aligned}$$

Let $\bar{u} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$.

Let $\bar{v} = \begin{bmatrix} v_1 \\ 0 \\ v_3 \end{bmatrix}$, where $v_2 = 0$.

Then,

$$\begin{aligned} \bar{n} \cdot \bar{v} &= 0 \\ \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ 0 \\ v_3 \end{bmatrix} &= 0 \\ 2v_1 + 4v_3 &= 0 \end{aligned}$$

Let $\bar{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

So the vector form is

$$\begin{aligned} \bar{x} &= \bar{p} + s\bar{u} + t\bar{v} \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + s \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \end{aligned}$$

And our parametric form is

$$
\begin{aligned}
x &= 2 + 3s + 2t \\
y &= 4 - 2s \\
z &= -1 - t
\end{aligned}
$$

2 Systems of Linear Equations

Definition 10. A **linear equation** in the n variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form: Feb 08

$$a_1x_2 + a_2x_2 + \cdots + a_nx_n = b$$

where the coefficients $a_1, \ldots, a_n$ and the constant term b is constant.

Definition 11. A finite set of linear equations is a **system of linear equations**. A **solution set** of a system of linear equations is the set of *all* solutions of the system. A system of linear equations is either "consistent" if it has a solution, or it is "inconsistent" if there is no such solution.

Theorem 1. A system of linear equations has **either**

1. A unique solution – consistent
2. Infinitely many solutions – consistent
3. No solution – inconsistent

Definition 12. Two linear systems are said to be **equivalent** if they have the same solution set.

Example 12. *See handout 3, problem 1*

$$
\begin{aligned}
2x + y &= 8 \\
x - 3y &= -3 \\
y &= 2, x = 3
\end{aligned}
$$

Example 13. *See handout 3, problem 2*

Example 14. *See handout 3, problem 1*

Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$$

be the matrix of coefficients, and let

$$\bar{b} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

Be the vector of constants.

Then,

$$[A \mid \bar{b}] = \begin{bmatrix} 2 & 1 \mid 8 \\ 1 & -3 \mid -3 \end{bmatrix}$$

is called the augmented matrix.

2.1 Direct methods of solving systems

Example 15. For the system Feb 10

$$\begin{aligned} 2x - y &= 3 \\ x + 3y &= 5 \end{aligned}$$

The coefficient matrix A is

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

The constant vector $\bar{b}$ is

$$\bar{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

The augmented matrix is

$$[A \mid \bar{b}] = \begin{bmatrix} 2 & -1 & \mid & 3 \\ 1 & 3 & \mid & 5 \end{bmatrix}$$

Definition 13 (Row echelon form of a matrix)**.** *See handout 4*

2.2 Gaussian and Gauss-Jordan Elimination

To solve a system of linear equations using Guassian Elimination: Feb 12

- Write out an augmented matrix for the system of linear equations
- Use elementary row operations to reduce the matrix to row echelon form
- Write out a system of equations corresponding to the row echelon matrix
- Use back substitution to find solution(s), if any exist, to the new system of equations

To solve the system of linear equations using Gauss-Jordan Elimination: reduce the augmented matrix to reduced row echelon form

- Write out the system of equations corresponding to the reduced row echelon matrix
- Use back substitution to find solution(s), if any exist, to the new system of equations

Example 16. *See handout 5*

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Example 17. *See handout 5*

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