Lecture 1
Asymptotic Analysis

Chapter 3, Appendix A

Analysis of Algorithms

- An *algorithm* is a finite set of precise instructions for performing a computation or for solving a problem.

- What is the goal of analysis of algorithms?
  - To compare algorithms mainly in terms of running time but also in terms of other factors (e.g., memory requirements, programmer's effort etc.)

- What do we mean by running time analysis?
  - Determine how running time increases as the size of the problem increases.

Input Size

- Input size (number of elements in the input)
  - size of an array
  - polynomial degree
  - # of elements in a matrix
  - # of bits in the binary representation of the input
  - vertices and edges in a graph

Polynomial-Time

- **Brute force**. For many non-trivial problems, there is a natural brute force search algorithm that checks every possible solution.
  - Typically takes 2^N time or worse for inputs of size N.
  - Unacceptable in practice.

> There exists constants $c > 0$ and $d > 0$ such that on every input of size N, its running time is bounded by $c\,N^d$ steps.

- **Desirable scaling property**. When the input size doubles, the algorithm should only slow down by some constant factor C. An algorithm is poly-time if this property holds.

Types of Analysis

- Worst case
  - Provides an upper bound on running time
  - An absolute guarantee that the algorithm would not run longer, no matter what the inputs are

- Best case
  - Provides a lower bound on running time
  - Input is the one for which the algorithm runs the fastest

$$\boxed{Lower\ Bound \le Running\ Time \le Upper\ Bound}$$

- Average case
  - Provides a prediction about the running time
  - Assumes that the input is random

How do we compare algorithms?

- We need to define a number of <u>objective measures</u>.

  (1) Compare execution times?

  Not good: times are specific to a particular computer !!

  (2) Count the number of statements executed?

  Not good: number of statements vary with the programming language as well as the style of the individual programmer.

Ideal Solution

- Express running time as a function of the input size *n* (i.e., $f(n)$).

- Compare different functions corresponding to running times.

- Such an analysis is independent of machine time, programming style, etc.

Example

- Associate a "cost" with each statement.
- Find the "total cost" by finding the total number of times each statement is executed.

Algorithm 1

	Cost
arr[0] = 0;	c_1
arr[1] = 0;	c_1
arr[2] = 0;	c_1
...	...
arr[N-1] = 0;	c_1

$c_1+c_1+...+c_1 = c_1 \times N$

Algorithm 2

	Cost
for(i=0; i<N; i++)	c_2
arr[i] = 0;	c_1

$(N+1) \times c_2 + N \times c_1 =$

$(c_2 + c_1) \times N + c_2$

Another Example

- ***Algorithm 3*** *Cost*

sum = 0;	c_1
for(i=0; i<N; i++)	c_2
for(j=0; j<N; j++)	c_2
sum += arr[i][j];	c_3

$$c_1 + c_2 \text{ x } (N+1) + c_2 \text{ x } N \text{ x } (N+1) + c_3 \text{ x } N^2$$

Asymptotic Analysis

- To compare two algorithms with running times $f(n)$ and $g(n)$, we need a **rough measure** that characterizes **how fast each function grows.**

- <u>*Hint:*</u> use *rate of growth*

- Compare functions in the limit, that is, **asymptotically!**
  - (i.e., for large values of n)

Rate of Growth

- Consider the example of buying *elephants* and *goldfish:*

  Cost: cost_of_elephants + cost_of_goldfish

  Cost ~ cost_of_elephants (approximation)

- The low order terms in a function are relatively insignificant for **large** n

$$n^4 + 100n^2 + 10n + 50 \quad \sim \quad n^4$$

i.e., we say that $n^4 + 100n^2 + 10n + 50$ and n^4 have the same **rate of growth**

Asymptotic Notation

- O notation: asymptotic "less than":
  - f(n)=O(g(n)) implies: f(n) "≤" g(n)
- Ω notation: asymptotic "greater than":
  - f(n)= Ω (g(n)) implies: f(n) "≥" g(n)
- Θ notation: asymptotic "equality":
  - f(n)= Θ (g(n)) implies: f(n) "=" g(n)

Big-O Notation

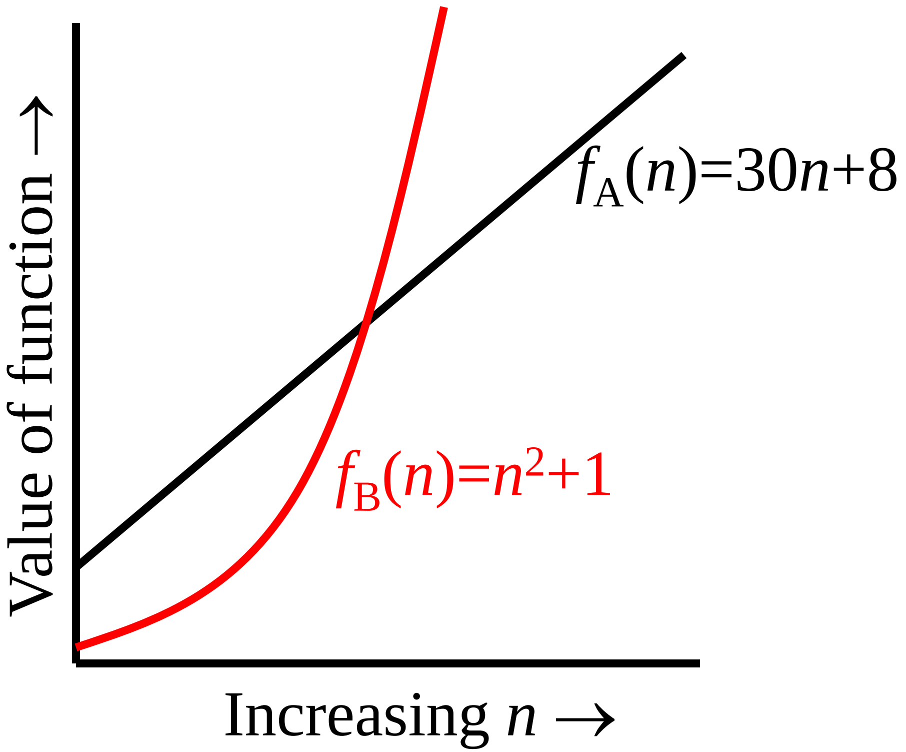
- We say $f_A(n)=30n+8$ is *order n*, or $O(n)$
  It is, at most, roughly *proportional* to n.

- $f_B(n)=n^2+1$ is *order* n^2, or $O(n^2)$. It is, at most, roughly proportional to n^2.

- In general, any $O(n^2)$ function is faster-growing than any $O(n)$ function.

Visualizing Orders of Growth

- On a graph, as you go to the right, a faster growing function eventually becomes larger...



More Examples …

- $n^4 + 100n^2 + 10n + 50$ is $O(n^4)$
- $10n^3 + 2n^2$ is $O(n^3)$
- $n^3 - n^2$ is $O(n^3)$
- constants
  - 10 is $O(1)$
  - 1273 is $O(1)$

Back to Our Example

Algorithm 1

	Cost
arr[0] = 0;	c_1
arr[1] = 0;	c_1
arr[2] = 0;	c_1
...	
arr[N-1] = 0;	c_1

$c_1+c_1+...+c_1 = c_1 \times N$

Algorithm 2

	Cost
for(i=0; i<N; i++)	c_2
arr[i] = 0;	c_1

$(N+1) \times c_2 + N \times c_1 =$

$(c_2 + c_1) \times N + c_2$

- Both algorithms are of the same order: *O(N)*

Example (cont'd)

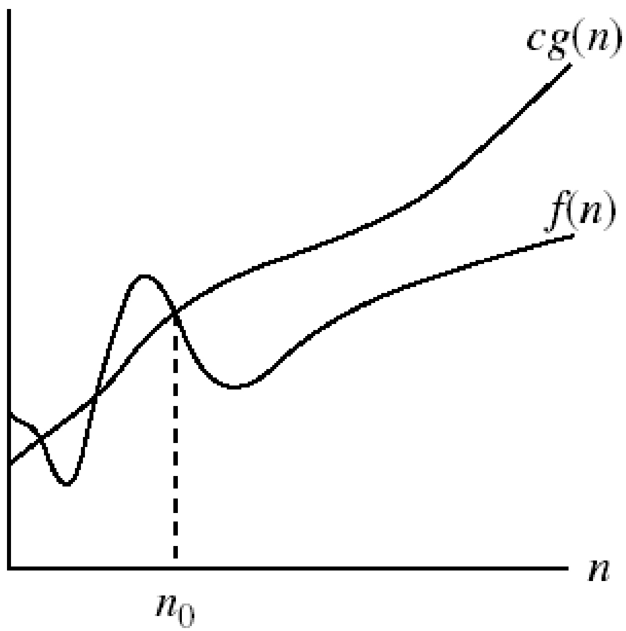
Algorithm 3	***Cost***
sum = 0;	c_1
for(i=0; i<N; i++)	c_2
for(j=0; j<N; j++)	c_2
sum += arr[i][j];	c_3

$$c_1 + c_2 \; x \, (N+1) + c_2 \; x \, N \; x \, (N+1) + c_3 \; x \, N^2 = O(N^2)$$

Asymptotic notations

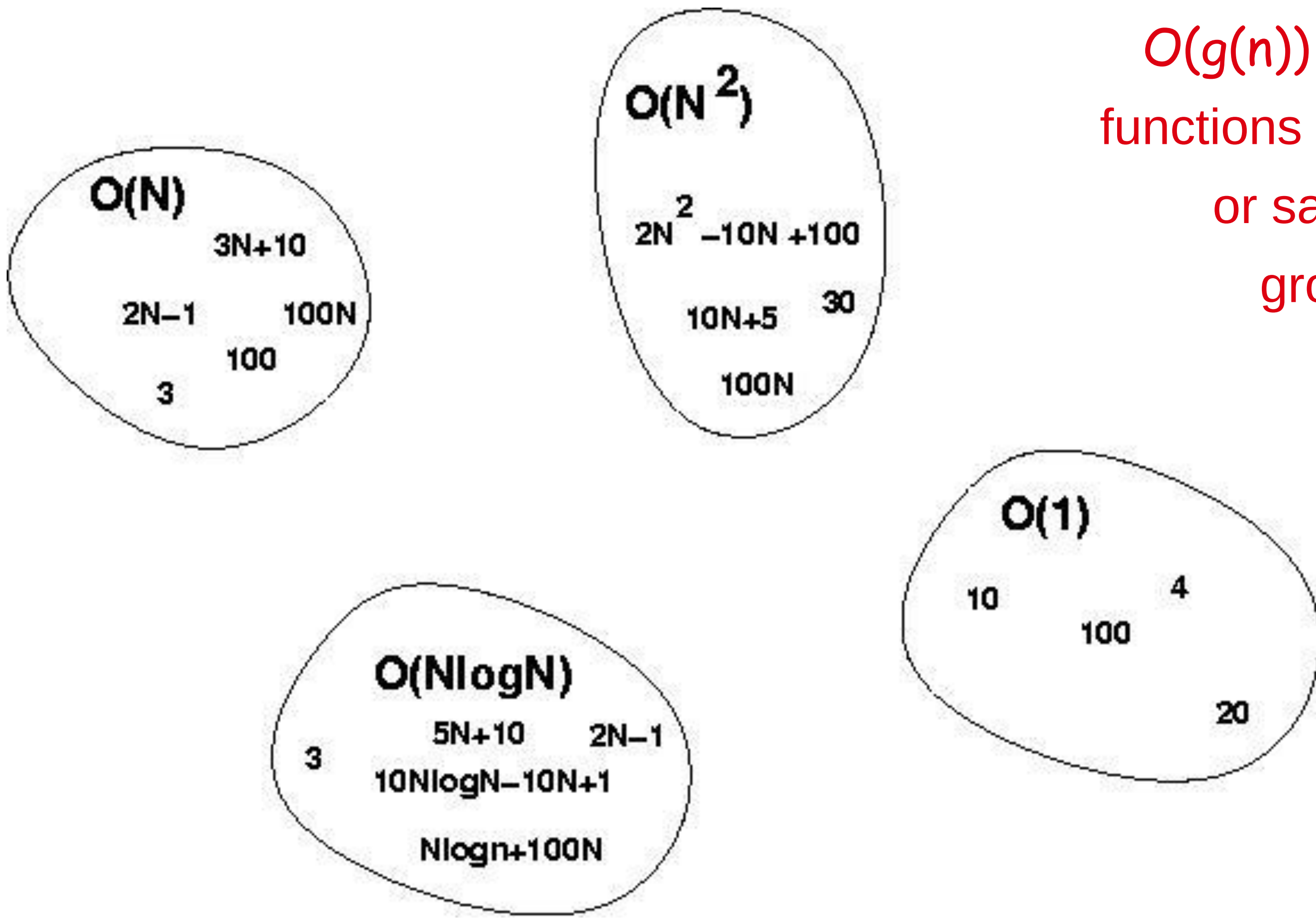
- *O-notation*

$$O(g(n)) = \{f(n) : \text{ there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\} .$$



$g(n)$ is an ***asymptotic upper bound*** for $f(n)$.

Big-O Visualization



$O(g(n))$ is the set of functions with smaller or same order of growth as $g(n)$

Examples

- $2n^2 = O(n^3)$: $\quad 2n^2 \le cn^3 \Rightarrow 2 \le cn \Rightarrow c = 1$ and $n_0 = 2$

- $n^2 = O(n^2)$: $\quad n^2 \le cn^2 \Rightarrow c \ge 1 \Rightarrow c = 1$ and $n_0 = 1$

- $1000n^2+1000n = O(n^2)$:

$1000n^2+1000n \le 1000n^2+ n^2 =1001n^2 \Rightarrow c=1001$ and $n_0 = 1000$

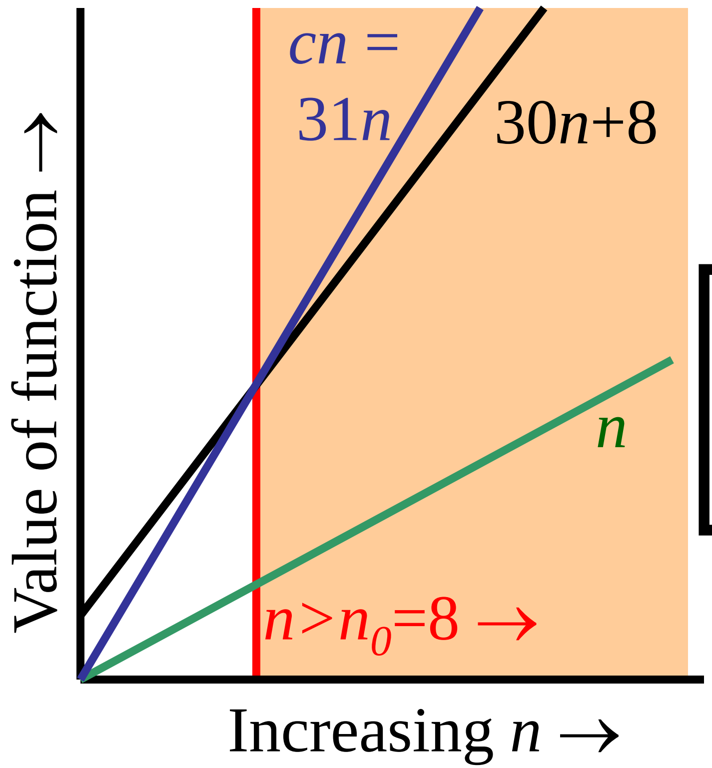
- $n = O(n^2)$: $\quad n \le cn^2 \Rightarrow cn \ge 1 \Rightarrow c = 1$ and $n_0 = 1$

More Examples

- Show that $30n+8$ is $O(n)$.
  - Show $\exists c, n_0$: $30n+8 \le cn$, $\forall n > n_0$.
    - Let $c=31$, $n_0=8$. Assume $n > n_0 = 8$. Then $cn = 31n = 30n + n > 30n+8$, so $30n+8 < cn$.

Big-O example, graphically

- Note $30n+8$ isn't less than n *anywhere* ($n>0$).
- It isn't even less than $31n$ *everywhere*.
- But it *is* less than $31n$ <u>everywhere to the right of $n=8$</u>.



$30n+8 \in O(n)$

No Uniqueness

- There is no unique set of values for n_0 and c in proving the asymptotic bounds

- Prove that $100n + 5 = O(n^2)$
  - $100n + 5 \le 100n + n = 101n \le 101n^2$

    for all $n \ge 5$

    $n_0 = 5$ and $c = 101$ is a solution
  - $100n + 5 \le 100n + 5n = 105n \le 105n^2$

    for all $n \ge 1$

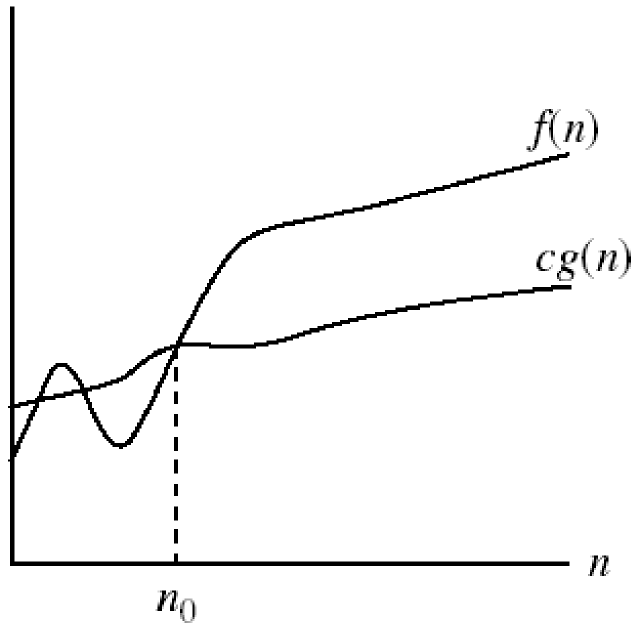
    $n_0 = 1$ and $c = 105$ is also a solution

Must find **SOME** constants c and n_0 that satisfy the asymptotic notation relation

Asymptotic notations (cont.)

- Ω - *notation*

$$\Omega(g(n)) = \{f(n) : \text{ there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\} .$$



$\Omega(g(n))$ is the set of functions with larger or same order of growth as $g(n)$

$g(n)$ is an ***asymptotic lower bound*** for $f(n)$.

Examples

- $5n^2 = \Omega(n)$

  $\exists$ c, n_0 such that: $0 \le cn \le 5n^2$

  $\Rightarrow c \le 5n \Rightarrow c = 1$ and $n_0 = 1$

- $100n + 5 \ne \Omega(n^2)$

  $\exists$ c, n_0 such that: $0 \le cn^2 \le 100n + 5$

  $100n + 5 \le 100n + 5n \ (\forall\ n \ge 1) = 105n$

  $cn^2 \le 105n \Rightarrow n(cn - 105) \le 0$

  Since n is positive $\Rightarrow cn - 105 \le 0 \Rightarrow n \le 105/c$

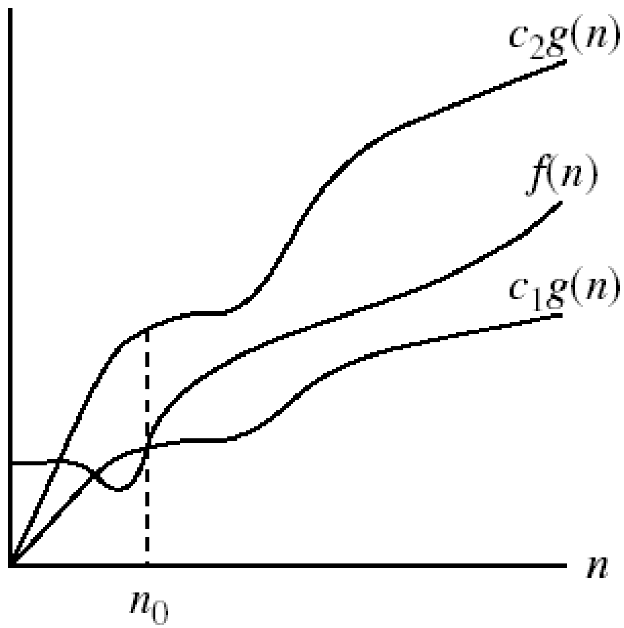
  $\Rightarrow$ contradiction: n cannot be smaller than a constant

- $n = \Omega(2n)$, $n^3 = \Omega(n^2)$, $n = \Omega(\log n)$

Asymptotic notations (cont.)

- Θ*-notation*

$$\Theta(g(n)) = \{f(n) : \text{there exist positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0\} .$$



$\Theta(g(n))$ is the set of functions with the same order of growth as $g(n)$

$g(n)$ is an ***asymptotically tight bound*** for $f(n)$.

Examples

- $n^2/2 - n/2 = \Theta(n^2)$
  - $\frac{1}{2} n^2 - \frac{1}{2} n \leq \frac{1}{2} n^2 \ \forall n \geq 0 \quad \Rightarrow \quad c_2 = \frac{1}{2}$
  - $\frac{1}{2} n^2 - \frac{1}{2} n \geq \frac{1}{2} n^2 - \frac{1}{2} n * \frac{1}{2} n \ (\forall n \geq 2) = \frac{1}{4} n^2$

    $\Rightarrow \quad c_1 = \frac{1}{4}$

- $n \neq \Theta(n^2)$: $c_1 n^2 \leq n \leq c_2 n^2$

  $\Rightarrow$ only holds for: $n \leq 1/c_1$

Examples

- $6n^3 \neq \Theta(n^2)$: $c_1 n^2 \leq 6n^3 \leq c_2 n^2$

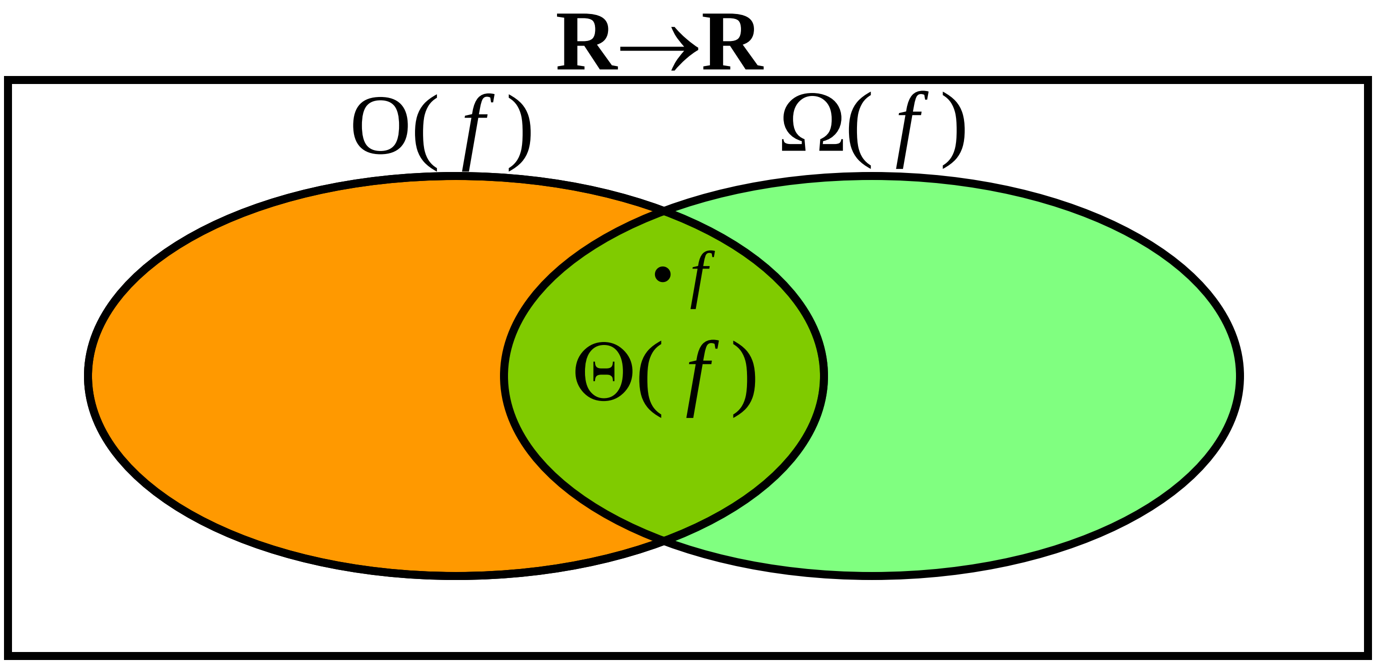
  $\Rightarrow$ only holds for: $n \leq c_2 / 6$

- $n \neq \Theta(\log n)$: $c_1 \log n \leq n \leq c_2 \log n$

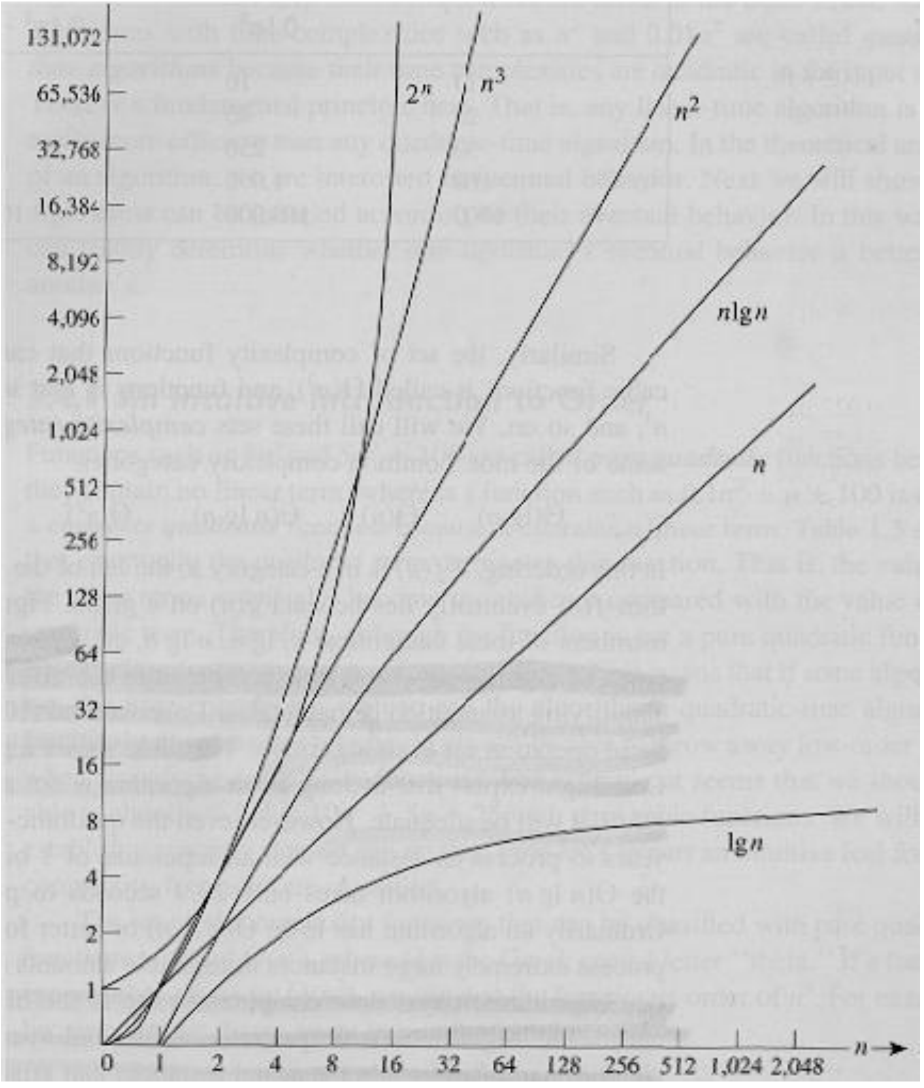
  $\Rightarrow c_2 \geq n/\log n, \forall\ n \geq n_0$ - impossible

Relations Between Different Sets

- Subset relations between order-of-growth sets.



Common orders of magnitude



Common orders of magnitude

n $f(n)$	$\lg n$	n	$n \lg n$	n^2	2^n	$n!$
10	0.003 μs	0.01 μs	0.033 μs	0.1 μs	1 μs	3.63 ms
20	0.004 μs	0.02 μs	0.086 μs	0.4 μs	1 ms	77.1 years
30	0.005 μs	0.03 μs	0.147 μs	0.9 μs	1 sec	8.4×10^{15} yrs
40	0.005 μs	0.04 μs	0.213 μs	1.6 μs	18.3 min	
50	0.006 μs	0.05 μs	0.282 μs	2.5 μs	13 days	
100	0.007 μs	0.1 μs	0.644 μs	10 μs	4×10^{13} yrs	
1,000	0.010 μs	1.00 μs	9.966 μs	1 ms		
10,000	0.013 μs	10 μs	130 μs	100 ms		
100,000	0.017 μs	0.10 ms	1.67 ms	10 sec		
1,000,000	0.020 μs	1 ms	19.93 ms	16.7 min		
10,000,000	0.023 μs	0.01 sec	0.23 sec	1.16 days		
100,000,000	0.027 μs	0.10 sec	2.66 sec	115.7 days		
1,000,000,000	0.030 μs	1 sec	29.90 sec	31.7 years		

Logarithms and Properties

- In algorithm analysis we often use the notation "log n" without specifying the base

Binary logarithm $\lg n = \log_2 n$

Natural logarithm $\ln n = \log_e n$

$$\lg^k n = (\lg n)^k$$

$$\lg \lg n = \lg(\lg n)$$

$$\log x^y = y \log x$$

$$\log xy = \log x + \log y$$

$$\log \frac{x}{y} = \log x - \log y$$

$$a^{\log_b x} = x^{\log_b a}$$

$$\log_b x = \frac{\log_a x}{\log_a b}$$

More Examples

- For each of the following pairs of functions, either f(n) is O(g(n)), f(n) is Ω(g(n)), or f(n) = Θ(g(n)). Determine which relationship is correct.
  - $f(n) = \log n^2$; $g(n) = \log n + 5$ — $f(n) = \Theta(g(n))$
  - $f(n) = n$; $g(n) = \log n^2$ — $f(n) = \Omega(g(n))$
  - $f(n) = \log \log n$; $g(n) = \log n$ — $f(n) = O(g(n))$
  - $f(n) = n$; $g(n) = \log^2 n$ — $f(n) = \Omega(g(n))$
  - $f(n) = n \log n + n$; $g(n) = \log n$ — $f(n) = \Omega(g(n))$
  - $f(n) = 10$; $g(n) = \log 10$ — $f(n) = \Theta(g(n))$
  - $f(n) = 2^n$; $g(n) = 10n^2$ — $f(n) = \Omega(g(n))$
  - $f(n) = 2^n$; $g(n) = 3^n$ — $f(n) = O(g(n))$

Properties

- *Theorem:*

$$f(n) = \Theta(g(n)) \Leftrightarrow f = O(g(n)) \text{ and } f = \Omega(g(n))$$

- Transitivity**:**
  - $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$
  - Same for O and Ω
- Reflexivity:
  - $f(n) = \Theta(f(n))$
  - Same for O and Ω
- Symmetry:
  - $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$
- Transpose symmetry:
  - $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$

Asymptotic Notations in Equations

- On the right-hand side
  - $\Theta(n^2)$ stands for some anonymous function in $\Theta(n^2)$

  $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$ means:

  There exists a function $f(n) \in \Theta(n)$ such that

  $2n^2 + 3n + 1 = 2n^2 + f(n)$

- On the left-hand side

  $2n^2 + \Theta(n) = \Theta(n^2)$

  No matter how the anonymous function is chosen on the left-hand side, there is a way to choose the anonymous function on the right-hand side to make the equation valid.

Common Summations

- Arithmetic series: $$\sum_{k=1}^{n} k = 1+2+\ldots+n = \frac{n(n+1)}{2}$$
- Geometric series: $$\sum_{k=0}^{n} x^k = 1+x+x^2+\ldots+x^n = \frac{x^{n+1}-1}{x-1} \left(x \neq 1\right)$$
  - Special case: $|x| < 1$: $$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$
- Harmonic series: $$\sum_{k=1}^{n} \frac{1}{k} = 1+\frac{1}{2}+\ldots+\frac{1}{n} \approx \ln n$$
- Other important formulas: $$\sum_{k=1}^{n} \lg k \approx n \lg n$$

$$\sum_{k=1}^{n} k^p = 1^p + 2^p + \ldots + n^p \approx \frac{1}{p+1} n^{p+1}$$

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Mathematical Induction

- A powerful, rigorous technique for proving that a statement S(*n*) is true for *every* natural number *n*, no matter how large.

- Proof:
  - **Basis step**: prove that the statement is true for $n = 1$
  - **Inductive step:** assume that $S(n)$ is true and prove that $S(n+1)$ is true for all $n \geq 1$

- Find case n "within" case $n+1$