Law of Total Probability and Bayes' Rule

Law of Total Probability:

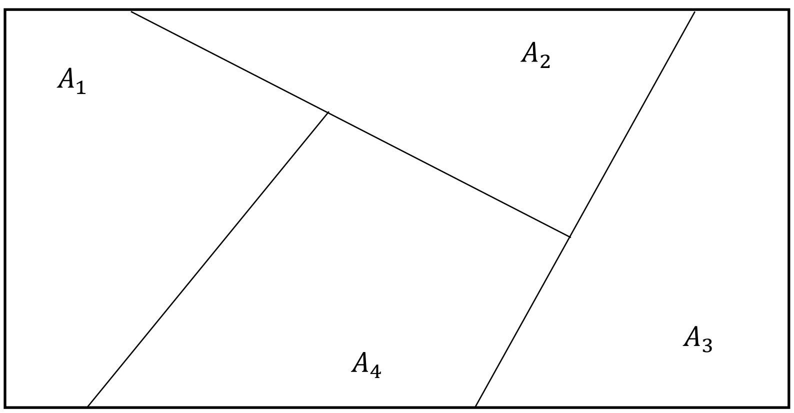
If $A_1, A_2, \ldots, A_n$ are mutually exclusive and exhaustive events, and B is any event, then

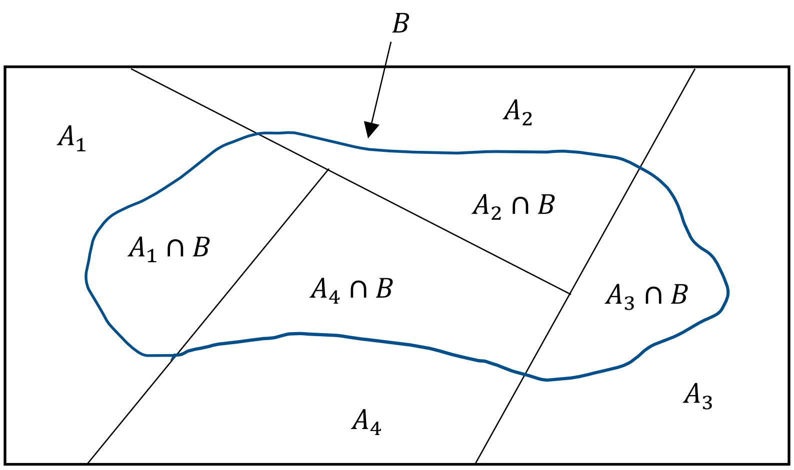
$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$$

Equivalently, if $P(A_i) \neq 0$ for each A_i,

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_n)$$

In the figure at the bottom, the mutually exclusive and exhaustive events A_1, A_2, A_3, A_4, divide the event B into mutually exclusive subsets.





Bayes' Rule:

Special Case: Let A and B be events with $P(A) \neq 0$, $P(A^C) \neq 0$, and $P(B) \neq 0$. Then

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A)+P\left(B|A^C\right)P\left(A^C\right)}$$

General Case: Let $A_1, A_2, \ldots, A_n$ be mutually exclusive and exhaustive events with $P(A_i) \neq 0$ for each A_i. Let B be any event with $P(B) \neq 0$. Then

$$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{P(B|A_1)P(A_1)+ P(B|A_2)P(A_2)+ \cdots + P(B|A_n)P(A_n)}$$

Ex.

During frequent trips to a certain city, a traveling salesman stays at hotel A 50% of the time, at hotel B 30% of the time, and at hotel C 20% of the time. When checking in, there is some problem with the reservation 3% of the time at hotel A, 6% of the time at hotel B, and 10% of the time at hotel C. Suppose the salesperson travels to this city.

(a) Find the probability that the salesperson stays at hotel A and has a problem with the reservation.

(b) Find the probability that the salesperson has a problem with the reservation.

(c) Suppose the salesperson has a problem with the reservation; what is the probability that the salesperson is staying at hotel A?