Exam 3
Zed Chance

1. THE SERIES $\sum_{k=0}^{\infty} \frac{1}{k^2+1}$ CONVERGES. Let $\sum_{k=0}^{\infty} \frac{1}{k^2+1} = a_k$

SHOW THE SERIES CONVERGES USING AS MANY SERIES TEST AS POSSIBLE. IF A TEST DOESN'T APPLY EXPLAIN WHY. IF A TEST IS INCONCLUSIVE, JUST SAY SO.

Comparison:

Consider $b_k = \sum_{k=0}^{\infty} \frac{1}{k^2}$, for all values of k, $a_k < b_k$.

Since $a_k < b_k$, if b_k converges, so will a_k.

Note: $\sum_{k=0}^{\infty} \frac{1}{k^2}$ is a p-series with $p=2$, since $p>1$ the series will converge.

So, since b_k converges, and $a_k < b_k$ for all values of k, a_k must converge also.

Limit comparison:

Consider $b_k = \sum_{k=0}^{\infty} \frac{1}{k^2}$, for all values of k, $a_k < b_k$. So if the limit of $\frac{a_k}{b_k}$ exists, then a_k and b_k will have same convergence/divergence.

SO: $\lim_{K\to\infty} \frac{\frac{1}{K^2+1}}{\frac{1}{K^2}}$

$= \lim_{K\to\infty} \frac{1}{K^2+1} \cdot \frac{K^2}{1}$

$= \lim_{K\to\infty} \frac{K^2}{K^2(1+\frac{1}{K^2})}$

$= \lim_{K\to\infty} \frac{1}{1+\cancel{\frac{1}{K^2}}^{\,0}}$

$= \underline{1}$

Since the limit exists, a_K and b_K will have same result of convergence.

Since b_K is a p-series with $p=2$, b_K converges.

$\therefore$ a_K must converge because b_K converges and the limit:

$$\lim_{K\to\infty} \frac{a_K}{b_K}$$

exists.

Ratio test

We can tell absolute convergence if the limit L is less than $\underline{1}$ where:

$$L = \lim_{K\to\infty} \frac{a_{K+1}}{a_K}$$

So,
$$\lim_{k\to\infty} \frac{\frac{1}{(k+1)^2+1}}{\frac{1}{k^2+1}}$$

$$= \lim_{k\to\infty} \frac{1}{(k+1)^2+1} \cdot \frac{k^2+1}{1}$$

$$= \lim_{k\to\infty} \frac{k^2+1}{k^2+2k+2}$$

$$= \lim_{k\to\infty} \frac{\cancel{k^2}\left(1+\frac{1}{k^2}\right)}{\cancel{k^2}\left(1+\frac{2}{k}+\frac{2}{k^2}\right)}$$

$$= \lim_{k\to\infty} \frac{1+\overset{\to 0}{\frac{1}{k^2}}}{1+\overset{\to 0}{\frac{2}{k}}+\overset{\to 0}{\frac{2}{k^2}}}$$

$$= 1$$

Since the limit equals 1, unfortunately the test is inconclusive.

2. CALCULATE THE FOLLOWING LIMITS

(a) $\lim_{K\to\infty} \sqrt[K]{K}$ (b) $\lim_{K\to\infty} \left[\frac{K}{K+1}\right]^K$

a) $\lim_{K\to\infty} K^{1/K}$ $\xrightarrow{} 0$

$\lim_{K\to\infty} K^0$

$= 1$

b)

on other side

b) $\lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k$

$= e^{\lim_{k\to\infty} \ln\left(\frac{k}{k+1}\right)^k}$

$= e^{\lim_{k\to\infty} k \ln\left(\frac{k}{k+1}\right)}$

$= e^{\lim_{k\to\infty} \frac{\ln\left(\frac{k}{k+1}\right)}{\frac{1}{k}}}$ ← Note: indeterminat form "$\frac{0}{0}$" Lets use l'Hospitals:

ASIDE

$\frac{\partial}{\partial x}\left(\ln\left(\frac{x}{x+1}\right)\right)$

$= \frac{x+1}{x} \cdot \frac{1(x+1) - x(1)}{(x+1)^2}$

$= \frac{\cancel{(x+1)}}{x} \cdot \frac{1}{(x+1)^{\cancel{2}}}$

$= \frac{1}{x^2+x}$

$\frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

So:

$= e^{\lim_{k\to\infty} \frac{\frac{1}{x^2+x}}{-\frac{1}{x^2}}}$

$= e^{\lim_{k\to\infty} - \frac{\cancel{x^2}}{\cancel{x^2}(1+\frac{1}{x})}}$

$= e^{-\lim_{k\to\infty} \frac{1}{1+\cancel{\frac{1}{x}}\to 0}}$

$= e^{-1}$

$\boxed{= \frac{1}{e}}$

$\frac{1}{2}$ $\frac{1}{1}$

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$\frac{1}{4}$ $\frac{1}{3}$

$\frac{1}{4-2} > \frac{1}{4}$

$\frac{1}{2} > \frac{1}{4}$

$\frac{1}{4+2}$ $\frac{1}{4}$

$\frac{1}{6} < \frac{1}{4}$

3. DETERMINE WHICH OF THE FOLLOWING SERIES CONVERGES OR DIVERGES.

~~(a)~~ $\displaystyle\sum_{K=10}^{\infty} \frac{(\sqrt{K}+2)^3}{(\sqrt{K}-2)^6}$ ~~(b)~~ $\displaystyle\sum_{K=1}^{\infty} \cot^{-1}(K)$

~~(c)~~ $\displaystyle\sum_{K=5}^{\infty} \frac{(-1)^{K+1}(K^2+1)}{(K^2-2)}$ ~~(d)~~ $\displaystyle\sum_{K=1}^{\infty} (-\tan^{-1}(-K))^K$

~~(e)~~ $\displaystyle\sum_{K=1}^{\infty} \frac{\sqrt{K}-1}{K^3 \ln K}$ ~~(f)~~ $\displaystyle\sum_{K=3}^{\infty} \frac{\ln(K)}{K}$

~~(g)~~ $\displaystyle\sum_{K=1}^{\infty} \frac{(-1)^{K+2}(5K^2+3)(4K)!}{(2K^2+3K-1)(4K)^{4K}}$ ~~(h)~~ $\displaystyle\sum_{K=2}^{\infty} \frac{(-1)^{K-1}}{\sqrt[3]{\ln K}}$

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a) WTS $\displaystyle\sum_{K=10}^{\infty} \frac{(\sqrt{K}+2)^3}{(\sqrt{K}-2)^6}$ CON by comparison

Proof

If I can show that a series greater than our series converge, then our series also converges.

$$\Sigma a_k < \Sigma b_K$$

① I propose a b_k of:

$$\sum_{K=10}^{\infty} \frac{(\sqrt{K}+2)^3}{(\sqrt{K}+2)^6}$$

Since the denomenator is bigger, this series is less than our given.

Lets evaluate b_k:

$$\sum_{k=10}^{\infty} \frac{(\sqrt{k}+2)^3}{(\sqrt{k}+2)^6}$$

$$= \sum_{k=10}^{\infty} \frac{1}{(\sqrt{k}+2)^3}$$

2) This series is less than

$$\sum_{k=10}^{\infty} \frac{1}{\sqrt{k}^3}$$

and this series is a p-series with $p=\frac{3}{2}$. since $p>1$ the series converges. So our series b_k converges also because it is less than the p-series.

3) So, since our a_k is less than our b_k, and b_k converges, a_k must also converge!

b) WTS $\sum_{k=1}^{\infty} \cot^{-1}(k)$ DIV

by Integral test

Proof

If I can show that the integral representation of the series diverges, then the series also diverges.

① So let $f(x) = \cot^{-1}(x)$

$$\int_1^{\infty} \cot^{-1}(x)\, dx$$

$$= \lim_{b\to\infty} \int_1^{b} \cot^{-1}(x)\, dx$$

$u = \cot^{-1}(x)$ $\qquad dv = dx$

$du = -\frac{1}{x^2+1} dx$ $\qquad v = x$

$$= \lim_{b\to\infty} \left[x\cot^{-1}(x) - \int_1^{b} x \cdot -\frac{1}{x^2+1}\, dx \right]$$

$$= \lim_{b\to\infty} \left[x\cot^{-1}(x) + \int_1^{b} \frac{x}{x^2+1}\, dx \right]$$

$w = x^2+1$

$dw = 2x\, dx$

$$= \lim_{b\to\infty} \left[x\cot^{-1}(x) + \frac{1}{2}\int_1^{b} \frac{1}{w}\, dw \right]$$

$$= \lim_{b\to\infty} \left[x\cot^{-1}(x) + \frac{1}{2}\ln(w) \right]_1^b \qquad w = x^2+1$$

$$= \lim_{b\to\infty} \left[x\cot^{-1}(x) + \frac{1}{2}\ln(x^2+1) \right]_1^b$$

$$= \lim_{b\to\infty} \left[b\cancelto{0}{\cot^{-1}(b)} + \frac{1}{2}\ln(b^2+1) - \left(1\cot^{-1}(1) + \frac{1}{2}\ln(2)\right) \right]$$

Note:

$$\lim_{b\to\infty} \cot^{-1}(b) = 0$$

$$= \lim_{b\to\infty} \left[\frac{1}{2}\cancelto{\infty}{\ln(b^2+1)} - \left(\frac{\pi}{4} + \ln 2\right) \right]$$

Note:

$$\lim_{b\to\infty} \ln(b^2+1) = \infty$$

$$= \infty$$

2 Since the integral reprenstation diverges, that means the series also diverges!

c) WTS $\sum_{k=5}^{\infty} \frac{(-1)^{k+1}(k^2+1)}{(k^2-2)}$ DIV

Proof

① Lets try a absolute convergence test:

$$\sum_{k=5}^{\infty} \left| \frac{(-1)^{k+1}(k^2+1)}{(k^2-2)} \right|$$

$$= \sum_{k=5}^{\infty} \frac{(k^2+1)}{(k^2-2)}$$

If the limit of this series is not 0 then the series diverges. Additionally if the series diverges then our given series also diverges.

② so:

$$\lim_{k \to \infty} \frac{(k^2+1)}{(k^2-2)}$$

$$= \lim_{k \to \infty} \frac{\cancel{k^2}(1+\frac{1}{k^2})^{\to 0}}{\cancel{k^2}(1-\frac{2}{k^2})^{\to 0}}$$

$$= 1$$

So the original series is not abs convergent. Lets look at the limit of the original series:

$$\lim_{k \to \infty} \frac{(-1)^{k+1}(k^2+1)}{(k^2-2)}$$

$$= \lim_{k\to\infty} \frac{(-1)^k(-1)(k^2+1)}{(k^2-2)}$$

$$= -\lim_{k\to\infty} \frac{(-1)^k(k^2+1)}{(k^2-2)}$$

$$= -\lim_{k\to\infty} \frac{(-1)^k \cancel{(k^2)}\left(1+\overset{\to 0}{\frac{1}{k^2}}\right)}{\cancel{k^2}\left(1-\overset{\to 0}{\frac{2}{k^2}}\right)}$$

$$= -\lim_{k\to\infty} (-1)^k$$

$$= -\infty$$

So, the series must diverge because the terms are not trending toward 0.

d) WTS $\sum_{k=1}^{\infty} (-\tan^{-1}(-k))^k$ DIV by limit test

Proof

If the limit of the series doesn't approach 0, then by the divergence test the series diverges.

① So:

$$\lim_{k \to \infty} (-\tan^{-1}(-k))^k$$

As $k \to \infty$

$-k \to -\infty$

$\tan^{-1}(-k) \to \frac{\pi}{2}$

$-\tan^{-1}(-k) \to -\frac{\pi}{2}$

$\left(-\frac{\pi}{2}\right)^k \to \infty$

$$= \lim_{k \to \infty} (-\tan^{-1}(-k))^k$$

$$= \infty$$

② $\therefore$ So because the limit is not 0, the series must diverge.

e) WTS $\sum_{k=1}^{\infty} \frac{\sqrt{k}-1}{k^3 \ln k}$ CON by comparison

Proof

If I can show that a series that is greater than our given seriel converges, then our given series will also converge.

(1) So, I propose this comparison:

$$\sum_{k=1}^{\infty} \frac{\sqrt{k}-1}{k^3 \ln k} < \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^3 \ln k} < \sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$$

Our given

This is bigger because the numerator is bigger!

This is bigger because the denomenator is smaller!

② So, the series

$$\sum_{k=1}^{\infty} \frac{1}{k^{5/2}}$$

is a p-series with $p = \frac{5}{2}$, with a $p > 1$ the series converges!

③ Since the greatest series converges, the two series less than that must also converge!

④ $\therefore$ By the comparison test, the series $\sum_{k=1}^{\infty} \frac{\sqrt{k} - 1}{k^3 \ln k}$

converges!

f) WTS $\sum_{k=3}^{\infty} \frac{\ln(k)}{k}$ DIV by Integral test

Proof

If the integral representation of a series diverges, then the series will also diverge.

Consider: $f(x) = \frac{\ln(x)}{x}$

Lets integrate:

① $\int_1^{\infty} \frac{\ln(x)}{x} dx$

$= \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x} dx$

$u = \ln x$

$du = \frac{1}{x} dx$

$= \lim_{b\to\infty} \int_1^b u \, du$

$= \lim_{b\to\infty} \left[\frac{1}{2}u^2\right]_1^b$

$= \lim_{b\to\infty} \left[\frac{1}{2}(\ln x)^2\right]_1^b$

$= \lim_{b\to\infty} \left[\frac{1}{2}(\ln b)^2 - \frac{1}{2}(\ln(1))^2\right]$

$$= \lim_{b \to \infty} \left[\frac{1}{2} (\ln b)^2 \right]$$

$$= \infty$$

② So, since the integral representation of the series diverges, so does the series itself!

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9) WTS $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+2}(5k^2+3)(4k)!}{(2k^2+3k-1)(4k)^{4k}}$ CON by ratio test

<u>Proof</u>

By the ratio test, if the limit L is less than 1, then the series is absolutely convergent where:

$$L = \lim_{k \to \infty} \frac{a_{k+1}}{a_k}$$

① Consider:

$$\sum_{k=1}^{\infty} a_{k+1} = \sum_{k=1}^{\infty} \frac{(-1)^{(k+1)+2}(5(k+1)^2+3)(4(k+1))!}{(2(k+1)^2+3(k+1)-1)(4(k+1))^{4(k+1)}}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+3}(5(k^2+2k+1)+3)(4k+1)!}{(2(k^2+2k+1)+3k+3-1)(4k+1)^{4k+4}}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+3}(5k^2+10k+5+3)(4k+1)(4k)!}{(2k^2+4k+2+3k+2)(4k+1)^{4k}(4k+1)^4}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+3}(5k^2+10k+8)(4k)!}{(2k^2+7k+4)(4k+1)^{4k}(4k+1)^3}$$

② Now that a_{k+1} is relatively simplified, lets multiply it by $\frac{1}{a_k}$ and see if the limit exists:

$$\lim_{k\to\infty} \frac{(-1)^{k+3}(5k^2+10k+8)\cancel{(4k)!}}{(2k^2+7k+4)(4k+1)^{4k}(4k+1)^3} \cdot \frac{(2k^2+3k-1)(4k)^{4k}}{(-1)^{k+2}(5k^2+3)\cancel{(4k)!}}$$

$$= \lim_{k\to\infty} \frac{(-1)(5k^2+10k+8)(2k^2+3k-1)(4k)^{4k}}{(2k^2+7k+4)(4k+1)^{4k}(4k+1)^3(5k^2+3)}$$

$$= -\lim_{k\to\infty} \frac{\cancel{k^2}(5+\frac{10}{k}+\frac{8}{k^2})k^2(2+\frac{3}{k}-\frac{1}{k^2})}{\cancel{k^2}(2+\frac{7}{k}+\frac{4}{k^2})(4k+1)^{4k+3}\,\cancel{k^2}(5+\frac{3}{k^2})}$$

$$= -\lim_{k\to\infty} \frac{(5)(2)}{(2)(4k+1)^{4k+3}}$$

$$= -5 \lim_{k\to\infty} \frac{1}{(4k+1)^{4k+3}}$$

$$= -\infty$$

③ $\therefore$ Since the limit is less than 1, the series converges absolutely!

Since the series converges absolutely, the series also converges.

h) WTS $\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{\sqrt[3]{\ln k}}$ CON using alternating series test.

Proof

If an alternating series has terms that are decreasing, and the limit of the series goes to 0 as it approaches infinity, then the series converges.

(1) Notice:

$$\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{\sqrt[3]{\ln k}}$$

The numerator makes the terms alternate signs.

② To show decreasing terms, lets compare a_{k+1} to a_k.

$$a_k = \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{\sqrt[3]{\ln k}}$$

$$a_{k+1} = \sum_{k=2}^{\infty} \frac{(-1)^{(k+1)-1}}{\sqrt[3]{\ln(k+1)}}$$

$$= \sum_{k=2}^{\infty} \frac{(-1)^{k}}{\sqrt[3]{\ln(k+1)}}$$

a_{k+1} still has the alternating $(-1)^k$ on the numerator, but the denomenator is always a little bigger than a_k because of the $+1$.

Since the next terms denomenator is always biggor, the terms are decreasing.

(3) The test also requires that the limit

$$\lim_{k\to\infty} a_k = 0$$

is zero

Remember, alternating series are defined by:

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k$$

So our a_k is $\frac{1}{\sqrt[3]{\ln k}}$

So lets check the limit,

$$\lim_{k\to\infty} \frac{1}{\sqrt[3]{\ln k}} \xrightarrow{} \infty$$

$$= 0$$

$\therefore$ By the alternating series test, because the limit of our a_k is 0, and the terms are decreasing, the series $\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{\sqrt[3]{\ln k}}$ converges!

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4. DETERMINE IOC FOR $\sum_{k=1}^{\infty} \frac{(-1)^k (k+1)}{(2k)^2 2^k} (x-2)^k$

Lets check interval of convergence using the ratio test:

$$\lim_{k\to\infty} \left| \frac{(-1)^{k+1}(k+1+1)(x-2)^{k+1}}{(2(k+1))^2 2^{(k+1)}} \cdot \frac{(2k)^2 2^k}{(-1)^k (k+1)(x-2)^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{\cancel{(-1)^k}\cancel{(-1)}(k+2)\cancel{(x-2)^k}(x-2)(2k)^2 \cancel{2^k}}{(2k+2)^2 \cancel{2^k} \cdot 2 \quad \cancel{(-1)^k}(k+1)\cancel{(x-2)^k}} \right|$$

$$= \lim_{k\to\infty} \frac{(k+2)(2k)^2}{(2k+2)^2 2(k+1)} \cdot |(x-2)|$$

$$= \lim_{k\to\infty} \frac{(k+2) 4k^2}{4k^2+8k+4(k+1)} \cdot |x-2|$$

$$= \lim_{k\to\infty} \frac{4k^3+8k^2}{4k^3+8k^2+4k+4k^2+8k+4} \cdot |x-2|$$

$$= \lim_{k\to\infty} \frac{\cancel{k^3}(4+\frac{8}{k})}{\cancel{k^3}(4+\frac{12}{k}+\frac{12}{k^2}+\frac{4}{k^3})}$$

$$= \lim_{k\to\infty} \frac{4}{4} \cdot |x-2|$$

$$= |x-2|$$

So our interval is

$$|x-2| < 1$$

$$-1 < x-2 < 1$$

$$1 < x < 3$$

Lets check endpoints

$x = 1$

$$\sum_{k=1}^{\infty} \frac{(-1)^k (k+1)}{(2k)^2 2^k} (1-2)^k$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k (k+1) \quad (-1)^k}{(2k)^2 2^k}$$

$$= \sum_{k=1}^{\infty} \frac{(k+1)}{(2k)^2 2^k}$$

ASIDE

$(-1)^k (-1)^k$

$= (-1)^{2k}$

Anything times 2 becomes even, so $(-1)^{2k}$ always evaluates to 1 for all $k > 0$

Lets try ratio test:

$$= \sum_{k=1}^{\infty} \frac{(k+1)+1}{(2(k+1))^2 \, 2^{(k+1)}} \cdot \frac{(2k)^2 2^k}{(k+1)}$$

$$= \sum_{k=1}^{\infty} \frac{(k+2)(2k)^2 \cancel{2^k}}{(2k+2)^2 \cancel{2^k} \, 2(k+1)}$$

$$= \sum_{k=1}^{\infty} \frac{(k+2) 2k^2}{(2k+2)^2 (k+1)}$$

$$= \sum_{k=1}^{\infty} \frac{2k^3 + 4k^2}{4k^2 + 8k + 4(k+1)}$$

$$= \sum_{k=1}^{\infty} \frac{\cancel{k^3}(2 + \frac{4}{k})}{\cancel{k^3}(4 + \frac{8}{k} + \frac{4}{k^2} + \frac{4}{k} + \frac{8}{k^2} + \frac{4}{k^2})}$$

$$= \frac{1}{2}$$

Since the limit is less than 1, th series converges, and we include the point.

Lets check the endpoint

$x = 3$

$$\sum_{k=1}^{\infty} \frac{(-1)^k (k+1)}{(2k)^2\, 2^k} (3-2)^k$$

ASIDE

$(3-2)^k$

$= 1^k$

$= 1$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k (k+1)}{(2k)^2\, 2^k}$$

Lets try alternating series test:

$$\lim_{k\to\infty} \frac{k+1}{(2k)^2\, 2^k}$$

$$= \lim_{k\to\infty} \frac{k+1}{4k^2\, 2^k}$$

$$= \lim_{k\to\infty} \frac{\cancel{k}(1+\frac{1}{k})}{\cancel{k}(4k\, 2^k)}$$

$$= \lim_{k\to\infty} \frac{1+\frac{1}{k}^{\to 0}}{4k\, 2^k}$$

$= 0$

Lets check for decreasing terms:

if $a_{k+1} < a_k$

$$\frac{(-1)^{k+1}(k+2)}{(2(k+1))^2\, 2^{k+1}} < \frac{(-1)^k (k+1)}{(2k)^2\, 2^k}$$

$$\frac{(-1)^k (-1)(k+2)}{(2k+2)\ 2^k\, 2} < \frac{(-1)^k (k+1)}{(2k)^2\, 2^k}$$

This ensures that a_{k+1} is less than a_k.

So the series passes the Alternating series test, so we include the endpoint in the interval.

$\therefore$ Overall our interval of convergence is $\boxed{1 \le x \le 3}$

This means we have a radius of convergence $R = 1$ and a center of 2.

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5. FIND A POWER SERIES REPRESENTATION FOR THE FUNCTION f DEFINED BY

$$f(x) = \frac{1}{2x-x^2}$$

CENTERED AT $a=1$. DETERMINE IOC FOR YOUR REPRESENTATION

$4x^2 - 6x^3$

$$f(x) = \frac{1}{2x-x^2}$$
$$= \frac{1}{x} \cdot \frac{1}{2-x}$$

let $g(x) = \frac{1}{2} \cdot \frac{1}{1-\frac{x}{2}}$

$g(x)$ is a geometric sum that has a $r = \frac{x}{2}$

This can be represented as

$$\sum_{k=0}^{\infty} ar^k = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k$$

So lets plug this into $f(x)$

$$f(x) = \frac{1}{x} \cdot \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k$$
$$= \frac{1}{2x} \sum_{k=0}^{\infty} \frac{x^k}{2^k}$$

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{x^{k-1}}{2^{k+1}}}$$

This is our power series representation of the function $f(x) = \frac{1}{2x-x^2}$.

The power series converges on the interval

$$\left|\frac{x}{2}\right| < 1$$

$$-1 < \frac{x}{2} < 1$$

$$-2 < x < 2$$

Lets check to see if we can add our endpoints in the interval:

$x = -2$

$$\sum_{k=0}^{\infty} \frac{(-2)^{k-1}}{2^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{\cancel{(-2)^k}}{(-2)\cancel{(-2)^k}(-2)}$$

$$= \sum_{k=0}^{\infty} \frac{1}{-4}$$

$\lim_{k \to \infty} \frac{1}{-4} = \infty$, so this series diverges, and cannot be included in our interval. Lets check $x = 2$

$$\sum_{k=0}^{\infty} \frac{(2)^{k-1}}{2^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{\cancel{2^k}}{2\,\cancel{2^k}\,2}$$

$$= \sum_{k=0}^{\infty} \frac{1}{4}$$

$\lim_{k \to \infty} \frac{1}{4} = \infty$, so this endpoint cannot be included in our interval. So the entire interval afterall is $\boxed{-2 < x < 2}$

6. USE THE MCLAURIN SERIES FOR ARCSINX TO GENERATE A POWER SERIES FOR $\pi/3$. USE THIS POWER SERIES TO GENERATE A FRACTION THAT IS WITHIN $\frac{1}{10{,}000}$ TH ON THE EXACT VALUE OF $\pi/3$.

$$\sin^{-1}(x) = \sum_{n=0}^{\infty} \frac{(2n)!\, x^{2n+1}}{(2^n n!)(2n+1)}$$

Lets try a Taylor polynomial out to 6 degrees:

$$P_6(x) = \frac{x^1}{1} + \frac{2!\, x^3}{2! \cdot 3} + \frac{4!\, x^5}{8! \cdot 5} + \frac{6!\, x^7}{24! \cdot 7} + \frac{8!\, x^9}{(16 \cdot 4)! \cdot 9} + \frac{10!\, x^{11}}{(32 \cdot 5)! \cdot 11}$$

Since:

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$ lets plug in $x = \frac{\sqrt{3}}{2}$

$$P_6\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^3}{3} + \frac{4!\left(\frac{\sqrt{3}}{2}\right)^5}{8! \cdot 5} + \frac{6!\left(\frac{\sqrt{3}}{2}\right)^7}{24! \cdot 7} + \frac{8!\left(\frac{\sqrt{3}}{2}\right)^9}{64! \cdot 9} + \frac{10!\left(\frac{\sqrt{3}}{2}\right)^{11}}{160! \cdot 11}$$

≈ 1.0825

no where near close enough...

The terms are only getting smaller so going out to more terms in the Taylor polynomial isn't going to get any closer to the actual value of ≈ 1.0471.

So I must be off...