Linear Combinations of Random Variables, Means and Variances

If $X_1, X_2, \ldots, X_n$ are random variables, then the mean of the sum $X_1 + X_2 + \cdots + X_n$ is given by

$$\mu_{X_1 + \cdots + X_n} = \mu_{X_1} + \cdots + \mu_{X_n}$$

*(e.g., **the mean of the sum is the sum of the means**)*

The sum $X_1 + X_2 + \cdots + X_n$ a special case of a **linear combination**.

Definition:

If $X_1, X_2, \ldots, X_n$ are random variables and $c_1, c_2, \ldots, c_n$ are constants, then the random variable

$$c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

is called a **linear combination** of $X_1, X_2, \ldots, X_n$.

Mean of a Linear Combination of Random Variables:

If X and Y are random variables, and a and b are constants, then

$$\mu_{aX+bY} = a\mu_X + b\mu_Y$$

More generally,

if $X_1, X_2, \ldots, X_n$ are random variables and $c_1, c_2, \ldots, c_n$ are constants, then the mean of the linear combination $c_1X_1 + c_2X_2 + \cdots + c_nX_n$ is given by

$$\mu_{c_1X_1 + c_2X_2 + \cdots + c_nX_n} = c_1\mu_{X_1} + c_2\mu_{X_2} + \cdots + c_n\mu_{X_n}$$

Variance of a Linear Combination of Independent Random Variables:

If $X_1, X_2, \ldots, X_n$ are *independent* random variables, then the variance of the sum $X_1 + X_2 + \cdots + X_n$ is given by

$$\sigma^2_{X_1+X_2+\cdots+X_n} = \sigma^2_{X_1} + \sigma^2_{X_2} + \cdots + \sigma^2_{X_n}$$

*(e.g., **the variance of the sum is the sum of the variances**)*

In general:

If $X_1, X_2, \ldots, X_n$ are *independent* random variables and $c_1, c_2, \ldots, c_n$ are constants, then the variance of the linear combination $c_1X_1 + c_2X_2 + \cdots + c_nX_n$ is given by

$$\sigma^2_{c_1X_1+c_2X_2+\cdots+c_nX_n} = c_1^2\sigma^2_{X_1} + c_2^2\sigma^2_{X_2} + \cdots + c_n^2\sigma^2_{X_n}$$

Two frequently encountered linear combinations are the sum and the difference of two random variables. It is interesting to note that, when the random variables are independent, the variance of the sum is the same as the variance of the difference.

e.g., If X and Y are *independent* random variables with variance σ_X^2 and σ_Y^2 , then the variance of the sum $X + Y$ is

$$\sigma^2_{X+Y} = \sigma_X^2 + \sigma_Y^2.$$

The variance of the difference $X - Y$ is

$$\sigma^2_{X-Y} = \sigma_X^2 + \sigma_Y^2.$$

Ex. A piston is placed inside a cylinder. The clearance is the distance between the edge of the piston and the wall of the cylinder and is equal to one-half the difference between the cylinder diameter and the piston diameter. Assume the piston diameter has a mean of 80.85 cm with a standard deviation of 0.02 cm. Assume the cylinder diameter has a mean of 80.95 cm with a standard deviation of 0.03 cm.

(a) Find the mean clearance.

(b) Assuming that the piston and cylinder are chosen independently, find the standard deviation of the clearance.