Simulation Assignment I

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1 Probability Integral Transformation

```
In [1]: import numpy as np
        import pandas as pd
        import matplotlib.pyplot as plt
        import seaborn as sns
        plt.style.use('ggplot')
        plt.rc('text', usetex=True)
        plt.rc('font', family='serif', size=15)
        %matplotlib inline

        import statsmodels.api as sm
        import scipy.stats as stats
        from scipy.stats import weibull_min
        from scipy.linalg import expm
        from progressbar import ProgressBar
```

```
In [4]: class ProbIntegralTransformer(object):

            def __init__(self, inv_cdf):
                self.inv_cdf = inv_cdf

            def draw(self, n):
                u = np.random.uniform(size=n)
                return np.array([self.inv_cdf(p) for p in u])

            def attach_histogram(self, ax, n_sample, bin_width,
                                 truncate=(-np.inf, np.inf),
                                 label='_nolegend_'):
                sample = self.draw(n_sample)
                lb, ub = truncate
                bins = np.arange(max(min(sample), lb), (
                    min(max(sample), ub)+bin_width), bin_width)
                ax.hist(sample, bins, normed=1,
                        alpha=0.3, color="grey", label=label)
                return ax, sample
```

```
In [6]: def weibull_inv(lamb, k):
            return lambda u: lamb*(-np.log(u))**(1/k)

        def weibull_pdf(x_vec, lamb, k):
            return np.array([(k/lamb * (x/lamb) ** (k-1)) * \
                np.exp(-(x/lamb)**k) * (x>=0) for x in x_vec])

        def max_brownian_bdge_inv(b, h):
            return lambda u: (b+np.sqrt(b**2-2*np.log(u)*h))/2

        def max_brownian_bdge_pdf(x_vec, b, h):
            return np.array([(x>=max(0,b))*np.exp(
                -2*x*(x-b)/h)*(4*x-2*b)/h for x in x_vec])

        def cauchy_inv(x0, gamma):
            return lambda u: gamma*np.tan(np.pi*u-np.pi/2)+x0

        def cauchy_pdf(x_vec, x0, gamma):
            return np.array([1/(
                np.pi*gamma*(1+((x-x0)/gamma)**2)) for x in x_vec])

        def gumbel_inv(mu, beta):
            return lambda u: mu-beta*np.log(-np.log(u))

        def gumbel_pdf(x_vec, mu, beta):
            return np.array([(1/beta)*np.exp(
                -(x-mu)/beta-np.exp(-(x-mu)/beta)) for x in x_vec])
```

1.1 Weibull Distribution

By Wikipedia, the CDF of Weibull distribution is

$$F_X(x) = \begin{cases} 1 - e^{-(x/\lambda)^k} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

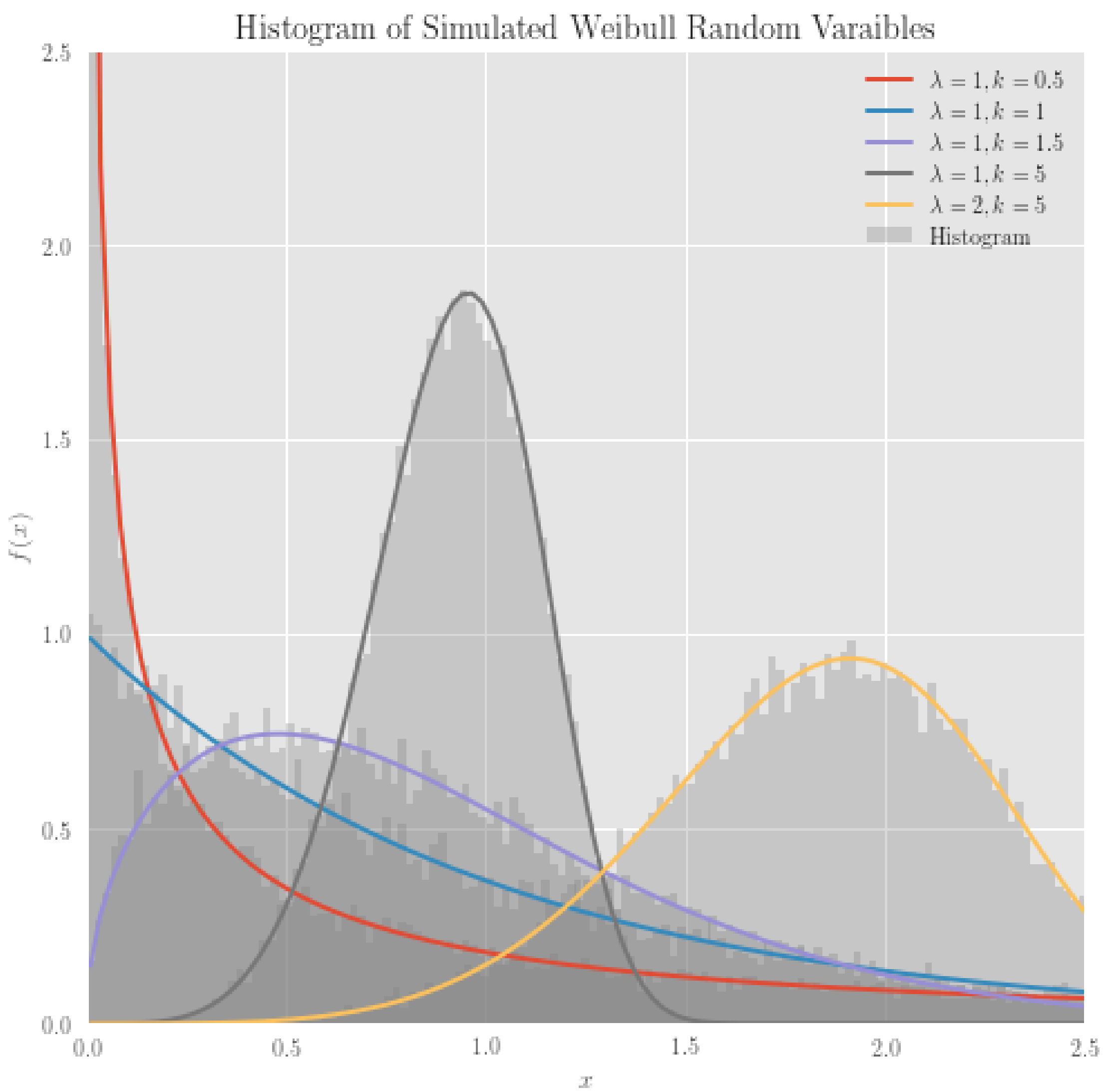
And the inverse CDF is

$$F^{-1}(u) = \lambda(-\log(1-u))^{\frac{1}{k}}, \quad u \in (0,1)$$

```
In [7]: weibull_params = [
             (1,0.5), (1, 1), (1, 1.5), (1, 5), (2, 5)]

        bar = ProgressBar()
        fig, ax = plt.subplots(1, 1, figsize=(8,8))

        for i, param in bar(enumerate(weibull_params)):
            weibull_sampler = ProbIntegralTransformer(
                weibull_inv(*param))
            label = 'Histogram' if i==4 else '_nolegend_'
```

```
    ax, sample = weibull_sampler.attach_histogram(
        ax, 20000, 0.02, label=label)
    x = np.linspace(0.01, 2.5, 100)
    ax.plot(x, weibull_pdf(x, *param), linewidth=2,
            label=r'$\lambda={}, k={}$'.format(*param))

ax.set_xlim((0,2.5))
ax.set_ylim((0,2.5))
ax.set_xlabel(r'$x$')
ax.set_ylabel(r'$f(x)$')
ax.set_title('Histogram of Simulated Weibull Random Varaibles')
_ = ax.legend()
```

```
| 4 Elapsed Time: 0:00:08
```



1.2 Distribution of Maximum of BM

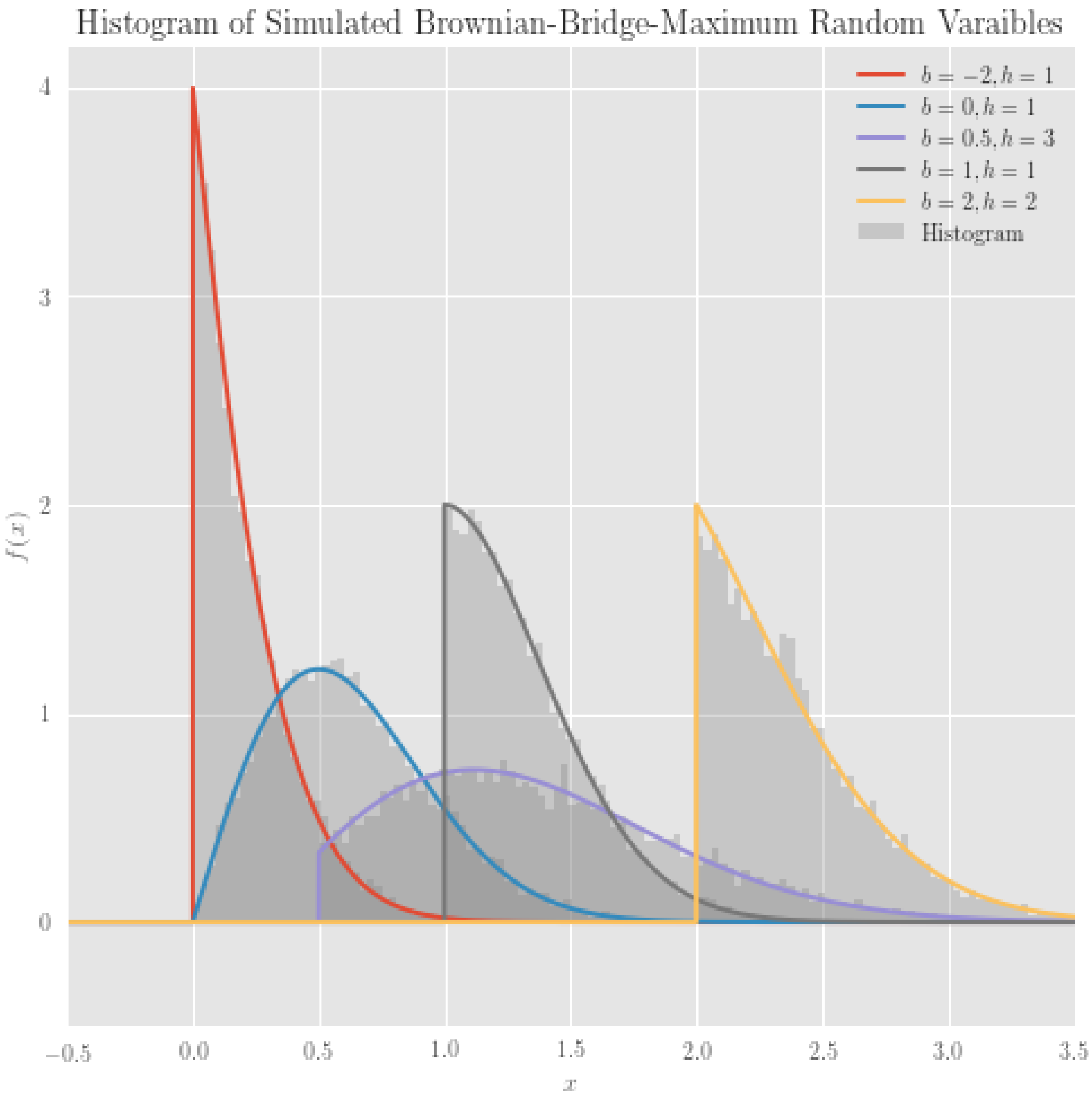
The inverse CDF of the given distribution is

$$F^{-1}(u) = \frac{b + \sqrt{b^2 - \frac{2\log(u)}{h}}}{2}, \quad u \in (0,1)$$

It's easy to see that $F^{-1}(u) \geq \max(0, b)$, since $\log(u) < 0$ with $0 < u < 1$. We can sample X with this formula.

```
In [8]: bb_params = [(-2,1), (0,1), (0.5,3), (1,1), (2,2)]
        bar = ProgressBar()
        fig, ax = plt.subplots(1, 1, figsize=(8,8))
        for i, param in bar(enumerate(bb_params)):
            max_brownian_bdge_sampler = ProbIntegralTransformer(
                max_brownian_bdge_inv(*param))
            label = 'Histogram' if i==4 else '_nolegend_'
            ax, sample = max_brownian_bdge_sampler.attach_histogram(
                ax, 10000, 0.03, label=label)
            x = np.linspace(-0.5, 4, 10000)
            ax.plot(x, max_brownian_bdge_pdf(x, *param), linewidth=2,
                    label=r'$b={}, h={}$'.format(*param))

        ax.set_xlim((-0.5,3.5))
        ax.set_ylim((-0.5,4.2))
        ax.set_xlabel(r'$x$')
        ax.set_ylabel(r'$f(x)$')
        ax.set_title('Histogram of Simulated Brownian-Bridge-Maximum Random Varaibles')
        _ = ax.legend()
```

```
| 4 Elapsed Time: 0:00:01
```



1.3 Cauchy Distribution

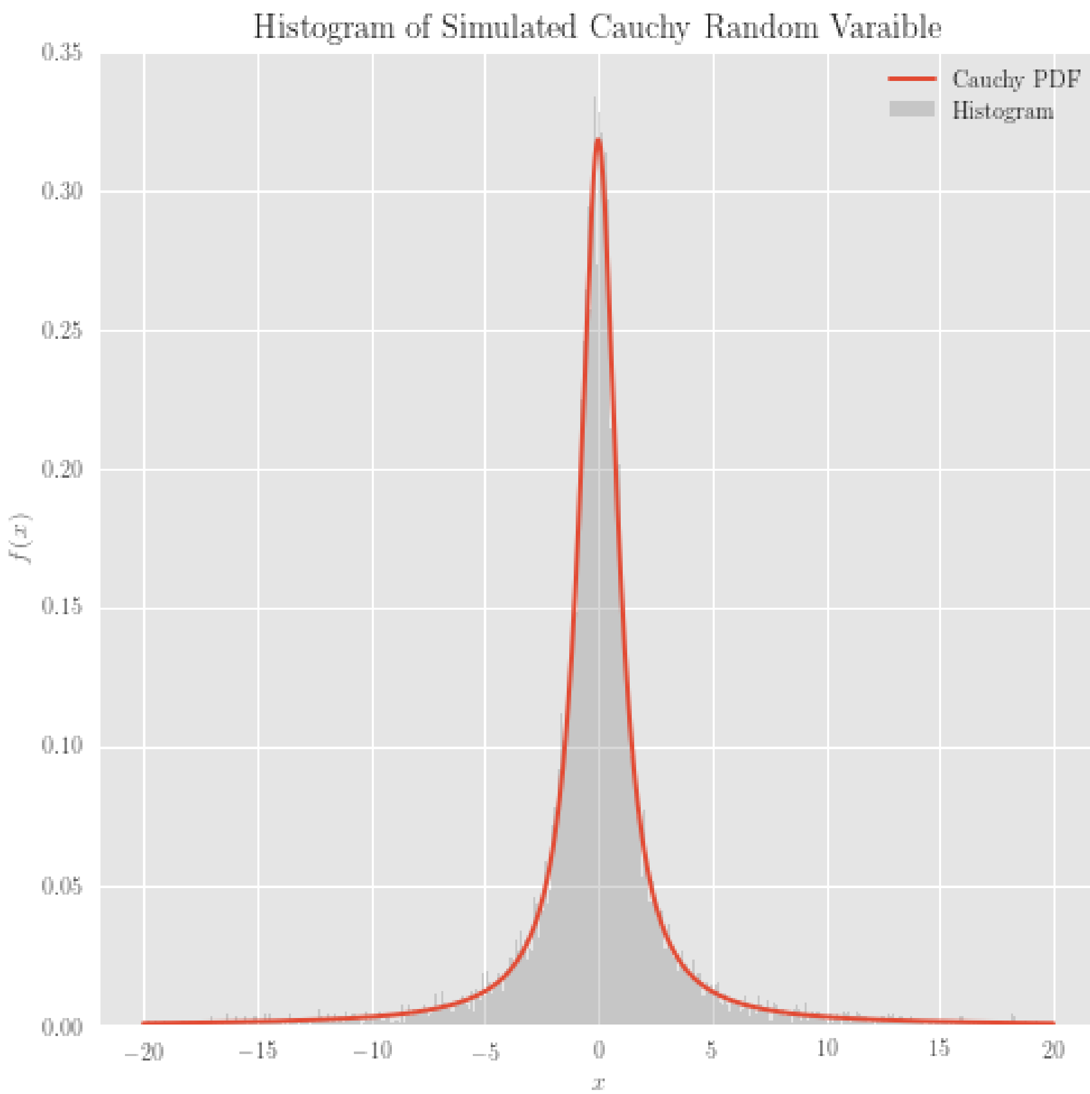
The inverse CDF of Cauchy distribution is

$$F^{-1}(u) = \tan(\pi u - \frac{\pi}{2}), \quad u \in (0, 1)$$

We sample X with this formula.

```
In [9]: x = np.linspace(-20, 20, 10000)
        fig, ax = plt.subplots(1, 1, figsize=(8,8))
        cauchy_sampler = ProbIntegralTransformer(cauchy_inv(0, 1))
        ax, sample = cauchy_sampler.attach_histogram(
            ax, 10000, 0.1, truncate=(-20,20), label="Histogram")
```

```
ax.plot(x, cauchy_pdf(x, 0, 1),
        linewidth=2, label="Cauchy PDF")
ax.set_xlabel(r'$x$')
ax.set_ylabel(r'$f(x)$')
ax.set_title('Histogram of Simulated Cauchy Random Varaible')
_ = ax.legend()
```



1.4 Standard Gumbel Distribution

(i) We sample with the inverse CDF:

$$F^{-1}(u) = -\log(-\log(u)), \quad u \in (0, 1)$$

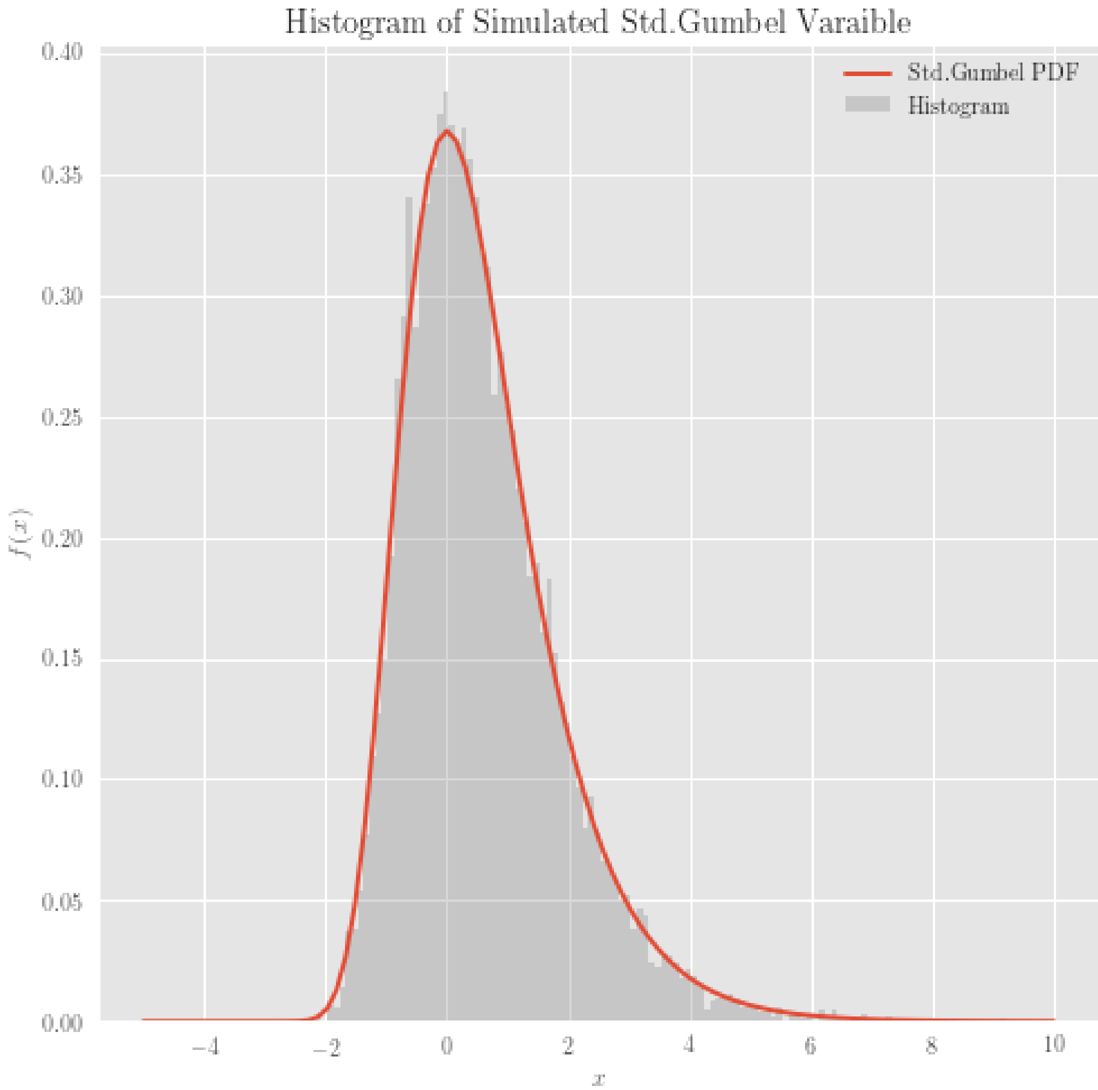
(ii) As instructed, let $Y_n := M_n - a_n$, the CDF of $Y_n = \max(X_1, ..., X_n) - a_n$ is

$$\begin{aligned}
F_{Y_n}(y) = P(M_n \leq y + a_n) &= P\left(\bigcap_{i=1}^{n}(X_i \leq y + a_n)\right) \\
&= \prod_{i=1}^{n} P(X_i \leq y + a_n) \\
&= \exp(-ne^{-(y+a_n)})
\end{aligned}$$

If Y_n is to be of standard Gumbel distributed, we need $F_{Y_n}(y) = \exp(-ne^{-(y+a_n)}) = \exp(-e^{-y}) \Rightarrow a_n = \log n$.

This implies that $\{M_n\}$ grows in a rate of $\log n$.

In [10]:
```
x = np.linspace(-5, 10, 100)
fig, ax = plt.subplots(1, 1, figsize=(8,8))
std_gumbel_sampler = ProbIntegralTransformer(gumbel_inv(0, 1))
ax, sample = std_gumbel_sampler.attach_histogram(
    ax, 10000, 0.1, label="Histogram")
ax.plot(x, gumbel_pdf(x, 0, 1),
        linewidth=2, label="Std.Gumbel PDF")
ax.set_xlabel(r'$x$')
ax.set_ylabel(r'$f(x)$')
ax.set_title('Histogram of Simulated Std.Gumbel Varaible')
_ = ax.legend()
```



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2 Bond Rating Simulation

In [2]:
```
Q = np.array([
        [-.1154, .1019, .0083, .0020, .0031, 0, 0, 0],
        [.0091, -.1043, .0787, .0105, .0030, .0030, 0, 0],
        [.0010, .0309, -.1172, .0688, .0107, .0048, 0, .0010],
        [.0007, .0047, .0713, -.1711, .0701, .0174, .0020, .0049],
        [.0005, .0025, .0089, .0813, -.2530, .1181, .0144, .0273],
        [0, .0021, .0034, .0073, .0568, -.1928, .0479, .0753],
        [0,0,.0142, .0142, .0250, .0928, -.4318, .2856],
        [0, 0, 0, 0, 0, 0, 0, 0]
    ])
```

```
M = np.array([
    [0,.8838, .0720, .0173, .0269, 0, 0, 0],
    [.0872, 0, .7545, .1007, .0288, .0288, 0,0],
    [.0085, .2637, 0, .5870, .0913, .0410, 0, .0085],
    [.0041, .0275, .4167, 0, .4097, .1017, .0117, .0286],
    [.0020, .0099, .0352, .3213, 0, .4668, .0569, .1079],
    [0, .0109, .0176, .0379, .2946, 0, .2484, .3906],
    [0, 0, .0329, .0329, .0579, .2149, 0, .6614],
    [0,0,0,0,0,0,0,1]
])
```

```
In [3]: class MarkovAliasTable(object):
            def __init__(self, transition_matrix):
                self.n = len(transition_matrix)
                self.transition_matrix = transition_matrix
                table_dim = (self.n, self.n-1)
                self.prob = np.zeros(table_dim)
                self.values = np.zeros(table_dim, dtype=np.int32)
                self.alias = np.zeros(table_dim, dtype=np.int32)
                self.__make_table()

            def __make_table(self):
                for i, row in enumerate(self.transition_matrix):
                    pmf = sorted([list(t) for t in zip(range(self.n), (
                        self.n-1)*row)], key=lambda t:-t[1])
                    for j in range(self.n-1):
                        (state, p), alias = pmf[-1], pmf[0][0]
                        self.prob[i,j] = p
                        self.values[i,j] = state
                        self.alias[i,j] = alias
                        pmf[0][1] -= 1-p
                        pmf.pop()
                        pmf = sorted(pmf, key=lambda t:-t[1])

            def draw(self, init_state):
                u = np.random.uniform()
                v = (self.n-1)*u
                i = int(np.ceil(v))
                w = i-v
                if w<=self.prob[init_state,i-1]:
                    return self.values[init_state,i-1]
                else: return self.alias[init_state,i-1]

            def draw_path(self, clock_model, T, init_state):
                clock, state = 0, init_state
                while clock <= T:
                    dt = clock_model(state)
```

```
                clock += dt
                if clock >= T: break
                state = self.draw(state)
            return state

        def estimate_prob(self, clock_model, T, init_state, n_paths):
            est = np.zeros(self.n)
            for _ in range(n_paths):
                final = self.draw_path(clock_model, T, init_state)
                est[final] += 1
            est /= n_paths
            return est

        def estimate_trans_matrix(self, clock_model, T, n_paths):
            bar = ProgressBar()
            est = np.zeros((self.n, self.n))
            for i in bar(range(self.n)):
                est[i,:] = self.estimate_prob(
                    clock_model, T, i, n_paths)
            return est
```

```
In [4]: def exponential_holding_time(state):
            exp_rates = [.1154, .1043, .1172, .1711, .2530, .1929, .4318, .0001]
            return np.random.exponential(1/exp_rates[state])

        def gamma_holding_time(state):
            gamma_rates = [.1154, .1043, .1172, .1711, .2530, .1929, .4318, .0001]
            return 0.5*np.random.gamma(2,1/gamma_rates[state])
```

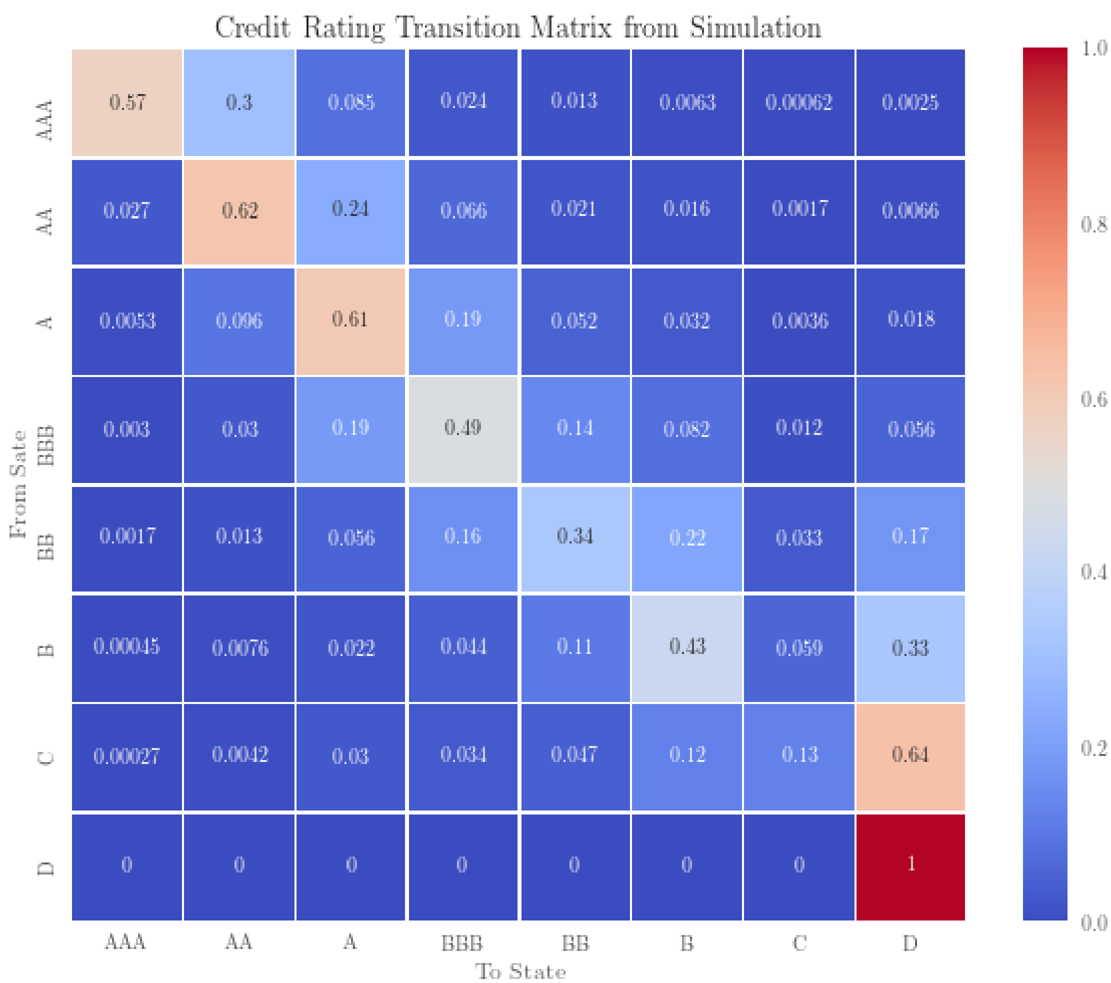
2.1 $\tau \sim$ Exponential(λ) Case

```
In [50]: T, n_paths = 5, 1000000
         credit_rating_names = ['AAA','AA','A','BBB','BB','B','C','D']
         credit_rating_simulator = MarkovAliasTable(M)
         P_hat = credit_rating_simulator.estimate_trans_matrix(
             exponential_holding_time, T, n_paths)
         df_p_hat = pd.DataFrame(
             P_hat, index=credit_rating_names,
             columns=credit_rating_names)
         df_p = pd.DataFrame(
             expm(Q*T), index=credit_rating_names,
             columns=credit_rating_names)
```

```
100% (8 of 8) |###########################| Elapsed Time: 0:01:08 Time: 0:01:08
```

```
In [43]: f, ax = plt.subplots(figsize=(10, 8))
         sns.heatmap(
             df_p_hat, annot=True,
             cmap="coolwarm",
             linewidths=.5, ax=ax)
         ax.set_xlabel('To State')
         ax.set_ylabel('From Sate')
         ax.set_title('Credit Rating Transition Matrix from Simulation')
```

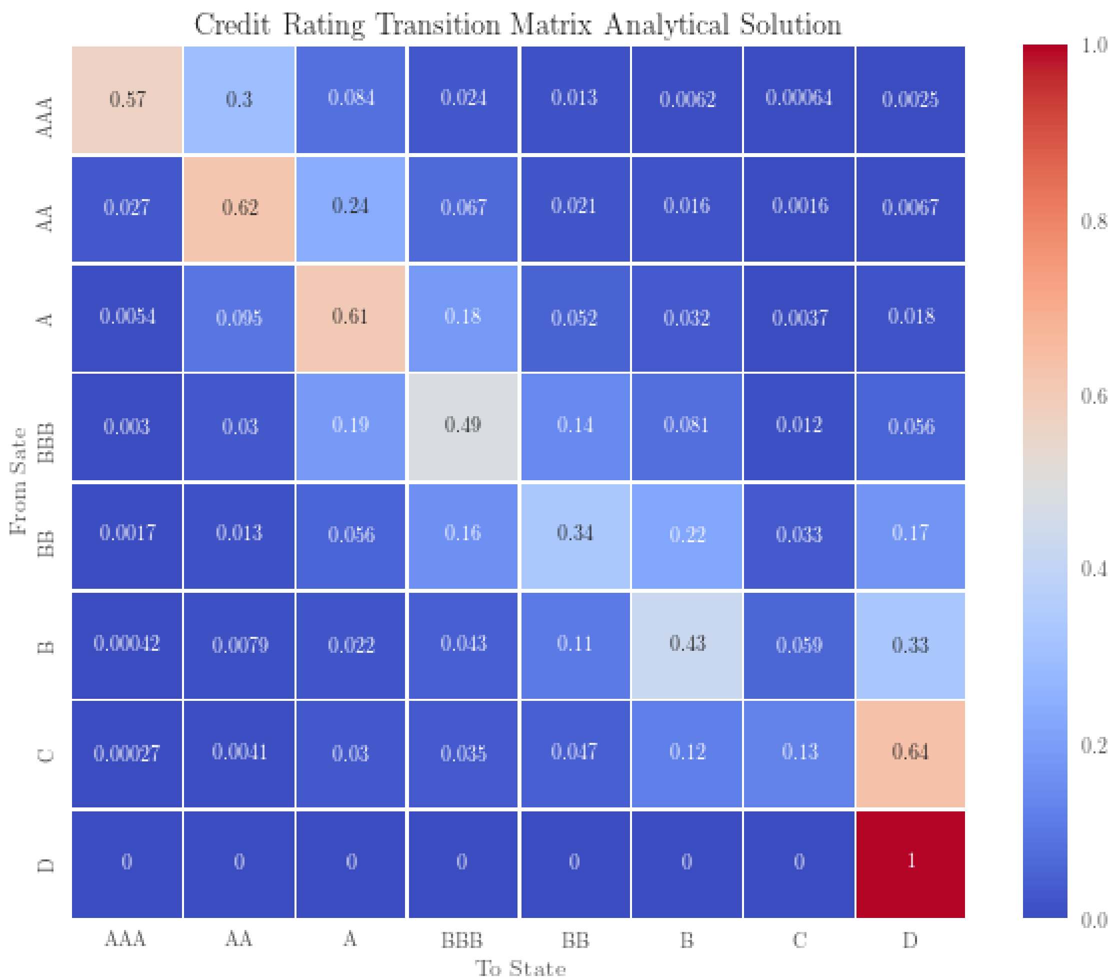
```
Out[43]: <matplotlib.text.Text at 0x114dc6320>
```




```
In [7]: f, ax = plt.subplots(figsize=(10, 8))
        sns.heatmap(
            df_p, annot=True,
            cmap="coolwarm",
            linewidths=.5, ax=ax)
        ax.set_xlabel('To State')
```

```
        ax.set_ylabel('From Sate')
        ax.set_title('Credit Rating Transition Matrix Analytical Solution')
```

Out[7]: <matplotlib.text.Text at 0x114411748>



In [51]:
```
# the exact solution calculated by exp(QT)
df_p
```

Out[51]:

	AAA	AA	A	BBB	BB	B	C \
AAA	0.568493	0.300650	0.084463	0.024180	0.012511	0.006203	0.000642
AA	0.027357	0.619661	0.240257	0.066931	0.021034	0.016447	0.001624
A	0.005355	0.095353	0.608999	0.184625	0.052265	0.031680	0.003656
BBB	0.003012	0.029545	0.190720	0.486208	0.141783	0.081408	0.011601
BB	0.001745	0.013086	0.056422	0.162734	0.338476	0.220439	0.033337
B	0.000421	0.007887	0.021583	0.043201	0.107411	0.430096	0.059091
C	0.000274	0.004136	0.029672	0.034582	0.046817	0.120885	0.127722
D	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

```
              D
AAA  0.002477
AA   0.006681
A    0.018066
BBB  0.055723
BB   0.173761
B    0.330310
C    0.635913
D    1.000000
```

In [52]:
```python
# The estimate via Monte Carlo simulation
df_p_hat
```

Out[52]:
```
          AAA        AA         A       BBB        BB         B         C  \
AAA  0.568884  0.300374  0.084725  0.024238  0.012518  0.006168  0.000661
AA   0.027497  0.619337  0.240777  0.066687  0.021087  0.016363  0.001625
A    0.005361  0.094974  0.609128  0.184794  0.052276  0.031665  0.003685
BBB  0.002882  0.029460  0.190791  0.486971  0.141442  0.081263  0.011550
BB   0.001735  0.013023  0.055963  0.163478  0.338321  0.219840  0.033213
B    0.000415  0.007838  0.021465  0.043599  0.107460  0.429637  0.058966
C    0.000250  0.004152  0.029692  0.034608  0.046714  0.120994  0.128291
D    0.000000  0.000000  0.000000  0.000000  0.000000  0.000000  0.000000

            D
AAA  0.002432
AA   0.006627
A    0.018117
BBB  0.055641
BB   0.174427
B    0.330620
C    0.635299
D    1.000000
```

In [53]:
```python
# Standard error
se_p_hat = np.sqrt(P_hat*(1-P_hat)/n_paths)
df_se_p_hat = pd.DataFrame(
    se_p_hat, index=credit_rating_names,
    columns=credit_rating_names)
df_se_p_hat
```

Out[53]:
```
          AAA        AA         A       BBB        BB         B         C  \
AAA  0.000495  0.000458  0.000278  0.000154  0.000111  0.000078  0.000026
AA   0.000164  0.000486  0.000428  0.000249  0.000144  0.000127  0.000040
A    0.000073  0.000293  0.000488  0.000388  0.000223  0.000175  0.000061
BBB  0.000054  0.000169  0.000393  0.000500  0.000348  0.000273  0.000107
BB   0.000042  0.000113  0.000230  0.000370  0.000473  0.000414  0.000179
B    0.000020  0.000088  0.000145  0.000204  0.000310  0.000495  0.000236
C    0.000016  0.000064  0.000170  0.000183  0.000211  0.000326  0.000334
```

```
D    0.000000  0.000000  0.000000  0.000000  0.000000  0.000000  0.000000

            D
AAA  0.000049
AA   0.000081
A    0.000133
BBB  0.000229
BB   0.000379
B    0.000470
C    0.000481
D    0.000000
```

In [54]:
```
# sanity check
# It appears that most of the absolute differences between
# exact and simulated P matrix are smaller 2 standard errors.
np.abs(P_hat - expm(Q*T)) <=2* se_p_hat
```

Out[54]:
```
array([[ True,  True,  True,  True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True],
       [False,  True,  True,  True,  True,  True,  True,  True],
       [ True,  True,  True, False,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True],
       [ True,  True,  True,  True,  True,  True,  True,  True]], dtype=bool)
```

2.2 $\tau \sim \frac{1}{2}\Gamma(2, \lambda_i)$ Case

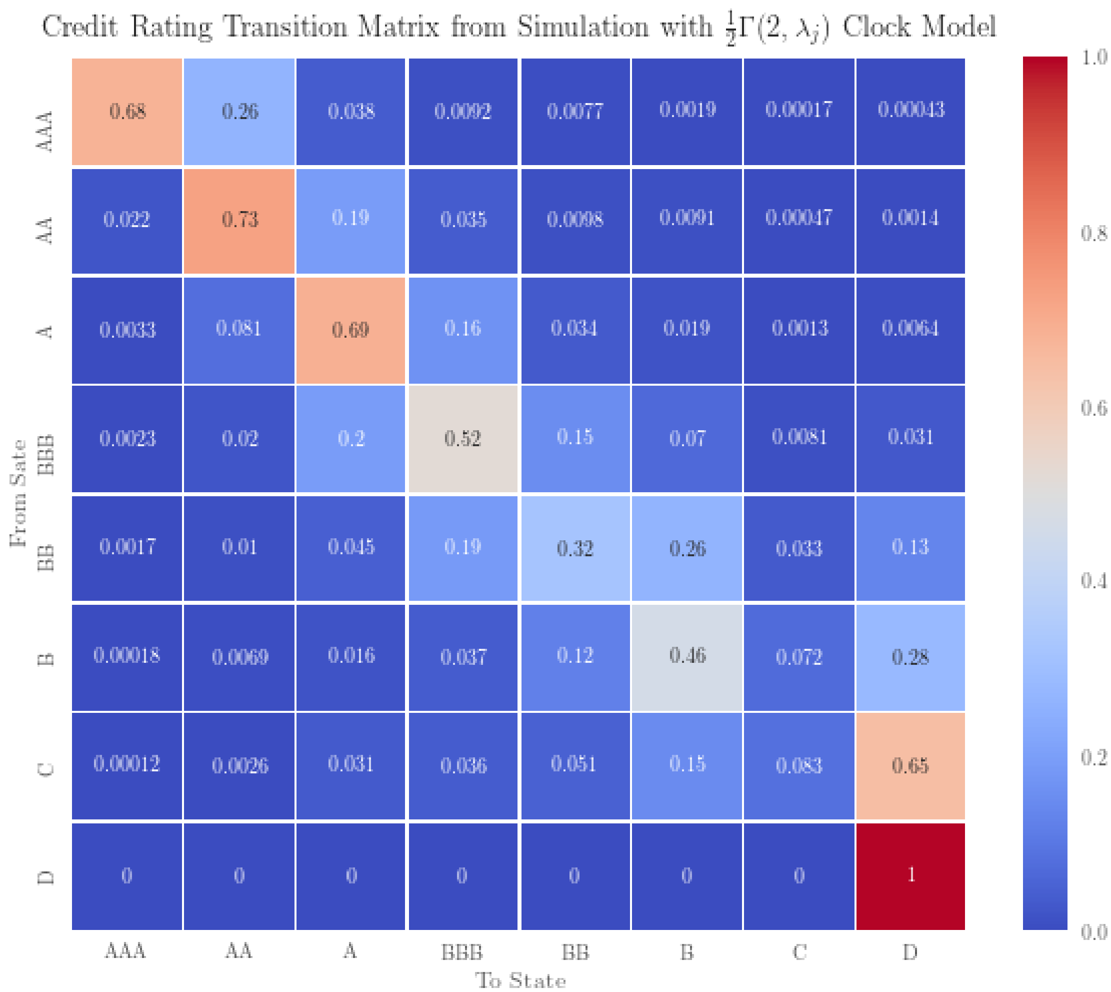
In [38]:
```
T = 5
credit_rating_names = ['AAA','AA','A','BBB','BB','B','C','D']
credit_rating_simulator = MarkovAliasTable(M)
P_hat_gamma = credit_rating_simulator.estimate_trans_matrix(
    gamma_holding_time, T, n_paths)
df_p_hat_gamma = pd.DataFrame(
    P_hat_gamma, index=credit_rating_names,
    columns=credit_rating_names)
```

```
100% (8 of 8) |###########################| Elapsed Time: 0:01:18 Time: 0:01:18
```

In [39]:
```
f, ax = plt.subplots(figsize=(10, 8))
sns.heatmap(
    df_p_hat_gamma, annot=True,
    cmap="coolwarm",
    linewidths=.5, ax=ax)
```

```
ax.set_xlabel('To State')
ax.set_ylabel('From Sate')
ax.set_title('Credit Rating Transition Matrix from Simulation '
             +r'with $\frac{1}{2}\Gamma(2,\lambda_j)$ Clock Model')
```

Out[39]: <matplotlib.text.Text at 0x114b04be0>



In [40]:
```
# The estimate via Monte Carlo simulation
df_p_hat_gamma
```

Out[40]:
```
          AAA        AA         A       BBB        BB         B         C  \
AAA  0.680262  0.262295  0.038076  0.009162  0.007715  0.001891  0.000168
AA   0.022481  0.727664  0.193886  0.035217  0.009832  0.009075  0.000467
A    0.003319  0.081349  0.690421  0.164152  0.034408  0.018589  0.001320
BBB  0.002329  0.019638  0.195105  0.520959  0.153372  0.069866  0.008118
BB   0.001662  0.009990  0.045198  0.190208  0.322348  0.264006  0.033377
B    0.000180  0.006854  0.016250  0.036910  0.123126  0.463220  0.071753
```

C	0.000124	0.002552	0.031314	0.035601	0.050524	0.149445	0.082540
D	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

	D
AAA	0.000431
AA	0.001378
A	0.006442
BBB	0.030613
BB	0.133211
B	0.281707
C	0.647900
D	1.000000

We can see that with $\tau \sim 0.5\Gamma(2, \lambda_j)$, the diagonal of the transition matrix has more probability mass, which means that the credit rating is more "stable" than the case in which we employ the exponentential clock model.

In [56]:
```
# The standard error
se_p_hat_gamma = np.sqrt(P_hat_gamma*(1-P_hat_gamma)/n_paths)
df_se_p_hat_gamma = pd.DataFrame(
    se_p_hat_gamma, index=credit_rating_names,
    columns=credit_rating_names)
df_se_p_hat_gamma
```

Out[56]:

	AAA	AA	A	BBB	BB	B	C
AAA	0.000466	0.000440	0.000191	0.000095	0.000087	0.000043	0.000013
AA	0.000148	0.000445	0.000395	0.000184	0.000099	0.000095	0.000022
A	0.000058	0.000273	0.000462	0.000370	0.000182	0.000135	0.000036
BBB	0.000048	0.000139	0.000396	0.000500	0.000360	0.000255	0.000090
BB	0.000041	0.000099	0.000208	0.000392	0.000467	0.000441	0.000180
B	0.000013	0.000083	0.000126	0.000189	0.000329	0.000499	0.000258
C	0.000011	0.000050	0.000174	0.000185	0.000219	0.000357	0.000275
D	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

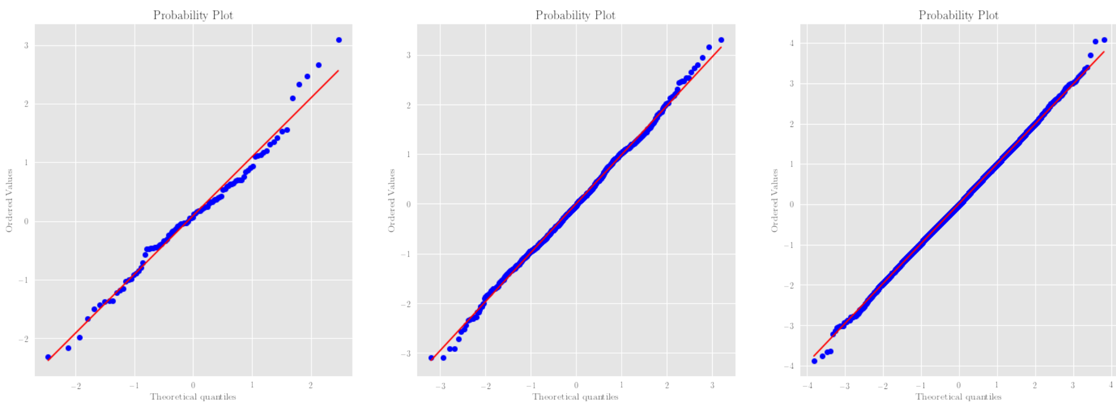
	D
AAA	0.000021
AA	0.000037
A	0.000080
BBB	0.000172
BB	0.000340
B	0.000450
C	0.000478
D	0.000000

3 Gaussian RV Generator

```
In [290]: def poor_mans(size):
              return np.array([sum(np.random.uniform(
                  size=12))-6 for _ in range(size)])
```

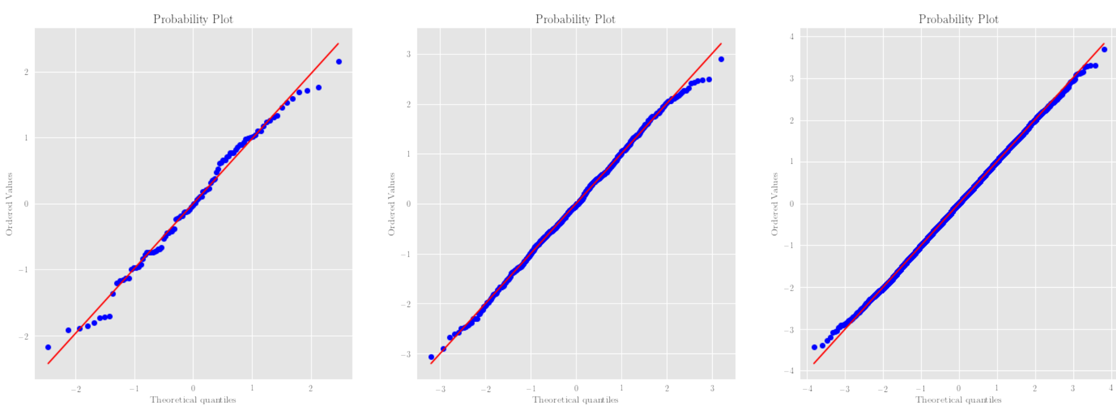
3.1 Python's Normal Generator

```
In [295]: f, axes = plt.subplots(1,3,figsize=(24, 8))
          for j in range(3):
              normal_sample = np.random.normal(size=10**(j+2))
              qq = stats.probplot(
                  normal_sample, dist="norm", plot=axes[j])
```



3.2 "Poor man's" Normal Generator

```
In [297]: f, axes = plt.subplots(1,3,figsize=(24, 8))
          for j in range(3):
              normal_sample = poor_mans(size=10**(j+2))
              qq = stats.probplot(
                  normal_sample, dist="norm", plot=axes[j])
```



Poor man's normal generator actually seems to do a good job.

```
In [ ]:
```