Optimization Assignment 6

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Problem 1. (a), (b): The optimal solution, amount of money (in dollar amount, not weights) invested in risky asset at each stage, are shown below.

Variables		
x0	x1	x2
		1125.03536
	750.007517	
		562.492835
500.001619		
		562.50185
	375.001575	
		281.240608

It turns out that the optimal solutions are the same for question **(a)** and **(b)**. In both cases, the expected utility (value of objective function) is 5.475.

Problem 2. (a) Take first order condition:

$$\boldsymbol{V}\boldsymbol{x} + \boldsymbol{R}(\boldsymbol{x} - \boldsymbol{x}_0) = 0 \quad \Rightarrow \quad \boldsymbol{x}^* = (\boldsymbol{V} + \boldsymbol{R})^{-1}\boldsymbol{R}\boldsymbol{x}_0 \tag{1}$$

(b-i) The value to go function at last stage T is

$$J_T(\boldsymbol{x}_{T-1}) = \min_{\boldsymbol{x}_T} \ \frac{1}{2}\boldsymbol{x}_T^\top \boldsymbol{V}\boldsymbol{x}_T + \frac{1}{2}(\boldsymbol{x}_T - \boldsymbol{x}_{T-1})^\top \boldsymbol{R}(\boldsymbol{x}_T - \boldsymbol{x}_{T-1}) \tag{2}$$

Which has exactly the same form as the one-stage problem in (a). So the solution is

$$
\begin{aligned}
\boldsymbol{x}_T^* &= (\boldsymbol{V}+\boldsymbol{R})^{-1}\boldsymbol{R}\boldsymbol{x}_{T-1} \\
J_T(\boldsymbol{x}_{T-1}) &= \frac{1}{2}\boldsymbol{x}_{T-1}^\top\Big[\boldsymbol{R}(\boldsymbol{V}+\boldsymbol{R})^{-1}\boldsymbol{V}(\boldsymbol{V}+\boldsymbol{R})^{-1}\boldsymbol{R}+\boldsymbol{R}(\boldsymbol{V}+\boldsymbol{R})^{-1}\boldsymbol{R}(\boldsymbol{V}+\boldsymbol{R})^{-1}\boldsymbol{R} \\
&\quad +\boldsymbol{R}-2\boldsymbol{R}(\boldsymbol{V}+\boldsymbol{R})^{-1}\boldsymbol{R}\Big]\boldsymbol{x}_{T-1} \\
&= \frac{1}{2}\boldsymbol{x}_{T-1}^\top\Big[\boldsymbol{R}(\boldsymbol{V}+\boldsymbol{R})^{-1}\boldsymbol{R}+\boldsymbol{R}-2\boldsymbol{R}(\boldsymbol{V}+\boldsymbol{R})^{-1}\boldsymbol{R}\Big]\boldsymbol{x}_{T-1} \\
&= \frac{1}{2}\boldsymbol{x}_{T-1}^\top\Big[\boldsymbol{R}-\boldsymbol{R}(\boldsymbol{V}+\boldsymbol{R})^{-1}\boldsymbol{R}\Big]\boldsymbol{x}_{T-1}
\end{aligned}
\tag{3}
$$

(b-ii) Ansatz: $\boldsymbol{x}_{t+1}^* = \boldsymbol{L}_{t+1}\boldsymbol{x}_t$, $J_{t+1}(\boldsymbol{x}_t) = \frac{1}{2}\boldsymbol{x}_t^\top\boldsymbol{K}_{t+1}\boldsymbol{x}_t$ for $t = 1, ..., T-1$. Then the Bellman equation at t is:

$$
J_t(\boldsymbol{x}_{t-1}) = \min_{\boldsymbol{x}_t} \;\; \frac{1}{2}\boldsymbol{x}_t^\top\boldsymbol{V}\boldsymbol{x}_t + \frac{1}{2}(\boldsymbol{x}_t-\boldsymbol{x}_{t-1})^\top\boldsymbol{R}(\boldsymbol{x}_t-\boldsymbol{x}_{t-1}) + \frac{1}{2}\boldsymbol{x}_t^\top\boldsymbol{K}_{t+1}\boldsymbol{x}_t \tag{4}
$$

First order condition yields

$$
\boldsymbol{V}\boldsymbol{x}_t + \boldsymbol{R}(\boldsymbol{x}_t-\boldsymbol{x}_{t-1}) + \boldsymbol{K}_{t+1}\boldsymbol{x}_t = 0 \tag{5}
$$

We have:

$$
\begin{aligned}
\boldsymbol{x}_t^* &= (\boldsymbol{V}+\boldsymbol{R}+\boldsymbol{K}_{t+1})^{-1}\boldsymbol{R}\boldsymbol{x}_{t-1} \\
J_t(\boldsymbol{x}_{t-1}) &= \frac{1}{2}\boldsymbol{x}_{t-1}^\top\Big[\boldsymbol{R}(\boldsymbol{V}+\boldsymbol{R}+\boldsymbol{K}_{t+1})^{-1}\boldsymbol{V}(\boldsymbol{V}+\boldsymbol{R}+\boldsymbol{K}_{t+1})^{-1}\boldsymbol{R} \\
&\quad +\boldsymbol{R}(\boldsymbol{V}+\boldsymbol{R}+\boldsymbol{K}_{t+1})^{-1}\boldsymbol{R}(\boldsymbol{V}+\boldsymbol{R}+\boldsymbol{K}_{t+1})^{-1}\boldsymbol{R} \\
&\quad +\boldsymbol{R}(\boldsymbol{V}+\boldsymbol{R}+\boldsymbol{K}_{t+1})^{-1}\boldsymbol{K}_{t+1}(\boldsymbol{V}+\boldsymbol{R}+\boldsymbol{K}_{t+1})^{-1}\boldsymbol{R} \\
&\quad +\boldsymbol{R}-2\boldsymbol{R}(\boldsymbol{V}+\boldsymbol{R}+\boldsymbol{K}_{t+1})^{-1}\boldsymbol{R}\Big]\boldsymbol{x}_{t-1} \\
&= \frac{1}{2}\boldsymbol{x}_{t-1}^\top\Big[\boldsymbol{R}(\boldsymbol{V}+\boldsymbol{R}+\boldsymbol{K}_{t+1})^{-1}\boldsymbol{R}+\boldsymbol{R}-2\boldsymbol{R}(\boldsymbol{V}+\boldsymbol{R}+\boldsymbol{K}_{t+1})^{-1}\boldsymbol{R}\Big]\boldsymbol{x}_{t-1} \\
&= \frac{1}{2}\boldsymbol{x}_{t-1}^\top\Big[\boldsymbol{R}-\boldsymbol{R}(\boldsymbol{V}+\boldsymbol{R}+\boldsymbol{K}_{t+1})^{-1}\boldsymbol{R}\Big]\boldsymbol{x}_{t-1}
\end{aligned}
\tag{6}
$$

That is:

$$
\boldsymbol{L}_t = (\boldsymbol{V}+\boldsymbol{R}+\boldsymbol{K}_{t+1})^{-1}\boldsymbol{R}; \qquad \boldsymbol{K}_t = \boldsymbol{R}-\boldsymbol{R}(\boldsymbol{V}+\boldsymbol{R}+\boldsymbol{K}_{t+1})^{-1}\boldsymbol{R} \tag{7}
$$

(c) Bellman equation at t is:

$$
V(\boldsymbol{x}_{t-1}) = \min_{\boldsymbol{x}_t} \;\; \frac{q}{2}\boldsymbol{x}_t^\top\boldsymbol{x}_t + \frac{r}{2}(\boldsymbol{x}_t-\boldsymbol{x}_{t-1})^\top(\boldsymbol{x}_t-\boldsymbol{x}_{t-1}) + \frac{k}{2}\boldsymbol{x}_t^\top\boldsymbol{x}_t \tag{8}
$$

First order condition yields

$$
q\boldsymbol{x}_t + r(\boldsymbol{x}_t-\boldsymbol{x}_{t-1}) + k\boldsymbol{x}_t = 0 \tag{9}
$$

It suffices to solve the scalar analog to **(b)**:

$$
\begin{aligned}
\boldsymbol{x}_t^* &= \frac{r}{q+r+k}\boldsymbol{x}_{t-1} \\
V(\boldsymbol{x}_{t-1}) &= \frac{1}{2}\left(r-\frac{r^2}{q+r+k}\right)\boldsymbol{x}_{t-1}^\top\boldsymbol{x}_{t-1}
\end{aligned}
\tag{10}
$$

That is:

$$
k = r - \frac{r^2}{q+r+k} \quad \Rightarrow \quad k^2 + kq - rq = 0 \tag{11}
$$

$$
k = \frac{-q \pm \sqrt{q^2+4rq}}{2} \tag{12}
$$