Asset Management HW3

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1 Problem 1

1.1 (a)

The Lagrangian of agent i's problem is

$$\mathcal{L}_i(\boldsymbol{x}, \lambda) = (\mathbb{E}\left[\boldsymbol{P}^1\right] - \boldsymbol{P}^0)^\top \boldsymbol{x} - \frac{\gamma_i}{2}\boldsymbol{x}^\top \boldsymbol{\Omega} \boldsymbol{x} - \lambda(m_i(\boldsymbol{P}^0)^\top \boldsymbol{x} - W_i) \tag{1}$$

$\boldsymbol{x}^i, \lambda_i$ solves the KKT condition:

$$\begin{cases} \nabla_{\boldsymbol{x}}\mathcal{L} = (\mathbb{E}\left[\boldsymbol{P}^1\right] - \boldsymbol{P}^0) - \gamma_i \boldsymbol{\Omega}\boldsymbol{x} - \lambda_i m_i \boldsymbol{P}^0 = \boldsymbol{0} \\ m_i(\boldsymbol{P}^0)^\top \boldsymbol{x} - W_i \le 0 \\ \lambda_i \ge 0 \\ \lambda_i(m_i(\boldsymbol{P}^0)^\top \boldsymbol{x} - W_i) = 0 \end{cases} \tag{2}$$

The first equality yields $\boldsymbol{x}^i = \frac{1}{\gamma_i}\boldsymbol{\Omega}^{-1}\left(\mathbb{E}\left[\boldsymbol{P}^1\right] - (1+\lambda_i m_i)\boldsymbol{P}^0\right)$. Assume the constraint is binding, then

$$\begin{aligned} & 0 = m_i(\boldsymbol{P}^0)^\top \boldsymbol{x}^i - W_i = \frac{1}{\gamma_i} m_i (\boldsymbol{P}^0)^\top \boldsymbol{\Omega}^{-1}\left(\mathbb{E}\left[\boldsymbol{P}^1\right] - (1+\lambda_i m_i)\boldsymbol{P}^0\right) - W_i \\ \Rightarrow\quad & \frac{\lambda_i}{\gamma_i} m_i^2 (\boldsymbol{P}^0)^\top \boldsymbol{\Omega}^{-1}\boldsymbol{P}^0 = \frac{1}{\gamma_i} m_i(\boldsymbol{P}^0)^\top \boldsymbol{\Omega}^{-1}\left(\mathbb{E}\left[\boldsymbol{P}^1\right] - \boldsymbol{P}^0\right) - W_i \\ \Rightarrow\quad & \lambda_i = \frac{m_i(\boldsymbol{P}^0)^\top \boldsymbol{\Omega}^{-1}\left(\mathbb{E}\left[\boldsymbol{P}^1\right] - \boldsymbol{P}^0\right) - \gamma_i W_i}{m_i^2 (\boldsymbol{P}^0)^\top \boldsymbol{\Omega}^{-1}\boldsymbol{P}^0} \end{aligned} \tag{3}$$

Dual feasibility requires $\lambda_i \ge 0 \Rightarrow m_i(\boldsymbol{P}^0)^\top \boldsymbol{\Omega}^{-1}\left(\mathbb{E}\left[\boldsymbol{P}^1\right] - \boldsymbol{P}^0\right) - \gamma_i W_i \ge 0 \;\; (\dagger)$. If instead we have $(\dagger) < 0$, we have $\lambda_i = 0$. Then $\boldsymbol{x}^i = \frac{1}{\gamma_i}\boldsymbol{\Omega}^{-1}\left(\mathbb{E}\left[\boldsymbol{P}^1\right] - \boldsymbol{P}^0\right)$, hence $m_i(\boldsymbol{P}^0)^\top \boldsymbol{x}^i - W_i = \frac{1}{\gamma_i} m_i(\boldsymbol{P}^0)^\top \boldsymbol{\Omega}^{-1}\left(\mathbb{E}\left[\boldsymbol{P}^1\right] - \boldsymbol{P}^0\right) - W_i < 0$ as the risk aversion coefficient $\gamma_i > 0$. The primal feasibility is satisfied. To conclude:

$$\begin{aligned} & \boldsymbol{x}^i = \frac{1}{\gamma_i}\boldsymbol{\Omega}^{-1}\left(\mathbb{E}\left[\boldsymbol{P}^1\right] - (1+\psi_i)\boldsymbol{P}^0\right) \\ & \text{where } \psi_i = \begin{cases} \frac{m_i(\boldsymbol{P}^0)^\top \boldsymbol{\Omega}^{-1}\left(\mathbb{E}\left[\boldsymbol{P}^1\right] - \boldsymbol{P}^0\right) - \gamma_i W_i}{m_i (\boldsymbol{P}^0)^\top \boldsymbol{\Omega}^{-1}\boldsymbol{P}^0} & \text{if } m_i(\boldsymbol{P}^0)^\top \boldsymbol{\Omega}^{-1}\left(\mathbb{E}\left[\boldsymbol{P}^1\right] - \boldsymbol{P}^0\right) - \gamma_i W_i \ge 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{4}$$

Consequently

$$\begin{aligned} \boldsymbol{x}^* &= \sum_{i=1}^I \boldsymbol{x}_i = \sum_{i=1}^I \frac{1}{\gamma_i}\boldsymbol{\Omega}^{-1}\left(\mathbb{E}\left[\boldsymbol{P}^1\right] - (1+\psi_i)\boldsymbol{P}^0\right) \\ &= \frac{1}{\gamma}\boldsymbol{\Omega}^{-1}\left(\mathbb{E}\left[\boldsymbol{P}^1\right] - (1+\psi)\boldsymbol{P}^0\right) \\ & \text{where } \gamma = \frac{1}{\sum_{i=1}^I 1/\gamma_i}; \quad \psi = \frac{\sum_{i=1}^I \psi_i/\gamma_i}{\sum_{i=1}^I 1/\gamma_i} \end{aligned} \tag{5}$$

1.2 (b)

Premultiply both sides of (5) by $\gamma\boldsymbol{\Omega}$:

$$\begin{aligned} & \gamma\boldsymbol{\Omega}\boldsymbol{x}^* = \mathbb{E}\left[\boldsymbol{P}^1\right] - (1+\psi)\boldsymbol{P}^0 \\ \Rightarrow\quad & \mathbb{E}\left[\boldsymbol{P}^1\right] = (1+\psi)\boldsymbol{P}^0 + \gamma\boldsymbol{\Omega}\boldsymbol{x}^* \\ \Rightarrow\quad & \frac{\mathbb{E}\left[P_s^1\right] - P_s^0}{P_s^0} = \psi + \frac{\gamma}{P_s^0}(\boldsymbol{\Omega}\boldsymbol{x}^*)_s \end{aligned} \tag{6}$$

Since P_s^0 is non-random, we conclude that $\mathbb{E}\left[r_s\right] = \frac{\mathbb{E}\left[P_s^1\right]-P_s^0}{P_s^0} = \psi + \frac{\gamma}{P_s^0}(\boldsymbol{\Omega}\boldsymbol{x}^*)_s$.

1.3 (c)

By definition of $\boldsymbol{h}$, we know $\boldsymbol{h}^\top\boldsymbol{1} = 1$. Let $\mu_s := (\boldsymbol{\Omega}\boldsymbol{x}^*)_s/P_s^0 = \sum_{j=1}^S \Omega_{sj}x_j^*/P_s^0$ for $s = 1, ..., S$.

$$\begin{aligned} \frac{\mathbb{E}\left[r_M\right]-\psi}{\mathbb{V}\mathrm{ar}\left[r_M\right]} &= \frac{\mathbb{E}\left[\boldsymbol{r}^\top\boldsymbol{h}\right]-\psi}{\mathbb{V}\mathrm{ar}\left[\boldsymbol{r}^\top\boldsymbol{h}\right]} = \frac{\boldsymbol{h}^\top\mathbb{E}\left[\boldsymbol{r}\right]-\psi}{\boldsymbol{h}^\top\mathbb{C}\mathrm{ov}\left[\boldsymbol{r}\right]\boldsymbol{h}} \\ &= \frac{\boldsymbol{h}^\top(\psi\boldsymbol{1}+\gamma\boldsymbol{\mu})-\psi}{\boldsymbol{h}^\top\mathbb{C}\mathrm{ov}\left[\boldsymbol{r}\right]\boldsymbol{h}} = \frac{\gamma\boldsymbol{h}^\top\boldsymbol{\mu}+\psi\overbrace{(\boldsymbol{h}^\top\boldsymbol{1}-1)}^{0}}{\boldsymbol{h}^\top\boldsymbol{V}\boldsymbol{h}} \\ &= \frac{\gamma\sum_{i=1}^S h_i\mu_i}{\sum_{i=1}^S\sum_{j=1}^S h_iV_{ij}h_j} = \gamma\frac{\sum_{i=1}^S\left(\frac{P_i^0x_i^*}{(\boldsymbol{P}^0)^\top\boldsymbol{x}^*}\right)\left(\sum_{j=1}^S\Omega_{ij}x_j^*\frac{1}{P_i^0}\right)}{\sum_{i=1}^S\sum_{j=1}^S\frac{P_i^0x_i^*}{(\boldsymbol{P}^0)^\top\boldsymbol{x}^*}V_{ij}\frac{P_j^0x_j^*}{(\boldsymbol{P}^0)^\top\boldsymbol{x}^*}} \\ &= \gamma(\boldsymbol{P}^0)^\top\boldsymbol{x}^*\frac{\sum_{i=1}^S\sum_{j=1}^S {\color{red}P_i^0}{\color{blue}x_i^*P_i^0V_{ij}P_j^0x_j^*}{\color{red}\frac{1}{P_i^0}}}{\sum_{i=1}^S\sum_{j=1}^S {\color{blue}x_i^*P_i^0V_{ij}P_j^0x_j^*}} \\ &= \gamma(\boldsymbol{P}^0)^\top\boldsymbol{x}^* \end{aligned} \tag{7}$$

1.4 (d)

Use the results of **(c)**:

$$\begin{aligned} RHS = \psi+\beta+s(\mathbb{E}\left[r_M\right]-\psi) &= \psi+\mathbb{C}\mathrm{ov}\left[r_s,r_M\right]\frac{\mathbb{E}\left[r_M\right]-\psi}{\mathbb{V}\mathrm{ar}\left[r_M\right]} \\ &= \psi+\mathbb{C}\mathrm{ov}[r_s,\Sigma_{j=1}^S h_jr_j]\gamma(\boldsymbol{P}^0)^\top\boldsymbol{x}^* = \psi+\gamma(\boldsymbol{P}^0)^\top\boldsymbol{x}^*\sum_{j=1}^S V_{sj}h_j \\ &= \psi+\gamma(\boldsymbol{P}^0)^\top\boldsymbol{x}^*\sum_{j=1}^S\frac{\Omega_{sj}}{P_s^0P_j^0}\frac{P_j^0x_j^*}{(\boldsymbol{P}^0)^\top\boldsymbol{x}^*} \\ &= \psi+\gamma\sum_{j=1}^S\frac{\Omega_{sj}x_j^*}{P_s^0} = \psi+\gamma\frac{1}{P_s^0}(\boldsymbol{\Omega}\boldsymbol{x}^*)_s = \mathbb{E}\left[r_s\right] \quad \text{(By the result of } \textbf{(b)}\text{)} \end{aligned} \tag{8}$$

Which completes the proof.

2 Problem 2

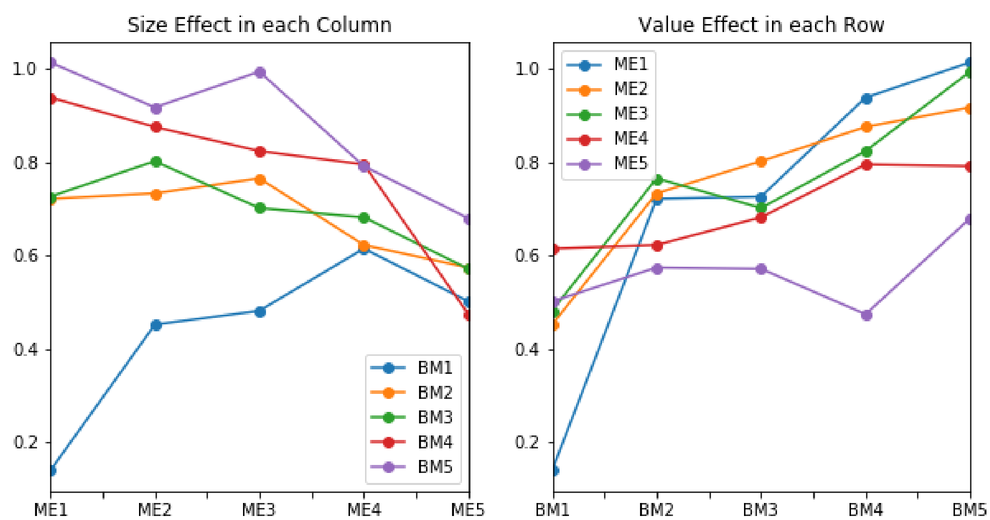
Unit for returns: percent, the same as what was used in Frenchs Data Library

2.1 (a)

Out[72]:

		BM1	BM2	BM3	BM4	BM5
ME1	mean	0.138941	0.721294	0.725931	0.938094	1.013257
	se	0.322117	0.280844	0.242438	0.229816	0.243004
	t-stat	0.431337	2.568314	2.994295	4.081942	4.169705
ME2	mean	0.451483	0.732897	0.801649	0.875069	0.916870
	se	0.293281	0.244021	0.220284	0.211684	0.244061
	t-stat	1.539423	3.003419	3.639155	4.133851	3.756730
ME3	mean	0.481265	0.765048	0.701462	0.823439	0.993619
	se	0.269862	0.221977	0.203838	0.198656	0.229668
	t-stat	1.783373	3.446511	3.441274	4.145044	4.326334
ME4	mean	0.614499	0.622239	0.681545	0.794719	0.791070
	se	0.242311	0.210575	0.202576	0.193848	0.230790
	t-stat	2.535990	2.954949	3.364389	4.099694	3.427656
ME5	mean	0.501318	0.573863	0.571117	0.473859	0.679433
	se	0.189887	0.181751	0.175723	0.189212	0.219410
	t-stat	2.640088	3.157413	3.250092	2.504379	3.096636

2.2 (b)



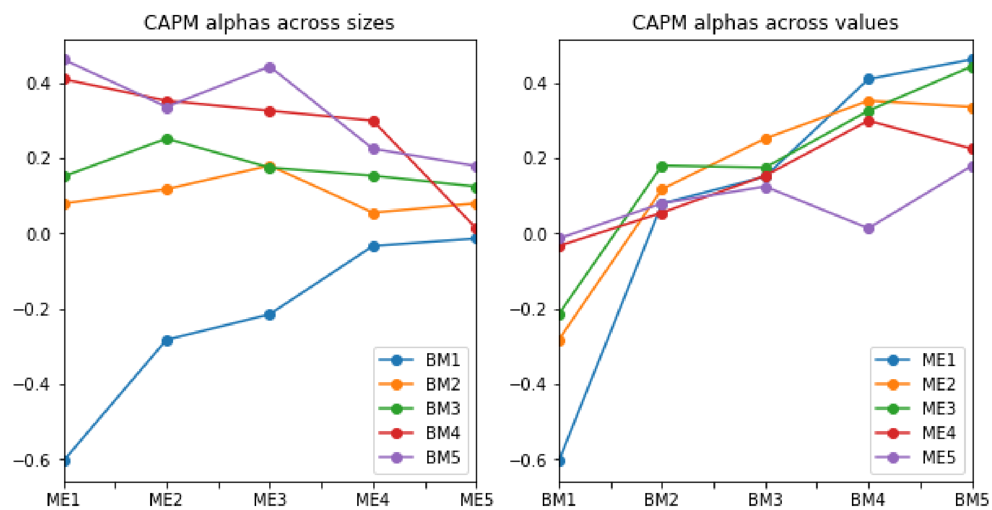
- For the same BM quantile group (above BM2), we can observe a size effect across different size groups, in the sense that the excess return increases as size group goes from BIG to SMALL. (Figure 1)
- The size effect described above is not evident for LoBM(BM1) group: the excess returns do not increase monotonically.
- For the same size quantile group, we can observe a value effect across different BM groups, in the sense that the excess return increases as BM group goes from LowBM to HighBM. (Figure 2)
- The value effect is stronger in the SMALL(ME1) size group, as the blue line in figure 2 has the steepest slope.

2.3 (c)

Out[69]:

		BM1	BM2	BM3	BM4	BM5
ME1	alpha	-0.604325	0.0792367	0.151235	0.410014	0.462256
	t-stat	-3.08398	0.456519	1.079	2.91167	3.03706
	p<0.05	(*)			(*)	(*)
ME2	alpha	-0.28214	0.117712	0.252293	0.352649	0.336371

	t-stat	-1.8987	0.975059	2.24046	3.1656	2.40059
	p<0.05			(*)	(*)	(*)
ME3	alpha	-0.214433	0.180408	0.17511	0.326279	0.443554
	t-stat	-1.73615	1.94766	1.88901	3.24615	3.41398
	p<0.05				(*)	(*)
ME4	alpha	-0.033038	0.0555156	0.153521	0.299895	0.225168
	t-stat	-0.352906	0.71254	1.73193	3.25954	1.82287
	p<0.05				(*)	
ME5	alpha	-0.0130529	0.0802354	0.124674	0.0141237	0.179359
	t-stat	-0.194115	1.27065	1.47073	0.136355	1.31604
	p<0.05					



2.4 (d)

Out[68]:

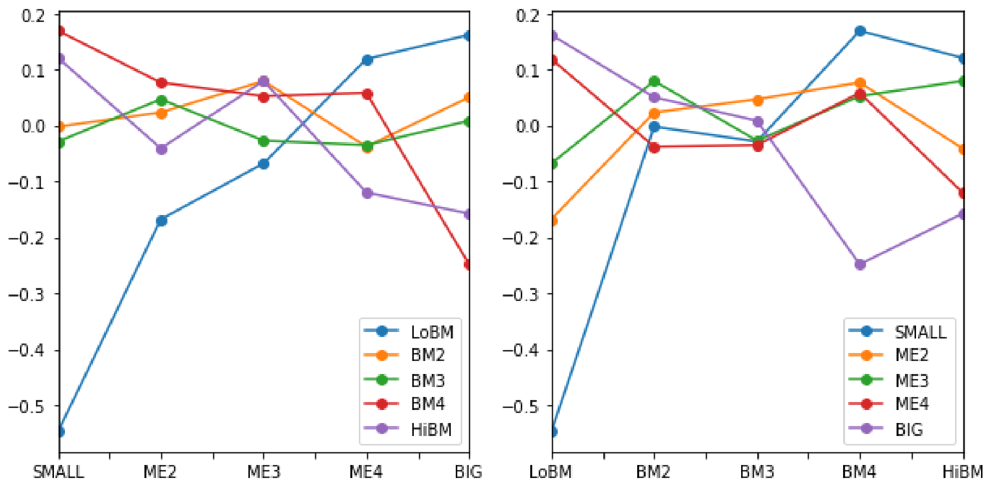
		BM1	BM2	BM3	BM4	BM5
ME1	alpha	-0.547419	-0.00188453	-0.0285962	0.169426	0.121075
	t-stat	-5.62311	-0.0260974	-0.523543	3.10497	2.11213
	p<0.05	(*)			(*)	(*)
ME2	alpha	-0.167721	0.0232988	0.0469446	0.0768339	-0.0407018
	t-stat	-2.46079	0.395727	0.793127	1.48421	-0.733885
	p<0.05	(*)				
ME3	alpha	-0.0678117	0.0795249	-0.0269168	0.0524966	0.079584
	t-stat	-1.09019	1.19467	-0.405653	0.817183	1.015
	p<0.05					
ME4	alpha	0.11902	-0.0376912	-0.0348759	0.0584575	-0.119813
	t-stat	1.91318	-0.521096	-0.468031	0.859294	-1.39163
	p<0.05					
ME5	alpha	0.162147	0.0504822	0.00859773	-0.248215	-0.157442
	t-stat	3.50773	0.881661	0.123204	-3.80089	-1.57558
	p<0.05	(*)			(*)	

Comments:

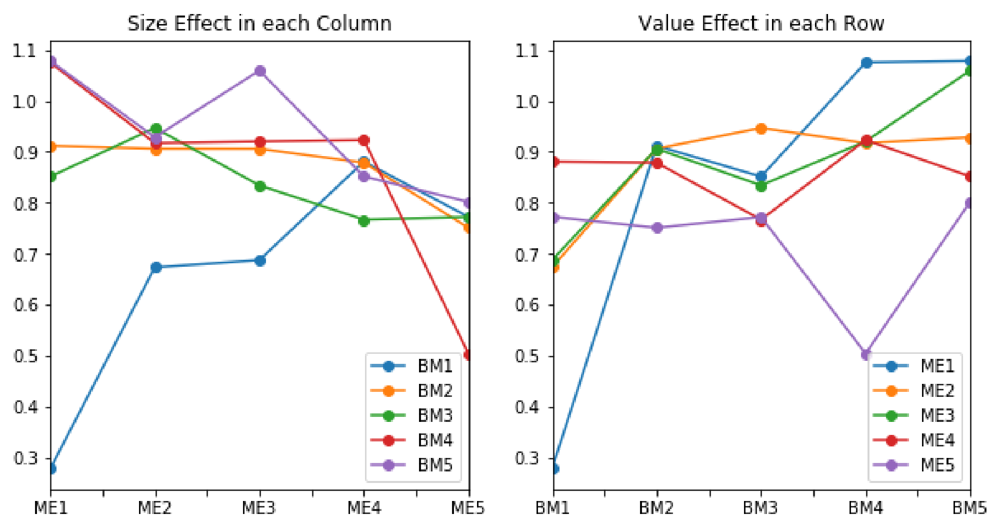
- The alphas that are statistically significant in part (c), i.e. the CAPM single factor model becomes either
  - Much smaller in scale, in terms of the absolute values. When we inspect the plots of alphas across different groups, we can see a much messy pattern as opposed to the CAPM alphas. OR

  – Becomes not statistically significant. The $(*)$ in table marks those alphas with p-value < 0.05. We can observe that many alphas in the top-right corrner (small size, high B/M) that are previously significant now becomes not significant to the confidence level of 0.95.

- These observations bespeaks the fact that a considerable amount of the CAPM alphas on the size$\times$value quantile portfolios can be explained by the SMB and HML risk factors.



2.5 (e)




```
# Repeat Part (a)
                BM1       BM2       BM3       BM4       BM5
ME1 mean    0.279520  0.911910  0.851436  1.074964  1.078133
    se      0.397315  0.346695  0.281501  0.270598  0.282844
    t-stat  0.703521  2.630298  3.024629  3.972553  3.811763
ME2 mean    0.673830  0.906238  0.946844  0.917159  0.928510
    se      0.350828  0.284240  0.255377  0.254190  0.302355
    t-stat  1.920685  3.188284  3.707637  3.608169  3.070930
ME3 mean    0.687848  0.906062  0.834394  0.920435  1.059032
    se      0.323769  0.259975  0.240980  0.243974  0.278055
    t-stat  2.124506  3.485189  3.462497  3.772677  3.808714
```

```
ME4 mean    0.880749  0.878720  0.767234  0.923185  0.851772
    se      0.291348  0.238622  0.247059  0.232234  0.283599
    t-stat  3.023010  3.682481  3.105468  3.975237  3.003433
ME5 mean    0.772807  0.751408  0.772299  0.505007  0.801949
    se      0.221474  0.210062  0.208792  0.244736  0.295633
    t-stat  3.489385  3.577085  3.698892  2.063480  2.712648


# Repeat Part (b): see the plots above.


# Repeat Part (c)
                  BM1        BM2        BM3        BM4        BM5
ME1 alpha   -0.693204  0.0743478   0.127975    0.41341   0.376434
    t-stat    -2.5771   0.311648   0.719195    2.25265    1.99988
    p<0.05        (*)                              (*)        (*)
ME2 alpha    -0.27519    0.12434   0.254925   0.235765   0.140363
    t-stat   -1.36331   0.789451    1.74087    1.58208   0.751792
    p<0.05
ME3 alpha   -0.214548   0.147307   0.146577   0.256493   0.327003
    t-stat   -1.23544    1.22032    1.21334     1.8511    1.93806
    p<0.05
ME4 alpha   0.0251955   0.170176  0.0642887   0.267753  0.0885353
    t-stat   0.190777    1.64981   0.514063    2.22401    0.53679
    p<0.05                                         (*)
ME5 alpha   0.0923624    0.11796   0.190769  -0.144086  0.0422177
    t-stat    1.17026    1.40275    1.70035   -0.98358   0.225891
    p<0.05


# Repeat Part (d)
                 BM1        BM2        BM3        BM4        BM5
ME1 alpha  -0.634844   0.047471  0.0315884   0.270725   0.175225
    t-stat  -4.60346   0.448571   0.424996    3.60445    2.33627
    p<0.05       (*)                              (*)        (*)
ME2 alpha  -0.203053  0.0659427   0.119476  0.0594593 -0.0984468
    t-stat  -2.24812   0.837165    1.53178   0.879905    -1.3781
    p<0.05       (*)
ME3 alpha  -0.108831  0.0794007  0.0143403   0.078121   0.101301
    t-stat   -1.3034   0.873981   0.162489   0.856674   0.945184
    p<0.05
ME4 alpha    0.11877  0.0891055 -0.0694418   0.124072  -0.125605
    t-stat   1.43847   0.965733  -0.680546      1.353   -1.07457
    p<0.05
ME5 alpha   0.189886   0.079747  0.0923893    -0.3241  -0.172125
    t-stat   3.71963    1.11945     1.1159   -3.70759   -1.28333
    p<0.05       (*)                              (*)
```

We used 1988 - 2018, i.e. recent 30 years for this section. **Differences:**

- In the recent period, the excess return profile across size× value double sorts becomes much more messy. As in the plots we can't find the downward/upward sloping average returns.
- Less portfolios from the double sorts have statisticallty significant CAPM alphas and FF3 alphas: the market becomes more efficient.

Similarities:

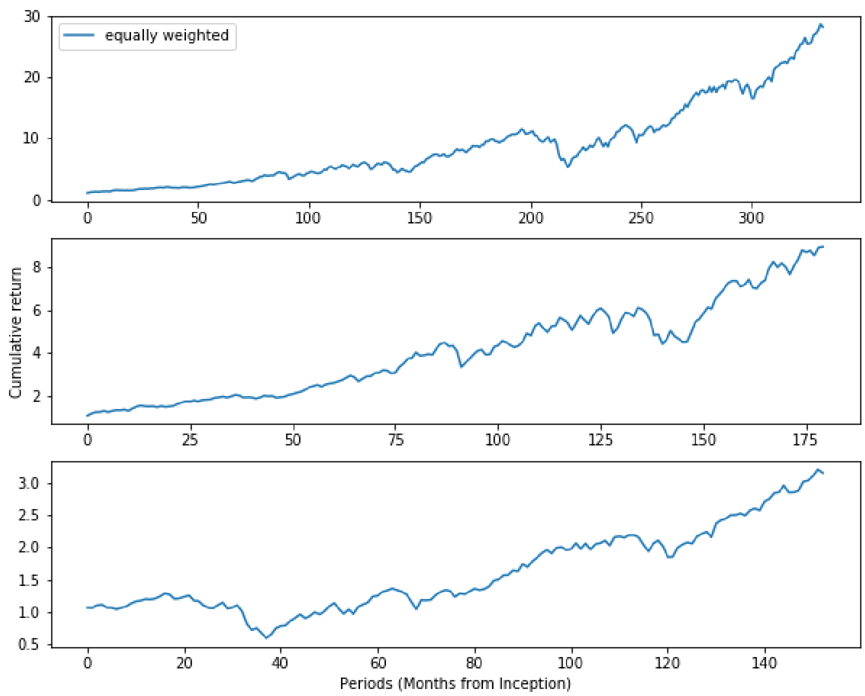
- The (Small size $\times$ high BM) portfolios in the top-right corner still has significant CAPM alphas, and that alpha reduces in a similar fashion as we account for FF3 factors.

3 Problem 3

Note: all the sharpe ratios are annualized, i.e. scaled by multiplying $\sqrt{12}$

3.1 (a)

```
Sharpe ratio (1991/01 - 2018/10): 0.6705
Sharpe ratio (1991/01 - 2005/12): 0.7862
Sharpe ratio (2006/01 - 2018/10): 0.5523
```



3.2 (b)

```
Sharpe ratio (1991/01 - 2018/10): 0.6916
Sharpe ratio (1991/01 - 2005/12): 0.9773
Sharpe ratio (2006/01 - 2018/10): 0.4414
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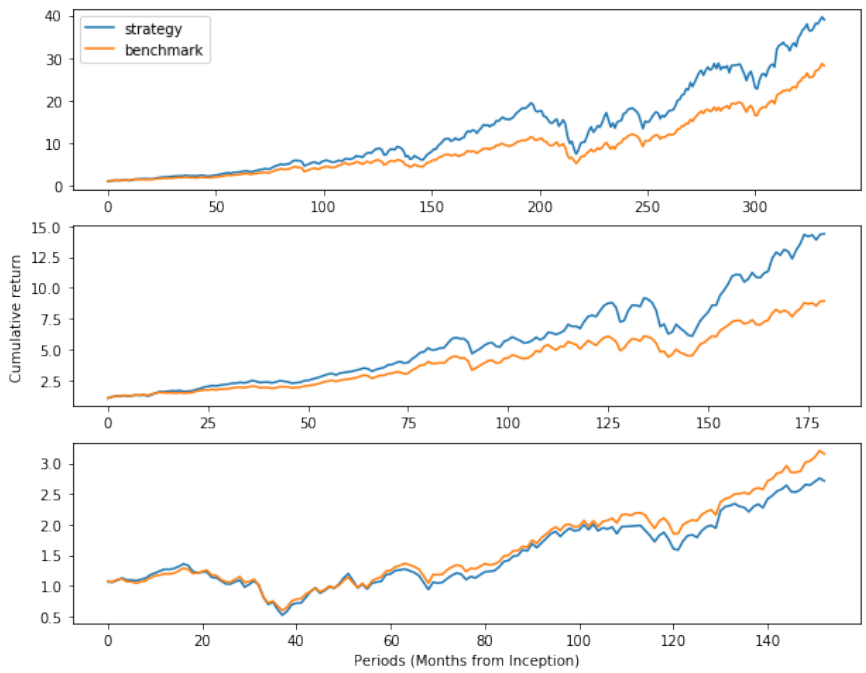

3.3 (c)

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Sharpe ratio (1991/01 - 2018/10): 0.7013
Sharpe ratio (1991/01 - 2005/12): 1.0266
Sharpe ratio (2006/01 - 2018/10): 0.4248
```
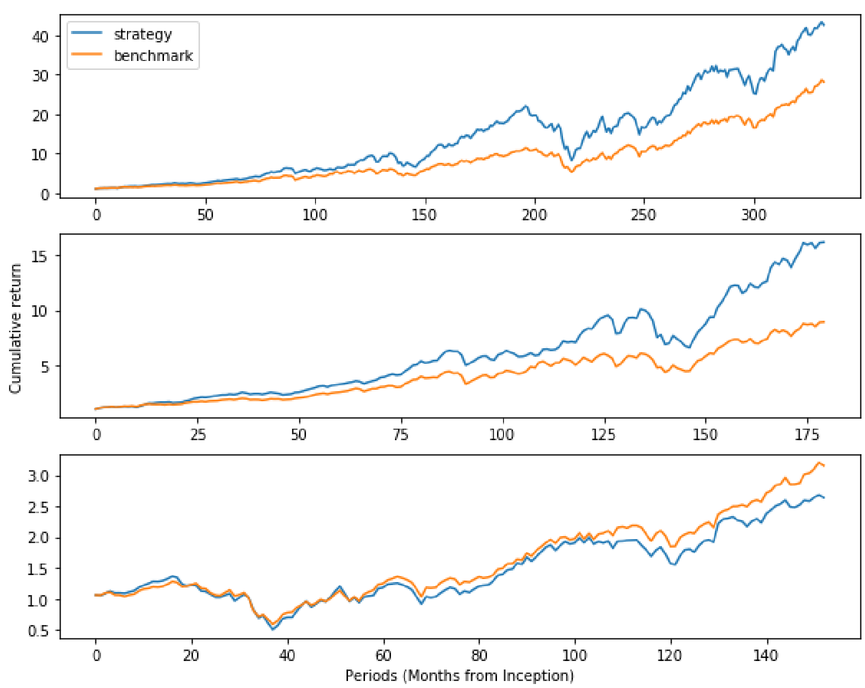
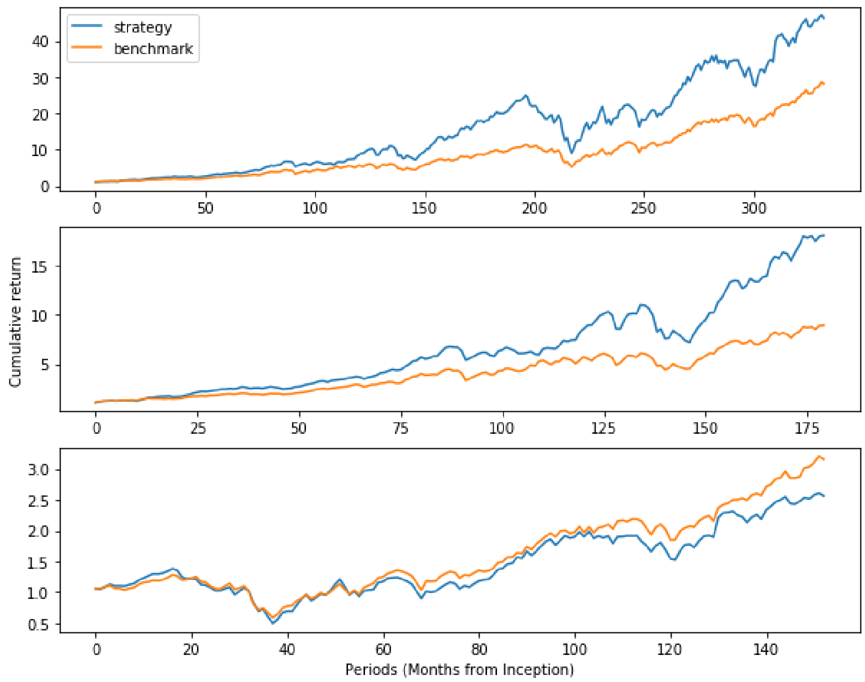



3.4 (d)

```
Sharpe ratio (1991/01 - 2018/10): 0.7074
Sharpe ratio (1991/01 - 2005/12): 1.0659
Sharpe ratio (2006/01 - 2018/10): 0.4083
```



3.5 (e)

Comments:

- In terms of aggressiveness, we have (d) > (c) > (b) > (a).
- More aggressive strategy has higher return and sharpe ratio in 1991-2018 ("overall"), and 1991-2005 ("good periods"); and outperforms the benchmark in those periods.
- More aggressive strategy has lower return and sharpe ratio in 2006-2018 ("bad/high volatility periods"); and underperforms the benchmark in those periods.

Limitations:

- The margin requirements ask for large amount of capital to implement more aggressive strategies.
- Short selling may not be easy for some countries/exchanges.
- The aggressive strategies have higher drawdowns in the "bad" periods, which may cause the fund to blow up.