Timeseries HW1

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Problem 1

1.a

$$
\begin{aligned}
\mathbb{E}\left[X_t\right] &= \mathbb{E}\left[a + bZ_t + cZ_{t-1}\right] = a \\
\mathbb{C}\mathrm{ov}\left[X_t, X_{t+h}\right] &= \mathbb{C}\mathrm{ov}\left[a + bZ_t + cZ_{t-1}, a + bZ_{t+h} + cZ_{t+h-1}\right] \\
&= b^2\mathbb{C}\mathrm{ov}\left[Z_t, Z_{t+h}\right] + c^2\mathbb{C}\mathrm{ov}\left[Z_{t-1}, Z_{t+h-1}\right] \\
&\quad + bc\mathbb{C}\mathrm{ov}\left[Z_{t+h}, Z_{t-1}\right] + bc\mathbb{C}\mathrm{ov}\left[Z_t, Z_{t+h-1}\right] \\
&= \begin{cases} (b^2+c^2)\sigma^2 & h = 0 \\ bc\sigma^2 & h = \pm 1 \\ 0 & \text{otherwise} \end{cases} \\
\mathbb{E}\left[X_t^2\right] &= \mathbb{V}\mathrm{ar}\left[X_t\right] + \mathbb{E}\left[X_t\right]^2 = a^2 + (b^2+c^2)\sigma^2 < \infty
\end{aligned}
\tag{1}
$$

Therefore, X_t is stationary.

1.b

$$
\begin{aligned}
\mathbb{E}\left[X_t\right] &= a + \sigma^2 \\
\mathbb{C}\mathrm{ov}\left[X_t, X_{t+h}\right] &= \mathbb{C}\mathrm{ov}\left[a + X_t^2, a + X_{t+h}^2\right] \\
&= \mathbb{C}\mathrm{ov}\left[X_t^2, X_{t+h}^2\right] \\
&= \begin{cases} 2\sigma^4 & h = 0 \\ 0 & \text{otherwise} \end{cases} \\
\mathbb{E}\left[X_t^2\right] &< \infty
\end{aligned}
\tag{2}
$$

Therefore, X_t is stationary.

1.c

$$
\begin{aligned}
\mathbb{E}\left[X_t\right] &= \mathbb{E}\left[Z_t Z_{t-1}\right] = 0 \\
\mathbb{C}\mathrm{ov}\left[X_t, X_{t+h}\right] &= \mathbb{C}\mathrm{ov}\left[Z_t Z_{t-1}, Z_{t+h} Z_{t+h-1}\right] \\
&= \mathbb{E}\left[Z_t Z_{t-1} Z_{t+h} Z_{t+h-1}\right] - \mathbb{E}\left[Z_t Z_{t-1}\right]\mathbb{E}\left[Z_{t+h} Z_{t+h-1}\right] \\
&= \mathbb{E}\left[Z_t Z_{t-1} Z_{t+h} Z_{t+h-1}\right] \\
&= \begin{cases} \mathbb{E}\left[Z_t^2\right]\mathbb{E}\left[Z_{t-1}^2\right] & h = 0 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \sigma^4 & h = 0 \\ 0 & \text{otherwise} \end{cases} \\
\mathbb{E}\left[X_t^2\right] &< \infty
\end{aligned}
\tag{3}
$$

Therefore, X_t is stationary.

Problem 2

Since $\{\epsilon_t\}$ is i.i.d. white noise, let $\mathbb{V}\text{ar}[\epsilon_t] = \sigma^2$.

$$\mathbb{E}[X_t] = \mathbb{E}[\epsilon_t + \epsilon_{t-1}\epsilon_{t-2}] = 0 \tag{4}$$

$$\begin{aligned}
\mathbb{C}\text{ov}[X_t, X_{t+h}] &= \mathbb{C}\text{ov}[\epsilon_t + \epsilon_{t-1}\epsilon_{t-2}, \epsilon_{t+h} + \epsilon_{t+h-1}\epsilon_{t+h-2}] \\
&= \mathbb{E}[(\epsilon_t + \epsilon_{t-1}\epsilon_{t-2})(\epsilon_{t+h} + \epsilon_{t+h-1}\epsilon_{t+h-2})] \\
&= \mathbb{E}[\epsilon_t\epsilon_{t+h} + \epsilon_{t-1}\epsilon_{t-2}\epsilon_{t+h} + \epsilon_{t+h-1}\epsilon_{t+h-2}\epsilon_t + \epsilon_{t-1}\epsilon_{t-2}\epsilon_{t+h-1}\epsilon_{t+h-2}] \\
&= \begin{cases} \mathbb{E}[\epsilon_t^2 + \epsilon_{t-1}^2\epsilon_{t-2}^2] & h = 0 \\ 0 & |h| > 0 \end{cases} \qquad (5) \\
&= \begin{cases} \sigma^2 + \sigma^4 & h = 0 \\ 0 & |h| > 0 \end{cases}
\end{aligned}$$

Therefore by definition, $\{X_t\}$ is weak white noise. But

$$\mathbb{E}[X_t|\mathcal{F}_{t-1}] = \mathbb{E}[\epsilon_t|\mathcal{F}_{t-1}] + \epsilon_{t-1}\epsilon_{t-2} = \epsilon_{t-1}\epsilon_{t-2} \neq 0 \tag{6}$$

Therefore, $\{X_t\}$ is not MDS.

Problem 3

$$\begin{aligned}
\mathbb{C}\text{ov}[X_t, X_{t+h}] &= \mathbb{C}\text{ov}[Y_t - Y_{t-1}, Y_{t+h} - Y_{t+h-1}] \\
&= \mathbb{C}\text{ov}[Y_t, Y_{t+h}] - \mathbb{C}\text{ov}[Y_{t-1}, Y_{t+h}] - \mathbb{C}\text{ov}[Y_t, Y_{t+h-1}] + \mathbb{C}\text{ov}[Y_{t-1}, Y_{t+h-1}] \\
&= \gamma_Y(h) - \gamma_Y(h+1) - \gamma_Y(h-1) + \gamma_Y(h) \qquad (7) \\
&= 2\gamma_Y(h) - \gamma_Y(h+1) - \gamma_Y(h-1) \\
&=: \gamma_X(h)
\end{aligned}$$

which is a function of h, independent of t. Moreover, $\mathbb{E}[X_t] = 0$. Therefore, X_t is also stationary.

Problem 4

4.1

```
library(quantmod)
```

```
## Loading required package: xts
```

```
## Loading required package: zoo
```

```
##
## Attaching package: 'zoo'
```

```
## The following objects are masked from 'package:base':
##
##     as.Date, as.Date.numeric
```

```
## Loading required package: TTR
```

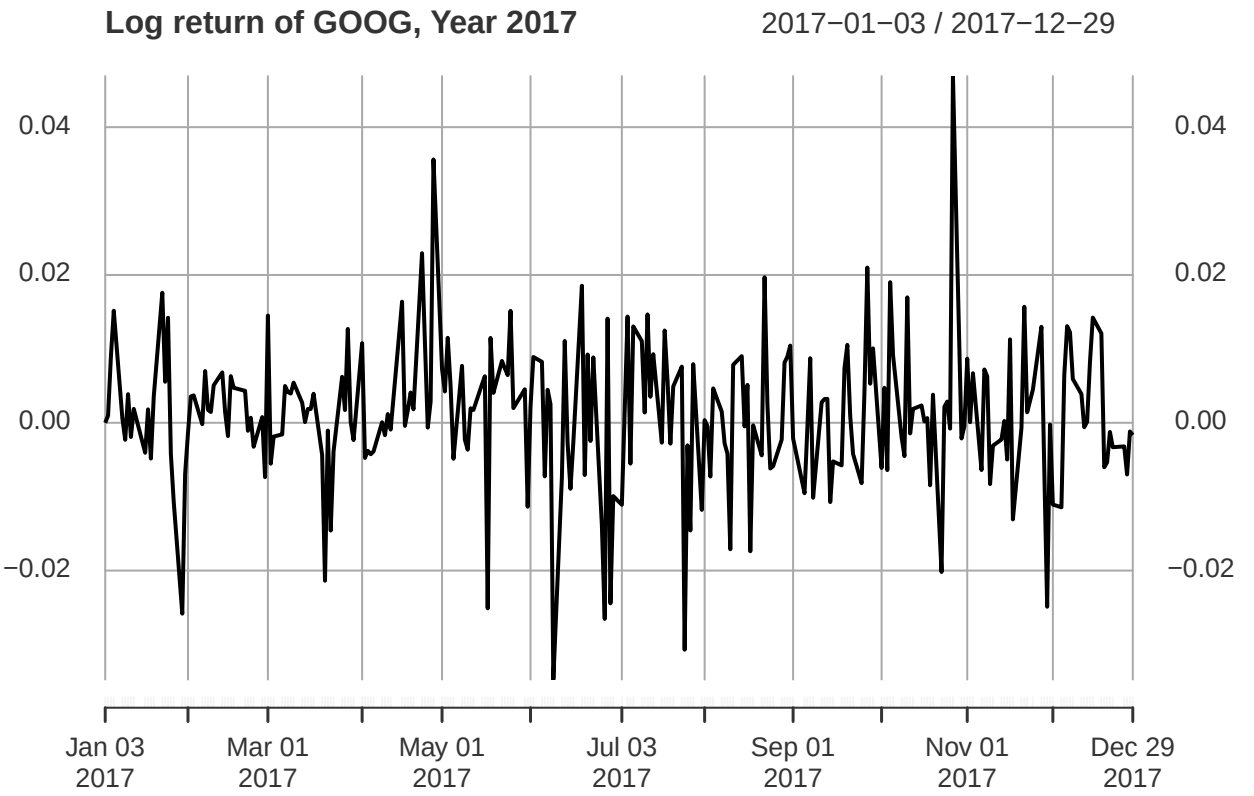
```
## Version 0.4-0 included new data defaults. See ?getSymbols.
```

```
goog = getSymbols('GOOG',from='2017-1-1',to='2017-12-31',auto.assign=F)
```

```
## 'getSymbols' currently uses auto.assign=TRUE by default, but will
## use auto.assign=FALSE in 0.5-0. You will still be able to use
## 'loadSymbols' to automatically load data. getOption("getSymbols.env")
## and getOption("getSymbols.auto.assign") will still be checked for
## alternate defaults.
##
## This message is shown once per session and may be disabled by setting
## options("getSymbols.warning4.0"=FALSE). See ?getSymbols for details.
```

```
##
## WARNING: There have been significant changes to Yahoo Finance data.
## Please see the Warning section of '?getSymbols.yahoo' for details.
##
## This message is shown once per session and may be disabled by setting
## options("getSymbols.yahoo.warning"=FALSE).
```

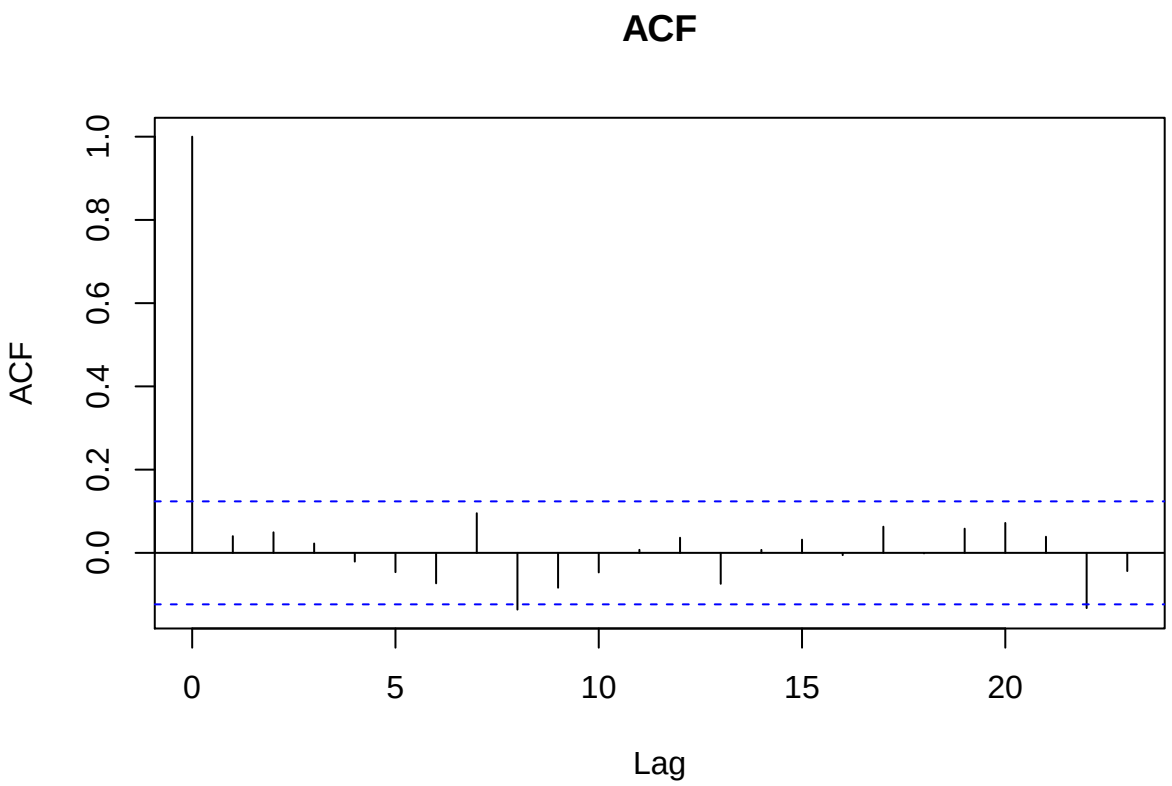
```
logret = dailyReturn(Ad(goog), type='log')
plot(logret, main='Log return of GOOG, Year 2017')
```



The log returns exhibit centain level of volatility clustering. However, there is no clear evidence of non-stationarity by just looking at the plot.

4.2

```
acf(logret, main='ACF')
```



4.3

```
Box.test(logret, lag=20, type="Ljung-Box")
```

```
##
##  Box-Ljung test
##
## data:  logret
## X-squared = 18.395, df = 20, p-value = 0.5614
```

The p-value is large. We fail to reject $H_0 : \{\rho_X(h) = 0 \text{ for some } h\}$. That is, there is no evidence of serial correlation up to lag 20.