Asset Management HW4

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Problem 2

2-(a)

$$
\begin{aligned}
& \mathcal{L}(\boldsymbol{h}, \boldsymbol{\lambda}) = \boldsymbol{h}^\top \boldsymbol{h} + \lambda_1(\boldsymbol{a}^\top \boldsymbol{h} - 1) + \lambda_2 \mathbf{1}^\top \boldsymbol{h} \\
\Rightarrow \quad & \nabla_{\boldsymbol{h}} \mathcal{L} = 2\boldsymbol{h} + \lambda_1 \boldsymbol{a} + \lambda_2 \mathbf{1} = 0 \\
\Rightarrow \quad & \begin{cases} \boldsymbol{h} = -\frac{1}{2}(\lambda_1 \boldsymbol{a} + \lambda_2 \mathbf{1}) \\ \lambda_1 \boldsymbol{a}^\top \boldsymbol{a} + \lambda_2 \boldsymbol{a}^\top \mathbf{1} = -2 \\ \lambda_1 \mathbf{1}^\top \boldsymbol{a} + \lambda_2 n = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \lambda_1 = \frac{2}{(\mathbf{1}^\top \boldsymbol{a})^2/n - \boldsymbol{a}^\top \boldsymbol{a}} \\ \lambda_2 = \frac{2 \cdot \mathbf{1}^\top \boldsymbol{a}/n}{\boldsymbol{a}^\top \boldsymbol{a} - (\mathbf{1}^\top \boldsymbol{a})^2/n} \end{cases} \\
\Rightarrow \quad & \boldsymbol{h}^* = \frac{1}{\boldsymbol{a}^\top \boldsymbol{a} - (\mathbf{1}^\top \boldsymbol{a})^2/n} \boldsymbol{a} - \frac{\mathbf{1}^\top \boldsymbol{a}/n}{\boldsymbol{a}^\top \boldsymbol{a} - (\mathbf{1}^\top \boldsymbol{a})^2/n} \mathbf{1} \\
& \quad = \frac{1}{n\sigma_{CS}^2(\boldsymbol{a})} \left(\boldsymbol{a} - \mu_{CS}(\boldsymbol{a}) \mathbf{1} \right) \\
& \quad = \frac{1}{n\sigma_{CS}(\boldsymbol{a})} \boldsymbol{z}
\end{aligned} \tag{1}
$$

Hence $C = \frac{1}{n\sigma_{CS}(\boldsymbol{a})}$ for general solution. $C = 1/n$ for zero cross-sectional mean and unit cross-sectional variance. In this special case: $\boldsymbol{h}^* = \frac{1}{n}\boldsymbol{a}$.

2-(b)

Let $\boldsymbol{b}_1^\top = (\mathbf{1}_k^\top \quad \mathbf{0}_{n-k}^\top)$, $\boldsymbol{b}_2^\top = (\mathbf{0}_k^\top \quad \mathbf{1}_{n-k}^\top)$, i.e. $\boldsymbol{B} = (\boldsymbol{b}_1 \quad \boldsymbol{b}_2)$. The neutrality constraints are: $\boldsymbol{b}_1^\top \boldsymbol{h} = \boldsymbol{b}_2^\top \boldsymbol{h} = 0$. One of them is redundant since we already have constraint $\mathbf{1}^\top \boldsymbol{h} = 0$, and

$\boldsymbol{b}_1 + \boldsymbol{b}_2 = \mathbf{1}$. Let dual variable $\boldsymbol{\lambda} = (\lambda_1 \quad \lambda_2 \quad \lambda_3)^\top$. The KKT condition changes to:

$$
\begin{aligned}
& \nabla_{\boldsymbol{h}} \mathcal{L} = 2\boldsymbol{h} + \lambda_1 \boldsymbol{a} + \lambda_2 \mathbf{1} + \lambda_3 \boldsymbol{b}_1 = 0 \\
\Rightarrow \quad & \begin{cases} \boldsymbol{h} = -\frac{1}{2} \begin{pmatrix} \boldsymbol{a} & \mathbf{1} & \boldsymbol{b}_1 \end{pmatrix} \boldsymbol{\lambda} \\ \begin{pmatrix} \boldsymbol{a}^\top \boldsymbol{a} & \boldsymbol{a}^\top \mathbf{1} & \boldsymbol{a}^\top \boldsymbol{b}_1 \\ \mathbf{1}^\top \boldsymbol{a} & n & k \\ \boldsymbol{b}_1^\top \boldsymbol{a} & k & k \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \end{cases} \\
\text{(Gaussian Elimination)} \quad \Rightarrow \quad & \begin{pmatrix} \boldsymbol{a}^\top \boldsymbol{a} - \frac{(\boldsymbol{a}^\top \boldsymbol{b}_1)^2}{k} - \frac{(\boldsymbol{a}^\top \boldsymbol{b}_2)^2}{n-k} & 0 & 0 \\ \boldsymbol{b}_2^\top \boldsymbol{a} & n-k & 0 \\ \boldsymbol{b}_1^\top \boldsymbol{a} & k & k \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \\
\Rightarrow \quad & \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} \frac{-2}{\boldsymbol{a}^\top \boldsymbol{a} - (\boldsymbol{a}^\top \boldsymbol{b}_1)^2/k - (\boldsymbol{a}^\top \boldsymbol{b}_2)^2/(n-k)} \\ -\frac{\boldsymbol{a}^\top \boldsymbol{b}_2}{n-k} \lambda_1 \\ \left[\frac{\boldsymbol{a}^\top \boldsymbol{b}_2}{n-k} - \frac{\boldsymbol{a}^\top \boldsymbol{b}_1}{k} \right] \lambda_1 \end{pmatrix}
\end{aligned} \tag{2}
$$

Therefore,

$$
\begin{aligned}
\boldsymbol{h}^* &= -\frac{1}{2}\lambda_1 \left(\boldsymbol{a} - \frac{\boldsymbol{a}^\top \boldsymbol{b}_2}{n-k} \mathbf{1} + \left(\frac{\boldsymbol{a}^\top \boldsymbol{b}_2}{n-k} - \frac{\boldsymbol{a}^\top \boldsymbol{b}_1}{k} \right) \boldsymbol{b}_1 \right) \\
&= -\frac{1}{2}\lambda_1 \left(\boldsymbol{a} - \frac{\boldsymbol{a}^\top \boldsymbol{b}_1}{k} \boldsymbol{b}_1 - \frac{\boldsymbol{a}^\top \boldsymbol{b}_2}{n-k} \boldsymbol{b}_2 \right) \\
&= -\frac{1}{2}\lambda_1 \left(\boldsymbol{a} - \frac{\sum_{j=1}^k a_j}{k} \boldsymbol{b}_1 - \frac{\sum_{j=k+1}^n a_j}{n-k} \boldsymbol{b}_2 \right) \\
&= -\frac{1}{2}\lambda_1 \widehat{\boldsymbol{a}} = C\widehat{\boldsymbol{a}}
\end{aligned} \tag{3}
$$

Where $C = -\frac{1}{2}\lambda_1 = 1 \Big/ \left[\boldsymbol{a}^\top \boldsymbol{a} - \frac{(\boldsymbol{a}^\top \boldsymbol{b}_1)^2}{k} - \frac{(\boldsymbol{a}^\top \boldsymbol{b}_2)^2}{n-k} \right] = 1 \Big/ \left[\sum_{j=1}^n a_j^2 - \frac{(\sum_{j=1}^k a_j)^2}{k} - \frac{(\sum_{j=k+1}^n a_j)^2}{n-k} \right]$

2-(c)

$$
\boldsymbol{a}^\top \boldsymbol{h}^{SMB} = \sum_{k=1}^4 \frac{1}{2|G_k|} \sum_{i \in G_k} a_i \text{sgn}(a_i) = \sum_{k=1}^4 \frac{1}{2|G_k|} \sum_{i \in G_k} |a_i| = \sum_{k=1}^4 \frac{1}{2} = 2 \tag{4}
$$

$$
\mathbf{1}^\top \boldsymbol{h}^{SMB} = \sum_{k=1}^4 \frac{1}{2|G_k|} \sum_{i \in G_k} \text{sgn}(a_i) = \frac{|S|}{2|S|} - \frac{|B|}{2|B|} = 0 \tag{5}
$$

$$
\boldsymbol{b}^\top \boldsymbol{h}^{SMB} = \sum_{k=1}^4 \frac{1}{2|G_k|} \sum_{i \in G_k} b_i \text{sgn}(a_i) = \sum_{k=1}^4 \frac{1}{2|G_k|} \sum_{i \in G_k} \text{sgn}(a_i b_i) = \frac{|SH|}{2|SH|} + \frac{|BL|}{2|BL|} - \frac{|SL|}{2|SL|} - \frac{|BH|}{2|BH|} = 0 \tag{6}
$$

2-(d)

We know from (c) that $\boldsymbol{a}^\top \boldsymbol{a} = \boldsymbol{b}^\top \boldsymbol{b} = n$, $\mathbf{1}^\top \boldsymbol{a} = |S| - |B|$, $\mathbf{1}^\top \boldsymbol{b} = |H| - |L|$, and $\boldsymbol{a}^\top \boldsymbol{b} = |SH| + |BL| - |SL| - |BH|$. Denote $A := |SL|, B := |SH|, C := |BL|, D := |BH|$. The KKT condition changes to:

$$
\begin{aligned}
&\nabla_{\boldsymbol{h}}\mathcal{L} = 2\boldsymbol{h} + \lambda_1\boldsymbol{a} + \lambda_2\boldsymbol{1} + \lambda_3\boldsymbol{b} = 0 \\
\Rightarrow\ &\begin{cases} \boldsymbol{h} = -\frac{1}{2}\begin{pmatrix}\boldsymbol{a} & \boldsymbol{1} & \boldsymbol{b}\end{pmatrix}\boldsymbol{\lambda} \\ \begin{pmatrix}\boldsymbol{a}^\top\boldsymbol{a} & \boldsymbol{a}^\top\boldsymbol{1} & \boldsymbol{a}^\top\boldsymbol{b} \\ \boldsymbol{1}^\top\boldsymbol{a} & n & \boldsymbol{1}^\top\boldsymbol{b} \\ \boldsymbol{b}^\top\boldsymbol{a} & \boldsymbol{b}^\top\boldsymbol{1} & \boldsymbol{b}^\top\boldsymbol{b}\end{pmatrix}\boldsymbol{\lambda} = \begin{pmatrix}-2\\0\\0\end{pmatrix} \end{cases} \\
\Rightarrow\ &\begin{pmatrix} A+B+C+D & A+B-C-D & B+C-A-D \\ A+B-C-D & A+B+C+D & B+D-A-C \\ B+C-A-D & B+D-A-C & A+B+C+D \end{pmatrix}\boldsymbol{\lambda} = \begin{pmatrix}-2\\0\\0\end{pmatrix} \\
\Rightarrow\ &\begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix} = \begin{pmatrix} -\frac{AB+BC+AD+CD}{2(ABC+ABD+ACD+BCD)} \\ -\frac{AB-CD}{2(ABC+ABD+ACD+BCD)} \\ -\frac{AD-BC}{2(ABC+ABD+ACD+BCD)} \end{pmatrix}
\end{aligned} \tag{7}
$$

Therefore

$$
\begin{aligned}
\boldsymbol{h}^* &= \frac{(AB+BC+AD+CD)\boldsymbol{a} + (AB-CD)\boldsymbol{1} + (AD-BC)\boldsymbol{b}}{4(ABC+ABD+ACD+BCD)} \\
h_i^* &= \begin{cases} \frac{ABC+ACD}{ABC+ABD+ACD+BCD}\cdot\frac{1}{2A} & i \in SL \\ \frac{ABD+BCD}{ABC+ABD+ACD+BCD}\cdot\frac{1}{2B} & i \in SH \\ \frac{ABC+ACD}{ABC+ABD+ACD+BCD}\cdot -\frac{1}{2C} & i \in BL \\ \frac{ABD+BCD}{ABC+ABD+ACD+BCD}\cdot -\frac{1}{2D} & i \in BH \end{cases}
\end{aligned} \tag{8}
$$

The coefficient before the required quantity is not a constant unless

$$
ABC + ACD = ABD + BCD \tag{9}
$$

That is, $\frac{1}{D} + \frac{1}{B} = \frac{1}{C} + \frac{1}{A} \iff \frac{1}{|BH|} + \frac{1}{|SH|} = \frac{1}{|BL|} + \frac{1}{|SL|}$. This is not necessarily the case. Hence, the proposition is disproved.

Problem 3

3-(a)

Out[4]:

	const	Mkt-RF	SMB	RMW	CMA
est	-0.042	0.018	0.011	0.129	1.025
t-stat	-0.486	0.849	0.381	3.169	22.795
p<0.05				(*)	(*)
Adj.R^2	0.500				

3-(b)

Out[5]:

	const	Mkt-RF	SMB	RMW	CMA	MOM
est	0.050	-0.000	0.013	0.149	1.008	-0.129
t-stat	0.587	-0.009	0.448	3.787	23.191	-6.822
p<0.05				(*)	(*)	(*)
Adj.R^2	0.535					

3-(c)

Out[6]:

	const	Mkt-RF	SMB	RMW	CMA	MOM
est	-0.014	0.066	0.014	-0.054	0.972	
t-stat	-0.110	2.136	0.324	-0.913	14.783	
p<0.05		(*)			(*)	
Adj.R^2	0.279					
est	0.363	-0.008	0.020	0.028	0.900	-0.528
t-stat	4.289	-0.413	0.706	0.712	20.699	-28.009
p<0.05	(*)				(*)	(*)
Adj.R^2	0.686					

3-(d)

Out[7]:

		const	Mkt-RF	SMB	RMW	CMA	MOM
HML	est	-0.042	0.018	0.011	0.129	1.025	
	t-stat	-0.486	0.849	0.381	3.169	22.795	
	p<0.05				(*)	(*)	
	Adj.R^2	0.500					
HML	est	0.050	-0.000	0.013	0.149	1.008	-0.129
	t-stat	0.587	-0.009	0.448	3.787	23.191	-6.822
	p<0.05				(*)	(*)	(*)
	Adj.R^2	0.535					
HML-DEV	est	-0.014	0.066	0.014	-0.054	0.972	
	t-stat	-0.110	2.136	0.324	-0.913	14.783	
	p<0.05		(*)			(*)	
	Adj.R^2	0.279					
HML-DEV	est	0.363	-0.008	0.020	0.028	0.900	-0.528
	t-stat	4.289	-0.413	0.706	0.712	20.699	-28.009
	p<0.05	(*)				(*)	(*)
	Adj.R^2	0.686					

Comments

- The results here are consistent with those in the article.
- For HML, we observe its large and statistically significant exposures on RMW, CMA, and MOM. The negative exposure to MOM indicates that the return of HML and the return of MOM is negatively related.
- HML has low, insignificant alpha when regressed on other factors. It bespeaks that the HML factor is kind of "redundant", in the sense that its risk premium can be almost fully expained by the risk premium of some return series living in the space spanned by other factors excluding HML.
- HML-DEV can "resurrect" HML, in the sense that it still has large, significant alpha (0.363%, with t-statistic = 4.289) when regressed to other factors. This indicates that HML-DEV is not "redundent" in the above sense.

3-(e)

1998 - 2018 Results

In [9]:
```
prd1 = df[(df.Date>=199800) & (df.Date<=201812)]
test_hml(prd1)
```

Out[9]:

	const	Mkt-RF	SMB	RMW	CMA	MOM
est	-0.352	0.185	0.063	0.454	0.863	
t-stat	-2.408	4.898	1.209	7.109	12.248	
p<0.05	(*)	(*)		(*)	(*)	
Adj.R^2	0.515					
est	-0.271	0.128	0.095	0.456	0.828	-0.128
t-stat	-1.917	3.366	1.887	7.445	12.181	-4.759
p<0.05		(*)		(*)	(*)	(*)
Adj.R^2	0.554					
est	-0.377	0.359	0.002	0.322	0.863	
t-stat	-1.636	6.046	0.022	3.198	7.771	
p<0.05		(*)		(*)	(*)	
Adj.R^2	0.265					
est	-0.015	0.108	0.144	0.330	0.706	-0.570
t-stat	-0.112	3.023	3.061	5.749	11.090	-22.576
p<0.05		(*)	(*)	(*)	(*)	(*)
Adj.R^2	0.762					

1978 - 2008 Results

In [10]:
```
prd1 = df[(df.Date>=197800) & (df.Date<=200812)]
test_hml(prd1)
```

Out[10]:

	const	Mkt-RF	SMB	RMW	CMA	MOM
est	0.023	-0.044	-0.087	0.173	0.911	
t-stat	0.208	-1.659	-2.339	3.754	16.527	
p<0.05			(*)	(*)	(*)	
Adj.R^2	0.553					
est	0.118	-0.047	-0.053	0.212	0.904	-0.127
t-stat	1.113	-1.847	-1.469	4.717	17.022	-5.450
p<0.05				(*)	(*)	(*)
Adj.R^2	0.586					
est	-0.008	-0.031	-0.192	-0.078	0.820	
t-stat	-0.051	-0.809	-3.551	-1.168	10.192	
p<0.05			(*)		(*)	
Adj.R^2	0.303					
est	0.378	-0.044	-0.056	0.080	0.791	-0.516
t-stat	3.622	-1.757	-1.582	1.801	15.124	-22.422
p<0.05	(*)				(*)	(*)
Adj.R^2	0.705					

1968 - 1988 Results

```
In [11]: prd1 = df[(df.Date>=196800) & (df.Date<=198812)]
         test_hml(prd1)
```

Out[11]:

	const	Mkt-RF	SMB	RMW	CMA	MOM
est	0.236	-0.082	0.049	-0.319	0.874	
t-stat	1.982	-3.127	1.250	-3.850	11.732	
p<0.05	(*)	(*)		(*)	(*)	
Adj.R^2	0.621					
est	0.276	-0.079	0.039	-0.293	0.877	-0.060
t-stat	2.294	-3.040	0.986	-3.521	11.841	-1.997
p<0.05	(*)	(*)		(*)	(*)	(*)
Adj.R^2	0.626					
est	0.403	-0.112	0.124	-0.518	0.761	
t-stat	2.470	-3.139	2.324	-4.565	7.464	
p<0.05	(*)	(*)	(*)	(*)	(*)	
Adj.R^2	0.487					
est	0.671	-0.094	0.055	-0.344	0.782	-0.412
t-stat	5.201	-3.372	1.304	-3.845	9.836	-12.677
p<0.05	(*)	(*)		(*)	(*)	(*)
Adj.R^2	0.689					

Comments

- The results in (d) is not always the case over different sample period.
- Over 1998 - 2018, HML-DEV does not have significant alpha, but HML has significant negative alpha when regressed to the other FF 4 factors.
- Over 1978 - 2008, the results in (d) hold in a similar way.
- Over 1968 - 1988, Both HML and HML-DEV are not redundent (have significant positive alphas). Most pricing factors turn out to be more effective in early times. Nevertheless, HML-DEV still has better (larger, and more significant) alphas compared with HML. So it's legitimate to say that the paper's main idea still holds in this period.

3-(f)

```
In [13]: rep5
```

Out[13]:

	const	Mkt-RF	SMB	RMW	CMA	MOM	HML-DEV
est	-0.231	0.006	-0.003	0.128	0.312	0.280	0.773
t-stat	-4.236	0.488	-0.155	5.096	8.657	15.407	29.913
p<0.05	(*)			(*)	(*)	(*)	(*)
Adj.R^2	0.813						

```
In [14]: rep6
```

Out[14]:

	const	Mkt-RF	SMB	RMW	CMA	MOM	HML
est	0.325	-0.008	0.010	-0.087	0.120	-0.429	0.773
t-stat	6.041	-0.639	0.567	-3.450	3.171	-34.512	29.913
p<0.05	(*)			(*)	(*)	(*)	(*)
Adj.R^2	0.874						

Comments

- HML-DEV has positive and significant alpha, while HML has negative significant alpha.
- Therefore, HML-DEC is more profitable when other factors are being controled.