Asset Management HW2

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1 Problem 1.

1.1 (a)

By the definition of benchmark portfolio, clearly we have $\mathbb{E}[r_B] = 0$. To compute α and tracking error, we define $\boldsymbol{r} = (r_1, r_2, ..., r_N)^\top$, $\tilde{\boldsymbol{r}} = (\mathbb{1}_{\{u_{11}=1\%\}}r_1, \mathbb{1}_{\{u_{21}=1\%\}}r_2, ..., \mathbb{1}_{\{u_{N1}=1\%\}}r_N)^\top$; $\mathbb{1}_{\{u_{i1}=1\%\}}$ is the indicator function that equals 1 if $u_{i1} = 1\%$, otherwise 0.

Since $\{u_{ij}\}$ are i.i.d., we have $\mathbb{E}\left[\mathbb{1}_{\{u_{ij}=1\%\}}r_i\right] = \mathbb{E}\left[\mathbb{1}_{\{u_{ij}=1\%\}}\left(1\% + \sum_{j=2}^M u_{ij}\right)\right] = 1\%/2$. $\mathbb{E}[r_i] = 0$. Besides, the independence of u suggests r_i are also i.i.d. across difference stock label i's. Hence $\mathbb{V}\mathrm{ar}[\boldsymbol{r}] = 0.01^2 M\boldsymbol{I}$

$$\begin{aligned}\alpha &= \mathbb{E}\left[\frac{2}{N}\mathbf{1}^\top\tilde{\boldsymbol{r}} - \frac{1}{N}\mathbf{1}^\top\boldsymbol{r}\right] = \frac{2}{N}\mathbf{1}^\top\mathbb{E}[\tilde{\boldsymbol{r}}] - \frac{1}{N}\mathbf{1}^\top\mathbf{0} \\ &= \frac{2}{N}\sum_{i=1}^N \frac{1\%}{2} + 0 = 1\%\end{aligned} \tag{1}$$

For the variance, we first compute the covariance matrix of $\tilde{\boldsymbol{r}}$. By the independence across stocks, it's also a diagonal matrix. The second moment is

$$\begin{aligned}\mathbb{E}\left[(\mathbb{1}_{\{u_{ij}=1\%\}}r_i)^2\right] &= \mathbb{E}\left[\mathbb{1}^2_{\{u_{ij}=1\%\}}(\sum_{j=1}^M u_{ij})^2\right] \\ &= \mathbb{E}\left[\mathbb{1}^2_{\{u_{ij}=1\%\}}\left(1\%^2 + \sum_{j=2}^M u_{ij}^2\right)\right] \\ &= \left(\frac{1}{2} + \frac{M-1}{2}\right)1\%^2 = \frac{M}{2}1\%^2\end{aligned} \tag{2}$$

Therefore, $\mathbb{V}\mathrm{ar}[\tilde{\boldsymbol{r}}] = \mathbb{E}\left[(\mathbb{1}_{\{u_{ij}=1\%\}}r_i)^2\right] - \mathbb{E}\left[\mathbb{1}_{\{u_{ij}=1\%\}}r_i\right]^2 = \frac{0.01^2(2M-1)}{4}\boldsymbol{I}$. Moreover, $\mathbb{E}\left[\mathbb{1}_{\{u_{ij}=1\%\}}r_i^2\right] = \frac{M}{2}\cdot 1\%^2$; so $\mathbb{C}\mathrm{ov}[\tilde{\boldsymbol{r}}, \boldsymbol{r}] = \frac{0.01^2 M}{2}\boldsymbol{I}$

$$\begin{aligned}\mathbb{V}\mathrm{ar}[r - r_B] &= \mathbb{V}\mathrm{ar}\left[\frac{2}{N}\mathbf{1}^\top\tilde{\boldsymbol{r}} - \frac{1}{N}\mathbf{1}^\top\boldsymbol{r}\right] \\ &= \frac{4}{N^2}\mathbf{1}^\top\mathbb{V}\mathrm{ar}[\tilde{\boldsymbol{r}}]\,\mathbf{1} - 2\times\frac{2}{N^2}\mathbf{1}^\top\mathbb{C}\mathrm{ov}[\tilde{\boldsymbol{r}}, \boldsymbol{r}]\,\mathbf{1} + \frac{1}{N^2}\mathbf{1}^\top\mathbb{V}\mathrm{ar}[\boldsymbol{r}]\,\mathbf{1} \\ &= \frac{4}{N}\frac{2M-1}{4}0.01^2 - \frac{4}{N}\frac{M}{2}0.01^2 + \frac{1}{N}M0.01^2 \\ &= \frac{M-1}{N}0.01^2\end{aligned} \tag{3}$$

Hence $\sigma(r - r_B) = 0.01\sqrt{\frac{M-1}{N}}$. The information ratio, by definition, is

$$IR = \frac{\alpha}{\sigma(r - r_B)} = \sqrt{\frac{N}{M-1}} \tag{4}$$

1.2 (b)

$$IC = \frac{\mathbb{C}\mathrm{ov}\left[u_{i1}, r_i\right]}{\sigma(u_{i1})\sigma(r_i)} = \frac{0.01^2}{0.01\sqrt{0.01^2 M}} = \frac{1}{\sqrt{M}} \tag{5}$$

The information ratio implied by fundamental law of AM is therefore $\sqrt{N/M}$, which does not equal to IR in (a) exactly.

There are a few things that can be said about this observation.

1. The fundamental law of AM is a rule of thumb. In Grinold and Kahn's derivation, many assumptions are made to prove the law, for example, the active holdings solve the mean-variance problem; the conditional expectation of residual returns (which they call *refined forecast*) can be estiamted by

$$\widehat{\mathbb{E}}\left[r|g\right] = \mathbb{E}\left[r\right] + \mathbb{C}\mathrm{ov}\left[r, g\right] \cdot \mathbb{V}\mathrm{ar}^{-1}\left[g\right] \cdot (g - \mathbb{E}\left[g\right])$$

   for some signal g. These assumptions does not hold exactly for our case.

2. The information ratio derived in (1) becomes close to $\sqrt{N/M}$ when both M and N are $\gg 1$.

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2 Problem 2

2.1 (a)

```
In [80]: N, M, runs = 100, 200, 20000
         sample = np.zeros((runs, 2))
         bar = ProgressBar()
         for i in bar(range(runs)):
             uu = .01*np.random.choice([-1,1],(M, N))
             u1 = uu[0,:]
             r = (2/N)*(u1>0)*(uu.sum(0))
             rb = (1/N)*(uu.sum(0))
             sample[i, :] = np.array([r.sum(), rb.sum()])
```

```
100% (20000 of 20000) |###################| Elapsed Time: 0:00:05 Time: 0:00:05
```

```
In [81]: r_active = sample[:, 0] - sample[:, 1]
         print(f'Simulated mean = {r_active.mean():.6f}, std = {r_active.std():.6f}')
         print(f'Theoretical mean = {0.01:.6f}, std = {0.01*np.sqrt((M-1)/N):.6f}')
```

```
Simulated mean = 0.009956, std = 0.014129
Theoretical mean = 0.010000, std = 0.014107
```

The simulated mean and standard deviation of $r - r_B$ are very close to the theoretical values, which comfirms the results from problem 1.

2.2 (b)

```
In [105]: N, M, runs = 100, 200, 100000
          sample = np.zeros((runs, 2))
          bar = ProgressBar()
          for i in bar(range(runs)):
              uu = .01*np.random.choice([-1,1],(M, N))
              u1 = uu[0,:]
              L = (u1>0).sum()
              weight_scale = 1/L if L>0 else 0
              r = weight_scale*(u1>0)*(uu.sum(0))
              rb = (1/N)*(uu.sum(0))
              sample[i, :] = np.array([r.sum(), rb.sum()])
```

```
100% (100000 of 100000) |#################| Elapsed Time: 0:00:29 Time: 0:00:29
```

```
In [106]: r_active = sample[:, 0] - sample[:, 1]
          print(f'Simulated mean = {r_active.mean():.6f}, std = {r_active.std():.6f}')
```

```
Simulated mean = 0.009940, std = 0.014216
```

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3 Problem 3.

3.1 (a)

Let $g(\boldsymbol{f})$ be the objective function.

$$\begin{aligned} g(\boldsymbol{f}) &= (\boldsymbol{r} - \boldsymbol{B}\boldsymbol{f})^\top \boldsymbol{\Delta}^{-1}(\boldsymbol{r} - \boldsymbol{B}\boldsymbol{f}) \\ &= \boldsymbol{r}^\top \boldsymbol{\Delta}^{-1}\boldsymbol{r} - 2\boldsymbol{f}^\top \boldsymbol{B}^\top \boldsymbol{\Delta}^{-1}\boldsymbol{r} + \boldsymbol{f}^\top \boldsymbol{B}^\top \boldsymbol{\Delta}^{-1}\boldsymbol{B}\boldsymbol{f} \\ \Rightarrow \quad \nabla_{\boldsymbol{f}} g &= -2\boldsymbol{B}^\top \boldsymbol{\Delta}^{-1}\boldsymbol{r} + 2\boldsymbol{B}^\top \boldsymbol{\Delta}^{-1}\boldsymbol{B}\boldsymbol{f} \end{aligned} \tag{6}$$

3.2 (b)

The first order condition yields:

$$\begin{aligned} & -2\boldsymbol{B}^\top \boldsymbol{\Delta}^{-1}\boldsymbol{r} + 2\boldsymbol{B}^\top \boldsymbol{\Delta}^{-1}\boldsymbol{B}\boldsymbol{f} = 0 \\ \Rightarrow \quad & \boldsymbol{f} = (\boldsymbol{B}^\top \boldsymbol{\Delta}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}^\top \boldsymbol{\Delta}^{-1}\boldsymbol{r} \end{aligned} \tag{7}$$

3.3 (c)

In this case, both $\boldsymbol{b}^\top \boldsymbol{\Delta}^{-1}\boldsymbol{b}$ and $\boldsymbol{b}^\top \boldsymbol{\Delta}^{-1}\boldsymbol{r}$ are scalars. So does $f = \boldsymbol{b}^\top \boldsymbol{\Delta}^{-1}\boldsymbol{r}/\boldsymbol{b}^\top \boldsymbol{\Delta}^{-1}\boldsymbol{b}$. The return of characteristic portfolio is

$$r_p = \boldsymbol{r}^\top \boldsymbol{x} = \frac{1}{\boldsymbol{b}^\top \boldsymbol{\Delta}^{-1}\boldsymbol{b}} \boldsymbol{r}^\top \boldsymbol{\Delta}^{-1}\boldsymbol{b} = f^\top = f \tag{8}$$

3.4 (d)

By assumption, let $\mathbf{\Delta} = \mathrm{diag}\{\delta_i\}$, in this case

$$\boldsymbol{x} = \frac{1}{\boldsymbol{b}^\top \mathbf{\Delta}^{-1} \boldsymbol{b}} \mathbf{\Delta}^{-1} \boldsymbol{b} = \frac{1}{\sum_{i=1}^n \delta_i^{-1}} \begin{pmatrix} \delta_1^{-1} b_1 \\ \vdots \\ \delta_n^{-1} b_n \end{pmatrix} \tag{9}$$

So the result holdings are just $+\delta_i^{-1}/\sum \delta_i^{-1}$ for stocks in long list ($b_i = 1$), and $-\delta_i^{-1}/\sum \delta_i^{-1}$ for stocks in short list ($b_i = -1$). If $\Delta = \delta \boldsymbol{I}$, the holdings become $\mathrm{sgn}(b_i)\frac{1}{n}$, i.e. $\boldsymbol{x} = \frac{1}{n}\boldsymbol{b}$, which is just the signs of $\boldsymbol{b}$ times a constant scaler $1/n$.