Optimization Assignment II

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Problem 1. **(a)** The KKT optimality conditions are:

$$\begin{cases} 2\boldsymbol{V}\boldsymbol{x} - \lambda\boldsymbol{a} = 0 \\ \boldsymbol{a}^\top\boldsymbol{x} = 1 \end{cases} \tag{1}$$

Where λ is dual variable. The first equation $\Rightarrow \boldsymbol{x} = \frac{\lambda}{2}\boldsymbol{V}^{-1}\boldsymbol{a}$. Premultiply both sides by $\boldsymbol{a}^\top$:

$$\boldsymbol{a}^\top\boldsymbol{x} = 1 = \frac{\lambda}{2}\boldsymbol{a}^\top\boldsymbol{V}^{-1}\boldsymbol{a}$$

Therefore

$$\lambda = \frac{2}{\boldsymbol{a}^\top\boldsymbol{V}^{-1}\boldsymbol{a}}; \quad \boldsymbol{x} = \frac{1}{\boldsymbol{a}^\top\boldsymbol{V}^{-1}\boldsymbol{a}}\boldsymbol{V}^{-1}\boldsymbol{a} \tag{2}$$

(b) The KKT optimality conditions are:

$$\begin{cases} \boldsymbol{\mu} - \gamma\boldsymbol{V}\boldsymbol{x} + y\boldsymbol{1} = 0 \\ \boldsymbol{1}^\top\boldsymbol{x} = 1 \end{cases} \tag{3}$$

The first equation $\Rightarrow \boldsymbol{x} = \gamma^{-1}\boldsymbol{V}^{-1}(\boldsymbol{\mu} + y\boldsymbol{1})$. Re-write $\boldsymbol{x}$ in the following way:

$$\boldsymbol{x} = \frac{\boldsymbol{1}^\top\boldsymbol{V}^{-1}\boldsymbol{\mu}}{\gamma}\frac{1}{\boldsymbol{1}^\top\boldsymbol{V}^{-1}\boldsymbol{\mu}}\boldsymbol{V}^{-1}\boldsymbol{\mu} + \frac{y\boldsymbol{1}^\top\boldsymbol{V}^{-1}\boldsymbol{1}}{\gamma}\frac{1}{\boldsymbol{1}^\top\boldsymbol{V}^{-1}\boldsymbol{1}}\boldsymbol{V}^{-1}\boldsymbol{1} \tag{4}$$

Besides, premultiply both sides of the first equation by $\boldsymbol{1}^\top$ yields:

$$\begin{aligned} \boldsymbol{1}^\top\boldsymbol{x} = 1 = \boldsymbol{1}^\top\gamma^{-1}\boldsymbol{V}^{-1}(\boldsymbol{\mu} + y\boldsymbol{1}) \\ \Rightarrow \quad y = \frac{\gamma - \boldsymbol{1}^\top\boldsymbol{V}^{-1}\boldsymbol{\mu}}{\boldsymbol{1}^\top\boldsymbol{V}^{-1}\boldsymbol{1}} \end{aligned} \tag{5}$$

Therefore

$$\begin{aligned} \boldsymbol{x} &= \frac{\boldsymbol{1}^\top\boldsymbol{V}^{-1}\boldsymbol{\mu}}{\gamma}\frac{1}{\boldsymbol{1}^\top\boldsymbol{V}^{-1}\boldsymbol{\mu}}\boldsymbol{V}^{-1}\boldsymbol{\mu} + \frac{\gamma - \boldsymbol{1}^\top\boldsymbol{V}^{-1}\boldsymbol{\mu}}{\gamma}\frac{1}{\boldsymbol{1}^\top\boldsymbol{V}^{-1}\boldsymbol{1}}\boldsymbol{V}^{-1}\boldsymbol{1} \\ &= \lambda\frac{1}{\boldsymbol{1}^\top\boldsymbol{V}^{-1}\boldsymbol{\mu}}\boldsymbol{V}^{-1}\boldsymbol{\mu} + (1-\lambda)\frac{1}{\boldsymbol{1}^\top\boldsymbol{V}^{-1}\boldsymbol{1}}\boldsymbol{V}^{-1}\boldsymbol{1} \end{aligned} \tag{6}$$

Where $\lambda = \frac{\boldsymbol{1}^\top\boldsymbol{V}^{-1}\boldsymbol{\mu}}{\gamma}$.

(c) Denote the two problems (1) and (2).
We've already obtained the optimal solution to (2) in question (b):

$$x_2^* = \lambda\frac{1}{\boldsymbol{1}^\top\boldsymbol{V}^{-1}\boldsymbol{\mu}}\boldsymbol{V}^{-1}\boldsymbol{\mu} + (1-\lambda)\frac{1}{\boldsymbol{1}^\top\boldsymbol{V}^{-1}\boldsymbol{1}}\boldsymbol{V}^{-1}\boldsymbol{1} \tag{7}$$

where $\lambda = \frac{\boldsymbol{1}^\top\boldsymbol{V}^{-1}\boldsymbol{\mu}}{\gamma}$. Now we are going to solve problem (1). The KKT optimality conditions for (1) are:

$$\begin{cases} \boldsymbol{V}\boldsymbol{x} - \xi\boldsymbol{\mu} + \theta\boldsymbol{1} = 0 \\ \boldsymbol{1}^\top\boldsymbol{x} = 1 \\ \bar{\mu} - \boldsymbol{\mu}^\top\boldsymbol{x} \leq 0 \\ \xi \geq 0 \\ \xi(\bar{\mu} - \boldsymbol{\mu}^\top\boldsymbol{x}) = 0 \quad \text{(Complementary slackness)} \end{cases} \tag{8}$$

The first equation $\Rightarrow x = \xi \boldsymbol{V}^{-1}\mu - \theta \boldsymbol{V}^{-1}\mathbf{1}$. This premultiplied by $\mathbf{1}^\top$ and $\boldsymbol{\mu}^\top$, respectively:

$$\begin{cases} \mathbf{1}^\top x = \xi \mathbf{1}^\top \boldsymbol{V}^{-1}\mu - \theta \mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1} = 1 \\ \boldsymbol{\mu}^\top x = \xi \boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\mu - \theta \boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\mathbf{1} \geq \bar{\mu} \end{cases} \tag{9}$$

Multiply the first equality by $\frac{\boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\mathbf{1}}{\mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1}}$:

$$\Rightarrow \quad \begin{cases} \left(\xi \mathbf{1}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu} - \theta \mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1}\right) \frac{\boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\mathbf{1}}{\mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1}} = \frac{\boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\mathbf{1}}{\mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1}} \\ \xi \boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu} - \theta \boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\mathbf{1} \geq \bar{\mu} \end{cases} \tag{10}$$

The second inequality minus the first *equality* (the sign of the inequality does not change), the terms with θ cancel out:

$$\xi \left(\boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu} - \mathbf{1}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu} \frac{\boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\mathbf{1}}{\mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1}} \right) \geq \bar{\mu} - \frac{\boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\mathbf{1}}{\mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1}} > 0 \tag{11}$$

Where the last strictly greater sign is obtained from the know condition about $\bar{\mu}$. By the dual feasibility: $\xi \geq 0$. Therefore, the only way to make RHS > 0 is $\xi > 0$, and $\left(\boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu} - \mathbf{1}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu} \frac{\boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\mathbf{1}}{\mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1}} \right) > 0$. Therefore, the complementary slackness equality evaluates to:

$$\bar{\mu} - \boldsymbol{\mu}^\top \boldsymbol{x} = 0 \tag{12}$$

Which suggests that the inequality constraint is binding. As a result, (11) takes the equal sign. We have:

$$\begin{aligned} &\xi \left(\boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu} - \mathbf{1}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu} \frac{\boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\mathbf{1}}{\mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1}} \right) = \bar{\mu} - \frac{\boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\mathbf{1}}{\mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1}} \\ \Rightarrow \quad &\xi = \frac{\bar{\mu}(\mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1} - \boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\mathbf{1})}{(\boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu})(\mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1}) - (\boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\mathbf{1})^2} \end{aligned} \tag{13}$$

And the optimal solution for problem (1) is therefore:

$$\begin{aligned} x_1^* &= \xi \boldsymbol{V}^{-1}\mu - \theta \boldsymbol{V}^{-1}\mathbf{1} \\ &= (\xi \mathbf{1}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu}) \frac{1}{\mathbf{1}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu}} \boldsymbol{V}^{-1}\mu - (\theta \mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1}) \frac{1}{\mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1}} \boldsymbol{V}^{-1}\mathbf{1} \\ &= (\xi \mathbf{1}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu}) \frac{1}{\mathbf{1}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu}} \boldsymbol{V}^{-1}\mu - (1 - \xi \mathbf{1}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu}) \frac{1}{\mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1}} \boldsymbol{V}^{-1}\mathbf{1} \\ &= \alpha \frac{1}{\mathbf{1}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu}} \boldsymbol{V}^{-1}\mu - (1 - \alpha) \frac{1}{\mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1}} \boldsymbol{V}^{-1}\mathbf{1} \end{aligned} \tag{14}$$

where $\alpha = \xi \mathbf{1}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu}$. Clearly, $x_1^* = x_2^*$ exactly $\iff \alpha = \lambda \iff$

$$\begin{aligned} &\xi \mathbf{1}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu} = \frac{\mathbf{1}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu}}{\gamma} \\ \Rightarrow \quad &\gamma = \frac{1}{\xi} = \frac{(\boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\boldsymbol{\mu})(\mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1}) - (\boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\mathbf{1})^2}{\bar{\mu}(\mathbf{1}^\top \boldsymbol{V}^{-1}\mathbf{1} - \boldsymbol{\mu}^\top \boldsymbol{V}^{-1}\mathbf{1})} \end{aligned} \tag{15}$$

Problem 2. (a) The minimum-variance long-only portfolio is solved by:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x} \\ s.t. \quad & \mathbf{1}^\top \boldsymbol{x} = 1 \\ & \boldsymbol{x} \succeq \mathbf{0} \end{aligned} \tag{16}$$

It has variance

$$\sigma_p^2 = 179.735026 \quad \text{BPs}^2 \tag{17}$$

(b) The betas of each stock are given by: $\boldsymbol{\beta} = \frac{1}{\boldsymbol{x}^{*\top} \boldsymbol{\Sigma} \boldsymbol{x}^*} \boldsymbol{\Sigma} \boldsymbol{x}^*$, where $\boldsymbol{x}^* = \mathbf{1}/20$ is the market portfolio weights vector. See the table below for numerical values.

Ticker	BOL	NE	AZO	FISV	DGX	SYK	STZ	TIF	SVU	MIL
Beta	0.6887	1.3330	0.6634	0.7017	1.3295	0.7484	1.3100	1.1785	0.7815	0.7301
Ticker	LEN	PAYX	RHI	NTAP	LH	R	FDO	MKC	XTO	ABC
Beta	1.0256	0.8383	0.8161	2.0953	1.5769	0.6097	0.8977	0.3364	1.7062	0.6332

Table 1: Beta of Stocks

(c) The required portfolio is solved by:

$$\begin{aligned} \max_{\boldsymbol{x}} \quad & \boldsymbol{\mu}^\top \boldsymbol{x} \\ s.t. \quad & \mathbf{1}^\top \boldsymbol{x} = 1 \\ & \boldsymbol{\beta}^\top \boldsymbol{x} = 1 \\ & \frac{\mathbf{1}}{10} \succeq \boldsymbol{x} \succeq \mathbf{0} \end{aligned} \tag{18}$$

The expected return of the required portfolio is 29.33 BPs.

(d) The required portfolio is solved by:

$$\begin{aligned} \min_{\boldsymbol{z},k} \quad & \boldsymbol{z}^\top \boldsymbol{\Sigma} \boldsymbol{z} \\ s.t. \quad & \boldsymbol{\mu}^\top \boldsymbol{z} = 1 \\ & \mathbf{1}^\top \boldsymbol{z} - k = 0 \\ & \boldsymbol{\beta}^\top \boldsymbol{z} - k = 0 \\ & \frac{k}{10}\mathbf{1} \succeq \boldsymbol{z} \succeq \mathbf{0} \end{aligned} \tag{19}$$

The maximum Sharpe ratio is 1.53746