Homework 3

46-923, Fall 2017

Due Thursday, November 16 at 3:00 PM

You should submit a single pdf file with your responses to the questions. There is nothing wrong with handwritten solutions; I am not asking you to learn Latex to complete the homework.

Please do not submit photos of your homework. Scanners are available for your use.

Question 1:

Assume that $X_1, X_2, \ldots, X_n$ are i.i.d.~with mean μ and variance σ^2. Both μ and σ^2 are unknown. Assume that the sample mean is $\overline{x} = 14.3$ and the sample standard deviation is $s = 4.2$.

a. Without any further information, can you construct a valid 95% confidence interval for μ if $n = 40$? If so, do it.
b. Without any further information, can you construct a valid 95% confidence interval for μ if $n = 10$? If so, do it.
c. If I told you that the X_i are normal, can you construct a valid 95% confidence interval for μ when $n = 10$? If so, do it.

Question 2:

Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. from the Poisson(λ) distribution.

a. Find the maximum likelihood estimator for λ.
b. What is the standard error of the estimator found in part (a)?
c. Suppose that a sample of size $n = 5$ is obtained:

$$1, 1, 2, 4, 5$$

Write out a statement reporting the best estimate of λ, along with the standard error of the estimator. This should be the type of statement you would include in a formal report.

Question 3:

Let $X_1, X_2, \ldots, X_n$ be i.i.d. from the following distribution:

$$f_X(x) = \begin{cases} \dfrac{\alpha\lambda^\alpha}{x^{\alpha+1}} & x > \lambda,\ \alpha > 0,\ \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

Assume that λ is a known constant, and show that the MLE for α is

$$\widehat{\alpha} = n \bigg/ \left(\sum_{i=1}^{n} \log\left(\frac{X_i}{\lambda}\right)\right).$$

Comment: This is the *Pareto distribution*.

Question 4:

Let $X_1, X_2, \ldots, X_n$ be a iid from the following distribution:

$$f_X(x;\theta) = \begin{cases} \dfrac{3x^2}{\theta^3} & 0 \le x \le \theta \\ \\ 0 & \text{otherwise} \end{cases}$$

Here are some facts regarding estimation of θ:

- Method of moments estimator is $\hat{\theta}_1 = \dfrac{4\bar{X}}{3}$
- The maximum likelihood estimator is $\hat{\theta}_2 = X_{(n)}$, where $X_{(n)}$ is the maximum of the X_i.

Here are some useful results about this distribution that will help you do this problem.

- $E(X) = \dfrac{3}{4}\theta$
- $V(X) = \dfrac{3}{80}\theta^2$
- $f_{X_{(n)}}(x) = \dfrac{3nx^{3n-1}}{\theta^{3n}}, \quad 0 \le x \le \theta$
- $V(X_{(n)}) = \dfrac{3n}{(3n+2)(3n+1)^2}\,\theta^2$

Now, do the following:

a. Show that $\hat{\theta}_1$ is an unbiased estimator for θ.

b. Show that $\hat{\theta}_2$ is *not* an unbiased estimator for θ, and find $\text{bias}(\hat{\theta}_2) = [E(\hat{\theta}_2) - \theta]$.

c. Show that for $n > 2$, even though $\hat{\theta}_1$ is unbiased and $\hat{\theta}_2$ is not, $\text{MSE}(\hat{\theta}_2) < \text{MSE}(\hat{\theta}_1)$