Optimization Assignment 5

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Problem 1. (a) Note that Y and $\mu+\sigma Z$ are identically distributed, where Z is standard normal.

$$\begin{aligned} &\mathrm{VaR}_\alpha(Y) = \inf\{y : \mathbb{P}\left(Y \geq y\right) = 1-\alpha\} = \inf\{y : \mathbb{P}\left(\frac{Y-\mu}{\sigma} < \frac{y-\mu}{\sigma}\right) = \alpha\} \\ &\Rightarrow \quad \frac{\mathrm{VaR}_\alpha(Y)-\mu}{\sigma} = \Phi^{-1}(\alpha) \quad \Rightarrow \quad \mathrm{VaR}_\alpha(Y) = \mu + \sigma\Phi^{-1}(\alpha) \end{aligned} \tag{1}$$

$$\begin{aligned} \mathrm{CVaR}_\alpha(Y) &= \mathbb{E}\left[Y | Y \geq \mu+\sigma\Phi^{-1}(\alpha)\right] = \mathbb{E}\left[\mu+\sigma Z | Z \geq \Phi^{-1}(\alpha)\right] = \frac{\mathbb{E}\left[\mu+\sigma Z ; Z \geq \Phi^{-1}(\alpha)\right]}{\mathbb{P}\left(Z \geq \Phi^{-1}(\alpha)\right)} \\ &= \frac{1}{1-\alpha}\left(\mu(1-\alpha) + \sigma\int_{\Phi^{-1}(\alpha)}^{\infty} \frac{z}{\sqrt{2\pi}} e^{-z^2/2} dz\right) = \mu + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha} \end{aligned} \tag{2}$$

(b) Note that Y and $e^{\mu+\sigma Z}$ are identically distributed, where Z is standard normal.

$$\begin{aligned} &\mathrm{VaR}_\alpha(Y) = \inf\{y : \mathbb{P}\left(Y \geq y\right) = 1-\alpha\} = \inf\{y : \mathbb{P}\left(\frac{\log Y-\mu}{\sigma} < \frac{\log y-\mu}{\sigma}\right) = \alpha\} \\ &\Rightarrow \quad \frac{\log \mathrm{VaR}_\alpha(Y)-\mu}{\sigma} = \Phi^{-1}(\alpha) \quad \Rightarrow \quad \mathrm{VaR}_\alpha(Y) = \exp\left(\mu + \sigma\Phi^{-1}(\alpha)\right) \end{aligned} \tag{3}$$

$$\begin{aligned} \mathrm{CVaR}_\alpha(Y) &= \mathbb{E}\left[Y | Y \geq e^{\mu+\sigma\Phi^{-1}(\alpha)}\right] = \mathbb{E}\left[e^{\mu+\sigma Z} | Z \geq \Phi^{-1}(\alpha)\right] = \frac{\mathbb{E}\left[e^{\mu+\sigma Z} ; Z \geq \Phi^{-1}(\alpha)\right]}{\mathbb{P}\left(Z \geq \Phi^{-1}(\alpha)\right)} \\ &= \frac{1}{1-\alpha}\left(\int_{\Phi^{-1}(\alpha)}^{\infty} e^{\mu+\frac{\sigma^2}{2}} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2+2\sigma z-\sigma^2}{2}} dz\right) = \frac{1}{1-\alpha} e^{\mu+\frac{\sigma^2}{2}} \Phi(\sigma - \Phi^{-1}(\alpha)) \end{aligned} \tag{4}$$

Problem 2. (a) We setup a linear program to solve VaR and cVaR, the results are:

$$\mathrm{VaR}_{0.9}(Y) = 40 \qquad \mathrm{cVaR}_{0.9}(Y) = 44.0693 \tag{5}$$

(b) We achieve the cVaR optimization by adding one more free variable to the program above: the portfolio weights $\boldsymbol{x}$. The optimal portfolio is

$$\boldsymbol{x}^* = \begin{pmatrix} 0.875 & 0.125 & 0 \end{pmatrix}^\top \tag{6}$$

And the corresponding portfolio VaR and cVaR are:

$$\mathrm{VaR}_{0.9}(Y) = 8.75 \qquad \mathrm{cVaR}_{0.9}(Y) = 24.5853 \tag{7}$$

Problem 3.

$$W_{t+1} = \begin{cases} W_t(1+f_1+f_2) & \text{If win both gambles at time } t \\ W_t(1+f_1-f_2) & \text{If win the first and lose the second gamble at time } t \\ W_t(1-f_1+f_2) & \text{If win the second and lose the first gamble at time } t \\ W_t(1-f_1-f_2) & \text{If lose both gambles } t \end{cases} \tag{8}$$

Therefore

$$W_n = W_0(1+f_1+f_2)^{\frac{k_3}{n}}(1-f_1+f_2)^{\frac{k_2}{n}}(1+f_1-f_2)^{\frac{k_1}{n}}(1-f_1-f_2)^{\frac{n-k_1-k_2-k_3}{n}} \tag{9}$$

Let $g = \log(W_n/W_0)^{1/n}$, $x = f_1 + f_2, y = f_1 - f_2$ we have:

$$\begin{aligned} \mathbb{E}[g] &= p_1p_2\log(1+f_1+f_2) + (1-p_1)p_2\log(1-f_1+f_2) + p_1(1-p_2)\log(1+f_1-f_2) \\ &\quad + (1-p_1)(1-p_2)\log(1-f_1-f_2) \\ &= p_1p_2\log(1+x) + (1-p_1)p_2\log(1-y) + p_1(1-p_2)\log(1+y) \\ &\quad + (1-p_1)(1-p_2)\log(1-x) \end{aligned} \tag{10}$$

First order conditions:

$$\begin{aligned} \nabla_x\mathbb{E}[g] &= \frac{p_1p_2}{1+x} - \frac{(1-p_1)(1-p_2)}{1-x} = 0 \quad \Rightarrow \quad x = \frac{p_1+p_2-1}{2p_1p_2-p_1-p_2+1} \\ \nabla_y\mathbb{E}[g] &= \frac{p_1(1-p_2)}{1+y} - \frac{(1-p_1)p_2}{1-y} = 0 \quad \Rightarrow \quad y = \frac{p_1-p_2}{-2p_1p_2+p_1+p_2} \end{aligned} \tag{11}$$

$$\begin{aligned} f_1 &= \frac{x+y}{2} = \frac{1}{2}\left(\frac{p_1+p_2-1}{2p_1p_2-p_1-p_2+1} + \frac{p_1-p_2}{-2p_1p_2+p_1+p_2}\right) \\ f_2 &= \frac{x-y}{2} = \frac{1}{2}\left(\frac{p_1+p_2-1}{2p_1p_2-p_1-p_2+1} - \frac{p_1-p_2}{-2p_1p_2+p_1+p_2}\right) \end{aligned} \tag{12}$$

In this problem, we have $p_1 = p_2 = p$, therefore

$$f_1 = f_2 = \frac{2p-1}{2(2p^2-2p+1)} = \frac{p^2-(1-p)^2}{2(p^2+(1-p)^2)} \tag{13}$$