Options Assignment 4

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Problem. 14
Solution. (a)

Figure 1: RN simulation of options

Parameters			Options Prices		
stock price	100			prices	error sd's
mu	0.07		Euro	8.413889	1.106453
sigma	0.38		Asian	4.909657	0.633384
			max/min	27.42326	0.827141
r	0.025		KO call	4.482573	1.012829
true m	0.00875		bi Asian	0.526698	0.049849
RN m	-0.04720				
m in sim	-0.04720		Black Scholes Price and Greeks		
			d1	0.127895	
delta t	0.003846		price	7.860534	
trd. days	65		Delta	0.550884	
expiry	0.25		gamma	0.020826	
pv dollar	0.993769		kappa	19.78464	
strike	100				
KO	97				
N (paths)	100				

(b) The RN price is

$$c_t^{RN} = \mathbb{E}^{RN}\left[c_T^*\right] e^{-r(T-t)}$$

Hence the Monte-Carlo sample (of size n) estimate of RN price is given by

$$\hat{c}_t^{RN} = \frac{e^{-r(T-t)}}{n} \sum_{i=1}^{n} c_T^* \tag{1}$$

The standard error is

$$SE(\hat{c}_t^{RN}) = \sqrt{\left(\frac{e^{-r(T-t)}}{n}\right)^2 \sum_{i=1}^{n} \mathbb{V}\mathrm{ar}\left[c_T^*\right]} = \frac{e^{-r(T-t)}}{n}\sqrt{n\hat{\sigma}^2} = \frac{e^{-r(T-t)}}{\sqrt{n}}\hat{\sigma} \tag{2}$$

Where n is the size of each Monte-Carlo sample, $\hat{\sigma}$ is the sample standard deviation. The numerical results are shown in the table above.

(c) For sure, the option valuations will change, because our sample estimate of c_t^{RN} (from Monte-Carlo), a.k.a. $\hat{c}_t^{RN}$, is itself a random variable. By CLT: the monte-carlo estimator has asymptotic distribution:

$$\hat{c}_t^{RN} \xrightarrow{p} \mathcal{N}\left(c_t^{RN}, \frac{\sigma_c^2}{n}\right) \tag{3}$$

Where c_t^{RN} is real RN price, σ_c^2 is a constant.
Our new monte carlo sample is displayed in the table below

Figure 2: RN simulation of options: another monte-carlo sample

Parameters			Options Prices			
stock price	100			prices	error sd's	
mu	0.07		Euro	8.549442	1.436051	
sigma	0.38		Asian	4.196099	0.646345	
			max/min	28.07866	1.065275	
r	0.025		KO call	4.395612	1.329218	
true m	0.00875		bi Asian	0.506822	0.049929	
RN m	-0.0472					
m in sim	-0.0472		Black Scholes Price and Greeks			
			d1	0.127895		
delta t	0.003846		price	7.860534		
trd. days	65		Delta	0.550884		
expiry	0.25		gamma	0.020826		
pv dollar	0.993769		kappa	19.78464		
strike	100					
KO	97					
N (paths)	100					

Standard Normals

day	path 1	path 2	path 3	path 4	path 5	path 6
1	2.192635	-2.56552	0.069527	-0.20665	-0.87762	-0.09643
2	-0.72609	-0.40002	-1.65994	-0.11473	-0.35783	-0.37563
3	-0.19587	-0.38338	0.711067	-1.35029	1.034972	-0.71808
4	-1.65527	0.444777	-1.30011	0.480845	1.180009	0.12239
63	0.284849	-0.02309	0.062837	-0.66821	-1.39049	-0.20905
64	0.583815	0.51561	1.01725	0.344072	0.316445	0.228322
65	-1.07571	-0.16154	0.196599	-0.11753	-0.34109	1.828486

Stock Price Paths

day	path 1	path 2	path 3	path 4	path 5	path 6
0	100	100	100	100	100	100
1	105.284	94.11601	100.1458	99.49613	97.93521	99.75489
2	103.479	93.21601	96.28634	99.20947	97.09518	98.85778
3	102.9837	92.36084	97.89567	96.08471	99.47447	97.18128
4	99.0258	93.3171	94.92448	97.16208	102.261	97.4443
63	103.4869	125.7255	83.5764	130.656	89.73299	101.8525
64	104.9015	127.2394	85.58866	131.6959	90.38827	102.3834
65	102.257	126.7329	85.97051	131.3078	89.64833	106.8723

	path 1	path 2	path 3	path 4	path 5	path 6
Euro	2.256993	26.73292	0	31.30776	0	6.872304
Asian	0.154769	13.55596	0	7.300724	5.052892	0
max/min	22.33779	34.87856	23.72147	44.15432	37.78395	29.42198
KO call	0	0	0	0	0	0
bi Asian	1	1	0	1	1	0

(d) The Black-Scholes price is the ground-truth RN price $c_t^{BS} = c_t^{RN}$, while our estimation from Monte-Carlo sample, $\hat{c}_t^{RN}$, is its estimate. Clearly $c_t^{MCS} = \hat{c}_t^{RN}$ is a random variable, there is no guarantee that it should be equal to c_t^{RN}, and the average amount of error is measured by the SE as we calculated in (a) and (b).

However, when the Monte-Carlo sample size $n \to \infty$, we will see the MCS price converge to Black-Scholes price in probability, i.e. for all $\epsilon > 0$:

$$\lim_{n\to\infty} \mathbb{P}\left(|c_t^{MCS} - c_t^{RN}| \geq \epsilon\right) = 0 \tag{4}$$

Problem. 20
Solution. The stock price and option price trees are given by:

Figure 3: Binomial valuation

Inputs									
Calculations									
Parameter Values			**Stock Price Tree**						
S	27								
sigma	0.2100				33.30931				
T-t	0.5			29.98919					
n	2		27		27				
dt	0.25			24.30876					
					21.88577				
u	1.1107								
d	0.9003		**European Call Price Tree**				**American Call Price Tree**		
true prob q	0.6								
					8.309308				4.099498
r				5.419162				3.113246	
R	1.018		3.442999		2				1.996395
				1.094751			1.728645		
pi_u	0.5474				0				0.102921
pi_d	0.4354							0.056336	
									0
X (European strike	25								
Y (Asian strike)	26								

So the price of European call at $t = 0$ is

$$c_{eu}(0) = 3.44300$$

The price of Asian call at $t = 0$ is

$$c_{asian}(0) = 1.72865$$

The replication delta (position in European option) is computed as

$$\Delta(t_i) = \frac{c_{asian}(u, t_{i+1}) - c_{asian}(d, t_{i+1})}{c_{eu}(u, t_{i+1}) - c_{eu}(d, t_{i+1})}$$

Where u means up state and d means down state.

Figure 4: Replication of Asian with European Call

Dynamic Replication			
		Delta	0.3333333
		Eu option v	1.806387
		cash positic	1.3068588
Delta	0.7068962	As option v	3.113246
Eu option v	2.433843		
cash positic	-0.705198		
As option v	1.728645	Delta	0.05146
		Eu option v	0.056336
		cash positic	0.00000
		As option v	0.05634

Denote cash position as B_t. Refer to the table above, we have:

$$\begin{aligned}&\Delta_0 = 0.706896; \quad B_0 = -0.705198\\&\Delta_1(u) = 0.33333; \quad B_1(u) = 1.3068588\\&\Delta_1(d) = 0.05146; \quad B_1(d) = 0\end{aligned} \tag{5}$$

Problem. 23

Solution. (a) We first calcuate u, d, with CRR calibration:

$$u = e^{\sigma\sqrt{t}} = 1.2214; \qquad d = e^{-\sigma\sqrt{t}} = 0.8187 \tag{6}$$

The state prices are

$$\pi_u = \frac{R-d}{R(u-d)} = 0.5653; \qquad \pi_d = \frac{u-R}{R(u-d)} = 0.3781 \tag{7}$$

Denote value of company, bond (prior to coupon payment), and equity as V, B, E respectively. Let the face value and coupon rate of bond be F and c. We have:

$$B_t = \begin{cases} \min(V_T, F(1+c)) & t = T \\ \pi_u B_{t+1,u} + \pi_d B_{t+1,d} & 0 \le t < T \end{cases}$$

Hence

$$E_t = \begin{cases} \max(0, V_T - F(1+c)) & t = T \\ V_t - B_t & 0 \le t < T \end{cases}$$

We assume that $\{V_t\}$ evolves by the binomial tree model. Denote $V_t^{[cum]}$ as cum dividend cum interest value, $V^{[ex]}$ as ex dividend ex interest value, we have:

$$\begin{aligned}&V_t^{[ex]} = V_t^{[cum]}(1-q) - Fc\\&V_{t,u}^{[cum]} = uV_t^{[ex]}\\&V_{t,d}^{[cum]} = dV_t^{[ex]}\end{aligned} \tag{8}$$

The bond price at time 0 is therefore calculated as

$$P_0 = \frac{B_0}{\text{principle}} \times 100\$$$

Where principle = 3 billion. We proceed by calibrate to the observed bond price $P_0 = 99.05$, and solve for the desired company value V_0. See the table below for details.

Figure 5: Calibrate to company value

Inputs								
Firm Value Process:			**Int. Rate and State Prices:**			**Bond:**		
V	4.195050145		R	1.06		principal	3	(billion)
in tree 1=	1000000000	(billion)	piu	0.5653		shares	30000000	
			pid	0.3781		coupon (%)	0.07	
sigma	0.2					coupon ($)	0.21	
maturity	2		**Stock:**			final	3.21	
n	2		# shares	0.05		face	100	
dt	1		div/shr	N/A		Treasury rate	0.06	
			cash divide	N/A				
u	1.2214		div rate	0.005		**Credit Spread**		
d	0.8187					ytm_0	0.075293	
						credit spread_0	0.015293	
						shares (partb)	1942.18464	
Ordinary Bond and Stock Trees:								
							uuV	5.970493438
		cum coupon, cum div			ex coupon ex div		bond	3.21
		uV	5.1238458		uV	4.8882266	equity	2.760493438
		bond	3.2383019		bond	3.028301887		
		equity	1.8855439		equity	1.859924702	udV	4.002141436
V	4.195050145						bond	3.21
bond value	2.97149997						equity	0.792141436
bond price	99.049999							
calc(ytm)	1.49324E-09							
ytm	0.075293							
equity	1.223550175						duV	3.917580315
s price	24.47						bond	3.21
		dV	3.4346166		dV	3.207443482	equity	0.707580315
		bond	3.0174799		bond	2.807479937		
		equity	0.4171366		equity	0.399963545	ddV	2.626032617
							bond	2.626032617
							equity	0

We obtain

$$E_0 = 4.19505 \quad \text{billion } \$$$

The yield to maturity at time 0 is therefore $y(0) = 7.5293\%$, the strike of the credit spread option, i.e. the credit spread at time 0 is

$$X = y(0) - r_f = 7.5293\% - 6\% = 1.5293\%$$

Subsequent ex-coupon ytms and credit spreads are:

$$\begin{aligned} y(1,u) &= 6\%; \quad cs(1,u) = y(1,u) - r_f = 0 \\ y(1,d) &= 14.3374\%; \quad cs(1,d) = y(1,d) - r_f = 8.3374\% \end{aligned} \tag{9}$$

We are now ready to price the credit spread option:

$$\begin{aligned} v(1,u) &= (cs(1,u) - X)^+ = 0 \\ v(1,d) &= (cs(1,u) - X)^+ = 0.068081 \\ v(0) &= v(1,u)\pi_u + v(1,u)\pi_d = 0.025744 \end{aligned} \tag{10}$$

See the table below. As a result, the fair value of the credit spread option at time 0 is:

$$v(0) = 0.025744 \ \$ \qquad \text{per 1\$ notional} \tag{11}$$

Figure 6: Binomial valuation of credit spread option

Payoff of the Credit Spread Option:				
year 0		maturity: year 1, ex coupon ex div		
option value	0.025744205	bond value	3.0283019	
		bond price	100.9434	
		calc(ytm)	0	
		ytm	0.06	
		credit spread	0	
		option value	0.000	
		bond value	2.8074799	
		bond price	93.582665	
		calc(ytm)	1.218E-13	
		ytm	0.1433742	
		credit spread	0.0833742	
		option value	0.068081	

(b) Since we are writing the option, to delta hedge this short position, essentially we want to replicate a **long** position.
Denote notional $N = 50$ million. The number of corporate bond at time 0 that one need to replicate credit spread option is given by:

$$\Delta_0 = \frac{N(v(1,u) - v(1,d))}{P^{[cum]}(1,u) - P^{[cum]}(1,d)} \tag{12}$$

Where $P^{[cum]}(1,s)$ is the cum coupon payoff of the bond at time 1, state s, i.e. the total cashflow of the bond at that state. It's calcuated as:

$$P^{[cum]}(1,s) = \frac{B_1(s)}{\text{principle}} + \text{coupon}$$

See the table below for calculations. We get $\Delta_0 = -0.4624607$, with unit price being the time-0 bond price, a.k.a. $P_0 = 99.05\$$.

Figure 7: Dynamic Hedging

		maturity, cum coupon	
		option value	0.000
		bond payoff	107.9433962
year 0	1.287210263	tbill payoff	106
notional (million)	50	bond value	-49.91958167
bond price	99.049999	tbill value	49.9195817
tbill price	100		
delta	-0.462460729		
bond notional	-45.8067347		
tbill notional	47.09394497	option value	3.404049
sanity check	0	bond payoff	100.5826646
		tbill payoff	106
		bond value	-46.5155323
		tbill value	49.9195817

So the total amount of money invested in corporate bond is

$$\Delta_0 P_0 = -45.806735 \quad \text{million } \$ \tag{13}$$

The total amount of money in t-bill is hence the option value minus the bond value, which is

$$Nc_0 - \Delta_0 P_0 = 47.093945 \ \text{ million \$} \tag{14}$$

Therefore, we short 45.807 millon dollars of corporate bond, and buy 47.093 millon dollars of T-bills to cover our position.

Problem. 32

Solution. (a) The market implied volatility is about

$$\sigma_i = 27.427\%$$

while assume our estimate of stock volatility $\sigma_a = 25\%$ is an oracle, then by BS formula, the fair option price should be $c(t, S_t; \sigma_a) = 70.296 < c(t, S_t; \sigma_i) = 76.5$.

Figure 8: BSM option price

Inputs:			**Calculations:**	
S	1300		h	0.1365
X	1300		N(h)	0.5543
			h-sig*sqrt	0.0115
r	0.057		N(h-sig*sqrt)	0.5046
expiry T-t	0.25			
pv	0.9859		**Implied**	
div	0.02		h	0.1360
qv	0.9950		N(h)	0.5541
			h-sig*sqrt	-0.0011
monthly vol	0.08660254		N(h-sig*sqrt)	0.4996
(ann) sigma	0.2500		market price	76.500
(implied) sigma	0.274267249		"fair" price	70.29635733

We therefore conclude that the option is **overvalued**.

(b) Denote implied volatility σ_i, actual volatility underlying the stock price dynamics σ_a, i.e. $dS_t = \mu S_t dt + \sigma_a S_t dW_t$. Futher assume that we dynamically hegde with delta $\Delta_h = \Delta(t, S_t, \sigma_h)$, i.e. we calculate the delta with BSM formula and another volatility parameter σ_h.

By (Carr et.al, 2005) [1], assume we can dynamically hedge continuously, and no transaction cost, the PnL of this volatility aribitrage stategy is given by

$$PnL(0,T) = c(t, S_t; \sigma_i) - c(t, S_t; \sigma_h) + \frac{1}{2}(\sigma_a^2 - \sigma_h^2) \int_0^T e^{-rt} S_t^2 \Gamma(t, S_t; \sigma_h) dt \tag{15}$$

The first term is deterministic spread between option prices, the second term is an integral, whose value is positive when $\sigma_a > \sigma_h$, but random.

As we assume our prediction 25% is the true volatility σ_a, we can obtain a deterministic PnL if we set $\sigma_h = \sigma_a = 25\%$, i.e. hedge with the actual volatility. Then the integral term vanishes, we are left with

$$PnL(0,T) = c(t, S_t; \sigma_i) - c(t, S_t; \sigma_a) \approx 76.5 - 70.296 = 6.204 \tag{16}$$

for each pair we trade. See the table below.

Figure 9: BSM option and portfolio greeks

Black-Scholes Prices and Greeks:								
	price	delta	omega	gamma	vega	theta		PnL
call	70.296357331	0.551522	10.199379	0.0024201	255.626579	-150.3346		48429.696
put	58.386500	-0.443490	-9.874492	0.0024201	255.626579	-103.1534		
stock	1300	1	1	0	0	0		
bond	1	0	0	0	0	0.0570		
Positions								
	price	delta	omega	gamma	vega	theta	position	value
call	76.500	0.551522	10.199379	0.002420	255.626579	-150.334628	-7806.654636	-597209.079619
stock	1300	1	1	0	0	0	4305.545446	5597209
bond	1	0	0	0	0.00	0.0570	-5000000.000000	-5000000.000000
portfolio	0	0	-75317	-18.8931	-1995588.42	888610.52		0.00

We set the cash limit to be 5 million, the portfolio is given by:

$$\begin{aligned} &-7806.6546 \quad \text{options} \\ &4305.5454 \quad \text{stocks} \\ &-5 \text{ million} \quad \text{(borrowed) from bank account} \end{aligned} \tag{17}$$

The PnL is

$$PnL = 48429.696 \ \ \$ \tag{18}$$

(c) The portfolio greeks are calculated in the table, with

$$\begin{aligned} &\Delta_p = 0 \\ &\Gamma_p = -18.8931 \\ &\Theta_p = 888610.52 \\ &\text{Vega}_p = -1995588.42 \end{aligned} \tag{19}$$

Interpretation: Denote portfolio value as Π, we have, approximately

$$d\Pi_t \approx \Delta_p dS_t + \frac{1}{2}\Gamma_p (dS_t)^2 + \Theta_p dt + \text{Vega}_p d\sigma \tag{20}$$

1. $\Delta_p = 0$: the portfolio is not sensitive to the directional (linear) change in stock price.
2. $\Gamma_p = -18.8931$: the portfolio is sensitive to the curvature (non-linear) change in stock price, with about -18.8931/unit change in S^2.
3. $\Theta_p = 888610.52$: the portflio grows linearly with time elapsed, with about 888610.52/unit change in t (years). Note that the telescoping summation of the combined effect of Θ_p and Γ_p is positive, which brings us a cumulated positive PnL.
4. $\text{Vega}_p = -1995588.42$: the portfolio is sensitive to the change of the real volatility in stock prices. Since our belief is $\sigma_a < \sigma_i$, an increase in σ_a is NOT in our favor, and the portfolio value will decrease.

References

[1] Carr, P. *FAQ's in Option Pricing Theory.* 2005, 40-43.