Options Assignment 5

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December 11, 2017

Problem. 17

Solution. We establish the following SDE system to simulate interest rate dynamics:

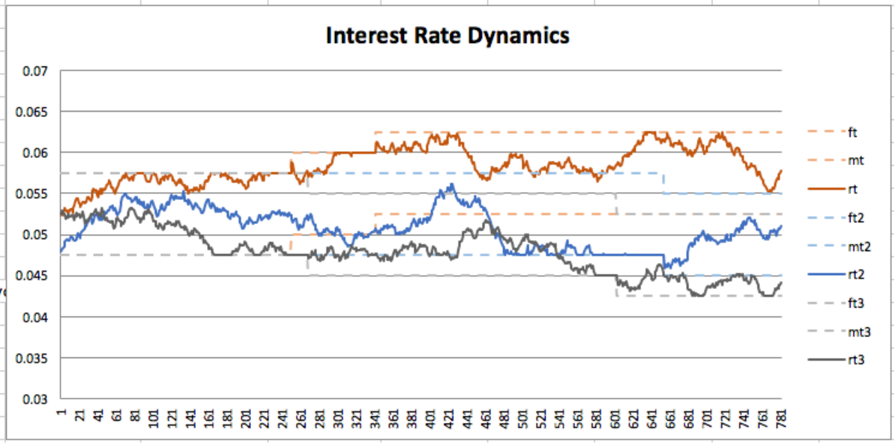
$$
\begin{aligned}
&r_t = r_t^* \mathbb{1}_{\{f_t \le r_t^* \le m_t\}} + f_t \mathbb{1}_{\{r_t^* < f_t\}} + m_t \mathbb{1}_{\{r_t^* > m_t\}} \\
&dr_t^* = \mathbb{1}_{\{r_t^* < f_t\}} \alpha (f_t - r_t^*) dt + \mathbb{1}_{\{r_t^* > m_t\}} \alpha (m_t - r_t^*) dt + \sigma dW_t \\
&f_t = \begin{cases} f_0 & t \le \frac{1}{12} \\ f_{t-\frac{1}{12}} + \gamma \mathbb{1}_{A_t} - \gamma \mathbb{1}_{B_t} & t > \frac{1}{12} \end{cases} \\
&m_t = f_t + b \\
&A_t := \bigcap_{s \in [t-\frac{1}{12}, t)} \{r_s^* > m_s\}; \quad B_t := \bigcap_{s \in [t-\frac{1}{12}, t)} \{r_s^* < f_s\}
\end{aligned}
\tag{1}
$$

Where r_t is the interest rate under FED's control, $r_t \in [f_t, m_t]$, m_t and f_t are the rate ceiling and floor. r_t^* is the shadow spot rate. W_t is a wiener process, the unit of time is year.
We choose parameters as suggested in the problem:

- $b = 1\%$ is the band width.
- $\sigma = 0.005$ is the annualized local volatility.
- $\alpha = 12$ is the mean-reversion speed.
- $\gamma = 0.25\%$ is the one-time band adjustment by FED.
- $f_0 = 4.75$ and $m_0 = 5.75$ are arbitrarily chosen.

Figure 1: Interest Rate Simulation

Fed Policy Band		
band	0.01	
f0	0.0475	
m0	0.0575	
lookback	22	(a month)
adapt	0.0025	
Timing		
T-t	3	
n	780	
dt	0.003846154	
Shadow Rate		
sigma	0.0050	(absolute v
alpha	12.0000	
r0	0.0525	



Problem. 24

Solution. **(a)** We first calcuate u, d, with CRR calibration:

$$u = e^{\sigma\sqrt{t}} = 1.2214; \quad d = e^{-\sigma\sqrt{t}} = 0.8187 \tag{2}$$

The state prices are

$$\pi_u = \frac{R-d}{R(u-d)} = 0.5377; \quad \pi_d = \frac{u-R}{R(u-d)} = 0.4192 \tag{3}$$

Denote value of company, bond (prior to coupon payment), and equity as V, B, E respectively. Let the face value and coupon rate of bond be F and c. We have:

$$B_t = \begin{cases} \min(V_T, F(1+c)) & t = T \\ \pi_u B_{t+1,u} + \pi_d B_{t+1,d} & 0 \le t < T \end{cases}$$

Hence

$$E_t = \begin{cases} \max(0, V_T - F(1+c)) & t = T \\ V_t - B_t & 0 \le t < T \end{cases}$$

We assume that $\{V_t\}$ evolves by the binomial tree model. Denote $V_t^{[cum]}$ as cum dividend cum interest value, $V^{[ex]}$ as ex dividend ex interest value, we have:

$$\begin{aligned} V_t^{[ex]} &= V_t^{[cum]}(1-q) - Fc \\ V_{t,u}^{[cum]} &= uV_t^{[ex]} \\ V_{t,d}^{[cum]} &= dV_t^{[ex]} \end{aligned} \tag{4}$$

The company value at time 0 is the sum of market values of bonds and equities

$$V_0 = E_0 + B_0 = 752 \ \text{ million}$$

We then calculate the fair price of bonds and equity via binomial pricing. See the following table.

Figure 2: Binomial Pricing for Capital Structure Arbitrage

Inputs								
Firm Value Process			**Interst Rates**			**Bond**		
V0	752		R	1.045		Principle	450	
unit	1000000		piu	0.537721998		coupon	0.075	
sigma	0.2		pid	0.419216		dollar coup	33.75	
maturity	2					final	483.75	
n	2		**Stock**			market val	455	
dt	1		# shares	22				
			share price	13.5				
u	1.221402758		div rate	0.01				
d	0.818730753							
Binomial Model							uuV	1069.41131
		Cum coupon cum div			Ex		Bond	483.75
		uV	918.4948741		uV	875.559925	Equity	585.661308
		Bond	496.6686603		Bond	462.91866		
		Equity	421.8262138		Equity	412.641265	udV	716.847837
V	752						Bond	483.75
Bond (market)	455						Equity	233.097837
Equity (market)	297							
Bond (binom)	473.1119564							
Euqity (binom)	278.8880436						duV	703.257657
overprice (stock	18.1119564						Bond	483.75
		dV	615.6855263		dV	575.778671	Equity	219.507657
		Bond	491.4945752		Bond	457.744575		
		Equity	124.1909511		Equity	118.034096	ddV	471.407705
							Bond	471.407705
							Equity	0

We saw the AFP for bonds and equities are:

$$\begin{aligned} E_{AFP} &= 278.8804 < E_{market} = 297 \\ B_{AFP} &= 473.1120 > B_{market} = 455 \end{aligned} \tag{5}$$

So we believe the **Equities** are overpriced, and **Bonds** are underpriced, both by the amount of 18.11196 million.

(b) Since We believe the **Equities** are over-priced relative to bonds, we will short stock and long bond. Our portfolio is structured as

Figure 3: Capital Structure Arbitrage Portfolio

Arbitrage Portfolio				
T = 0; (-) cash inflow			T = 1	
Mkt Bond price/$f	1.011111111		Bond price/fac	1.103708134
Mkt Stock price	13.5		Stock price	19.17391881
Bond position	1		Bond value	1.103708134
Bond value	1.011111111		Stock value	-0.016295595
Stock position	-0.000849883		Tbill	-1.087412538
Stock value	-0.011473426		Net position	0
Tbill	-1.040586161			
Net cash outflow	-0.040948476		Bond price/fac	1.092210167
			Stock price	5.645043233
Interpretation:			Bond value	1.092210167
Short T-bill	-1040586.161		Stock value	-0.004797629
Long corp bond	1011111.111		Tbill	-1.087412538
	(1000000 face)		Net position	0
Short Stock	-11473.42602			
	(850 shares)			
Net Cash outflow	-40948.47614			

At time 0, we long 1 million face of Corporate bond, which is financed by shorting 849.88 shares and shorting 1.040586 million dollars in T-Bills. This portfolio has a net 40948.47 dollars cash inflow at $t = 0$. And at $t = 1$, the total value will be sure to become 0, hence we close all positions without cost, and take a

$$\Pi = 40948.47 \quad \$$$

riskless profit.

(c) The accuracy of our volatility estimate will definitely affect the outcome of this trade, in that the amount in which the stock is over-priced relative to bonds is a function of real volatility.

Figure 4: Outcomes for different real volatility

(Bad)						**(Good)**	
T = 1, If sigma = 0.3			**T = 1, If sigma = 0.2**			**T = 1, If sigma = 0.1**	
Bond price/fac	1.103708134		Bond price/face$	1.103708134		Bond price/fa	1.103708134
Stock price	23.56478014		Stock price	19.17391881		Stock price	15.20090319
Bond value	1.103708134		Bond value	1.103708134		Bond value	1.103708134
Stock value	-0.020027316		Stock value	-0.016295595		Stock value	-0.012918995
Tbill	-1.087412538		Tbill	-1.087412538		Tbill	-1.087412538
Net position	-0.00373172		Net position	0		Net position	0.0033766
Bond price/fac	0.997079018		Bond price/face$	1.092210167		Bond price/fa	1.103708134
Stock price	4.927715636		Stock price	5.645043233		Stock price	8.353139913
Bond value	0.997079018		Bond value	1.092210167		Bond value	1.103708134
Stock value	-0.004187984		Stock value	-0.004797629		Stock value	-0.007099195
Tbill	-1.087412538		Tbill	-1.087412538		Tbill	-1.087412538
Net position	-0.094521505		Net position	0		Net position	0.0091964

- If $\sigma = 0.2$, then the stocks are overpriced by exactly the amount as we predicted. So our portfolio will make exactly a PnL of $\Pi = 40948.47$ dollars.
- If $\sigma < 0.2$, say $\sigma = 0.1$, this will be a "good" situation for us, as the stocks are overpriced by more than the predicted amount. So our PnL will be greater. In the $\sigma = 0.1$ case, our PnL will be

state-dependent:

$$\Pi_u = 44325.08 \text{ \$}, \qquad \Pi_d = 50144.88 \text{ \$}$$

- If $\sigma > 0.2$, say $\sigma = 0.3$, this will be a "bad" situation for us, as the stocks are overpriced by more than the predicted amount. So our PnL will be smaller, and it's possible to have a negative PnL, i.e. we'll lose money if we close all the positions at $t = 1$. In the $\sigma = 0.3$ case, our PnL will be state-dependent:

$$\Pi_u = 37216.76 \text{ \$}, \qquad \Pi_d = -53573.02 \text{ \$}$$

If the relative mispricing instead worsens over the next year, then closing out all the positions will incur a negative PnL. Indeed, if I choose to hold on to my potitions until the relative mispricing is finally corrected, then I will recover from this drawdown at year 1. However, since I was forced to close out all positions due to a margin call, the loss is inevitable.

Problem. 27

Solution. Let n, d be our two states (no-default, default), let $B(t, s)$ be the bond value at time t, state s, par F, let $CDS(t, s)$ be the value of CDS at time t, state s, premium X and default payment P. Using binomial model, we have

$$B(t,s) = \begin{cases} \pi_n B(t+1,n) + \pi_d B(t+1,d) + 0.07F & s = n, 0 < t < T \\ \pi_n B(t+1,n) + \pi_d B(t+1,d) & t = 0 \\ 0.7F & s = d \\ 1.07F & s = n, t = T \end{cases}$$

and

$$CDS(t,s) = \begin{cases} \pi_n CDS(t+1,n) + \pi_d CDS(t+1,d) - X & s = n, 0 < t \leq T \\ \pi_n CDS(t+1,n) + \pi_d CDS(t+1,d) & t = 0 \\ P & s = d \end{cases}$$

We first calibrate state prices such that the model bond price is equal to the market value, i.e. finding π_n, π_d such that $B(0) = 465$. We get

$$\pi_n = 0.7461; \qquad \pi_d = 0.2224$$

Then we solve for premium X such that the CDS worth 0 at time 0, we find

$$X = 2.9806$$

Therefore, the quoted premium after markup is

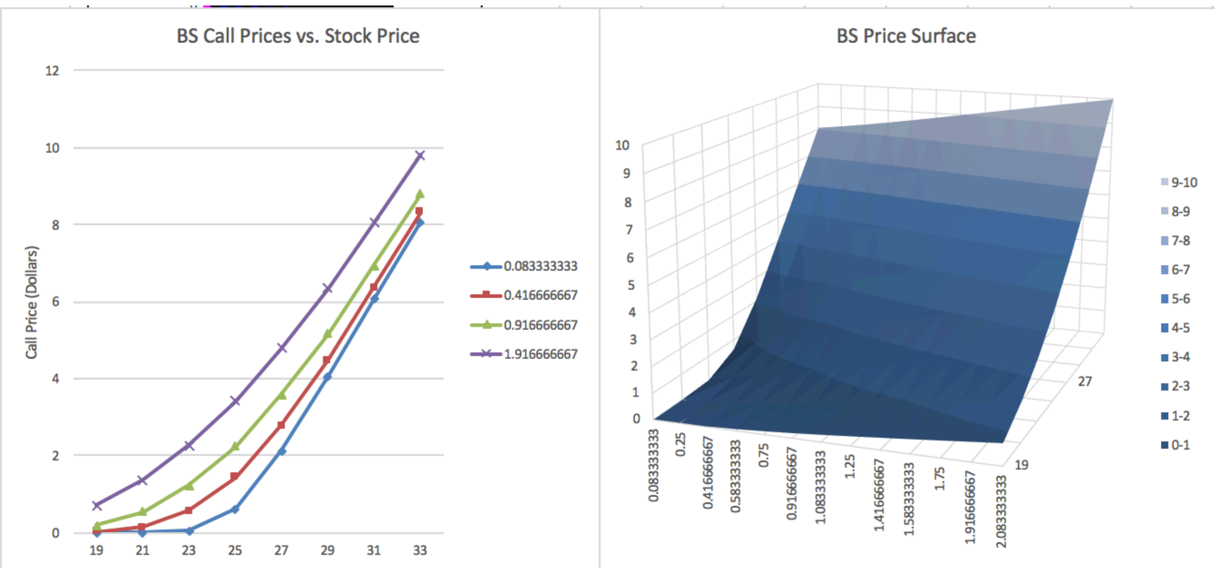
$$X^* = X(1 + 0.0025) = 2.9880$$

Figure 5: Credit Default Swap Valuation

Int. Rate and State Prices:			**ABCo Bond:**			**Default Swap Terms:**		
R	1.0325		principal	500		payment if default	10	
			coupon (%)	0.07		breakeven premium X	2.9806	(from Solver)
piu (no default)	0.7461	(from Solver)	coupon ($)	35		mark-up	0.0025	
pid (default)	0.2224		final	535		quoted prem	2.9880	
RN prob no def	0.7704		Recovery Rate	0.7				
RN prob default	0.2296		Mkt Price	465				
Bond and Swap Value Trees:						No Default		
				No Default		Bond	535	
				Bond	519.80			
				Swap CF	-2.9806			
		No Default		Swap Value	-2.9806	Default		
		Bond	508.46			Bond	385.00	
		Swap CF	-2.9806					
		Swap Value	-2.9806					
				Default				
				Bond	385.00			
Model Bond	465			Swap CF	10			
Swap Value	-1E-06			Swap Value	10			
		Default						
		Bond	385.00					
		Swap CF	10					
		Swap Value	10					

Problem. 34
Solution. (1)

Figure 6: B-S call price surface



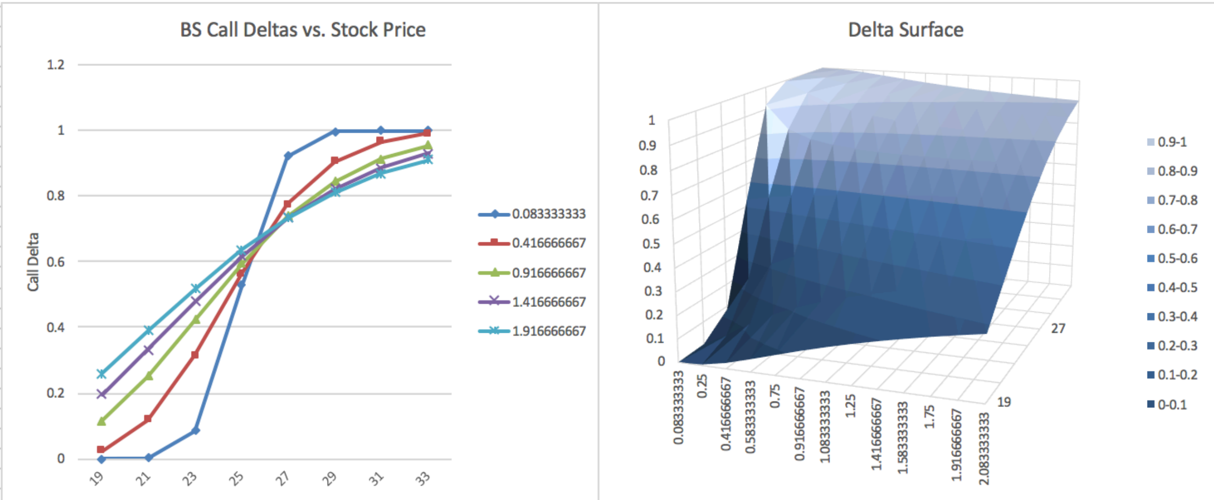
Intuition:

- Fix $T-t$, the call price $\nearrow$ with $S_t \nearrow$, simply because the payoff of the call, $(S_T-K)^+$ is positively related with S_T, and higher S_t is likely to evolve into higher S_T given fixed time to maturity.
- Fix S_t, the call price $\nearrow$ with $T-t \nearrow$. Longer time to maturity means that there is more untertain variation in the final stock price at S_T, due to the "symmetry" of brownian motion, the probability of going up is same as the probability of going down. The upside extreme variation will cause high option payoff, while the downside extreme variation will **NOT** cause negative payoff, since

$(S_T - K)^+$ has a limited downside risk. Therefore, longer time to maturity $\Rightarrow$ more variation in stock $\Rightarrow$ higher value of call.

(2)

Figure 7: B-S call Delta surface



Intuition:

- Fix $T-t$, $\Delta \nearrow$ with $S_t \nearrow$. In Black-Scholes framework with no dividend, the value of Δ is exactly

$$\mathbb{P}\,(\text{The option is in the money at maturity})$$

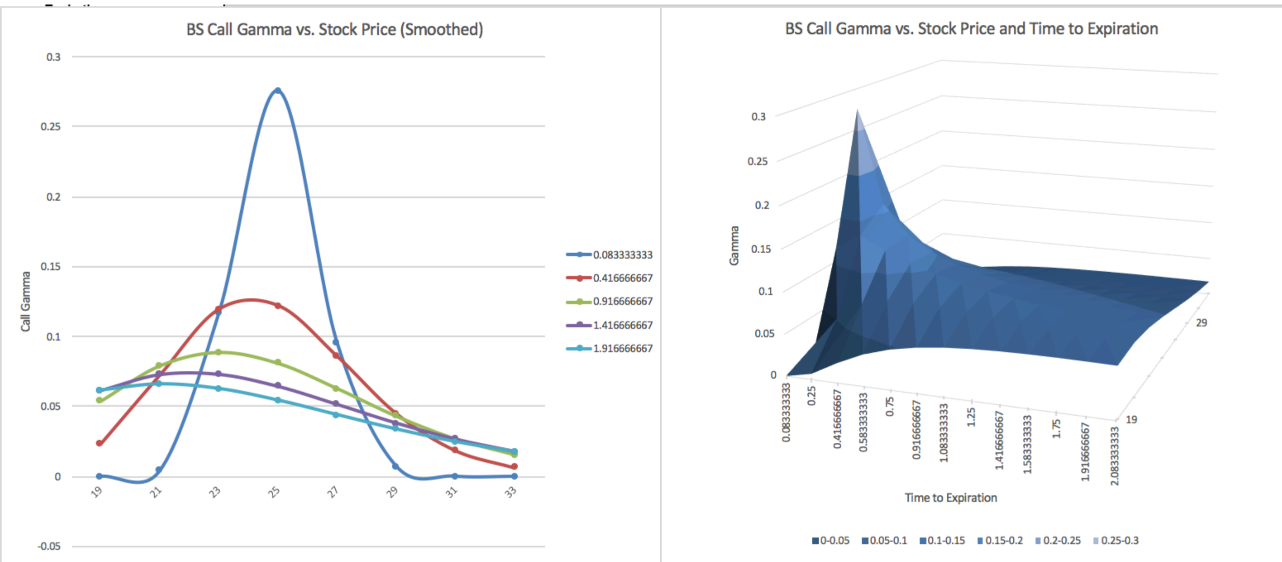
  For sure, higher $S_t \Rightarrow$ higher in-the-moneyness of the option $\Rightarrow$ more likely to expire in the money $\Rightarrow$ higher Δ.
- Fix S_t, Δ **steepens** around the center K, with $T-t \nearrow$. In particular when $T-t=0$, Δ converges to the *Heaviside Step Function*, i.e.

$$\Delta(S; T-t=0) = \mathbb{1}_{\{S>K\}}$$

  This is still due to the meaning of Delta as the probability of the option expires in the money. If $T-t=0$, whether or not the option expires in the money is deterministic: it's just $\mathbb{1}_{\{S_T>K\}}$. On the otherhand if time to maturity is long, there will be much variation in the stock price, and one is not sure about whether or not the option will expire in-the-money. Therefore the curve of Δ flattens around K.

(3)

Figure 8: B-S call Gamma surface



Intuition:

- Fix $T-t$, Γ first goes up, then goes down with $S_t \nearrow$. In particular, it peaks up around K. The plot of Γ can be figured out by its definition: the derivative of Δ with respect to S. Since $\Delta \nearrow$ with $S \Rightarrow \Gamma$ is always positive. Since Δ is the steepest around $K \Rightarrow \Gamma$ peaks up around K.
- Fix S_t, Γ **peaks higher** around the center K, with $T-t \nearrow$. In particular when $T-t=0$, Δ converges to the *Dirac* δ function, i.e. This is still due to the meaning of Delta as the probability of the option expires in the money. Once again, such behavior is clear when we regard Γ as the derivative of Δ. Since Δ flattens with longer time to maturity $\Rightarrow \Gamma$ also flattens. Since Δ steepens around K with shorter time to maturity, and converges to *Heaviside Step Function* when $T-t \to 0$ $\Rightarrow \Gamma$ peaks up higher around K with shorter time to maturity, and converges to the derivative of Heaviside function, which is the Dirac's Delta function when $T-t \to 0$.

Problem. 36
Solution. (a) In the Black-Scholes world, the price of any derivatives is a function of only 2 *variables*: the underlying price S_t and time t. While there are many other factors that affect the option valuation, like σ, r, and contractual terms, these are characterized as *constants*. Therefore, there is no risk associated with other factors other than S_t and t.
To this end we can use Ito's lemma, the value of any option $c = c(t, S_t|\sigma, r, K, T, ...) = c(t, S_t)$

$$dc(t, S_t) = \frac{\partial c}{\partial t}dt + \frac{\partial c}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}dS_t^2$$

Therefore, the change of option price only depends on $\frac{\partial c}{\partial t}, \frac{\partial c}{\partial S}, \frac{\partial^2 c}{\partial S^2}$. In other words, only these greeks are priced.

(b)

Figure 9: Portfolio Greeks

Current inputs:								
Stock	41		pv	0.9925				
Strike x	41		qv	1				
r	0.03		h	0.1304	h2	-0.044643		
q	0		N(h)	0.5519	N(h2)	0.4822		
T-t	0.25							
sigma	0.3500							
Black Scholes Prices and Greeks (now):								
	price	delta	gamma	theta	vega			
call	3.003866	0.551858	0.0551313	-6.2651	8.1091			
put	2.697516	-0.448142	0.0551313	-5.0442	8.1091			
stock	41	1	0	0.0000	0			
bonds	1	0	0	0.0300	0			
Price of Delta								
	value	delta	gamma	theta	vega	position		Price of 1 Delta
stock	41	1	0	0	0	1		
portfolio change	41	1	0	0	0			41
Price of Theta								
	value	delta	gamma	theta	vega	position		Price of 1 Theta
bonds	1	0	0	0.03	0	33.33333		
portfolio change	33.33333333	0	0	1	0			33.33333333
Price of Gamma								
	value	delta	gamma	theta	vega	position		Price of 1 Gamma
calls	54.49	10.01	1.00	-113.64	147.09	18.13852	(# calls)	
stock	-410.41	-10.01	0.00	0.00	0.00	-10.01	(# shares)	
bonds	3,787.96	0.00	0.00	113.64	0.00	$3,787.96	($s in t-bills)	
portfolio (now):	$3,432.04	0.00	1.00	0.00	147.09			3432.04

- To add 1 Δ to portfolio without changing any other priced greek of the portfolio, we can buy 1 stock. So the price of 1 Delta is $S = 41$.
- To add 1 Θ to portfolio without changing any other priced greek of the portfolio, we can buy $1/r$ dollars of bond. So the price of 1 Theta is $1/r = 33.333$.
- To add 1 Γ to the portfolio without changing any other priced greek of the portfolio, we can buy a Delta and Theta hedged options that has Gamma of 1. See the table above. The cost of this trade is the price of 1 Gamma, which is 3432.04 dollars.

Problem. 38

Solution. We looked at the option chain of SPDR S&P500 ETF that expires at Feb.16 2018, approximately 3 months from now.

The SPY price is around 265.73 as of Dec.9 2017. And the call option prices that corresponds to different strikes are listed in the table. We solve the Black-Scholes implied volatility, and display them in a plot. We can observe that there we have volatility skew in this option chain. The option price with higher strike implies a lower volatility around 5.55%, while the option price with lower strike implies a higher volatility around 6.40%

Figure 10: SPDR S&P500 ETF Options

SPY (SPDR S&P 500) Options
As of 12/09/2017
Source: Interactive Broker

Parameters:		
S_t	265.73	
r	0.0128	(3m t
pv dollar	0.9966088	
delta T	0.0038462	
remaining days	69	
T-t	0.2653846	
strike	265	
KO	260	
vol_t=a+b*S_t+c*		
a	1.5244042	
b	-0.0055	
c	0.0456502	
N (paths)	1000	

Option Chain

Name	strike	Mkt Price (Mid)	BS ISD	BS Price	ISD target	h
SPY Feb 16'18 262-Call	262	6.25	0.0640	6.2509	7.61929E-07	0.5486
SPY Feb 16'18 263-Call	263	5.65	0.0654	5.6500	1.34279E-10	0.4240
SPY Feb 16'18 264-Call	264	4.85	0.0621	4.8500	2.25928E-12	0.3262
SPY Feb 16'18 265-Call	265	4.16	0.0603	4.1600	2.31669E-12	0.2135
SPY Feb 16'18 266-Call	266	3.58	0.0597	3.5805	2.49882E-07	0.0928
SPY Feb 16'18 267-Call	267	3.03	0.0587	3.0308	5.70103E-07	-0.0302
SPY Feb 16'18 268-Call	268	2.48	0.0569	2.4800	8.508E-14	-0.1598
SPY Feb 16'18 269-Call	269	2.01	0.0555	2.0100	6.92367E-11	-0.2946

MCS European Calls

MCS Price	SE	sq errors
6.2997	0.1641	0.0025
5.5610	0.1544	0.0079
4.8617	0.1439	0.0001
4.2013	0.1328	0.0017
3.5830	0.1211	0.0000
3.0007	0.1092	0.0009
2.4826	0.0964	0.0000
1.9983	0.0835	0.0001
	total SSE	0.0131

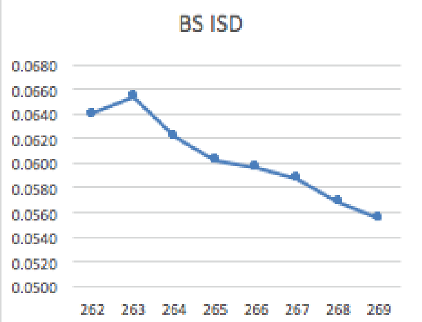


Table 1: BS Implied Volatility Skew

Option Chain Name	Strike	Mkt Price (Mid)	BS Implied Vol
SPY Feb 16'18 262-Call	262	6.25	0.0640
SPY Feb 16'18 263-Call	263	5.65	0.0654
SPY Feb 16'18 264-Call	264	4.85	0.0621
SPY Feb 16'18 265-Call	265	4.16	0.0603
SPY Feb 16'18 266-Call	266	3.58	0.0597
SPY Feb 16'18 267-Call	267	3.03	0.0587
SPY Feb 16'18 268-Call	268	2.48	0.0569
SPY Feb 16'18 269-Call	269	2.01	0.0555

Then we calibrate a local volatility model that matches Monte-Carlo price to market prices. We choose the model in the functional form of

$$v(t, S_t) = a + bS_t + ct$$

And obtain parameter estimates:

$$\begin{aligned} \hat{a} &= 1.5244 \\ \hat{b} &= -0.0055 \\ \hat{c} &= 0.4565 \end{aligned} \tag{6}$$

I.e. the volatility tends to increase with time, and increase when the underlying price drops.