Options Assignment 3

Ze Yang (zey@andrew.cmu.edu)

November 25, 2017

Problem. 2 (*Futures and forward cash flows*)

Solution. (a) The variation margin cash flow of future contract is given by

$$C_t = F_{t,T} - F_{t-\Delta t,T}$$

Since future contracts are linear in there cash flows, the cash flow for 20 longs is just $20C_t$, and that for 2 shorts is $-2C_t$.

Figure 1: Variation Margin Cashflow of Future Contracts

Day	Market Futures/foward prices	Futures Contract Variation Cash Flows	Long 20 Futures Contract Variation Cash Flows	Short 2 Futures Contract Variation Cash Flows
0 (initial)	50.2500	0.0000	0.0000	0.0000
1	51.5574	1.3074	26.1483	-2.6148
2	49.9032	-1.6543	-33.0850	3.3085
3	49.7240	-0.1791	-3.5830	0.3583
4	50.5355	0.8115	16.2294	-1.6229
5	49.7192	-0.8163	-16.3265	1.6327
6	49.2056	-0.5136	-10.2721	1.0272
7	47.9776	-1.2279	-24.5583	2.4558
8	46.6715	-1.3061	-26.1220	2.6122
9	45.9922	-0.6793	-13.5861	1.3586
10	45.4607	-0.5315	-10.6302	1.0630
11	44.0451	-1.4156	-28.3123	2.8312
12	43.6695	-0.3756	-7.5117	0.7512
13	43.4029	-0.2666	-5.3329	0.5333
14	43.4853	0.0824	1.6478	-0.1648
15	43.2446	-0.2407	-4.8142	0.4814
16	43.0298	-0.2147	-4.2945	0.4295
17	42.7886	-0.2412	-4.8243	0.4824
18	43.6475	0.8589	17.1772	-1.7177
19	43.5873	-0.0602	-1.2040	0.1204
20	43.4619	-0.1254	-2.5084	0.2508
21	43.1264	-0.3355	-6.7101	0.6710
22	44.4062	1.2798	25.5962	-2.5596
23	44.9770	0.5709	11.4177	-1.1418
24	46.5907	1.6137	32.2733	-3.2273
25 (delivery)	46.1337	-0.4571	-9.1412	0.9141

(b) The variation margin cash flow of forward contract is:

$$C_t = \begin{cases} S_T - F_0 & t = T \\ 0 & \text{otherwise} \end{cases}$$

Figure 2: Variation Margin Cashflow of Forward Contracts

Day	Market Futures/foward prices	Forward Contract Cash Flows
0 (initial)	50.2500	0.0000
1	51.5574	0.0000
2	49.9032	0.0000
3	49.7240	0.0000
4	50.5355	0.0000
5	49.7192	0.0000
6	49.2056	0.0000
7	47.9776	0.0000
8	46.6715	0.0000
9	45.9922	0.0000
10	45.4607	0.0000
11	44.0451	0.0000
12	43.6695	0.0000
13	43.4029	0.0000
14	43.4853	0.0000
15	43.2446	0.0000
16	43.0298	0.0000
17	42.7886	0.0000
18	43.6475	0.0000
19	43.5873	0.0000
20	43.4619	0.0000
21	43.1264	0.0000
22	44.4062	0.0000
23	44.9770	0.0000
24	46.5907	0.0000
25 (delivery)	46.1337	-4.1163

Problem. 18 (*Binomial*)

Solution. (a) We have:

$$\pi_u = \frac{R-d}{R(u-d)} = 0.3127; \qquad \pi_d = \frac{u-R}{R(u-d)} = 0.6774 \tag{1}$$

Let $c_t(S_t)$ be the value of american call at time t, we have:

$$c_t(S_t, D_t) = \max\left\{\max(0, S_t^{[cum]} - K), \pi_u c_{t+1}(uS_t^{[ex]}) + \pi_d c_{t+1}(dS_t^{[ex]})\right\} \tag{2}$$

Where $S_t^{[cum]}$ is the cum dividend stock price, $S_t^{[ex]}$ is the ex dividend stock price. $S_t^{[ex]} = S_t^{[cum]} - D_t$. The binomial tree calulation is presented in the table below.
We obtain:

$$c_0 = 1.2569$$

I.e. the 1-year American call option worth 1.2569 dollars today.

(b) Denote the value of asian call option along path w as $Ac_n(w)$, the number of periods is N. We have:

$$Ac_n = \begin{cases} \max(0, \frac{1}{N}\sum_{i=0}^{N} S_i^{[cum]}(w)) & n = N \\ \pi_u Ac_{n+1}((w,u)) + \pi_d Ac_{n+1}((w,d)) & 0 \le n < N \end{cases} \tag{3}$$

The binomial tree calulation is presented in the table below.
We obtain:

$$Ac_0 = 0.2830$$

I.e. the 1-year Asian call option worth 0.2830 dollars today.

Figure 3: American Option

Inputs						
Calculations						
Parameter Values			**Stock Price Tree**	Cum Div	Ex Div	
S	43					
sigma	0.2624					51.6078
T-t	1			49.02	45.27	
n	2					43.0065
dt	0.5		43			
						42.294
u	1.1400			40.85	37.1	
d	0.9500					35.245
true prob q	0.6					
			American Call			
r						
R	1.010					6.6078
					4.0200	
pi_u	0.3127					0
pi_d	0.6774			1.2569		
						0
K (strike)	45				0.0000	
	45					0
Cash Dividend	3.75					

Figure 4: Asian option

Asian Call			
			2.8759
		0.9052	
			0.0088
	0.2830		
			0.0000
		0	
			0.0000

Problem. 19 (*Binomial(2)*)

Solution. (a) For the dynamic replication strategy, the stock and bond positions are given by:

$$\Delta_t = \frac{p_{t+1,u} - p_{t+1,d}}{S_t(u-d)} \tag{4}$$

$$B_t = \frac{u p_{t+1,d} - d p_{t+1,u}}{R(u-d)} \tag{5}$$

where the value of American put option price at time t is:

$$p_t = \max\left\{\max(K - S_t, 0), S_t\Delta_t + B_t\right\} \tag{6}$$

i.e. the maximum between replicating portfolio and the early exercise payoff. The trees of option price and dynamic replication portfolio are presented below.
The dynamic strategy is described by the right blue tree, with

$$\begin{aligned} &\Delta_0 = -0.47502, \quad B_0 = 25.4844 \\ &\Delta_{1,u} = 0, \quad B_{1,u} = 0, \quad \Delta_{1,d} = -1, \quad B_{1,d} = 45.5437 \end{aligned} \tag{7}$$

And the price at time 0 is $c_0 = 1.7334$.

Figure 5: American option (Dynamic Replication)

Stock Price Tree							
		61.07014					
	55.25855						
50		50					
	45.24187						
		40.93654					
American Put Price Tree (Replication)				Dynamic Rep. Positions Tree			
		0			0		Delta
	0				0		Bond
1.733386		0		-0.47502			
	4.758129			25.48443			
		9.063462			-1		
					48.54369		

(b) We first calculate the state prices:

$$\pi_u = \frac{R-d}{R(u-d)} = 0.6066; \qquad \pi_d = \frac{u-R}{R(u-d)} = 0.3643 \tag{8}$$

And the value of American put option price at time t is:

$$p_t = \max\left\{\max(K - S_t, 0), \quad p_{t+1,u}\pi_u + p_{t+1,d}\pi_d\right\} \tag{9}$$

Plug the equations into the binomal tree, we obtain: $p_0 = 1.7334$.

(c) We first calculate the risk-neutral probabilities:

$$q_u^{RN} = \frac{R-d}{u-d} = 0.6248; \qquad q_d^{RN} = \frac{u-R}{u-d} = 0.3752 \tag{10}$$

And the value of American put option price at time t is:

$$p_t = \max\left\{\max(K - S_t, 0), \quad \tfrac{1}{R}\left(p_{t+1,u}q_u^{RN} + p_{t+1,d}q_d^{RN}\right)\right\} \tag{11}$$

Plug the equations into the binomal tree, we obtain: $p_0 = 1.7334$.

(d) Now with the dividends, the value of American put option price at time t is:

$$p_t = \max\left\{\max(K - S_t^{[ex]}, 0), \quad p_{t+1,u}\pi_u + p_{t+1,d}\pi_d\right\} \tag{12}$$

For the call:

$$c_t = \max\left\{\max(S_t^{[cum]} - K, 0), \quad c_{t+1,u}\pi_u + c_{t+1,d}\pi_d\right\} \tag{13}$$

See the trees below.

Figure 6: American option (Dividend)

Stock Price Tree (Dividend)							
			58.8598				
	55.25855	53.25855					
			48.19033				
50							
			48.34224				
	45.24187	43.74187					
			39.57928				
American Put Price Tree (With Dividend)				American Call Price Tree (With Dividend)			
		0				8.859796	
	0.659264				5.374121		
		1.809675				0	
2.679728				3.259801			
		1.657756				0	
	6.258129				0		
		10.42072				0	

We got $c_0 = 3.2598$, $p_0 = 2.6797$. The put call parity for European option is

$$c_t - p_t = S_t - D_t - Ke^{-r(T-t)}$$

Where D_t is the cash dividend paid at time t. Clearly this equality does not hold for American option. Because with early exericise, this relation becomes an inequality:

$$S_t - D_t - K \leq c_t - p_t \leq S_t - Ke^{-r(T-t)}$$

Problem. 21 (*Binomial/Capital Structure*)

Solution. We first calcuate u, d, with CRR calibration:

$$u = e^{\sigma\sqrt{t}} = 1.2969; \quad d = e^{-\sigma\sqrt{t}} = 0.7711 \tag{14}$$

The state prices are

$$\pi_u = \frac{R-d}{R(u-d)} = 0.4822; \quad \pi_d = \frac{u-R}{R(u-d)} = 0.4859 \tag{15}$$

Denote value of company, bond (prior to coupon payment), and equity as V, B, E respectively. Let the face value and coupon rate of bond be F and c. We have:

$$B_t = \begin{cases} \min(V_T, F(1+c)) & t = T \\ \pi_u B_{t+1,u} + \pi_d B_{t+1,d} & 0 \leq t < T \end{cases}$$

Hence

$$E_t = \begin{cases} \max(0, V_T - F(1+c)) & t = T \\ V_t - B_t & 0 \leq t < T \end{cases}$$

We assume that $\{V_t\}$ evolves by the binomial tree model. Denote $V_t^{[cum]}$ as cum dividend cum interest value, $V^{[ex]}$ as ex dividend ex interest value, we have:

$$\begin{aligned} V_t^{[ex]} &= V_t^{[cum]} - D_t - Fc \\ V_{t,u}^{[cum]} &= uV_t^{[ex]} \\ V_{t,d}^{[cum]} &= dV_t^{[ex]} \end{aligned} \tag{16}$$

See the binomial tree below for calculation details. We set $E_0 = 250$ as target and solve for desired coupon rate c^*. And we obtain:

$$c^* = 6.435\%$$

Figure 7: Capital Structure Problem

Inputs								
Firm Value Process:			**Int. Rate and State Prices:**			**Bond:**		
V	750		R	1.033		principal	500	
in tree 1=	1000000					coupon (%)	0.0643501	
			piu	0.4822		coupon ($)	32.175062	
sigma	0.26		pid	0.4859		final	532.17506	
maturity	2							
n	2		**Stock:**					
dt	1		# shares	20				
			div/shr	1				
u	1.2969		dividend	20.0				
d	0.7711							
Ordinary Bond and Stock Trees:							uuV	1193.853
		cum coupon, cum div			ex coupon ex div		bond	532.1751
		uV	972.6976		uV	920.5225	equity	661.6783
		bond	547.3494		bond	515.17431		
		equity	425.3482		equity	405.34819	udV	709.7703
V	750						bond	532.1751
bond	500						equity	177.5953
calc(ytm)	1.08E-13							
ytm	0.064							
equity	250						duV	682.3326
s price	12.50						bond	532.1751
		dV	578.2887		dV	526.11363	equity	150.1575
		bond	485.8823		bond	453.70719		
		equity	92.40643		equity	72.406433	ddV	405.6607
							bond	405.6607
							equity	0

I.e. the company should sell the par-500 bond for annual coupon rate 6.435%.

Figure 8: Capital Structure Problem(b)

Inputs								
Firm Value Process:			**Int. Rate and State Prices:**			**Bond:**		
V	750		R	1.033		principal	475	
in tree 1=	1000000					coupon (%)	0.0571577	
			piu	0.4822		coupon ($)	27.149887	
sigma	0.26		pid	0.4859		final	502.14989	
maturity	2							
n	2		**Stock:**					
dt	1		# shares	20				
			div/shr	1				
u	1.2969		dividend	20.0				
d	0.7711							
Ordinary Bond and Stock Trees:							uuV	1200.371
		cum coupon, cum div			ex coupon ex div		bond	502.1499
		uV	972.6976		uV	925.54768	equity	698.2207
		bond	513.2582		bond	486.10831		
		equity	459.4394		equity	439.43937	udV	713.645
V	750						bond	502.1499
bond	475						equity	211.4951
calc(ytm)	2.54E-12							
ytm	0.057							
equity	275						duV	688.8499
s price	13.75						bond	502.1499
		dV	578.2887		dV	531.1388	equity	186.7
		bond	468.2614		bond	441.11147		
		equity	110.0273		equity	90.027329	ddV	409.5354
							bond	409.5354
							equity	0

(b) With same analysis, we now set $E_0 = 275$, $F = 475$ and solve for c. For this one,

$$c^* = 5.7158\%$$

I.e. the company should sell the par-475 bond for annual coupon rate 5.7158%.

Figure 9: Capital Structure Problem(Putable Bond)

Inputs								
Firm Value Process:			Int. Rate and State Prices:			Bond:		
V	750		R	1.033		principal	500	
in tree 1=	1000000					coupon (%)	0.0431582	
			piu	0.4822		coupon ($)	21.579103	
sigma	0.26		pid	0.4859		final	521.5791	
maturity	2							
n	2		Stock:					
dt	1		# shares	20				
			div/shr	1				
u	1.2969		dividend	20.0				
d	0.7711							
Putable Bond and Stock Trees:							uuV	1207.596
put price	485						bond	521.5791
		uV	972.6976		uV	931.11846	equity	686.0164
		bond	526.4959		bond (live)	504.91685		
V	750	equity	446.2016		bond	504.91685	udV	717.9404
bond (live)	500				equity	426.20162	bond	521.5791
bond	500						equity	196.3613
calc	1.01E-09							
ytm	0.043158						duV	696.0748
equity	250						bond	521.5791
s price	12.5	dV	578.2887		dV	536.70959	equity	174.4957
		bond	506.5791		bond (live)	452.56721		
		equity	71.70959		bond	485	ddV	413.8308
					equity	51.709586	bond	413.8308
							equity	0

(c) With same analysis, we now set $E_0 = 250$, $F = 500$ and solve for c. For this one,

$$B_t = \begin{cases} \min(V_T, F(1+c)) & t = T \\ \max\{K, \quad \pi_u B_{t+1,u} + \pi_d B_{t+1,d}\} & 0 \le t < T \end{cases}$$

where K is the put price, $K = 485$.
We obtain

$$c^* = 4.3158\%$$

I.e. the company should sell the par-500 putable bond for annual coupon rate 4.3158%.

Problem. 31 (*Black Scholes Hedging*)

Solution. Denote Δ_t, B_t the stock, bond position in replicating portfolio. c_t the Black-Scholes option price, Π_t the value of replicating portfolio, ϵ_t the tracing error, and δt the hedging interval width. At time t, the incoming portfolio value:

$$\Pi_t = \Delta_{t-\delta t} S_t + B_{t-\delta t} e^{r\delta t}$$

$$h_t := h(t, S_t) = \frac{\log(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$\Delta_t = N(h_t); \qquad B_t = \Pi_t - S_t \Delta_t$$

$$c_t = S_t N(h_t) - K e^{-t(T-t)} N(h_t - \sigma(T-t))$$

where $N(z)$ is the standard normal cdf.

$$\epsilon_t = \Pi_t - c_t$$

The dynamic hedging process is presented in the table below.

Figure 10: Dynamic Delta Hedge

Inputs:			**Calculations:**			**Example of Delta Hedging:**				
S	35		h	0.1617		**Rebalancing once per day:**				
X	35		N(h)	0.5642		rep. port. liq. value		5.5963		
						actual c*		5.5010		
r	0.052		h-sig*sqrt	0.0117		diff		0.0953		
expiry T-t	0.25		N(h-sig*sqr	0.5047						
pv	0.9871									
monthly vol	0.086603									
(ann) sigma	0.3000									
Hedging Calculations with Daily Rebalancing:										
delta t		0.0038								
sigma in price sim		0.3000								
true mu		0.07								
true m in sim		0.025								
day	**z path**	**S path**	**Time Left**	**incoming port. value**	**new h(S,t)**	**outgoing # shrs**	**outgoing pv borrow**		**Black-Scholes Option Value**	**Tracking Error**
0		35	0.25	2.3128	0.1617	0.5642	-17.4348		2.3128	0.0000
1	0.1362	35.0922	0.2462	2.3613	0.1781	0.5707	-17.6649		2.3456	0.0157
2	-1.1415	34.3581	0.2423	1.9388	0.0338	0.5135	-15.7035		1.9281	0.0108
3	-0.5527	34.0098	0.2385	1.7569	-0.0380	0.4848	-14.7326		1.7352	0.0217
4	-0.5186	33.6865	0.2346	1.5972	-0.1066	0.4575	-13.8159		1.5639	0.0332
61	1.1956	39.3585	0.0154	4.4818	3.1942	0.9993	-34.8491		4.3868	0.0951
62	0.2192	39.5231	0.0115	4.6393	3.8062	0.9999	-34.8810		4.5441	0.0952
63	0.3867	39.8123	0.0077	4.9216	4.9246	1.0000	-34.8907		4.8263	0.0952
64	-0.6040	39.3712	0.0038	4.4735	6.3455	1.0000	-34.8977		4.3782	0.0953
65	1.5155	40.5010	0.0000	5.5963					5.5010	0.0953

In the final date T, we have

$$\begin{aligned} \Pi_T &= 5.5963 \\ c_T &= 5.5010 \\ \epsilon_T &= 0.0953 \end{aligned} \tag{17}$$